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Internal Wave Generation Over Rough, Sloped Topography: An Experimental Study

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Internal Wave Generation Over Rough, Sloped Topography:

An Experimental Study

Lauren E. Eberly

A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

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ABSTRACT

Internal Wave Generation Over Rough, Sloped Topography: An Experimental Study

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Master of Science

Internal waves exist everywhere in stratified fluids - fluids whose density changes with depth. The two largest bodies of stratified fluid are the atmosphere and ocean. Internal waves are generated from a variety of mechanisms. One common mechanism is wind forcing over repeated sinusoidal topography, like a series of hills. When modeling these waves, linear theory has been employed due to its ease and low computational cost. However, recent research has shown that non-linear effects, such as boundary layer separation, may have a dramatic impact on wave generation. This research has consisted of experimentation on sloped, sinusoidal hills. As of yet, no experimental research has been done to characterize internal wave generation when repeated sinusoidal hills lie on a sloped surface such as a continental slope or a foothill. In order to perform this experiment, a laboratory was built which employed the synthetic schlieren method of wave visualization. Measurements were taken to find wind speed, boundary layer thickness, and density perturbation. From these data, an analysis was performed on wave propagation angle, wave amplitude, and pressure drag. The result of the analysis shows that when wind blows across a series of sloped sinusoidal hills, fluid becomes trapped in the troughs of the hills resulting in a lower apparent forcing amplitude. The generated waves contain less energy than linear predictions. Additionally, the sloped hills produce waves which propagate at an angle away from the viewer. A necessary correction, which shifts from the reference frame of the observer to the reference plane of the waves is described. When this correction is applied, it is shown that linear theory may only be applied for low Froude numbers. At high Froude numbers, the effect of the boundary layer is great enough that the wave characteristics deviate significantly from linear theory predictions. The analyzed data agrees well with previous studies which show a similar deviation from linear theory.

Keywords: internal waves, stratified flow, synthetic schlieren, topography, continental slope
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\( \alpha \)  The hills’ slope
\( P \)  Light intensity of current pixel at time \( t \)
\( P_{0,0} \)  Light intensity of current pixel at \( t_0 \)
\( P_{0,1} \)  Light intensity of pixel above \( P_{0,0} \) at \( t_0 \)
\( P_{0,-1} \)  Light intensity of pixel below \( P_{0,0} \) at \( t_0 \)
CHAPTER 1. INTRODUCTION

1.1 Introduction

With growing global concern over environmental issues such as pollution, air quality, and climate change, the need to understand the mechanics behind oceanic and atmospheric flow is more urgent than ever. The study of environmental flow over topography is extremely important to meteorology, physical oceanography, and environmental engineering as geographic forcing is a main contributor to oceanic and atmospheric dynamics. While much research has been devoted to understanding physical transport in ocean and atmospheric currents, understanding energy transport is crucial as well. One of the most significant mechanisms for transporting energy is through internal wave propagation. These waves are found ubiquitously in the ocean and atmosphere [1].

Internal gravity waves, though similar to more commonly understood interfacial waves, have some important distinctions. Both types of waves are driven by gravity - in fact, internal waves are often referred to as internal gravity waves - and exhibit similar breaking characteristics but internal waves differ in being three dimensional and by transporting energy 90 degrees from phase motion [2]. They are generated by a variety of forcing mechanisms however this thesis will focus mainly on forcing due to flow over topographic features. Figure 1.1 is a satellite photo of an internal wave. Wind is forced up and over an island mountain and clouds are formed in the troughs of the internal waves.

1.2 Internal Wave General Description

Internal waves are only found in density stratified fluid - a fluid whose density changes with depth. When the density increases linearly with height, the fluid is said to be linearly and stably stratified. In both the ocean and atmosphere, density changes are very small [3].
When a disturbance is made to the fluid, there is a possibility of internal wave generation. To understand this generation, it’s important to introduce a term known as the buoyancy frequency. This is the frequency of oscillation of the fluid when no forcing is present (imagine vertically displacing some of the fluid and letting it fall back to its original location). The resultant frequency of oscillation is known as $N$. When forcing occurs below this harmonic frequency, wave generation and propagation will occur. In general, internal waves will propagate away from the source of forcing.

Internal waves can be described by some well known properties however, due to their three-dimensional nature, they may be hard to visualize. A stratified fluid can be described as containing isopycnals, or lines of constant density. The isopycnals increase in density with depth. An internal wave can be described as the forced disturbance of an isopycnal from its resting position and the resultant oscillation as it moves back toward its stable position [2]. In a fluid with constant stratification, the density displacement will occur along lines radiating outward from the source of forcing. These are called phase lines and can be described as an area of positive or negative density displacement [4]. The wavelength of an internal wave can be described as the distance between two positive phase lines (here positive refers to a positive density change or high density moving upward to a location of low density) and the strength of the displacement along a phase line describes the amplitude of the wave. These waves have both a group and phase speed. The first being the speed of the full group of waves and the second being the speed of any individual wave crest. One very interesting difference between internal waves and surface waves is that the
phases of the internal wave do not propagate in the same direction, rather, the phases propagate perpendicularly to the wave propagation [2].

1.3 Motivation

Environmental flow, as the name suggests, refers to any type of naturally occurring fluid flow. This includes the dynamics of the ocean, atmosphere, rivers, glaciers, and even magma flows due to volcanic eruptions. Of particular interest to this thesis are flows in the ocean and atmosphere. Oceanic and atmospheric flows occur on a variety of scales ranging from meters to thousands of kilometers [3].

Energy transport is responsible for circulation in the atmosphere and ocean, wind, currents, mixing, and cloud formation among other atmospheric and oceanic dynamics. It is important to note the distinction between advection of fluid, as in a wind, and energy transport due to wave motion. In the latter case, although particles can rotate through significant distances, very little net particle translation occurs - in fact, the average motion over a characteristic period is null - though large amounts of energy can be transported through wave motion. This type of motion is of great importance to climatologists in describing the magnitude of weather events, the aerospace industry where efficiency in flight patterns is of interest, and even the power industry where knowing the best locations for energy extraction through use of wind mills, wave farms, or under-water generators could mean the difference between alternative and traditional power sources.

If the amplitude of a wave remains fixed but the wavelength becomes shorter (possibly through a background wind or viscous interactions with the earth’s surface boundary layer among other explanations) the wave steepens [5]. A wave can only remain at a certain level of steepness before becoming unstable at which point the wave overturns and breaks. The full details of internal wave breaking are not yet well known but it is certain that internal wave breaking is responsible for much of the energy diffusion in the atmosphere and ocean [6]. Turbulence from wave breaking also has biological impacts ranging from mixing plankton and other micro organisms to affecting avian flight patterns [7]. Mixing caused by internal wave breaking is of crucial importance for understanding pollution dispersion. Clear-air turbulence experienced in airplanes is a source for both customer discomfort and mechanical wear. Wave breaking causes turbulence in the local area but can have far-reaching effects as turbulence can itself generate internal waves as well [7].
Any disturbance to a stratified fluid can generate an internal wave but one of the most common environmental sources of generation is flow over topography [8]. Significant progress has been made in understanding internal wave generation due to topographic flow. Experiments have been successful in estimating internal wave generation during flow over a single hill and series of hills [9] [10]. Experiments have also been executed which simulate flow over smooth continental slopes which show that internal waves are generated during this process as well. Computer simulations of stratified flow over both idealized and realistic topography support linear theory [11]. Despite this progress, no experimental studies have been accomplished where the topography is oriented at an angle to both the isopycnals - lines of constant density- and the flow. Angled topographic obstacles are more accurate at describing flow over oceanic continental shelves where the shelves are considered 'rough' [12]. Here, roughness refers to the presence of regular, idealized topography.

Laboratory investigation is critical as linear theory is not completely accurate in describing the generation of internal waves in realistic situations. The disparity between linear theory and physical observation is greatest when the wave steepness reaches a critical level or when boundary layer separation occurs [13]. Previous experimental work on bottom topography suggests that linear theory over-predicts the amplitude of generated lee waves as it does not account for effects due to boundary layer separation [10]. Experimental trials are also desirable, in some ways, to observational data as field data is extremely costly. Also, since the earth is very large and its topography complex, as well as flow characteristics varying considerably depending on time of day, month, year, century, etc., experimental analysis provides the scientist the ability to control otherwise uncontrollable variables (wind speed, density gradient, topographic features, etc.) [14].

Thorpe (1992) set forth a linear theory solution for flow over regular, sloped, sinusoidal topography [15]. Since linear theory does not encompass boundary layer effects or breaking, there are obvious limitations to the application of the solution. Aguilar (2005) performed a series of experiments set to measure the effects of boundary layer growth and separation on internal wave generation and propagation due to stratified flow over regular non-sloped sinusoidal topography [10]. She found that boundary layer separation diminishes the amplitude of generated internal waves. A more realistic set up would be to study sloped sinusoidal topography similar to the shapes of many continental slopes.
1.4 Objectives

The objectives of this thesis research are as follows:

- Design a laboratory in which to study internal waves. This laboratory is intended to be a permanent fixture at Brigham Young University and will serve as the setting for future research endeavors.

- Characterize internal wave generation over rough sloped topography. This characterization should include measurements of frequency, wavenumber, amplitude, propagation angle, and pressure drag. Additionally, attempts to characterize non-linear effects like boundary layer separation and recirculation will be made.

- Compare experimental data with previously performed studies. It is the hope that a relationship is seen between boundary layer separation and decrease in wave amplitude as well as a relationship between the slope of the hills and the angle of wave propagation.

Chapter 2 will provide an overview of geophysical flow research including various wave generation mechanisms. A discussion of topography will show the importance of studying the geometry chosen for this research. Emphasis will be given on the necessity of studying non-linear dynamics. Chapter 3 will discuss the methods by which data were taken. To build the laboratory, careful attention will be paid to the current standards used by scientists currently studying internal wave generation. An appropriate visualization technique will be chosen and analysis will be performed to produce a development of the density displacement field over time. Both qualitative and quantitative results will be gained shown in chapters 4 and 5 respectively. Qualitative results will include dye line analysis of flow near the forcing mechanism as well as a diagnostic analysis of wave propagation angle. Once it has been shown that boundary layer separation is a likely occurrence at high flow speeds, a comparison to linear theory using wave amplitude, wave frequency, and drag will be made. The effect of the slope will be analyzed in detail in Chapter 5. It will be made clear that waves are generated in a frequency regime where linear theory would not predict wave generation. A likely cause for this phenomenon will be discussed in detail in Chapter 5.
CHAPTER 2. LITERATURE REVIEW

2.1 Internal Wave Theory

Internal gravity waves are found ubiquitously in stably stratified fluids; that is to say internal waves are found in fluids which have some density gradient over altitude or depth [16]. The stratification can be caused by compressibility effects, temperature gradients, or solvent concentration and is defined as $\frac{\partial \rho}{\partial z}$.

Internal waves are generated by interruptions in the flow field. When a volume of fluid of density $\rho_0$ is forced upward from its initial position to a location where the surrounding fluid is of density $\rho_{\text{min}}$, gravity pulls the mass back downward. As the volume falls, it gains momentum and continues past its initial location. When the volume comes to rest, it finds itself surrounded by fluid of density $\rho_{\text{max}}$. Buoyancy effects push the volume upward. Again the volume gains momentum and travels past its initial position - this time back upward. Figure 2.1 shows a free body diagram on a parcel of fluid. The fluid is being pulled downward by gravity and upward by buoyancy. The darker fluid represents higher density. Depending on the fluid parcel’s location in the fluid, one force, either the buoyant force or the force of gravity, will prevail over the other and cause an acceleration on the parcel. Due to this motion, an oscillation is induced [1]. If unforced, this frequency of oscillation is $N$, the natural frequency of the fluid, and is defined as,

$$N^2 = -\frac{g}{\rho_0} \frac{d\rho}{dz}$$  \hspace{1cm} (2.1)

Internal waves are generated by perturbing a stratified fluid at a frequency less than the buoyancy frequency and are described, in part, by the generated oscillation.

The natural frequency is derived in the following way using the free-body diagram shown in Figure 2.1. If the volume of fluid (the circle with density $\rho_0$ in Figure 2.1) is defined as $\mathcal{V}$ at an elevation of $z$ in a body of fluid with density profile $\rho(z)$, the weight of the fluid may be defined
as $\rho(z)g\Psi$. The volume is displaced by a distance $h$ such that the surrounding fluid has density $\rho(z+h)$. The buoyant force on the volume can be defined as the weight of an equally sized volume at the new location: $\rho(z+h)g\Psi$. The net force acting on the volume of fluid is $[\rho(z+h) - \rho(z)]g\Psi$.

By equating the sum of the forces to the mass of the fluid times its acceleration, we get,

$$
\rho(z)\Psi \frac{d^2h}{dt^2} = [\rho(z+h) - \rho(z)]g
$$

(2.2)

The Boussinesq approximation assumes that the density differences are very small and essentially negligible unless they are multiplied by gravitational acceleration. Therefore the $\rho(z)$ term on the left hand side may be replaced by an average or reference density, $\rho_0$. A Taylor series expansion is applied to the right-hand side and the equation is divided by $\Psi$.

$$
\frac{d^2h}{dt^2} = g \frac{d\rho}{\rho_0 \frac{dz}{h}}
$$

(2.3)

The buoyancy, or natural, frequency can now be defined as $N$ - the frequency of fluid oscillation when no forcing is present as described in Equation 2.1. Another way to visualize this would be the frequency of settling after a disturbance has been made to the fluid.
The value of the buoyancy frequency, also the natural frequency of the system, can be used to predict the generation and propagation of waves. For positive values of \( N \), the fluid is said to be stably stratified; that is an oscillating fluid volume will continue to oscillate until either viscosity or another force dampens or interrupts the oscillation. For negative values of \( N \), the oscillation amplitude will increase exponentially and fluid mixing will inevitably occur. This fluid is top heavy and not stably stratified.

A comparison of a forced oscillation to the natural frequency can also be used to predict wave generation. When the forcing frequency is less than or equal to the natural frequency, waves will be generated and will propagate away from the forcing location. When the forcing frequency exceeds the natural frequency, waves may initially be generated but do not propagate vertically. These are said to be evanescent waves [2]. For the purposes of this study, evanescent waves are not considered.

### 2.2 Internal Wave Generation Mechanisms

Internal waves can be generated from any disturbance to a stably stratified fluid. There are a few common environmental generators, namely gravity intrusions, plumes and thermals, winds over topography, winds over surface roughness, shear between atmospheric layers, and turbulence.

#### 2.2.1 Gravity Intrusions

Gravity intrusions occur when a volume of fluid flows horizontally into a volume of fluid of greater or lesser density than the first fluid. This commonly happens in the atmosphere when cold air falls rapidly downward and then spreads horizontally or in the ocean where fresh river water flows into saline ocean water and spreads laterally (or radially depending on the surrounding geography) along the upper surface of the ocean [17].

Figure 2.2 shows a mass of dense fluid at A which falls in a less dense fluid. The dense fluid spreads out laterally along the floor of the container displacing less dense fluid above. Waves are generated at the front of the intrusion and the red lines represent phase lines.

Gravity intrusions can also occur along the interface of two different density fluids or in a stably stratified fluid. In the later case, the intrusion flows horizontally at the depth where the
2.2.2 Plumes and Thermals

The atmosphere is made up of discrete layers. The stratosphere is so named because of the constant stratification normally found there. Interactions between the layers of the atmosphere can cause enough disturbance to generate an internal wave [19]. Namely, a plume which rises through the troposphere, but whose upward motion is dampened by the sharp increase in stratification of
the stratosphere, can generate waves along the tropopause (the interface between the troposphere
and stratosphere). These plumes can either be man made - like smoke rising from a stack - or nat-
urally occurring - like an updraft of warm air [20]. Figure 2.4 shows a schematic of atmospheric
density to altitude on the right and a buoyant plume on the left. As an aside, atmospheric sci-
entists almost always prefer to show density changes in the atmosphere by presenting a temperature
profile, like the one shown in Figure 2.4. This is done for two reasons. First, since static pressure
decreases linearly with altitude, the temperature profile has been found to be proportional to the
density profile and, second, temperature is far easier to measure directly than density. Thus, since
temperature and density are directly related and the changes in density gradient with height are
cased by changes in temperature, not pressure, a temperature profile is sufficient to show areas of
density gradient change in the atmosphere.

When the plume hits the tropopause (the height of which is shown by the diagram on the
right of the figure), the plume rises slightly into the stratosphere before spreading radially along the
tropopause as a gravity current. Internal waves are generated at the nose of the gravity current [21].
As the plume or thermal rises through the troposphere, it gains mass and increases slightly in
density due to entrainment of background fluid. When the plume hits the tropopause, it has enough
momentum to deflect the tropopause but not enough to push through into the stratosphere due to
an increase in stratification.

The plume gets pushed downward and oscillates around its area of neutral buoyancy. This
oscillatory motion can generate circular waves which spread radially. As the plume spreads along
the tropopause, it becomes a gravity current and more waves are generated at the nose of the
current.

2.2.3 Mixed Region Collapse

When a volume of stratified fluid becomes thoroughly mixed, the volume will collapse and
spread laterally along its elevation of neutral buoyancy. This mixing can naturally occur due to up
and down drafts in the atmosphere, plumes or thermals, air or ocean currents, or breaking internal
waves [22]. Waves are generated in two stages of the collapse. First, during the collapse phase,
waves are generated due to the displacement of the mixed fluid. As the fluid spreads laterally,
essentially a gravity current, waves are generated at the nose of the current [23].
Figure 2.4: A plume rises through the troposphere as shown by the figure on the left. The figure on the right describes the temperature changes of the troposphere and atmosphere at the same height scale as the figure on the left. Waves are shown by the straight lines radiating away from the gravity current.

Figure 2.5 shows the two stages in which waves are generated. State A represents a column of well mixed fluid which lies in a stratified body. The arrows show the direction of fluid motion. As the region collapses, fluid is displaced above and below the region and internal waves are generated (long red lines). The mixed region spreads along its level of neutral buoyancy as a gravity current. Waves are also generated in this phase at the nose of the gravity current as represented by the short red lines.

2.2.4 Turbulence

While the above wave-generation mechanisms are given to the reader for the sake of offering a thorough background on internal waves, one generation mechanism that is highly relevant to the research which will be presented is turbulence. It was mentioned previously that waves can overturn and break thus causing turbulence. If we look at the opposite scenario, we find that turbulence itself can generate internal waves. A common situation is a turbulent, mixed layer laying above a deep region (a region whose depth is much greater than the thickness of the turbulent layer) of stably stratified fluid. If turbulent eddies are present, the chaotic motion of the eddies will alter
Figure 2.5: The mass of mixed fluid, state A, falls in toward its level of neutral buoyancy. As it falls, it spreads and becomes a radially propagating gravity current. Waves are generated due to the initial collapse (long red lines) and due to the gravity current (short red lines).

The depth of the mixed layer (though the average depth of the mixed layer will remain constant over time). Since the interface between the mixed layer and the stratified layer is being warped by turbulence, the distortion causes isopycnal perturbations and, consequently, wave generation and propagation. Interestingly, it has been found that when turbulence produces internal waves, the waves tend to have the same propagation angle and amplitude independent of the towing speed of the forcing topography [24].

The study done by Munroe and Sutherland [24] produced turbulence where the largest eddies measured about 6cm. A series of four rectangular hills was towed through a mixed region between speeds of $U = 0.8 - 5.0cm/s$. PIV analysis in the mixed region was used to show the existence of turbulent eddies in the wake of the four rectangular hills. This study showed that there was no relationship between tow speed and wave properties (however, the tow speeds in each case were sufficiently fast for turbulent eddies to be present). A weak correlation between turbulent length scale and wave frequency exists since the wavelength roughly sets the diameter of the recirculating region [10], [25]. From these studies, it has been concluded that when turbulent eddies are present in a mixed region, here particularly due to a forcing function like a series of rectangular hills, wave propagation angle and amplitude will always fall within a certain range and will be independent of flow speed and turbulent length scale. A table showing propagation angles
and amplitudes for turbulence-generated waves is given in Table 2.1. The four studies show that waves propagate at angles $35 < \theta < 60$ and at amplitudes of $0.003 < A_\zeta/\lambda_x < 0.04$ where $A_\zeta$ represents the displacement amplitude of the waves and $\lambda_x$ is the horizontal forcing wavelength (i.e. the wavelength of the hills).

Table 2.1: Propagation angles and wave amplitudes are given for turbulence-generated waves based on four different studies.

<table>
<thead>
<tr>
<th>Generation Mechanism</th>
<th>$\theta$ (°)</th>
<th>$A_\zeta/\lambda_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linden et al [26]</td>
<td>35</td>
<td>N/A</td>
</tr>
<tr>
<td>Dohan and Sutherland [27]</td>
<td>42-55</td>
<td>0.02-0.04</td>
</tr>
<tr>
<td>Sutherland and Linden [28]</td>
<td>46-60</td>
<td>0.003-0.03</td>
</tr>
<tr>
<td>Munroe and Sutherland [24]</td>
<td>39-59</td>
<td>0.015-0.020</td>
</tr>
</tbody>
</table>

2.3 Topography

Whether in the ocean or atmosphere, all stratified flow interacts with topography. These interactions often lead to internal wave generation. As was discussed previously, a displacement of an isopycnal (a line of constant density) must occur to generate an internal wave. In the case of stratified flow over topography, the up or down-slope motion of the fluid acts as the forcing which generates waves. This forcing can be large as in the case of mountains, shallow, as in the case with hills or slopes, rough and chaotic, like forest-covered cliffs and gullies, or periodic, like a range of mountains or hills.

Of particular interest are the terms roughness, orography, and slope. Roughness refers to either the geology or flora which appears on the surface of larger scale geometries. Unfortunately, ”roughness” is still a rather ambiguous term in the field of environmental flow and can be used either to describe when small-scale geology appears on large-scale geography or to discuss boundary-layer thickness due to friction on the surface. In other words, roughness can either refer to surface conditions many orders of magnitude smaller than the overall geometry or geography that is less than an order of magnitude - or even on the scale of - the overall geometry. When
the roughness is close to the scale of the overall geometry, the roughness itself can cause significant pressure differences such that that boundary layer separation occurs. For the purposes of this thesis, roughness will strictly refer to the latter case.

Orography here refers to a large geographic geometry that is on the order of tens to hundreds of meters [2]. This is a general term which includes smooth geometries like hills, sharp geometries like cliffs and mountains, as well as slopes. Slopes, like the continental slope, are simply smooth, inclined planes as shown in Figure 2.6 (A).

Slopes, like the continental slope, have a scale of kilometers to thousands of kilometers. The types of slopes that will be discussed here have two important characteristics: they are shallow (4° – 20°) and are very long in the lateral direction [29]. Figure 2.6 shows a typical example of an orographic slope (A) - such as one might find in the ocean - as well as some sinusoidally approximated topographic hills (B). The reader will note that the hills are much smaller in scale than the slope. The geometry shown in (C), a slope with hills, will be the primary focus of this thesis. A more realistic example of environmental inclines includes small-scale topographic roughness.
2.3.1 Rough Slopes

The most prominent slopes are continental slopes which occur in the ocean around the edges of the tectonic plates. These slopes are typically found in the deep ocean where the density stratification of the ocean is both linear and continuous. Continental slopes in the ocean are cut with valleys and crags and covered with hills, mountains, and various other geological structures [30]. Thus realistic continental slopes are considered to have rough surfaces. The surface of particular interest to this thesis is repeating hills which models slopes which are covered with a series of hills. Figure 2.6 (C) above shows the case where a slope is covered by sinusoidal topography.

There are two flow possibilities in dealing with rough slopes. One is the up and down-slope direction which is caused by barotropic tide motion. In this case, the tide pushes the fluid up the slope and the flow is parallel to the crests of the hills. The other possibility is the along-slope direction which is caused by current motion. In this case the fluid flows up and down the sinusoidal hills. When looking at a slope covered in sinusoidal hills, it is far more interesting to look at the current flow scenario because the isopycnals are being displaced in an oscillatory fashion as they flow over the hill train. For the purposes of this thesis, the along-slope scenario will be looked at exclusively.

When the isopycnal closest to the hills is deflected, the isopycnals above are deflected in turn. In this sense, the hills become the forcing function for constant wave generation. A disturbance near the topographic forcing (either an isolated mountain as shown in Figure 2.7 or a series of hills) will propagate upward. This cascading of disturbances makes the wave. In Figure 2.7 waves are generated for four different flow speeds - $U$ increases from $U_1$ to $U_4$. The isopycnals are observed to have wave-like structures - these represent internal waves. The red circle indicates an area where an isopycnal has folded over on itself creating a density instability. When this occurs, waves can overturn and break causing mixing.

In the study of stratified flow over topography, it is useful to introduce parameters which relate the hill geometry to the stratification. Let us first introduce a useful parameter known as the reduced gravity:

\[ g' = g \frac{\Delta \rho}{\rho_0} \]  

(2.4)
Figure 2.7: Wave generation and propagation due to stratified flow over an isolated hill. Lines represent lines of constant density (isopycnals). The top left figure to the bottom right figure show internal wave generation for four different background velocities ($U_4 > U_1$) [2]. The red circle is added to isolate a location of wave instability.

where $\rho_0$ is a reference density and the change in $\rho$ is the density change over the depth of interest.

If $U_0$ is taken as the free-stream flow speed, $N$ is the buoyancy frequency, $\lambda$ is the characteristic hill wave length, and $k = 2\pi/\lambda$, then the Froude number for the flow is:

$$Fr = \frac{|U_0k|}{N}.$$  \hspace{1cm} (2.5)

The Froude number relates the inertial forces to the buoyancy forces. If $Fr > 1$ the waves are said to be evanescent and do not propagate as the perturbations are forced too quickly for the fluid to respond. Thus it is expected that waves should only propagate for forcing frequencies of $Fr < 1$.

It is also useful to introduce a parameter which relates the hill’s height scale to the density scale. The Long number is defined as follows:

$$Lo = \frac{NH}{U_0}.$$  \hspace{1cm} (2.6)

where $H$ is the height of the hills by measuring the trough to crest distance. The Long number is significant in describing flow separation. When $N$ is large, the weight and inertia of the fluid effectively damps out movement up and down the hill’s slope and traps fluid in the troughs of the hills. The fluid instead sees a flat surface, rather than a sinusoidal surface. When $Lo$ is sufficiently large due to large $H$, boundary layer effects become prominent [10].
When $Lo$ exceeds a value of 0.5, boundary layer separation occurs early on the lee side of the hills such that a recirculation region with diameter equal to the trough-to-crest hill height sets up. This trapped layer creates an effective hill height, $H_{eff}$ less than the actual hill height, $H$. Since the wave amplitude is directly related to the amplitude of the disturbance forcing, when waves are generated by topography, either repeated or singular, the amplitude of the generated waves is limited [10]. Through laboratory investigation, it was found that the amplitude of propagating waves is related to the topography hill height through the relationship $A_\xi = 0.35H/2$ [10] for $Fr > 1$.

Figure 2.8 shows three different flow scenarios over consecutive, sinusoidal mountains. For the low $Re$ case, boundary layer separation occurs well down slope in the lee of the hills such that a small recirculation region is present. For the high $Re$ flow, the fluid has filled in the trough to the point where the ambient fluid no longer “sees” mountains. The effective hill height decreases with increasing ambient flow speed to the point where surface geometry is completely negated.

Similar to Table 2.1, topography-generated waves can be distinguished by their propagation angle and amplitude as shown in Table 2.2.

<table>
<thead>
<tr>
<th>Study</th>
<th>$\theta(\circ)$</th>
<th>$A_\zeta/\lambda_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aguilar and Sutherland 1 [25]</td>
<td>52-58</td>
<td>0.015-0.021</td>
</tr>
<tr>
<td>Aguilar and Sutherland 2 [10]</td>
<td>45-85</td>
<td>0.01-0.035</td>
</tr>
</tbody>
</table>

2.4 Instability and Breaking

When waves reach a critical amplitude, there is a possibility for wave breaking. There does not exist a formula which can tell with certainty when and where a wave will break or if the wave will dissipate instead [31]. Rather, statistical models have been produced which approximate regimes where waves become unstable and regimes where the instability is great enough that it can be assumed the waves will definitely overturn [24]. Linear models are generally used for predicting wave breaking and instability - critical elements of a regional weather and ocean models. If it can
Figure 2.8: As \( L_o \) increases (top to bottom panel), the apparent height of the hills decreases as the location of boundary layer separation moves up the lee of the hills for this left to right flow. Solid lines represent streamlines of the fluid and dashed lines represent isopycnals.

It can be shown that linear models are overpredicting wave breaking, drastic improvements would be made to ocean and atmospheric models which would manifest, for example, in higher accuracy in weather prediction or more detailed momentum transport models in the ocean.

The instability regime includes waves that may break as well as waves that dissipate, or are absorbed. This regime is defined as the area above the curve defined in Equation 2.7.
\[ A_{\text{unstable}} = \frac{1}{2\pi \sqrt{2}} \sin 2\theta \] (2.7)

Where the letter \( A \) represents wave amplitude (in this case, the amplitude at which instabilities occur) and \( \theta \) is the angle of propagation.

Wave instability may also be defined as fluid with greater density lying above fluid with less density and thus the waves are assumed to have reached a critical instability where wave breaking and overturning is assured [32]. This overturning regime is defined as the area above the curve in Equation 2.8.

\[ A_{\text{overturn}} = \frac{1}{2\pi} \cot \theta \] (2.8)

If the non-linear effects from boundary layer separation are significant, they should prevent the waves from reaching these critical amplitudes [33]. The significance is due to recirculation which lowers the effective hill height and thus the displacement amplitude of the propagating wave. Since linear theory has been employed to predict areas of wave breaking and wave instability over topographic flow, a disparity between expectation and observation has become apparent because of boundary layer separation and recirculation. Where linear theory would predict wave breaking, waves are observed as remaining stable and at amplitudes far less than predicted. If linear theory were to hold true for all \( 0 < Fr < 1 \) then the wave amplitude should remain at the original forcing amplitude (the crest to trough height of the forcing hills). It is the expectation that the amplitudes of all experimentally generated waves in this thesis will lie well below both instability curves due to non-linear effects becoming significant at high \( Fr \) and subsequent decreases in wave amplitude. If this is shown to be the case, it could help highlight a major flaw in ocean and atmosphere models and provide greater incentives for investment in non-linear models.

2.5 Drag

Using linear theory, the pressure drag produced by hills would require integrating pressure over the geometry [34]. Using this, the theoretical drag on inviscid flow over hills can be represented in Equation 2.9 [25],

\[ \text{Drag} \]
\begin{equation}
F_{D,\text{theory}} = \frac{1}{4} \rho_0 N^2 \sin 2\theta A_{\zeta,\text{linear}}^2 \tag{2.9}
\end{equation}

where $A_{\zeta,\text{linear}}$ is half the trough to peak height of the hills and $\theta$ is the propagation angle of the waves (both of these terms will be discussed in detail in Chapter 3).

However, in the case where viscosity is present and boundary layer separation induces trapped flow in the valleys of the hills, it would be safe to assume that the measured drag would therefore decrease \cite{35}. It is first important to introduce a parameter which relates non-linearity to stratification. Recalling that the Long number is defined as $Lo = NH/U$, an empirical relationship between Long number and actual drag was first shown in Welch et al \cite{36} and is as follows:

\begin{equation}
\frac{F_{D,\text{meas}}}{F_{D,\text{theory}}} \approx \frac{R}{\pi} \left\{ \arccos(1 - 2R) - 2(1 - 2R)\sqrt{R(1 - 2R)} \right\} \tag{2.10}
\end{equation}

where

\begin{equation}
R = \begin{cases} 
1 & Lo \leq Lo_c \\
\frac{Lo_c}{Lo} & Lo > Lo_c 
\end{cases} \tag{2.11}
\end{equation}

$Lo_c$ refers to the critical Long number. Values of $Lo > Lo_c$ represent situations where boundary layer separation is significant enough that fluid is trapped in the valleys of the hills and resultant internal waves have a lower amplitude than linear theory predictions \cite{37}, \cite{38}.

### 2.6 Review

This work is aimed at investigating some of the intricacies of internal wave generation over both slopes and regular topography. Most importantly, the effects of non-linear phenomenon - especially boundary layer growth and separation - will be qualified and quantified. The hope is to be able to offer some insight into internal wave generation over naturally occurring topographic features like hills on continental slopes in the ocean or foothills in the atmosphere. Additionally, the laboratory created for this thesis has been used for another experimentation-based thesis. While other research done in the field of internal waves at Brigham Young University involves the interaction of waves with background flow, vorticity, or other waves, this research is unique in being focused on the principal mechanisms behind wave generation.
A fully linear solution of wave generation over rough, sloped topography has been worked out by Thorpe et al [15] and the effects of non-linearities on wave generation over non-sloped topography have been shown experimentally by Aguilar et al [10], [25]. However, no studies have been done on realistic wave generation (i.e. wave generation that includes non-linear variables) over rough, sloped topography. The aim of this thesis is not only to qualify and quantify waves generated over such a topography but also to provide guidance into measuring these properties in future experiments.
CHAPTER 3. METHODS

3.1 Physical Set-Up

All experiments were performed in an acrylic tank having internal dimensions 243cm long, 91cm high, and 11.5cm wide. The length and depth were chosen in order to allow significant wave propagation prior to wave reflection off the side and bottom walls. Even so, experiments had to be recorded over a relatively short time period (80-160s) in order to observe the waves before they reflect off the bottom surface of the tank. Since internal waves propagate three dimensionally, it was important to restrict wave motion to two-dimensional propagation as much as possible since the chosen visualization method depends on density variations in the $y-z$ plane. In Chapter 2, wave numbers in three dimensions were introduced - $k, l, m$. While it is inarguable that internal waves are inherently three dimensional, restricting motion to the $x-z$ plane is an acceptable simplification to make. This is in accordance with previous studies [10], [25] where a two-dimensional plane was chosen for analyzation. Thus the width was chosen to be small so that $x-y$-plane density variation was negligible.

3.1.1 Double Bucket Method

Creating stratification in a laboratory setting requires use of the double bucket method [39]. This method utilizes two buckets, A and B. Both buckets are of equal volume at about 30 gallons. Both buckets have a 2” diameter hole on the bottom center surface which allows them to be connected together through piping. Along the pipe which connects the two buckets is a single ball valve.

Initially the ball valve is open and both buckets are filled to equal height. The valve is closed and salt is poured into bucket A. The amount of salt and water used varies depending on the desired buoyancy frequency but for the majority of experiments done for this thesis, 80 gallons of
water was used per bucket (160 gallons total) and 30 lbs of salt was added to bucket A. A pump is turned on allowing water to be pulled from and recirculated into bucket A allowing for mixing. This process takes approximately 4-6 hours to allow all of the salt to dissolve. Once bucket A’s salt has dissolved, the mixing process is stopped and the buckets sit for 12-24 hours to allow any air dissolved in the water to escape. If this is not done, bubbles form on the interior walls of the tank. A schematic of the double-bucket method is shown in Figure 3.1.

The mass of the salt is measured and an equal mass of fresh water is added to bucket B. This ensures that the static pressure on either side of the valve is equal such that when the valve is opened, no flow between the buckets occurs. At this point, bucket A now contains salt water and bucket B contains an equal volume of fresh water.

Water is pulled from bucket A and flows into the tank where the experiments will take place. As water is pulled from bucket A, fresh water from bucket B flows into A to maintain equilibrium. Since the saline solution in bucket A is being diluted due to the fresh water intake from bucket B, the water being pumped out of bucket A decreases in salinity over time. Fluid is pumped from the buckets into the tank containing the experiments via irrigation nozzles sitting in Styrofoam boxes. Figure 3.2 shows how a Styrofoam box remains on the upper surface of the tank. At initial times, the water is very saline. If the tank is filled slowly enough to prevent mixing
Figure 3.2: At time $t=1$, the Styrofoam box lies on the surface of the water and the drip irrigation nozzle expels a high-salt-content solution. As the water level rises, the Styrofoam box remains on the surface of the water and expels decreasing amounts of salt until at $t=\text{final}$, a near fresh-water concentrate is expelled at the upper surface.

Figure 3.3: Pink Styrofoam boxes with sponge bottoms were placed in the tank. A tube ending in a drip irrigation nozzle is placed inside the box. The Styrofoam ensures the box will remain floating on the surface of the water through the duration of the experiment and the nozzle prevents salt water from flowing into the tank too quickly.

along the upper surface, a stratified volume will be achieved. Since the stratification is stable, the fluid will remain stratified for up to five days before diffusion becomes significant.

The salt water solution flows into the tank through a drip irrigation nozzle which lies in a floating sponge and Styrofoam box in the tank. The sponge and Styrofoam box ensure that the drip irrigation nozzle deposits new fluid at the top surface of the water in the tank. Figure 3.3 shows a picture of one of the Styrofoam boxes used in the experimental set up along with the drip irrigation nozzle which rests inside it. The drip irrigation nozzle and sponge are necessary to ensure that the water flows into the tank at a slow flow rate. Preliminary designs which allowed the water to flow
to quickly into the tank resulted in mixed regions rather than a constant stratification. Due to the slow rate of fill, the process to fill takes approximately 10-12 hours.

### 3.1.2 Stratification

Often a perfectly linear stratification was not achieved immediately following filling. It was found that allowing an additional 24-48 hours to allow for some diffusion to take place would effectively blend any discontinuous bands of differing density.

2mL samples of the water were taken every 3cm of depth and then analyzed using an Anton Paar DMA 4100 M density meter. Fitting a line of best fit to the measured density vs. depth directly gives the buoyancy frequency of the background density field. Though the buoyancy frequency was slightly different for each trial, most experiments were done with an $N$ value between 1 and 1.5 $s^{-1}$.

To confirm that stratification is constant at all locations in the tank, columns of density data were taken using the procedure described above at eight different locations shown in Figure 3.4. Figure 3.5 shows the results by plotting the measured density verses the depth for the various locations. The standard of deviation for these data is $SD = 0.011525$ which is sufficiently small to conclude that the density changes only in the $z$ direction.

In some experiments, a line of potassium permanganate was added along the top layer to serve for visualizing the boundary layer. For these experiments, the stratification process was not altered save for a small amount of dye which was injected into the dripper nozzle during the final ten minutes of filling.
Three model hill sets were used each having a hill amplitude of $H = 1.30\,cm$. An issue arises when attempting to model stratification in the ocean. While vertical density changes in the atmosphere are generally large, the buoyancy frequency in the deep ocean can be far less than unity. If such a stratification were achieved in a laboratory setting, the density perturbations would not be sufficiently large to be detected by the synthetic schlieren system. So rather than match buoyancy frequency, Froude numbers realistic to what is found in both the atmosphere and ocean, were chosen. The majority of experiments were conducted with Froude numbers in the range $0.2 < Fr < 1.5$ though some trials were performed outside this range. These Froude numbers
were chosen to match Froude numbers used in similarly conducted experiments as well as Froude numbers commonly found in geophysical flows [10] [25].

This height scale was chosen based on former experiments by Aguilar et al [10], [25]. Each set of hills had three different continental slopes: 0°, 10°, and 20° as is commonly found in the ocean [2]. The crest-to-crest wavelength of the hills was 13.7cm and each model hill block contained four hills. Four repeating hills were chosen such that the hills would span the field of view and fully developed topography-generated waves would only be seen. If only a single hill were chosen, for example, waves created from the wake at the front and back of the hill would be seen in the field of view. Using many repeated hills allows for a time period where only topography-generated waves are seen since the front and back wake are out of view. Perturbations to the density field were obtained by towing the hills upside-down across the surface of the water as shown in Figure 3.6.

A counterweight was attached to the aft end of the hill set to maintain tension in the line. Another weight was placed on the top of the hills to ensure that the proper angle was achieved. Without the weight the 10° and 20° degree sloped sets would float at an irregular angle. A motor attached to a board which was then placed on top of the tank wound the line around a spool such that when the motor was turned on, the hills were pulled across the surface of the water.

Despite the tank’s length, only a small portion of the flow field was recorded. This was to ensure the wave-field had reached steady flow. The model hills were attached to a pulley system. Towing speeds were determined according to the stratification strength and the desired angle of propagation.

### 3.2 Synthetic Schlieren

All results were measured using the synthetic schlieren method. A Photron FASTCAM-APX RS model 250K digital camera was placed approximately 1.3m in front of the tank (though often this distance was adjusted) such that the viewing area was approximately 2ftx2ft. In order to avoid as much paralax error as possible, the camera was placed far from the tank and a telephoto lens was used to zoom to the appropriate viewing window.

The synthetic schlieren method is a way to non-intrusively measure a density perturbation field [4]. A screen of printed horizontal lines is placed behind the tank. A change in density through
Figure 3.7: A single light ray passes through a fluid with a propagating internal wave. At time $t_1$ the light ray passes through a different index of refraction than that at $t_2$. The light ray color corresponds to the time as indicated above.

A point will result in a change in index of refraction at that point [40]. Thus a changing density field will bend light at different angles through time resulting in a distortion of the background image [41], [42], [43]. A simple analysis can be done on the background image by tracking the apparent displacement of the lines on the screen [44].

Figure 3.7 shows how when a wave propagates through a tank of stratified fluid, the index of refraction will change over time. As a light ray fluctuates due to index of refraction changes, the ray will hit different parts of the camera lens resulting in an apparent motion of the object on which the lens is focused (in this case, the line screen).

### 3.2.1 Light Ray Tracing

All analysis is based on the assumption that light will bend toward an area of larger index of refraction [4]. Fermat’s variational principle describes the behavior of light in an inhomogeneous medium in the following way

$$\partial \int n(x, y, z) ds = 0$$

Where $n(x, y, z)$ describes the refractive index field and $s$ is oriented along the light ray. We shall define the coordinate field such that $x$ is along the length of the tank, $y$ is across the width, and $z$ is the vertical depth.
Only light rays with components in the y direction are of importance due to both the two-dimensionality of the experiment and the location of the camera. Thus, the light ray paths can be described by \( x = \xi y \) and \( z = \zeta y \). Combining this restriction to Fermat’s variational principle gives:

\[
\frac{d^2 \xi}{dy^2} = \left[ 1 + \left( \frac{d\xi}{dy} \right)^2 + \left( \frac{d\zeta}{dy} \right)^2 \right] \frac{1}{n} \frac{\partial n}{\partial x} \tag{3.2}
\]

\[
\frac{d^2 \zeta}{dy^2} = \left[ 1 + \left( \frac{d\xi}{dy} \right)^2 + \left( \frac{d\zeta}{dy} \right)^2 \right] \frac{1}{n} \frac{\partial n}{\partial z} \tag{3.3}
\]

Use of the synthetic schlieren technique requires only light rays parallel to the y direction. Thus the terms \((d\zeta/dy)^2\) and \((d\xi/dy)^2\) may be neglected. Solving for the light-ray paths gives:

\[
\xi = \xi_i + y \tan \phi_\xi - \frac{1}{2} y^2 \frac{1}{n} \frac{\partial n}{\partial x} \tag{3.4}
\]

\[
\zeta = \zeta_i + y \tan \phi_\zeta - \frac{1}{2} y^2 \frac{1}{n} \frac{\partial n}{\partial z} \tag{3.5}
\]

Where \( \zeta_i \) and \( \xi_i \) describe the incident light-ray location and \( \tan \phi_\xi = d\xi/dy(y = y_0) \) and \( \tan \phi_\zeta = d\zeta/dy(y = y_0) \) describe the horizontal and vertical components, relative to the y direction, at which light rays enter the tank.

Knowing these light-ray paths is essential for calculating the apparent shift of the pattern on a screen placed at a distance B behind a tank of width W. Also, the refractive index field, \( n(x,y,z) \), can be described as \( n = n_0 + n_{\text{base}} + n' \) where \( n_0 \) is the reference index of refraction for the medium (as related to the reference density used in buoyancy frequency calculations), \( n_{\text{base}} \) is a spatial variation in background refractive index associated with stratification, and \( n' \) is the variation due to flow. Both \( n_{\text{base}} \) and \( n' \) are small in comparison to \( n_0 \). Back-tracing the rays received by the camera, the apparent shift in light-ray origin \((\Delta \xi, \Delta \zeta)\) is given by

\[
\Delta \xi = \frac{1}{2} W (W + 2B) \frac{1}{n_0} \frac{\partial n'}{\partial x} \tag{3.6}
\]

\[
\Delta \zeta = \frac{1}{2} W (W + 2B) \frac{1}{n_0} \frac{\partial n'}{\partial z} \tag{3.7}
\]
Figure 3.8: Experimental setup and visualization method. Three scenarios are shown. As the change in index of refraction increases, the rays bend at increasingly dramatic angles when they pass into the tank fluid.

Since there is a linear relationship between the index of refraction of salt water and its density, the density perturbation may be written as:

$$\nabla n = \frac{dn}{d\rho} \nabla \rho = \beta \frac{n_0}{\rho_0} \nabla \rho$$  \hspace{1cm} (3.8)

Where

$$\beta = \frac{\rho_0}{n_0} \frac{dn}{d\rho} = \text{cst}$$  \hspace{1cm} (3.9)

3.2.2 Displacement Field

The analysis shown in the previous section allows us to solve for the apparent displacement of the light ray in terms of density perturbation as:

$$\Delta \xi = \frac{1}{2} W (W + 2B) \frac{\beta}{\rho_0} \frac{\partial \rho'}{\partial z}$$  \hspace{1cm} (3.10)

The next step in the analysis is to record $\Delta \xi$ and $\Delta \zeta$ for the field of interest. This is done using a screen of lines placed some distance behind the tank so as to allow a more dramatic angle through which the light rays bend. As the density field changes, the lines appear to shift a distance as shown in Equations 3.6 and 3.7. As seen from Figure 3.8, when a ray passes through an area of greater index of refraction perturbation, as in ray C, the ray will bend more dramatically as it...
passes into the fluid [45]. Using a high-definition digital camera and frame grabber, we were able to acquire a series of still frames. A software package, Digiflow, was used for analysis.

3.2.3 Digiflow

A program written specifically for synthetic schlieren analysis called Digiflow was used to calculate $\Delta \xi$ and $\Delta \zeta$ [46]. However, some alterations had to be made to the program. Since traditional PIV relies on brightly illuminated particles against a black background and our setup used black dots on a white background, all images had to be inverted. Similar to traditional PIV, the code outputs velocity fields. By multiplying the velocity fields by the time step, $\Delta \xi$ and $\Delta \zeta$ were found. Rearranging Equations 3.6 and 3.7 gives the density gradient for every point $(x, z)$.

\[
\begin{align*}
\frac{\partial \rho'}{\partial z} &= \frac{2\Delta \xi \rho_0 / \beta}{W(W + 2B)} \quad (3.11) \\
\frac{\partial \rho'}{\partial x} &= \frac{2\Delta \zeta \rho_0 / \beta}{W(W + 2B)} \quad (3.12)
\end{align*}
\]

When Digiflow reads in an image, it stores the pixel intensity value for every pixel in the image. Since dark pixels represent a dark schlieren line, measuring how the dark pixels move can translate to apparent motion of the screen. Equation 3.13 shows how $\Delta \zeta$ can be found by comparing pixel intensity values.

\[
\Delta \zeta = \left[ \frac{(P - P_{0,0})(P - P_{0,1})}{(P_{0,1} - P_{0,0})(P_{0,1} - P_{0,-1})} - \frac{(P - P_{0,0})(P - P_{0,1})}{(P_{0,-1} - P_{0,0})(P_{0,-1} - P_{0,1})} \right] \Delta z \quad (3.13)
\]

Where $P$ is the light intensity at a current pixel at time $t$, $P_{0,0}$ is the light intensity at the current pixel at $t_0$, $P_{0,1}$ is the light intensity at the pixel above $P_{0,0}$ at $t_0$, and $P_{0,-1}$ is the pixel intensity at the pixel below $P_{0,0}$ at $t_0$. Using Equation 3.13 Digiflow can calculate the $N^2$ field by using Equation 3.14.

\[
\Delta N^2 = \frac{-2\Delta \zeta g}{\beta W(W + 2n_{\text{water}} / n_{\text{wall}}T)} \left( \frac{L - (1 - n_{\text{air}} / n_{\text{water}})W - 2(1 - n_{\text{air}} / n_{\text{wall}}T)}{L - B - (1 - n_{\text{air}} / 2n_{\text{water}})W - 2(1 - n_{\text{air}} / 2n_{\text{wall}}T)} \right) \quad (3.14)
\]
3.3 MatLab Analysis

The Digiflow analysis provides two essential types of data for extracting the information of interest: a series of .jpg images showing the evolution of the wavefield over time and accompanying datafiles providing the $N^2$ displacement for every pixel on the .jpg image [46]. In other words, this image contain pixels whose intensity value represents the amount of buoyancy frequency change as compared to the stratified volume at rest when no waves are present.

The images (usually around 4,500 per data set) are then imported into Matlab. When an image is imported into Matlab, each pixel is assigned an intensity value on the 256 gray scale. This poses a bit of a problem as each pixel must instead correspond to an $\Delta N^2$ value to be useful in calculations. So while the image files were useful to the user for quickly isolating areas where internal lee waves were present, the data files were crucial for translating the images to a map of density displacement. Thus is is important to map the Digiflow-produced $\Delta N^2$ field. A series of data files containing the $\Delta N^2$ value at ever pixel were produced by Digiflow in addition to a series of images. Every twentieth image is called in Matlab and compared to its original data set. The minimum and maximum values in the data set are compared to the minimum and maximum pixel intensity value (1-256) such that the colormap of the images corresponds to a $\Delta N^2$ value. The end result yields a Matlab image file where each pixel represents a $\Delta N^2$ value.

Images are chosen from the data sets where lee waves - waves generated in the downwind section of the hills - are apparent. These ‘apparent’ waves had phase lines that connected to the crests and troughs of the forcing hills, were of an expected wave length and of an expected frequency. These images undergo a fast Fourier transform to acquire the energy spectrum of the waves as well as the horizontal and vertical wave numbers.

Since the lines on the screen are horizontal, it is not expected that there be high resolution in the horizontal direction. Thus finding the frequency to calculate the horizontal wavenumber is necessary. In order to do this, a time series is created. Each frame from the experiment is read in series and a single column of data is copied into a file such that the end result is an image of time vs depth with pixels representing $\Delta N^2_t$ (the change in buoyancy frequency over time). Performing a Fourier transform on this image gives the energy spectrum of frequencies. The process for finding wavelengths and frequency by using a Fourier transform will be discussed in further detail in Sections 5.2 and 5.3.
Figure 3.9: A line drawn from A to B is reflected to the horizontal plane. A point normal to the plane ABC is named D. The plane ABD describes the plane in which waves will be found. The plane enclosed by the red line, tilted forward by $\alpha$, represents a subset of ABD.

### 3.3.1 Amplitude

A second analysis is performed where a single column of data taken at the same x-location is compiled into a time series image. That is, at the end of this analysis, an image of time vs depth is produced with contours of $\Delta N^2/\Delta t$. This is known as the $N_t$ field or the field which represents the change in buoyancy frequency over time. The minimum and maximum pixel-intensity values correspond to the minimum and maximum amplitudes of the time-series waves. Again, by using the pixel-to-$\Delta N^2$ correlation described in the above step, a value for $A_{N_t}$ is obtained.

Using Equation 3.15, the vertical displacement amplitude, $A_{\zeta}$ is found.

$$A_{N_t} = 2\pi N^3 \sin \theta \frac{A_{\zeta}}{\lambda_x}$$  \hspace{1cm} (3.15)

In this case, $\theta$ refers to the propagation angle of the waves which can be found by the following equation:

$$\theta = \arctan \frac{m}{k}$$  \hspace{1cm} (3.16)
3.4 Slope Theory

An idealized sinusoidal topography has a slope of angle $\alpha$ and is covered with a sinusoidal set of hills. The flow is along the slope as shown in Figure 3.9. It will be shown that when water flows over this hills, there are two possible phase planes along which the waves can travel [15]. That is to say phases will travel across the planes parallel to the crest lines of the hills and the wave group will travel perpendicular to the hill crest lines away from the hills. These planes describe the planes in which waves will be present if the hill set was infinite. However, the hills used in the experiment (similar to real hills) are finite in width. A subset is drawn show in red on Figure 3.9 where the plane of interest is tilted forward by the angle of incline, $\alpha$. This is also drawn such that the energy and group speed, $c_g$ propagates normally away from the topographic surface.

A line is drawn along the crest of one of the hills with end points A and B. This line is projected onto the horizontal plane creating line AC. If the phase plane is extended through the topography, it intersects the slope at point D. The plane defined by ABC contains the constant phases of the wave train. The group speed, $c_g$, lies in the plane ABD, normal to line AD and away from the slope. Since we assume away-from-slope propagation of the waves, a positive $z$-component is assigned to the vector $c_g$ [15].

To bring back a concept introduced earlier, we will discuss the dispersion relation as it relates to waves generated by this specific topography. In a reference frame where $X$ and $Y$ lie on the horizontal plane and $Z$ is oriented vertically upward, we denote the $X$, $Y$, and $Z$ wavenumbers as $K$, $L$, and $M$ respectively. Note, the reason for marking these as capital rather than the more commonly seen lower-case letters is to differentiate the two different orientations. The dispersion relation becomes

$$\omega^2 = \frac{N^2(K^2 + L^2)}{K^2 + L^2 + M^2}$$  \hspace{1cm} (3.17)

It may seem a daunting task to re-solve the wave equations using this rather complicated geography as a bottom boundary condition. Thankfully, there is an elegant and simple solution. We keep the existing wave equations and simply change the reference frame. For the new geographic-oriented reference frame, $x$ lies normal to plane ABD from Figure 3.9, $z$ lies in ABD, perpendicular to line AB, and $y$ remains unchanged [15].
In the rotated orientation, the wave numbers are defined as $k, l, \text{ and } m$. To translate from the slope-oriented system to the surface-oriented system, the following relationships are used:

\[ K = k \cos(\alpha) - m \sin \alpha \]  
\[ L = l \]  
\[ M = k \sin \alpha + m \cos \alpha \]

### 3.5 Reflections in Internal Waves

When an internal wave encounters an interface (be it solid or fluid) a reflection of the wave will occur [47]. However, unlike light wave reflection, internal waves always reflect with symmetry about the vertical (normal to the density gradient) without regard to the geometry encountered [48].
Figure 3.11 shows two examples of internal wave reflections. The container shown in (A) contains non-vertical walls however the waves reflect with respect to the gradient, not to the angle of the container walls. All the experiments done in this thesis were done in a rectangular container as shown in (B).

When a wave is generated, its angle to the vertical can be described as follows:

$$\theta = \cos^{-1} \frac{\omega}{N}. \quad (3.21)$$

If we assume the slope off of which the waves are reflecting has a slope of $\alpha$, reflections can be broken down into two scenarios:

$$|\alpha| < \frac{\pi}{2} - |\theta| \quad (3.22)$$

$$|\alpha| > \frac{\pi}{2} + |\theta|. \quad (3.23)$$

In the former case, waves propagate upward to fluid of less density. In the latter case, waves reflect downward. When waves reflect off a vertical surface, i.e. $\alpha = \pi/2$, reflected waves will propagate downward into the medium [49]. It is important to note that theory does not predict a change in either $\theta$ or $\omega$ due to reflection [50].
3.6 Wave Angles

When measuring the wave angle, it is important to measure the wave angle with respect to the resultant wave plane ABD as shown in Figure 3.9, not with respect to the viewer. In the case of no-slope hills, the waves propagate solely in the x-z plane. There is no change in propagation over the y direction. However, when the forcing hills are placed at an angle, the resultant wave plane also shifts. Here the waves propagate at an angle into the face of the tank as shown in Figure 3.12. Finding the wave angle of interest is relatively simple.

Since it is known that, in the y-z plane, the waves will propagate away from the slope at 90°, using simple trigonometry gives the true length of the wave ray relative to depth. When the data images are considered, a triangle can be drawn where the legs are the depth of the tank and the x-distance traveled and the top angle can be measured directly from the image. Often times the wave reflected off the front and back of the tank and dispersed before reaching the bottom of the tank so, in these cases, a line was drawn extending the initial wave prior to reflection. The line of reflection off the front or back of the tank was able to be seen and is described in further detail in Chapter 4. Since the bottom foot is the same dimension either relative to the viewer or
relative to the plane of propagation, once this measurement is found, and using the distance the
wave travels from the set of hills to the face of the tank, the true angle $\theta$ can be found. Due to
the skewed viewpoint of the observer, even high-speed waves can appear to have a shallow angle
of propagation. According to linear theory, it is expected that the phase planes propagate away
from the topographic slope at an angle normal to the slope [15]. Prior to this work, the effects of
non-linear phenomenon like boundary layer separation on this phase plane angle were unknown.
These effects are discussed in detail in Chapters 4 and 5.

3.7 Error Analysis

Several sources of error were found for the this method though the error was determined to
be small enough to maintain confidence in the accuracy of the results.

When taking density samples to measure the stratification strength in the tank, effort was
made to take a 2mL sample every 2 to 4cm of depth. However, the sample-taking syringe pulled
fluid from both above and below the sample depth. If an approximation is made that sample mass
lies entirely in a sphere of volume 2mL centered at the sample site, then there will be an uncertainty
in the sample depth of $\pm 0.0014m$. Additionally, the ruler used to measure the sample depth had an
accuracy of 1mm.

Anton Paar reports an accuracy of $\pm 0.000005 \, g/cm^3$ for the model density meter used in
sample collection. The line of best fit which was fitted to the depth vs. density data had an $R^2$
value of 0.98 minimum or the tank was drained and the experiment was scrapped.

Digiflow required that the following distances be measured: screen to tank, tank inner-
thickness, tank wall thickness, tank to camera. All of these measurements were made with a ruler
which had an accuracy of $\pm 1mm$. The smallest measurement taken was more than an order of
magnitude greater than the ruler’s resolution and the majority of the measurements were three
orders of magnitude larger. In order to mitigate computational error, Digiflow underwent three
interpolative passes to remove extraneous vectors and fill in missing pixels. Digiflow used an
interrogation window of 30pixels. Missing values were filled by fitting a gaussian curve over a
7x7 pixel area. Outliers whose value was 50% greater than its surrounding values were fixed using
a gaussian interpolation over an area of 11x11 pixels. Each pixel in the final image represented
approximately 1/1745m, though this value changed slightly depending on where the camera was
situated. Each line was approximately 4 pixels thick or about 2mm - the thickness of the actual lines.

By applying a propagation of error analysis to Digiflow’s process of finding $\Delta \zeta$ and $\Delta N^2$, the total uncertainty in processing can be found

$$u_R = \pm \left[ \sum_{i=1}^{V} (\theta_i u_{x_i})^2 \right]^{1/2}.$$  \hspace{1cm} (3.24)

Applying the uncertainty equation, Equation 3.24, to Equation 3.14 according to the uncertainty of each term given in Table 3.1, the uncertainty for finding the $N^2$ was found to be 31.7%. While this value is high, it does not take into account various error reducing measurements like multiple passes to remove spurious vectors and reduction of window size per pass. The creator of the program estimated the error to be closer to 5%.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g$</td>
<td>$9.8 m/s^2$</td>
<td>N/A</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.184</td>
<td>N/A</td>
</tr>
<tr>
<td>$L$</td>
<td>Varies per experiment (usually around 3.5m)</td>
<td>$\pm 0.001$</td>
</tr>
<tr>
<td>$W$</td>
<td>0.018m</td>
<td>$\pm 0.001m$</td>
</tr>
<tr>
<td>$T$</td>
<td>0.12m</td>
<td>$\pm 0.001m$</td>
</tr>
<tr>
<td>$B$</td>
<td>0.75m</td>
<td>$\pm 0.001m$</td>
</tr>
<tr>
<td>$n_{air}$</td>
<td>1.0002</td>
<td>N/A</td>
</tr>
<tr>
<td>$n_{water}$</td>
<td>1.3332</td>
<td>N/A</td>
</tr>
<tr>
<td>$n_{acrylic}$</td>
<td>1.4914</td>
<td>N/A</td>
</tr>
<tr>
<td>$P$</td>
<td>27</td>
<td>$\pm 0.5$</td>
</tr>
<tr>
<td>$P_{0.0}$</td>
<td>27</td>
<td>$\pm 0.5$</td>
</tr>
<tr>
<td>$P_{0.1}$</td>
<td>26</td>
<td>$\pm 0.5$</td>
</tr>
<tr>
<td>$P_{0.2}$</td>
<td>25</td>
<td>$\pm 0.5$</td>
</tr>
</tbody>
</table>

When scaling the 256grayscale matlab images to represent a density displacement field, one image in every twenty was chosen to check the scaling limits as discussed earlier. Additionally, when finding the vertical wave number, between 10-20 images containing lee waves were analyzed. The final results were averaged. When finding the frequency, every image in the data set was
used with a time step of 1/60s. When the observed propagation angle was compared against the calculated propagation angle (not theoretical but calculated from the vertical and horizontal wave numbers), there was a maximum error of 5%.

3.8 Review

Model, sloped, sinusoidal hills will be towed through a stably stratified fluid in a controlled environment. The wave motion will be visualized using the synthetic schiren method. The experiment will be filmed in front of a screen of horizontal lines - wave motion will distort the lines such that they appear to move. A program, Digiflow, will be used to translate the apparent line motion to density displacement. Using a single frame, which shows a density perturbation field at a single point in time, the wavenumbers can be found through use of a Fourier transform. By creating a time series, an image which shows the propagation of phases, a Fourier transform can be done to find the wave frequency. Since waves will propagate away from the inclined hills, the waves are predicted to propagate into the walls and the propagation angles of the hills will appear to be shallower than the actual propagation angles. By knowing the location of wave reflection off the front or back of the tank, some simple trigonometry can be used to find the actual propagation angles.
CHAPTER 4. QUALITATIVE RESULTS

4.1 Synthetic Schlieren Results

The synthetic schlieren method was found to be highly useful in visualizing and measuring generated internal waves. Raw images were filmed using a high resolution camera at 60fps over a period of 20 to 79 seconds. Often, deflection of the background screen was not apparent to the naked eye. Figure 4.1 shows three raw data images for Froude numbers of 0.6, 0.8, and 1.8 respectively - all three were taken using the same hill slope of 0°.

It is interesting to note how deflection to background screen is not noticeable for the low Fr image. If the reader compares Figure 4.1 B to Figure 4.3 B they will see that while the lines may not be obviously deflected in the raw data image (Figure 4.1 B), Digiflow is able to calculate a visible $\Delta N^2$ as shown in the results image (Figure 4.3 B).

The reader will surely note that the hills are being dragged across the surface of the water rather than across the bottom like what would be seen in the ocean. As it turns out, it does not matter if the hills are towed across the top or the bottom of the tank as long as the density gradient is constant. If we recall the buoyancy frequency, this parameter depends on the gradient of the density alone - not on a z location. Thus for ease of experimentation, the hills were inverted and dragged across the surface of the water. Figures 4.1 and 4.2 show raw data images of experiments using the 0° sloped hills and the 20° sloped hills. The apparent displacement of the lines can be seen especially well in Figure 4.2. Between the two lines have been drawn on Figure 4.2 A (drawn for clarification) the reader will see periodic distortions to the background screen. When this image is compared to Figure 4.5 B the reader will see that the periodic distortions of the background screen correspond to the phases in the results image.

Raw data images like Figures 4.1 and 4.2 were analyzed using the program Digiflow. Use of the program required accurate measurements for the camera-to-tank distance, tank-to-screen distance, inner-tank dimension, tank thickness, and tank material. Often in synthetic schlieren ex-
Figure 4.1: Raw data for 0° hills at A) $Fr = 0.6$, B) $Fr = 0.8$, C) $Fr = 1.8$.

Figure 4.2: Raw data for 20° hills at A) $Fr = 0.6$, B) $Fr = 0.8$, C) $Fr = 1.8$.

Experiments, the tank thickness and material is disregarded since the minor deflection of light through the tank walls is negligible in comparison to the deflection caused by the density perturbation field. However, since this experiment used an experiment thickness (inner tank diameter) that was not significantly larger than the thickness of the walls, this measurement was crucial.

Figures 4.3, 4.4, and 4.5 show the results of using Digiflow to analyze 0°, 10°, and 20° hills at Froude numbers of 0.6, 0.8, and 1.8 respectively. These figures show wave generation as evident by the phase lines (the light and dark lines extending diagonally downward from the hills). In all 9 cases, the hills are being towed from left to right. The low $Fr$ waves propagate at a shallow angle. The mid $Fr$ waves propagate at a steeper angle into the tank. Here, it is important to note that linear theory would suggest that the waves in Figure 4.3 B, Figure 4.4 B, and Figure 4.5 B should all propagate at the same angle to the vertical. However, there seems to be a decrease in propagation angle as the incline angle of the forcing hills increases. If the angle correction
described in Chapter 3 were to be employed, it would be the expectation that the propagation angles of the 10° hills and the 20° hills would be equal.

The reader will recall that linear theory does not predict waves would be apparent when the forcing frequency is above the buoyancy frequency. However, all three hill geometries produced waves above $Fr = 1$. Using the quantitative analysis, a possible and likely explanation for wave generation above $Fr = 1$ will be discussed in Chapter 5.

Due to the noise in the original image as well as the inclusion of the hill and surrounding objects, the colorbar on the Digiflow image was scaled such that the area of interest was washed out to bands of light blue and green. While the images produced by Digiflow were not terribly useful, the data files produced by the program were. These files contained a calculated density perturbation value per pixel in the original image - no scaling was done to calculate these values.

To conclude, the synthetic schlieren method was deemed essential for this experiment. The calculations and subsequent data files containing the density perturbation field data was crucial.
4.2 Propagation angle

The first experiments performed were a diagnostic test of the accuracy of the system. A non-inclined set of hills identical to those used by Aguilar et al [10] was towed across the tank at various speeds. Figure 4.6 shows a plot of the resultant wave angle per towing speed. The waves angles show good agreement with the equation

\[ \cos \theta = \frac{\omega}{N} \] (4.1)

Where \( \theta \) is the angle of propagation with respect to the horizontal, \( \omega \) is the wave frequency, and \( N \) is the buoyancy frequency.

Since there is no dependence of the wave propagation angle on hill angle, one would expect a near perfect correlation with theory as is the case here except for frequencies approaching the buoyancy frequency (about \( N^2 = 1 \) for the majority of the experiments). The propagation angle in the experiments never fell below 37°. This may very well be in line with the hypothesis that at high forcing frequencies the apparent hill height decreases to near zero causing possible turbulence. Table 2.1 shows the range of propagation angles expected for turbulence-generated internal waves. It is important to note that the angles represented above are the corrected propagation angles as described in 3.6.

4.3 Dye Line Analysis

In order to visualize the boundary layer growth, some experiments utilized a single dye line spread 5mm from the surface of the salt water. The tank was filled normally (With an \( N^2 \) value
Figure 4.6: The measured wave propagation angle vs. the normalized towing frequency for all three geometries is shown above.

of 1) but dye was injected into the dripper nozzle for the last 10 minutes of filling leaving a line of dye which extended across the tank. The dye line was used as a visualization aid for flow very near the hills surface.

Figure 4.7 shows flow with a single dye line over sinusoidal hills. It was found that, in accordance with previous experiments on non-sloped hills, the effective hill height does indeed decrease over each hill due to trapped, stagnant or recirculating fluid in the troughs of the hills. At a subcritical $Fr = 0.5$, the effective hill height was already very small as evidenced by the dye line conforming to the shape of the hills but with a much lower amplitude. That is to say, the shape of the topography can be seen in the dye line meaning the streamlines are following the topographic shape. This is exactly what was expected for $Fr$ in the transition-to-nonlinear-theory region. At a critical $Fr = 1$, the dye line showed that flow over the hills was nearly flat - no fluid is flowing into the troughs of the hills suggest a recirculation region. The above fluid did not ”see” topography but rather a flat slope. At the supercritical $Fr = 1.43$, the dye line suggested turbulence may be present
Figure 4.7: Dye is used to visualize flow over sinusoidal hills. Three cases are shown, $Fr = 0.5$, $Fr = 1.0$ and $Fr = 1.43$. In each case, the hills lie below a black, thick line of dye (as indicated in the picture). Green lines are added (post processing) to the $Fr = 0.5$ case for clarification.

evidenced by peaks and divots in the dye line which did not correspond at all with underlying geometry as they did for the subcritical case. This results in waves which have wavenumbers and propagation angles which are not related to the underlying topography.

It is likely that at high $Fr$ numbers, the boundary layer will separate and turbulence generated waves are apparent. Consistently, when turbulence was apparent in the separated boundary layer over the hills, waves with propagation angles within the ranges shown in Table 2.1 were visible.

In the case of supercritical flow, flow for which the Froude number exceeds unity, linear theory predicts that no waves would be produced. However, waves were seen in all cases for both low and high $Fr$. Using the dye line analysis as a diagnostic, it may be safe to say that both the sloped hills and the non-sloped hills grew a turbulent boundary layer thick enough to produce turbulence-generated internal waves (such a boundary layer is seen in the top example of Figure 4.7).

4.4 Boundary Layer

Since the dye line analysis is not conclusive enough to form a firm conclusion of the existence of turbulence due to boundary layer separation, raw data images of flow above $Fr = 1$ were analyzed. Figure 4.8 shows a synthetic schlieren image of flow over sinusoidal hills. The rough
Figure 4.8: A raw data image is shown where boundary layer separation is likely occurring. Small-scale, chaotic distortions to the schlieren lines create a rough look in and above the troughs of the hills.

Appearance of the schlieren lines in the troughs and immediately above the hills (as opposed to the smooth distortions that indicate waves) is indicative of boundary-layer separation and, possibly, turbulence [10] [25]. Further analysis could be performed for the region immediately above the hills if a screen of dots were used rather than lines - this would provide for greater resolution of horizontal motion.
CHAPTER 5. QUANTITATIVE RESULTS

5.1 Slope Correction

If the reader will recall Equations 3.18, 3.19, and 3.20, there is a convenient way to translate waves seen by the observer to their sloped propagation plane. Due to the hills’ slope, all generated internal waves propagated at an angle into the front or back face of the tank rather than straight downward. Therefore, the angle of propagation as seen by the Digiflow analyzed data was quite a bit smaller than the actual angle of propagation. By following the correction detailed in Chapter 3, the actual propagation angle was found.

The theory set forth by Thorpe suggested that the main difference between waves generated from sloped topography and non-sloped topography is that sloped-topography waves propagate at an angle to the vertical [15]. Therefore, once the reference frame is corrected, it is expected that the amplitudes of the sloped-topography waves follow the same relationship to frequency as the amplitudes of non-sloped-topography waves.

Figure 5.1 shows the relationship between the observed propagation angle and the actual angle of propagation. Again, the observed angles are always less than the actual angle of propagation because the plane along which the phases are propagating is not oriented vertically downward. For every Froude number below $Fr = 1$, there are two data points - the upper point is the actual angle and the lower point is the observed angle. Figure 5.2 shows the difference between the observed and corrected propagation angles for both the $10^\circ$ and $20^\circ$ hill sets. As the Froude number increases, the difference between the observed and corrected angles decreases. The theory developed by Thorpe [15] suggests that the propagation phase planes should always propagate normally from the surface of the topography with no dependence on flow speed. However, it is seen that for increasing $Fr$ the phase planes turn downward such that the expected $10^\circ$ or $20^\circ$ angle difference (for the respective hill inclines) is not seen. Rather, the expected difference is seen at low $Fr$ and no difference is seen at high $Fr$. 
Figure 5.1: A relationship between observed wave propagation angle and corrected propagation angle for the 10° hills is shown.

Figure 5.2: The difference between the observed propagation angles and the corrected propagation angles for both the 10° and 20° hills is shown.

An analysis of the difference between the observed angles of propagation and the actual angels of propagation, as seen in Figure 5.2, shows that as the Froude number approaches unity, the waves start propagating more downward as evidenced by the line of reflection moving further down the tank wall (see Figure 5.3). This would suggest that while the waves are being generated due to topographic forcing at low Froude numbers, when boundary layer effects become dominant,
the waves are generated mostly by turbulence forcing. This is further validated by the observation that the difference between the observed propagation angle and the corrected propagation angle decreases with increasing Fr. Linear theory predicts a constant angle difference equal to the angle of incline. The decreasing difference could be explained by the influence of a secondary wave-generation mechanism like turbulent forcing.

Waves generated by the 10° hill sets propagated at an angle into the tank, meaning the observer viewed an angle of propagation that was shallower than the actual angle of propagation. When the angle was corrected, the propagation angles matched well with previous studies [26] [28] [27] [18] [10]. As the hills approached a Froude number of unity, the difference between the viewed propagation angle and the actual propagation angle decreased. This supports the hypothesis that as boundary layer effects become more prominent, the wave generation mechanism shifts from topographic forcing to turbulence forcing. Above \( Fr = 1 \), the waves propagate vertically downward and there is no difference between the apparent wave angle and the actual wave angle.

5.2 Vertical and Horizontal Wave Numbers

Digiflow-analyzed images were selected from each data set which clearly showed topographically generated waves in the lee of the hills. Since many of these images contained blank space, spurious waves, or lines caused by the waves reflecting off the front and back faces of the tank (as described in section 3.6), the images were carefully cropped so as to only contain the waves of interest.

As shown in Figure 5.3, often, after a reflection, the waves will lose coherency. Above the reflection line, coherent waves were clearly seen where below the reflection line, the waves were less coherent. In this case “reflection” refers to the internal waves reflecting off the front and back faces of the tank. When this occurred, the waves closest to the topography were kept and the rest were cropped out. The importance of cropping out spurious waves will be made more clear below. The cropped wave image is Fourier transformed in space to find the horizontal and vertical wave numbers. The cropped image and energy peaks are seen in Figures 5.4 and 5.5. The cropped image is then Fourier transformed to find the energy spectrum of the wave field. The vertical wave number can be found by the location of the peak on the vertical energy spectrum plot as shown in Figure 5.5.
Figure 5.3: A Digiflow analyzed image is given where a reflection is apparent. The reflection occurs as indicated by the red line. Below the line, less coherent, 'spurious’ waves are seen.

The reader will surely note that the resolution on the horizontal wave number is far less than the resolution on the vertical wave number. Since horizontal lines were used as the background screen, Digiflow was not able to calculate horizontal pixel displacement with high accuracy. Due to the low resolution of the horizontal wave number, a wave frequency could not be calculated with much confidence. Therefore a second analysis was performed where a time series was created.

5.3 Wave Frequency

Time series plots were made showing the $N_t$ field, or a field where each pixel value represents a change in N over time rather than over space as shown in the previous section. An example of a time series plot is given in Figure 5.6. The amplitude of this field, $A_{N_t}$ can be found directly from comparing the maximum and minimum values of the field. A Fourier transform of the
Figure 5.4: An image showing a Digiflow analyzed image after cropping out spurious waves.

Figure 5.5: The horizontal and vertical energy spectrum of a highly coherent set of waves. Only a single peak is apparent for both.

The time series plots were constructed by taking a column of data from the same x location of each frame in the data set. The resulting image, time by depth, shows contours of phase. The steeper the phase lines appear, the faster the phases propagate through the wave. Performing a Fourier transform on a time series plot produces a frequency spectrum as shown in Figure 5.7. In
Figure 5.6: The time series is shown with coordinates of time vs. depth.

Figure 5.7: The frequency power spectrum is shown. The strongest frequency is represented by the peak in the figure. $\omega$ can be found directly from this plot at the maximum power location.

cases where the resolution of the frequency was low (as in Figure 5.7), the frequency was verified using the angle of propagation.

### 5.4 Phase Speed

At this point, we can look at the expected wave frequency in comparison to the measured wave frequency. For the purposes of this thesis, from this point forward, the quantity $\omega/N$ will represent the measured normalized wave frequency and $Fr$ will represent the normalized frequency of forcing. (The reader should recall that $Fr \equiv \omega/N$ so from henceforth $Fr_m = \omega_{measured}/N$.)
Figure 5.8 shows the relationship between the intended Froude number and the measured, normalized frequency. Up to a Froude number of 0.7, the generated waves followed linear theory almost exactly. Beyond Fr = 0.7, the measured, normalized frequency remains in a band of approximately 0.45 < ω/N < 0.65. A surprising phenomenon happens above Fr = 1. Linear theory suggests that waves become evanescent (or non-propagating) as the forcing frequency approaches the natural frequency of the fluid. However, waves were consistently observed at high Fr numbers. This is strong evidence for a secondary wave-generation mechanism beside topographic forcing. In other words, the forcing of the waves above the intended Fr = 1 is most likely not caused by topography but rather by turbulence. There doesn’t seem to be a correlation between the hills’ slope and divergence from linear theory in this case.

### 5.5 Amplitude

Figure 5.9 shows the speed vs. wave amplitude. As was noted in Aguilar et al [10], there is a deviation from theory as the hill speed increases thus validating the theory that boundary layer growth decreases the apparent height of the forcing hill and, by extension, decreases the amplitude of the resultant gravity waves even when those hills are on a slope or incline. For low values of Fr
Figure 5.9: The amplitude of the generated waves normalized by the hill height is compared to the Froude number. A theoretical curve representing the relationship between the forcing frequency and expected wave frequency is shown as well.

where the boundary layer is not very thick and has certainly not separated, the measured amplitudes agree very closely with linear theory. In all three cases, the waves deviated from theory starting around $Fr > 0.7$ however no significant difference was found between the three geometries. That is to say, it can be reasonably concluded that the slant of the hills had little to no effect on where the boundary layer separated and that the shapes of the hills were effectively changed in the same way.

For supercritical values of $Fr$ - when the Froude number is greater than unity - the waves level off at approximately $0.31H/2$. This number correlates well to Aguilar who found supercritical lee waves to be $0.35H/2$ and Sutherland who found supercritical lee waves to be $0.33H/2$ [10].

5.6 Turbulence

One likely explanation for the divergence from linear theory is the presence of non-linear phenomenon such as boundary layer separation and turbulence. The dye line analysis along with raw data images of the flow between the hills at high $Fr$ suggest that boundary layer separation resulting in turbulence may play a significant role in wave generation above $Fr = 1$. Without having a method by which to measure turbulence in the troughs of the hills, a diagnostic analysis
may be done by comparing known wave properties of turbulence-generated waves to properties of waves generated in the current study’s experiments.

Figures 5.10 and 5.11 show how the amplitudes and propagation angles for the experiments performed for this thesis lie in the range of respective values for studies involving turbulence-generated waves. For the sake of clarity, the two other studies mentioned in Table 2.1 are not shown however the reader can compare the properties of the possible-turbulence-generated waves in this study to known-turbulence-generated waves in the other studies [27], [26]. While many of the experimental amplitudes fell within the range given in Munroe [24], the propagation angles fit better within the range given in Sutherland et al [28]. Since no explanation was given as to why there is such variance in the four studies [24], [28], [27], [26], other than statistical scatter (the studies with larger ranges had more data points), it is safe to say that all experiments fell within the ranges given by the four studies listed previously.

When taking this comparison in conjunction with the qualitative evidence of boundary layer separation and turbulent eddies (as given in Figures 4.7 and 4.8), one may draw the conclusion that the waves generated near and above $Fr = 1$ are likely caused by turbulent eddies forming above the hill set.
Figure 5.11: Propagation angles of generated waves are shown for their various $Fr$. The lines represent the boundaries of propagation angles of turbulence-generated waves for two studies [24] and [28].

### 5.7 Drag

It is well concluded by this point that the boundary layer separation and resultant recirculation zone in the trough significantly affects the amplitude of the generated waves for $Fr > 1$. However, it’s important to understand that the overall energy of the waves also decreases. A simplistic way of visualizing this would be to consider how for equal wavelengths and frequencies, the wave with the highest amplitude will have the most energy. Since the boundary layer decreases the wave amplitude but does not affect its frequency or wavelength, it’s safe to conclude the waves carry less momentum than linear theory would predict, and thus their generation produces less drag on the hill.

As the valleys collect stagnant or recirculating water and the apparent hill height goes to zero, the flow sees the hills as a flat block rather than a sinusoidal surface. With far less surface area over which the flow must propagate, and with an increasing turbulent boundary layer, the pressure drag on the hills decreases significantly as the Froude number approaches the critical level.

The Long number of the flow is of primary interest in this case as it is a parameter which measures the effect of stratification on boundary layer separation. As was discovered previously
Figure 5.12: Measured drag is compared against the Long number. Only sub-critical cases were considered. By means of comparison, two idealized $F_d/F_{d,\text{ theory}}$ curves, one where the critical Long number is equal to 0.5 and one where $Lo_c = 0.75$ are plotted as well.

in Aguilar et al [10], non-angled sinusoidal hills have a critical Long number between roughly $0.5 < Lo_c < 0.75$. Other similar experiments found $Lo_c$ to be closer to 0.77 [51] or 0.67 [36].

Figure 5.12 shows a relationship between Long number and drag - note that the majority of the points lie between $0.4 < Lo_c < 0.85$. As non-linear effects become greater, the drag over the hills decreases. The experimental results were compared with two commonly regarded critical Long numbers, $Lo = 0.5$ and $Lo = 0.75$. Consistent with literature, we don’t find a great correlation with the theoretical drag curves but we do see an overall trend of higher Long numbers relating to less drag [10] [25]. As the Long number becomes greater and boundary layer separation comes to the foreground as a main contributor to flow dynamics, the measured drag falls sharply away from the linear theoretical drag where neither boundary layer separation nor flow trapping occur.

5.8 Wave Stability and Generation Mechanisms

For Figure 5.9, the wave amplitudes were normalized by dividing the amplitude by $2H$. By normalizing the amplitude by the horizontal wavelength instead, an analysis of wave breaking can take place.

In Figure 5.13 we compare the normalized wave amplitude to their angles of propagation. All points here lie well under the curve ”Critical Amplitude” meaning all waves were stable and not at risk for breaking. About a third of the points fall between $37^\circ$ and $44^\circ$ and a normalized am-
Figure 5.13: Normalized amplitude as a function of wave propagation angle. The line shows the area above which waves are known to be unstable. Since all points fall under the curve, it is safe to assume wave stability in all cases.

Amplitude between 0.15 and 0.4. Dohan and Sutherland [27] showed that turbulence-generated waves would fall in this range where Aguilar [25] showed that boundary-trapped lee waves generated by periodic topography were more likely to be found for $55^\circ < \theta < 65^\circ$ and $0.01 < A_\xi/\lambda_x$. With the exception of a few outliers, the data collected in this study reinforces that sinusoidal hills, whether sloped or non-sloped, generate waves due to topographic forcing or turbulent eddies. Outliers were found in other studies similar to the ones seen in this case. The reader will note that the outliers fall in the high range for $\theta$ right before non-linear effects begin to take hold. Since the points still fall under the critical amplitude umbrella, we can safely say they show the extent to which linear theory can be pushed.
CHAPTER 6. CONCLUSIONS

It was found that when waves are generated by sloped topography, the phase planes do not propagate straight downward but rather at an angle into the front or back of the tank. This results in the observer viewing a propagation angle which is less than the actual propagation angle. However, as Froude number increases, the waves tend to propagate more vertically until after the critical $Fr=1$ there is no difference in propagation angle between the observed and actual angles. This means the waves propagate completely vertically downward after $Fr = 1$.

It was found that the difference between the observed and corrected angles of propagation did not follow linear theory for all values of $Fr$. Rather, as $Fr$ increased, the propagation planes turned downward until after a critical $Fr$, the propagation planes were oriented vertically and no correction was needed. This is further evidence to support the theory that a new wave generation mechanism is significantly affecting the flow starting above $Fr = 0.5$.

When a correction to the propagation angle is made, it is found that the propagation angle follows closely to linear theory at low $Fr$. When the $Fr$ reaches approximately 0.7, the propagation angles drop down to between 35° and 55°. Above $Fr = 1$ where linear theory predicts no propagating waves would be seen, waves were still observed.

A dye line analysis elucidates some of the unpredicted phenomena. At low $Fr$, the boundary layer separates and a recirculation region sets up in the troughs of the hills. Waves propagate at their predicted propagation angle but with a lower amplitude due to the lowered effective forcing amplitude. The underlying shape of the hills is still be seen by the fluid but the amplitude has been greatly decreased. At a critical $Fr$, the boundary layer has separated directly behind the crests of the hills and a recirculation region has filled in the hill troughs entirely. Ambient fluid sees a flat wedge. At high $Fr$, peaks not correlating with the underlying hill geometry are seen suggesting the boundary layer is turbulent. This is consistent with the observation that waves propagating
above $Fr = 1$ propagate independent of hill slope and propagate within the regime for turbulence generated waves.

The amplitudes of the waves, both those generated from sloped hills and those generated from non-sloped hills, followed the same pattern. The amplitudes matched linear theory predictions up to approximately $Fr = 0.7$. Between $Fr = 0.7$ and $Fr = 1$ the amplitudes dropped from linear theory predictions to approximately $0.33H/2$ and remained at that averaged value for high values of $Fr$.

The drag was also calculated and it was found that there was a decrease in drag as the $Fr$ increased. As fluid filled the cavities of the hills, the effective Long number decreased. Consequently, since the ambient fluid saw an effectively flatter geometry, the drag decreased. Supercritical Froude numbers were not considered for this comparison since boundary layer separation occurs, for these cases, due to adverse pressure gradients in the lee of the hills.

All waves were found to be well below the critical stability amplitude and no breaking was seen.
REFERENCES


