Modulation and Synchronization for Aeronautical Telemetry

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Modulation and Synchronization for Aeronautical Telemetry

Christopher G. Shaw

A dissertation submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

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March 2014

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ABSTRACT

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Aeronautical telemetry systems have historically been implemented with constant envelope modulations like CPM. Shifts in system constraints including reduced available bandwidth and increased throughput demands have caused many in the field to reevaluate traditional methods and design practices. This work examines the costs and benefits of using APSK for aeronautical telemetry instead of CPM. Variable rate turbo codes are used to improve the power efficiency of 16- and 32-APSK. Spectral regrowth in nonlinear power amplifiers when driven by non-constant envelope modulation is also considered. Simulation results show the improved spectral efficiency of this modulation scheme over those currently defined in telemetry standards.

Additionally, the impact of transitioning from continuous transmission to burst-mode is considered. Synchronization loops are ineffective in burst-mode communication. Data-aided feedforward algorithms can be used to estimate offsets in carrier phase, frequency, and symbol timing between the transmitter and the receiver. If a data-aided algorithm is used, a portion of the transmitted signal is devoted to a known sequence of pilot symbols. Optimum pilot sequences for the three synchronization parameters are obtained analytically and numerically for different system constraints. The alternating sequence is shown to be optimal given a peak power constraint.

Alternatively, synchronization can be accomplished using blind algorithms that do not rely on a priori knowledge of a pilot sequence. If blind algorithms are used, the observation interval can be longer than for data-aided algorithms. There are combinations of pilot sequence length and packet length where data-aided algorithms perform better than blind algorithms and vice versa. The conclusion is that a sequential arrangement of blind algorithms operating over an entire burst performs better than a CRB-achieving data-aided algorithm operating over a short pilot sequence.

Keywords: aeronautical telemetry, turbo-codes, APSK, synchronization, data-aided estimation, blind estimation, symbol timing, carrier frequency, carrier phase
I wish to thank Dr. Michael Rice for his patience and guidance. He is an exemplary teacher, mentor, and researcher and a good friend. I am thankful for my parents. They have taught me, both in words and by example, the value of education and hard work. I am deeply grateful for my wife, Julie. Without her encouragement and support this would not be possible. Her faith in me is truly inspirational.
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Chapter 1

Introduction

Wireless communications is a field of electrical engineering with no shortage of examples of the classical engineering trade-offs. Aeronautical telemetry is a special case of wireless communication with certain defining features. The word “aeronautical” implies that at least one endpoint is airborne, in our case the transmitter. The word “telemetry” connotes a unidirectional downlink. Historically the critical design goal of an aeronautical telemetry system is achieving long range while satisfying size, weight, and power (SWaP) constraints on the transmitter. Until recently bandwidth constraints have not been a driving factor in this arena. Given these system constraints, aeronautical telemetry systems use various forms of continuous-phase modulation (CPM). The key benefit of CPM is immunity to nonlinear amplitude distortion when a solid-state power amplifier (PA) is driven in full saturation. This allows the telemetry system to achieve maximum power efficiency.

Recently, the telemetry community has experienced a significant shift in system constraints. Bandwidth is becoming increasingly more valuable for the following reasons:

- Available bandwidth is decreasing: Historically, aeronautical telemetry operated in Lower L-band (1435-1535 MHz), Upper L-band (1755-1850 MHz), Lower S-band (2200-2290 MHz), and Upper S-band (2310-2390 MHz). The lower portion of upper S-band was reallocated in two separate FCC auctions in 1997: 2320-2345 MHz was assigned to digital audio radio (today’s Sirius-XM satellite radio) and 2305-2320 MHz and 2345-2360 MHz were assigned to wireless communication services. In addition, the goals of the National Broadband Plan [1] to reallocate 500 MHz of government spectrum to the commercial sector is motivating all government users to reexamine how efficiently their spectral allocations are being used.

- The number of users has increased with more test ranges and more test units per range.
The demand for more data and higher throughput has increased for each user. Figure 1.0 shows how data rates for aeronautical telemetry systems have increased year by year [2].

This shift in system constraints has caused the telemetry community, and the standards committees, to reexamine incumbent design decisions. Among current considerations are more spectrally efficient modulation schemes and network-centric approaches to bandwidth sharing.

Interestingly, what has happened to the telemetry community is not unique. As wireless communication becomes more popular, the number of users increases along with the throughput demands per user. Many other industries under the umbrella of wireless communication have had to shift paradigms from power constrained designs to bandwidth constrained designs. The conclusions we draw in this dissertation are more broadly applicable than to the narrow field of aeronautical telemetry.
Historically, the trade-offs in aeronautical telemetry favored using a constant-envelope modulation such as CPM with an RF power amplifier operating in full saturation. In Chapter 2 we reconsider shifting constraints and explore the use of linear modulation for aeronautical telemetry. The key benefit derived from replacing CPM with linear modulation is an increase in spectral efficiency. There are two consequences accompanying the decrease in power efficiency. First, linear modulation schemes achieve greater spectral efficiency by packing more points into the symbol constellation. This requires a higher signal-to-noise ratio (SNR) to achieve a fixed bit-error rate (BER). Additionally, linear modulation schemes are not immune to PA nonlinearity. This requires PA “backoff” further reducing the SNR available at the receiver. We propose two methods for mitigating the power efficiency losses while maintaining the spectral efficiency gains of linear modulation. First, we use Amplitude-Phase shift keying (APSK) which is a form of linear modulation with a lower peak-to-average power ratio than rectangular QAM. The lower peak-to-average ratio requires less backoff for given levels of distortion and spectral regrowth compared to rectangular QAM. Second, we use turbo codes to regain some of the SNR lost due to backoff. The conclusion we draw in Chapter 2 is that turbo coded APSK is a good alternative to the standardized telemetry modulation schemes. Accounting for backoff, turbo-coded APSK achieves three times the spectral efficiency of SOQPSK with the same transmitter power constraints.

In addition to calling for more spectrally efficient modulation, new telemetry standards also call for time-division multiplexing of limited bandwidth resources. Without continuous transmission, synchronization between the transmitter and receiver becomes much more difficult. In Chapters 3 and 4 we examine synchronization techniques for burst-mode communication systems. The receiver needs to account for the carrier phase offset, carrier frequency offset, and the symbol timing offset between itself and the transmitter. Synchronization techniques are divided into two categories: data-aided and blind.

Data-aided synchronization algorithms allocate a portion of the transmitted burst to a sequence of pilot symbols known to the receiver. The data-aided receiver estimates the offsets using the received signal and a priori knowledge of the pilot sequence. In Chapter 3 we compute the optimum pilot sequences for data-aided synchronization.
In contrast to data-aided algorithms, a blind synchronization algorithm relies on knowledge of the statistics of the transmitted signal but not on a sequence of pilot symbols. In general, data-aided algorithms perform better than blind algorithms for a given observation interval. But a blind algorithm can use the entire burst to estimate the offsets while a data-aided algorithm is limited to the shorter pilot. In Chapter 4 we explore the tradeoffs between data-aided and blind synchronization algorithms and conclude that some blind algorithms can outperform our data-aided ML estimator for pilot sequence lengths and packet lengths of practical interest.

1.1 Related Publications

Chapters 2, 3, and 4 are based on peer-reviewed published articles [3], [4], and [5], respectively. They are presented here representing what the editors consider to be valuable and novel contributions to their respective journals. Each chapter includes its own literature review.

The following works have been published or are in review for publication in peer-reviewed academic journals and magazines:


The following works have been published in manuscript-reviewed conference proceedings:


The following works have been published in abstract-reviewed conference proceedings:


Chapter 2

Turbo-coded APSK for Aeronautical Telemetry

2.1 Introduction

All wireless communication systems require an RF power amplifier to drive the transmit antenna. All RF amplifiers are nonlinear devices even though many have operating regions in which they behave nearly linearly. The best power efficiency, measured by the ratio of RF output power to DC input power, is obtained when the RF power amplifier is operating at the maximum input amplitude. In this case, the RF power amplifier is highly nonlinear and is said to be operating in “full saturation.”

In aeronautical telemetry, system constraints place a premium on DC current available to the telemetry transmitter. Historically, the best way to meet this constraint has been to operate an RF power amplifier in full saturation and to adopt a modulation scheme immune to the nonlinear behavior of the RF power amplifier. Consequently, the aeronautical telemetry standards have been based on constant envelope modulations — PCM/FM (Tier 0), Shaped Offset QPSK (Tier 1), and ARTM CPM (Tier 2) [6] — all of which are variations on continuous phase modulation (CPM).

It is well known that linear memoryless modulations based on bandwidth efficient Nyquist pulse shapes offer a better operating point than CPM in the spectral efficiency versus power efficiency space. This is illustrated in Figure 2.3. Observe that even simple BPSK and QPSK offer better power efficiency (i.e., lower $E_b/N_0$ to achieve a $10^{-6}$ bit error rate) and higher spectral efficiency (measured in bits/second/Hz) than the Tier 0 and Tier 1 modulations from aeronautical telemetry. However, these improvements are realized only with a linear RF power amplifier.

The most common approach to forcing a nonlinear RF power amplifier to behave like a linear RF power amplifier is to use “backoff” — reduce the input signal level to a point
in the linear region of the RF power amplifier operating characteristic. This approach has two negative consequences. First, for a fixed RF power output level, the power efficiency is reduced and the weight/volume of the RF power amplifier increases. (That is, if 10 W of RF transmitted power is required for the link and 6 dB of back-off is required for linear operation, an RF power amplifier capable of producing 50 W of RF power is required!)

The impact of backoff can be relaxed somewhat if the question is reformulated as follows: Suppose the transmitted RF power is allowed to decrease. What is the impact of backoff? When cast in this way, backoff does not increase weight/volume, but reduces the transmitted RF power (and hence, reduces the distance over which the telemetry link can be maintained). In this way, improvements in spectral efficiency and power efficiency are obtained at the expense of range. This penalty can be reduced by using an error correcting code. In effect, the error correcting code reduces the range penalty, but achieves this at the expense of spectral efficiency.

Now the question becomes this: A linear memoryless modulation offers the possibility of improved spectral efficiency and power efficiency, but does so at the cost of available RF power. The RF power penalty can be reduced (or even eliminated) through the use of error correcting codes. But the error correcting codes do so at the expense of spectral efficiency. When this is all put together, is a system based on a coded linear modulation with backoff superior to a Tier 1 system with no backoff?

The answer to this question depends on the specifics of the RF power amplifier, the modulation, and the power of the code. For this reason, RF power amplifiers currently in use in aeronautical telemetry are examined in the context of this question. The modulation we examine is APSK, which has been proposed as part of the second generation DVB system because of its attractive performance with nonlinear RF power amplifiers. We consider simple turbo-codes because of their ability to provide good bit error rate performance at relatively low signal-to-noise ratios.

2.2 System Model

Our proposed system is illustrated in Figure 2.0. A stream of bits is input to the transmitter and passed to a rate $k/n$ turbo encoder which adds redundancy to the bitstream.
by generating $n$ output bits for every $k$ input bits ($k/n < 1$). The output of the encoder is then permuted by a random interleaver. This permuted bitstream is passed to the APSK modulator which divides the bits into blocks of 4 bits for 16-APSK or 5 bits for 32-APSK. Each block of bits is mapped to a signal in the 16- or 32-APSK signal set represented by the constellations shown in Figures 2.1 and 2.2 respectively. We use an SRRC pulse shape with 50% excess bandwidth. The signal is mixed and then amplified using an RF power amplifier and then transmitted over a channel to the receiver. For the purposes of this paper we assume that there is an additive white Gaussian noise (AWGN) channel between the transmitter and receiver.

At the receiver, the signal goes through a soft-demodulator which will be discussed in Section 2.2.2. The bit metrics output by the soft-demodulator are deinterleaved and fed into the turbo decoder which calculates an estimate of the transmitted bits.

### 2.2.1 Amplitude-Phase Shift Keying

Amplitude-phase shift keying (APSK), recently included in the ETSI second generation Digital Video Broadcasting (DVB-S2) standard [7], is almost as power efficient as corresponding quadrature amplitude modulation (QAM) constellations assigned to a square grid [8]. This can be seen in Figure 2.3. APSK also has the advantage of being more re-
Figure 2.2: 16-APSK constellation as defined in [7].

silient to channel AM/AM and AM/PM nonlinearities since the constellation points lie on concentric circles [9].

For our telemetry system, we use a pseudo-Gray mapping to assign blocks of bits to APSK constellation points as defined in [7]. For the 16-APSK constellation shown in Figure 2.1, we let \( \rho = r_2/r_1 = 2.57 \), \( \phi_1 = \pi/4 \), and \( \phi_2 = \pi/12 \) also as defined in [7]. Similarly, for the 32-APSK constellation shown in Figure 2.2, we let \( \rho_1 = r_2/r_1 = 2.53 \), \( \rho_2 = r_3/r_1 = 4.30 \), \( \phi_1 = \pi/4 \), \( \phi_2 = \pi/12 \), and \( \phi_3 = \pi/8 \).

2.2.2 Turbo Code

The turbo code we use is based on the original parallel concatenation of convolution codes (PCCC) idea [10]. The constituent recursive systematic convolutional (RSC) code is the (37,21) code found in [11] and illustrated in Figure 2.4. The interleaver is an S-random interleaver 2048 bits long. The turbo decoder runs for 10 iterations and then computes an
a posteriori probability for each bit. The turbo code is punctured using a rate compatible puncture pattern to achieve various code rates.

At the transmitter, it is easy to see how to map code bits to 16- or 32-ary symbols. However, at the receiver, the turbo decoder requires a bit metric for each code bit. The approach used by Zehavi in [12], called bit-interleaved coded modulation (BICM), solves this problem. BICM includes an interleaver between the encoder and modulator, a soft demodulator at the receiver, and a deinterleaver between the soft-demodulator and the decoder as shown in Figure 2.0. A typical maximum-likelihood hard decision detector for APSK uses a filter matched to the pulse shape to project the received signal onto the signal space spanned by the APSK signal set. BICM uses the same projection. The bit metrics are then calculated as $p(r|b_i = 0)$ and $p(r|b_i = 1)$. Since we assume the channel adds Gaussian noise, we can calculate the density function given that the symbol $s_j$ was transmitted, $p(r|s_j)$, by using the constellation point for $s_j$ as the mean in the Gaussian pdf. In order to calculate
Figure 2.4: Spectral efficiency versus $E_b/N_0$ required to achieve a BER below $10^{-6}$ for various forms of linear modulation. ARTM Tier 1 achieves $10^{-6}$ BER at 14 dB $E_b/N_0$.

$p(r|b_i = 0)$ the demodulator must add together $p(r|s_j)$ for all $s_j$ that have a zero in the $i$th position. Similarly, $p(r|b_i = 1) = \sum_{s_j \in A_1^i} p(r|s_j)$ where $A_1^i$ is the set of constellation points that have a one in the $i$th position. This illustrates the need for the bit-wise interleaver between coding and modulation. If a bad symbol is received, several consecutive bits may be corrupted at the output of the demodulator. After the deinterleaver, these bits are likely no longer adjacent and the decoder is better able to recover the corrupted data. This becomes even more important in the case of fading channels so that adjacent bits in the coded bitstream experience uncorrelated fades.

2.2.3 Nonlinear Power Amplifier

An ideal amplifier has a constant gain over the entire band of interest for any input signal level. Ideal amplifiers do not exist because there is a limited amount of DC power available. Even if an amplifier is perfectly efficient, the output signal power cannot exceed
that delivered by the power supply. In other words, in any real-world power amplifier, there is a point at which an increase in the input signal level will not have a corresponding increase in the output signal level. The amplifier is “nonlinear” because the gain of the device is a function of the input amplitude. A very simple nonlinear model for an amplifier is one with a linear operating region and a saturation operating region. The AM/AM curve for this model is illustrated by the red curve in Figure 2.5. As long as the peaks in the input signal are lower than the input level corresponding to output saturation, then the amplifier is well modeled as a constant gain. However, real power amplifiers do not have a sharp transition from the linear region to the saturation region.

A more accurate model for the gain of a solid-state power amplifier was first proposed by Rapp in [13]. This model is

\[ g(A) = \frac{vA}{1 + \left[\frac{vA}{A_0}\right]^{2p}} \]  

(2.1)

where \( v \) is the small signal (linear) gain, \( A \) is the input signal amplitude, \( A_0 \) is the saturated output amplitude, and \( p \) is a model parameter. Higher values of \( p \) result in a sharper transition from the linear region to saturation. The phase distortion, or AM/PM, of solid-state amplifiers is usually small enough to be ignored [14].
We have used this to model the AM/AM characteristic of four solid-state power amplifiers used in telemetry. Our goal is to find the parameters $v$ and $p$ that minimize the sum of the squared error between our model and the measured data. Figure 2.5 shows the results of this search for one of the power amplifiers where $v = 1.79$ and $p = 3.00$. Notice that the saturation output amplitude is 50 Volts and there is a more gradual transition than the simplified model. Higher values of $p$ make the transition sharper. Table 2.0 gives the parameters for four different amplifiers.

2.3 Results

We have simulated the performance of our turbo-coded APSK system when using the nonlinear power amplifier models described in Section 2.2.3. Results have been generated for both 16 and 32-APSK as defined in [7] with various code rates (1/3, 1/2, 2/3, 4/5, 7/8,
Table 2.1: Power Amplifier Parameters

<table>
<thead>
<tr>
<th>Amplifier</th>
<th>$v$</th>
<th>$p$</th>
<th>$V_{\text{sat}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>1.79</td>
<td>3.00</td>
<td>50</td>
</tr>
<tr>
<td>$B$</td>
<td>1.49</td>
<td>3.78</td>
<td>30</td>
</tr>
<tr>
<td>$C$</td>
<td>1.72</td>
<td>3.60</td>
<td>32</td>
</tr>
<tr>
<td>$D$</td>
<td>1.94</td>
<td>2.50</td>
<td>55</td>
</tr>
</tbody>
</table>

8/9) and output backoff levels. We define 0 dB of backoff to mean the RMS amplitude of the input signal is placed at the point where the output is fully saturated. 1 dB of backoff means the RMS amplitude of the input signal is placed at the point where the output is 1 dB below saturation. In all instances, the noise power spectral density at the receiver remains constant such that when the amplifier is driven in saturation (0 dB backoff), the received bit-energy to noise power spectral density ratio ($E_b/N_0$) is 14 dB, the same $E_b/N_0$ that yields a BER of $10^{-6}$ for ARTM Tier 1. This allows a fair comparison between the two systems and ensures that an increase in backoff reduces the $E_b/N_0$ at the receiver. We want to find the set of configurations that can achieve a BER below $10^{-6}$ in this environment and find the spectral efficiency of those configurations. However, since spectral efficiency is calculated as bit rate normalized by bandwidth, we need to consider the bandwidth of the signal output from the power amplifier.

2.3.1 Spectral Regrowth

Although the signal fed into the power amplifier may use a pulse shape with finite bandwidth, such as a square-root raised cosine (SRRC) pulse, the resulting output signal will not possess the same spectral properties. This is illustrated in Figure 2.6, which shows the power spectral density of a 16-APSK signal using an SRRC pulse shape with 50% excess bandwidth after passing through power amplifier $A$. We use 16-APSK rather than square 16-QAM because APSK has a lower peak-to-average ratio than square QAM. Thus, for a given average symbol energy ($E_{\text{avg}}$), 16-APSK has a lower peak symbol energy than 16-QAM. We have shown in Figure 2.6 the PSD of the output of power amplifier $A$ for different amounts of output backoff. Notice that the bandwidth of the signal after passing through the amplifier is much wider than the bandwidth of the original pulse shape. This will decrease
Figure 2.7: Spectral regrowth for 0, 3, and 6 dB backoff for amplifier A. The input is a 16-APSK signal using an SRRC pulse shape with 50% excess bandwidth.

the spectral efficiency measurement since we normalize bit rate by bandwidth. Notice also that increasing the backoff lowers the sidelobes and makes the bandwidth narrower.

In Figure 2.3, we calculated spectral efficiency using the actual bandwidth of the SRRC pulse shape. However, the output of the power amplifier occupies a much larger, and possibly infinite, bandwidth. Another measure of bandwidth must be used in order to calculate the spectral efficiency of the system with the nonlinear power amplifier models. Two ways of measuring bandwidth are the -50 dBc and the -60 dBc bandwidth. In other words, outside of the -50 dBc bandwidth, the signal only has power 50 dB below an unmodulated carrier. As can be seen in Figure 2.6, the -50 dBc or -60 dBc bandwidth decreases as backoff is increased. In fact, if the power amplifier is backed off at least 2.5 dB from saturation, the -50 dBc bandwidth is the same as the input signal. Similarly, if the amplifier is backed off
Table 2.2: Backoff Requirements for Unchanged Bandwidth

<table>
<thead>
<tr>
<th>Input and PA</th>
<th>For -50 dBc</th>
<th>For -60 dBc</th>
<th>For 99% Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>16-QAM on A</td>
<td>3.3 dB</td>
<td>5.5 dB</td>
<td>1.5 dB</td>
</tr>
<tr>
<td>16-APSK on A</td>
<td>2.5 dB</td>
<td>4.5 dB</td>
<td>1.1 dB</td>
</tr>
<tr>
<td>32-APSK on A</td>
<td>3.7 dB</td>
<td>5.3 dB</td>
<td>1.6 dB</td>
</tr>
<tr>
<td>16-QAM on B</td>
<td>3.3 dB</td>
<td>5.2 dB</td>
<td>1.1 dB</td>
</tr>
<tr>
<td>16-APSK on B</td>
<td>2.5 dB</td>
<td>4.3 dB</td>
<td>0.8 dB</td>
</tr>
<tr>
<td>32-APSK on B</td>
<td>3.3 dB</td>
<td>5.0 dB</td>
<td>1.1 dB</td>
</tr>
<tr>
<td>16-QAM on C</td>
<td>3.3 dB</td>
<td>5.5 dB</td>
<td>1.4 dB</td>
</tr>
<tr>
<td>16-APSK on C</td>
<td>2.6 dB</td>
<td>4.4 dB</td>
<td>1.2 dB</td>
</tr>
<tr>
<td>32-APSK on C</td>
<td>3.6 dB</td>
<td>5.2 dB</td>
<td>1.6 dB</td>
</tr>
<tr>
<td>16-QAM on D</td>
<td>3.9 dB</td>
<td>6.6 dB</td>
<td>1.3 dB</td>
</tr>
<tr>
<td>16-APSK on D</td>
<td>3.6 dB</td>
<td>5.4 dB</td>
<td>1.1 dB</td>
</tr>
<tr>
<td>32-APSK on D</td>
<td>4.1 dB</td>
<td>6.4 dB</td>
<td>1.4 dB</td>
</tr>
</tbody>
</table>

at least 4.5 dB from saturation the -60 dBc bandwidth remains unchanged from input to output. A third measure of bandwidth is the 99% power bandwidth and is the bandwidth that contains 99% of the signal power. Table 2.1 shows how much backoff is necessary to keep the -50 dBc, -60 dBc, or 99% power bandwidth unchanged from input to output for the four amplifier models. Included are the results for square 16-QAM for reference. More backoff is not always better, however, since increasing backoff means decreasing transmitted signal power. This corresponds to a lower SNR at the receiver. We will show in Section 2.3.2 that there is a tradeoff between SNR and distortion. We want to back off enough to minimize the nonlinear distortion in the amplifier but not too much to drop the SNR below the threshold of the turbo code’s error-correction capabilities.

2.3.2 Performance Comparison

Figure 2.7 shows simulation results for 32-APSK with a rate 4/5 turbo code with power amplifier A. All of the other code rates when used with 16 or 32-APSK on the four different amplifiers generate results similar in shape to this one. Notice that there is an optimal amount of backoff. On the left side of the curve, the BER is high due to the nonlinear distortion in the amplifier. As the backoff is increased the BER improves since the nonlinear distortion in the signal of interest is reduced so that correct symbol decisions are made. On the other hand, the right side of the curve represents the error-correcting capability of the
turbo code. As backoff is increased, the transmitted signal power is reduced and, since the receiver noise power is constant, the lower SNR at the receiver causes a high BER. In all of our simulations for different code rates and different amplifier models, 3 dB of backoff minimizes the bit-error rate. Notice, however, that the curve in Figure 2.7 does not dip below a BER of $10^{-6}$. Thus, 32-APSK with a rate 4/5 turbo code is not a candidate solution to outperform ARTM Tier 1. The set of configurations that do achieve a BER below $10^{-6}$ at 3 dB of backoff are listed in Table 2.2 along with their spectral efficiency. Keep in mind that ARTM Tier 1 has a spectral efficiency of 0.67 bits/sec/Hz.

2.4 Conclusions

The goal of this work is to develop a system for aeronautical telemetry that uses linear memoryless modulation and examine the performance of this system compared to ARTM
Table 2.3: Configurations that Outperform ARTM Tier 1

<table>
<thead>
<tr>
<th>PA</th>
<th>M-APSK</th>
<th>R</th>
<th>-50 dBc BW</th>
<th>Spectral efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>16</td>
<td>1/3</td>
<td>1.52 Hz/Hz</td>
<td>0.88 bits/sec/Hz</td>
</tr>
<tr>
<td>A</td>
<td>16</td>
<td>1/2</td>
<td>1.52 Hz/Hz</td>
<td>1.32 bits/sec/Hz</td>
</tr>
<tr>
<td>A</td>
<td>16</td>
<td>2/3</td>
<td>1.52 Hz/Hz</td>
<td>1.75 bits/sec/Hz</td>
</tr>
<tr>
<td>A</td>
<td>16</td>
<td>4/5</td>
<td>1.52 Hz/Hz</td>
<td>2.11 bits/sec/Hz</td>
</tr>
<tr>
<td>A</td>
<td>32</td>
<td>1/3</td>
<td>2.93 Hz/Hz</td>
<td>0.56 bits/sec/Hz</td>
</tr>
<tr>
<td>A</td>
<td>32</td>
<td>1/2</td>
<td>2.93 Hz/Hz</td>
<td>0.85 bits/sec/Hz</td>
</tr>
<tr>
<td>B</td>
<td>16</td>
<td>1/3</td>
<td>1.52 Hz/Hz</td>
<td>0.88 bits/sec/Hz</td>
</tr>
<tr>
<td>B</td>
<td>16</td>
<td>1/2</td>
<td>1.52 Hz/Hz</td>
<td>1.32 bits/sec/Hz</td>
</tr>
<tr>
<td>B</td>
<td>16</td>
<td>2/3</td>
<td>1.52 Hz/Hz</td>
<td>1.75 bits/sec/Hz</td>
</tr>
<tr>
<td>B</td>
<td>16</td>
<td>4/5</td>
<td>1.52 Hz/Hz</td>
<td>2.11 bits/sec/Hz</td>
</tr>
<tr>
<td>B</td>
<td>32</td>
<td>1/3</td>
<td>2.30 Hz/Hz</td>
<td>0.72 bits/sec/Hz</td>
</tr>
<tr>
<td>B</td>
<td>32</td>
<td>1/2</td>
<td>2.30 Hz/Hz</td>
<td>1.09 bits/sec/Hz</td>
</tr>
<tr>
<td>B</td>
<td>32</td>
<td>2/3</td>
<td>2.30 Hz/Hz</td>
<td>1.45 bits/sec/Hz</td>
</tr>
<tr>
<td>C</td>
<td>16</td>
<td>1/3</td>
<td>1.48 Hz/Hz</td>
<td>0.90 bits/sec/Hz</td>
</tr>
<tr>
<td>C</td>
<td>16</td>
<td>1/2</td>
<td>1.48 Hz/Hz</td>
<td>1.35 bits/sec/Hz</td>
</tr>
<tr>
<td>C</td>
<td>16</td>
<td>2/3</td>
<td>1.48 Hz/Hz</td>
<td>1.80 bits/sec/Hz</td>
</tr>
<tr>
<td>C</td>
<td>16</td>
<td>4/5</td>
<td>1.48 Hz/Hz</td>
<td>2.16 bits/sec/Hz</td>
</tr>
<tr>
<td>C</td>
<td>32</td>
<td>1/3</td>
<td>2.81 Hz/Hz</td>
<td>0.57 bits/sec/Hz</td>
</tr>
<tr>
<td>C</td>
<td>32</td>
<td>1/2</td>
<td>2.81 Hz/Hz</td>
<td>0.89 bits/sec/Hz</td>
</tr>
<tr>
<td>D</td>
<td>16</td>
<td>1/3</td>
<td>1.52 Hz/Hz</td>
<td>0.88 bits/sec/Hz</td>
</tr>
<tr>
<td>D</td>
<td>16</td>
<td>1/2</td>
<td>1.52 Hz/Hz</td>
<td>1.32 bits/sec/Hz</td>
</tr>
<tr>
<td>D</td>
<td>16</td>
<td>2/3</td>
<td>1.52 Hz/Hz</td>
<td>1.75 bits/sec/Hz</td>
</tr>
<tr>
<td>D</td>
<td>16</td>
<td>4/5</td>
<td>1.52 Hz/Hz</td>
<td>2.11 bits/sec/Hz</td>
</tr>
<tr>
<td>D</td>
<td>32</td>
<td>1/3</td>
<td>2.66 Hz/Hz</td>
<td>0.62 bits/sec/Hz</td>
</tr>
<tr>
<td>D</td>
<td>32</td>
<td>1/2</td>
<td>2.66 Hz/Hz</td>
<td>0.94 bits/sec/Hz</td>
</tr>
</tbody>
</table>

We use 16 and 32-APSK because they have a better peak-to-average power ratio than square QAM and thus perform better when using a nonlinear power amplifier. However, the power amplifier still must be driven below saturation so we use a turbo code to recover the lost power efficiency. We have modeled several different power amplifiers and simulated the performance of this telemetry system for various code rates. If we fix power efficiency and BER, we can compare the spectral efficiency of this system with the spectral efficiency of ARTM Tier 1. We have shown that several configurations match the power efficiency and BER performance of ARTM Tier 1 with significant increases, some more than 3 times, in spectral efficiency. The increase in spectral efficiency can be used to increase the bit rate of the telemetry system or shrink the bandwidth, allowing more users.
Other work has been done in [15] to further examine the effects of spectral regrowth. This work studies the effects of adjacent channel interference on turbo-coded APSK and concludes that despite the spectral regrowth produced by the power amplifier channels do not need to be spaced further than the requirements of the SRRC pulse shape. Using this definition of bandwidth, we obtain slightly better results than shown in Table 2.2.
Chapter 3

Optimum Pilot Sequences for Data-Aided Synchronization

3.1 Introduction

Synchronization is a fundamental task in any digital communication system. In wireless communications, the primary synchronization parameters are carrier phase offset, carrier frequency offset, and symbol timing offset between the transmitter and the receiver. In continuous transmission systems, feedback synchronization loops can be used to compensate for these offsets. In burst mode communication systems, feedback loops may take too long to lock for reasonable loop bandwidths. For this reason, data-aided feedforward block estimators are more commonly used. A data-aided synchronization algorithm allocates a portion of the burst (at the expense of information throughput) to a sequence of pilot symbols known at the receiver. The receiver uses the received signal, corrupted by carrier phase, carrier frequency, and symbol timing offsets and additive noise, to estimate these offsets. The question we address in this chapter is: Which pilot sequence is best?

One way to characterize the performance of an estimator is to examine the residual error between the true values of the synchronization parameters and the estimated values. If the mean of the estimate is equal to the true value of the estimate, then the estimator is unbiased. An estimator is efficient if its residual error variance is as small as possible. The Cramér-Rao bound (CRB) is a lower bound on the estimator error variance for any unbiased estimator. An efficient estimator achieves the CRB.

With three synchronization parameters, several options exist for estimation algorithms. A receiver could estimate each of the parameters independently, sequentially, or jointly. In general, a joint estimator will be more complex but will have a lower CRB than independent or sequential estimators. In this chapter, the method we use to determine
which pilot sequence is “best” is finding the pilot sequence that minimizes the CRB for a joint estimator of the synchronization parameters.

The true CRB for single parameter estimators and two parameter estimators are available. Specifically, the CRB for symbol timing estimation can be found in [16]. The joint CRB for symbol timing and carrier phase estimation can be found in [17]. The joint CRB for carrier phase and carrier frequency estimation can be found in [18, 19, 20]. The modified CRB (MCRB), discussed in [21, 22, 23], is often used to average across all possible data sequences and remove its dependency from the CRB. We will not use this technique because we want to keep the effect of the data sequence in the CRB so we can find the CRB-minimizing sequence. The true CRB for all three synchronization parameters with data dependency intact is found in [24, 25].

The notion of defining “best” as the sequence that minimizes the CRB has been used before. Minn and Xing [20] used the joint CRB for carrier phase and carrier frequency but only found the sequence that minimized the CRB for carrier frequency. They use a hybrid peak and average constraint to get a similar sequence to our frequency sequence but some of the energy is smeared to adjacent symbols if the peak constraint is violated. Tavares and Tavares [26] use this technique for the three pairwise combinations (phase-frequency, phase-timing, frequency-timing) using a Gaussian pulse shape. The sequences we obtain for signals using an SRRC pulse are different from the sequences obtained in [26] for a Gaussian pulse.

The alternating sequence is commonly used for timing synchronization because it provides the maximum number of waveform transitions. In [17], it is shown that the alternating sequence yields a lower CRB than random data. The alternating sequence is used in [27] because the alternating sequence through an SRRC pulse shape can be approximated by a sine wave which simplifies the maximum likelihood estimator. We show in [24] that the alternating sequence minimizes the CRB for all three synchronization parameters if the sequence search space is constrained to a constellation and if the pulse shape is SRRC with 50% excess bandwidth.
In this paper we make the following contributions:

• We review the work done in [24, 25] to show that the alternating sequence minimizes the CRB for all three synchronization parameters if the sequence search space is constrained to a constellation. If the search space is constrained to the complex plane with an average energy constraint, we find three different sequences that minimize the CRB for the three synchronization parameters.

• We derive the maximum likelihood joint estimator and show that the estimator error variance for each of the three parameters achieves the CRB (i.e. the maximum likelihood joint estimator is efficient).

• We examine the performance degradation of the ML estimator with a preamble of pilot symbols immediately followed by unknown random data. The estimator suffers from an error variance floor and does not achieve the CRB at high SNR.

In Section 3.2 we present our signal model with the joint CRB presented in Section 3.3. We present the optimum pilot sequences in Section 3.4. The constellation constrained search is discussed in Section 3.4.1 and the unconstrained searches are discussed in Sections 3.4.2 and 3.4.3. We present the maximum likelihood joint estimator in Section 3.5 and the performance losses from appending random data to the end of the pilot in Section 3.6. We draw our conclusions in Section 3.7.

3.2 Signal Model

The complex baseband representation of a linearly modulated signal is

\[ s(t) = \sum_{l=0}^{L_p-1} a(l)p(t - lT_s) \]  

(3.1)

where \( a(l) = a_I(l) + ja_Q(l) \) is a sequence of pilot symbols, \( T_s \) is the symbol time, \( p(t) \) is any real square-root Nyquist I pulse shape [28] with support on \(-LT_s \leq t \leq LT_s\), and \( L_p \) is the length of the pilot. The signal \( s(t) \) has support on \(-LT_s \leq t \leq (L + L_p - 1)T_s\). The sequence of symbols, \( a(l) \), may be drawn from an \( M \)-ary constellation such as MPSK or MQAM, but,
to maintain generality, we will not make that restriction until later in the discussion. The
complex baseband representation of the received signal is
\[
r(t) = e^{j(\omega t + \theta)} \sum_{l=0}^{L_p-1} a(l)p(t - lT_s - \tau) + w(t) \tag{3.2}
\]
where \(\theta\) is the phase rotation, \(\omega\) is the residual frequency shift, \(\tau\) is the timing offset, and
\(w(t)\) is additive white Gaussian noise (AWGN). We assume the phase, frequency, and timing
offsets are constant for the duration of the burst. Thus \(\theta, \omega,\) and \(\tau\) are not functions of time \(t\).

We assume that the receiver uses an analog-to-digital (A/D) converter either at IF
or complex baseband. Without loss of generality, we assume the sample rate at the syn-
chronizer input is \(N\) times the symbol rate where \(N = T_s/T\) and \(T\) is the sample time in
seconds/sample. After sampling (and mixing in the case of IF sampling), the \(n\)-th sample
of the received signal is
\[
r(nT) = e^{j(\Omega n + \theta)} \sum_{l=0}^{L_p-1} a(l)p((n - lN - \tau/T)T) + w(nT) \tag{3.3}
\]
where \(\Omega = \omega T\) rads/sample and \(-NL \leq n \leq N(L_p - 1) + NL\).

Now construct the following:
\[
\begin{bmatrix}
    r((-NL)T) \\
    r((-NL + 1)T) \\
    \vdots \\
    r((N(L_p - 1) + NL)T)
\end{bmatrix},
\tag{3.4}
\]
\[
D_\Omega =
\begin{bmatrix}
    e^{j\Omega(-NL)} & 0 & \cdots & 0 \\
    0 & e^{j\Omega(-NL+1)} & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & e^{j\Omega(N(L_p-1)+NL)}
\end{bmatrix},
\tag{3.5}
\]

23
\[ a = \begin{bmatrix} a(0) \\ a(1) \\ \vdots \\ a(L_p - 1) \end{bmatrix}, \quad (3.6) \]

and

\[ w = \begin{bmatrix} w((-NL)T) \\ w((-NL + 1)T) \\ \vdots \\ w((NL(L_p - 1) + NL)T) \end{bmatrix}. \quad (3.7) \]

Also define a matrix \( P_{\tau} \) where the \( ik \)-th entry is

\[ p((i - kN - NL - \tau/T)T), \quad (3.8) \]

\( i \) is the sample index and \( k \) is the symbol index. We can now write (3.3) in matrix form as follows:

\[ \mathbf{r} = e^{j\theta} D\Omega P_{\tau} \mathbf{a} + \mathbf{w}. \quad (3.9) \]

The goal of a synchronization algorithm is to generate estimates for phase offset, frequency offset, and timing offset (\( \hat{\theta}, \hat{\Omega}, \hat{\tau} \)) using the observation \( \mathbf{r} \). If the synchronizer is data-aided, then the algorithm can also use knowledge of the transmitted symbols \( \mathbf{a} \). The question we address in this work is which \( \mathbf{a} \) is best. Our definition of “best” is the sequence \( \mathbf{a} \) that minimizes the Cramér-Rao bound for the estimates of each of the synchronization parameters.

### 3.3 Cramér-Rao Bound

Because \( w(t) \) is a white, zero-mean, complex-valued Gaussian random process, the sequence \( w(nT), -NL \leq n < NL + N(L_p - 1) \), is a sequence of uncorrelated zero-mean complex-valued Gaussian random variables. The real and imaginary components of \( w(nT) \) are each zero-mean Gaussian random variables with variance \( \sigma^2 \). Consequently, the log-
likelihood function for $\theta$, $\Omega$, and $\tau$ is

$$\Lambda(\theta, \Omega, \tau, a) = K - \frac{1}{2\sigma^2} \left( r - e^{j\theta}D_\Omega P_\tau a \right)^H \left( r - e^{j\theta}D_\Omega P_\tau a \right)$$

(3.10)

where $K$ is a constant that is not a function of $\theta$, $\Omega$, or $\tau$. The Fisher information matrix for the joint estimator of $\theta$, $\Omega$, and $\tau$ is

$$J = \begin{bmatrix} J_{\theta,\theta} & J_{\theta,\Omega} & J_{\theta,\tau} \\ J_{\Omega,\theta} & J_{\Omega,\Omega} & J_{\Omega,\tau} \\ J_{\tau,\theta} & J_{\tau,\Omega} & J_{\tau,\tau} \end{bmatrix}.$$ 

(3.11)

The entries in $J$ are derived in [24, 25] and are given in the Appendix by (A.24).

The inverse of the Fisher information matrix is a $3 \times 3$ matrix whose diagonal entries are the Cramér-Rao lower bounds for the error variances of the joint estimators. It can be seen from (A.24) that the CRB is a function of $\sigma^2$, $\tau$, the pulse shape, and $a$. A system designer will not have control over the true value of $\tau$, but in [24] we showed that the CRB is not a strong function of $\tau$ anyway. The impact of $\sigma^2$ simply scales the CRB by the signal-to-noise ratio. One could increase the amplitude of the pilot signal and improve the estimates. But if a training power budget is fixed, carefully choosing a pilot sequence is the only way to minimize the CRB and improve estimator performance. The remainder of this paper will be focused on how to choose $a$ to minimize the CRB. We will discuss pulse shapes in Section 3.4.1.

### 3.4 Pilot Sequences

Equipped with the CRB as a function of the pilot sequence from Section 3.3, we are able to use standard optimization techniques to find the global minimum of the CRB.
function.

\[ \begin{align*}
\text{argmin}_a \text{CRB}_\theta(a) & \quad \text{subject to} \quad a^H a \leq E_{avg} L_p \\
\text{argmin}_a \text{CRB}_\Omega(a) & \quad \text{subject to} \quad a^H a \leq E_{avg} L_p \\
\text{argmin}_a \text{CRB}_\tau(a) & \quad \text{subject to} \quad a^H a \leq E_{avg} L_p.
\end{align*} \tag{3.12} \]

This must be a constrained optimization because the CRB decreases as the energy in \(a\) increases. In the following, we will evaluate the results of the optimization problem using two different constraints. First, in Section 3.4.1 we will require the elements of the pilot sequence to be drawn from a discrete and finite set of constellation points such as QAM or PSK. Second, in Sections 3.4.2 and 3.4.3 we relax that requirement and impose a constraint on the total energy in the pilot sequence and conduct a search over the \(L_p\) dimensional complex plane.

### 3.4.1 Constellation Constrained Optimization

In this section we will find the pilot sequence symbols [the \(a(l)\) in (3.1)] that minimize the joint CRB for carrier phase, carrier frequency, and symbol timing estimators subject to a constellation constraint on the symbols in the pilot sequences. If the constellation is small and the pilot sequence is short, the search space is small and we can apply the brute-force search method of discrete optimization. We have applied this exhaustive search technique to BPSK, QPSK, 8-PSK, and 16-QAM with a square-root raised cosine (SRRC) pulse shape with 50% excess bandwidth for sequence lengths \(L_p = \{1, 2, \ldots, 10\}\) and found the sequence that minimizes the CRB for each of \(\theta, \Omega,\) and \(\tau\). In all of these test cases for each of the synchronization parameters, the CRB-minimizing sequence is what we will call the “alternating sequence” as illustrated in Figure 3.0. That is, for a given constellation, the CRB-minimizing sequence for \(\theta, \Omega,\) and \(\tau\) alternates between a pair of antipodal symbols with the greatest amplitude for \(L_p\) symbols. This CRB-minimizing sequence is a unique minimum (to within a constant phase rotation because all of these constellations exhibit rotational symmetry).
Figure 3.1: The alternating sequence, which minimizes the CRB for timing and the CRBs for frequency and phase if $\alpha = 0.5$. $L_p = 10$.

It can be seen from (A.24) that the CRB is a function of the pulse shape, or, in the case of the SRRC pulse shape, the excess bandwidth parameter $\alpha$. For symbol timing synchronization, the CRB-minimizing sequence is always the alternating sequence regardless of the excess bandwidth. For frequency and phase, however, different sequences may yield a lower CRB than the alternating sequence for different values of $\alpha$. For example, Figure 3.1 shows the sequence that minimizes the CRB for phase and frequency when $\alpha = 0.75$ and Figure 3.2 shows the sequence that minimizes the CRB for phase and frequency when $\alpha = 1$. The alternating sequence minimizes the CRB for phase and frequency for $0.1 \leq \alpha \leq 0.33$ and for $0.48 \leq \alpha \leq 0.65$. For the remainder of this paper, we will use an SRRC pulse shape with 50% excess bandwidth.

While we cannot prove that the alternating sequence is globally optimal for all pilot sequence lengths, $L_p$, there is a pattern in sequences of $L_p = \{1, 2, \ldots, 10\}$ that would suggest this is the case. The alternating sequence is commonly used for symbol timing
**Figure 3.2:** The phase and frequency CRB minimizing sequence when $\alpha = 0.75$. $L_p = 10$.

**Figure 3.3:** The phase and frequency CRB minimizing sequence when $\alpha = 1$. $L_p = 10$. 
synchronization possibly because it is known to yield good performance when using timing tracking loops [29]. Morelli, et al. used this sequence in [27] because it can be approximated by a sine wave after passing through a pulse shaping filter. The resulting CRB in [27] is an approximation of the true and more general CRB derived in Section 3.3. The benefits of using the alternating sequence for carrier phase and frequency estimation as illustrated above have not previously been shown.

### 3.4.2 Unconstrained Optimization for Symbol Timing

If we relax the constraint that the pilot sequence symbols [the \( a(l) \) in (3.1)] must be drawn from a discrete constellation, and instead perform a power constrained search over the \( L_p \)-dimensional space of complex numbers, we must employ more sophisticated search algorithms than the brute-force method from Section 3.4.1. The structure of the Fisher information matrix in (A.24) leads us to use different optimization techniques for the pilot sequence that minimizes the CRB for symbol timing than the sequences that minimize the CRB for carrier frequency and phase. We know from the discussion on nuisance parameters in [30, p.232] that joint estimators with nonzero off-diagonal entries in the Fisher information matrix cannot perform better than corresponding single parameter estimators. For our signal model, this result implies that, if there is a nonempty set of sequences that give the Fisher information matrix a block diagonal structure, the optimum sequence will be in that set of sequences. We can show that this is true for the CRB for symbol timing synchronization. Partition \( \mathbf{J} \) as follows:

\[
\mathbf{J} = \begin{bmatrix}
J_{\theta,\theta} & J_{\theta,\Omega} & J_{\theta,\tau} \\
J_{\Omega,\theta} & J_{\Omega,\Omega} & J_{\Omega,\tau} \\
J_{\tau,\theta} & J_{\tau,\Omega} & J_{\tau,\tau}
\end{bmatrix}.
\]

Using the matrix inversion lemma [31] the (3,3) element of \( \mathbf{J}^{-1} \) may be expressed as

\[
(J^{-1})_{3,3} = \left( J_{\tau,\tau} - [J_{\tau,\theta} \ J_{\tau,\Omega}] \begin{bmatrix}
J_{\theta,\theta} & J_{\theta,\Omega} \\
J_{\Omega,\theta} & J_{\Omega,\Omega}
\end{bmatrix}^{-1} \begin{bmatrix}
J_{\theta,\tau} \\
J_{\Omega,\tau}
\end{bmatrix}\right)^{-1}.
\]
The following observations will be used:

**Lemma 1** $J_{\tau,\tau}$ is a positive constant that is maximized, subject to the constraint $a^H a = E_{\text{avg}} \times L_p$, if

$$a = \sqrt{E_{\text{avg}} L_p} u_1$$

where $u_1$ is the (real-valued) normalized eigenvector corresponding to the largest eigenvalue of $\dot{P}_\tau^T \dot{P}_\tau$.

**Proof:** $J_{\tau,\tau} = (2/\sigma^2) a^H \dot{P}_\tau^T \dot{P}_\tau a$ is a quadratic form involving the symbol vector $a$ and the positive semidefinite symmetric matrix $\dot{P}_\tau^T \dot{P}_\tau$. The remainder of the proof may be found in [31, Theorem 6.5].

**Lemma 2** The second term inside the parentheses of (3.14) is non-negative. It is zero if and only if $J_{\theta,\tau} = J_{\tau,\theta} = J_{\Omega,\tau} = J_{\tau,\Omega} = 0$.

**Proof:** From the definitions (A.24), we have $J_{\theta,\tau} = J_{\tau,\theta}$ and $J_{\Omega,\tau} = J_{\tau,\Omega}$. Consequently, the second term inside the parentheses of (3.14) is a quadratic matrix-vector expression. The matrix in the quadratic form is, itself, a Fisher information matrix which, as shown in [32], is positive definite. As such, the result is a real-valued, non-negative scalar. It follows from the definition of a positive definite matrix that the quadratic form will equal zero if and only if $\begin{bmatrix} J_{\tau,\theta} & J_{\tau,\Omega} \end{bmatrix} = 0$.

The entries $J_{\theta,\tau}$ and $J_{\tau,\theta}$ depend on $\dot{P}_\tau^T P_\tau$. It is well known that if $\dot{P}_\tau^T P_\tau$ is symmetric, then any complex-valued sequence $a$ renders the quadratic form $a^H \dot{P}_\tau^T P_\tau a$ real-valued so that $J_{\theta,\tau} = J_{\tau,\theta} = 0$. Similarly, $J_{\Omega,\tau}$ and $J_{\tau,\Omega}$ depend on $\dot{P}_\tau^T C P_\tau$. Again, if $\dot{P}_\tau^T C P_\tau$ is symmetric, any complex-valued sequence $a$ renders $J_{\Omega,\tau} = J_{\tau,\Omega} = 0$. In this application, it is rarely the case that $\dot{P}_\tau^T P_\tau$ and $\dot{P}_\tau^T C P_\tau$ are symmetric. This leads us to the following lemma.

**Lemma 3** For nonsymmetric $\dot{P}_\tau^T P_\tau$ and $\dot{P}_\tau^T C P_\tau$, $J_{\theta,\tau} = J_{\tau,\theta} = J_{\Omega,\tau} = J_{\tau,\Omega} = 0$ if the symbols in $a$ lie on a line through the origin in the complex plane.

**Proof:** $J_{\theta,\tau}, J_{\tau,\theta}, J_{\Omega,\tau},$ and $J_{\tau,\Omega}$ are zero if the expression inside the Re\{\cdot\} operator is purely imaginary. This is the case when $a^H M a$ is real for any real-valued square matrix $M$. If
\[ a = e^{j\phi}b \] where \( b \) is a real-valued vector, then \( a^H Ma = b^T Mb \) is a real number. Thus,

\[
\frac{2}{\sigma^2} \text{Re} \{ j b^T Mb \} = 0.
\]

Note that any sequence that satisfies the hypothesis of Lemma 3 renders \( J_{\theta,\tau} = J_{\tau,\theta} = J_{\Omega,\tau} = J_{\tau,\Omega} = 0 \) and the Fisher information matrix inherits a block diagonal structure:

\[
J = \begin{bmatrix}
J_{\theta,\theta} & J_{\theta,\Omega} & 0 \\
J_{\Omega,\theta} & J_{\Omega,\Omega} & 0 \\
0 & 0 & J_{\tau,\tau}
\end{bmatrix}.
\]  

(3.15)

Now the \((3,3)\) element of \( J^{-1} \) may be expressed as

\[
(J^{-1})_{3,3} = (J_{\tau,\tau})^{-1} = \frac{\sigma^2}{2} \frac{1}{a^H \hat{P}_\tau^T \hat{P}_\tau a}.
\]  

(3.16)

Lemmas 2 and 3 show that any real-valued sequence \( a \) will block-diagonalize the Fisher information matrix. Tavares and Tavares show in [26] that, for pairwise CRBs involving timing estimation, a time symmetric pilot sequence makes \( J_{\theta,\tau} = J_{\tau,\theta} = J_{\Omega,\tau} = J_{\tau,\Omega} = 0 \) when using a Gaussian pulse shape instead of an SRRC pulse shape. Other authors [16, 22] show that the same off-diagonal entries are zero if the observation is long and the data is random. We have shown, instead, that any real-valued sequence will do. Lemma 1 shows which real-valued sequence minimizes the \((3,3)\) element of \( J^{-1} \). Equipped with these three Lemmas, we can prove the following theorem:

**Theorem 1** The real-valued sequence

\[
a = \sqrt{E_{avg} L_p} u_1
\]  

(3.17)

minimizes the Cramér-Rao bound for timing estimation subject to the constraint \( a^H a = E_{avg} \times L_p \) where \( u_1 \) is the normalized eigenvector corresponding to the largest eigenvalue of \( \hat{P}_\tau^T \hat{P}_\tau \).
Figure 3.4: Pilot sequence that minimizes the CRB for $\tau$. $L_p = 10$ and $p(t)$ is an SRRC pulse shape with 50% excess bandwidth.

Proof: The Cramér-Rao bound for timing is given by (3.14) and is minimized when the term inside the parentheses is maximized. By Lemma 1, the first term inside the parentheses of (3.14) is maximized by the given real-valued sequence. By Lemmas 2 and 3, the second term inside the parentheses of (3.14) is minimized at zero if $a$ is real. Thus, the sequence

$$a = \sqrt{E_{\text{avg}}L_p}u_1$$

simultaneously maximizes the first term and minimizes the second term in (3.14).

For $L_p = 10$ and a SRRC pulse shape with 50% excess bandwidth, the timing optimal pilot sequence is shown in Figure 3.3. Notice that this sequence is similar to the alternating sequence but with an envelope that concentrates more energy in the middle rather than the ends of the training period.

The benefit of using the optimal sequence from (3.17) instead of the alternating sequence varies with the sequence length $L_p$. This can be seen in Figure 3.4. The CRB
difference increases as $L_p$ increases from 1 to 8. The difference decreases as $L_p$ increases for $L_p > 8$. The maximum CRB difference occurs at $L_p = 8$ and is 0.4 dB. Caveat Lector — the maximum difference of 0.4 dB occurring at $L_p = 8$ is a function of the pulse shape used in this example. Different pulse shapes may yield different values for the maximum difference and the corresponding $L_p$. While the benefit of using the timing optimum sequence is not overwhelming, the additional complexity may be negligible if the transmitter and receiver are capable of dealing with arbitrary complex-valued symbols.

### 3.4.3 Unconstrained Optimization for Carrier Phase and Frequency

The structure of the Fisher information matrix allowed us to find an expression for the optimal pilot sequence for symbol timing offset estimation. The estimation error for carrier phase offset and carrier frequency offset are correlated regardless of the pilot sequence used. In other words, the upper left $2 \times 2$ matrix in (3.15) cannot be block-diagonalized. Other
authors [26, 19, 22] have found a block diagonal form for the Fisher information matrix for the two parameter case of joint phase and frequency estimation with known timing offset. Having a known timing offset changes the signal model in two ways. First, for Nyquist pulse shapes like SRRC, the received signal can be sampled at one sample per symbol. Second, the timing offset is used as an extra degree of freedom to force the off-diagonal elements of the $2 \times 2$ Fisher information matrix to zero. These two factors (one sample per symbol and an extra degree of freedom) make the inverse of the matrix trivial to compute. In the $3 \times 3$ case we are exploring this is not possible. It is perhaps due to this more complicated Fisher information matrix structure that other authors have not explored the 3 parameter joint estimator. However, we show in the following that, while closed-form solutions are not feasible for this signal model, we can find numerical results that satisfy intuition and significantly reduce the CRB for carrier phase and carrier frequency estimation.

For carrier phase offset estimation, we want to find

$$\arg\min_a (J^{-1})_{1,1} \quad \text{subject to } a^H a \leq E_{avg} L_p,$$

(3.18)

and, for carrier frequency offset estimation,

$$\arg\min_a (J^{-1})_{2,2} \quad \text{subject to } a^H a \leq E_{avg} L_p.$$

(3.19)

The pilot sequence that minimizes the CRB for carrier frequency offset estimation for $L_p = 10$ is shown in Figure 3.5. This sequence concentrates all of the available energy into the first and last symbols and leaves all of the other symbols zero. For other values of $L_p$, the CRB-minimizing sequence exhibits similar properties. For any sequence length, the first and last symbols have amplitude $\sqrt{L_p/2}$. For sequences of odd length, the last symbol in the CRB-minimizing sequence is the negative of the first symbol. The benefits from using the unconstrained optimal sequence for frequency estimation increase as sequence length increases (see Figure 3.6). This is primarily due to the increased energy available for the nonzero symbols when the sequence length increases.

The pilot sequence that minimizes the CRB for carrier phase offset estimation is shown in Figure 3.7. For any sequence length, the sequence that minimizes the CRB for
carrier phase offset estimation concentrates all of the available energy into the first symbol and leaves all of the other symbols zero. Again, the benefits from using the unconstrained optimal sequence for phase estimation increase as sequence length increases and, in the case of phase estimation, can be quite significant (see Figure 3.8).

For practical reasons, the sequences shown in Figures 3.5 and 3.7 might not be useful. Because all of the available energy is concentrated in one or two symbols, these sequences yield waveforms that exhibit a high peak-to-average ratio. If the transmitter’s RF power amplifier does not have a sufficiently high dynamic range, the transmitted signal will be severely distorted and the benefits of using these pilot sequences forfeited. Minn and Xing [20] use a hybrid peak and average power constraint in their search for an optimum pilot sequence for carrier frequency offset estimation. The results of their work show that some of the energy in the optimum sequence spreads to adjacent “symbols” as the peak-to-average ratio of the constraint decreases. In this paper we have explored the two limiting extremes on
the power constraint: peak only (constellation constrained) and average only. Given a set of
design parameters with limited dynamic range, the alternating sequence may be a “better”
choice for carrier frequency and phase offset estimation than their corresponding optimum
sequences.

The CRBs for phase, frequency, and timing estimation are shown in Figures 3.15, 3.16, and 3.17, respectively, for all of the sequences discussed. Naturally, the lowest CRB
in each figure is achieved by using the appropriate CRB-minimizing sequence. The timing
CRB when using the timing optimum pilot sequence is almost 0.4 dB better than when
using the alternating sequence. Using the phase optimum sequence yields a poor timing
CRB. However, the phase CRB when using the phase optimum sequence is about 5 dB
better than the alternating sequence and almost 7 dB better than using the timing optimum
sequence. Similar gains are achieved for the frequency CRB when using the frequency
optimum sequence. Selecting the “best” sequence for a given scenario must be made with
the application in mind. For example, 64-QAM requires a much more accurate phase estimate than BPSK so a system designer would choose a sequence that yields good phase estimator performance perhaps at the expense of another parameter.

By way of summary, under a power constraint, the CRB-minimizing sequence is different for each of the three synchronization parameters. For symbol timing estimation, the optimum sequence has zero crossings at each symbol time which is reminiscent of the alternating sequence, but the envelope of the timing optimum sequence is not constant. The shape of the envelope is a function of the pulse shape. On the other hand, the CRB-achieving phase estimator comprises one large amplitude pilot symbol rather than any pattern of symbols. Giving the estimator additional looks at the phase offset is not as good as giving the estimator one better (in terms of SNR) look at the phase offset. We can similarly interpret the frequency optimum sequence as the sequence that gives the best two looks at phase, separated in time as far as possible. This situation is closely related to the conclusions drawn in [33, 34, 35] which show that the CRB for carrier frequency offset estimation is lower with disjoint blocks of pilot symbols at the beginning and end of the burst than with one contiguous block of pilot symbols.

3.5 Maximum Likelihood Estimation

In this section we compare the performance of the maximum likelihood (ML) joint estimator using the pilot sequences from Section 3.4 with the CRB. Because we will be evaluating the estimator in simulation, computational complexity is of minor interest but not ultimately important.

The maximum likelihood joint estimator for $\theta$, $\Omega$, and $\tau$ finds the parameters $\hat{\theta}$, $\hat{\Omega}$ and $\hat{\tau}$ that maximize the log-likelihood function (3.10):

$$
\hat{\theta}, \hat{\Omega}, \hat{\tau} = \arg\max_{\theta, \Omega, \tau} \{ \Lambda(\theta, \Omega, \tau, a) \} 
$$

$$
= \arg\min_{\theta, \Omega, \tau} \left\{ (r - e^{j\theta}D_\Omega P_\tau a)^H (r - e^{j\theta}D_\Omega P_\tau a) \right\}. 
$$

Formulated in this way, the estimator (3.21) describes a 3-dimensional search. The complexity of this search can be reduced by solving for $\hat{\theta}$ and substituting into (3.21). The ML
estimate $\hat{\theta}$ must satisfy

$$0 = \frac{\partial \Lambda(\theta, \Omega, \tau, a)}{\partial \theta} \bigg|_{\theta = \hat{\theta}} = \frac{2}{\sigma^2} \text{Re} \left\{ -je^{-j\hat{\theta}} (D_{\Omega}P_{\tau}a)^{\text{H}}r \right\}. \quad (3.22)$$

Solving (3.22) for $\hat{\theta}$ produces an expression for $\hat{\theta}$ in terms of the other two parameters $\Omega$ and $\tau$:

$$\hat{\theta} = \arg \left\{ (D_{\Omega}P_{\tau}a)^{\text{H}}r \right\}. \quad (3.23)$$

Using the relationship

$$e^{j\hat{\theta}} = \frac{(D_{\Omega}P_{\tau}a)^{\text{H}}r}{\left| (D_{\Omega}P_{\tau}a)^{\text{H}}r \right|} \quad (3.24)$$
for the complex exponential in (3.21) produces a less-complex version of the ML estimator based on a 2-dimensional search as defined by the following two equations:

$$\hat{\Omega}, \hat{\tau} = \arg\min_{\Omega, \tau} \left\{ \left| \begin{bmatrix} I - \frac{\langle D_{\hat{\Omega}} P_{\tau} a \rangle (D_{\hat{\Omega}} P_{\tau} a)^H}{r} \end{bmatrix} r \right|^2 \right\}, \quad (3.25)$$

$$\hat{\theta} = \arg \left\{ \langle D_{\hat{\Omega}} P_{\tau} a \rangle^H r \right\} \quad (3.26)$$

where $|v|^2 = v^H v$ for a column vector $v$ and where $D_{\hat{\Omega}}$ is defined in the same way as $D_{\Omega}$ except using $\hat{\Omega}$ in place of $\Omega$ and $P_{\hat{\tau}}$ is defined in the same way as $P_{\tau}$ except using $\hat{\tau}$ in place of $\tau$. The estimator (3.25) computes the estimates in two steps. In the first step, a 2-dimensional search is performed to identify the ML estimates for $\Omega$ and $\tau$. In the second step, the ML estimate for $\theta$ is computed based on the results of the first step. Most of the commonly used search techniques, such as those outlined in [36, Chapter 10] may be used.
The difficulty of the search is the construction of the matrices $D_\Omega$ and $P_\tau$ for each trial value of $\Omega$ and $\tau$.

The performance of the ML estimator for carrier phase, frequency, and symbol timing is shown in Figures 3.15, 3.16, and 3.17, respectively. For these simulations we use the MATLAB unconstrained minimization function `fminunc`, $N = 2$, $L_p = 10$, and $L = 6$. As expected, the ML estimator achieves the CRB for sufficiently high $E/N_0$. Included for reference are the results of the ML estimator using a Zadoff-Chu\(^1\) (CAZAC) sequence [37, 38]. These sequences are popular in channel sounding and as pilot sequences for estimating channel conditions in wireless communication systems such as 3G and 4G/LTE standards.

Contour plots of the argument in (3.25) shed some light on the reasons behind the similarities and differences in estimator performance. Contour plots of (3.25) for the sequences discussed are given in Figures 3.9 through 3.13 for the noise-free case and for $\Omega = 0.1 \times 2\pi$ and $\tau/T_s = 0.123$. We observe from these that a “steep” gradient in the timing dimension corresponds to a lower timing CRB. Similarly, a “steep” gradient in the frequency dimension corresponds to a lower frequency CRB. “Broad” minima correspond to higher CRBs.

We can see from Figure 3.11 that the frequency optimum sequence gives a very steep gradient in the frequency dimension but there are several local minima. This means that the frequency CRB is low but an estimator might have difficulty determining the global minimum. This is a well-known problem for frequency estimators with non-adjacent pilot symbols [33, 34, 35]. Similarly, Figure 3.12 shows that the timing dimension has another local minimum at integer multiples of the symbol time when using the timing optimum sequence. The phase optimum sequence shown in Figure 3.10 is not a good sequence for timing or frequency estimation but it only has one local minimum which allows for a simple gradient descent search algorithm. This is because the phase optimum sequence has only one nonzero symbol. Lastly, notice in Figure 3.13 that when using the CAZAC sequence, the timing error and frequency error are correlated. This is because the CAZAC sequence is not real-valued and does not satisfy Lemma 3 in Section 3.4.2.

\(^1\)The simulation results are based on a length-10 Zadoff-Chu [37] sequence with parameter $M = 3$.  

40
3.6 The Impact of Unknown Data

The signal model in (3.1) represents a transmitted signal composed of a sequence of pilot symbols filtered by a band limited pulse shape. Until this point in our discussion, we have included the leading and trailing transients of the transmitted signal in our observation vector \( \mathbf{r} \) defined in (3.3) and (3.4). In this section, we will refer to this signal model as our “original” signal model. Another way to represent an equivalent transmitted signal is to assume all symbols before and after the sequence of pilot symbols are zero-valued. Most data aided synchronization algorithms of practical importance transmit unknown data symbols either immediately before or immediately after the pilot symbols (or both) which is different from our model. In this section we will study a packet structure illustrated in Figure 3.14 with a preamble of pilot symbols immediately followed by unknown data symbols. We will refer to this signal model as our “augmented” signal model. How does this practical consideration change our analysis from previous sections?
Figure 3.11: Contour plot of the argument of (3.25) with $\Omega = 0.1 \times 2\pi$ and $\tau/T_s = 0.123$ with the phase optimum sequence.

If we refer to the unknown data symbols as “nuisance parameters,” we could employ different techniques to average out the dependence on the data and compute different variants of the CRB. To compute the “true” CRB as in [23], we can assume a uniform prior distribution of constellation points and then evaluate the log-likelihood function using the law of total probability. This method, to our knowledge, is intractable. Alternatively, we could compute the modified CRB (MCRB) [21] by leaving the data dependence in until the Fisher information matrix is computed and then average across uniformly likely data sequences. The MCRB is a lower bound on the true CRB. While this technique is tractable for this augmented signal model, it is not clear that minimizing an unachievable lower bound produces the desired result. What is clear is that the performance of the new arrangement will not be better than the original model.

The approach taken here is to examine the performance of the estimator derived in Section 3.5 (that achieves the CRB using the original signal model) when used with
the augmented signal model. The observation interval for the augmented signal model is $-NL \leq n \leq N(L_p - 1)$. The justification for this change is illustrated in Figure 3.14. The leading transients of the data are included in the observation vector and the trailing transients of the pilot are excluded.

The simulated performance of the ML estimator for carrier phase, frequency, and symbol timing is also shown in Figures 3.15, 3.16, and 3.17, respectively. We note the following observations.

- With the augmented signal model, the estimator reaches an error variance floor which is due to the interference or “self-noise” caused by the nonlinear characteristic of the ML estimator [see (3.25)] operating in the presence of unknown data symbols.

- In Figures 3.15–3.17, the optimum phase sequence has the lowest error variance floor. This is because the optimum phase sequence has all of its energy in the first symbol which is as far in time as possible from the interfering data symbols. If the pilot block
Figure 3.13: Contour plot of the argument of (3.25) with $\Omega = 0.1 \times 2\pi$ and $\tau/T_s = 0.123$ with the timing optimum sequence.

were a post-amble instead of a preamble, the results would be quite different, unless the phase sequence concentrated all of its energy at the end of the pilot block. [The phase-optimum pilot sequence places all of the available energy at the location defined by $t = 0$. See (3.2), (3.3), and (3.5).]

- The sequences with the highest error variance floor are different in Figures 3.15–3.17. The alternating, timing-optimum, and CAZAC sequences have the highest phase error variance floor (see Figure 3.15); all of them except for the phase-optimum sequence have approximately the same (highest) frequency error variance floor (see Figure 3.16); and the frequency-optimum and CAZAC sequences have the highest timing error variance floor (see Figure 3.17). Curiously, the CAZAC sequence is the only sequence common to all three cases.

- For values of $E/N_0$ where performance is dominated by additive noise (below about 20 dB for the parameters used in our examples), the ML estimator achieves essentially the
same error variance performance for the original and augmented signal models. The
exception is the frequency-optimum sequence which suffers from self-noise at a lower
$E/N_0$ than the other sequences.

The dominant feature of the ML estimator performance with the augmented signal model is
self-noise. If the self-noise could be eliminated, these results suggest that the performance
of the ML estimator under the augmented and original signal models is the same. Con-
sequently, the “best” sequences derived from the original (tractable) signal model are the
“best” sequences for the augmented signal model. Fortunately, the self-noise can be reduced
or even eliminated by increasing the length of the pilot to make $L_p \gg L$, prefiltering [39, 40],
or some combination of the two.
Figure 3.15: The top represents the “original” signal model given in (3.3) with leading and trailing pilot transients included in the observation vector. The bottom represents the “augmented” signal model discussed in Section 3.6. The observation vector is truncated by $NL$ samples so that the leading transients of the first $L$ unknown data symbols are included and the trailing transients of the last $L$ pilot symbols are excluded.
Figure 3.16: The CRB and ML estimator performance for carrier phase offset estimation versus $E/N_0$. The lines denote the CRB for each of the sequences discussed in Section 3.4. The outlined markers denote the error variance of the ML estimator described in Section 3.5. The filled markers denote the error variance of the ML estimator using the augmented signal model discussed in Section 3.6.
Figure 3.17: The CRB and ML estimator performance for carrier frequency offset estimation versus $E/N_0$. The lines denote the CRB for each of the sequences discussed in Section 3.4. The outlined markers denote the error variance of the ML estimator described in Section 3.5. The filled markers denote the error variance of the ML estimator using the augmented signal model discussed in Section 3.6.
Figure 3.18: The CRB and ML estimator performance for symbol timing estimation versus $E/N_0$. The lines denote the CRB for each of the sequences discussed in Section 3.4. The outlined markers denote the error variance of the ML estimator described in Section 3.5. The filled markers denote the error variance of the ML estimator using the augmented signal model discussed in Section 3.6.
3.7 Conclusions

We derived the 3-parameter joint CRB for carrier phase, carrier frequency, and symbol timing estimation as a function of a sequence of pilot symbols. We computed the sequences that minimize the CRB under two different constraints: a constellation constraint, and an average energy constraint. Under the constellation constraint, the CRB-minimizing sequence is the alternating sequence shown in Figure 3.0 if the pulse shape is an SRRC with 50% excess bandwidth. Under the average energy constraint, the sequence that minimizes the CRB for timing is shown in Figure 3.17, the sequence that minimizes the CRB for frequency is shown in Figure 3.5, and the sequence that minimizes the CRB for phase is shown in Figure 3.7. We derived the maximum likelihood estimator and showed that it achieves the CRB for sufficiently high $E/N_0$. If we modify our signal model to include the leading transients of the unknown data symbols as interference in our pilot sequence observation, the ML estimator performance is characterized by an error variance floor due to self-noise.

There are several opportunities for future work to be done on this topic. Whereas we have stated that the derivation of the “true” CRB of an estimator based on the augmented signal model presented in Section 3.6 is intractable, numerical techniques may be applied to approximate the optimum pilot signals in this scenario. Also, the practical implications of using the pilot sequences in Figures 3.5 and 3.7 with large peak-to-average ratios should also be considered although the details will likely be application specific. The effects of a fading channel can be incorporated into the derivation of optimum pilot sequences and the model perhaps expanded to include channel estimation.
Chapter 4

Data-Aided versus Blind Synchronization Techniques

4.1 Introduction

Synchronization is a critical component of all wireless digital communication systems. A receiver must account for the carrier offset and the symbol timing offset between itself and the transmitter. In continuous transmission systems, feedback synchronization loops can be used to compensate for these offsets. On the other hand, feedforward estimators are more commonly used in burst-mode communication systems to avoid the acquisition time required by tracking loops. Feedforward estimators can be divided into two categories: blind and data-aided. Data-aided estimators allocate some portion of the transmitted symbol sequence to a sequence of pilot symbols known a priori at the receiver. Because these symbols are known at the receiver, they convey no information and only aid in synchronization. In contrast, blind estimators only employ knowledge of the statistics of the transmitted signal and not any known symbol sequence. The benefit of data-aided estimation is reduced mean-squared error compared to blind estimation for a fixed sequence length. However, holding sequence length fixed is not a fair comparison of estimator performance. In data-aided synchronization, the pilot sequence will be only a fraction of the transmitted burst. A blind estimator will be able to operate on the entire burst rather than the small fraction allocated to pilot symbols (see Figure 4.0). The goal of this paper is to explore this trade-off and help a system designer decide which synchronization algorithm(s) to use.

To facilitate this trade-off we compare the performance of data-aided and blind feedforward estimators for timing and frequency. Because it is a practical impossibility to exhaustively study all known estimators, we focus on the performance of representative examples.

• Data-Aided Estimator: The optimum pilot sequences for the joint maximum likelihood (ML) estimation of timing, frequency, and phase were derived and analyzed in [4]. It
was shown that the “alternating” sequence is the best sequence for joint estimation. Other data-aided estimators have been described in [19, 27] and [41, 42, 43, 44, 45, 46, 47, 48, 49]. Because the “alternating” pilot sequence operating with the joint estimator presented in [4] is best, it is used as the data-aided estimator in the comparisons of Section 4.4.

- **Blind Joint Estimator:** A blind technique that jointly estimates timing and frequency was presented by Gini and Giannakis [50] with a mathematical analysis in [51]. The estimator is based on the second-order cyclostationary statistics of the matched filter outputs. Because the Gini and Giannakis work is a discrete-time derivation and improves upon similar approaches presented in [52, 53, 54, 55], it is used as the representative case of blind joint estimation of timing and frequency in the comparisons of Section 4.4.

- **Blind Timing and Frequency Estimator:** An alternative approach to blind joint estimation of timing and frequency is to estimate each timing and frequency offset separately.

  1. A blind timing estimator based on an ad-hoc approach was described in [56], modified ML-based approaches were described in [57, 58], and on cyclostationary properties were described in [59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69]. Because, in general, the approaches based on cyclostationarity perform better in the presence of a frequency offset, we focus our attention there. Of these, the Oerder-Meyr estimator [59] is the most popular. This popularity is due in large measure to its excellent performance and low complexity. For this reason, the Oerder-Meyr estimator [59] estimator is used as the representative case for blind estimation of timing in the comparisons of Section 4.4.

  2. Blind frequency estimators based on ML techniques have been described in [70, 71, 72, 73, 74]. An expanded analysis of [74] is given in [41]. In Section 4.4, this estimator is used to represent the class of ML-based blind frequency estimators. Blind frequency estimators based on cyclostationarity are found in [75, 76]. The “delay-and-multiply” technique for blind frequency estimation is closely related to maximum likelihood methods and cyclostationary statistical methods. Examples
of frequency estimators based on this technique are found in [77, 75, 41]. The
delay-and-multiply estimator found in [41] is also simulated in Section 4.4.

Some work has been done on blind estimation for carrier phase offset estimation [78,
79, 80]. Because most forms of linear modulation involve a constellation with rotational
symmetry (e.g., MPSK or MQAM) blind phase offset estimation is usually accompanied
by a phase ambiguity. Phase ambiguity resolution requires either a few pilot symbols or
differential detection. For the duration of this paper, we will not focus on phase estimation
because there is no truly blind algorithm to compare with the ML data-aided estimator.

In summary, the blind estimators we will simulate in Section 4.4 are: the Oerder-Meyr
(OM) blind timing estimator, the Mengali perfect timing (MPT) blind frequency estimator,
the Mengali unknown timing (MUT) blind frequency estimator, and the Gini-Giannakis
joint cyclostationary timing and frequency (CST and CSF) blind estimators. More detailed
explanations of each of these algorithms are given in Section 4.3.

4.2 Signal Model

The complex baseband representation of a linearly modulated signal is

\[ s(t) = \sum_{l=0}^{L-1} a(l)p(t - lT_s) \]  

(4.1)

where \( a(l) = a_I(l) + j a_Q(l) \) is a sequence of symbols, \( T_s \) is the symbol time, \( p(t) \) is any
real square-root Nyquist I pulse shape [28] with support on \( -L T_s \leq t \leq L T_s \), and \( L \) is the
length of the packet or burst. The signal \( s(t) \) has support on \( -L T_s \leq t \leq (L + L - 1) T_s \).
The elements of \( a(l) \) are drawn from an \( M \)-ary constellation such as MPSK or MQAM. The
complex baseband representation of the received signal is

\[ r(t) = e^{j(\omega t + \theta)} \sum_{l=0}^{L-1} a(l)p(t - lT_s - \tau) + w(t) \]  

(4.2)

where \( \theta \) is the phase rotation, \( \omega \) is the residual frequency shift, \( \tau \) is the timing offset, and
\( w(t) \) is additive white Gaussian noise (AWGN). We assume the phase, frequency, and timing
offsets are constant for the duration of the burst. Thus $\theta$, $\omega$, and $\tau$ are not functions of time $t$.

We assume that the receiver uses an analog-to-digital (A/D) converter either at IF or complex baseband. Without loss of generality, we assume the sample rate at the synchronizer input is $N$ times the symbol rate where $N = T_s/T$ and $T$ is the sample time in seconds/sample. After sampling (and mixing in the case of IF sampling), the $n$-th sample of the received signal is

$$r(nT) = e^{j(\Omega n + \theta)} \sum_{l=0}^{L-1} a(l)p((n - lN - \tau/T)T) + w(nT) \quad (4.3)$$

where $\Omega = \omega T$ rads/sample and $-NL \leq n \leq N(L - 1) + NL$.

Some estimators operate directly on the received samples modelled in (4.3) while others operate on the output of a matched filter. The matched filter output is

$$y(nT) = r(nT) * p(-nT). \quad (4.4)$$

In the following, we will discuss one particular blind frequency estimator that assumes accurate timing synchronization has already been carried out. The input to that algorithm is the output of the matched filter

$$y(kT_s) = y(nT), \ n = (k + \hat{\tau})N \quad (4.5)$$

where the evaluation at $n = (k + \hat{\tau})N$ is usually performed by interpolation [29].

The goal of a synchronization algorithm is to generate estimates for frequency offset and timing offset ($\hat{\Omega}$, $\hat{\tau}$) using an observed signal, either the received signal or the matched filter output. If the synchronizer is data-aided, then the algorithm can also use knowledge of the transmitted symbols.

### 4.3 Synchronization Algorithms

For a given observation length, a data-aided estimator achieves a lower mean-squared error (MSE) than a blind estimator. But this benefit is realized at the cost of overhead since
the pilot symbols must be known at the receiver and cannot be information bearing. On the other hand, a blind estimator can use the entire packet for its observation. This is illustrated in Figure 4.0. The observation interval for a data-aided estimator is $L_p$ and the observation interval for a blind estimator is $L_b$. We want to determine how much overhead is needed for the data-aided estimator to outperform the blind estimator. In other words, what is the relationship between $L_p$ and $L_b$?

In the remainder of this section, we summarize the various estimators used in simulation in Section 4.4.

4.3.1 Maximum-Likelihood Joint Estimation

The data-aided estimator we will use in our comparisons is the Maximum-Likelihood (ML) estimator developed in [4]. With (4.3) as a starting point, we switch to matrix-vector
notation for simplicity and write:

\[ r = \begin{bmatrix} r((-N\mathcal{L})T) \\ r((-N\mathcal{L} + 1)T) \\ \vdots \\ r((N(L_p - 1) + N\mathcal{L})T) \end{bmatrix}, \quad (4.6) \]

\[
D_{\Omega} = \begin{bmatrix} e^{j\Omega(-N\mathcal{L})} & 0 & \ldots & 0 \\
0 & e^{j\Omega(-N\mathcal{L}+1)} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & e^{j\Omega(N(L_p-1)+N\mathcal{L})} \end{bmatrix}, \quad (4.7)
\]

and

\[
a = \begin{bmatrix} a(0) \\ a(1) \\ \vdots \\ a(L_p - 1) \end{bmatrix}. \quad (4.8)
\]

Also define a matrix \( P_\tau \) where the \( ik \)-th entry is

\[
p((i - kN - N\mathcal{L} - \tau/T)T), \quad (4.9)
\]

\( i \) is the sample index and \( k \) is the symbol index.

The ML estimator for carrier frequency and symbol timing is based on the 2-dimensional search [4]

\[
\hat{\Omega}_{\text{ML}}, \hat{\tau}_{\text{ML}} = \arg\min_{\Omega, \tau} \left\{ \left\| \begin{bmatrix} I - (D_{\Omega}P_\tau a)(D_{\Omega}P_\tau a)^H \\ (D_{\Omega}P_\tau a)^H r \end{bmatrix} r \right\|^2 \right\}. \quad (4.10)
\]

It is shown in [4] that the ML estimator in (4.10) with the ‘alternating’ sequence achieves the CRB over a broad range of signal-to-noise ratio (SNR). The SNR values we will be simulating in Section 4.4 are in the range where the ML estimator achieves the CRB. The pilot sequence we will use in Section 4.4 is the CRB-minimizing alternating sequence [4].
4.3.2 Oerder-Meyr Timing Estimation

Oerder and Meyr [59] introduced perhaps the most well-known blind algorithm for symbol timing estimation. Assuming a matched filter for the detection filter, the Oerder-Meyr (OM) estimate is

\[ \hat{\tau}_{OM} = \frac{1}{2\pi} \text{arg}(X) \] (4.11)

where

\[ X = L_bN - 1 \sum_{n=0}^{L_bN-1} |y(nT)|^2 e^{j2\pi n/N} \] (4.12)

where \( y(nT) \) is the matched filter output given by (4.4). The OM estimator leverages the fact that the phase of the first DFT component of \( |y(nT)|^2 \) is linearly related to the timing offset. While not required, the complexity of the OM estimator is greatly reduced when \( N = 4 \) samples/symbol because the complex exponential in (4.12) reduces to \( \pm 1 \) and \( \pm j \) at every time index. Consequently, \( X \) may be computed using two real-valued accumulators.

4.3.3 Mengali Frequency Estimation

Mengali, et al, describe two open-loop carrier frequency estimators in [41]. The first algorithm (MPT) requires the receiver to have accurate symbol timing information. It also requires the symbol constellation to be \( M \)-ary PSK and operates on the output of the matched filter. The MPT frequency estimate is

\[ \hat{\Omega}_{MPT} = \frac{1}{M} \text{arg}\left\{ \sum_{k=1}^{L_b-1} [y(kT_s)g^*((k - 1)T_s)]^M \right\} \] (4.13)

where \( y(k) \) is the matched filter output sampled at one sample per symbol with no timing offset. Descriptively, the algorithm removes the bulk phase by performing the operation in the square brackets in (4.13), and then removes the modulation by raising the product to the \( M \)-th power. For QPSK, \( M = 4 \).

As previously discussed, this algorithm operates at one sample per symbol assuming timing synchronization has been done. In Section 4.4, we will simulate how this estimator performs in two different systems. The first is a system where there is no timing offset
between transmitter and receiver. The second is a system where symbol timing estimation is done with the Oerder-Meyr algorithm and interpolation and downsampling are done with an ideal interpolator.

The second open-loop frequency estimation algorithm given in [41] does not require the receiver to have knowledge of the symbol timing offset (MUT). The algorithm is based on the delay-and-multiply method and operates at \( N = 4 \) samples/symbol.

\[
\hat{\Omega}_{\text{MUT}} = 4 \arg \left\{ \sum_{n=1}^{4L_b-1} r(nT)r^*((n-1)T) \right\}.
\]

(4.14)

### 4.3.4 Cyclostationary Joint Estimation

Gini and Giannakis [50] proposed a blind estimation algorithm for symbol timing estimation and a blind estimation algorithm for carrier frequency estimation based on cyclostationary statistics. Because both algorithms rely on the second-order cyclostationarity of the signal transmitted in (4.1), the first step is to compute the sample estimate of the cyclic correlation function:

\[
\hat{\mathcal{M}}_{2x}(k; \epsilon) = \frac{1}{NL_b} \sum_{n=0}^{NL_b-\epsilon-1} y(nT)y^*((n+\epsilon)T)e^{-j2\pi kn/N}
\]

(4.15)

where \( k = 0, 1, \ldots, N - 1 \). From there, the cyclostationary frequency estimate can be computed as

\[
\hat{\Omega}_{\text{CSF}} = \frac{1}{2L_g} \sum_{\epsilon=1}^{L_g} \frac{1}{\epsilon} \arg \left\{ \hat{\mathcal{M}}_{2x}(1; \epsilon)\hat{\mathcal{M}}_{2x}(-1; \epsilon) \right\}
\]

(4.16)

where \( L_g \) is an averaging factor.

The cyclostationary timing estimate is

\[
\hat{\tau}_{\text{CST}} = -\frac{1}{4\pi(L_g+1)} \sum_{\epsilon=0}^{L_g} \left[ \arg \left\{ \hat{\mathcal{M}}_{2x}(1; \epsilon)e^{-j\pi\epsilon/N} \right\} - \arg \left\{ \hat{\mathcal{M}}_{2x}(-1; \epsilon)e^{j\pi\epsilon/N} \right\} \right].
\]

(4.17)

### 4.4 Numerical Results

The performance of the estimators described in Section 4.3 was evaluated using simulations. The mean-squared error (MSE) was used to quantify the performance. As discussed
in Section 4.1, we want to know the observation intervals where the data-aided estimation algorithms have the same MSE as blind estimation algorithms. This represents a threshold where a longer packet gives the performance edge to the blind algorithm or a longer pilot gives the edge to the data-aided algorithm.

Our simulations are all based on QPSK transmission over an AWGN channel. For data-aided estimation, we use the “alternating” sequence from [4] for the pilot sequence. The data payload is uniformly distributed random QPSK symbols. We use an SRRC pulse shape with truncation length $L = 6$ symbols. We simulate a fixed frequency offset of 5% of the symbol rate and a fixed timing offset of $\tau = 0.123 \ T_s$. For the cyclostationary estimators we use the same averaging factor $L_g = 16$ as in [50].

Figure 4.1 compares the performance of the ML timing estimator from Section 4.3.1 with the different blind timing estimators, specifically, the Oerder-Meyr (OM) estimator from Section 4.3.2 and the cyclostationary timing (CST) estimator from Section 4.3.4. MSE performance is plotted against $E_s/N_0$ where $E_s$ is the average energy per symbol. Included are simulation results for different values of excess bandwidth. Notice that the blind estimation algorithms track the CRB with a constant offset for a portion of the SNR range and then hit an MSE floor caused by self-noise. That offset between the blind estimator and the CRB can be overcome by lengthening the observation interval for the blind estimator. The Oerder-Meyr algorithm suffers from a lower MSE floor than the cyclostationary algorithm. In contrast to the two blind algorithms, the ML estimator tracks the CRB for the full range of SNR we are simulating [4]. Lastly, notice that the estimates improve as excess bandwidth increases.

The comparison in Figure 4.1 is unfair because $L_p = L_b = 512$ for both the blind and data-aided estimators. It is not simulating the scenario found shown in Figure 4.0. Figure 4.2 remedies the unfairness and shows on the vertical axis how many pilot symbols are necessary to achieve the same MSE as the blind estimator with an observation interval given on the horizontal axis. In other words, the curve shown in Figures 4.2 is the threshold. The region above the curve is where data-aided estimation outperforms (in the MSE sense) blind estimation. Similarly, the region below the curve is where blind estimation outperforms data-aided estimation. The SNR operating point for Figure 4.2 is 20 dB. Figure 4.3 shows
the threshold for an SNR operating point of 40 dB, where the cyclostationary estimator has hit its error floor but the Oerder-Meyr estimator is still tracking the CRB with an offset. Similarly, the results for an SNR operating point of 60 dB are given in Figure 4.4. Notice that the OM threshold with 50% excess bandwidth and 40 dB SNR shown in Figure 4.3 can be accurately fitted by a line with slope 1/4. This means that the Oerder-Meyr blind timing estimator can only be outperformed by a data-aided estimator with more than 25% of the packet allocated to pilot symbols, or 25% overhead.

Figure 4.5 compares the performance of the ML frequency estimator from Section 4.3.1 with the different blind frequency estimators from Sections 4.3.3 and 4.3.4. While simulations have been performed at various amounts of excess bandwidth, the results for frequency estimation were indistinguishable. To keep the plots uncluttered, only the results for 50% excess bandwidth are included. The MPT estimator is simulated in two different environments. The first assumes there is no timing offset between transmitter and receiver. We
Figure 4.3: Comparison of data-aided, Oerder-Meyr (OM), and cyclostationary (CST) symbol timing estimators. The horizontal axis is the length of the burst. The vertical axis is pilot sequence length. The curves represent how many pilot symbols are necessary to outperform the blind estimator for different values of excess bandwidth. $E/N_0 = 20$ dB.
Figure 4.4: Comparison of data-aided, Oerder-Meyr (OM), and cyclostationary (CST) symbol timing estimators. The horizontal axis is the length of the burst. The vertical axis is pilot sequence length. The curves represent how many pilot symbols are necessary to outperform the blind estimator for different values of excess bandwidth. $E/N_0 = 40$ dB.
Figure 4.5: Comparison of data-aided, Oerder-Meyr (OM), and cyclostationary (CST) symbol timing estimators. The horizontal axis is the length of the burst. The vertical axis is pilot sequence length. The curves represent how many pilot symbols are necessary to outperform the blind estimator for different values of excess bandwidth. $E/N_0 = 60$ dB.
denote this scenario as “Genie → MPT.” The second simulates a sequential approach where
timing synchronization is carried out first, with an Oerder-Meyr timing estimator and ideal
interpolator and downsampler. The resulting signal (at one sample per symbol) is input
to the MPT blind frequency estimator. This scenario is denoted “OM → MPT.” Notice
that blind frequency estimation performs much worse (in the MSE sense) than data-aided
frequency estimation. Of the three blind estimators simulated, only the Mengali estimator
with perfect timing information (MPT) tracks the CRB for any values of SNR. Without any
information on symbol timing, the best blind estimator (Mengali with unknown timing) can
only achieve an MSE of $10^{-3}$. The blind frequency estimators also suffer from self-noise error
floors the same way blind timing estimators do. Figure 4.6 shows the threshold operating
point for the MPT estimator in both simulated environments at an SNR operating point of
40 dB. The results will be similar for any SNR operating point in the range where the MPT
estimator tracks the CRB with a constant offset, about 20 dB through about 50 dB. Outside
of this SNR range, the data-aided ML estimator will perform better and the threshold will
shift down. The threshold plots for the MUT and CSF estimators are not shown because
the MUT and CSF estimators can be outperformed by a data-aided estimator using very
few (less than 3) pilot symbols for any SNR operating point.

4.5 Conclusions

With the simulation results of Section 4.4, we can draw the following conclusions:

1. The Oerder-Meyr estimator performs well for symbol timing estimation. Better perfor-
   mance can only be achieved by a data-aided estimator with more than 25% of the burst
   allocated to pilot symbols. In most communication systems, this amount of overhead
   is too high.

2. If the modulation scheme is MPSK, the MPT algorithm is a good choice for carrier
   frequency estimation. The MPT algorithm operates at one sample per symbol so
   symbol timing synchronization needs to be completed first.

3. For non-PSK modulation schemes, data-aided frequency estimation is the best option
   with a short pilot sequence.
Figure 4.6: CRB and mean-square error performance of data-aided and blind carrier frequency estimators. $L_p = L_b = 512$.

The Oerder-Meyr blind timing estimator performs remarkably well for a broad range of SNR. For symbol timing synchronization, it is a better option than data-aided estimation in terms of MSE performance, packet overhead, and computational complexity. When combined with the MPT blind frequency estimator for MPSK, the tandem of blind estimators performs very well. A data-aided algorithm can only perform better with more than 25% of the packet allocated to pilot symbols.
Figure 4.7: Comparison of data-aided and MPT blind carrier frequency estimators. The horizontal axis is the length of the burst. The vertical axis is pilot sequence length. The curves represent how many pilot symbols are necessary to outperform the blind estimator. Includes results for two systems using the MPT estimator. $E/N_0 = 40$ dB.
Chapter 5

Conclusions and Future Work

5.1 Conclusions

This work has surveyed many of the issues faced by the aeronautical telemetry community as bandwidth constraints have tightened. The contributions of this dissertation are the following:

- It has been shown that APSK is a suitable modulation for aeronautical telemetry even if backoff is applied to the transmit RF power amplifier. Variable rate turbo codes can overcome the loss in power efficiency and the overall system achieves a significant increase in spectral efficiency. These improvements can be used to reduce bandwidth, increase users, or increase throughput per user.

- It has been shown that turbo-codes can help to mitigate the effects of nonlinear distortion in addition to increasing effective SNR at the receiver.

- The alternating sequence is shown to be the optimum pilot sequence for data-aided synchronization for all three synchronization parameters. The alternating sequence has been a common choice of pilot sequence for symbol timing synchronization among system developers and in the literature but its optimality was unknown until now. The alternating sequence had not previously been used for carrier phase and carrier frequency synchronization.

- If the peak power constraint is reduced to an average power constraint and the pilot symbols allowed to take on arbitrary complex values, then an entirely new class of pilot sequences becomes available. The unconstrained optimum pilot sequence for symbol timing can be derived analytically and it exhibits similarities to the alternating
sequence. The unconstrained optimum pilot sequences for carrier phase and carrier frequency can be obtained numerically and significant gains can be realized when they are used in data-aided synchronization.

- As an intermediate step to obtaining the optimal pilot sequences, the CRB for data-aided synchronization has been derived. Other forms of the CRB can be found in the literature but the data-dependency is always removed. In the derivation presented here, the data-dependency is left explicit and the resulting CRB is used as a tool in obtaining the CRB minimizing pilot sequence.

- Equipped with a CRB minimizing pilot sequence and a CRB achieving ML estimator, the performance of blind estimation algorithms can be evaluated. Whereas, data-aided estimators operate over a short observation interval, blind estimators can use the entire burst or packet for an observation interval. Not all blind estimators perform well but the Oerder-Meyr timing estimator yields very good performance for reasonable burst lengths. The sequential combination of the Oerder-Meyr estimator for timing synchronization and Mengali’s frequency estimator for PSK modulation can outperform data-aided algorithms without any overhead. For other forms of modulation, like APSK, a good solution includes the Oerder-Meyr algorithm for timing synchronization and a data-aided algorithm for carrier phase and carrier frequency synchronization with a short alternating sequence pilot.

5.2 Future Work

There are several opportunities for further research and development based on this work. Of course, one of the more enticing avenues is to build a real-world system that employs the techniques presented and observe its performance.

Turbo codes present some challenges to any wireless communication system. There can be a large amount of latency resulting from the iterative decoding algorithm. Latency also originates from the multiplicity of interleavers on the transmitter and receiver. Some applications of aeronautical telemetry are very concerned with “end events” where the transmitter may not have time to push the most important bits through several interleavers. For-
ward error-correction may not be the right thing to do in these scenarios. Turbo codes also tend to be more popular in research than in practice because many specific implementations are protected by patents and other licensing restrictions. Other, more open, methods of error-correction, such as LDPC codes could yield similar results to those presented here.

The discussion in Chapter 2 included modeling several solid-state power amplifiers. Solid-state amplifiers exhibit high levels of amplitude distortion (AM/AM) when driven in saturation but pass phase information (AM/PM) relatively undistorted. Other amplifier structures such as TWTA amplifiers do not share this property. Further work can be done to model the distortion of TWTA amplifiers, often used in satellite communication systems.

All of the work presented in this dissertation uses an additive white Gaussian noise (AWGN) channel model. For aeronautical telemetry, this can be a good model for relatively high altitude line-of-sight channels. The AWGN channel is inadequate for scenarios experiencing multipath propagation. Given the wideband nature of telemetry signals and the typical multipath environments (e.g. low elevation angle scenarios, flight line scenarios), the multipath channel is usually a frequency selective channel. The notion of synchronization can become intertwined with channel estimation when fading channels are added to the system. Perhaps a joint estimator for synchronization and channel estimation could be brought into the framework presented in Chapter 3 and pilot sequences that minimize a different CRB will result.
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Appendix A

The Joint Cramér-Rao bound for Data-Aided Synchronization

The received signal is modeled as (see (3.3) and (4.3))

\[ r = e^{j\theta} D_\Omega P_r a + w \]  

where

\[ r = \begin{bmatrix} r((-NL)T) \\ r((-NL+1)T) \\ \vdots \\ r((N(L_p-1)+NL)T) \end{bmatrix}, \]  

\[ D_\Omega = \begin{bmatrix} e^{j\Omega(-NL)} & 0 & \cdots & 0 \\ 0 & e^{j\Omega(-NL+1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{j\Omega(N(L_p-1)+NL)} \end{bmatrix}, \]  

\[ a = \begin{bmatrix} a(0) \\ a(1) \\ \vdots \\ a(L_p-1) \end{bmatrix}, \]  

and

\[ w = \begin{bmatrix} w((-NL)T) \\ w((-NL+1)T) \\ \vdots \\ w((N(L_p-1)+NL)T) \end{bmatrix}. \]
Also, the matrix $P_\tau$ where the $ik$-th entry is

$$p((i - kN - NL - \tau/T)T),$$  \hspace{1cm} (A.6)

$i$ is the sample index and $k$ is the symbol index.

The log-likelihood function is

$$\Lambda(\theta, \Omega, \tau, a) = \frac{K - 1}{2\sigma^2} \left( r - e^{j\theta}D_\Omega P_\tau a \right)^H \left( r - e^{j\theta}D_\Omega P_\tau a \right)$$

(A.7)

$$= K - \frac{1}{2\sigma^2} \left( r^H r - e^{-j\theta}a^H P_\tau^T D_\Omega^H r - e^{j\theta}r^H D_\Omega P_\tau a + a^H P_\tau^T P_\tau a \right).$$

The first step is to compute the three first partial derivatives of the likelihood function. The first partial derivative with respect to carrier phase is

$$\frac{\partial \Lambda}{\partial \theta} = -\frac{1}{2\sigma^2} \left( je^{-j\theta}a^H P_\tau^T D_\Omega^H r - je^{j\theta}r^H D_\Omega P_\tau a \right)$$

(A.8)

$$= -\frac{1}{\sigma^2} \text{Re} \left\{ je^{-j\theta} (D_\Omega P_\tau a)^H r \right\}.$$

The first step in obtaining the first partial derivative with respect to carrier frequency is to find the first partial derivative of the matrix $D_\Omega$ with respect to carrier frequency.

$$\frac{\partial}{\partial \Omega} D_n = \frac{\partial}{\partial \Omega} e^{j\Omega n} = jne^{j\Omega n} = jnD_n,$$

$$\frac{\partial}{\partial \Omega} D_\Omega = jCD_\Omega = jD_\Omega C,$$

$$\frac{\partial}{\partial \Omega} D_\Omega^H = -jCD_\Omega^H = -jD_\Omega^H C.$$

(A.9)

The first partial derivative with respect to carrier frequency is

$$\frac{\partial \Lambda}{\partial \Omega} = -\frac{1}{2\sigma^2} \left( je^{-j\theta}a^H P_\tau^T D_\Omega^H Cr - je^{j\theta}r^H CD_\Omega P_\tau a \right)$$

(A.10)

$$= -\frac{1}{\sigma^2} \text{Re} \left\{ je^{-j\theta} (CD_\Omega P_\tau a)^H r \right\}.$$
The first partial derivative with respect to symbol timing is
\[
\frac{\partial \Lambda}{\partial \tau} = -\frac{1}{2\sigma^2} \left( e^{-j\theta} a^H P^T_{\tau} D^H_{\Omega} r - e^{j\theta} r^H D_{\Omega} \dot{P}_{\tau} a + a^H P^T_{\tau} \dot{P}_{\tau} a + a^H \dot{P}_{\tau}^T P_{\tau} a \right) 
\]

\[
= -\frac{1}{\sigma^2} \text{Re} \left\{ e^{-j\theta} \left( D_{\Omega} \dot{P}_{\tau} a \right)^H r + a^H P^T_{\tau} \dot{P}_{\tau} a \right\}. 
\]

(A.11)

To summarize, the three first derivatives are
\[
\frac{\partial \Lambda}{\partial \theta} = -\frac{1}{\sigma^2} \text{Re} \left\{ j e^{-j\theta} (D_{\Omega} P_{\tau} a)^H r \right\}, 
\]

\[
\frac{\partial \Lambda}{\partial \Omega} = -\frac{1}{\sigma^2} \text{Re} \left\{ j e^{-j\theta} (CD_{\Omega} P_{\tau} a)^H r \right\}, 
\]

\[
\frac{\partial \Lambda}{\partial \tau} = -\frac{1}{\sigma^2} \text{Re} \left\{ -e^{-j\theta} \left( D_{\Omega} \dot{P}_{\tau} a \right)^H r + a^H P^T_{\tau} \dot{P}_{\tau} a \right\}. 
\]

(A.12)

where
\[
\dot{P}_{ij} = \frac{\partial}{\partial \tau} \text{p}((i - jN - NL - \tau/T)T), 
\]

(A.13)

and
\[
C = \begin{bmatrix}
-NL \\
-NL + 1 \\
\vdots \\
N(L_p - 1) + NL
\end{bmatrix}. 
\]

(A.14)

Continuing, we need to compute six partial second derivatives since the 3 \times 3 Fisher information matrix is symmetric.

The partial second derivatives with carrier phase are
\[
\frac{\partial^2 \Lambda}{\partial \theta^2} = -\frac{1}{2\sigma^2} \left( e^{-j\theta} a^H P^T_{\tau} D^H_{\Omega} r + e^{j\theta} r^H D_{\Omega} P_{\tau} a \right) 
\]

\[
= -\frac{1}{\sigma^2} \text{Re} \left\{ e^{-j\theta} (D_{\Omega} P_{\tau} a)^H r \right\}, 
\]

(A.15)

\[
\frac{\partial^2 \Lambda}{\partial \theta \partial \Omega} = \frac{\partial^2 \Lambda}{\partial \Omega \partial \theta} = -\frac{1}{2\sigma^2} \left( e^{-j\theta} a^H P^T_{\tau} D^H_{\Omega} C r + e^{j\theta} r^H C D_{\Omega} P_{\tau} a \right) 
\]

\[
= -\frac{1}{\sigma^2} \text{Re} \left\{ e^{-j\theta} (CD_{\Omega} P_{\tau} a)^H r \right\}, 
\]

(A.16)
\[ \frac{\partial^2 \Lambda}{\partial \theta \partial \tau} = \frac{\partial^2 \Lambda}{\partial \tau \partial \theta} = -\frac{1}{2\sigma^2} \left( je^{-j\theta} a^H P_T^T D_H^H r - je^{j\theta} r^H D_\Omega \dot{P}_r a \right) \]
\[ = -\frac{1}{\sigma^2} \text{Re} \left\{ \left( D_\Omega \dot{P}_r a \right)^H \right\} . \]  
(A.17)

The partial second derivatives with carrier frequency are

\[ \frac{\partial^2 \Lambda}{\partial \Omega^2} = -\frac{1}{2\sigma^2} \left( e^{-j\theta} a^H P_T^T D_H^H C C r + e^{j\theta} r^H C C D_\Omega P_\tau a \right) \]
\[ = -\frac{1}{\sigma^2} \text{Re} \left\{ e^{-j\theta} (C C D_\Omega P_\tau a)^H r \right\} \]  
(A.18)

and

\[ \frac{\partial^2 \Lambda}{\partial \Omega \partial \tau} = \frac{\partial^2 \Lambda}{\partial \tau \partial \Omega} = -\frac{1}{2\sigma^2} \left( je^{-j\theta} a^H P_T^T D_H^H C r - je^{j\theta} r^H C C D_\Omega \dot{P}_r a \right) \]
\[ = -\frac{1}{\sigma^2} \text{Re} \left\{ j e^{-j\theta} \left( C C D_\Omega \dot{P}_r a \right)^H r \right\} . \]  
(A.19)

The last partial second derivative is with respect to symbol timing:

\[ \frac{\partial^2 \Lambda}{\partial \tau^2} = -\frac{1}{2\sigma^2} \left( e^{-j\theta} a^H P_T^T D_H^H r - e^{j\theta} r^H D_\Omega \dot{P}_r a + a^H P_T^T \dot{P}_r a + 2a^H \dot{P}_T^T \ddot{P}_r a + a^H \dddot{P}_T^T \ddot{P}_r a \right) \]
\[ = -\frac{1}{\sigma^2} \text{Re} \left\{ -e^{-j\theta} \left( D_\Omega \dot{P}_r a \right)^H r + a^H P_T^T \dot{P}_r a + a^H \dot{P}_T^T \ddot{P}_r a \right\} . \]  
(A.20)
To summarize, the nine partial second derivatives are

\[
\begin{align*}
\frac{\partial^2 \Lambda}{\partial \theta^2} &= -\frac{1}{\sigma^2} \text{Re} \left\{ e^{-j \theta} (D_\Omega P_\tau a)^H r \right\} \\
\frac{\partial^2 \Lambda}{\partial \theta \partial \Omega} &= -\frac{1}{\sigma^2} \text{Re} \left\{ e^{-j \theta} (C D_\Omega P_\tau a)^H r \right\} \\
\frac{\partial^2 \Lambda}{\partial \theta \partial \tau} &= -\frac{1}{\sigma^2} \text{Re} \left\{ j e^{-j \theta} (D_\Omega P_\tau a)^H r \right\} \\
\frac{\partial^2 \Lambda}{\partial \Omega^2} &= -\frac{1}{\sigma^2} \text{Re} \left\{ e^{-j \theta} (C D_\Omega P_\tau a)^H r \right\} \\
\frac{\partial^2 \Lambda}{\partial \Omega \partial \tau} &= -\frac{1}{\sigma^2} \text{Re} \left\{ j e^{-j \theta} (C D_\Omega P_\tau a)^H r \right\} \\
\frac{\partial^2 \Lambda}{\partial \tau^2} &= -\frac{1}{\sigma^2} \text{Re} \left\{ e^{-j \theta} (D_\Omega \ddot{P}_\tau a)^H r + a^H P_\tau^T \ddot{P}_\tau a + a^H \dddot{P}_\tau \dot{P}_\tau a \right\}.
\end{align*}
\]

There are two different definitions for the Fisher information matrix $J$. The first is more general and it is the expected value of the outer product of the vector of first derivatives. The second is only valid if the log-likelihood function is twice differentiable which we have just shown is the case for our estimation problem. In this formulation, the entries of the Fisher information matrix are the negative expected values of the second derivatives of the log-likelihood function.

In each of the second derivatives listed in (A.21), the only remaining random element is the vector of received samples $r$ which is defined in (A.1). The expectation operator slides through each of the entries and operates on $r$.

\[
E \{ r \} = E \{ e^{j \theta} D_\Omega P_\tau a + w \} = e^{j \theta} D_\Omega P_\tau a
\]

because $w$ is zero-mean Gaussian noise.
\[
J = \begin{bmatrix}
J_{\theta,\theta} & J_{\theta,\Omega} & J_{\theta,\tau} \\
J_{\Omega,\theta} & J_{\Omega,\Omega} & J_{\Omega,\tau} \\
J_{\tau,\theta} & J_{\tau,\Omega} & J_{\tau,\tau}
\end{bmatrix}.
\]

(A.23)

\[
\begin{align*}
J_{\theta,\theta} &= -E \left\{ \frac{\partial^2 \Lambda}{\partial \theta^2} \right\} = \frac{1}{\sigma^2} a^H \hat{P}_\tau^T \hat{P}_\tau a \\
J_{\theta,\Omega} &= -E \left\{ \frac{\partial^2 \Lambda}{\partial \theta \partial \Omega} \right\} = \frac{1}{\sigma^2} a^H \hat{P}_\tau^T \hat{C} \hat{P}_\tau a \\
J_{\theta,\tau} &= -E \left\{ \frac{\partial^2 \Lambda}{\partial \theta \partial \tau} \right\} = \frac{1}{\sigma^2} \text{Re} \left\{ ja^H \hat{P}_\tau^T \hat{P}_\tau a \right\} \\
J_{\Omega,\theta} &= -E \left\{ \frac{\partial^2 \Lambda}{\partial \Omega \partial \theta} \right\} = \frac{1}{\sigma^2} a^H \hat{P}_\tau^T \hat{C} \hat{P}_\tau a \\
J_{\Omega,\Omega} &= -E \left\{ \frac{\partial^2 \Lambda}{\partial \Omega^2} \right\} = \frac{1}{\sigma^2} a^H \hat{P}_\tau^T \hat{C} \hat{C} \hat{P}_\tau a \\
J_{\Omega,\tau} &= -E \left\{ \frac{\partial^2 \Lambda}{\partial \Omega \partial \tau} \right\} = \frac{1}{\sigma^2} \text{Re} \left\{ ja^H \hat{P}_\tau^T \hat{C} \hat{P}_\tau a \right\} \\
J_{\tau,\theta} &= -E \left\{ \frac{\partial^2 \Lambda}{\partial \tau \partial \theta} \right\} = \frac{1}{\sigma^2} \text{Re} \left\{ ja^H \hat{P}_\tau^T \hat{P}_\tau a \right\} \\
J_{\tau,\Omega} &= -E \left\{ \frac{\partial^2 \Lambda}{\partial \tau \partial \Omega} \right\} = \frac{1}{\sigma^2} \text{Re} \left\{ ja^H \hat{P}_\tau^T \hat{C} \hat{P}_\tau a \right\} \\
J_{\tau,\tau} &= -E \left\{ \frac{\partial^2 \Lambda}{\partial \tau^2} \right\} = \frac{1}{\sigma^2} a^H \hat{P}_\tau^T \hat{P}_\tau a.
\end{align*}
\]