Springback Force Considerations in Compliant Haptic Interfaces

Dallin R. Swiss
Brigham Young University

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Springback Force Considerations in Compliant Haptic Interfaces

Dallin R. Swiss

A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

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ABSTRACT

Springback Force Considerations in Compliant Haptic Interfaces

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Master of Science

This thesis investigates the potential benefits and challenges of using compliant mechanisms in the design of haptic interfaces. The benefits and challenges are presented with an emphasis on their inherent springback behavior and an active compensation approach. Design criteria for compliant mechanism joint candidates are reviewed and several joints are surveyed. Quantitative calculations of axial stiffness and maximum stress for five candidates are presented. Generalized analytical models of springback force and compensation torque are created to simulate the implementation of each joint candidate in a two degree-of-freedom planar pantograph. We use these models in the development and discussion of an analytical approach to predict the motor torques needed to actively compensate for the effects of springback.

This approach relies on virtual work analyses of the haptic pantograph to determine the springback forces, compensation torques, haptic workspace, and available haptic force after compensation. A key to estimating the available haptic force is knowing that the force capability is different depending on the local springback force. If a component of the desired haptic force aligns with the springback force, then the two can work together, thus increasing the maximum magnitude of available haptic force beyond the nominal amount. Analytical and experimental results are presented. A detailed method of implementation is given along with a hardware demonstration of active compensation.

Keywords: haptics and haptic interfaces, compliant joint/mechanism, kinematics
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### NOMENCLATURE

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<tr>
<td>A, B, C, or D</td>
<td>Nested coefficients used in describing the results of virtual work analyses</td>
</tr>
<tr>
<td>b or w</td>
<td>Flexure width</td>
</tr>
<tr>
<td>E</td>
<td>Young’s modulus</td>
</tr>
<tr>
<td>G</td>
<td>Shear modulus</td>
</tr>
<tr>
<td>g</td>
<td>Gap across the center of the open-cross joint</td>
</tr>
<tr>
<td>F or F</td>
<td>Force</td>
</tr>
<tr>
<td>Hf</td>
<td>Distance from the center to the far connection of thin beams in butterfly joint</td>
</tr>
<tr>
<td>hf</td>
<td>Distance from the center to the close connection of thin beams in butterfly joint</td>
</tr>
<tr>
<td>h or t</td>
<td>Flexure thickness</td>
</tr>
<tr>
<td>I</td>
<td>Area moment of inertia</td>
</tr>
<tr>
<td>i</td>
<td>Electrical current</td>
</tr>
<tr>
<td>J</td>
<td>Jacobian matrix</td>
</tr>
<tr>
<td>K or Q</td>
<td>Geometry based constant</td>
</tr>
<tr>
<td>k or k</td>
<td>Axial stiffness or torsional stiffness</td>
</tr>
<tr>
<td>L or l</td>
<td>Flexure length</td>
</tr>
<tr>
<td>M, T, or τ</td>
<td>Moment or torque</td>
</tr>
<tr>
<td>n</td>
<td>Number or quantity</td>
</tr>
<tr>
<td>Pi, P, or Z</td>
<td>Position or position vector of the i-th joint</td>
</tr>
<tr>
<td>q</td>
<td>Generalized coordinate for virtual work analysis</td>
</tr>
<tr>
<td>R</td>
<td>Radius</td>
</tr>
<tr>
<td>Sy</td>
<td>Yield strength of a given material</td>
</tr>
<tr>
<td>V</td>
<td>Potential energy terms</td>
</tr>
<tr>
<td>δW</td>
<td>Virtual work</td>
</tr>
<tr>
<td>R</td>
<td>Link lengths</td>
</tr>
<tr>
<td>α or β</td>
<td>Inner angles of triangle</td>
</tr>
<tr>
<td>δ or Δ</td>
<td>Change in value of referenced variable</td>
</tr>
<tr>
<td>φ</td>
<td>Half the angle between two vertical sides of the butterfly flexure triangle</td>
</tr>
<tr>
<td>ν</td>
<td>Poisson’s ratio</td>
</tr>
<tr>
<td>σmax or τmax</td>
<td>Maximum stress in the joint</td>
</tr>
<tr>
<td>θ or Θ</td>
<td>Angular deflection</td>
</tr>
<tr>
<td>∞</td>
<td>Infinity</td>
</tr>
<tr>
<td>π</td>
<td>Pi = 3.1415926...</td>
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### Abbreviations

<table>
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<tr>
<th>Abbreviation</th>
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<tr>
<td>ABS</td>
<td>Acrylonitrile butadiene styrene</td>
</tr>
<tr>
<td>ADLIF</td>
<td>Anti-symmetric double leaf-type isosceles-trapezoidal flexure</td>
</tr>
<tr>
<td>AHF</td>
<td>Available haptic force</td>
</tr>
<tr>
<td>ASTM</td>
<td>American Society for Testing and Materials</td>
</tr>
<tr>
<td>CM</td>
<td>Compliant mechanism</td>
</tr>
<tr>
<td>CNC</td>
<td>Computer numeric control</td>
</tr>
<tr>
<td>CORE</td>
<td>Compliant rolling-contact element</td>
</tr>
<tr>
<td>CR</td>
<td>Center-Moment compliant revolute</td>
</tr>
<tr>
<td>cRIO</td>
<td>CompactRIO or compact reconfigurable input output</td>
</tr>
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DC
FDM
FPGA
I/O
NI
OCR
Q-LIFT
RT
SLFP
VI

Subscripts, superscripts, and other indicators

\[(t)\] indicates \[\] is a function of time, in the \(t\) domain

\[(\theta)\] indicates \[\] is a function of angle, in the \(\theta\) domain

\[0\] indicates \[\] is evaluated at time \(t\) equal to zero

\[c\] indicates \[\] is referring to compensation forces

\[F\] indicates \[\] is related to forces

\[h\] indicates \[\] is referring to haptic forces and torques

\[h\] indicates \[\] is referring to half-way location in forward kinematic analysis

\[i\] or \[\#\] indicates \[\] is referring to components of the corresponding number

\[i \rightarrow j\] indicates \[\] is directional reference from location \(i\) to location \(j\)

\[l\] indicates \[\] is referring to properties of the flexible portion

\[M\] indicates \[\] is related to moments

\[pl\] indicates \[\] is referring to the planets of CORE bearing

\[r\] indicates \[\] is referring to required forces

\[s\] indicates \[\] is referring to springback forces

\[v\] indicates \[\] is related to potential energy sources

\[sn\] indicates \[\] is referring to the sun of CORE bearing

\[rng\] indicates \[\] is referring to the outer ring of CORE bearing

\[min\] indicates \[\] is the minimum value of the referenced variable

\[max\] indicates \[\] is the maximum value of the referenced variable

\[ref\] indicates \[\] is a reference value

\[x\] or \[y\] indicates \[\] is referring to properties in the \(x\) or \(y\) direction

\[\times\] indicates \[\] is raised to the \(x\) power

\[-T\] indicates \[\] is the inverse transpose

\[\prime\] indicates \[\] is an equivalent alternative or primed coefficient

\[\|\|\] indicates \[\] is the norm of referenced vector
CHAPTER 1. INTRODUCTION

1.1 Motivation

Modern technology has allowed us to have virtual experiences with stunning visuals and audio effects, but only recently have we seen touch and force incorporated into these experiences. This is the field of haptics, which comes from the Greek word απτο [ap-tow], meaning "touch." Haptic interfaces are robotic force-feedback devices that “allow users to ‘feel’ virtual objects in a simulated environment” [1]. These devices track the motion of a user and project it into a virtual or remote environment. A computer then calculates the interaction of the motion with the environment determining the response (e.g., the normal forces from pressing on a virtual wall, the tactile sensation of friction along a virtual surface, etc.). These forces and sensations can be communicated back to the user through the device in near real-time, creating the illusion that the user is directly connected with the virtual world. Haptic technologies have the potential to significantly expand and improve the way we communicate, convey, and interact with information and machines. The areas affected by haptics are vast and growing. This includes aerospace [2], assistive technology [3], [4], education/entertainment [5]–[7], gaming [8], manufacturing/assembly [9], medical simulation/training [10], rehabilitation [11], scientific visualization [12], surgical robotics [13], [14], telerobotics [15], [16], and more.

For example, researchers at the Italian Center for Aerospace Research Virtual Reality Laboratory have been using computer-based training systems, combined with virtual reality techniques and haptic interaction to simulate machine assembly and maintenance designed for the aerospace industry. Their study utilized a virtual maintenance environment and a hand-based haptic device [2]. A more recent study in 2014, aimed at bridging the gap between robotic technology and health care, concluded that robots and haptic feedback can provide great advantages to health care in areas such as robotic assisted surgery, rehabilitation, prosthetics, and companion robotic
systems. However, the current cost of equipment, maintenance, and supplies limits their use to a few large institutions and research centers throughout the world [3].

Haptic interfaces are designed to improve task performance, expand capabilities, enhance communication, and make an experience more enjoyable and satisfying [17]. However, to make haptic technologies more widespread (including wearable and portable applications) and to realize their full potential, considerable improvements must be made in the performance and cost of haptic interfaces. For example, a study in 2010 suggested that the added cost of haptics-enabled training tools, specifically the Simbionix LapMentor II for laparoscopic surgery, did not add enough value to justify them over non-haptic VR simulators [18]. By making improvements that reduce cost and improve performance, such as those mentioned in [2], [19], tight-budget organizations and designer/developers may be more inclined to invent and incorporate haptics into their applications.

In many mechanical systems in which performance, cost, and weight are important factors, designers incorporate flexible compliant mechanisms (CMs) into designs. CMs are mechanical devices that gain their motion through flexibility rather than through traditional joints, such as pin hinges and bearings. Some of the common CMs are the small-length flexural pivot, living hinge, cross-axis flexural pivot, and others. CMs have been used with high success in many fields due to various factors, including decreased part count, assembly time, weight, friction, and wear; increased performance and reliability; and potential for miniaturization [20]. It is important to note that in the design and control of haptic devices, having an accurate and stable physical model is critical. Without accuracy, the experience feels unrealistic and the device will not provide a transparent virtual interface. Without stability, the hardware and control effort will fail to properly render the environment. Instability leads to singularities and infinite virtual forces, which should not be present in the haptic experience.

Developing an accurate and stable model requires both science and judgment. Designers must balance between the intended rendering and the capabilities of the hardware. Care must be taken to select an appropriate fidelity of the physical model while not wasting effort on insignificant factors. Part of our work contributes to improving the accuracy of the models used to render haptic environments by incorporating compliant mechanisms. Compliant haptic interfaces partake of the aforementioned benefits, including the significant reduction in friction. Eliminating friction provides an important advantage since some aspects of friction are difficult to model. While there
are simple models to account for friction, small nonlinearities can lead to instabilities and errors in the haptic rendering. By significantly reducing friction, designs become simpler, and accuracy and realism of haptic interfaces increase [21].

One obstacle in utilizing compliant mechanisms in haptic devices is their inherent tendency to experience mechanical energy storage and springback behavior. Because of their structural flexibility, springback is felt as the mechanism tries to return to its undeflected position and orientation. In haptic devices, as a person moves the interface, springback forces interfere with the haptic forces related to the feel of the virtual environment. Although statically balanced CMs have been suggested in other works as a way to nullify springback in haptic interfaces [22]–[26], the focus of this thesis is on understanding springback, comparing compliant revolute joints for haptic interfaces, and obtaining models of their behavior that would be suitable for use in active springback compensation. Specifically, we consider and compare alternative compliant revolute joints as replacements for traditional rotary bearings in a 2 degree-of-freedom five-bar planar mechanism. This device is referred to as a pantograph, and our research strives to model, verify, and demonstrate the effects of incorporating compliant mechanisms in the design of this haptic interface.

1.2 Related Research

While there is much literature exploring the development, implementation, and success of various compliant mechanisms, little research has been done that explores the potential benefits of incorporating CMs into haptic designs. In 2005, Trease et al. [27] presented a qualitative survey of compliant joints, both translational and revolute, for use in general mechanical systems (not limited to haptic interfaces). They mentioned that the small clearances between mating rigid parts lead to backlash in assemblies. Small relative motions between parts cause friction and wear, which degrade accuracy and reliability. They recommended CMs with their flexible joints, also know as flexures, as alternatives for traditional joints, to link the rigid portions together. They suggested five criteria by which joints could be evaluated, all of which are dependent on material properties and geometry. They are:
• Range of Motion. The range through which a compliant joint can deflect or rotate without exceeding its yield stress or joint limits. When the yield stress is reached, the deformations will become plastic and irreversible.

• Axial Drift. A measure of the extent to which the compliant joint follows the path of translation or rotation of its traditional mechanical counterpart. For revolute joints, this measures the off-axis translation of the axis of rotation for a given deflection. Minimizing and eliminating drift can be accomplished through symmetry in the joint. Axial drift is particularly important for the accuracy of forward and inverse kinematics calculations.

• Off-axis Stiffness. The ratio of desired stiffness in one direction to undesired stiffness in another. For a revolute joint, this is a comparison of the off-axis rotational and/or translational stiffness to the axial stiffness. The axial stiffness, or torsional stiffness, is the stiffness in the direction of intended motion. High off-axis stiffness is usually a desirable characteristic.

• Stress Concentration. CMs often involve transitions from thin inner members to grounded sections. Depending on the geometry, stress concentrations can arise in these regions which reduce the fatigue life of the joint. These can be minimized by using fillets between cross-sectional regions and fixing surface cracks before they reach their critical length.

• Compactness. This refers to both the cross-sectional area occupied in the plane of motion and the total volume of the joint.

Using their criteria for optimal joint candidates, Trease et al. created the end-moment compliant revolute joint and center-moment compliant revolute (CR) joint, which were precursors to the open-cross revolute (OCR) joint. In total, they evaluated five translational joints and eleven rotational joints. The majority of their survey provided qualitative assessments, allowing them to compare joints without respect to scale. Direct quantitative comparisons are possible and they expressed an intent to pursue them in the future, but acknowledged that they are less generalizable. Nonetheless, they provided a quantitative comparison of three compliant revolute joints under various equivalent conditions (i.e., equivalent longest length, volume, range of motion, and torsional stiffness) looking at torsional stiffness and range of motion. The joints they compared were the split-tube joint, free-flex joint (cross-pivot), and CR joint. In all four cases, the split-tube joint had
greater range of motion than the CR joint, while the CR joint performed better in torsional stiffness. The free-flex joint often had best performance in both categories but at the cost of compactness and axial drift. A more detailed discussion of the other rotational joints reviewed in their survey will be presented as part of Chapter 2.

Gillespie et al. developed the cTouch, which is a compliant five-bar mechanism for rendering haptic environments in two dimensions [28]. Their research was motivated by the potential advantages of incorporating compliant joints into haptic devices. They emphasized that low-end haptic devices, such as those used in education and gaming, are cost inhibited because of the price of high-precision components such as bearings. High-end haptic devices also suffer from the presence of coulomb friction forces, burdening the physical model of the device [29]. In place of traditional joints, the researchers designed a CM referred to as an OCR joint. Thus they replaced the traditional mechanism with a monolithic design with each joint integrated into the total structure. The OCR joint, a large-deflection hinge, is primarily characterized by its open-cross design. In comparison to a traditional bearing, their joint would not have the same relatively infinite out-of-plane stiffness and range of motion, but the OCR joint design offered a greater range of motion and higher off-axis stiffness than many alternative flexure joints. Their prototype mechanism was manufactured in acrylonitrile butadiene styrene (ABS) using fused deposition modeling (FDM).

They presented individual values for the analytical, finite-element analysis predicted, and experimentally determined joint stiffnesses. There was some degree of hysteresis and inconsistency in joint stiffness predictions, which they attributed to known variations of stiffnesses of FDM parts made of ABS. Due to the disagreement of their analytical stiffness predictions to experimental results, which ranged between 0.9-115% higher than predicted, they developed an approach to experimentally characterize and compensate for springback in the joints which they called self-characterization. This was accomplished using readings on the encoders for measuring displacement and the current-controlled amplifier to set the electrical current. They implemented closed-loop position control to move the interface point to various positions throughout the mechanism’s workspace. Once the position became steady, the current $i(\theta)$ was recorded. The commanded current needed to hold the mechanism in place effectively compensated the joint restoring torques. The recorded values populated a look-up table for use in creating a blank design space for haptic renderings. They also created a simplified analytical model for the torque-position compensation.
relationship based on the experimentally determined stiffness values, but this was used strictly for comparison of their results using self-characterization. Their self-characterized springback force values were about 33% higher than their analytical predictions, but achieved a 95% compensation effect.

Gillespie et al. discussed the magnitude of available haptic force (AHF) from a compensated system stating the maximum occurred at the center where the mechanism is undeflected. They emphasized that AHF diminishes as the interface point gets closer to the boundaries of the workspace [28]. This interpretation assumes that springback must continually be compensated. However, in this thesis we will show that if a component of the desired haptic force aligns with the springback force, the two can work together, thus increasing the magnitude of AHF beyond the nominal amount in that direction. The work presented in this thesis continues their effort to explore the nuances of compliant haptic interfaces and clarifies some of the concepts they first introduced.

1.3 Overview of Thesis

In Chapter 1 we introduced the background, related research, and motivation for our work. In Chapter 2 we will revisit Trease’s survey and compare a selection of other joints against haptics-centric criteria. In addition to the joints considered in their survey, several others have been published by various authors since 2005 [30]–[38]. In Chapter 3 we will discuss the development and validation of an analytical model for the springback behavior of a compliant pantograph. In Chapter 4 we will discuss a broader analysis of the AHF and a hardware demonstration of active compensation, where springback is allowed to aid in the haptic rendering. In Chapter 5 we will review and conclude the scope and contributions of this thesis.

In this thesis, we survey and compare several compliant joints. While the criteria introduced by Trease can be quantified for each candidate, this is not the purpose of our analysis. Calculations of some joint characteristics that are related to the criteria, such as axial stiffness and maximum stress, are presented. We use these values in the development and discussion of an analytical approach to predict the motor torques needed to actively compensate for the effects of springback. This approach is based on geometry and joint stiffnesses and relies on virtual work analyses of the haptic pantograph to determine the springback forces, compensation torques, as well as the haptic workspace and AHF after compensation.
CHAPTER 2. COMPARISON OF COMPLIANT REVOLUTE JOINTS

2.1 Joint Candidates

Compliant joints vary in many ways; including shape, complexity, and functionality. For our compliant haptic interface, we examine several rotational joints seeking to determine which will perform well in the five criteria outlined by Trease et al. in their 2005 survey [27], which consisted of range of motion, axial drift, off-axis stiffness, stress concentrations, and compactness. We examine the following joints: (a) small-length flexural pivot (SLFP) [39]; (b) living hinges, (c) fixed-pinned, (d) fixed-fixed, (e) initially curved fixed-pinned [20]; (f) curved-beam flexural pivot [40]; (g) curved-compliant annulus-shaped flexure pivot [33], [41]; (h) compliant rolling-contact element (CORE) joint, (i) CORE bearing, (j) compliant contact-aided revolute joint [30], [31]; (k) cross-axis flexural pivot [42]; (l) cartwheel flexure; (m) tubular cross-axis flexural pivot; (n) double blade rotary pivot [38]; (o) torsional joint with rectangular CS; (p) center moment compliant revolute joint, (q) end moment compliant revolute joint [27]; (r) open-cross revolute (OCR) joint [28]; (s) quadra blade rotary joint [38]; (t) split tube flexure [43]; (u) ADLIF: a large-displacement beam-based flexure; (v) butterfly flexure [34], [37]. Following the qualitative fashion of their survey, Table 2.1 presents our evaluation of each joint as having either poor (-), normal (0), or good (+) performance in each criteria. The benchmarks of what is poor, normal, or good are based on engineering judgment after having reviewed the source literature on all the joints. A qualitative evaluation allows for comparison of inherent form without respect to scale. In addition to Trease’s criteria, we evaluate the joints for simplicity and ease of manufacturing. Including these additional categories adds a dimension of practicality to the total score of each joint. For example, while the SLFP scores poorly in the majority of Trease’s criteria, it scores high in simplicity and ease of manufacturing. Top performing candidates include: (g), (i), (j), and (t).

Qualitative surveys allow for general comparison, but they lack some of the detail and clarity of a quantitative study. As part of a quantitative comparison, we narrow down our list to five
Qualitative evaluation of joint candidates. The first five criteria were outlined in the 2005 survey by Trease et al. [27]. The scores for the SLFP, cross-axis flexural pivot, split-tube flexure, and OCR joint in the first five categories were given in their survey, but the rest of the candidates were not. A qualitative evaluation allows for comparison of inherent form without respect to scale. Five joints are marked bold for later reference and discussion. The evaluation legend
(-: poor, 0: normal, +: good).

<table>
<thead>
<tr>
<th>Joint</th>
<th>Range of Motion</th>
<th>Axial Drift</th>
<th>Stress Concentration</th>
<th>Off-axis Stiffness</th>
<th>Compactness</th>
<th>Simplicity</th>
<th>Ease of Manufacturing</th>
<th>Total Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) SLFP</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td></td>
<td>-1</td>
</tr>
<tr>
<td>(b) living hinges</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>(c) fixed-pinned</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>(d) fixed-fixed</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+</td>
<td></td>
<td>-1</td>
</tr>
<tr>
<td>(e) initially curved fixed-pinned</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+</td>
<td></td>
<td>-2</td>
</tr>
<tr>
<td>(f) curved-beam flexural pivot</td>
<td>0</td>
<td>0</td>
<td>+</td>
<td>-</td>
<td>0</td>
<td>-</td>
<td></td>
<td>-1</td>
</tr>
<tr>
<td>(g) curved-compliant annulus-shaped flexure pivot</td>
<td>+</td>
<td>+</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td>+3</td>
</tr>
<tr>
<td>(h) CORE joint</td>
<td>+</td>
<td>-</td>
<td>0</td>
<td>+</td>
<td>0</td>
<td>0</td>
<td></td>
<td>+1</td>
</tr>
<tr>
<td>(i) CORE bearing</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>0</td>
<td></td>
<td>+3</td>
</tr>
<tr>
<td>(j) compliant contact-aided revolute joint</td>
<td>+</td>
<td>+</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td>+3</td>
</tr>
<tr>
<td>(l) cross-axis flexural pivot</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>0</td>
<td>-</td>
<td></td>
<td>-2</td>
</tr>
<tr>
<td>(k) cartwheel flexure</td>
<td>+</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td></td>
<td>-1</td>
</tr>
<tr>
<td>(m) tubular cross-axis flexural pivot</td>
<td>+</td>
<td>+</td>
<td>0</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>(n) double blade rotary pivot</td>
<td>+</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td></td>
<td>-1</td>
</tr>
<tr>
<td>(o) torsional joint with rectangular cross-section</td>
<td>-</td>
<td>+</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td>-1</td>
</tr>
<tr>
<td>(p) center moment compliant revolute joints</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td>+2</td>
</tr>
<tr>
<td>(q) end moment compliant revolute joints</td>
<td>+</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>(r) OCR joint</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>0</td>
<td>-</td>
<td></td>
<td>+2</td>
</tr>
<tr>
<td>(s) quadra blade rotary joint</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>0</td>
<td>-</td>
<td></td>
<td>-1</td>
</tr>
<tr>
<td>(t) split tube flexure</td>
<td>+</td>
<td>0</td>
<td>+</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td>+4</td>
</tr>
<tr>
<td>(u) ADLIF: a large-displacement beam-based flexure</td>
<td>+</td>
<td>0</td>
<td>0</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>(v) butterfly flexure</td>
<td>+</td>
<td>0</td>
<td>0</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>
candidates shown in Fig. 2.1: (a), (i), (r), (t), and (v). For each candidate joint, the blue, yellow,

Figure 2.1: Joint candidates: (A) SLFP, (B) split-tube flexure, (C) OCR joint, (D) butterfly flexural pivot, and (E) CORE bearing. The blue, yellow, and gray regions represent the fixed, flexible, and free portions respectively.

and gray regions represent the fixed, flexible, and free portions respectively. Each is chosen not exclusively by their score but for specific reasons, which we will discuss. We do not include candidates (g) or (j) in our narrowed selection. Although they have high scores, models for the stiffness and stress of (g) are not readily available without significant development effort, and (j) has many similarities to (i) but lacks (i)’s advantages of symmetry. Also, the contact hinge in the center of (j) could allow for off-axis translation, a property undesirable for our use. The five candidates and the reasons for their selection are provided below.

- (A) SLFP. In the field of CMs, it is among the first and simplest of compliant joints. This joint provides a baseline for comparison of the other joints. While its total score is below that of some joints, it is the easiest to model and manufacture.
• (B) Split-tube flexure. This joint also provides a baseline for comparison of the other joints. While its performance is rated only “normal” in some categories, its total score is highest overall, and is considered “a well-behaved revolute flexure” [43].

• (C) OCR Joint. The results obtained in [28] by Gillespie et al. were based on a mechanism with OCR joints. The joint performs well in the qualitative assessment and allows us to verify our model’s results against their data.

• (D) Butterfly flexural pivot. This exotic flexure is included because of its different and unique design. It is a more sophisticated descendant of the cartwheel hinge and cross-axis flexural pivots.

• (E) CORE Bearing. By combining several CORE joints, the CORE bearing exhibits a special behavior only seen in the CORE family of joints. The restoring torque is the same for any given angular deflection. This property will be explored further in our discussion of the quantitative results.

Analytical equations for the axial stiffness and maximum stress of each joint candidate are presented from their respective source literature. In all instances, $G$ refers to the shear modulus and $E$ is the modulus of elasticity. $I$ refers to the area moment of inertia for the cross section of interest. $M$ refers to the maximum moment in the joint for a given angular deflection $\theta$, and is calculated by $M = k(\theta - \theta_0)$, where $\theta_0$ is the initial angle and $k$ is the torsional stiffness. These equations are used to calculate the performance of each candidate in some of the metrics presented above, as well as in the design criteria for haptic interfaces, namely relative workspace size in Chapter 2 and AHF after compensation in Chapter 4.

2.1.1 Small Length Flexural Pivot

The SLFP [39] is a simple notch flexure with a thin short filleted section that connects two rigid links to each other or one rigid link to ground. The joint can be seen in Fig. 2.2. Its axial stiffness is calculated from

$$k = \frac{(EI)I}{l},$$

(2.1)
where \( I = \frac{bh^3}{12} \). Referring to the small flexible region, \( b \) is the width, \( h \) is the thickness, and \( l \) is the length. The maximum stress in the joint is

\[
\sigma_{\text{max}} = \frac{(Mh)_t}{2I}.
\]

### 2.1.2 Split-tube Flexure

The split-tube flexure [43] is a well-behaved revolute flexure intended for large displacements. The joint can be seen in Fig. 2.3. It is composed of a hollow thin-walled tube with a slit along its longitudinal axis. Links are attached directly to the outer walls. Its axial stiffness is cal-
culated from
\[ k = \frac{2\pi GRt^3}{3L}, \quad (2.3) \]
where \( R \) is the radius, \( t \) is the wall thickness, and \( L \) is the length. The maximum stress in the joint is
\[ \tau_{\text{max}} = \frac{3M}{2\pi R t^2}. \quad (2.4) \]

### 2.1.3 Open-Cross Revolute Joint

The open-cross revolute joint, a large-deflection hinge, is primarily characterized by its open-cross design. It was developed by Gillespie et al. and presented in [28, Fig. 4]. The joint can be seen in Fig. 2.4. Eight beams provide range of motion through their torsion and bending. By having an open-cross, rather than a closed cruciform cross-section, stress concentrations are eliminated along the length of the beams and axial stiffness is significantly reduced without greatly affecting off-axis stiffness. However, the unique geometry of the flexure makes its physical manufacture more challenging. Its stiffness is calculated from
\[ k = \frac{24EI(w + g)^2}{L^3} + \frac{8GK}{L}, \quad (2.5) \]
where $I = wt^3/12$, and, referring to the eight beams, $w$ is the width; $t$ is the thickness, which is typically smaller than $w$; $g$ is the gap across the center of the cross; $L$ is the length;

$$K = \frac{wt^3}{16} \left( \frac{16}{3} - \frac{3.36t}{w} \left( 1 - \frac{t^4}{12w^4} \right) \right)$$  \hspace{1cm} (2.6)

is a geometry-based constant. The maximum stress in the joint is

$$\sigma_{\text{max}} = \frac{1.733\theta}{L^2Q} \sqrt{2.25(ERt)^2(w+g)^2 + 3(KGL)^2},$$  \hspace{1cm} (2.7)

where $Q = w^2t^2/(3w + 1.8t)$.

### 2.1.4 Butterfly Flexural Pivot

The butterfly flexural pivot [34], [37] is an exotic quadri-leaf-type isosceles-trapezoidal flexural pivot or Q-LITF. The joint can be seen in Fig. 2.5. The joint has four LITF pivots, two on top, and two on bottom. It is a special derivative of the cartwheel flexure designed for high-

![Figure 2.5: Schematic of butterfly flexural pivot and relevant dimensions.](image-url)
compression large-deflection situations. Its stiffness is calculated from

\[ k = \frac{K_1K_2}{2(K_1 + K_2)}, \quad (2.8) \]

where

\[ K_i = \frac{8EI_i(H_i^2 + H_i h_{if} + h_{if}^2)\cos(\phi_i)}{(H_i - h_{if})^3} \quad (2.9) \]

is a geometry-based constant. The index \( i \) refers to either the upper or lower halves of the butterfly. \( H_i \) is the distance from the center of the joint to the far connection of the inner thin beams, \( h_{if} \) is the distance from the center of the joint to the close connection of the inner thin beams, \( t \) is the thickness, \( b \) is the width, and \( \phi \) is half the angle between the two vertical sides of the triangle. The maximum stress in the joint is

\[ \sigma_{i,max} = \frac{E t_i (2H_i + h_{if}) \cos(\phi_i) \theta_i}{(H_i - h_{if})^2}, \quad (2.10) \]

where \( \theta_i = M_i/K_i \).

### 2.1.5 CORE Bearing

The CORE bearing [30], [31] resembles a planetary gear chain with a sun, outer ring, and planets of equal radius. The joint can be seen in Fig. 2.6. Each planet is connected to the sun and outer ring by thin flexures. \( R_{pl}, R_{sn}, \) and \( R_{rng} \) are the radii of the planets, sun, and outer ring respectively. When deflected, the equivalent radius for each body is used to calculate the torque of the ring on a planet. For the configuration shown in Fig. 2.1.E, the equivalent radii are

\[ R'_{1,pl} = R'_{2,pl} = \infty, \quad (2.11) \]

\[ R'_{1,sn} = \left( \frac{1}{R_{sn}} + \frac{1}{R_{pl}} \right)^{-1}, \quad (2.12) \]

and

\[ R'_{2,rng} = \left( \frac{1}{R_{pl}} + \frac{1}{R_{rng}} \right)^{-1}. \quad (2.13) \]
The indices, 1 and 2, refer respectively to the inner and outer thin flexures. The torque exerted on a planet when the bearing is deflected is

\[ T_{pl} = EI_1 \left( \frac{1}{R'_{1pl}} + \frac{1}{R'_{1sn}} \right) + EI_2 \left( \frac{1}{R'_{2pl}} + \frac{1}{R'_{2rng}} \right). \]  

(2.14)

This is used in calculating the total torque exerted on the sun,

\[ T_{out} = n_{pl} T_{pl} \frac{R_{sn}(R_{sn} + 2R_{pl})}{R_{pl}(2R_{sn} + 2R_{pl})}. \]  

(2.15)

where \( n_{pl} \) is the number of planets in the bearing. \( T_{out} \) is used in representing the axial stiffness of the bearing,

\[ k = \frac{T_{out}}{\delta \theta_i}. \]  

(2.16)

The maximum stress in the joint is

\[ \sigma_{max} = \frac{M h_i}{2I}, \]  

(2.17)

where \( I = bh_i^3/12 \). Referring to the thin connecting flexures, \( h_i \) is the width, and \( b \) is the thickness of the joint out of the plane.
COREs are unique in that the restoring torque is the same for any given angular deflection. In Chapter 3, we will discuss the effect of constant restoring torque on springback force. More immediately though, in this chapter we will see that constant torque equates to constant stress, enabling the range of motion to be exceptionally large. So long as the stress levels in the joints are set below the yield stress of the material, the range of motion for such a mechanism is bounded only by its mechanical stops. To model this behavior, the stiffness in the bearing decreases by the inverse of the deflection as seen in eq. (2.16).

2.2 Pantographs

The compliant pantograph modeled throughout the remainder of this thesis follows the configuration of the Pantograph Mk II [44] and cTouch [28], but uses the design parameters of the latter (see Appendix A for details). There are at least two pantograph designs types: the traditional pantograph and the half-pantograph. The two are often referred to by the common term “pantograph,” but actually a traditional pantograph contains two loops, while a half-pantograph has only one. In this thesis, when we refer to the term “pantograph,” technically we are discussing a half-pantograph.

2.3 Forward and Inverse Kinematics of the Haptic Pantograph

The design and analysis of a compliant pantograph mechanism is very similar to that of a pantograph with traditional bearings, however the compliant pantograph experiences springback in its joints. We will focus on the aspects that are common to both types, while the differences between them will be addressed later in Chapter 3. Both can be represented as having five links, including the ground link, with five joints. Two of these joints are fixed to ground and we refer to these as the base joints 2 and 4. The remaining three are referred to as the distal joints 3, 5, and \( P_3 \), with the middle joint \( P_3 \) being the interface point that the user would grip. The mechanism has two degrees-of-freedom, with actuators at each of the base joints to provide two independent input torques. A handle or knob is connected to the interface point, and by moving it around, the user can interact with the virtual environment. Our haptic design is impedance based rather than admittance based, meaning that it tracks position as opposed to tracking force; reading the position of \( P_3 \) into
a haptic rendering algorithm. A computer then calculates the interaction of the motion with the environment determining the response forces to communicate back to the user.

The first component of our active compensation strategy is the position relationships that result from forward and inverse kinematics analyses. Solutions to the \textit{forward kinematics} problem entail finding the location of the interface point $P_3$, when the link lengths $R$, and base angles $\theta_2$ and $\theta_4$ are known. Solutions to the \textit{inverse kinematics} problem involve finding the base angles $\theta_2$ and $\theta_4$, when the link lengths $R$, and interface point $P_3$ are known. Schematics depicting these parameters can be seen in Fig. 2.7 and Fig. 2.8. The full derivation and resulting equations of the

![Figure 2.7: Forward kinematics. The interface point, $P_3$, is determined from the link lengths and base angles, $\theta_2$ and $\theta_4$, at the intersection of the two circles.](image1)

![Figure 2.8: Inverse kinematics. The base angles, $\theta_2$ and $\theta_4$, are determined from the link lengths, interior angles, and interface point, $P_3$.](image2)
forward and inverse kinematics relationships are omitted here for brevity but can be found in both Appendix B and [44]. It should be noted that while axial drift, as described earlier in Chapter 1, will cause small deviations in the kinematic relationships, they are ignored for our current interests. Thus, axial drift is not included our current models.

2.4 Haptic Workspace

The total haptic workspace, a region for which the stress in any joint is less than the yield stress for the material, is a useful criteria in evaluating the expected performance of the design. It defines the area available for rendering virtual environments without incurring damage to the hardware. The workspace size is calculated by applying the forward and inverse kinematic relationships to determine each joint’s deflection and stress for a given location of the interface point. The joint stresses are compared to the material’s yield stress, revealing the boundary of the workspace for the mechanism. To compare the performance of each candidate, their differences will be normalized by using the axial stiffness as a fixed control. The geometry of each joint has been optimized so that the base and distal joints have consistent axial stiffness values of 2.864 and 1.25 mN-m/deg respectively (see Appendix A for details). It is acknowledged that there is more than one solution of geometric parameters to achieve these stiffness values. In our case, the objective is to use constrained optimization, specifically MATLAB’s fmincon() function, to minimize the stress with respect to angular deflection while constraining the axial stiffness to the values mentioned above. Unfortunately, the solution set for the CORE bearing that set its axial stiffness to match the other candidates make it impractical for actual use. However, the results are still informative.

Fig. 2.9 outlines sequentially the resulting allowable workspace area for each joint. The order of workspace size from greatest to least is: (1) CORE bearing, (2) SLFP, (3) butterfly flexural pivot, (4) split-tube flexure, and (5) OCR joint. The sizes are also provided in detail in Table 2.2. The workspace lines are determined by identifying the locations where the maximum stress in at least one of the joints exceeds the yield stress for the material, which for our simulations is ABS (See Appendix A for details). Due to the constant moment condition, the CORE bearing is limited only by its joint limits. Its workspace area extends beyond the domain boundaries for the plot, thus no bounding lines are shown. However, this information alone does not determine which joint candidate will be best suited for haptic interfaces. It is important to consider the intended
Figure 2.9: Available haptic workspace for each candidate. The workspace lines are determined by identifying the locations where the maximum stress in at least one of the joints exceeds the yield stress for the material, ABS. The order of workspace size from greatest to least: (1) CORE bearing, (2) SLFP, (3) butterfly flexural pivot, (4) split-tube flexure, and (5) OCR joint. It should be noted that for the given dimensions, outline (1) does not have a workspace limit based on yield stress, thus no bounding line is shown.

<table>
<thead>
<tr>
<th>Joint</th>
<th>Workspace ($\text{mm}^2$)</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. CORE bearing</td>
<td>N/A</td>
<td>The CORE bearing is not limited by yield stress</td>
</tr>
<tr>
<td>2. SLFP</td>
<td>6541.3</td>
<td></td>
</tr>
<tr>
<td>3. Butterfly flexural pivot</td>
<td>3707.3</td>
<td>Has a slightly different shape than the others. The top and bottom regions are more restricted.</td>
</tr>
<tr>
<td>4. Split-tube flexure</td>
<td>3387.3</td>
<td></td>
</tr>
<tr>
<td>5. OCR joint</td>
<td>2657.5</td>
<td></td>
</tr>
</tbody>
</table>

application and performance in all of the metrics suggested by Trease et al. [27]. While a candidate may perform well in one aspect, it does not guarantee best performance in all categories.

It is somewhat surprising to see the OCR joint having the smallest area, when according to the qualitative evaluation it had better range of motion than many others. This of course re-enforces the remarks stated earlier about the differences between qualitative and quantitative comparisons. The qualitative assessments are still valid, because they gauge general capabilities.
and design potential without respect to scale. The quantitative results given here have provided us with specific insights about this particular set of conditions, namely the comparison of workspace size for these five joints. Another important consideration for these results is that our model of the compliant pantograph is purely two dimensional, and does not account for off-axis stiffness or motion. Nevertheless, the SLFP dominates the others in simplicity, ease of manufacturing, and usable workspace size.

2.5 Conclusion

In this chapter, we reviewed several joint candidates and performed a qualitative comparison using the criteria established by Trease et al [27]. From the results of our survey, we selected five candidates for further study. Each was selected for various reasons explained within the chapter. Analytical equations for axial stiffness and stress were presented for the SLFP, Split-Tube Flexure, OCR Joint, Butterfly Flexural Pivot, and CORE Bearing. We introduced the pantograph mechanisms and the forward and inverse kinematics. For each of our joint candidates, we modeled the available workspace of a pantograph using exclusively one type of CM as joint replacements. This revealed the SLFP has the largest practical workspace, being 2.46 times greater than that of the OCR joint.

While the workspace size is valuable information, this information alone does not determine which joint candidate will be best suited for haptic interfaces. We did not undertake a quantitative comparison of axial drift or off-axis stiffness, both of which can be important characteristics in selecting an optimal joint replacement.
CHAPTER 3. SPRINGBACK FORCES AND COMPENSATION TORQUES

A primary focus of this thesis is on understanding and modeling springback forces in CMs in order to actively compensate for them in haptic applications. Here we will derive force-displacement relationships for the interface point of the compliant pantograph based on kinematic and virtual work analyses. We verify the accuracy of the force and torque relationships in three ways: a comparison against analytical data from Gillespie et al., a comparison of force to torque using the Jacobian transform, and a comparison with our own experimental measurements.

3.1 Modeling Compliant Pantograph for Use in Haptic Interfaces

Under well-defined conditions, modeling the compliant pantograph through traditional large-deflection analysis would be possible, but it is difficult to extend to exotic geometries. Instead, we rely on a more efficient method of modeling the displacement behavior of CMs, known as pseudo-rigid-body modeling. This approach allows most compliant joint designs to be modeled as a pin joint and torsional spring [20]. This method simplifies the analysis of large, nonlinear deflections so that the traditional forward and inverse kinematics analyses from Chapter 2 can be applied to the compliant version. This provides the ground work for two virtual work analyses, which derive the relationships for output springback force, \( F = F(R, k, \delta \theta) \), and input compensation torque, \( T = T(R, k, \delta \theta) \), both as functions of geometry, joint stiffness, and position. The function parameters are the link lengths \( R = R_1, \ldots, R_5 \), joint stiffnesses \( k = k_2, \ldots, k_ip \), and changes in angle at each joint \( \delta \theta = \delta \theta_2, \ldots, \delta \theta_5 \).

3.1.1 Virtual Work Analysis

Fig. 3.1 and 3.2 show the two cases considered in our analyses. Relationships between the location of the interface point \( (P_3) \) and the output forces \( (F_x \text{ and } F_y) \) felt by the user at the interface point, or the compensation torques \( (T_2 \text{ and } T_4) \) required to equalize the springback forces, are
Figure 3.1: Diagram of virtual work analysis involving only forces. The output springback forces are modeled at the interface point, $P_3$. Following steps 1-5 and 10-14 of the method presented by Howell [20], the relationships $F_x$ and $F_y$ as functions of $\theta_2$ and $\theta_4$ are derived. The spirals represent torsional springs at each joint with stiffness $k$, and are used in steps 10-11 of the analysis.

Figure 3.2: Diagram of virtual work analysis involving only moments. The compensation torques are modeled at the base joints, $\theta_2$ and $\theta_4$. Following steps 6-14 of the method presented by Howell [20], the relationships $T_2$ and $T_4$ as functions of $\theta_2$ and $\theta_4$ are derived. The spirals represent torsional springs at each joint with stiffness $k$, and are used in steps 10-11 of the analysis.
needed to determine the collective springback behavior of the joints. This is accomplished through
the application of the principle of virtual work \[45\]. Following the 14 step method outlined by
Howell in \[20\], the virtual work of forces, moments, and potential energy sources on the mech-
anism are summed and set to zero to solve for the unknowns $F_x$, $F_y$, $T_2$, and $T_4$. An abbreviated
explanation is provided below, but the detailed derivation and results can be found in Appendix C.

Our system has only two degrees of freedom, $\theta_2$ and $\theta_4$, thus limiting our analysis to two
generalized coordinates. This means that we have only two known inputs. We are interested in four
unknown relationships: two forces and two moments. With four unknown and two known vari-
ables, it becomes necessary that the 14 steps be divided into two partially redundant analyses. By
separating into analyses with only forces or only torques, we can develop the desired relationships
mentioned above. Since there are only two generalized coordinates, $q_1 = \theta_2$ and $q_2 = \theta_4$, the output
springback forces and input compensation torques are examined separately, as shown in Fig. 3.1
and 3.2.

### 3.2 Springback Forces and Compensation Torques

The equations for $F_x$, $F_y$, $T_2$, and $T_4$ that result from applying the principle of virtual work
are very complicated and highly nonlinear, making their direct interpretation infeasible. The only
practical way to present them is through the use of nested coefficients. As an example, $F_x$ is pro-
vided below.

$$F_x = -k_4(\theta_4 - \theta_{40}) + k_p A_5 \left( A_1 - \frac{A_4}{A_8} \right) + \left( k_2(\theta_2 - \theta_{20}) - k_p A_5 \left( \frac{A_3}{A_7} - A_2 \right) \right)$$

$$+ k_3(A_2 + 1)(\theta_2 - \theta_{20} + \theta_{30} - A_{18}) - \frac{k_5 A_3 A_6}{A_7} A_{11} \frac{A_9 + k_5 A_6 (A_1 - 1)}{A_{10} A_{11} A_{12} - A_{13} - A_{14} + A_{15} + A_{16} + A_{17}} (3.1)$$

The several $A_i$ terms, along with the relationships for $F_y$, $T_2$, and $T_4$, can be found in Appendix C.
Numerical results for $F_x$, $F_y$, $T_2$, and $T_4$, plotted over a region of interest, produce well defined trends that can be more easily interpreted and are useful in assessing the performance of the design. The models are evaluated using MATLAB scripts and displayed using MATLAB’s GUIDE (graphical user interface development environment). A snapshot of the custom simulation environment can be seen in Fig. 3.3. MATLAB source code for our work can be found in Appendix D.

The experimental results of Gillespie et al. in [28] provide a significant opportunity for a validity check on our models. Their mechanism was 3D printed out of ABS using fused deposition modeling, so our simulation also uses the material properties of ABS. In order to generate equivalent results, we temporarily deviate from the fixed axial stiffness values introduced in Chapter 2. Instead, we set the joint stiffnesses to equal the experimentally determined stiffness values from [28], given in Table 3.1. Fig. 3.4 can be compared to [28, Fig. 14] shown below in Fig. 3.5.

<table>
<thead>
<tr>
<th>$k_n$</th>
<th>(mNm/deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base Joints</td>
<td>n=1</td>
</tr>
<tr>
<td></td>
<td>n=2</td>
</tr>
<tr>
<td></td>
<td>n=3</td>
</tr>
<tr>
<td>Distal Joints</td>
<td>n=4</td>
</tr>
<tr>
<td></td>
<td>n=5</td>
</tr>
</tbody>
</table>

In gathering their data, Gillespie et al. implemented closed-loop position control to move the interface point to various positions throughout the mechanism workspace. Once the position became steady, the current $i(\theta)$ was recorded. The commanded current needed to hold the mechanism in place effectively compensated the joint restoring torques. A simplified analytical model was created for the torque-displacement relationship based on their experimentally determined stiffness values, but this was used strictly for comparison of their results using self-characterization. Their self-characterized values were about 33% higher than their analytical predictions. Our analytical
Figure 3.3: Custom simulation environment developed using MATLAB’s GUIDE (graphical user interface development environment). All simulations and equations are evaluated in MATLAB scripts and results are displayed here in the GUI or in separate MATLAB figures.
Figure 3.4: Contour plot of the magnitude of the springback force (N) for OCR-based device felt at locations throughout the workspace. The springback force is felt in the direction of the gradient. In other words, the interface point will move inward from a deflected position along the gradient toward the center. The joint candidate used in producing this data was the OCR, providing results comparable to [28, Fig. 14]. The region of interest defined by the black lines represents locations within the yield stress limits of the joints.

Figure 3.5: Contour plot of self-characterized end-effector force as published by Gillespie et al. in [28, Fig. 14] ©2008 IEEE. The range of values and general behavior of our simulation matches their data closely. Slight differences exist between results. Our model predicts a slightly steeper slope in the Y direction.
predictions match their experimental data better, both in range of values and general behavior. Slight differences do exist between results though; our model predicts a slightly steeper gradient in the Y direction.

Returning now to the comparison of joints introduced in Chapter 2, since the axial stiffness values were controlled to be the same across the five joint candidates, a simulation of the output springback forces and input compensation torques will be identical, with the exception of the CORE bearing. In Fig. 3.6, the magnitude of the output springback force, felt by the user at the interface point, is plotted over a focused range. The range of motion is set by the yield stress in each joint, which varies for each candidate. For all the candidates, higher springback forces are experienced at the edges of the workspace. The joint candidate used in producing these data was the SLFP. From the results, it is apparent that moving the interface point from side to side (in the X direction) changes springback gradually, while moving the point forward and backward (in the Y direction) changes springback more rapidly. The gradient of the surface indicates the direction

Figure 3.6: Contour plot of the magnitude of the springback force (N) for SLFP-based device felt at locations throughout the workspace. While the joint candidate used in producing this data was the SLFP, since the axial stiffnesses were the same for all candidates, all but the CORE bearing generate identical plots. The region of interest defined by the black lines represents locations within the yield stress limits of the SLFP joints.
of the springback forces as shown in Fig. 3.7. The theoretical tendency is for the interface point to return to its origin along the path of the gradient until the springback and deflection are both zero. Several streamlines demonstrate the different free-responses starting from rest.

Several streamlines demonstrate the different free-responses starting from rest. In actuality though, it is reasonably expected that overshoot and oscillation will occur in the free response when the deflected mechanism is allowed to return to its undeflected state.

The springback forces for the CORE bearing based device can be seen in Fig. 3.8. The differences between Fig. 3.6 and Fig. 3.8 further illustrate the unique nature of the CORE bearing. The regions of springback forces are the result of the constant moment characteristic in each bearing. The magnitude of each region depends on the sign of the deflections, determining whether the moments add together or cancel out. The transitions from one region to another indicate locations of the interface point where at least one joint in the pantograph is in equilibrium. This implies that for a given position, the potential energy stored in at least one of the joints is zero. These five unique curves are the equilibrium paths for each of the five joints as indicated by the joint number. The
Figure 3.8: Magnitude of the springback force based on CORE bearings. The different regions indicate combinations of the moments experienced in each joint. The boundaries represent locations of the interface point where the pantograph is in partial equilibrium. Along a boundary line, at least one of the joints is undeflected as indicated by the joint number.

divided region behavior may have great potential for use in haptic interfaces since the regions can be clearly identified by treating the mechanism as the superposition of five separate four-bar mechanisms, tracing the path of each coupler point. The springback value is relatively constant within each region. However, addressing and smoothing the discontinuity from one region to another may be problematic.

While the springback forces are informative, the compensation torques are the key to producing a haptic interface relying on CMs. The surface plots shown in Fig. 3.9 for the SLFP displays the torques which must be exerted on base joint 2 and 4 in order to maintain a given position of the interface point. Torques from both base joints 2 and 4 must be exerted in unison in order to compensate the springback force and maintain position of the interface point. While the joint candidate used in producing this data was the SLFP, since the axial stiffnesses were the same for all candidates, all but the CORE bearing generate identical plots. The data reveal a slight curvature and a zero-torque curve. This curve signifies the positions of the interface point at which the pantograph is in partial equilibrium for that base joint. The two zero-torque curves could be determined
by setting one of the input torques equal to zero and solving the other for the deflection of the interface point.

### 3.3 Validation of $F_x$ and $F_y$

In robotics and kinematics, the mapping between input torques and output forces is accomplished through the Jacobian. Specifically in our case, the springback forces and the compensation torques are related to each other by the inverse of the transpose of the Jacobian matrix,

$$
\begin{bmatrix}
F_x \\
F_y
\end{bmatrix}
= J^{-T} \tau =
\begin{bmatrix}
\frac{\partial x_3}{\partial \theta_2} & \frac{\partial x_3}{\partial \theta_4} \\
\frac{\partial y_3}{\partial \theta_2} & \frac{\partial y_3}{\partial \theta_4}
\end{bmatrix}^{-T}
\begin{bmatrix}
T_2 \\
T_4
\end{bmatrix},
$$

(3.2)

The Jacobian is a combination of derivative terms available through the kinematics analysis. The four derivative terms listed in the matrix are omitted here for brevity, but are provided in the end of Appendix B. Using this relationship, we compare the simulated springback forces against the simulated compensation torques to reveal that they are identical. This simple analytical verification gives further evidence that the relationships for $F_x$, $F_y$, $T_2$, and $T_4$ from virtual work are correct.

In addition, load testing was conducted on a prototype pantograph with SLFP joints. The SLFP was chosen for its large available workspace, simplicity in design modeling, and ease of
manufacturing. While the other joints would be interesting to test and validate, the SLFP allowed for a faster timeline for testing and results. The prototype was designed and then manufactured on a CNC mill out of 1/4 inch polypropylene sheet and can be seen in Fig. 3.10. Drawings of the pantograph design can be found in Appendix E. Note that this is a monolithic mechanism without rotary joints. Polypropylene was selected for its high strength-to-elasticity ratio, making it an excellent candidate for a CM prototype. Young’s modulus of the prototype was determined experimentally by performing tensile tests on four dog bones cut from the same sheet as the prototype following the guidelines outlined in ASTM D638, “Standard Test Method for Tensile Properties of Plastics.” The dimensions of the dog bones are provided in Fig. 3.11. The samples were loaded at different rates: one at 50 mm/min, two at 500 mm/min, and one at 1000 mm/min. The average modulus of the four tests was 1.5 GPa.

The objective in our load tests was to validate our simulation model for $F_x$ and $F_y$ against experimental measurements. Force data were acquired by mounting the prototype in an Instron and then actuating the base with constant velocity (500 mm/min) along X or Y paths. These paths were selected from the available haptic workspace for the corresponding SLFP prototype. The joints in
the mechanism were designed with the intent of having axial stiffness values of 2.864 and 1.25 mNm/deg in the base and distal joints respectively. However, due to manufacturing imperfections, each joint was slightly off its mark. The dimensions measured in mm and the calculated joint stiffness in mNm/deg are provided in Table 3.2 below. These unique dimensions and stiffness values affect the symmetry typically seen in our previous simulations, especially the prediction of available workspace. This new predicted boundary of available workspace served as the test conditions for our force validation measurements. As mentioned before, X and Y paths were selected, so that for a given X or Y reference position, the interface point could move back and forth along a Y or X path respectively to acquire force data. The paths were spaced 0.25 inches apart. While there is nothing special to this amount of path spacing, it provided ample sampling of the springback

Table 3.2

Dimensions and stiffnesses of joints in SLFP-based pantograph prototype. Although the stiffness values for base and distal joints were intended to be 2.864 and 1.25 mNm/deg respectively, due to manufacturing imperfections, the predicted joint stiffnesses differed slightly and are given here.

<table>
<thead>
<tr>
<th>Joint</th>
<th>Width (mm)</th>
<th>Length (mm)</th>
<th>Thickness (mm)</th>
<th>Stiffness (mNm/deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.25</td>
<td>8.65</td>
<td>6.16</td>
<td>3.034</td>
</tr>
<tr>
<td>3</td>
<td>1.15</td>
<td>19.95</td>
<td>6.16</td>
<td>1.025</td>
</tr>
<tr>
<td>4</td>
<td>1.19</td>
<td>8.59</td>
<td>6.16</td>
<td>2.636</td>
</tr>
<tr>
<td>5</td>
<td>1.25</td>
<td>19.95</td>
<td>6.16</td>
<td>1.316</td>
</tr>
<tr>
<td>$P_3$</td>
<td>1.14</td>
<td>19.97</td>
<td>6.16</td>
<td>0.997</td>
</tr>
</tbody>
</table>
forces. The gathered data sufficiently explored the available workspace to highlight the capability of our simulation model. For example, at \( x = 0 \), the interface point was deflected through two complete saw-tooth profiles between \(-1.25 < \Delta y < 1\) inches starting from \( \Delta y = 0 \) in the positive direction. The deflections provided measurements for \( F_y \) at \( x = 0 \). This was repeated throughout the non-symmetric available haptic workspace for both \( X \) and \( Y \) direction forces. The reference locations and low to high boundaries are given in Tables 3.3 and 3.4. The order of tests within the

Table 3.3

Experimental design for testing \( F_y \) springback force. For each reference location, the interface point was displaced through two full cycles of saw-tooth movement between the min and max \( X \) values. The order of tests was randomized, in order to satisfy proper practices for designing experiments.

<table>
<thead>
<tr>
<th>Test #</th>
<th>Random Seed</th>
<th>Test Order</th>
<th>( X_{\text{ref}} )</th>
<th>( Y_{\text{min}} )</th>
<th>( Y_{\text{max}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8185</td>
<td>11</td>
<td>-1.5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.8845</td>
<td>12</td>
<td>-1.25</td>
<td>-0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>3</td>
<td>0.7189</td>
<td>10</td>
<td>-1</td>
<td>-0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>4</td>
<td>0.2922</td>
<td>6</td>
<td>-0.75</td>
<td>-0.5</td>
<td>0.75</td>
</tr>
<tr>
<td>5</td>
<td>0.6839</td>
<td>9</td>
<td>-0.5</td>
<td>-0.75</td>
<td>0.75</td>
</tr>
<tr>
<td>6</td>
<td>0.2620</td>
<td>4</td>
<td>-0.25</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>0.2062</td>
<td>2</td>
<td>0</td>
<td>-1.25</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>0.6396</td>
<td>8</td>
<td>0.25</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>0.1066</td>
<td>1</td>
<td>0.5</td>
<td>-0.75</td>
<td>0.75</td>
</tr>
<tr>
<td>10</td>
<td>0.2889</td>
<td>5</td>
<td>0.75</td>
<td>-0.75</td>
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</tr>
<tr>
<td>11</td>
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<td>3</td>
<td>1</td>
<td>-0.5</td>
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<tr>
<td>12</td>
<td>0.9830</td>
<td>13</td>
<td>1.25</td>
<td>-0.25</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>0.3635</td>
<td>7</td>
<td>1.5</td>
<td>-0.25</td>
<td>-0.25</td>
</tr>
</tbody>
</table>

two groupings, \( F_y \) and \( F_x \) forces, were randomized. This was done to satisfy proper practices for designing experiments. Each test was video recorded for record keeping and future reference.

The prototype was bolted to a fixture through two holes at the base of the pantograph and scalloped tracks in the test fixture, a series of partially overlapping holes. The setup and fixture can be seen in Fig. 3.12. The scalloped design allowed for positioning of the pantograph at the correct spacing for a path. A 22 lb load cell was attached to the pantograph via a rod end with a ball-bearing hole that could provide freedom to rotate during loading, but rigidity in the \( X \), \( Y \), and
Experimental design for testing $F_x$ springback force. For each reference location, the interface point was displaced through two full cycles of saw-tooth movement between the min and max Y values. The order of tests was randomized, in order to satisfy proper practices for designing experiments.

<table>
<thead>
<tr>
<th>Test #</th>
<th>Random Seed</th>
<th>Test Order</th>
<th>$Y_{ref}$</th>
<th>$X_{min}$</th>
<th>$X_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>0.1314</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-0.25</td>
</tr>
<tr>
<td>15</td>
<td>0.3652</td>
<td>3</td>
<td>0.75</td>
<td>0.75</td>
<td>-0.75</td>
</tr>
<tr>
<td>16</td>
<td>0.8474</td>
<td>10</td>
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<td>0.5</td>
<td>-1</td>
</tr>
<tr>
<td>17</td>
<td>0.7560</td>
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<td>0.25</td>
<td>-1.25</td>
</tr>
<tr>
<td>18</td>
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<td>5</td>
<td>0</td>
<td>0</td>
<td>-1.5</td>
</tr>
<tr>
<td>19</td>
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<td>-0.25</td>
<td>-0.25</td>
<td>-1.25</td>
</tr>
<tr>
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<td>9</td>
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<td>-0.5</td>
<td>-1</td>
</tr>
<tr>
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<td>-0.75</td>
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<td>-0.5</td>
</tr>
<tr>
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<td>-1</td>
<td>-1</td>
<td>-0.25</td>
</tr>
<tr>
<td>23</td>
<td>0.6199</td>
<td>6</td>
<td>-1.25</td>
<td>-1.25</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 3.12: Springback force validation test setup. The prototype was bolted to a fixture by the two holes at the base and scalloped tracks in the test fixture. The scalloped design allowed for positioning of the pantograph at the correct spacing for a path.
Z directions. The load cell was a SMT1-22 made by Interface Inc. The sensor has a non-linear accuracy of ±0.05% full scale output, and hysteresis accuracy of ±0.03% full scale output. The rated output is 2.0 mV/V. The results of the tests are presented in Fig. 3.13. These 3D scatter plots of springback force comparing simulated and experimental results. The green points represent the experimentally measured force at a location. The black points represent the simulated force at the same locations. The simulated results follow the experimental data very well, and the average absolute error between the two for $F_x$ is 0.0647 N, for $F_y$ is 0.0794 N.

Figure 3.13: 3D scatter plots of springback force comparing simulated and experimental results. The left portion shows $F_x$ while the right shows $F_y$. The green points represent the experimentally measured force at a location. The black points represent the simulated force at the same locations. The simulated results follow the experimental data very well, and the average absolute error between the two for $F_x$ is 0.0647 N, for $F_y$ is 0.0794 N. These small average absolute errors confirm to us that our model accurately predicts the X and Y direction springback behavior. In addition, we can also infer that the model accurately predicts the springback force magnitude and direction, since these are derived from $F_x$ and $F_y$.

3.4 Conclusion

In this chapter, we introduced a model for the springback behavior of a compliant pantograph. Two virtual work analyses led to the formulation of equations for $F_x$, $F_y$, $T_2$, and $T_4$, which
are derived in Appendix C. A custom MATLAB simulation was developed to facilitate simulations of springback force and compensation torque given a set of link lengths, axial stiffnesses, and joint displacements. Simulation results were compared against the experiment results presented by Gillespie et al. in [28], and were found similar in both magnitude and behavior. Additional simulations were presented corresponding to the optimized candidates from Chapter 2. Specifically, we presented contour plots of springback force and compensation torques, as well as a vector field showing the theoretical direction of motion of the interface point given an initial displacement. The springback force plot for the CORE bearing revealed the existence of five zero-stress curves. This led to the insight that a five-bar mechanism comprises the motion of five unique four-bar mechanisms.

Our models for springback force, namely $F_x$ and $F_y$, were experimentally validated through randomized force-displacement tests. A prototype compliant pantograph was machined from polypropylene, and was subjected to several displacement paths. The force was measured throughout the motion and the data formed a cross-sectional slice of the springback force surface. By comparing the data to simulated results of the same displacements, it was found that the average absolute error between the two for $F_x$ was 0.0647 N, for $F_y$ was 0.0794 N.

The development of an effective model of springback force and compensation torque fulfills the primary requirement of active compensation in a compliant haptic interface. While this model has been verified for 2D planar motion, extending its predictions to 3D is an interesting question for future work.
CHAPTER 4. AVAILABLE HAPTIC FORCE AFTER COMPENSATION

4.1 Background and Redefinition

The available torque after compensation is a measure of how much remains of the torque capacity for each actuator after compensating for the compliant joints. This remaining amount can be transformed by the Jacobian relationship in eq. (3.2) into a measure of the available haptic force (AHF) after compensation. As mentioned before, springback forces are generally undesired in haptic interface design because they are parasitic forces that detract from the representation of the desired virtual or remote environment. An illustration of such parasitic forces is the friction encountered in traditional joints. Without high quality bearings and proper lubrication, the user will experience general resistance to motion through that joint. However, if the intended virtual experience is to simulate return-to-home forces, the springback can be advantageous. For example, representing the resistance of muscle tissue in a surgical simulation can benefit from high springback forces. If the muscle tissue were designed to exist at the edge of the haptic workspace, simulating the collision response for the user could be aided by having high springback forces. Thus the inherent elasticity of compliant mechanisms can aid in the simulation.

Gillespie et al. presented a plot of AHF throughout their workspace as a single value for each location [28, Fig. 18], as seen in Fig. 4.1. From the values shown in their plot, we conclude that they assume, during compensation, a portion of the actuator’s potential capacity is unavailable, thus limiting the maximum magnitude of haptic forces that can be rendered at that location. However, this is only true for directions that oppose the springback force. If a component of the desired haptic force aligns with the springback force, then the two can work together, thus increasing the maximum magnitude of AHF beyond the nominal amount. Even in cases where the virtual environment does not focus exclusively on return-to-home forces, springback can still be used to increase the capability of compliant haptic interfaces. Although these benefits are maximized as
Figure 4.1: Maximum magnitude of available haptic force as published by Gillespie et al. in [28, Fig. 18] ©2008 IEEE. This contour plot illustrates their perspective that the maximum AHF diminished as the interface point moved closer to the boundaries of the workspace.

As the springback force grows, the increase in capacity or efficiency is not limited to just the boundary of the haptic workspace, but is available wherever springback forces persist.

We will now expound on the benefits of incorporating springback forces into rendering haptic forces. Consider the compliant pantograph under discussion, shown in Fig. 4.2. In the undeflected state, it is capable of rendering any force vector from a set of haptic forces bounded by some force region; a force vector could be applied to the interface point in any direction and magnitude within the bounding region. As shown in Fig. 4.2, the red lines represent the bounding region of all possible forces at this location using a given set of actuators. The exact size of the region is dependent on the mechanism geometry and maximum torque capacity of the actuators. By applying eq. (3.2) to all possible combinations of the actuator torques, with the domain \( \{(T_2, T_4) \in \mathbb{R} \mid -T_{\text{max}} < T < T_{\text{max}} \} \), a range of \( \mathbf{F} \) is generated for a given location. In a deflected state, the springback force limits the AHF in directions away from home. This can be seen for a given deflected state in Fig. 4.3. The red and black lines create three distinct regions referred to as 1, 2, and 3. Region 1 describes the boundary of haptic forces that are no longer available due to the springback forces. Region 2 describes the haptic forces that have become available thanks to the springback forces. Together, regions 2 and 3 (the area bounded by the red lines) describe the
Figure 4.2: Available haptic force region after compensation at the undeflected location. The red lines describe the boundary of all possible haptic forces that can be rendered from this location; a force vector could be applied to the interface point in any direction and magnitude within the bounding region. The exact size of the region is dependent on geometry and actuator characteristics.

Figure 4.3: Available haptic force after compensation at a deflected location. The red and black lines create three distinct regions referred to as 1, 2, and 3. Region 1 describes the boundary of haptic forces that are no longer available due to the springback forces. Region 2 describes the haptic forces that have become available thanks to the springback forces. Together, regions 2 and 3 (the area bounded by the red lines) describe the boundary of all possible haptic forces that can be rendered at this location. By taking advantage of the springback force, the boundary of AHF is shifted in the direction of the springback force. For this specific deflected location at $x = 0$ and $y = 135$, there would be an 8.1% increase in the average AHF, and a 13.4% increase in the maximum AHF.
boundary of all possible haptic forces that can be rendered at this location. By taking advantage of the springback force, the boundary of AHF is shifted in the direction of the springback force. When compared with the traditional pantograph (one without springback forces), this novel method of leveraging springback in haptic force rendering actually produces moderate to significant increases in our mechanism’s mean and maximum capabilities. For this specific deflected location at $x = 0$ and $y = 135$ mm shown in Fig. 4.3, there is an 8.1% increase in the average AHF, and a 13.4% increase in the maximum AHF. The increases throughout the workspace is shown in Fig. 4.4.

Figure 4.4: Percent increase in mean available haptic force (Top) and maximum available haptic force (Bottom). The contour plots describe the percent increase, after compensation, of a compliant pantograph versus a traditional pantograph. A surface plot (Bottom-Right) of the percent increase in maximum AHF reveals the unique changes in slope and concavity, especially those where $150 < y < 170$ mm and $y < 150$ mm.

contour plots describe respectively the percent increase in mean and max AHF after compensation of a compliant pantograph versus a traditional pantograph for a given location of the interface point. The increase is more sensitive where the interface point goes up and down and is generally
insensitive to side-to-side displacements. A surface plot of the percent increase in maximum AHF reveals the unique changes in slope and concavity, especially those where $150 < y < 170$ mm and $y < 150$ mm.

4.2 Hardware Demonstration

To demonstrate the usefulness of our springback models, we implement our model on real hardware to perform active compensation of a compliant haptic pantograph. Our purpose in developing a hardware demonstration is not necessarily to provide more experimental measurements. Precise experimental validation of the springback force equations was provided in Chapter 3. It is appropriate to infer that AHF is valid because AHF is derived from those equations. Consequently, our purpose is to share our insights and the implementation used to enable active compensation.

The demonstration of active compensation for our compliant haptic pantograph relies on a combination of commercial off-the-shelf parts and some custom built mechanical parts and software design. Our haptic interface consists of mechanical, electrical, and software components as shown in Fig. 4.5. A detailed list of the outlined components is given in Table 4.1. The list describes, by part and quantity, several components including physical parts, motors, encoders, motor drivers, and computer I/O hardware to run the demonstration. We will speak in some detail about each item.

4.2.1 Software

The main objective of software is to compensate for the springback force through an awareness of position. Position information is critical in enabling the haptic experience. Our open-loop process is shown in Fig. 4.6. This is implemented in a MathScript Node running in LabVIEW (see Appendix D.3.1 for source code). Position information is critical in enabling the haptic experience. The relative angles of the base joints are measured by the encoders. In the top path, $\theta_2, \theta_4$ become $x_3, y_3$ by means of the forward kinematic relationship. The location of the interface point, $x_3, y_3$, determine what haptic forces the user experiences from the virtual/remote environment. This is transformed from haptic forces, $F_{x,h}, F_{y,h}$ to haptic torques, $T_{2,h}, T_{4,h}$ by means of the Jacobian ma-
Figure 4.5: Compliant haptic pantograph demonstration with outlines and labels. A picture taken of the prototype and setup. The color scheme has no significance other than to distinguish one outline from the other. Each outline is labeled according to the parts list in Table 4.1.

Table 4.1: Parts list for demonstration of active compensation in compliant haptic pantograph.

<table>
<thead>
<tr>
<th>Label</th>
<th>Mechanical</th>
<th>Material</th>
<th>Quantity</th>
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<tbody>
<tr>
<td>M1</td>
<td>Custom Compliant Pantograph</td>
<td>Polypropylene</td>
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</tr>
<tr>
<td>M2</td>
<td>Custom Base</td>
<td>Aluminum</td>
<td>1</td>
</tr>
<tr>
<td>M3</td>
<td>Custom Motor Clamp</td>
<td>Aluminum</td>
<td>2</td>
</tr>
<tr>
<td>M4</td>
<td>Custom Interface Point</td>
<td>Aluminum</td>
<td>1</td>
</tr>
<tr>
<td>M5</td>
<td>Ball Joint Rod End</td>
<td>Steel</td>
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<table>
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</thead>
<tbody>
<tr>
<td>E1</td>
<td>NI cRIO-9074</td>
<td>1</td>
</tr>
<tr>
<td>E2</td>
<td>NI 9263 Analog Output Module</td>
<td>1</td>
</tr>
<tr>
<td>E3</td>
<td>NI 9411 Differential Digital Input Module</td>
<td>1</td>
</tr>
<tr>
<td>E4</td>
<td>BE15A8J Brushless DC Servo Amplifier</td>
<td>2</td>
</tr>
<tr>
<td>E5</td>
<td>Maxon Brushless DC Motor</td>
<td>2</td>
</tr>
<tr>
<td>E6</td>
<td>HEDL-5540 Encoder</td>
<td>2</td>
</tr>
<tr>
<td>E7</td>
<td>Power Supply</td>
<td>1</td>
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</tbody>
</table>

<table>
<thead>
<tr>
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<th>Software</th>
</tr>
</thead>
<tbody>
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<td>S1</td>
<td>LabVIEW Software</td>
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Figure 4.6: Block diagram of open-loop process for demonstration of active compensation. Position information is critical in enabling the haptic experience. The relative angles of the base joints are measured by the encoders. In the top path, $\theta_2, \theta_4$ become $x_3, y_3$ by means of the forward kinematic relationship described in Appendix B. The location of the interface point, $x_3, y_3$, determine what haptic forces the user experiences from the virtual/remote environment. This is transformed from haptic forces, $F_x, F_y$, to haptic torques, $T_2, T_4$ by means of the Jacobian matrix. However, the Jacobian matrix is position-dependent, and thus must be updated for the current location. Along the bottom path, the compensation torques $T_2, T_4$ are calculated from the equations established through the virtual work analysis. The haptic torques and compensation torques are summed together and sent through the motors, pantograph, and on to the user.

4.2.2 Mechanical Components

Many of the mechanical parts of the prototype are custom designed and machined. Some of them are generic, low tolerance parts, but the compliant pantograph itself is more particular. A close up of the demonstration unit is shown in Fig 4.7. Several drawings are available in Appendix E. A custom compliant pantograph was machined on a CNC mill out of 1/4 inch polypropylene sheet. The CM chosen for the prototype was the SLFP because of its simplicity, ease of manufacture, and large haptic workspace. A custom base was machined out of aluminum. Custom motor clamps were machined out of aluminum. They were pressed over the rigid links $R_2, R_4$, of the pantograph. They were aligned just right to place the axis of motion through the center of the clamping region. Once in place, a set screw tightened the clamp around the drive shaft allowing the motion of
the motors to exert direct torque at the location of the base joints. A close up of the motor arm connector, before the motor shaft is mounted in place, is shown in Fig. 4.8. A custom interface assembly allowed the user to control the compliant pantograph at point $P_3$, however, this location is shared by a compliant flexure. In order for the user to exert and receive forces at that location,
the custom interface assembly attached to the nearby rigid link $R_3$ and transferred the forces to the location of $P_3$, as seen in Fig. 4.9. A commercial ball joint rod end provided a means for the user to grab hold and control the haptic pantograph. The ball bearing in the rod end allowed it freedom to rotate while it translated.

![Figure 4.9: End effector assembly shown from below. The custom interface assembly allows the user to control the compliant pantograph at point $P_3$, however, this location is shared by the pseudo pivot of the compliant joint.](image)

4.2.3 Electrical Components

All of the electrical parts of the prototype were commercially available without any major modifications. The main controller was a National Instruments (NI) CompactRIO (cRIO) 9074. This cRIO has an embedded real-time (RT) industrial controller with reconfigurable I/O and a field-programmable gate array (FPGA). It also features a backplate for our two NI modules: NI 9263 Analog Output Module, and NI 9411 Differential Digital Input Module. The output module provides analog output voltage circuitry to command the motor amplifiers. The maximum range of output voltages is $\pm 10$ V on up to four channels. The input module provides differential input circuitry to read the A and B channels of our encoders. Two BE15A8 Brushless DC Servo Amplifier receive the command voltage from the output module and engage the motors through current control. The input voltage range is $\pm 10$ V, matching the output module’s range appropriately. Each amplifier interfaces with eight motor lines (five for the hall sensor and three for the motor portion). The supply voltage for the motors (24 V) is also routed through the amplifiers. We selected
two Maxon brushless DC motors for this application for their backdrivability compared with other brushed motors and/or heavily geared alternatives. Backdrivability contributes to the transparency or the haptic interface and is particularly important. The motors chosen have a maximum output torque of 25 in-oz ($\approx 169$ mN-m) which is adequate to satisfy the maximum compensation torques presented in Chapter 3. Each motor has an HEDL-5540 encoder attached to the rear of its drive shaft. The encoders provide the position awareness essential for active compensation. They were sampled at the 400 MHz rate by the FPGA under the direction of the RT controller. At 500 pulses per revolution, and 4x quadrature sampling, they provided a resolution spacing of 0.18 deg per count.

### 4.2.4 Required Force

Rendering haptic forces is achieved by comparing the springback force and desired haptic force vectors to identify how much force to contribute from the motors. Fig. 4.10 shows how springback, haptic, and compensation forces can be compared to determine the required force. This is the actual effort commanded from the motors to render the desired haptic force vector. The required force can be calculated in two ways. By Method 1, it is the difference between the haptic and springback forces. This approach identifies how much of the haptic force can be provided by the existing springback force and how much is needed from the motors. By Method 2, the required force is the sum of the compensation force and the desired haptic force.

The result of either method is the same vector, both in magnitude and direction. This is true because the compensation force is equal to but opposite in direction of the springback force. While the two methods have identical results, Method 2 is slightly more efficient. If we assume that active compensation is always engaged, with or without the presence of haptic forces, both methods already have available to them the values of $T_2$ and $T_4$ introduced in Chapter 3. By summing the compensation and haptic force vectors, rather than differencing the haptic and springback forces, Method 2 avoids the unnecessary computational cost of calculating springback forces. Method 2 was implemented in our prototype (See Appendix D for details).
Figure 4.10: Comparison of springback, haptic, and compensation forces to determine the required force. The required force can be calculated in two ways. By Method 1, it is the difference between the haptic and springback forces. This approach identifies how much of the haptic force can be provided by the existing springback force and how much is needed from the motors. By Method 2, the required force is the sum of the compensation force and the desired haptic force. The result of either method is the same vector, both in magnitude and direction. This is true because the compensation force is equal to but opposite in direction of the springback force.

4.2.5 Results and Discussion

The prototype was used to demonstrate the effectiveness of active compensation and the experience of a user interacting with a virtual enclosure. We will present experimental data in support of this demonstration. First, we present a time history of motor angles and the XY position of the interface point shown in Fig. 4.11. These plots support and inform the interpretation of upcoming figures. It is easier to interpret these two position histories when paired with a top down graph of the workspace shown in Fig. 4.12. The blue line shows the path of the interface point as the user moved throughout the workspace. Enclosing a subset of the available workspace, the black dashed lines show the location of virtual walls. Force vectors overlay the blue path, showing the springback forces as black arrows and the required forces as green arrows. It is important to reiterate that the required force is the sum of both the compensation force and the desired haptic force. In other words, this figure illustrates Method 1 mentioned previously. The user traced a path that interacted with each wall, passing through the center on each pass. As the interface point moved from the arbitrary starting point toward the top wall, the springback forces increase. However, as the inter-
face point collides with and penetrates the virtual wall, the required force changes magnitude and direction in order to render an appropriate haptic response.

The walls are modeled as boundaries with very high stiffness. The ideal wall would have infinite stiffness, but this is not possible in practice. As the interface point penetrates the wall, a collision is detected and the penetration depth is scaled by the stiffness to determine the desired haptic response. The resulting haptic response is shown in Fig. 4.13. Notice that the forces experienced by the user throughout the compensated workspace is near zero. However, when the user is in contact with the virtual walls the forces are pronounced and perpendicular to the wall surface. All three force types are superimposed in the close up view.

Time histories for the components, $F_x$ and $F_y$, corresponding to the aforementioned paths are provided in Fig. 4.14. Both plots show the springback force, required force, and haptic force. We reiterate that the haptic force is the desired outcome, it can also be thought of as the sum of the springback and required force. The top plot shows $F_x$ and the bottom plot shows $F_y$. The four large responses, two in $F_y$ and two in $F_x$, correspond to periods where the interface point was
Figure 4.12: Top down view of a path traced by the interface point including springback and required forces. The blue line shows the path of the interface point. The black dashed lines show the location of virtual walls. Force vectors experienced along the path are shown: springback forces as black arrows, and required forces as green arrows. The user traced a path that interacted with each wall, passing through the center on each pass. As the interface point moved from the arbitrary starting point toward the top wall, the springback forces increase. However, as the interface point collides with and penetrates the virtual wall, the required force changes magnitude and direction in order to render an appropriate haptic response.

Figure 4.13: Top down view of a path traced by the interface point including haptic forces experienced by the user and a close-up view. The left shows just haptic forces throughout the entire path, while the right shows all three force types superimposed on a close-up region of the path. Forces throughout the compensated workspace are near zero. When the user is in contact with the virtual walls the forces are pronounced and perpendicular to the wall surface.
Figure 4.14: Time histories for $F_x$ and $F_y$. The top plot shows $F_x$ and the bottom plot shows $F_y$. Both plots show the springback force, required force, and haptic force. The four large responses, two in $F_y$ and two in $F_x$, correspond to periods where the interface point was determined to be in contact with the virtual wall. The jagged profiles of the required force and thus haptic force in these regions come from varying penetration depths of the virtual wall. This is connected to the human user’s response to the open loop virtual experience. It is interesting to note that from $2 < \text{Time (s)} < 2.6$ and $4.2 < \text{Time (s)} < 4.8$, $F_y$ is not completely compensated. This comes from the slight nonlinearities in our springback force model, which become more pronounced as the interface point moves further from home in the Y-direction. However, within the available workspace, the error is small and can be ignored.

determined to be in contact with the virtual wall. The jagged profiles of the required force and thus haptic force in these regions come from varying penetration depths of the virtual wall. This is connected to the human user’s response to the open loop virtual experience. It is interesting to note that from $2 < \text{Time (s)} < 2.6$ and $4.2 < \text{Time (s)} < 4.8$, $F_y$ is not completely compensated. This comes from the slight nonlinearities in our springback force model, which become more pronounced as the interface point moves further from home in the Y-direction. However, within the available workspace, the error is small and can be ignored.
Time histories for the motor torques, corresponding to the aforementioned path are provided in Fig. 4.15. These motor torques are based on the commands sent from the cRIO to the motor drivers. The command vs actual output torque was calibrated beforehand for each motor. The data provides another perspective on motor behavior during compensation and during compensation plus haptic rendering. It is interesting to note that, for our pantograph, the motors exert in opposite directions to create the $F_y$ component forces, but they work in the same directions to create the $F_x$ component forces.

Another factor to consider is the update frequency of the motor commands. The motor commands seemed to update at two different rates depending on whether the response was compensation only, or compensation and haptic forces. For compensation only, the motors updated at an average of 242 Hz with a standard deviation of 24 Hz. For compensation and haptic forces—when contacting a wall—the motors updated at an average of 173 Hz with a standard deviation of 43 Hz. This decrease in update frequency speaks to the importance of having optimized code running the controls. Since active compensation relies on a series of calculations, as opposed to a referenced look-up table, the calculations need to be efficiently structured. Poorly organized code and/or unnecessary overhead processes can slow the update frequency well below 10 Hz, making the lag and aliasing in the virtual environment impossible to ignore, where the motors noticeably step between commanded behavior. Speed is crucial to a transparent haptic experience.
4.3 Conclusion

In this chapter, we introduced the concept of available haptic force after compensation. It is important to reiterate that the AHF is not just an assessment of the remaining capacity of the motors as described by Gillespie et al. in [28, Fig. 18], but is dependent on both the direction and magnitude of the desired haptic force relative to the local springback force. We discussed situations and examples in which springback force would aid in rendering the virtual environment. When springback force is incorporated as a component of the desired haptic force, it shifts the range of potential haptic forces in the direction of the springback force. Simulated results showed that this actually creates an increase in both the average and maximum AHF. Plots of the percent increases were provided.

We also presented a hardware demonstration of active compensation. The parts and components were discussed and specific details are provided in Appendix E. The required force needed to render an appropriate haptic force was presented in two ways. Position, torque, and calculated force data were monitored while a user manipulated the prototype. The forces they experienced during the test were presented and discussed. It is important to state that the efficiency of the software code plays a significant and non-trivial role in the refresh rate of the rendering. Inefficient and wasteful calculations steps can create significant lag in the forces experienced by the user.

Active compensation relies on the development a valid model of the springback force and compensation torques. And with that requirement comes greater understanding of the forces present in the haptic simulation. Active compensation is a viable and elegant alternative to look-up tables or self-characterized methods.
CHAPTER 5. CONCLUSIONS AND FUTURE WORK

This thesis offers insights into the use and implementation of compliant mechanisms in haptic interfaces. The advantages and challenges have been thoroughly discussed. A solution referred to as active compensation has been developed, demonstrated and validated. This approach compensates for the springback behavior of compliant mechanisms, which is an obstacle to compliant haptics. Our method was based on analytical models of springback force and compensation torque, allowing our compliant haptic pantograph to render a transparent virtual experience. The analysis to compute the springback forces in our model relied on the forward and inverse kinematics, Jacobian matrix, stiffness characteristics of the joints, and results of virtual work. Analytical simulations and experimental data were presented and the results included the quantification of springback forces, compensation torques, and available haptic workspace. Our simulated springback force data follow our experimental data very well. The average absolute error between the two for $F_x$ was 0.0647 N, and for $F_y$ was 0.0794 N.

Although utilizing the actuators to compensate springback behavior can seem detrimental to the available haptic force, it really depends on the magnitude and direction of the desired haptic force. The most significant impact of active compensation on available haptic force is that the available haptic force gets shifted in the direction of the springback force. The real key is knowing that the haptic force capability is different depending on the local springback forces. If a component of the desired haptic force aligns with the springback force, then the two can work together, thus increasing the maximum magnitude of available haptic force beyond the nominal amount. The results and methods of this work form a foundation for a model-based compensation approach for haptic interfaces.

The major contributions of this thesis are as follows. We:

- Expanded the joint survey efforts of Trease et al. [27] to include several additional compliant mechanisms and two new criteria: simplicity and ease of manufacture.
• Developed and validated an analytical model for the springback force and compensation torques for a compliant pantograph.

• Discussed and corrected the incomplete assumptions regarding available haptic force. Gille-spie et al. presented a single value approach to assessing the available haptic force for any location [28, Fig. 18], which values were neither the minimum nor the maximum force available at a location. The available haptic force actually comprises a range of forces. Depending on the direction and magnitude of the desired haptic response, the influence of the local springback force can either take away from or add to the haptic rendering.

• Created a hardware demonstration of active compensation and a transparent virtual environment using a compliant haptic interface.

Our hardware demonstration gives us great anticipation for the future work. While this thesis thoroughly explores the kinematic implications of compliant haptics, it lacks assessment in several important haptic parameters. These include:

• An assessment of the bandwidth of the haptic display for CM based haptic interfaces.

• The development of Z-width plots for the CM based haptic interfaces.

• A discussion of the passivity of CM based haptic device.

• The calculation of the apparent mass of the mechanism at the interface point.

Other future work should:

• Create a list of the virtual environments that have a return-to-home force requirement.

• Inquire about possible solutions to the inverse problem for available haptic workspace.

• Solve for the maximum possible force for CM based haptic interfaces.

• Explore the use of compliant joints in other haptic interface configurations, including 3D devices.
The inverse problem proposes the question, given a desired workspace, can we determine the joint characteristics and mechanism geometry to achieve it. The maximum force is proportional to the actuators, but since the links of the mechanism are more prone to failure, this is not a trivial question.

In conclusion, the successful integration of compliant mechanisms into haptic interfaces is an important step toward new and improved devices and experiences.
REFERENCES


APPENDIX A. DIMENSIONS AND MATERIAL PROPERTIES

The common dimensions of each pantograph modeled through this work are shown in Fig. A.1 and Table A.1.

![Pantograph Diagram](image)

Figure A.1: The link lengths and joint angles of the pantograph modeled throughout this thesis.

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<th>Link Lengths</th>
<th>Initial Angles</th>
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<tr>
<td>$R_2$</td>
<td>94 mm</td>
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<tr>
<td>$R_3$</td>
<td>150 mm</td>
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<tr>
<td>$R_4$</td>
<td>94 mm</td>
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<tr>
<td>$R_5$</td>
<td>150 mm</td>
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Table A.1: Geometry values relevant to the mechanisms and joint candidates

The dimensions used in the joint candidates can be seen in Table A.2
Table A.2

Material properties and geometry values for the mechanism and joint candidates. Subscripts of 1 refer to the base joints, and 2 refer to the distal joints. Several figures are referenced below.

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<tr>
<td>$v$</td>
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<td>$S_y$</td>
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<td>$S_y$</td>
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<td>$b$</td>
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<tr>
<td>$t$</td>
<td>0.439 mm</td>
</tr>
<tr>
<td>$L_1$</td>
<td>10 mm</td>
</tr>
<tr>
<td>$L_2$</td>
<td>15.38 mm</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Butterfly Flexural Pivot - ABS, see Fig. A.5</th>
<th>CORE Bearing - ABS, see Fig. A.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>10 mm</td>
</tr>
<tr>
<td>$h_f$</td>
<td>5 mm</td>
</tr>
<tr>
<td>$t$</td>
<td>0.5 mm</td>
</tr>
<tr>
<td>$\phi$</td>
<td>50 mm</td>
</tr>
<tr>
<td>$b_1$</td>
<td>7.21 mm</td>
</tr>
<tr>
<td>$b_2$</td>
<td>3.147 mm</td>
</tr>
<tr>
<td>$h_{1,2}$</td>
<td>0.654 mm</td>
</tr>
<tr>
<td>$b_1$</td>
<td>170 mm</td>
</tr>
<tr>
<td>$b_2$</td>
<td>74.197 mm</td>
</tr>
<tr>
<td>$R_s$</td>
<td>220 mm</td>
</tr>
<tr>
<td>$R_p$</td>
<td>220 mm</td>
</tr>
<tr>
<td>$R_r$</td>
<td>661.31 mm</td>
</tr>
</tbody>
</table>

Figure A.2: Schematic of Small Length Flexure Pivot and relevant dimensions.
Figure A.3: Schematic of Split-Tube Flexure and relevant dimensions.

Figure A.4: Schematic of Open-Cross Revolute Joint and relevant dimensions.
Figure A.5: Schematic of Butterfly Flexural Pivot and relevant dimensions.

Figure A.6: Schematic of CORE Bearing and relevant dimensions.
APPENDIX B. KINEMATIC ANALYSES

The analysis of a compliant pantograph mechanism can be represented as a mechanism having five links, including the ground link, with five joints. Two of these joints are fixed to ground and we refer to these as the base joints 2 and 4. The remaining three are referred to as the distal joints 3, 5, and \( P_3 \), with the middle joint \( P_3 \) being the interface point that the user would grip. The mechanism has two degrees-of-freedom.

The first component of our active compensation strategy is the position relationships that result from forward and inverse kinematics analyses. It should be noted that while axial drift due to the compliant joints will cause small deviations in the kinematic relationships, it is assumed that they are negligible to our current interests. Thus, axial drift is not included our current models.

B.1 Forward Kinematics

Solutions to the forward kinematics problem entail finding the location of the interface point \( P_3 \), when the link lengths \( R \), and base angles \( \theta_2 \) and \( \theta_4 \) are known. Schematics depicting these parameters can be seen in Fig. B.1. The derivation provided here, and resulting equations

![Figure B.1: Forward kinematics. The interface point, \( P_3 \), is determined from the link lengths and base angles, \( \theta_2 \) and \( \theta_4 \), at the intersection of the two circles.](image)
of the forward kinematics relationships are based on the approach presented in [44]. The origin is placed at the lower right hand joint, such that the positive z-axis comes out of the page. Several positions can be defined based on geometry, assuming that the interface point, \( P_3 \), is located at the intersection of two circles, each centered at \( P_2 \) and \( P_4 \).

The first step is to define the position for the distal joints, \( P_2 \) and \( P_4 \). This requires that we also define the position of \( P_1 \).

\[
P_1 = (x_1, y_1) = [-R_1, 0] \quad (B.1)
\]

\[
P_2 = (x_2, y_2) = [x_1 + R_2 \cos(\theta_2), y_1 + R_2 \sin(\theta_2)] \quad (B.2)
\]

\[
P_4 = (x_4, y_4) = [R_4 \cos(\theta_4), R_4 \sin(\theta_4)] \quad (B.3)
\]

Next, in order to solve for the intersection of circles, we define the vector that connects \( P_2 \) and \( P_4 \) starting at \( P_2 \).

\[
P_{2 \rightarrow 4} = (x_{2 \rightarrow 4}, y_{2 \rightarrow 4}) = P_4 - P_2 \quad (B.4)
\]

Now, we will define the distance along \( P_{2 \rightarrow 4} \) that connects \( P_2 \) to the base of \( P_h \), where \( P_h \) is the perpendicular vector that connects \( P_{2 \rightarrow 4} \) and \( P_3 \). We will call this distance, \( R_{2 \rightarrow h} \). Assume that two circles were defined, one at \( P_2 \), and the other at \( P_4 \). Since we are only interested in the magnitude along \( P_{2 \rightarrow 4} \) to reach \( P_h \), we can simplify the derivation by temporarily setting our origin and coordinate axis at \( P_2 \) where \( P_4 \) lies along the \( x' \) coordinate axis. We are only interested in the magnitude along \( P_{2 \rightarrow 4} \). Positive \( x' \) would go along \( P_{2 \rightarrow 4} \) toward \( P_4 \), and positive \( y' \) would perpendicular in the direction of \( P_3 \). The definition each circle would be as follows:

\[
(R_{2 \rightarrow h})^2 + R_h^2 = R_3^2 \quad (B.5)
\]

\[
\left( R_{2 \rightarrow h} - \|P_{2 \rightarrow 4}\| \right)^2 + R_h^2 = R_5^2 \quad (B.6)
\]

By subtracting \( B.6 \) from \( B.5 \), the \( R_h^2 \) terms cancel. Then we can expand the squared difference and the \( R_{2 \rightarrow h}^2 \) terms cancel leaving just a single, non-squared \( R_{2 \rightarrow h} \). We then rearrange and solve for,

\[
R_{2 \rightarrow h} = \frac{R_3^2 - R_5^2 + \|P_{2 \rightarrow 4}\|^2}{2\|P_{2 \rightarrow 4}\|}. \quad (B.7)
\]
The position of this $P_h$ along $P_{2→4}$, with respect to our original coordinate frame is given as,

$$
P_h = (x_h, y_h) = P_2 + \frac{R_{2→h}}{||P_{2→4}||} P_{2→4}.
$$  \hspace{1cm} (B.8)

Now, we will define the distance between $P_h$ and $P_3$. We will call this distance, $R_{h→3}$. This can be done by solving the right angle triangle of one of our circles, specifically the one centered at $P_2$.

$$
R_{h→3} = \sqrt{R_3^2 - (R_{2→h})^2}
$$  \hspace{1cm} (B.9)

Finally, the position of the interface point is given by,

$$
P_3 = \left[ x_h \mp \frac{R_{h→3}}{||P_{2→4}||} y_{2→4}, y_h \mp \frac{R_{h→3}}{||P_{2→4}||} x_{2→4} \right].
$$  \hspace{1cm} (B.10)

It is important to recognize that the useful solution for $P_3$ uses the negative sign in both $x$ and $y$ components. Using the positive sign would result in the mechanism being flipped into the closed configuration.

### B.2 Inverse Kinematics

Solutions to the inverse kinematics problem involve finding the base angles $\theta_2$ and $\theta_4$, when the link lengths $R$, and interface point $P_3$ are known. Schematics depicting these parameters can be seen in Fig. B.2. The derivation provided here, and resulting equations of the inverse kinematics

![Figure B.2: Inverse kinematics. The base angles, $\theta_2$ and $\theta_4$, are determined from the link lengths, interior angles, and interface point $P_3$.](image)
relationships are based on the approach presented in [44]. The origin is placed at the lower right hand joint, such that the positive z-axis comes out of the page. The angles are easily solved for by dividing the pantograph into three triangular regions and applying the law of cosines.

\[ \theta_2 = \alpha_2 + \beta_2, \]  

(B.11)

where

\[ \alpha_2 = \text{atan2} \left( \frac{y_{1 \rightarrow 3}}{x_{1 \rightarrow 3}} \right) \]  

(B.12)

and

\[ \beta_2 = \arccos \left( \frac{R_{22}^2 - R_{32}^2 + (R_{1 \rightarrow 3})^2}{2R_2(R_{1 \rightarrow 3})} \right). \]  

(B.13)

In similar fashion,

\[ \theta_4 = \pi - \alpha_4 - \beta_4, \]  

(B.14)

where

\[ \alpha_4 = \arccos \left( \frac{R_{42}^2 - R_{52}^2 + \|P_3\|^2}{2R_4\|P_3\|} \right) \]  

(B.15)

and

\[ \beta_4 = \text{atan2} \left( \frac{y_3}{-x_3} \right). \]  

(B.16)

These equations provide the inverse solution so long as \( \alpha_4 > 0 \) and \( \beta_2 > 0 \).

### B.3 Jacobian Relationship

The Jacobian matrix can be determined by differentiating the position of the interface point, \( P_3(x_3, y_3) \) as supplied by the forward kinematics solution, with respect to the input angles, \( \theta_2 \) and \( \theta_4 \). The springback forces and the compensation torques discussed in 3 are related to each other by the inverse of the transpose of the Jacobian.

\[
\begin{bmatrix}
F_x \\
F_y
\end{bmatrix} = J^{-T} \tau = \left[ \frac{\partial x_3}{\partial \theta_2} \quad \frac{\partial x_3}{\partial \theta_4} \right]^{-T} \begin{bmatrix} T_2 \\ T_4 \end{bmatrix},
\]  

(B.17)
The differentiation of each term was performed symbolically using MATLAB’s symbolic `diff()` function (See Appendix D.1.1 for source code). The resulting equations have been verified and are presented here.

\[
\frac{\partial x_3}{\partial \theta_2} = \frac{A_1 A_5}{2A_4} - R_2 \sin(\theta_2) - \frac{A_1 A_3 A_5}{2A_4^2} - \frac{A_6 \left( \frac{A_1 A_3}{2A_4} - \frac{A_1 A_3^2}{4A_4^2} \right)}{2 \sqrt{A_4} \left( R_3^2 - \frac{A_3^2}{4A_5^2 + 4A_6^2} \right)} + \frac{R_2 A_2 \cos(\theta_2)}{\sqrt{A_4}} + \frac{R_2 A_3 \sin(\theta_2)}{2A_4} - \frac{A_1 A_2 A_6}{2A_4^{3/2}} \tag{B.18}
\]

and

\[
\frac{\partial y_3}{\partial \theta_2} = R_2 \cos(\theta_2) - \frac{A_1 A_6}{2A_4} - \frac{A_1 A_2 A_5}{2A_4^{3/2}} - \frac{A_5 \left( \frac{A_1 A_3}{2A_4} - \frac{A_1 A_3^2}{4A_4^2} \right)}{2 \sqrt{A_4} \left( R_3^2 - \frac{A_3^2}{4A_5^2 + 4A_6^2} \right)} - \frac{R_2 A_3 \cos(\theta_2)}{2A_4} + \frac{R_2 A_2 \sin(\theta_2)}{\sqrt{A_4}} + \frac{A_1 A_3 A_6}{2A_4^2}, \tag{B.19}
\]

where

\[
A_1 = 2R_2 A_5 \sin(\theta_2) + 2R_2 A_6 \cos(\theta_2), \quad \tag{B.20}
\]

\[
A_2 = \sqrt{R_3^2 - \frac{A_3^2}{4A_4}}, \quad \tag{B.21}
\]

\[
A_3 = A_5^2 + A_6^2 + R_3^2 - R_5^2, \quad \tag{B.22}
\]

\[
A_4 = A_5^2 + A_6^2, \quad \tag{B.23}
\]

\[
A_5 = R_1 - R_2 \cos(\theta_2) + R_4 \cos(\theta_4), \quad \tag{B.24}
\]

\[
A_6 = R_2 \sin(\theta_2) - R_4 \sin(\theta_4). \quad \tag{B.25}
\]
\[ \frac{\partial x_3}{\partial \theta_4} = \frac{B_1 A_3 A_5}{2A_4^2} - \frac{B_1 A_5}{2A_4} + \left( A_6 \left( \frac{B_1 A_3}{2A_4} - \frac{B_1 A_3^2}{4A_4^2} \right) \right) / \left( 2\sqrt{A_4} \sqrt{R_3^2 - \frac{A_3^2}{4A_5^2 + 4A_6^2}} \right) - \frac{R_4 A_2 \cos(\theta_4)}{\sqrt{A_4}} - \frac{R_4 A_3 \sin(\theta_4)}{2A_4} + \frac{B_1 A_2 A_6}{2A_4^{3/2}} \] (B.26)

and

\[ \frac{\partial y_3}{\partial \theta_4} = \frac{B_1 A_6}{2A_4} + \frac{B_1 A_2 A_5}{2A_4^{3/2}} + \left( A_5 \left( \frac{B_1 A_3}{2A_4} - \frac{B_1 A_3^2}{4A_4^2} \right) \right) / \left( 2\sqrt{A_4} \sqrt{R_3^2 - \frac{A_3^2}{4A_5^2 + 4A_6^2}} \right) + \frac{R_4 A_3 \cos(\theta_4)}{2A_4} - \frac{R_4 A_2 \sin(\theta_4)}{\sqrt{A_4}} - \frac{B_1 A_3 A_6}{2A_4^{2}}, \] (B.27)

where

\[ B_1 = 2R_4 A_5 \sin(\theta_4) + 2A_6 R_4 \cos(\theta_4). \] (B.28)
APPENDIX C. VIRTUAL WORK ANALYSES

“The force-displacement characteristics of compliant mechanisms can be found by applying the principle of virtual work” [20]. In this appendix, we derive two important force-displacement relationships.

1. $F_x$ and $F_y$. The springback forces at the interface point, $P_3$, caused by a given angular displacement of the base joints, $\theta_2$ and $\theta_4$.

2. $T_2$ and $T_4$. The compensation torques required at the base joints in order to compensate the elastic springback of the compliant joints caused by a given angular displacement of the base joints, $\theta_2$ and $\theta_4$.

We follow a method outlined by Howell involving 14 steps:

1. Choose a generalized coordinate, $q$

2. Define the applied force vectors, $\mathbf{F}$

3. Define a position vector from the origin to the each force vector in step 2, $\mathbf{Z}$

4. Find the virtual displacement by differentiating the position vectors with respect to the generalized coordinate, $\delta \mathbf{Z} = (d\mathbf{Z}/dq)\delta q$

5. Calculate the virtual work due to forces by taking the dot product of the force vectors and virtual displacements, $\delta W_F = \mathbf{F} \cdot \delta \mathbf{Z}$

6. Define the applied moment vectors, $\mathbf{M}$

7. Define the angular displacement vectors of each moment in step 6, $\Theta = (\theta - \theta_0)\hat{k}$

8. Find the virtual angular displacements by differentiating the angular displacement with respect to the generalized coordinate, $\delta \Theta = (d\Theta/dq)\delta q$
9. Calculate the virtual work due to moments by taking the dot product of the moment vectors and virtual angular displacements, \( \delta W_M = \mathbf{M} \cdot \delta \Theta \)

10. Define potential energy terms already included, \( V \)

11. Calculate the virtual work due to potential energy by differentiating with respect to the negatively signed generalized coordinate, \( \delta W_V = -(dV/dq)\delta q \)

12. Calculate the total virtual work by adding the terms together, \( \delta W = \delta W_F + \delta W_M + \delta W_V \)

13. Set the total virtual work to zero, \( \delta W = 0 \)

14. Solve for the desired unknown

Our system has only two degrees of freedom, \( \theta_2 \) and \( \theta_4 \), thus limiting our analysis to two generalized coordinates. This means that we have only two known inputs. We are interested in four unknown relationships: two force related and two moment related. With four unknowns and two knowns, it becomes necessary that the 14 steps be divided into two partially redundant analyses. By separating into analyses with only forces or only torques, we can develop the desired relationships mentioned above.

C.1 Virtual Work of Forces

Here we will present the application of steps 1-5. Since there are two generalized coordinates, there will be two response for some steps. The following terms are valuable in solving for the springback forces, \( F_x \) and \( F_y \). These terms are illustrated in Fig. C.1.

1. \( q_1 = \theta_2 \) and \( q_2 = \theta_4 \)

2. \( \mathbf{F} = [F_x, F_y] \)

3. \( \mathbf{Z}_1 = \mathbf{P}_3 \)

4. (q1) \( dZ_1 = \frac{d\mathbf{P}_3}{d\theta_2} \delta \theta_2 \)

   (q2) \( dZ_1 = \frac{d\mathbf{P}_3}{d\theta_4} \delta \theta_4 \)

5. \( \delta W_F = \mathbf{F} \cdot \delta \mathbf{Z}_1 \)
The output springback forces are modeled at the interface point, $P_3$. Following steps 1-5 and 10-14 of the method presented by Howell [20], the relationships $F_x$ and $F_y$ as functions of $\theta_2$ and $\theta_4$ are derived. The spirals represent torsional springs at each joint with stiffness $k$, and are used in steps 10-11 of the analysis.

C.2 Virtual Work of Moments

We continue with the application of steps 6-9. The following terms are valuable in solving for the compensation torques, $T_2$ and $T_4$. These terms are illustrated in Fig. C.2.
6. \( M = T_2 \) and \( T_4 \)

7. (a) \( \Theta_2 = \theta_2 - \theta_{20} \hat{k} \),

(b) \( \Theta_3 = \theta_3 - \theta_{30} \hat{k} \),

(c) \( \Theta_4 = \theta_4 - \theta_{40} \hat{k} \),

(d) \( \Theta_5 = \theta_5 - \theta_{50} \hat{k} \)

8. (q1) \( \delta \Theta_2 = \frac{d\theta_2}{d\theta_2} \delta \theta_2 \hat{k} = \delta \theta_2 \hat{k} \), and \( \delta \Theta_4 = \frac{d\theta_4}{d\theta_2} \delta \theta_2 \hat{k} \)

(q2) \( \delta \Theta_2 = \frac{d\theta_2}{d\theta_4} \delta \theta_4 \hat{k} = \delta \theta_4 \hat{k} \)

9. \( \delta W_M = M \cdot \delta \Theta = T_2 \cdot \delta \Theta_2 + T_4 \cdot \delta \Theta_4 \)

C.3 Virtual Work of Potential Energy

Steps 10-11 introduce the virtual work done by potential energy terms. These terms are essential for both force and moment analyses since the relationships of interest primarily equate the energy stored in the compliant joints to the outside influence of forces and torques. Assuming each compliant joint can be represented as a pin joint with a torsional spring, potential energy is stored in each joint proportional to the square of the angular deflection of the joint.

10. (a) \( V_2 = \frac{(k_2 \Theta_2^2)}{2} \)

(b) \( V_3 = \frac{k_3 (\Theta_3 - \Theta_2)^2}{2} \)

(c) \( V_4 = \frac{(k_4 \Theta_4^2)}{2} \)

(d) \( V_5 = \frac{k_5 (\Theta_5 - \Theta_4)^2}{2} \)

(e) \( V_p = \frac{k_p (\Theta_3 - \Theta_5)^2}{2} \)

(f) \( V = V_2 + V_3 + V_4 + V_5 + V_p \)

11. (q1) \( \delta W_V = -\frac{dV}{d\theta_2} \delta \theta_2 \)

(q2) \( \delta W_V = -\frac{dV}{d\theta_4} \delta \theta_4 \)
C.4 Results

Our analysis concludes with the application of steps 12-14. There will be one set of equations pertaining to the force analysis, and another set pertaining to the moment analysis.

12. (F) \[ \delta W = \delta W_F + \delta W_V \]
   (M) \[ \delta W = \delta W_M + \delta W_V \]

13. (F) (q1) \[ 0 = \left( \mathbf{F} \cdot \frac{d\mathbf{P}_3}{d\theta_2} \right) - \left( \frac{d\mathbf{V}}{d\theta_2} \right) \]
   (q2) \[ 0 = \left( \mathbf{F} \cdot \frac{d\mathbf{P}_3}{d\theta_4} \right) - \left( \frac{d\mathbf{V}}{d\theta_4} \right) \]
   (M) (q1) \[ 0 = \left( \mathbf{T}_2 + \mathbf{T}_4 \right) \frac{d\theta_4}{d\theta_2} - \left( \frac{d\mathbf{V}}{d\theta_2} \right) \]
   (q2) \[ 0 = \left( \mathbf{T}_2 \frac{d\theta_2}{d\theta_4} + \mathbf{T}_4 \right) - \left( \frac{d\mathbf{V}}{d\theta_4} \right) \]

Step 14, is to solve these equations of step 13 for the desired unknowns, \( F_x, F_y, T_2, \) and \( T_4. \) This is done using MATLAB’s symbolic functions and solver (see Appendix D.1.2 and D.1.3 for source code). By solving for one of the desired variables and substituting it into the remaining equation, it is possible to generate an analytical solution in terms of \( \theta_2 \) and \( \theta_4. \) The equations for \( F_x, F_y, T_2, \) and \( T_4 \) that result from applying the principle of virtual work are very complicated and highly nonlinear, making their direct interpretation infeasible. They are provided below as a set of nested equations. The main component is presented first, followed by several building block relationships.

\[
F_x = - \left[ k_4 (\theta_4 - \theta_{40}) + k_p A_5 \left( A_1 - \frac{A_4}{A_8} \right) + \left( k_2 (\theta_2 - \theta_{20}) - k_p A_5 \left( \frac{A_3}{A_7} - A_2 \right) \right. \right. \\
+ k_3 (A_2 + 1)(\theta_2 - \theta_{20} + \theta_{30} - A_{18}) - \frac{k_5 A_3 A_6}{A_7} \left( \frac{A_{11}}{A_9} + k_5 A_6 (A_1 - 1) \right) \left. \right] \\
- \left[ \frac{k_3 A_4 (\theta_2 - \theta_{20} + \theta_{30} - A_{18})}{A_8} \right] \left/ \left( \frac{A_{10} A_{11}}{A_9} - A_{12} - A_{13} - A_{14} + A_{15} + A_{16} + A_{17} \right) \right. \right)
\]  

(C.1)
where

\[ A_1 = \left( A_{33} - A_{34} + A_{35} - A_{36} + A_{37} + A_{38} - R_4 \cos(\theta_4) \right) / (A_7 A_{19}) \]
\[ - \left( A_{20} (R_4 \sin(\theta_4) + A_{12} + A_{13} + A_{14} - A_{15} - A_{16} - A_{17}) \right) / \left( A_7 A_{19}^2 \right) \]  \quad (C.2)

\[ A_2 = \left( A_{21} - A_{22} + A_{23} - A_{24} + A_{25} + A_{26} \right) / \left( A_8 (A_{39} + A_{40}) \right) \]
\[ + \left( (-A_{41} + A_{42}) (-A_{27} - A_{28} + A_{29} + A_{30} + A_{31} - A_{32}) \right) / \left( A_8 (A_{39} + A_{40})^2 \right) \]  \quad (C.3)

\[ A_3 = \frac{A_9}{A_{19}} - \frac{A_{10} A_{20}}{A_{19}^2} \]  \quad (C.4)

\[ A_4 = \frac{A_{11}}{A_{39} + A_{40}} + \frac{(-A_{41} + A_{42}) (-A_{12} - A_{13} - A_{14} + A_{15} + A_{16} + A_{17})}{(A_{39} + A_{40})^2} \]  \quad (C.5)

\[ A_5 = \pi + \theta_{30} - \theta_{50} + \arctan \left( \frac{A_{20}}{A_{19}} \right) - A_{18} \]  \quad (C.6)

\[ A_6 = \pi - \theta_{4} + \theta_{40} - \theta_{50} + \arctan \left( \frac{A_{20}}{A_{19}} \right) \]  \quad (C.7)

\[ A_7 = \frac{A_{20}^2}{A_{19}^2} + 1 \]  \quad (C.8)

\[ A_8 = \frac{(-A_{41} + A_{42})^2}{(A_{39} + A_{40})^2} + 1 \]  \quad (C.9)

\[ A_9 = A_{21} - A_{22} + A_{23} - A_{24} + A_{25} + A_{26} - R_2 \cos(\theta_2) \]  \quad (C.10)

\[ A_{10} = R_2 \sin(\theta_2) + A_{27} + A_{28} - A_{29} - A_{30} - A_{31} + A_{32} \]  \quad (C.11)

\[ A_{11} = A_{33} - A_{34} + A_{35} - A_{36} + A_{37} + A_{38} \]  \quad (C.12)

\[ A_{12} = \frac{A_{44} A_{52}}{2 A_{46} \sqrt{A_{50}}} \]  \quad (C.13)

\[ A_{13} = \frac{A_{46} A_{48} A_{52}}{A_{45}} \]  \quad (C.14)

\[ A_{14} = \frac{A_{48} A_{49} A_{51}}{2 A_{50}^2} \]  \quad (C.15)
\[ A_{15} = \frac{R_4 A_{49} \sin(\theta_4)}{2A_{50}} \]  
(C.16)

\[ A_{16} = \frac{R_4 A_{46} \cos(\theta_4)}{\sqrt{A_{50}}} \]  
(C.17)

\[ A_{17} = \frac{A_{48} A_{51}}{2A_{50}} \]  
(C.18)

\[ A_{18} = \arctan\left(\frac{-A_{41} + A_{42}}{A_{39} + A_{40}}\right) \]  
(C.19)

\[ A_{19} = R_2 \cos(\theta_2) - R_1 - R_4 \cos(\theta_4) + A_{39} + A_{40} \]  
(C.20)

\[ A_{20} = R_2 \sin(\theta_2) - R_4 \sin(\theta_4) - A_{41} + A_{42} \]  
(C.21)

\[ A_{21} = \frac{A_{43} A_{51}}{2A_{46} \sqrt{A_{50}}} \]  
(C.22)

\[ A_{22} = \frac{A_{47} A_{49} A_{52}}{2A_{50}^2} \]  
(C.23)

\[ A_{23} = \frac{R_2 A_{49} \cos(\theta_2)}{2A_{50}} \]  
(C.24)

\[ A_{24} = \frac{R_2 A_{46} \sin(\theta_2)}{\sqrt{A_{50}}} \]  
(C.25)

\[ A_{25} = \frac{A_{47} A_{52}}{2A_{50}} \]  
(C.26)

\[ A_{26} = \frac{A_{46} A_{47} A_{51}}{A_{45}} \]  
(C.27)

\[ A_{27} = \frac{A_{43} A_{52}}{2A_{46} \sqrt{A_{50}}} \]  
(C.28)

\[ A_{28} = \frac{A_{47} A_{49} A_{51}}{2A_{50}^2} \]  
(C.29)

\[ A_{29} = \frac{R_2 A_{49} \sin(\theta_2)}{2A_{50}} \]  
(C.30)

\[ A_{30} = \frac{R_2 A_{46} \cos(\theta_2)}{\sqrt{A_{50}}} \]  
(C.31)

\[ A_{31} = \frac{A_{47} A_{51}}{2A_{50}} \]  
(C.32)

\[ A_{32} = \frac{A_{46} A_{47} A_{52}}{A_{45}} \]  
(C.33)
\[ A_{33} = \frac{A_{44}A_{51}}{2A_{46}\sqrt{A_{50}}} \]  
(C.34)

\[ A_{34} = \frac{A_{48}A_{49}A_{52}}{2A_{50}^2} \]  
(C.35)

\[ A_{35} = \frac{R_4A_{49}\cos(\theta_4)}{2A_{50}} \]  
(C.36)

\[ A_{36} = \frac{R_4A_{46}\sin(\theta_4)}{\sqrt{A_{50}}} \]  
(C.37)

\[ A_{37} = \frac{A_{48}A_{52}}{2A_{50}} \]  
(C.38)

\[ A_{38} = \frac{A_{46}A_{48}A_{51}}{A_{45}} \]  
(C.39)

\[ A_{39} = \frac{A_{49}A_{51}}{2A_{50}} \]  
(C.40)

\[ A_{40} = \frac{A_{46}A_{52}}{\sqrt{A_{50}}} \]  
(C.41)

\[ A_{41} = \frac{A_{49}A_{52}}{2A_{50}} \]  
(C.42)

\[ A_{42} = \frac{A_{46}A_{51}}{\sqrt{A_{50}}} \]  
(C.43)

\[ A_{43} = \frac{A_{47}A_{49}}{2A_{50}} - \frac{A_{47}A_{49}^2}{4A_{50}^2} \]  
(C.44)

\[ A_{44} = \frac{A_{48}A_{49}}{2A_{50}} - \frac{A_{48}A_{49}^2}{4A_{50}^2} \]  
(C.45)

\[ A_{45} = 2A_{50}^{3/2} \]  
(C.46)

\[ A_{46} = \sqrt{R_3^2 - \frac{A_{49}^2}{4A_{50}}} \]  
(C.47)

\[ A_{47} = 2R_2A_{51}\sin(\theta_2) + 2R_2A_{52}\cos(\theta_2) \]  
(C.48)

\[ A_{48} = 2R_4A_{51}\sin(\theta_4) + 2R_4A_{52}\cos(\theta_4) \]  
(C.49)

\[ A_{49} = A_{51}^2 + R_3^2 - R_5^2 + A_{52}^2 \]  
(C.50)

\[ A_{50} = A_{51}^2 + A_{52}^2 \]  
(C.51)
\[ A_{51} = R_1 - R_2 \cos(\theta_2) + R_4 \cos(\theta_4) \]  
(C.52)

\[ A_{52} = R_2 \sin(\theta_2) - R_4 \sin(\theta_4) \]  
(C.53)

\[ F_y = \left[ -B_1 + B_2 + B_3 - B_4 + \left( k_4(\theta_4 - \theta_{40}) + k_p B_7 \left( B_5 - \frac{B_6}{B_{21}} \right) + \frac{B_{12}(B_1 - B_2 - B_3 + B_4)}{B_{19}} \right) \right] \]

\[ + k_5 B_9 (B_5 - 1) - \frac{k_3 B_6 \left( \theta_2 - \theta_{20} + \theta_{30} - \arctan \left( \frac{B_{34}}{B_{35}} \right) \right)}{B_{21}} \]  
\( B_{20} \]

\[ \left/ \left( \frac{B_{12} B_{20}}{B_{19}} - B_{13} - B_{14} - B_{15} + B_{16} + B_{17} + B_{18} \right) \right/ B_{19} \]  
(C.54)

where

\[ B_1 = k_2 (\theta_2 - \theta_{20}) \]  
(C.55)

\[ B_2 = k_p B_7 \left( \frac{B_8}{B_{11}} - B_{10} \right) \]  
(C.56)

\[ B_3 = \frac{k_5 B_8 B_9}{B_{11}} \]  
(C.57)

\[ B_4 = k_3 (B_{10} + 1) \left( \theta_2 - \theta_{20} + \theta_{30} - \arctan \left( \frac{B_{34}}{B_{35}} \right) \right) \]  
(C.58)
\[ B_5 = \left( \frac{B_{24} - B_{25} + B_{26} - R_4 \cos(\theta_4) + \frac{B_{39}B_{44}}{2B_{42}} - \frac{R_4B_{38} \sin(\theta_4)}{\sqrt{B_{42}}} + \frac{B_{38}B_{39}B_{43}}{B_{30}}}{(B_{11}B_{22})} \right) - \left( \frac{B_{23} \left( R_4 \sin(\theta_4) + B_{13} + B_{14} + B_{15} - B_{16} - B_{17} - B_{18} \right)}{(B_{11}B_{22})^2} \right) \] (C.59)

\[ B_6 = \frac{B_{12}}{B_{35}} - \left( \frac{B_{34} \left( B_{13} + B_{14} + B_{15} - B_{16} - B_{17} - B_{18} \right)}{(B_{35})^2} \right) \] (C.60)

\[ B_7 = \pi + \theta_3 - \theta_5 + \arctan \left( \frac{B_{23}}{B_{22}} \right) - \arctan \left( \frac{B_{34}}{B_{35}} \right) \] (C.61)

\[ B_8 = \frac{B_{19}}{B_{22}} - \frac{B_{20}B_{23}}{B_{22}^2} \] (C.62)

\[ B_0 = \pi - \theta_4 - \theta_5 + \theta_0 + \arctan \left( \frac{B_{23}}{B_{22}} \right) \] (C.63)

\[ B_{10} = \left( \frac{B_{27} - B_{28} + B_{29} + \frac{B_{40}B_{44}}{2B_{42}} - \frac{R_2B_{38} \sin(\theta_2)}{\sqrt{B_{42}}} + \frac{B_{38}B_{40}B_{43}}{B_{30}}}{(B_{21}B_{35})} \right) - \left( \frac{B_{34} \left( B_{31} + B_{32} - B_{33} - \frac{B_{40}B_{43}}{2B_{42}} - \frac{R_2B_{38} \cos(\theta_2)}{\sqrt{B_{42}}} + \frac{B_{38}B_{40}B_{44}}{B_{30}} \right)}{(B_{21}B_{35})^2} \right) \] (C.64)

\[ B_{11} = \frac{B_{23}^2}{B_{22}^2} + 1 \] (C.65)

\[ B_{12} = B_{24} - B_{25} + B_{26} + \frac{B_{39}B_{44}}{2B_{42}} - \frac{R_4B_{38} \sin(\theta_4)}{\sqrt{B_{42}}} + \frac{B_{38}B_{39}B_{43}}{B_{30}} \] (C.66)

\[ B_{13} = \frac{B_{36}B_{44}}{2B_{38}\sqrt{B_{42}}} \] (C.67)

\[ B_{14} = \frac{B_{38}B_{39}B_{44}}{B_{30}} \] (C.68)

\[ B_{15} = \frac{B_{39}B_{41}B_{43}}{2B_{42}^2} \] (C.69)

\[ B_{16} = \frac{R_4B_{41} \sin(\theta_4)}{2B_{42}} \] (C.70)

\[ B_{17} = \frac{R_4B_{38} \cos(\theta_4)}{\sqrt{B_{42}}} \] (C.71)

\[ B_{18} = \frac{B_{36}B_{43}}{2B_{42}} \] (C.72)
\[ B_{19} = -R_2 \cos(\theta_2) + B_{27} - B_{28} + B_{29} + \frac{B_{40}B_{44}}{2B_{42}} - \frac{R_2 B_{38} \sin(\theta_2)}{\sqrt{B_{42}}} + \frac{B_{38}B_{40}B_{43}}{B_{30}} \]  
\[ (C.73) \]

\[ B_{20} = R_2 \sin(\theta_2) + B_{31} + B_{32} - B_{33} - \frac{B_{40}B_{43}}{2B_{42}} - \frac{R_2 B_{38} \cos(\theta_2)}{\sqrt{B_{42}}} + \frac{B_{38}B_{40}B_{44}}{B_{30}} \]  
\[ (C.74) \]

\[ B_{21} = \frac{B_{34}^2}{B_{35}^2} + 1 \]  
\[ (C.75) \]

\[ B_{22} = R_2 \cos(\theta_2) - R_1 - R_4 \cos(\theta_4) + \frac{B_{41}B_{43}}{2B_{42}} + \frac{B_{38}B_{44}}{\sqrt{B_{42}}} \]  
\[ (C.76) \]

\[ B_{23} = R_2 \sin(\theta_2) - R_4 \sin(\theta_4) + \frac{B_{38}B_{43}}{\sqrt{B_{42}}} - \frac{B_{41}B_{44}}{2B_{42}} \]  
\[ (C.77) \]

\[ B_{24} = \frac{B_{36}B_{43}}{2B_{38}\sqrt{B_{42}}} \]  
\[ (C.78) \]

\[ B_{25} = \frac{B_{39}B_{41}B_{44}}{2B_{42}^2} \]  
\[ (C.79) \]

\[ B_{26} = \frac{R_4B_{41} \cos(\theta_4)}{2B_{42}} \]  
\[ (C.80) \]

\[ B_{27} = \frac{B_{37}B_{43}}{2B_{38}\sqrt{B_{42}}} \]  
\[ (C.81) \]

\[ B_{28} = \frac{B_{40}B_{41}B_{44}}{2B_{42}^2} \]  
\[ (C.82) \]

\[ B_{29} = \frac{R_2B_{41} \cos(\theta_2)}{2B_{42}} \]  
\[ (C.83) \]

\[ B_{30} = 2B_{42}^{3/2} \]  
\[ (C.84) \]

\[ B_{31} = \frac{B_{37}B_{44}}{2B_{38}\sqrt{B_{42}}} \]  
\[ (C.85) \]

\[ B_{32} = \frac{B_{40}B_{41}B_{43}}{2B_{42}^2} \]  
\[ (C.86) \]

\[ B_{33} = \frac{R_2B_{41} \sin(\theta_2)}{2B_{42}} \]  
\[ (C.87) \]

\[ B_{34} = \frac{B_{38}B_{43}}{\sqrt{B_{42}}} - \frac{B_{41}B_{44}}{2B_{42}} \]  
\[ (C.88) \]

\[ B_{35} = \frac{B_{41}B_{43}}{2B_{42}} + \frac{B_{38}B_{44}}{\sqrt{B_{42}}} \]  
\[ (C.89) \]
\[
B_{36} = \frac{B_{39}B_{41}}{2B_{42}} - \frac{B_{39}B_{41}^2}{4B_{42}^2} \quad (C.90)
\]
\[
B_{37} = \frac{B_{40}B_{41}}{2B_{42}} - \frac{B_{40}B_{41}^2}{4B_{42}^2} \quad (C.91)
\]
\[
B_{38} = \sqrt{R_3^2 - \frac{B_{41}^2}{4B_{42}}} \quad (C.92)
\]
\[
B_{39} = 2R_4B_{43}\sin(\theta_4) + 2R_4B_{44}\cos(\theta_4) \quad (C.93)
\]
\[
B_{40} = 2R_2B_{43}\sin(\theta_2) + 2R_2B_{44}\cos(\theta_2) \quad (C.94)
\]
\[
B_{41} = R_3^2 - R_5^2 + B_{43}^2 + B_{44}^2 \quad (C.95)
\]
\[
B_{42} = B_{43}^2 + B_{44}^2 \quad (C.96)
\]
\[
B_{43} = R_1 - R_2\cos(\theta_2) + R_4\cos(\theta_4) \quad (C.97)
\]
\[
B_{44} = R_2\sin(\theta_2) - R_4\sin(\theta_4) \quad (C.98)
\]
\[
T_2 = k_2(\theta_2 - \theta_{2,0}) - k_p\left(\frac{C_2}{C_3} - C_1\right) \left(\pi + \theta_{3,0} - \theta_{5,0} + \arctan\left(\frac{C_{17}}{C_{16}}\right) - \arctan\left(\frac{C_4}{C_5}\right)\right)
+ k_3(C_1 + 1) \left(\theta_2 - \theta_{2,0} + \theta_{3,0} - \arctan\left(\frac{C_4}{C_5}\right)\right)
- k_5C_2 \left(\pi - \theta_4 + \theta_{4,0} - \theta_{5,0} + \arctan\left(\frac{C_{17}}{C_{16}}\right)\right) / C_3 \quad (C.99)
\]

where

\[
C_1 = \left(\frac{C_{24}C_{20}}{2C_{22}} + C_6 + C_9 - C_{10} + C_{13} - C_{14}\right) / \left(\frac{C_4^2}{C_5} + C_5\right)
+ \left(C_4\left(\frac{C_{20}C_{23}}{2C_{22}} - C_7 - C_8 - C_{11} + C_{12} + C_{15}\right)\right) / \left(C_4^2 + C_5^2\right) \quad (C.100)
\]
\[ C_2 = \left( \frac{C_{24}C_{20}}{2C_{22}} - R_2 \cos(\theta_2) + C_6 + C_9 - C_{10} - C_{13} - C_{14} \right) / C_{16} \]

\[ - \left( C_{17} \left( R_2 \sin(\theta_2) - \frac{C_{20}C_{23}}{2C_{22}} + C_7 + C_8 + C_{11} - C_{12} - C_{15} \right) \right) / C_{16}^2 \]  

(C.101)

\[ C_3 = \frac{C_{17}^2}{C_{16}^2} + 1 \]  

(C.102)

\[ C_4 = \frac{C_{19}C_{23}}{\sqrt{C_{22}}} - \frac{C_{21}C_{24}}{2C_{22}} \]  

(C.103)

\[ C_5 = \frac{C_{21}C_{23}}{2C_{22}} + \frac{C_{19}C_{24}}{\sqrt{C_{22}}} \]  

(C.104)

\[ C_6 = \frac{C_{18}C_{23}}{2C_{19}\sqrt{C_{22}}} \]  

(C.105)

\[ C_7 = \frac{C_{18}C_{24}}{2C_{19}\sqrt{C_{22}}} \]  

(C.106)

\[ C_8 = \frac{C_{19}C_{20}C_{24}}{2C_{22}^{3/2}} \]  

(C.107)

\[ C_9 = \frac{C_{19}C_{20}C_{23}}{2C_{22}^{1/2}} \]  

(C.108)

\[ C_{10} = \frac{C_{20}C_{21}C_{24}}{2C_{22}^2} \]  

(C.109)

\[ C_{11} = \frac{C_{20}C_{21}C_{23}}{2C_{22}^2} \]  

(C.110)

\[ C_{12} = \frac{R_2C_{21}\sin(\theta_2)}{2C_{22}} \]  

(C.111)

\[ C_{13} = \frac{R_2C_{21}\cos(\theta_2)}{2C_{22}} \]  

(C.112)

\[ C_{14} = \frac{R_2C_{19}\sin(\theta_2)}{\sqrt{C_{22}}} \]  

(C.113)

\[ C_{15} = \frac{R_2C_{19}\cos(\theta_2)}{\sqrt{C_{22}}} \]  

(C.114)

\[ C_{16} = R_2 \cos(\theta_2) - R_1 - R_4 \cos(\theta_4) + \frac{C_{21}C_{23}}{2C_{22}} + \frac{C_{19}C_{24}}{\sqrt{C_{22}}} \]  

(C.115)

\[ C_{17} = R_2 \sin(\theta_2) - R_4 \sin(\theta_4) + \frac{C_{19}C_{23}}{\sqrt{C_{22}}} - \frac{C_{21}C_{24}}{2C_{22}} \]  

(C.116)
\[ C_{18} = \frac{C_{20}C_{21}}{2C_{22}} - \frac{C_{20}C_{21}^2}{4C_{22}^2} \]  
(C.117)

\[ C_{19} = \sqrt{R^2 - \frac{C_{21}^2}{4C_{22}}} \]  
(C.118)

\[ C_{20} = 2R_2C_{23}\sin(\theta_2) + 2R_2C_{24}\cos(\theta_2) \]  
(C.119)

\[ C_{21} = R_3^2 - R_5^2 + C_{23}^2 + C_{24}^2 \]  
(C.120)

\[ C_{22} = C_{23}^2 + C_{24}^2 \]  
(C.121)

\[ C_{23} = R_1 - R_2\cos(\theta_2) + R_4\cos(\theta_4) \]  
(C.122)

\[ C_{24} = R_2\sin(\theta_2) - R_4\sin(\theta_4) \]  
(C.123)

\[ T_4 = k_4(\theta_4 - \theta_{40}) + k_p \left( D_1 - \frac{D_2}{D_3} \right) \left( \pi + \theta_{30} - \theta_{50} + \arctan \left( \frac{D_5}{D_4} \right) - \arctan \left( \frac{D_{16}}{D_{17}} \right) \right) \]
\[ + k_5(D_1 - 1) \left( \pi - \theta_4 + \theta_{40} - \theta_{50} + \arctan \left( \frac{D_5}{D_4} \right) \right) \]
\[ - \left[ k_3D_2 \left( \theta_2 - \theta_{20} + \theta_{30} - \arctan \left( \frac{D_{16}}{D_{17}} \right) \right) \right] / D_3 \]  
(C.124)

where

\[ D_1 = \left( \frac{D_{20}D_{24}}{2D_{22}} - R_4\cos(\theta_4) + D_6 + D_9 - D_{10} + D_{13} - D_{14} \right) / \left( \frac{D_5^2}{D_4} + D_4 \right) \]
\[ - \left( D_5 \left( R_4\sin(\theta_4) - \frac{D_{20}D_{23}}{2D_{22}} + D_7 + D_8 + D_{11} - D_{12} - D_{15} \right) \right) / \left( D_5^2 + D_4^2 \right) \]  
(C.125)

\[ D_2 = \left( \frac{D_{20}D_{24}}{2D_{22}} + D_6 + D_9 - D_{10} + D_{13} - D_{14} \right) / D_{17} \]
\[ + \left( D_{16} \left( \frac{D_{20}D_{23}}{2D_{22}} - D_7 - D_8 - D_{11} + D_{12} + D_{15} \right) \right) / D_{17}^2 \]  
(C.126)
\[ D_3 = \frac{D_{16}^2}{D_{17}^2} + 1 \]  
\[ D_4 = R_2 \cos(\theta_2) - R_1 - R_4 \cos(\theta_4) + \frac{D_{21}D_{23}}{2D_{22}} + \frac{D_{19}D_{24}}{\sqrt{D_{22}}} \]  
\[ D_5 = R_2 \sin(\theta_2) - R_4 \sin(\theta_4) + \frac{D_{19}D_{23}}{\sqrt{D_{22}}} - \frac{D_{21}D_{24}}{2D_{22}} \]  
\[ D_6 = \frac{D_{18}D_{23}}{2D_{19}\sqrt{D_{22}}} \]  
\[ D_7 = \frac{D_{18}D_{24}}{2D_{19}\sqrt{D_{22}}} \]  
\[ D_8 = \frac{D_{19}D_{20}D_{24}}{2D_{22}^{3/2}} \]  
\[ D_9 = \frac{D_{19}D_{20}D_{23}}{2D_{22}^{3/2}} \]  
\[ D_{10} = \frac{D_{20}D_{21}D_{24}}{2D_{22}^2} \]  
\[ D_{11} = \frac{D_{20}D_{21}D_{23}}{2D_{22}^2} \]  
\[ D_{12} = \frac{R_4D_{21}\sin(\theta_4)}{2D_{22}} \]  
\[ D_{13} = \frac{R_4D_{21}\cos(\theta_4)}{2D_{22}} \]  
\[ D_{14} = \frac{R_4D_{19}\sin(\theta_4)}{\sqrt{D_{22}}} \]  
\[ D_{15} = \frac{R_4D_{19}\cos(\theta_4)}{\sqrt{D_{22}}} \]  
\[ D_{16} = \frac{D_{19}D_{23}}{\sqrt{D_{22}}} - \frac{D_{21}D_{24}}{2D_{22}} \]  
\[ D_{17} = \frac{D_{21}D_{23}}{2D_{22}} + \frac{D_{19}D_{24}}{\sqrt{D_{22}}} \]  
\[ D_{18} = \frac{D_{20}D_{21}}{2D_{22}} - \frac{D_{20}D_{21}^2}{4D_{22}^2} \]  
\[ D_{19} = \sqrt{R_3^2 - \frac{D_{21}^2}{4D_{22}}} \]
\[ D_{20} = 2R_4D_{23} \sin(\theta_4) + 2R_4D_{24} \cos(\theta_4) \]  
(C.144)

\[ D_{21} = R_3^2 - R_5^2 + D_{23}^2 + D_{24}^2 \]  
(C.145)

\[ D_{22} = D_{23}^2 + D_{24}^2 \]  
(C.146)

\[ D_{23} = R_1 - R_2 \cos(\theta_2) + R_4 \cos(\theta_4) \]  
(C.147)

\[ D_{24} = R_2 \sin(\theta_2) - R_4 \sin(\theta_4) \]  
(C.148)
APPENDIX D. SOURCE CODE

Source code is provided here to document the scripts used in deriving the virtual work solutions, calculating response values, and executing the hardware demonstration. The scripts were written and developed using MATLAB 2014b.

D.1 Model Derivation

D.1.1 Jacobian.m

```matlab
% File: Jacobian.m
% Author: Dallin Swiss
% Date: June 24, 2014
% Description: This is code for the research. It determines the jacobian relationship between force and torque

format compact
clear
clc

% Variables
% Link Lengths
R1 = sym('R1'); % [m]
R2 = sym('R2'); % [m]
R3 = sym('R3'); % [m]
R4 = sym('R4'); % [m]
R5 = sym('R5'); % [m]

% Angle between ground joints
```
th1 = pi; % [rad] the ground joints are always level
th2 = sym('th2'); % [rad]
th3 = sym('th3'); % [rad]
th4 = sym('th4'); % [rad]
th5 = sym('th5'); % [rad]

% Trignometric abbreviations
c1 = -1;
s1 = 0;
c2 = cos(th2);
s2 = sin(th2);
c3 = cos(th3);
s3 = sin(th3);
c4 = cos(th4);
s4 = sin(th4);
c5 = cos(th5);
s5 = sin(th5);

%% Solve for intersection of circles
% Define position vectors
P1 = [R1*c1, R1*s1];
P2 = [P1(1)+R2*c2, P1(2)+R2*s2];
P4 = [R4*c4, R4*s4];

% Vector from points 2->4
P24 = P4-P2;
R24 = sqrt(P24(1)^2+P24(2)^2);

% Vector along P24 that stops at the intersection of C3 & C5
R2h = (R3^2-R5^2+R24^2)/(2*R24);
Ph = P2+R2h/R24*P24;

% Vector to the interface point (P3)
R3h = sqrt(R3^2-R2h^2);
P3 = [Ph(1)-R3h/R24*P24(2), Ph(2)+R3h/R24*P24(1)];
Determine the Jacobian

The jacobian forms a relationship between the desired output forces and the required input torques needed to generate the forces. It is a 2x2 matrix:

\[
J = \begin{bmatrix}
\frac{dx3}{d\theta2} & \frac{dx3}{d\theta4} \\
\frac{dy3}{d\theta2} & \frac{dy3}{d\theta4}
\end{bmatrix}
\]

\[
dx32 = \text{diff}(P3(1),'th2');
\]
\[
dx34 = \text{diff}(P3(1),'th4');
\]
\[
dy32 = \text{diff}(P3(2),'th2');
\]
\[
dy34 = \text{diff}(P3(2),'th4');
\]

Form the Jacobian

\[
J = \begin{bmatrix}
dx32 & dx34 \\
dy32 & dy34
\end{bmatrix}
\]

Determine the transpose of the Jacobian

\[
J\_trans = \begin{bmatrix}
dx32 & dy32 \\
dx34 & dy34
\end{bmatrix}
\]

Display the equations

\[
\text{pretty(dx32)}
\]
\[
\text{pretty(dx34)}
\]
\[
\text{pretty(dy32)}
\]
\[
\text{pretty(dy34)}
\]

Write symbolic equation to file

\[
\text{fileID1 = fopen('dx32.txt','wt');}
\]
\[
\text{fileID2 = fopen('dx34.txt','wt');}
\]
\[
\text{fileID3 = fopen('dy32.txt','wt');}
\]
\[
\text{fileID4 = fopen('dy34.txt','wt');}
\]
\[
\text{dx32\_equation = char(dx32);}
\]
\[
\text{dx34\_equation = char(dx34);}
\]
\[
\text{dy32\_equation = char(dy32);}
\]
\[
\text{dy34\_equation = char(dy34);}
\]
D.1.2 Virtual Work Forces.m

% File: Virtual_Work_Forces.m
% Author: Dallin Swiss
% Date: May 31, 2014
% Description: This is code for the research. It determines the
% relationship between input position and output forces

format compact
clear
clc

%% Variables
% Link Lengths
R1 = sym('R1'); % [m]
R2 = sym('R2'); % [m]
R3 = sym('R3'); % [m]
R4 = sym('R4'); % [m]
R5 = sym('R5'); % [m]

% Angle between ground joints
th1 = pi; % [rad] the ground joints are always level
th2 = sym('th2'); % [rad]
th3 = sym('th3'); % [rad]
th4 = sym('th4'); % [rad]
th5 = sym('th5'); % [rad]

% Trigonometric abbreviations
cl = -1;
s1 = 0;
c2 = cos(th2);
s2 = sin(th2);
c3 = cos(th3);
s3 = sin(th3);
c4 = cos(th4);
s4 = sin(th4);
c5 = cos(th5);
s5 = sin(th5);

%% Solve for intersection of circles
% Define position vectors
P1 = [R1*c1, R1*s1];
P2 = [P1(1)+R2*c2, P1(2)+R2*s2];
P4 = [R4*c4, R4*s4];

% Vector from points 2->4
P24 = P4-P2;
R24 = sqrt(P24(1)ˆ2+P24(2)ˆ2);

% Vector along P24 that stops at the intersection of C3 & C5
R2h = (R3ˆ2-R5ˆ2+R24ˆ2)/(2*R24);
Ph = P2+R2h/R24*P24;

% Vector to the interface point (P3)
R3h = sqrt(R3ˆ2-R2hˆ2);
P3 = [Ph(1)-R3h/R24*P24(2), Ph(2)+R3h/R24*P24(1)];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% ---------------------- Warning ----------------------%
% The range of th3 and th5 around limited only to the %
% region directly in front of the device. This ought %
% to be sufficient for most workspaces. However, it %
% could be expanded if necessary, but the analysis %
% need logic which is not supported for symbolic %
% solutions.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Solution angles

% Joint angles


% % Quadrant 1
% th3 = atan((P3(2)-P2(2))/(P3(1)-P2(1)));   
% th5 = atan((P3(2)-P4(2))/(P3(1)-P4(1)));   

% % Quadrant 2
th3 = atan((P3(2)-P2(2))/(P3(1)-P2(1)));   
th5 = atan((P3(2)-P4(2))/(P3(1)-P4(1)))+pi;  

% % Quadrant 3
% th3 = atan((P3(2)-P2(2))/(P3(1)-P2(1)))+pi;
% th5 = atan((P3(2)-P4(2))/(P3(1)-P4(1)))+pi;

% Quadrant 4
% th3 = atan((P3(2)-P2(2))/(P3(1)-P2(1)))-pi;
% th5 = atan((P3(2)-P4(2))/(P3(1)-P4(1)));

% Debugging: Checking the numeric values for theta3 & theta 5

% % Exact Results for theta3
% th3_exact = th3;
% th3_exact = subs(th3_exact,R1,5);
% th3_exact = subs(th3_exact,R2,5);
% th3_exact = subs(th3_exact,R3,7.071);
% th3_exact = subs(th3_exact,R4,5);
% th3_exact = subs(th3_exact,R5,7.071);

% % th3_exact = subs(th3_exact,th2,120*pi/180);
% th3_exact = subs(th3_exact,th4,60*pi/180);

% % display(eval(th3_exact)*180/pi)
% % Exact Results for theta5
% th5_exact = th5;
% th5_exact = subs(th5_exact,R1,5);
% th5_exact = subs(th5_exact,R2,5);
% th5_exact = subs(th5_exact,R3,7.071);
% th5_exact = subs(th5_exact,R4,5);
% th5_exact = subs(th5_exact,R5,7.071);
%
% th5_exact = subs(th5_exact,th2,120*pi/180);
% th5_exact = subs(th5_exact,th4,60*pi/180);
%
% display(eval(th5_exact)*180/pi)

% Find the kinematic coefficients needed in the virtual work
% Differentiate th3 and th5 with respect to theta2
dth32 = diff(th3,'th2');
dth52 = diff(th5,'th2');

% Differentiate th3 and th5 with respect to theta2
dth34 = diff(th3,'th4');
dth54 = diff(th5,'th4');

% Virtual work (round 1)
% Step 1: generalized coordinate q1 = th2
% Step 2: Define the force vector
Fx = sym('Fx');
Fy = sym('Fy');
F = [Fx,Fy];
% Step 3: Define the position vector from the origin to the force
Z1 = P3;
% Step 4: Differentiate the position vector with respect to q1
dZ1 = diff(Z1,'th2');
% Step 5: Determine the virtual work due to the Force
dWf1 = F(1)*dZ1(1)+F(2)*dZ1(2);
% Step 6: Define the moments
M2 = sym('M2');
M4 = sym('M4');

% Step 7: Define the angular displacement
th2o = sym('th2o');
th3o = sym('th3o');
th4o = sym('th4o');
th5o = sym('th5o');
TH2 = (th2-th2o);
TH3 = (th3-th3o);
TH4 = (th4-th4o);
TH5 = (th5-th5o);

% Step 8: Differentiate the angular displacement
dTH2_1 = diff(TH2,'th2');
dTH4_1 = diff(TH4,'th2');

% Step 9: Determine the virtual work due to the Moments
dWm1 = M2*dTH2_1+M4*dTH4_1;

% Step 10: Define the potential energy terms
k2 = sym('k2');
k3 = sym('k3');
k4 = sym('k4');
k5 = sym('k5');
kp = sym('kp');
V2 = 1/2*k2*(TH2)^2;
V3 = 1/2*k3*(TH3-TH2)^2;
V4 = 1/2*k4*(TH4)^2;
V5 = 1/2*k5*(TH5-TH4)^2;
Vp = 1/2*kp*(TH3-TH5)^2;
V = V2+V3+V4+V5+Vp;

% Step 11: Differentiate the potential energy terms
dWv1 = -diff(V,'th2');

% Step 12: Add the Virtual Works
W1 = dWf1+dWv1;

%% Virtual work (round 2)
% % Step 1: generalized coordinate q2 = th4
% % Step 2: Define the force vector
166  \% Fx = sym('Fx');
167  \% Fy = sym('Fy');
168  \% F = [Fx,Fy];
169  \% Step 3: Define the position vector from the origin to the force
170  Z2 = F3;
171  \% Step 4: Differentiate the position vector with respect to q1
172  dZ2 = diff(Z2,'th4');
173  \% Step 5: Determine the virtual work due to the Force
174  dWf2 = F(1)*dZ2(1)+F(2)*dZ2(2);
175  \% \% Step 6: Define the moments
176  \% M2 = sym('M2');
177  \% M4 = sym('M4');
178  \% \% Step 7: Define the angular displacement
179  \% th2o = sym('th2o');
180  \% th3o = sym('th3o');
181  \% th4o = sym('th4o');
182  \% th5o = sym('th5o');
183  \% TH2 = (th2-th2o);
184  \% TH3 = (th3-th3o);
185  \% TH4 = (th4-th4o);
186  \% TH5 = (th5-th5o);
187  \% Step 8: Differentiate the angular displacement
188  dTH2_2 = diff(TH2,'th4');
189  dTH4_2 = diff(TH4,'th4');
190  \% Step 9: Determine the virtual work due to the Moments
191  dWm2 = M2*dTH2_2+M4*dTH4_2;
192  \% \% Step 10: Define the potential energy terms
193  \% k2 = sym('k2');
194  \% k3 = sym('k3');
195  \% k4 = sym('k4');
196  \% k5 = sym('k5');
197  \% kp = sym('kp');
198  \% V2 = 1/2*k2*(TH2)^2;
199  \% V3 = 1/2*k3*(TH3-TH2)^2;
200  \% V4 = 1/2*k4*(TH4)^2;
201  \% V5 = 1/2*k5*(TH5-TH4)^2;
\[ V_p = \frac{1}{2} k_p (H_3 - H_5)^2; \]

\[ V = V_2 + V_3 + V_4 + V_5 + V_p; \]

% Step 11: Differentiate the potential energy terms
\[ dW_v = -\text{diff}(V, 'th4'); \]

% Step 12: Add the Virtual Works
\[ W_2 = dW_f + dW_v; \]

%% Solve
\[ \text{solve}(W_1, F_y); \]
\[ \text{subs}(W_2, F_y, \text{solve}(W_2, F_x)); \]
\[ \text{solve}(\text{subs}(F_x, F_y, \text{solve}(W_2, F_x)), F_x); \]
\[ \text{subs}(F_y, F_x, \text{solve}(W_2, F_x)); \]

% Display Force relationship in a pretty format. By the way, this is a
% masive and nasty expression. It’s huge!
\[ \text{pretty}(\text{solve}(W_1, F_x)); \]
\[ \text{pretty}(\text{solve}(W_2, F_x)); \]

%% Substitute in real values
\[ \text{subs}(F, \text{solve}(W_1, F_x)); \]
\[ \text{subs}(F, \text{solve}(W_2, F_x)); \]
\[ \text{subs}(F, \text{solve}(W_2, F_x)); \]
\[ \text{subs}(F, \text{solve}(W_2, F_x)); \]
\[ \text{subs}(F, \text{solve}(W_2, F_x)); \]

% F = \text{subs}(F, R_1, 51.0/1000);
% F = \text{subs}(F, R_2, 94.0/1000);
% F = \text{subs}(F, R_3, 150.0/1000);
% F = \text{subs}(F, R_4, 94.0/1000);
% F = \text{subs}(F, R_5, 150.0/1000);
% F = \text{subs}(F, th_2, 142.03*\pi/180);
% F = \text{subs}(F, th_3, 49.67*\pi/180);
% F = \text{subs}(F, th_4, 40.0*\pi/180);
% F = \text{subs}(F, th_5, 130.33*\pi/180);
% F = \text{subs}(F, k_2, 5);
% F = \text{subs}(F, k_3, 5);
% File: Virtual_Work_Moments.m
% Author: Dallin Swiss
% Date: May 31, 2014
% Description: This is code for the research. It determines the
% relationship between input position and output torques

format compact
clear
clc

%% Variables
% Link Lengths
R1 = sym('R1'); % [m]

D.1.3 Virtual_Works_Moments.m

% F = subs(F,k4,5);
% F = subs(F,k5,5);
% F = subs(F,kp,5);

% eval(F)

%% Write symbolic equation to file
% fileID1 = fopen('Fx_(q4).txt','wt');
% fileID2 = fopen('Fy_(q4).txt','wt');

% Fx_equation = char(Fx_in_terms_of_Th2_Th4);
% Fy_equation = char(Fy_in_terms_of_Th2_Th4);

% fprintf(fileID1,'%s',Fx_equation);
% fprintf(fileID2,'%s',Fy_equation);

% fclose(fileID1);
% fclose(fileID2);
R2 = sym('R2'); % [m]
R3 = sym('R3'); % [m]
R4 = sym('R4'); % [m]
R5 = sym('R5'); % [m]

% Angle between ground joints
th1 = pi; % [rad] the ground joints are always level
th2 = sym('th2'); % [rad]
th3 = sym('th3'); % [rad]
th4 = sym('th4'); % [rad]
th5 = sym('th5'); % [rad]

% Trigonometric abbreviations
c1 = -1;
s1 = 0;
c2 = cos(th2);
s2 = sin(th2);
c3 = cos(th3);
s3 = sin(th3);
c4 = cos(th4);
s4 = sin(th4);
c5 = cos(th5);
s5 = sin(th5);

% Solve for intersection of circles
% Define position vectors
P1 = [R1*c1, R1*s1];
P2 = [P1(1)+R2*c2, P1(2)+R2*s2];
P4 = [R4*c4, R4*s4];

% Vector from points 2->4
P24 = P4-P2;
R24 = sqrt(P24(1)^2+P24(2)^2);

% Vector along P24 that stops at the intersection of C3 & C5
R2h = (R3^2-R5^2+R24^2)/(2*R24);
\[ \text{Ph} = \frac{P2+R2h}{R24*P24}; \]

% Vector to the interface point (P3)
\[ R3h = \sqrt{R3^2-R2h^2}; \]
\[ P3 = \left[ \text{Ph}(1)-\frac{R3h}{R24*P24(2)}, \text{Ph}(2)+\frac{R3h}{R24*P24(1)} \right]; \]

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Warning ------------------------------------------%
% The range of \( \theta_3 \) and \( \theta_5 \) around limited only to the %
% region directly in front of the device. This ought %
% to be sufficient for most workspaces. However, it %
% could be expanded if necessary, but the analysis %
% need logic which is not supported for symbolic %
% solutions.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Solution angles
% Joint angles

% % Quadrant 1
% \[ \theta_3 = \arctan\left(\frac{P3(2)-P2(2)}{P3(1)-P2(1)}\right); \]
% \[ \theta_5 = \arctan\left(\frac{P3(2)-P4(2)}{P3(1)-P4(1)}\right); \]
% % Quadrant 2
% \[ \theta_3 = \arctan\left(\frac{P3(2)-P2(2)}{P3(1)-P2(1)}\right); \]
% \[ \theta_5 = \arctan\left(\frac{P3(2)-P4(2)}{P3(1)-P4(1)}\right)+\pi; \]
% % Quadrant 3
% \[ \theta_3 = \arctan\left(\frac{P3(2)-P2(2)}{P3(1)-P2(1)}\right)+\pi; \]
% \[ \theta_5 = \arctan\left(\frac{P3(2)-P4(2)}{P3(1)-P4(1)}\right)+\pi; \]
% % Quadrant 4
% \[ \theta_3 = \arctan\left(\frac{P3(2)-P2(2)}{P3(1)-P2(1)}\right)-\pi; \]
% \[ \theta_5 = \arctan\left(\frac{P3(2)-P4(2)}{P3(1)-P4(1)}\right); \]

% Debugging: Checking the numeric values for \( \theta_3 \) & \( \theta_5 \)
% % Exact Results for \( \theta_3 \)
% \[ \text{th3} = \text{th3}; \]
% \[ \text{th3} = \text{subs}(\text{th3}, R1, 5); \]
% \[ \text{th3} = \text{subs}(\text{th3}, R2, 5); \]
% th3_exact = subs(th3_exact,R3,7.071);
% th3_exact = subs(th3_exact,R4,5);
% th3_exact = subs(th3_exact,R5,7.071);
%
% th3_exact = subs(th3_exact,th2,120*pi/180);
% th3_exact = subs(th3_exact,th4,60*pi/180);
%
% display(eval(th3_exact)*180/pi)
%
% % Exact Results for theta5
% th5_exact = th5;
% th5_exact = subs(th5_exact,R1,5);
% th5_exact = subs(th5_exact,R2,5);
% th5_exact = subs(th5_exact,R3,7.071);
% th5_exact = subs(th5_exact,R4,5);
% th5_exact = subs(th5_exact,R5,7.071);
%
% th5_exact = subs(th5_exact,th2,120*pi/180);
% th5_exact = subs(th5_exact,th4,60*pi/180);
%
% display(eval(th5_exact)*180/pi)

% Find the kinematic coefficients needed in the virtual work
% Differentiate th3 and th5 with respect to theta2
dth32 = diff(th3,'th2');
dth52 = diff(th5,'th2');

% Differentiate th3 and th5 with respect to theta2
dth34 = diff(th3,'th4');
dth54 = diff(th5,'th4');

% Virtual work (round 1)
% Step 1: generalized coordinate q1 = th2
% Step 2: Define the force vector
Fx = sym('Fx');
Fy = sym('Fy');
F = [Fx, Fy];

% Step 3: Define the position vector from the origin to the force
Z1 = P3;

% Step 4: Differentiate the position vector with respect to q1
dZ1 = diff(Z1,'th2');

% Step 5: Determine the virtual work due to the Force
dWf1 = F(1)*dZ1(1)+F(2)*dZ1(2);

% Step 6: Define the moments
M2 = sym('M2');
M4 = sym('M4');

% Step 7: Define the angular displacement
th2o = sym('th2o');
th3o = sym('th3o');
th4o = sym('th4o');
th5o = sym('th5o');
TH2 = (th2-th2o);
TH3 = (th3-th3o);
TH4 = (th4-th4o);
TH5 = (th5-th5o);

% Step 8: Differentiate the angular displacement
dTH2_1 = diff(TH2,'th2');
dTH4_1 = diff(TH4,'th2');

% Step 9: Determine the virtual work due to the Moments
dWm1 = M2*dTH2_1+M4*dTH4_1;

% Step 10: Define the potential energy terms
k2 = sym('k2');
k3 = sym('k3');
k4 = sym('k4');
k5 = sym('k5');
kp = sym('kp');
V2 = 1/2*k2*(TH2)^2;
V3 = 1/2*k3*(TH3-TH2)^2;
V4 = 1/2*k4*(TH4)^2;
V5 = 1/2*k5*(TH5-TH4)^2;
Vp = 1/2*kp*(TH3-TH5)^2;
V = V2+V3+V4+V5+Vp;
% Step 11: Differentiate the potential energy terms

\[ dWv1 = -\text{diff}(V, 'th2'); \]

% Step 12: Add the Virtual Works

\[ W1 = dWm1 + dWv1; \]

%% Virtual work (round 2)
% % Step 1: generalized coordinate q2 = th4
% % Step 2: Define the force vector
% Fx = sym('Fx');
% Fy = sym('Fy');
% F = [Fx, Fy];
% % Step 3: Define the position vector from the origin to the force
% Z2 = P3;
% % Step 4: Differentiate the position vector with respect to q1
% dZ2 = diff(Z2, 'th4');
% % Step 5: Determine the virtual work due to the Force
% dWf2 = F(1)*dZ2(1)+F(2)*dZ2(2);
% % Step 6: Define the moments
% M2 = sym('M2');
% M4 = sym('M4');
% % Step 7: Define the angular displacement
% th2o = sym('th2o');
% th3o = sym('th3o');
% th4o = sym('th4o');
% th5o = sym('th5o');
% TH2 = (th2-th2o);
% TH3 = (th3-th3o);
% TH4 = (th4-th4o);
% TH5 = (th5-th5o);
% % Step 8: Differentiate the angular displacement
% dTH2_2 = diff(TH2, 'th4');
% dTH4_2 = diff(TH4, 'th4');
% % Step 9: Determine the virtual work due to the Moments
% dWm2 = M2*dTH2_2+M4*dTH4_2;
% % Step 10: Define the potential energy terms
% k2 = sym('k2');
% k3 = sym('k3');
% k4 = sym('k4');
% k5 = sym('k5');
% kp = sym('kp');
% V2 = 1/2*k2*(TH2)^2;
% V3 = 1/2*k3*(TH3-TH2)^2;
% V4 = 1/2*k4*(TH4)^2;
% V5 = 1/2*k5*(TH5-TH4)^2;
% Vp = 1/2*kp*(TH3-TH5)^2;
% V = V2+V3+V4+V5+Vp;

% Step 11: Differentiate the potential energy terms
dWv2 = -diff(V,'th4');

% Step 12: Add the Virtual Works
W2 = dWm2+dWv2;

% Display Virtual Work
% pretty(W1)
% pretty(W2)

% Solve
M2_in_terms_of_Th2_Th4 = solve(W1,M2);
M4_in_terms_of_Th2_Th4 = solve(W2,M4);

% Display Force relationship in a pretty format. By the way, this is a
% masive and nasty expression. It's huge!
pretty(M2_in_terms_of_Th2_Th4)
pretty(M4_in_terms_of_Th2_Th4)

% Substitute in real values
% M = [M2_in_terms_of_Th2_Th4; M4_in_terms_of_Th2_Th4];
% M = subs(M,R1,51.0/1000);
% M = subs(M,R2,94.0/1000);
% M = subs(M,R3,150.0/1000);
% M = subs(M,R4,94.0/1000);
% M = subs(M,R5,150.0/1000);
%
% M = subs(M,th2o,139.6*pi/180);
% M = subs(M,th3o,49.67*pi/180);
% M = subs(M,th4o,40.0*pi/180);
% M = subs(M,th5o,130.33*pi/180);
%
% M = subs(M,th2,142.03*pi/180);
% M = subs(M,th4,37.97*pi/180);
%
% M = subs(M,k2,5);
% M = subs(M,k3,5);
% M = subs(M,k4,5);
% M = subs(M,k5,5);
% M = subs(M,kp,5);
%
% eval(M)

%% Write symbolic equation to file
% fileID1 = fopen('M2_(q4).txt','wt');
% fileID2 = fopen('M4_(q4).txt','wt');
%
% M2_equation = char(M2_in_terms_of_Th2_Th4);
% M4_equation = char(M4_in_terms_of_Th2_Th4);
%
% fprintf(fileID1,'%s',M2_equation);
% fprintf(fileID2,'%s',M4_equation);
%
% fclose(fileID1);
% fclose(fileID2);

D.2 MATLAB Simulation

D.2.1 Analytical_solution.m

function analytical_solution(handles)
%% Variables
% Link Lengths
R1 = str2double(get(handles.edit_R1,'string')); % [m]
R2 = str2double(get(handles.edit_R2,'string')); % [m]
R3 = str2double(get(handles.edit_R3,'string')); % [m]
R4 = str2double(get(handles.edit_R4,'string')); % [m]
R5 = str2double(get(handles.edit_R5,'string')); % [m]
R = [R1,R2,R3,R4,R5];

% Angle between ground joints
th1 = pi; % [rad] the ground joints are always level

%% Forward Kinematic Solution
if get(handles.rb_FK,'value') == 1
    % Input Variables
    % Angles
    % th1 = pi; % [rad] the ground joints are always level
    th2 = str2double(get(handles.edit_th2,'String'))*pi/180; % [rad]
    th4 = str2double(get(handles.edit_th4,'String'))*pi/180; % [rad]

    % Trignometric abbreviations
    c1 = -1;
    s1 = 0;
    c2 = cos(th2);
    s2 = sin(th2);
    c4 = cos(th4);
    s4 = sin(th4);

    % Solve for intersection of circles
    % Define position vectors
P1 = [R1*c1, R1*s1];
P2 = [P1(1)+R2*c2, P1(2)+R2*s2];
P4 = [R4*c4, R4*s4];

% Vector from points 2->4
P24 = P4-P2;
R24 = sqrt(P24(1)^2+P24(2)^2);

% Vector along P24 that stops at the intersection of Circles C3 & C5
R2h = (R3^2-R5^2+R24^2)/(2*R24);
Ph = P2+R2h/R24*P24;

% Vector to the interface point (P3)
R3h = sqrt(R3^2-R2h^2);
P3 = [Ph(1)-R3h/R24*P24(2), Ph(2)+R3h/R24*P24(1)];

% Solution angles
th3 = atan2(P3(2)-P2(2),P3(1)-P2(1));

% Set solution angles to the screen
set(handles.static_th3,'String',round(th3*180/pi*100)/100); % [rad]
set(handles.static_th5,'String',round(th5*180/pi*100)/100); % [rad]

% Solve for the boundary lines using intersection of circles
% Vector along P1 that stops at the intersection of Circles C3 & C5
R1bh = ((R2+R3)^2-(R4+R5)^2+R1^2)/(2*R1);
Pbh = -R1bh/R1*P1;

% Vector to the interface point (P3)
R3bh = sqrt((R2+R3)^2-R1bh^2);
Pb = [Pbh(1)+R3bh/R1*P1(2), Pb(2)-R3bh/R1*P1(1)];

% Create angle vectors
th_L = linspace(acos((R1bh-R1)/(R4+R5)),pi,50)';
th_R = linspace(0,acos(R1bh/(R2+R3)),50)';
\text{Pc\_L} = [(R4+R5)\ast \cos(\text{th\_L}), (R4+R5)\ast \sin(\text{th\_L})];
\text{Pc\_R} = [P1(:,1)+(R2+R3)\ast \cos(\text{th\_R}), (R2+R3)\ast \sin(\text{th\_R})];

\% Shift Plot
P1(1) = P1(1)+R1/2;
P2(1) = P2(1)+R1/2;
P4(1) = P4(1)+R1/2;
Pb(1) = Pb(1)-R1/2;
Pc\_L(:,1) = Pc\_L(:,1)+R1/2;
Pc\_R(:,1) = Pc\_R(:,1)+R1/2;

\% Plot the resulting mechanism
\% Create plot data for left side
x1 = [R1/2, P1(1)];
y1 = [0, P1(2)];
x2 = [P1(1), P2(1)];
y2 = [P1(2), P2(2)];
x3 = [P2(1), P2(1)+R3\ast \cos(\text{th3})];
y3 = [P2(2), P2(2)+R3\ast \sin(\text{th3})];
\% Create plot data for right side
x4 = [R1/2, P4(1)];
y4 = [0, P4(2)];
x5 = [P4(1), P4(1)+R5\ast \cos(\text{th5})];
y5 = [P4(2), P4(2)+R5\ast \sin(\text{th5})];
\% Create plot data for boundary
xb = Pb(1);
yb = Pb(2);
xc\_L = Pc\_L(:,1);
yc\_L = Pc\_L(:,2);
xc\_R = Pc\_R(:,1);
yc\_R = Pc\_R(:,2);

axes(handles.axes1)
cla \% Clear the axes
axis equal \% 1:1 aspect ration
hold on
grid on
plot(x1,y1,'k--','LineWidth',2)
plot(x2,y2,'b','LineWidth',4)
plot(x3,y3,'r','LineWidth',4)
plot(x3(2),y3(2),'ko','LineWidth',6)
plot(x4,y4,'m','LineWidth',4)
plot(x5,y5,'g','LineWidth',4)
plot(xb,yb,'kp','LineWidth',4)
plot(xc_L,yc_L,'r')
plot(xc_R,yc_R,'r')

% Set axis size
axis([-1.2*(R4+R5),1.2*(R2+R3),-1.2*(R2+R4)/2,1.2*(R2+R3)])

% Calculate the force to create this position
FKin_values = FKin_numeric(th2,th4,R,handles);

% Joint Stiffnesses
joint_stiffness(handles);
k2 = str2double(get(handles.edit_k12,'string')); % [m]
k3 = str2double(get(handles.edit_k23,'string')); % [m]
k4 = str2double(get(handles.edit_k14,'string')); % [m]
k5 = str2double(get(handles.edit_k45,'string')); % [m]
kp = str2double(get(handles.edit_k35,'string')); % [m]
k = [k2,k3,k4,k5,kp];

% Spot to include stress calculation

k = abs(k);
F = force_given_angles_numeric_v02(R,k,FKin_values);
F_mag = sqrt(F(1)^2+F(2)^2);
F_dir = atan2(F(2),F(1));
M = moment_given_angles_numeric_v02(R,k,FKin_values);

% Print springback forces and moments to the GUI
set(handles.static_FmagSB,'string',sprintf('%0.2f',F_mag));
set(handles.static_FdirSB,'string',sprintf('%0.2f',(F_dir*180/pi)));
set(handles.static_M2SB,'string',sprintf('%0.2f',M(1)));
set(handles.static_M4SB,'string',sprintf('%0.2f',M(2)));

F_mag = F_mag*20; % Magnify the F_mag value for plotting
FplotX = [x3(2),x3(2)-F_mag*cos(F_dir)];
FplotY = [y3(2),y3(2)-F_mag*sin(F_dir)];
plot(FplotX,FplotY,'k','LineWidth',4)

%% Determine the haptic torques based on input haptic forces
% Calculate the jacobian
J = Jacobian_numeric_v02(R,FKin_values);

% Get haptic forces
FxH = str2double(get(handles.edit_FxH,'string'));
FyH = str2double(get(handles.edit_FyH,'string'));
FH = [FxH;FyH];

% Calculate required torques/moments to generate desired Haptic forces
MH = J'*FH;

% Display on GUI
set(handles.static_M2H,'string',MH(1));
set(handles.static_M4H,'string',MH(2));

% Check the two forces
% FS is the same as F but is obtained by transforming the moments
% through the Jacobian. The fact that there is little error between the
% two forces leads me to believe that the solution is correct.
FS = (J')\M;
% Because F and FS are the same, they cancel and I am left with only
% the haptic forces, FH.
FR = F-FS+FH;

% Sum the moments to determine the necessary moments to make it work
% This is the combination of the compensation moment and haptic moment
MR = M+MH;

% Display on GUI
set(handles.static_FxR,'string',sprintf('%0.2f',FR(1)));
set(handles.static_FyR,'string',sprintf('%0.2d',FR(2)));
set(handles.static_M2R,'string',sprintf('%0.2f',MR(1)));
set(handles.static_M4R,'string',sprintf('%0.2f',MR(2)));

% Inverse Kinematic Solution
elseif get(handles.rb_IK,'value') == 1
    % Input Variables
    P1 = [R1*cos(th1), R1*sin(th1)];
    x03 = str2double(get(handles.edit_Px,'String'))-R1/2;
    y03 = str2double(get(handles.edit_Py,'String'));
    if y03 == 0
        y03 = -0;
    end
    P03 = [x03, y03];
    R03 = norm(P03);
    P13 = [P03(1)-P1(1), P03(2)-P1(2)];
    R13 = norm(P13);

    % Solve for inner angles of 3 triangles
    % Solve for theta2
    alpha1 = acos((R2^2-R3^2+R13^2)/(2*R2*R13));
    beta1 = atan2(P13(2), P13(1));
    th2 = alpha1+beta1;

    % Solve for theta4
    alpha2 = acos((R4^2-R5^2+R03^2)/(2*R4*R03));
beta2 = atan2(P03(2), P03(1));

th4 = (beta2 - alpha2);

if (~isreal(th2) || ~isreal(th4))
    display('ERROR: Point out of range')
    set(handles.figure1,'Color',[1,0,0]) % Set figure background to red
    pause(0.05);
    set(handles.figure1,'Color',[0.8,0.8,0.8]) % Reset background ...
    to gray
else
    % Set solution angles to the screen
    set(handles.static_th2,'String',round(th2*180/pi*100)/100); % [rad]
    set(handles.static_th4,'String',round(th4*180/pi*100)/100); % [rad]
    set(handles.edit_th2,'String',round(th2*180/pi*100)/100); % [rad]
    set(handles.edit_th4,'String',round(th4*180/pi*100)/100); % [rad]

    P2 = [P1(1)+R2*cos(th2), P1(2)+R2*sin(th2)];
    P4 = [R4*cos(th4), R4*sin(th4)];

    % Solution angles
    % Joint angles
    th3 = atan2(y03-P2(2),x03-P2(1));
    th5 = atan2(y03-P4(2),x03-P4(1));

    % Set solution angles to the screen
    set(handles.static_th3,'String',round(th3*180/pi*100)/100); % [rad]
    set(handles.static_th5,'String',round(th5*180/pi*100)/100); % [rad]

    % Solve for the boundary lines using intersection of circles
    % Vector along P1 that stops at the intersection of Circles C3 ... & C5
    R1bh = ((R2+R3)^2-(R4+R5)^2+R1^2)/(2*R1);
    Pbh = -R1bh/R1*P1;

    % Vector to the interface point (P3)
    R3bh = sqrt((R2+R3)^2-R1bh^2);
\[
P_b = [P_{bh}(1) + R_{3bh}/R_1 \cdot P_1(2), P_{bh}(2) - R_{3bh}/R_1 \cdot P_1(1)];
\]

```matlab
% Create angle vectors
th_L = linspace(acos((R1bh-R1)/(R4+R5)),pi,50)';
th_R = linspace(0,acos(R1bh/(R2+R3)),50)';
Pc_L = [(R4+R5)*cos(th_L), (R4+R5)*sin(th_L)];
Pc_R = [P1(:,1)+(R2+R3)*cos(th_R), (R2+R3)*sin(th_R)];

%% Shift Plot
P1(1) = P1(1)+R1/2;
P2(1) = P2(1)+R1/2;
P4(1) = P4(1)+R1/2;
Pb(1) = Pb(1)-R1/2;
Pc_L(:,1) = Pc_L(:,1)+R1/2;
Pc_R(:,1) = Pc_R(:,1)+R1/2;

%% Plot the resulting mechanism
% Create plot data for left side
x1 = [R1/2, P1(1)];
y1 = [0, P1(2)];
x2 = [P1(1), P2(1)];
y2 = [P1(2), P2(2)];
x3 = [P2(1), P2(1)+R3*cos(th3)];
y3 = [P2(2), P2(2)+R3*sin(th3)];
% Create plot data for right side
x4 = [R1/2, P4(1)];
y4 = [0, P4(2)];
x5 = [P4(1), P4(1)+R5*cos(th5)];
y5 = [P4(2), P4(2)+R5*sin(th5)];
% Create plot data for boundary
xb = Pb(1);
yb = Pb(2);
xc_L = Pc_L(:,1);
yc_L = Pc_L(:,2);
xc_R = Pc_R(:,1);
yc_R = Pc_R(:,2);
```
axes(handles.axes1)
cla % Clear the axes
axis equal % 1:1 aspect ration
hold on
grid on
plot(x1,y1,'k--','LineWidth',2) % Ground link
plot(x2,y2,'b','LineWidth',4) % Link 2 - bottom left
plot(x3(2),y3(2),'ko','LineWidth',4) % Interface point - top
 % All blue links
    plot(x3,y3,'b','LineWidth',4) % Link 3 - top left
    plot(x4,y4,'b','LineWidth',4) % Link 4 - bottom right
    plot(x5,y5,'b','LineWidth',4) % Link 5 - top right
 % Multi-color links
    plot(x3,y3,'r','LineWidth',4) % Link 3 - top left
    plot(x4,y4,'m','LineWidth',4) % Link 4 - bottom right
    plot(x5,y5,'g','LineWidth',4) % Link 5 - top right
    plot(xb,yb,'kp','LineWidth',4) % Intersection of kinematic ...
    % curves
    plot(xc_L,yc_L,'r') % Left kinematic bounding curve
    plot(xc_R,yc_R,'r') % Right kinematic bounding curve
    plot(0,175.27,'cp','LineWidth',10) % Location of home

% Set axis
    % Dynamics axes
    axis([-1.2*(R4+R5),1.2*(R2+R3),-1.2*(R2+R4)/2,1.2*(R2+R3)])
    axis([-125,125,-20,250]) % fixed axes
    axis([-125,125,-20,250]) % fixed axes
xlabel('X (mm)')
ylabel('Y (mm)')

% Calculate the force to create this position
FKin_values = FKin_numeric(th2,th4,R,handles);
% Joint Stiffnesses

joint_stiffness(handles);
k2 = str2double(get(handles.edit_k12,'string')); % [m]
k3 = str2double(get(handles.edit_k23,'string')); % [m]
k4 = str2double(get(handles.edit_k14,'string')); % [m]
k5 = str2double(get(handles.edit_k45,'string')); % [m]
kp = str2double(get(handles.edit_k35,'string')); % [m]
k = [k2,k3,k4,k5,kp];

% Spot to include stress calculation

k = abs(k);

F = force_given_angles_numeric_v02(R,k,FKin_values);
F_mag = sqrt(F(1)^2+F(2)^2);
F_dir = atan2(F(2),F(1));

M = moment_given_angles_numeric_v02(R,k,FKin_values);

% Print springback forces and moments to the GUI
set(handles.static_FmagSB,'string',sprintf('%0.2f',F_mag));
set(handles.static_FdirSB,'string',sprintf('%0.2f',(F_dir*180/pi)));
set(handles.static_M2SB,'string',sprintf('%0.2f',M(1)));
set(handles.static_M4SB,'string',sprintf('%0.2f',M(2)));

F_mag = F_mag*20; % Magnify the F_mag value for plotting
FplotX = [x3(2),x3(2)-F_mag*cos(F_dir)];
FplotY = [y3(2),y3(2)-F_mag*sin(F_dir)];

% Determine the haptic torques based on input haptic forces
% Calculate the jacobian
J = Jacobian_numeric_v02(R,FKin.values);

% Get haptic forces
FxH = str2double(get(handles.edit_FxH,'string'));
FyH = str2double(get(handles.edit_FyH,'string'));
FH = [FxH;FyH];

% Calculate required moments to generate desired Haptic forces
MH = J'*FH;

set(handles.static_M2H,'string',MH(1));
set(handles.static_M4H,'string',MH(2));

%% Check the two forces
% FS is the same as F but is obtained by transforming the moments
% through the Jacobian. The fact that there is little error ...
% between the
% two forces leads me to believe that the solution is correct.
FS = (J')\M;
% Because F and FS are the same, they cancel and I am left with ...
% only
% the haptic forces, FH.
FR = F-FS+FH;

%% Sum the moments to determine the necessary moments to make ...
% it work
MR = M+MH;

set(handles.static_FxR,'string',sprintf('%0.2f',FR(1)));
set(handles.static_FyR,'string',sprintf('%0.2f',FR(2)));
set(handles.static_M2R,'string',sprintf('%0.2f',MR(1)));
set(handles.static_M4R,'string',sprintf('%0.2f',MR(2)));

%% Check Manipulability measures
[U,S,~] = svd(J); % singular value decomposition of the Jacobian
sig = diag(S); % Extracting the diagonal values of matrix S
\[ w = \text{prod}(\text{sig}); \text{ % Determining the manipulability measure for the current position} \]

\[ \% Calculate the the Major and minor axes of the manipulability ... \]
\[ \text{ellipse} \]
\[ \text{ME} = \text{zeros}(2, \text{length}(\text{sig})); \text{ % manipulability ellipse} \]
\[ \text{MFE} = \text{zeros}(2, \text{length}(\text{sig})); \text{ % manipulability-force ellipse} \]
\[ \text{ang} = \text{zeros}(1, \text{length}(\text{sig})); \]
\[ \text{for} \ i = 1: \text{length}(\text{sig}) \]
\[ \hspace{1em} \text{ME}(; , i) = \text{U}(; , i) \times \text{sig}(i); \]
\[ \hspace{1em} \text{MFE}(; , i) = \text{U}(; , i) / \text{sig}(i); \]
\[ \hspace{1em} \text{ang}(i) = \text{atan2} (\text{ME}(2,i), \text{ME}(1,i)); \]
\[ \text{end} \]
\[ \% \text{display(ME)} \]
\[ \% \text{display(MFE)} \]
\[ \% \% Plot manipulability ellipse \]
\[ \text{h} = 300 \times \text{norm} (\text{ME}(; , 1)); \text{ % first axis with arbitrary scaling} \]
\[ \text{v} = 300 \times \text{norm} (\text{ME}(; , 2)); \text{ % second axis with arbitrary scaling} \]
\[ \text{area ME} = \pi \times \text{norm} (\text{ME}(; , 1)) \times \text{norm} (\text{ME}(; , 2)); \]
\[ \text{x0} = \text{x3}(2); \text{ % x0,y0 ellipse centre coordinates} \]
\[ \text{y0} = \text{y3}(2); \]
\[ \text{t} = [-\pi:0.01:}\pi; \]
\[ \text{x} = \text{h} \times \text{cos}(t); \]
\[ \text{y} = \text{v} \times \text{sin}(t); \]
\[ \% \text{rotate ellipse by angle of axes} \]
\[ \text{for} \ i = 1: \text{length(x)} \]
\[ \hspace{1em} \text{Rotation} = [\cos (\text{ang}(i)), -\sin (\text{ang}(i)); \]
\[ \hspace{2em} \sin (\text{ang}(i)), \cos (\text{ang}(i))]; \]
\[ \hspace{1em} \text{RotationPair} = \text{Rotation} \times [\text{x}(i); \text{y}(i)]; \]
\[ \hspace{1em} \text{x}(i) = \text{RotationPair}(1) + \text{x0}; \]
\[ \hspace{1em} \text{y}(i) = \text{RotationPair}(2) + \text{y0}; \]
\[ \text{end} \]
\[ \% \text{plot(x,y,'k','LineWidth',1)} \]
\[ \% \text{display(area ME)} \]
% Plot manipulability-force ellipse
hf = 2*norm(MFE(:,1)); % first axis
vf = 2*norm(MFE(:,2)); % second axis
area_MFE = pi*norm(MFE(:,1))*norm(MFE(:,2));
x0 = x3(2); % x0,y0 ellipse centre coordinates
y0 = y3(2);
t = -pi:0.01:pi;
x = hf*cos(t);
y = vf*sin(t);
% rotate ellipse by angle of axes
for i=1:length(x)
    Rotation = [cos(ang(1)), -sin(ang(1));
                sin(ang(1)), cos(ang(1))];
    RotationPair = Rotation*[x(i);y(i)];
    x(i) = RotationPair(1)+x0;
    y(i) = RotationPair(2)+y0;
end
% plot(x,y,'k','LineWidth',1)
% display(area_MFE)

% Determine the available force
Tmax = str2double(get(handles.edit_Tmax,'string'));

% Create Torque vectors for computing available force
Tstep = 0.01;
T2 = -Tmax:Tstep:Tmax;
T4 = T2;
FAvail = zeros(4*length(T2)-3,2);
% Calculate the forces within the diamond
for i=1:length(T2)
    for j=1:length(T4)
        Th = [T2(i);T4(j)];
        Fh = (J')\Th;
        FAvail((i-1)*length(T4)+j,1) = Fh(1);
        FAvail((i-1)*length(T4)+j,2) = Fh(2);
    end
for i=1:length(T2)
    Th = [T2(1);T4(i)];
    Fh = (J') \ Th;
    FAvail((length(T2)+1)-i,1) = Fh(1);
    FAvail((length(T2)+1)-i,2) = Fh(2);
end

for i=2:length(T2)
    Th = [T2(i);T4(1)];
    Fh = (J') \ Th;
    FAvail(length(T2)+(i-1),1) = Fh(1);
    FAvail(length(T2)+(i-1),2) = Fh(2);
end

for i=1:length(T2)
    Th = [T2(end);T4(i)];
    Fh = (J') \ Th;
    FAvail(2*length(T2)-1+i,1) = Fh(1);
    FAvail(2*length(T2)-1+i,2) = Fh(2);
end

for i=1:length(T2)
    Th = [T2(i);T4(end)];
    Fh = (J') \ Th;
    FAvail(4*length(T2)-1-i,1) = Fh(1);
    FAvail(4*length(T2)-1-i,2) = Fh(2);
end

% display(FAvail)
Fmag = sqrt(FAvail(:,1).^2+FAvail(:,2).^2);
Fmean = mean(Fmag);
Fmax = max(Fmag);
Fmag2 = sqrt((FAvail(:,1)-F(1)).^2+(FAvail(:,2)-F(2)).^2);
Fmean2 = mean(Fmag2);
Fmax2 = max(Fmag2);
MeanIncrease = Fmean2/Fmean*100;
MaxIncrease = Fmax2/Fmax*100;
display(MeanIncrease);
display(MaxIncrease);

% The average increase is not 0. For the given shape (a diamond)
% the average increase goes up assuming that the new location moves
% in the direction of the hotdog.

%% Plot available Force zone
plot(10*FAvail(:,1)+x0,10*FAvail(:,2)+y0,'k','LineWidth',1)
plot(10*(FAvail(:,1)-F(1))+x0,10*(FAvail(:,2)-F(2))+y0,...
     'r','LineWidth',1)
end
end

D.3  Hardware Demonstration

D.3.1  LabVIEW_Demo.m

This code ran in a Math Script Node in the LabVIEW VI

%% File:   LabVIEW_Demo.m
% Author: Dallin Swiss
% Date:   May 1, 2015
% Description: This is code for the LabVIEW demo. This code will ... 
in a
% LabVIEW mathscript node. Essentially, this code will
% coordinate the different operating states of the demo. It
% will read encoder positions and calculate the appropriate
% motor torques for the current settings.

% Read in values (variables and flags)

th2 = Encoder1*2*pi+(139.6*pi/180);   % Encoder1 [rad]
th4 = Encoder2*2*pi+(40.4*pi/180);   % Encoder2 [rad]
%% Initialize variables

% Link Lengths
R1 = 0.051; % [m]
R2 = 0.094; % [m]
R3 = 0.150; % [m]
R4 = R2; % [m]
R5 = R3; % [m]

% Joint Stiffnesses
k2 = 3.539 /1000*180/pi; % [N*m/rad]
k3 = 1.545 /1000*180/pi; % [N*m/rad]
k4 = k2;
k5 = k3;
kp = k3;

% Calculate Forward Kinematics Values

% Trignometric abbreviations
cl = -1; % th1 = pi. This is only for comment sake
s1 = 0;
c2 = cos(th2);
s2 = sin(th2);
c4 = cos(th4);
s4 = sin(th4);

% Solve for intersection of circles
% Define position vectors
P1 = [R1*c1, R1*s1];
P2 = [P1(1)+R2*c2, P1(2)+R2*s2];
P4 = [R4*c4, R4*s4];

% Vector from points 2->4
P24 = P4-P2;
R24 = sqrt(P24(1)^2+P24(2)^2);

% Vector along P24 that stops at the intersection of C3 & C5
R2h = (R3^2-R5^2+R24^2)/(2*R24);
Ph = P2+R2h/R24*P24;

% Vector to the interface point (P3)
R3h = sqrt(R3^2-R2h^2);
P3 = [Ph(1)-R3h/R24*P24(2), Ph(2)+R3h/R24*P24(1)];
P3x = P3(1)*1000+25.5;
P3y = P3(2)*1000;

% Solution angles
th3 = atan2(P3(2)-P2(2),P3(1)-P2(1))*180/pi; % [deg]

th5 = atan2((P3(2)-P4(2)),(P3(1)-P4(1)))*180/pi; % [deg]

% The following lines are commented out because quadrant 3 is typically
% outside the yield strength of the compliant pantograph.
% % This is necessary to correct the direction of the resultant ...
%     force for
%     % % the majority of quadrant 3
%     % if (th2 <= 0 && th4 <= 0 && (th3 <= -90 || th3 >= 90) && th5 <= -90)
%     %     th2 = th2 + 2*pi;
%     %     th4 = th4 + 2*pi;
%     % end
%     %
%     % % This is necessary to correct the direction of the resultant ...
%     %     force for
%     % % a small piece of quadrant 3
%     % if (th2 >=0 && th3 <= -90 && th4 <= 0 && th5 <= -90)
%     %     th2 = th2 + 2*pi; % [rad]
%     %     th4 = th4 + 2*pi; % [rad]
%     % end
%     %
%     % % Reevaluating Trignometric abbreviations
% c2 = cos(th2);
% s2 = sin(th2);
% c4 = cos(th4);
% s4 = sin(th4);

% % Calculation abbreviations
% Rc2 = R2*c2;
% Rs2 = R2*s2;
% Rc4 = R4*c4;
% Rs4 = R4*s4;

% % Calculate Compensation Torques
% Declare torque vector
M1 = 0;
M2 = 0;

% If compensation is enabled, calculate compensation torques
if (Compensate)
    R = [R1 R2 R3 R4 R5];
k = [k2 k3 k4 k5 kp];
FKin_values = [Rc2 Rs2 Rc4 Rs4 th2 th3 th4 th5];

    [M1, M2] = moments(R, k, FKin_values);
    [F1, F2] = forces(R, k, FKin_values);
end

% % Calculate haptic forces
if (Boundary) % Boundary is true when boundary is engaged
    switch bType
    case 0 % Square (orientation angle == 0)
      % Boundary definitions
      sRight = 20;
      sLeft = -sRight;
      sTop = 185;
      sBottom = 165;
% Determine whether interface point is outside the boundary
if (P3x > sRight || P3x < sLeft || P3y > sTop || P3y < sBottom)
    hitWall = true;
else
    hitWall = false;
end

if (hitWall)
    SW = false;
    SE = false;
    NE = false;
    NW = false;
    W = false;
    E = false;
    S = false;
    N = false;
% Calculate normal direction
if (P3x > sRight && P3y > sTop) % Pointing SW
    dx = P3x-sRight;
    dy = P3y-sTop;
    angle = atan2(dy,dx);
    eX = -cos(angle);
    eY = -sin(angle);
    SW = true;

% Penetration
P_depth = sqrt(dx^2+dy^2); % (mm)

elseif (P3x < sLeft && P3y > sTop) % Pointing SE
    dx = sLeft-P3x;
    dy = P3y-sTop;
    angle = pi-atan2(dy,dx);
    eX = -cos(angle);
    eY = -sin(angle);
    SE = true;
\begin{verbatim}
% Penetration
P_depth = sqrt(dx^2+dy^2);  % (mm)

elseif (P3x < sLeft && P3y < sBottom) % Pointing NE
    dx = sLeft-P3x;
    dy = sBottom-P3y;
    angle = pi+atan2(dy,dx);
    eX = -cos(angle);
    eY = -sin(angle);
    NE = true;

    % Penetration
    P_depth = sqrt(dx^2+dy^2);  % (mm)

elseif (P3x > sRight && P3y < sBottom) % Pointing NW
    dx = P3x-sRight;
    dy = sBottom-P3y;
    angle = atan2(dy,dx);
    eX = -cos(angle);
    eY = sin(angle);
    NW = true;

    % Penetration
    P_depth = sqrt(dx^2+dy^2);  % (mm)

elseif (P3x > sRight) % Pointing W
    dx = P3x-sRight;
    dy = 0;
    eX = -1;
    eY = 0;
    W = true;

    % Penetration
    P_depth = sqrt(dx^2+dy^2);  % (mm)
\end{verbatim}
elseif (P3x < sLeft) % Pointing E
    dx = sLeft-P3x;
    dy = 0;
    eX = 1;
    eY = 0;
    E = true;

    % Penetration
    P_depth = sqrt(dx^2+dy^2); %(mm)

elseif (P3y > sTop) % Pointing S
    dx = 0;
    dy = P3y-sTop;
    eX = 0;
    eY = -1;
    S = true;

    % Penetration
    P_depth = sqrt(dx^2+dy^2); %(mm)

elseif (P3y < sBottom) % Pointing N
    dx = 0;
    dy = sBottom-P3y;
    eX = 0;
    eY = 1;
    N = true;

    % Penetration
    P_depth = sqrt(dx^2+dy^2); %(mm)

end

% Decide whether force is assisted by springback
if (fType) % True = Static, False = Dynamic
    % Determine motor torques for haptic force
    % Calculate the jacobian
J = jacobian(R,FKin_values);

% Get haptic forces
Fmag = 1.5;  % (N)
FxH = Fmag*eX;  % (N)
FyH = Fmag*eY;  % (N)
FH = [FxH;FyH];

% Calculate torques/moments to generate desired ...
    Haptic forces
MH = J'*FH;

% Add the haptic torque to compensation torque
M1 = M1+MH(1);
M2 = M2+MH(2);
else  % False = Dynamic
    % This is more realistic because the wall pushes back
    % in proportion to how far you penetrate the wall.
    % Determine motor torques for haptic force
    % Calculate the jacobian
    J = jacobian(R,FKin_values);

    % Get haptic forces
    kWall = 1;  % Wall stiffness
    Fmag = kWall*P_depth;
    FxH = Fmag*eX;  % (N)
    FyH = Fmag*eY;  % (N)
    FH = [FxH;FyH];

    % Calculate torques/moments to generate desired ...
        Haptic forces
    MH = J'*FH;

    % Add the haptic torque to compensation torque
    M1 = M1+MH(1);
M2 = M2+MH(2);
end

else
    SW = false;
    SE = false;
    NE = false;
    NW = false;
    W = false;
    E = false;
    S = false;
    N = false;
end

case 1 % Circle
    % Incomplete. Never got around to finish.
end
else
    SW = false;
    SE = false;
    NE = false;
    NW = false;
    W = false;
    E = false;
    S = false;
    N = false;
end

% Convert torques to output voltage
Nm2ozin = 141.612;
OzIn2V_motor1 = 1.275;
OzIn2V_motor2 = 1.0943;
M1 = M1*Nm2ozin/OzIn2V_motor1;
M2 = M2*Nm2ozin/OzIn2V_motor2;
% Convert angles to degrees for display

th2 = th2 * 180 / pi;

th4 = th4 * 180 / pi;
APPENDIX E. HARDWARE SPECIFICATIONS AND CAD DRAWINGS

E.1 Hardware Specifications

E.1.1 Mechanical Components

Many of the mechanical parts of the demonstration prototype are custom designed and machined. Some of them are generic, low tolerance parts, but the compliant pantograph itself is more particular.

(M1) Custom Compliant Pantograph. Machined on a CNC mill out of 1/4 inch polypropylene sheet. In order to milling out the shape of the pantograph, especially the thin flexures, requires that the material be adhered to a sacrificial material. In our case, the polypropylene was glued to a sheet of plywood using spray adhesive. After the adhesive cures, the plywood is mounted in the CNC mill and the shape is machined. It is based on the dimensions given in drawing: SLFP_v3_v2 shown in Fig. E.2 - E.5. The CM chosen is the SLFP. This CM is chosen for the prototype because of its simplicity, ease of manufacture, and large haptic workspace. When completed, the actual dimensions turned out to those given in Table 3.2 in Chapter 3.

(M2) Custom Base. Machined on a CNC mill out of aluminum block. It is based on the dimensions given drawing: Motor_Table shown in Fig. E.6 - E.7. In addition to providing support for the motors and the pantograph, the metallic base also provides a heat sink for cooling. As the motor operate, the heat is conducted throughout the base away from the motors. While no fan is currently used for cooling purposes, the base provide the potential for such if needed.

(M3) Custom Motor Arm Connector. The motors described later under the electrical components have smooth drive shafts. In order to attach the motors to the pantograph at the location of the base joints, two custom motor arm connectors are used. They are based on the dimensions
given in drawing: Motor_Arm shown in Fig. E.1. Machined on a hand mill out of aluminum square stock, they slide over the rigid links $R_2$, $R_4$, of the pantograph. They are aligned just right to place the axis of motion through the center of the clamping region. Once in place, a set screw tightens the clamp around the drive shaft allowing the motion of the motors to exert direct torque at the location of the base joints.

(M4) Custom Interface Assembly. Machined on a hand mill, the user is supposed to control the compliant pantograph at point $P_3$, however, this location is shared by a compliant flexure. In order for the user to exert and receive forces at that location, the custom interface assembly attaches to the nearby rigid link $R_3$ and transfers the forces to the location of $P_3$. It is based on the dimensions given in drawings: EE Clip and EE Handle shown in Fig. E.8 and E.9 respectively. Since $P_3$ is partially located on $R_3$, the rigid body motion is preserved. The interface assembly provides the screw hole for the rod end.

(M5) Ball Joint Rod End. The user grabs hold of the rod end to control the haptic pantograph. The ball bearing in the rod end allows it freedom to rotate as it translates. The rod end was purchased from McMaster Carr, part no. 60645K121. Notes include a zinc-plated steel housing with a chrome-plated steel ball.

E.1.2 Electrical Components

All of the electrical parts of the prototype were commercially available without any major modifications.

(E1) NI cRIO-9074. The main controller was a National Instruments CompactRIO (cRIO) 9074. This cRIO has an embedded real-time (RT) industrial controller with reconfigurable I/O and a field-programmable gate array (FPGA). The cRIO executes the control code from the LabVIEW software on both its RT and FPGA platforms. It interfaces with LabVIEW through an Ethernet connection. The RT processor on the 9074 module operates at 400 MHz and has eight slots for modules. See National Instruments, www.ni.com, for more information.

(E2) NI 9263 Analog Output Module. This module provides analog output voltage circuitry to command the motor amplifiers. The maximum range of output voltages is ±10 V on up to
four channels. Since our application has only two motors, only two channels are used. See National Instruments, www.ni.com, for more information.

(E3) NI 9411 Differential Digital Input Module. This module provide differential input circuitry to read the A and B channels of our encoders. There is capacity to read up to six sets of channels; we use only two. See National Instruments, www.ni.com, for more information.

(E4) BE15A8 Brushless DC Servo Amplifier. This motor driver receives the command voltage from component E2 and engages the motors through current control. The input voltage range is ±10 V, matching the output of E2 appropriately. Each amplifier interfaces with eight motor lines (five for the hall sensor and three for the motor portion). The input voltage for the motors (24 V) is also routed through the amplifiers. See Advanced Motion Controls, www.a-m-d.com, for more information.

(E5) Maxon Brushless DC Motor. We selected brushless motors for this application for their back-drivability compared with available brushed motors and/or heavily geared alternatives. Back-drivability contributes to the transparency or the haptic interface and is particularly important. The motors available to us have a maximum output torque of 25 in-oz (169 mN-m) which is adequate to satisfy the maximum compensation torques presented in Chapter 3. The motor constants were determined experimentally with motor 2 at 1.275 V/in-oz and motor 4 at 1.0943 V/in-oz. While the motors are identical models, variations in motor constants varied. This variation may be attributable to differences in the amplifier gains between the motors. However, the motors and amplifiers are hand calibrated with the LabVIEW VI to provide consistent identical torque responses on a torque watch gauge.

(E6) HEDL-5540 Encoder. The encoders provide the position awareness essential to any active compensation strategy. The encoders are sampled at the 400 MHz rate by the FPGA under the direction of the RT controller. At 500 pulses per revolution, and 4x quadrature sampling, they provide a resolution spacing of 0.18 deg. See Avago Technologies, www.avagotech.com, for more information. The pin-out is provided in Appendix E.

(E7) Power Supply. Standard power supply providing 24 VDC. The system requires approximately 0.25 A at rest and an average current of 0.5 A when engaged in compensation. The
current increases as the commanded torque grows. Power is routed directly to the motor drivers which then routes it on to the motors.
E.2 CAD Drawings

Drawing scale listed in title block may not be accurate. Use dimensions listed on the drawing. Do not scale drawings.
Figure E.2: Drawing of compliant haptic pantograph, sheet 1 of 4

Compliant Haptic Pantograph

Polypropylene

Do not scale drawing.
Figure E.3: Drawing of compliant haptic pantograph, sheet 2 of 4
Figure E.4: Drawing of compliant haptic pantograph, sheet 3 of 4
Figure E.6: Drawing of motor base, sheet 1 of 2
Figure E.8: Drawing of end effector clip, sheet 1 of 1
Figure E.9: Drawing of end effector handle, sheet 1 of 1
Several parts were designed and machined as components of the fixture for the experimental validation of the springback force in Chapter 3.
Figure E.11: Drawing of spacer, sheet 1 of 1
Figure E.12: Drawing of variable mounting plate, sheet 1 of 1
Figure E.13: Drawing of placement stick, sheet 1 of 1