2011-12-09

A Conceptual Framework for Student Understanding of Logarithms

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A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Arts

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December 2011

ABSTRACT

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In the past, frameworks for what it means for students to understand elementary mathematical concepts like addition have been well-researched. These frameworks are useful for identifying what students must understand to have a good grasp of the concept. Few such research-based frameworks exist for secondary mathematical topics. The intent of this study was to create such a framework for what it means for students to understand logarithms, a topic that has been under-researched up to this point. Four task-based interviews were conducted with each of four different preservice secondary mathematics teachers in order to test a preliminary framework I had constructed to describe what it means for students to understand logarithms. The framework was adjusted according to the findings from the interviews to better reflect what it means for students to have a good understanding of logarithms. Also, a common practice taught to students learning logarithms, switching from logarithmic form to exponential form, was found to possibly have negative effects on student understanding of logarithms. The refined, research-based framework for what it means for students to understand logarithms is described in full in this document. The implications of the results of this study for mathematics teachers as well as for mathematics education researchers are also discussed.

Keywords: mathematics education, logarithms, understanding, framework, mathematics teaching

ACKNOWLEDGEMENTS

I would like to thank my school support team: Dan Siebert, for being a great advisor and believing in me, as well as for telling me to slow down my life; Keith Leatham, for knowing so much about APA and teaching me the hard way, for being on my committee, and for convincing me to apply for the Master's Program in the first place; Hope, for lending me books I don't return in a timely fashion and for being on my committee; Loraine Hanson, for being a great first "roommate" (office mate); Jon Call, for letting me bounce ideas off him and for being a great friend; Kathy Lee Garrett, for reminding me about all the deadlines I would surely have missed without her, and all the other staff, faculty, and graduate students in the Math Education department for their concern and support for me.

I would also like to thank my family support team: Chris Williams, for being a supportive husband and helping me finish what I started; Mom, for proofreading my thesis, bringing me chicken soup, and being a great mother; Dad, for passing on his "math genes" to me, for reading my thesis, and for making me feel smart when I took a hard math class; and Alif, Nikki, Justin, Ben, Amy, Jake, Elly, Jonah, Erica, Doug, and Peter for being there to play when I couldn't work anymore.

I would also like to thank my interview subjects for letting me pick their brains and learn from them, as well as for giving me four or more hours of their precious time during a busy semester.

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CHAPTER I: INTRODUCTION

Understanding is highly valued in the field of mathematics education. It has been said that "many students follow rules and execute procedures they do not understand, making it impossible for them to modify or extend their skills to fit new situations or to monitor their performance and catch errors when they occur" (Hiebert 2003). In other words, it is likely that only students who *understand* the mathematics they are using will be able to consistently check their own work for errors and extend their knowledge beyond problem types they have already seen worked out from start to finish. Another reason understanding is important is that students who have an understanding of mathematics are more likely to have a productive disposition – that is, they will probably enjoy math more and thus be more engaged in mathematics than those who do not understand what they are doing (National Research Council, 2001). Further, students who understand a mathematical concept, rather than just having memorized a procedure for it, are not so likely to forget it (Skemp, 1978).

Much research has been done exploring what children must understand about basic mathematical concepts like counting, addition, and solving one and two-step equations. For example, a significant amount of work has gone into identifying children's conceptions involving multiplication (e.g. Carpenter, Fennema, Franke, Levi, & Empson, 1999). Students ought to understand that multiplication and division problems involve a number of equal-sized groups of objects, a certain number of objects within each group, and a total number of objects. They must also be able to tell which number they must solve for in order to perform the right operation. Solving for the total number of objects is done by multiplication, while solving for the number of groups or the number of objects in each group is measurement or partitive division, respectively. There has been research done on how students think about the four basic whole number operations (adding, subtracting, multiplication, and division), and implications for teaching these operations (such as Baroody, 2003; Carpenter et al., 1999; Fuson et al., 1997). However, in higher levels of mathematics, some mathematical topics have not been as carefully researched as topics from the early grades. A few topics are exceptions to this rule. Polynomial functions, limits, derivatives, and integrals are among the secondary mathematical topics that have received significant attention in mathematics education research. One of the areas that has not been carefully researched is logarithms; little research has been done on what students do understand about logarithms, and no research has been done on what students *should* understand about logarithms.

Students generally do not have a good understanding of logarithms. Teachers who have taught mathematics classes that include logarithms in the curriculum will attest to this widespread problem. In searching for studies revolving around student understanding of logarithms, I have been unable to find any studies that have shown a group of students who demonstrate an adequate (or better) understanding of logarithms. Even students who appear proficient at completing logarithm problems while studying them often cannot do similar exercises a few weeks later (Kastberg, 2002; Weber, 2002b). Students tend to remember the rules incorrectly and use these mis-remembered rules without making sure they are correct, perhaps in part because they do not know how to check for correctness (Kastberg, 2002; Kenney, 2005). This is consistent with the opening quotation by Hiebert (2003), because he wrote that students who memorize rules and procedures without understanding cannot extend their knowledge or check for correctness.

There are at least two reasons that students should have a firm grasp on logarithms: for future science, technology, engineering, and mathematics (STEM) classes and for solving problems in the real world. Logarithms are used heavily in calculus classes and are also used in many other upper-level math and science classes. In careers, logarithms are used for the Richter scale (of earthquakes), pH, musical octaves, brightness of stars, volume in decibels, population growth, radioactive decay/carbon dating, compound interest, computer programming, and even the melting of glaciers. Therefore, it is important that students obtain a good understanding of logarithms, since many students will use them again in their future education and/or careers.

In order to reliably assess students' understanding of logarithms, we must first be able to articulate what it means to understand logarithms. It would be difficult to be able to know whether a student has a good grasp of logarithms without first having a good description of what true understanding of logarithms would look like. Further, without such a description, it would be difficult to try to correct a students' faulty understanding. In other topics of mathematics (such as arithmetic, as mentioned previously), conceptual frameworks have been constructed to help explore what concepts must be in place for a student to have a "good understanding" of a topic.

Previous to this research, a detailed framework describing what it means to understand logarithms did not exist. The purpose of my study was to create a research-based conceptual framework for logarithms. I began my research by creating a preliminary framework, and the main purpose of my study was to test and refine this framework and alter it as needed to create a good framework for defining what it means to understand logarithms (this framework is described in detail in chapter two of this document). My preliminary framework was based around thought-experiments on what it means to understand logarithms as well as on existing literature (what little exists). Because the primary purpose of my study was to test the validity of this framework and alter it as needed, I needed to test the framework on students who were already familiar with logarithms in order to avoid missing some of the understanding students might have. Therefore, I chose to test my framework by using task-based interviews with undergraduate mathematics education majors. The participants of my study were four students who formed a focus group from a class that explored the concepts behind mathematical topics in depth, including logarithms. The students were familiar with logarithms prior to the study and gained more understanding throughout the course of my study. The result of this study has led to my revised framework of what it means for a student to have a good understanding of logarithms.

The rest of my thesis is structured as follows: first, I explain the framework as it was prior to testing, and explain how it came to be. Next, I describe the existing literature related to my topic and explain how my study goes beyond what has been written previously. Then I explain the methodology of my study, including the setting and subjects, how I collected data, and how I analyzed the data. After that, I describe the results of my study, including the revised version of my framework which was adjusted throughout the process of data analysis. Finally, I explain the theoretical and pedagogical implications of my study for the fields of mathematics education and for mathematics educators.

CHAPTER II: THEORETICAL FRAMEWORK

This chapter explains the theoretical framework I developed prior to data collection. Over the course of data analysis, the framework changed many times and is thus revisited in the results section of this document. However, it is important to explain the framework I was working with as I collected data in order to make sense of my data collection and results.

In order to test student understanding of logarithms, this understanding must be described in detail. In an attempt to study the various elements of knowledge of logarithms, Berezovski and Zazkis (2006) split "understanding logarithms" into three categories: logarithms as numbers, logarithms as operations, and logarithms as functions. Reading other studies (e.g. Kastberg, 2002) led me to believe this framework was not complete because I found that students who had been studied experienced difficulties with logarithms that were not explained by the framework developed by Berezovski and Zazkis (2006). To further define what it means to understand logarithms, I modified the framework given by Berezovski and Zazkis (2006) as explained below. I also fleshed out the framework to explain more precisely what is included in each category of the framework.

My theoretical framework is based on four categories:

1. Logarithms as objects
2. Logarithms as processes
3. Logarithms as functions
4. Logarithms in contextual problems

Although I have listed these categories in a numerical order, this was done only for the sake of referencing the categories throughout this document and is not indicative of an actual order of importance or how students might develop understanding in the different categories. I believe that a student with a thorough understanding of logarithms would have fully developed the knowledge required to be proficient in each of the four categories. However, some students may have either a partial or no understanding of some (or all) of the categories, thus having a partial understanding of logarithms in general. The categories are presented here in detail. For each of the four categories, the origins of the category are explained. A list of conceptions within the category (prior to data analysis) is shown in a table. I also explain more in depth what I believe to be the important conceptions (as listed in the tables) tied to each category.

Before I describe my framework, it should be explained what the criteria were for adding a concept to my framework. The first criterion for including a concept in my framework was that a lack of understanding of the concept would lead to difficulties in solving certain types of problems involving logarithms. The other criterion for including a concept in my framework was that it would have to apply to several kinds of problems involving the use of logarithms. In other words, the concept would need to be useful for solving many different types of problems involving logarithms, rather than just a key "trick" for one specific problem or problem type. Whereas the first criterion is based on concepts that students cannot be successful without understanding, the second criterion is based on concepts people might use to their advantage when solving many types of problems. Thus, the first criterion is something of a “bare minimum” meaning that a student must understand the concept in order to succeed at problems involving logarithms, whereas the second goes beyond that to include concepts that are beyond the bare minimum but are still widely applicable.

Since I had not yet collected data when I created my initial framework, I relied on past experiences from teaching as well as examples from research to help me find instances where students' difficulties in solving a particular type of problem pointed to key concepts they did not understand. For example, having seen a student try to solve an equation by dividing both sides by "log" prompted me to include understanding logarithmic notation in my framework. To get at the second criterion, I relied mainly on thought experiments to pinpoint the concepts that I had used widely or that I had seen used widely for solving many types of problems involving logarithms.

Logarithms as Objects

The idea that students must understand logarithms as objects originated from Berezovski and Zazkis (2006). In their framework, they included the idea that students must understand that logarithms are numbers and don't need to "be finished." Some students, when solving equations, do not feel that $x = log_3 2$ is an acceptable answer because they have learned to simplify logarithms and believe that logarithms must be in the form of a whole number or fraction in order to be considered a number. These students probably do not understand that many logarithmic expressions are irrational numbers. The reason I extended this category from "logarithms as numbers" to "logarithms as objects" is that I believed there is more to the problem than students not understanding that a logarithm is a number. In this section, I describe the components I believed to be missing, and thus included in my framework.

The object definition for logarithms is as follows: $log_a b$ is *the exponent* you must raise a to in order to get b. This definition helps make the rules of logarithms more transparent, as it relates them to the rules of exponents. Consider, for example, $log_2(4 \cdot 8)$. How do you get $4 \cdot 8$ using powers of 2? Well, $4 = 2^2$ and $8 = 2^3$ so $4 \cdot 8 = 2^2 \cdot 2^3$. Since 2^2 and 2^3 have the same base, multiplying them is equivalent to adding the exponents, which gives 2^5, and so $log_2(4 \cdot 8) = log_2 2^5 = 5$. If we know that multiplying exponents with the same base allows us to simply add the exponents, and that logarithms are really exponents, it makes sense that $log_a(b \cdot c) = log_a b + log_a c$. Understanding logarithms as objects is what allows students to use the rules of logarithms meaningfully to perform operations on logarithms rather than to use the rules by rote memorization. Because I have seen many students who do not display such understanding, I realized that object understanding must be a part of a full understanding of logarithms. That is, I realized the necessity of this category through observing the absence of it. Also, students who learn the rules by rote memorization often mis-remember them. The students interviewed in Kastberg (2002) displayed this lack of understanding by mis-remembering many of the rules of logarithms and being unable to correctly reconstruct them.

Table 1 includes the summary points of what is meant by understanding logarithms as objects. The term "object" is often used in mathematics education to describe a particular way of understanding certain mathematical concepts. Sfard (1991) used the term to describe a way of viewing a mathematical structure as something which can be operated on. In writing about logarithms as objects, I mean that students can think about and operate on logarithms as objects (as in the definition given in O-def).

Table 1
Evidence of Object Understanding

Label	Description of Understanding
O-def	The student is able to think of $log_a b$ as *the exponent* you must raise a to in order to get b.
O-num	The student recognizes that a logarithmic expression such as ln2 is a number, and does not need to be approximated with a decimal.
O-rule	The student is able to flexibly change forms of a logarithmic expression using the rules of logarithms (power, sum/product, difference/quotient, change of base). They recognize these forms as equivalent.
O-not	The student must know the notational conventions of logarithms and how they relate to the order of operations (such as writing $2log_3 6$ instead of $log_3 6 \cdot 2$, because they have different meanings).
O-part	The student knows that a logarithmic expression must have both a base and an argument (in addition to "log" or "ln") in order for it to make sense and be complete (i.e. they do not try to separate the "parts" of a logarithmic expression because they know that the logarithmic expression is meaningless without all of its "parts"). This includes knowing the implied bases for "ln" and "log" with no base written.

Students with an object understanding of logarithms will understand this definition (O-def) and will be able to see the logarithm of a number as an exponent and as a number. In order to understand logarithms as numbers, students must be able to recognize that a logarithmic expression *is a number* – that is, something like $log_3 4$ is a number (although irrational) and not necessarily something to be "figured out." Students must understand why it is not necessary to convert such an expression into a decimal approximation, but they must also recognize that such an approximation does exist. They should also be able to recognize what the number means – i.e. that $log_2 8$ is the power you must raise 2 to in order to get 8, and thus recognize, for example, that this particular expression could be rewritten as simply the number 3. However, their understanding of logarithms as a number with this meaning should extend beyond simple cases into understanding cases such as $log_2 7$ is the power you must raise 2 to in order to get 7 (which is an irrational number).

Using the rules that pertain to logarithms (such as change of base, addition, etc.) is something that students ought to be able to do in order to have a complete understanding of logarithms. Being able to use these rules to operate on logarithms and to change the form of a logarithm is part of this category of understanding. An example of what sort of mathematics a student with a robust object conception of logarithm might be able to do is to recognize the equivalence of and to convert back and forth between any of the following expressions:

$$lne^5, \qquad 5lne, \qquad 5ln(e * 1), \qquad 5(lne + ln1), \qquad 5\ln\left(\frac{e^2}{e}\right), \qquad 5(lne^2 - lne), \qquad 5$$

They may describe these operations in words such as "if you are subtracting two logs you can take the log of the first divided by the second," thus indicating that they realize the equivalence of subtracting logarithmic expressions and dividing expressions within a logarithm.

Students also ought to be able to understand what the written conventions mean in terms of logarithms. For example, although a letter concatenated with a number usually means "multiply," $ln10$ does not mean $l * n * 10$, and it cannot be rewritten as $10ln$. Students should know that they can perform operations on logarithms in the same way that they can perform operations on other numbers, but that they cannot separate the "log" from the base or the argument – those are part of the object. I once saw a student attempt to divide both sides of an equation by ln, which is why I felt this notational issue was important to include in my framework.

Logarithms as Processes

Berezovski and Zazkis (2006) included the category "logarithms as operations" in their conceptual framework. However, "taking a logarithm" is not as simple as performing an operation like addition or multiplication. Therefore, I have renamed this category "logarithms as processes," because I feel it is more indicative that taking a logarithm is a somewhat complex process. I describe this process as converting the argument to an exponential expression with the same base as the base of the logarithm, and then knowing that the expression is equal to the exponent of the argument. While students may not know how to compute a decimal approximation for most logarithms (those that result in irrational numbers), they should be able to understand the basic principles of what it means to "take the logarithm" and thus be able to tell which two integers the logarithm is between. Berezovski and Zazkis (2006) noted that their category of logarithms as operations could be subdivided into two parts: operational fluency and operational meaning. In my framework, this category incorporates both parts.

Table 2 includes the summary points of what is meant by understanding logarithms as processes. Often students are told that to find the logarithm, they should convert the argument into an exponential expression with the same base as the logarithm, and to take the exponent as the answer (see example given in P-def). This is a process-based idea about logarithms. Part of the process understanding of logarithms is this idea of using a process to approximate a logarithm; a related idea is that *taking a logarithm* is just applying this process to a number (or expression).

Table 2

Evidence of Process Understanding

Label	Description of Understanding
P-def	The student is able to think of $log_a b$ as a process of taking the argument (b), converting it to the form of a^x, and writing x as the answer. Using this knowledge, they are able to simplify expressions such as $log_2 16$ by employing the process as $log_2 16 = log_2 2^4 = 4$.
P-est	The student can use their knowledge from P-def to estimate values of logarithmic expressions such as $log_3 29$ by recognizing that 29 is close to (and just higher than) 27, and since $log_3 27 = 3$, $log_3 29$ must be slightly greater than 3.
P-elim	The student can think of "taking a logarithm" as eliminating the base of an exponential expression to solve an exponential equation such as $3^x = 17$.
P-exp	The student can explain that it is necessary to "take the logarithm" when trying to find the value of an exponent in an expression.
P-div	The student relates the action of taking a logarithm to repeated division and recognizes when repeated division is an appropriate method for solving a problem (i.e. when you can divide a whole number of times to get to 1 or else when only an approximation is needed).
P-root	The student can relate the actions of taking a square (or other) root and taking a logarithm, explaining similarities and differences (i.e. that in roots you are trying to find the *base* of the exponential expression, whereas in logarithms you are trying to find the *exponent*).

When students are using P-def, they convert the argument to a number with the same base as the logarithm to obtain the exponent that is "the answer." On logarithms that have whole-number answers, this process will allow students to find the whole number that the logarithm is equivalent to. On logarithms that are not equal to whole numbers, this use is still beneficial, because it allows the student to determine the two consecutive integers the logarithm falls between.

Students who understand logarithms as operations will often use the language "taking the log." They recognize that taking the logarithm of a number or expression is an operation just as multiplying or dividing by a number or expression is an operation. When students use the log button on a calculator, they should understand that the calculator is performing an operation on the number they entered into the calculator. Students with an operational understanding would also be able to solve an equation such as $e^{3x-1} = 1$ by taking the natural logarithm of both sides, simplifying the equation to $3x - 1 = 0$ and then solving for x. The justification of *why* we can take the natural log of both sides of an equation and maintain equivalency, however, lies within the realm of functions, which is the next category of my framework (the reason being that the logarithm is a one-to-one function).

Another idea students ought to understand about logarithms as a process is that you can find the value of a (whole number) logarithmic expression by repeatedly dividing the argument by the base, and keeping track of how many times you divide until you reach 1. They should already know that raising a number to a whole number power is repeated multiplication. However, if you are “going backwards” (i.e. starting with the result of the exponentiation) there are two questions you might ask: what was the number I was raising to a power, and what was the power I raised it to. Students must see the difference between these two processes (taking a root and taking a logarithm, respectively) in order to fully understand logarithms as a process.

Logarithms as Functions

Along with the previous categories, this category was borrowed from Berezovski and Zazkis (2006). Berezovski and Zazkis (2006) recognized that students ought to understand logarithms as functions, but did not elaborate much more on the subject (most likely to due to time or space constraints). Because I also believe it is important for students to understand logarithms as functions, I have included this category in my framework. However, because Berezovski and Zazkis (2006) did not elaborate on what they meant by understanding logarithms as functions, I constructed this category according to my own thought experiments and previous experience in teaching students. It is shown later that this category is the one that changed the most as a result of data collection and analysis.

Table 3 includes the summary points of what is meant by understanding logarithms as functions. Students who understand logarithms as functions recognize that $f(x) = log_a(x)$ is a function as readily as they recognize that $f(x) = 3x + 2$ is a function. They associate logarithms with the mathematical concept of function in general and all that goes with it.

Table 3
Evidence of Function Understanding

Label	Description of Understanding
F-def	The student understands that "plugging in" an x-value within the domain of the function will produce a single y-value. They can see that this property is what makes logarithms functions.
F-graph	The student knows what the graph of a logarithmic function generally looks like and understands how they might create a logarithmic graph by plotting individual values.
F-d/r	The student understands that logarithmic functions have a restricted domain, and unrestricted range, and can explain that that is the case because you cannot obtain zero or a negative number by raising a positive number to a power, but there is no restriction on what power you may raise a number to.
F-asym	The student can explain why there is a vertical asymptote, but no horizontal asymptote, on the logarithmic graph, using reasoning about domain and range as in the previous point.
F-inv	The student can relate logarithmic function and exponential functions as inverses.
F-ineq	The student can connect their knowledge of domain in logarithms to understanding the limitations on solutions for equations and inequalities that involve logarithms.

With regards to graphing logarithms, they should be able to use the function to find and plot ordered pairs and graph the function. They should recognize the general shape of a graph of a logarithm. They should know the general shape and essential parts (asymptote/intercepts) of the "$log_a(x)$" graph and be flexible with transforming the basic "$log_a(x)$" graph to represent other logarithmic functions.

Students should also be able to explain the relationship between exponential functions and logarithmic functions, not just as a memorized rule that they are inverses, but explaining what that means in terms of the functions. One way they might explain this is as follows:

If $f(x) = a^x$ maps x to y, then $f^{-1}(x) = log_a x$ maps y to x.

Again, it is not enough for students to memorize this, but they should actually understand it and be able to say it in their own words.

F-ineq was created to address the idea that students who understand logarithms as functions are be able to use their knowledge of domain to solve equations and inequalities that involve logarithms such as $log_5(x-1) < 2$ which has a solution set of $(1,26)$. In order to solve such problems, students first must be able to perform operations on logarithms (in the example, essentially just raising 5 to the power of each side), which requires an object understanding. However, when deciding which solutions are mathematically legitimate, students must be able to consider the domain of the logarithmic function involved and use the domain to determine whether a solution produced by algebraic manipulation can be a solution to the inequality, given the domain restrictions.

Logarithms in Contextual Problems

This category was not included in the framework presented by Berezovski and Zazskis (2006). I decided to include this category based on the literature and based on experience. In her study, Kastberg (2002) described students' responses to contextual problems involving logarithms. Their lack of understanding was what prompted me to include this category in my framework. From personal experience and from reading literature, something that students are missing is the ability to recognize the contexts in which logarithms would be useful and how to use logarithms in such contexts.

Table 4 includes the summary points of what is meant by understanding logarithms in contextual problems. Students exhibiting this type of understanding ought to be able to do at least two things: recognize contexts where using logarithms would be helpful and solve contextual problems using logarithms. Kastberg (2002) brought to my attention that students who are able to *solve* contextual problems involving logarithms may not *recognize* that the problems involve logarithms, indicating a lack of understanding. On the other hand, Watters and Watters (2006) brought to my attention the idea that students might *recognize* that a given problem must be solved with logarithms, but may not be able to *solve* the problem, again indicating a lack of understanding. Although it is admirable when students can utilize problem-solving skills to solve logarithmic problems without using logarithms, students with a firm understanding of logarithms in contextual problems would be more efficient at solving such problems because they would both recognize the usefulness of logarithms in solving the problem and be able to utilize their knowledge to actually solve the problem.

Table 4
Evidence of Contextual Understanding

Label	Description of Understanding
C-reas	The student is able to reason about (and solve) real-world problems involving logarithmic properties, whether or not the problem contains typical characteristics of symbolic logarithm problems such as “logarithm” or “ln.”
C-rec	The student is able to recognize when a real-world problem is most easily solved using a logarithm, whether or not the problem contains typical characteristics of symbolic logarithm problems such as “logarithm” or “ln.”
C-real	The student can explain *why* logarithms are useful in the real world, as opposed to the mathematics classroom (e.g. to change a scale; to solve for exponent variables in exponential relationships).
C-sym	The student is able to relate information given in a real world problem involving logarithms to symbolic notation (for example, they might write an exponential or logarithmic equation to represent information in a problem that could be solved using such an equation).

Students who have a firm grasp on how logarithms relate to real life contexts also ought to be able to explain how a logarithm might be useful in certain circumstances, such as to change a scale from one where numbers are spread very far apart (such as from .00001 to 1,000,000) to one with numbers that are much closer together (such as the Richter scale) or to solve a real life problem that can be modeled with an exponential expression where the exponent is variable (such as with interest problems). Further, they should be able to actually *use* logarithms to solve these problems, which implies that they must be able to transform the situation into something symbolic that they can manipulate mathematically to solve the problem, and then interpret the results in terms of the original problem.

CHAPTER III: LITERATURE REVIEW

To review the existing literature in the field as it pertains to my study, I have broken this literature review into four sections, which parallel the four categories of my theoretical framework: (1) logarithms as objects, (2) logarithms as processes, (3) logarithms as functions, and (4) logarithms in contextual problems. In each of those four sections I summarize the existing literature related to the concepts found in that category of the framework. It should be noted, however, that there is very little research on how students understand logarithms; the small amount of literature I have found relating to each of the four categories of my framework were most often examples of students failing to understand logarithms. From these examples (as well as from my own thought experiments), I extrapolated what students might need to understand in order to be successful with logarithms.

It should also be noted that all of the studies that have previously been done regarding student understanding of logarithms have been deficit studies. What I mean by this is, although not many studies have been done regarding student understanding of logarithms, those that have been done have shown that students lack understanding with regards to logarithms (Kastberg, 2002; Kenney, 2005; Watters & Watters, 2006; Weber, 2002a, Weber, 2002b). Deficit studies are important because they can reveal what students do not understand. Thus, deficit studies helped me construct my initial framework because seeing how deficient understanding of certain concepts affected students' ability to solve problems helped me realize that those concepts were important and should be included in my framework. However, my study differed from these deficit studies in that the focus was on both *deficit* and *existence*. This was done partly by examining deficient understanding which allowed me to identify concepts that, when not understood, can cause students to be unable to solve problems involving logarithms. However, I also looked for examples of students exhibiting good understanding, because identifying the concepts that helped students to be proficient at solving and explaining problems involving logarithms allowed me to consider other important concepts for my framework. Since my study took into account both what students *did* and *did not* understand about logarithms, I was able to identify important concepts for my framework in two ways (deficit and existence) rather than just one (deficit).

Logarithms as Objects

Kastberg (2002) reported that students failed to see logarithmic expressions as objects. The students in her study perceived "log" as a command to operate rather than part of the expression. She found that students sometimes correctly remembered rules and sometimes incorrectly remembered them, but they tended to believe that a problem was not finished until it was in decimal form. Kenney (2005) asked students to solve the following equation for x: $log_5(x) + log_5(x+4) = 1$. Both students interviewed believed they should "cancel out" the logs because the logs were of the same base, leaving them with $x + x + 4 = 1$. Though they had been recently tested on logarithms, both students failed to recognize that adding logarithmic expressions is equivalent to multiplying the expressions inside the logs (x and x+4). This indicates a misunderstanding of logarithmic expressions as objects, because they believed that they had to get rid of the logarithms before performing any operations. Students with an object conception of logarithms ought to be able to operate on logarithms, using the rules of logarithms, without "removing the logarithm" first.

With regards to the rules of logarithms, Weber (2002a) wrote, "as time passes, one's knowledge of symbolic rules will generally decay. If one has a deep understanding of the concepts involved, these rules can be reconstructed. If not, the rules cannot be recovered" (p.

1025). If we take Weber's assertion to be true, then mis-remembering rules and failing to check them for validity (perhaps because they do not know how) could indicate a lack of understanding. Weber found that students in a pilot study who were taught in a way that focused on concepts could reconstruct rules such as $b^x b^y = b^{x+y}$, while students enrolled in a more traditional class could not reconstruct such rules, and mis-remembered them without correction. Kastberg (2002) found that students who were successful with computational logarithm problems mis-remembered rules a few weeks later, such as remembering $log_a b = \frac{\log(a)}{\log(b)}$ instead of correctly remembering $log_a b = \frac{\log(b)}{\log(a)}$ or remembering $\log(a) + \log(b) = \log(a + b)$ instead of correctly remembering $\log(a) + \log(b) = \log(ab)$. Although being proficient at the rules of logarithms is an important part of understanding logarithms as objects, it is insufficient for students to simply memorize the rules. If students do not understand the rules of logarithms they will probably not remember them or be able to reconstruct them once they forget exactly what the rules are.

In examining student understanding of logarithms and logarithmic expressions as numbers, Berezovski and Zazkis (2006) expressed doubt that facility with calculating logarithmic expressions involving only numbers either with a calculator or by hand indicates an understanding of logarithms as numbers. They suggested that students may have learned a procedure when presented with such types of problems, but that these students may not recognize that, for example, $-\log_3(2)$ is a number, and does not need to be operated on in order to become a number. Kastberg's research (2002) supports the idea that students who can solve problems do not necessarily perceive logarithmic expressions as numbers. For example, one student referred to the process of finding a numeric value for the expression $log_4 5$ as solving an equation (p. 101). The student then correctly computed a decimal approximation for $log_4 5$, but did not seem to recognize that $log_4 5$ was already a number, instead labeling it an equation. I believe that students can and should understand that logarithmic expressions such as $-\log_3(2)$ are numbers.

Logarithms as Processes

In some cases, a process orientation is the most helpful way to view logarithms, as demonstrated by the following vignette from Berezovski and Zazkis (2006). In this instance, students were trying to find the whole number equivalent to $5log_3 9$. After some discussion about how to use the change of base rule and input the values correctly on the calculator, one student explained that you just need to convert 9 to 3^2 and the problem becomes much simpler. This student demonstrated (though her classmates did not) that she had some understanding of logarithms as processes.

In viewing logarithms as processes, Smith and Confrey (1994) wrote that logarithms are built from multiplication as a primitive structure in itself, not multiplication as extrapolated from addition. They called this primitive structure "splitting" and claimed that by providing students with contextual problems based on the splitting concept, they were able to demystify some of the rules of logarithms for students (Confrey & Smith, 1995). They explained that if you view multiplication as a structure parallel to, instead of building from, addition, then rules like $\log(a) + \log(b) = \log(ab)$ are grounded in the understanding that addition in one structure is equivalent to multiplication in the other. While Confrey and Smith (1995) moved for *less* extrapolation (logarithms founded on multiplication, which is a primitive structure), Hurwitz (1999) moved for *more*: logarithms are founded on the exponential function (as its inverse) which is founded on multiplication which, in turn, is founded on addition. Hurwitz claimed that if students are shown that the exponential function "puts on an exponent," then the idea that logarithms, as the inverse of the exponential function, "lift off the exponent" will build upon previous student knowledge and give students a foundation from which to build. Hurwitz explained "lifting off" as, for example, in $g_8(8^{4/3})$, applying the "liftoff function" gives 4/3, because you have lifted off the exponent. She also reinforced her method through notation by writing **(l)**ift**(o)**fffunctionis ($\boldsymbol{g}_{b(x)}$), circling the l, o, and $g_{b(x)}$.

Logarithms as Functions

Students sometimes struggle to see logarithms as functions. Hurwitz (1999) suggested this may be due in part to the notation, because $\log(x)$ does not look like many of the common functions, such as polynomials. A student named Jamie also commented on the fact that just seeing "logarithm" confused her, and believed that the fact that it was a word, instead of a number, was what threw her and others off (Kastberg, 2002 p. 75). Another student in the same study also drew the graph of the logarithm as including both the logarithmic function and the exponential function, and believed that the two graphs together made up the graph of the logarithmic function. This student was a straight A student in her mathematics classes, yet she did not seem to recognize that her graph could not possibly be a function because there were x-values that corresponded to more than one y-value. It may also be that if asked if such a graph was a function, she would say no, it doesn't pass the vertical line test, and she just does not conceive of the logarithmic function as a function.

Logarithms in Contextual Problems

Berezovski and Zazkis (2006) posed the question "Which number is larger, 25^{625} or 26^{620}?" and found that more than half of the students (who had just completed a unit on logarithms) did not attempt to use logarithms to solve this problem. This seems to indicate a lack of understanding of logarithms in context, because one of the primary purposes of using logarithms in contexts is to make extremely large numbers more usable.

Wood (2005) observed that students “have a particularly difficult time relating to” logarithms (p. 167). He suggested this may result from a lack of true application problems, and suggested several real world applications that teachers might use to help students relate better to logarithms, such as the decibel scale, the Richter scale, and stock analysis. Watters and Watters (2006) found that neither freshmen enrolled in biochemistry nor upper-level students in the same program were very successful at solving pH problems that required the ability to reason with logarithms. This is the only study I could find that tested logarithmic understanding of upper-level college students who ought to have been able to solve problems with logarithms. On the other hand, Kastberg (2002) found that her subjects (college algebra students) were usually able to problem-solve their way through logarithmic problems in context, as long as they didn’t know the problem involved logarithms. The students did not recognize that logarithms could be used to solve such problems, so they solved them by relating the problems to exponents (which they were more comfortable with than logarithms) and were successful, if not efficient, in solving the problems. In order to have a good understanding of logarithms in context, I believe that students ought to *recognize* that logarithms will help you solve the problem (as the students in the first two studies did, but Kastberg’s did not), and be able to *solve* the problems correctly (as the students in the first two studies did not, but Kastberg’s did).

In summary, what it means to understand logarithms has not been well researched. There have been several studies that have pointed to a lack of understanding in this area, but none have resulted in a research-based explanation of what understanding logarithms can and should look like. Being able to describe what it means to understand logarithms is important because in order to teach all concepts and assess for a full knowledge of logarithms, there must be some way to describe what *should* be understood by students. Therefore, my study aims to fulfill this need, at least in a preliminary way. The primary purpose of my study was essentially to define, test, and refine a conceptual model of what it means to understand logarithms.

CHAPTER IV: METHODOLOGY

This chapter describes the methodology I used to complete my study, including the setting in which the study took place, the subjects I studied, the instruments I used to collect data, how the instruments were used to collect data, and how I analyzed the data. The purpose of my study was to test a framework. Zandieh (2000) wrote,

> How do you know if a framework is useful? For a concept as multifaceted as derivative [or, I would argue, logarithms] it is not appropriate to ask simply whether or not a student understands the concept. Rather one should ask for a description of a student's understanding of the concept of derivative [or logarithms] – what aspects of the concept a student knows and the relationships a student sees between these aspects (p. 104).

This is a good summary of my purpose in this study. Everything in my methodology was set up to find out what students know, or should know, about logarithms – what constitutes a good understanding of logarithms. The students' responses were then compared to my framework, and my framework was adjusted where the data indicated the framework was lacking.

Setting and Subjects

The participants of this study were preservice secondary mathematics teachers enrolled in a mathematics education course at a western university. This course has, as prerequisites, first and second semester calculus, so the students in this class had already learned about logarithms and used them fairly extensively. The course was designed to provoke preservice teachers to think deeply about mathematical topics they have learned in school mathematics and the topic of logarithms was one of the units the students studied in depth in the class.

Choosing to study preservice mathematics teachers, rather than algebra students or mathematics professors, was an important decision. I could have chosen to study students who were learning about logarithms for the first time (novices), but I ruled that out almost immediately because I believed their knowledge would not be robust enough to form an adequate representation of what constitutes good understanding of logarithms. On the other hand, I could have chosen to study experts, such as mathematics professors or algebra teachers. However, it would be fairly unlikely that they would struggle with any of the tasks in my interviews, and seeing a person struggle with a task can help pinpoint essential understandings. For example, seeing a student who cannot solve a problem which requires the student to view logarithms as numbers is good evidence that knowing that logarithms are numbers is very important. Thus, choosing preservice mathematics teachers allowed me to test my framework on “pseudo-experts” rather than novices or experts, which provided good data because they were expert enough to demonstrate good understanding of logarithms, but novice enough to make revealing mistakes.

Choosing students from this particular class was also helpful because for this class they were regularly required to explain their thinking, so they were fairly good at explaining their thought processes to me. Also, the fact that students were studying logarithms in class meant that they were frequently thinking about logarithms, which meant we did not need to take time in interviews reviewing things they may have forgotten about logarithms. Finally, it was important that the class was studying logarithms at the time of the study because the students experienced a natural progression in their understanding of logarithms. Although I was not testing for a progression in understanding, the progression caused the students to give a broader range of answers, using less advanced thinking at the start and more advanced thinking over time. Thus, although I only studied four students, I was able to see a somewhat broad spectrum of understanding of logarithms.

From this class, I chose a focus group of four students to participate in interviews. I chose to study only four students because I wanted to interview each student four times, so studying more than four students would have been impractical given time constraints. However, I did not want to study fewer than four students, because I wanted to ensure the students I chose would give a range of answers and display multiple ways of thinking about logarithms.

There have been other studies that have used task-based interviews to explore student conceptions of specific areas of mathematics (e.g. Rubel, 2007; Watson & Moritz, 2000). The two mentioned studies chose their interview subjects through analysis of a written survey administered to a group of people from which they drew their subjects. Because I wanted to ensure I chose my focus group wisely, I chose my interview subjects by analyzing a survey administered to the entire class, just as the two mentioned studies did.

To choose a good focus group, I wrote a short survey (see appendix A) to measure the students' willingness to participate, knowledge of logarithms, and ability to explain their thinking. Once I collected the surveys (which were filled out by the entire class), I began a process of elimination. First, I eliminated the students who indicated they were not willing to be interviewed. Next, I eliminated students who showed little or no evidence of understanding – either they left most questions blank, or mainly wrote memorized facts. After this, I sorted through the remaining surveys to identify students who were able to justify or explain themselves. Lastly, I sorted through the remaining surveys to ensure that a variety of thinking was represented until I had narrowed it down to just four people. Since the survey was written with the intent to briefly examine the students' understanding of each of the four categories, I was able to look for students who were strong in different categories to ensure variety in my focus group. So, for example, I chose one student who could not remember the rules of logarithms very well, but showed good understanding in the object and function categories, while another student remembered all the rules but showed less understanding in the function category.

As part of my study, I attended the class with the students throughout the unit on logarithms. Although I was not collecting data during class time, I went so that I would be familiar with what they were learning in class, and I would occasionally incorporate ideas that came up in class into the interviews. For example, in class, the students had discussed possible process meanings for logarithms, so in the first interview I asked my interview subjects to explain and evaluate the different ideas that had come up in class. However, the reason I attended class was so that I was familiar with the ways my subjects had been recently discussing logarithms; this would hopefully allow me to understand them better during interviews. Again, I did not collect data during class time; I wanted to get a deep understanding of the conceptions my subjects had about logarithms, and I felt I could best achieve this goal in a one-on-one setting where I could continually ask the student to explain until I felt I understood what they meant.

Instruments

As mentioned previously, I wrote a survey (included as Appendix A) to choose my focus group of four students from the class. The survey included a question about whether the student would be willing to be interviewed along with several questions to get an idea of how well they understood logarithms and how well they expressed themselves. The questions were open-ended with the purpose in mind that hopefully those students who were able to express themselves thoroughly on a written survey would also be able to express their thoughts verbally in interviews.

During the unit, I interviewed each of the four focus students four times for approximately one hour per interview. I interviewed each student about once per week for four weeks. I wanted to do several interviews over a period of time for several reasons. The first is that I believed there was too much material to cover in just one interview. The next reason is that I believed I might be able to collect different information at different times from the same students, since their knowledge of logarithms was changing as they learned about them in class. In the first interview or two, I was able to see more student mistakes (indicating missing concepts) with some student successes, and later I was able to see more student successes (indicating concepts that might be important and widely useful) and some student mistakes. The last reason I wanted to do several interviews over time is that I wanted to be able to adjust later interval protocols as needed to include other things that might come up in class or in interviews.

The interviews were semi-structured in nature. That is, each interview protocol consisted of a series of questions and tasks, but the interviews were flexible enough to explore what the student was thinking, and if the student's thinking led away from the protocol, we could explore that thought before moving on. All four interview protocols are included as Appendix B of this document. To ensure that I tested each category of my framework, I designed my tasks to bring out various aspects of the framework. Of the interview protocols, I have noted what types of understanding I thought the questions might elicit. In reality, however, the students did not always answer questions in the way I expected, and sometimes revealed understanding that I had not anticipated the question might reveal or even understanding not yet written into my framework.

I decided against focusing each interview on only one category of the framework because I was concerned that the understanding displayed in the first interview might not be on par with the understanding displayed in the last interview. I thought this might cause me to miss out on some great conceptions they had about the category in the first interview simply because their understanding of that category grew after the interview. Therefore, I planned for the first interview to focus on object and process conceptions of logarithms (the first two categories of my framework), the second interview to focus on functions and contextual problems (the other two categories of my framework), the third interview to focus on object and process again, and the fourth interview to focus on functions and contextual problems again. This would allow me to revisit each category twice, so that any new conceptions they developed over the course of the interviews would hopefully be revealed.

The first and second interviews went as planned, the third interview did primarily focus on object and process meanings of logarithms, but the fourth interview ended up as kind of a "clean up" interview. What I mean by this is that although I had intended interview four to focus on function and contextual understanding of logarithms, the protocol was adjusted to address any of the understandings in my framework I felt I had not yet sufficiently covered. I have included an interview protocol for interview four in Appendix B; however, the interview protocol for interview four differed slightly for each subject according to what I still had not covered with them in previous interviews. Most of the interview four protocol was the same for all four students, and the commonality is what I have included as my interview protocol. However, for each student there were one or two minor adjustments to the interview.

Data Sources and Data Collection

As mentioned previously, I conducted four clinical interviews with each of the four focus students that were selected through analysis of the surveys. These interviews were videotaped for analysis. The idea of conducting clinical interviews to obtain my data originated from reading

Kastberg (2002). Kastberg conducted clinical interviews with her subjects (shortly after they had studied logarithms in a college algebra class) in order to explore their understanding of logarithms. Although her purpose differed from mine (she focused on what college algebra students understood after instruction, while I focused on developing a general model for understanding logarithms), she utilized these interviews well to explore the students' understanding and draw out what the students' conceptions were. Of course, the ability to draw out student conceptions depends on interview design and interviewing skills. The questions that are asked in an interview are vitally important, and they must all be targeted at the same thing: discovering the answer to research questions. Although many of the tasks in my interviews had a specific answer, there were several ways of going about the task. I also made sure to ask a lot of questions about the way the student was thinking. Some of the questions in my interview protocols might have been suitable for a classroom test, but my purpose in asking them was different. Truran and Truran (1998) wrote, "questions in a clinical interview are designed to elicit information about a [subject]'s understanding; they are designed to elicit information to which the interviewer does *not* know the answer" (p. 70, emphasis in original). Thus, the purpose in asking these questions was not to see if the students knew the answer, but rather to see how they thought about the question.

Ellemore-Collins and Wright (2008) suggested that videotaping individual student interviews frees the interviewer from writing and allows the interviewer to focus on the task at hand – understanding the interviewee and asking appropriate questions to push at that understanding. There are also other benefits of videotaping interviews. Rewinding the video allows the researcher to go back and make sure he has understood the interviewee or to watch the same interview multiple times with different things in mind, such as evidence of each of the four components in my framework (Ellemore-Collins & Wright, 2008; Powell, 2003). Aside from these benefits, I wanted to use a whiteboard in my interviews. While I could have had the students work on paper and kept the paper for records, having them work on the whiteboard while I interviewed allowed me to view in real-time exactly when they wrote things. This proved to be important to my study because often the students (or I) would gesture at things that had been written on the board as part of a verbal explanation or question. This was, of course, captured on the video, whereas it could not have been if I had simply recorded the audio and used paper for written work.

Data Analysis

I analyzed the data for this study both during data collection and afterwards. The way I analyzed the data during data collection was different (and much simpler) than the way I analyzed the data after data collection was complete. In the next two sections, I explain the process I used to analyze the data for both cases.

Data Analysis during Data Collection

During the time I was collecting data, I did not have time to fully analyze the data. I wanted the interviews to be close together, which meant that I was interviewing four students each week. Still, I had planned to adjust the interview protocols according to what I found as I collected data, so I knew I needed to analyze the data, at least minimally, while I was collecting it. To do this, I wrote a short memo about certain things that stuck out to me during the interviews directly after conducting each one. For example, after my first interview with Sarah[1], I wrote in my memo that I believed Sarah's first impulse on most problems involving logarithms

[1] Participants' names are gender-preserving pseudonyms.

was to use the graph of the logarithm. In that memo, I also noted some ways I believed Sarah understood logarithms.

Also, I attended the class from which I recruited these students and occasionally incorporated topics from class into interview protocols. Most notably, I incorporated two ideas from class (stretching/compacting and distortion) into interview four. These ideas were incorporated because I did not have anything written in my framework about them at the time, but they seemed like very important ideas about logarithms and I wanted to include them in my study. Another instance where I incorporated a topic from class into our interviews was when the class was first discussing what the process meaning for logarithms might be. In the interviews, I asked the students to describe each of the possible process meanings we had discussed in class and explain how each was useful (or if it wasn't, explain why).

Data Analysis after Data Collection

I analyzed the interviews one student at a time. I started with the student who demonstrated the most advanced reasoning about logarithms and identified four "episodes" during which he exhibited particularly good understanding. I transcribed these episodes fully to analyze. However, I found that a full transcription was not the most helpful way to test my framework using the interviews. In order to make sense of how the interviews fit with my framework, I needed my data to be organized according to the framework. Since the ideas elicited from the students were not necessarily in the order my framework was written in, organizing the data chronologically proved unfruitful. Thus, I needed to analyze the data in a format that let me keep the structure of the framework.

In the end, I started over with my data analysis using the following method. First, I created a table of all four categories of my theoretical framework in one Microsoft Word document. Each type of understanding (including all the sub-categories) had its own cell in the table. Then, I slowly watched the interviews for one student. As I watched, I paused each time I noticed the student demonstrating understanding of some sort. I tried to determine if the understanding fit with a concept already in my framework. If it was, I summarized what the student was saying/writing along with a timestamp in the corresponding cell of the table. If the understanding did not seem to fit into my framework, I would summarize the idea in the same document, but below the table. Also, I noted in the table when the student exhibited a lack of understanding. This was done because deficient understanding of a concept can provide evidence of the concept's importance. Specifically, if a student's failure to understand some concept inhibits their general understanding of logarithms, that concept is important enough to be included in my framework. Thus, at the end of analyzing all four interviews for the first student, I had a completed table with timestamped examples for each category and sub-category from the interviews of where the student exhibited either good understanding or lack of understanding.

After I analyzed the first student's interviews, I reviewed my theoretical framework and made some minor changes. In the instance of the first student, the changes were to tentatively add two subcategories to the framework (which I later decided was a good change, and kept them both after rewording them). Analysis of the other three students was very similar: I created a document for that student, created the table with the framework, filled it out where possible with timestamped examples, noted at the bottom of the document any understanding or misunderstanding that appeared not to fit exactly in the framework, and revised the framework as needed at the end. After revising the framework, I did not go back and re-analyze the previously analyzed students, because my purpose in this study was not to analyze the students according to my completed framework, but rather to complete my framework according to my analysis of the students. Even if only one student exhibited a type of understanding which was not originally included in my framework, and the concept appeared to be important for understanding logarithms, that was enough evidence to indicate my framework needed to change.

CHAPTER V: RESULTS

In this chapter, I present the results of my study in three sections. The first section is about my framework in general, including how and why it was adjusted. The second section is about a particular way of dealing with logarithms that many students exhibit, which I believe textbooks and teachers promote, called *switching forms*. The third section explains why my framework, while good at separating mediocre from good understanding of logarithms, fails to separate good understanding from exceptional understanding.

The Framework

In this section I discuss my theoretical framework, including how it was adjusted throughout data analysis. First, I present the changes I made to the framework along with an explanation for each change. Then, I explain how using the framework was useful for assessing students understanding of logarithms. Finally, I explain how category four of my framework (logarithms in context) seems to be weaker than the other categories.

Adjustments to the Framework

In this section I first describe the criteria I used for adjusting my framework. Next, I show some in depth examples of how I decided to make changes to the framework. Finally, I list and describe all of the changes made to my framework.

Criteria for adjusting the framework.

Throughout data collection and data analysis, I looked for instances in interviews that might indicate a change in my framework was needed. There were three types of changes I made to the framework. One type of change I made to my framework was to reword a concept description. Sometimes I did this just for clarity, if my colleagues noted that the original description was not concise or difficult to understand. Other times, I reworded the concept to broaden it so it would be useful in more situations. A second way I changed my framework was to combine two concepts. I did this in two cases where the pair of concepts were so interrelated that it made more sense for them to be together than apart. The third way I changed my framework was to add a concept. Sometimes I would see a student exhibit understanding of a concept of logarithms in a way that did not fit within my framework. In this case, I would note the concept and analyze later whether or not it should be included.

After finding a possible signal that my framework might need adjustment, I had to analyze whether and how to adjust it. Generally, I followed the same guidelines I used to create my framework in the first place. That is, I looked for situations in which students could not solve problems because they were missing some type of understanding (a minimum standard), as well as for types of understanding which applied to several different problem types or situations (above minimum standard, but still widely applicable). I also decided not to include some things in my framework if I felt they were too general and not specific to logarithms. For example, I did not include the idea that students ought to be able to check their answers for correctness in my framework, because although it is vital for students to be able to do so, checking one's work is a very general math skill not particular to logarithms. On the other hand, I did include the idea that exponentials and logarithms are inverses, because although the idea of inverses is general, the idea that exponentials and logarithms are inverses is particularly related to logarithms.

Examples of changes to the framework.

The three types of changes I made to my framework were rewording a concept for clarity or to broaden the concept, combining two related concepts, and adding in a new concept. To illustrate how I used the data to make these changes, I provide an example for each of the types of changes below.

The first type of change I made to my framework was to reword a concept for clarity or to make it broader. One example of this type of change can be seen in O-part from my framework. Originally, O-part read “The student knows that a logarithmic expression must have both a base and an argument (in addition to ‘log’ or ‘ln’) in order for it to make sense and be complete (i.e. they do not try to separate the ‘parts’ of a logarithmic expression because they know that the logarithmic expression is meaningless without all of its ‘parts’). This includes knowing the implied bases for ‘ln’ and ‘log’ with no base written.” Some of my colleagues, upon reading my framework, expressed confusion at this concept because it was long-winded and not very particular. I also realized from analyzing my data that simply knowing there are these three parts to a logarithm is insufficient. When I asked the students about the different parts of the logarithmic expression, some had names for each part, and some didn’t, but they had all developed a meaning for each part. The students repeatedly referred to the different parts of a logarithmic expression and used their meanings to justify their thinking, so I changed O-part to include having a correct meaning for each part of a logarithm. After rewording the concept to make it more concise and to include the idea that students must have an accurate meaning for each part of a logarithmic expression, O-part was revised to read “The student has developed an accurate meaning for the base, argument, and the terms ‘log’ and ‘ln’ and does not try to separate these parts (such as by dividing by ‘log’).”

The second type of change I made to my framework was to combine two related concepts into one concept. One example is from the process category, which originally had the two related categories P-elim and P-exp. P-elim originally read “the student can think of ‘taking a logarithm’ as eliminating the base of an exponential expression to solve an exponential equation such as $3^x = 17$.” P-exp originally read, “The student can explain that it is necessary to ‘take the logarithm' when trying to find the value of an exponent in an expression." Prior to data analysis, I had thought P-elim was the way a student would generally solve an exponential equation, while I imagined P-exp as kind of an inverse operation idea that could be used on an exponential expression. Watching the students in my study solve exponential equations revealed many different ways of thinking about exponents and logarithms. Some ways they solved exponential equations included graphing, applying a logarithm to both sides, using the object or process meaning to solve the equation in their head, and guess and check methods. I also watched one student fail to recognize that solving a particular story problem would be easiest if she took log base three of the number in the story, but she did produce an alternate way to solve the problem (repeated division). As a result of these instances, I realized while it is not *necessary* to take a logarithm to find an exponent either in an exponential equation or an exponential expression, a student ought to see how it could be *helpful*. Thus, I combined both categories into one, called P-exp, and it now reads "the student understands why you might take a logarithm when trying to find the value of an exponent in an expression or equation."

The third type of change I made to my framework was to add a new concept to my framework. Although watching the student interviews showed that the students had many insightful ways to solve problems, I had to analyze whether the things they were doing already fit into my framework (either as one concept or a combination of multiple concepts) or if it was something not in my framework. If I determined that the concept they used was not already in my framework, I then had to analyze whether it belonged in my framework. That is, it had to be useful in many types of problems, and it had to be specifically related to logarithms. One concept I added to my framework as a result of watching the students use it was the concept I called F-1to1: "the student understands that logarithmic functions are one-to-one and understands why this is important. For example, the student may argue that $\log(2x - 1) = \log(x^2)$ can be written as $2\text{x} - 1 = x^2$ because the logarithm function is one-to-one and so the arguments ($2\text{x} - 1$ and x^2) in this equation must be equal." The example in the concept description is an example a student actually used in an interview. This example reminded me of how some students try to "divide both sides by log," except it was mathematically correct. Some possible uses of the knowledge that a logarithm is one-to-one might be to justify applying a logarithm to a data set, to solve equations, to prove rules, or to reason about graphical problems. The concept was also specifically related to logarithms because, unlike polynomial functions, *all* logarithmic functions are one-to-one and this, along with the other concepts in the function category, helps us classify the logarithmic function.

Having discussed some examples in which I decided to make changes to my framework, it may also be helpful to show an example of where I decided not to change a concept in my framework. One concept from my framework that I chose not to change was C-rec, which reads "the student is able to recognize when a real-world problem is most easily solved using a logarithm, whether or not the problem contains typical characteristics of symbolic logarithm problems such as 'logarithm' or 'ln.'" The importance of this concept was made evident by the student responses to question five in the first interview, which read in part, "given the following sequence: 3, 9, 27, 81, …, where 3 is the first term, 9 is the second term, etc., how could you find out which term of the sequence is 1594323?" Three of the four students immediately recognized they could take log base three of 1594323 to find the answer, but Holly said she should divide by 3 until she got to 1, and count how many times she divided. When prompted, Holly could not think of another way to solve the problem. Although Holly's method is a correct solution to the problem, it would be inefficient in practice to divide such a large number by three repeatedly until she got to 1, even with the aid of a calculator. Holly's failure to recognize that using a logarithm would be efficient in this particular story problem supported my original belief that being able to recognize when a logarithm is useful is important for understanding logarithms. I did not see any reason to reword the concept or combine it with another concept, so the concept remained unchanged in my final version of my framework.

The revised framework.

To illustrate the changes made to my framework I have created comparative tables for the original and revised frameworks for understanding logarithms. In each table, the first column provides a label for the piece of understanding. The second column provides the description of that piece of understanding as it was in the original framework. If the cell in the second column is blank, it means that piece of understanding came about as a result of data analysis and was not included in the original framework. The third column provides a description of the piece of understanding in the revised framework. Where there was no change, the cell in the third column is simply marked "same," and where two categories were combined it is noted in the table with the words "grouped together as."

Object understanding revised.

Table 5 shows the changes that were made in the object category of understanding logarithms between the original and revised framework. Three concepts remained unchanged because there were many examples of the students using these concepts to their advantage during the interviews.

Table 5
Evidence of Object Understanding - Revised

Label	Description of Understanding (Original)	Revised Description
O-def	The student is able to think of $log_a b$ as *the exponent* you must raise a to in order to get b.	Same
O-num	The student recognizes that a logarithmic expression such as ln2 is a number, and does not need to be approximated with a decimal.	The student recognizes that a logarithmic expression such as ln2 is a number, and does not need to be approximated with or written as a decimal.
O-rule	The student is able to flexibly change forms of a logarithmic expression using the rules of logarithms (power, sum/product, difference/quotient, change of base). They recognize these forms as equivalent.	Same
O-not	The student must know the notational conventions of logarithms and how they relate to the order of operations (such as writing $2log_3 6$ instead of $log_3 6 \cdot 2$, because they have different meanings).	Same
O-part	The student knows that a logarithmic expression must have both a base and an argument (in addition to "log" or "ln") in order for it to make sense and be complete (i.e. they do not try to separate the "parts" of a logarithmic expression because they know that the logarithmic expression is meaningless without all of its "parts"). This includes knowing the implied bases for "ln" and "log" with no base written.	The student has developed an accurate meaning for the base, argument, and the terms "log" and "ln" and does not try to separate these parts (such as by dividing by "log").

For the concept O-num, I added "or written" to clarify that students are comfortable leaving their answers to problems as a logarithmic expression, such as $log_2 39$. On one problem, Sarah affirmed she would leave logs in her answer for her homework, because it was "close enough." The led me to believe Sarah did not recognize that an expression such as $log_2 39$ is a number as it is written, so I added in "or written" to O-num.

O-part was reworded to be more clear and concise, and I added in the idea that students ought to know the meanings for the different parts of the logarithmic expression. My reasoning for this change was explained in detail on p. 37.

Process understanding revised.

Table 6 shows the changes for the process category between the original framework and the revised framework. There were more changes in this category than in the object category, but most of the changes were for clarification purposes. P-def remained unchanged because there were many cases in which the students used P-def to their advantage, as well as a few cases where students became confused because they failed to use P-def when it would have been advantageous. This particular case is explored in depth starting on p. 57. P-est also remained unchanged because of the many times I saw the students use it to their advantage when solving problems in the interviews.

Table 6

Evidence of Process Understanding - Revised

Label	Description of Understanding (Original)	Revised Description
P-def	The student is able to think of $log_a b$ as a process of taking the argument (b), converting it to the form of a^x, and writing x as the answer. Using this knowledge, they are able to simplify expressions such as $log_2 16$ by employing the process as $log_2 16 = log_2 2^4 = 4$.	Same
P-est	The student can use their knowledge from P-def to estimate values of logarithmic expressions such as $log_3 29$ by recognizing that 29 is close to (and just higher than) 27, and since $log_3 27 = 3$, $log_3 29$ must be slightly greater than 3.	Same
P-elim	The student can think of “taking a logarithm” as eliminating the base of an exponential expression to solve an exponential equation such as $3^x = 17$.	Grouped together as P-exp: The student understands why you might take a logarithm when trying to find the value of an exponent in an expression or equation.
P-exp	The student can explain that it is necessary to "take the logarithm” when trying to find the value of an exponent in an expression.	
P-div	The student relates the action of taking a logarithm to repeated division and recognizes when repeated division is an appropriate method for solving a problem (i.e. when you can divide a whole number of times to get to 1 or else when only an approximation is needed).	The student can use the ideas of repeated division (of the argument) or repeated multiplication (of the base) to solve logarithm problems and can assess the accuracy of such a method (exact for whole numbers, but not for things like $log_3 29$).
P-root	The student can relate the actions of taking a square (or other) root and taking a logarithm, explaining similarities and differences (i.e. that in roots you are trying to find the *base* of the exponential expression, whereas in logarithms you are trying to find the *exponent*).	The student understands the relationship between roots, exponents, and logarithms. For example, when you have $x^2 = 9$, you take the square root of both sides because you are solving for the base of the exponential, but with $2^x = 9$ you would take log base 2 of both sides because you are solving for the exponent.

The first change in the process category is that P-elim and P-exp were grouped together into one category, because whether the student is working with an equation (as in P-elim) or an expression (as in P-exp), the student should realize that taking the logarithm is an option when there is a variable in an exponent. I explained my reasons for combining these categories more fully on p. 37 of this document.

The next change in the process category was the change to P-div. First, I added in repeated multiplication. Initially, I did not include repeated multiplication in this category because I associated repeated multiplication with exponents. However, as I analyzed the interviews, I realized students often used repeated division and repeated multiplication interchangeably to express the same idea. For example, in interview 1, question 5, when I asked Julie for alternate ways to find out which term in the sequence 1594323 was, she initially explained she would take log base three of the number. When I asked for other methods, she listed both repeated division and repeated multiplication as alternate ways to solve the problem. The repeated multiplication idea is more of a "building up" of the logarithmic expression rather than repeated division, which can be thought of more as "taking apart" the logarithmic expression. I also reworded the end of the concept description for clarification and conciseness.

The last change in the process category was the change to P-root. Although "taking a logarithm" and "taking a root" (as in the original) seemed to fit the process category more, I chose to change the concept because understanding the relationship between logarithms and roots is a broader idea. The idea I was trying to express was that you have to understand that you use logarithms and roots to solve for different parts of an exponential expression, e.g. solving $a = x^y$ for x or for y.

Function understanding revised.

Table 7 shows the changes for the function category between the original framework and the revised framework. As you can see, this category had more significant changes than the previous two categories. Although F-def, the definition, remained the same, everything else in this category changed and there were some new pieces of understanding added to the category. I chose not to change F-def because, although I didn't often see the students refer to this definition, it is this definition of a logarithm as a function that allowed the students to use the other concepts in this category of the framework. That is, concepts such as domain and range of a logarithm only make sense when viewing a logarithm as a function.

Table 7

Evidence of Function Understanding - Revised

Label	Description of Understanding (Original)	Revised Description
F-def	The student understands that "plugging in" an x-value within the domain of the function will produce a single y-value. They can see that this property is what makes logarithms functions.	Same
F-graph	The student knows what the graph of a logarithmic function generally looks like and understands how they might create a logarithmic graph by plotting individual values.	The student can picture a general version of the graph of a logarithm as a whole *and* can plot/imagine plotting the graph by points.
F-c/s		The student understands the nature of the logarithmic function with regard to compacting and stretching.[2]
F-d/r	The student understands that logarithmic functions have a restricted domain, and unrestricted range, and can explain that that is the case because you cannot obtain zero or a negative number by raising a positive number to a power, but there is no restriction on what power you may raise a number to.	Grouped together as F-d/r: The student understands why logarithmic functions have restricted domain but unrestricted range, either by using the object definition (O-def) or process definition (P-def) of logarithms. Similarly the student understands why there is a vertical asymptote, but no horizontal asymptote, on the logarithmic graph by using either O-def or P-def.
F-asym	The student can explain why there is a vertical asymptote, but no horizontal asymptote, on the logarithmic graph, using reasoning about domain and range as in the previous point.	
F-inv	The student can relate logarithmic function and exponential functions as inverses.	The student can relate logarithmic function and exponential functions as inverses graphically and symbolically.
F-ineq	The student can connect their knowledge of domain in logarithms to understanding the limitations on solutions for equations and inequalities that involve logarithms.	The student can solve equations and inequalities involving logarithms and explain the solution in terms of logarithms (including discarding answers outside of the domain). This entails understanding exponentials and logarithms as inverse operations (that they "undo" each other).
F-1to1		The student understands that logarithmic functions are one-to-one and understands why this is important. For example, the student may argue that $\log(2x-1)=\log(x^2)$ can be written as $2\mathrm{x}-1=x^2$ because the logarithm function is one-to-one and so the arguments ($2\mathrm{x}-1$ and x^2) in this equation must be equal.

[2] The idea of compacting and stretching (F-c/s) is described in detail on pp. 46-47.

The change to F-graph was essentially a rewording intended to highlight the difference between thinking of the graph as a whole or as a collection of points that could be graphed. I realized that both orientations were necessary for a robust understanding of logarithms when I heard one student's explanation of "how the logarithm graph was created."

> Sarah: They[3] had all the natural number outputs, and then they figured out the points [such as (1,0), (2,1), (4,2), and (8,3) on the graph of $y = log_2 x$]. Then they figured out how to do other ones that weren't as easy, and it kept up the pattern, so they figured out that it was a continuous curve. And so then they could put whatever input for x they wanted and find the right output.

Although I imagine this is not actually the way the graph of the logarithm was created, the idea of the graph of a logarithm as a collection of an infinite number of points is different from the idea of the graph of a logarithm as a whole, and each of these ideas can be useful depending on the situation. For example, thinking of the graph as a whole might be more helpful when working with transformations, while imagining the graph as a collection of an infinite number of points could be useful when trying to solve a system of equations to discover where two functions intersect. The idea of imagining plotting the graph led, for Sarah, to the idea that the graph is made up of an infinite number of points. Thus, in the revised version of this piece of understanding, I included the idea of imagining plotting points.

F-c/s was added as a result of the classroom instruction on logarithms. As my advisor and I met weekly to discuss my thesis, he explained the idea to me that he had never heard taught before. The idea is, essentially, that the logarithmic function, on the right side of the x-intercept, takes an interval of x-values and compacts them into a smaller y-interval, and it does not distribute them "evenly." Likewise, the logarithmic function, on the left side of the x-intercept, takes an interval of x-values and stretches them into a larger y-interval, and it does not distribute

[3] From context, I assume by "they" Sarah meant a group of mathematicians exploring the properties of the logarithmic function.

them evenly. One idea that relates to this is that students often think of the logarithmic function in a linear way. For example, one student believed $log_2 6$ would be exactly in the middle of $log_2 4$ and $log_2 8$ (linear thinking) until I asked whether $log_2 10$ was exactly in the middle of $log_2 4$ and $log_2 16$ because 10 was equidistant from 4 and 16. This, along with other misunderstandings displayed by students, led me to believe the very nature of the logarithmic graph, as different from other graphs like lines and roots, is an important and difficult concept for students to grasp.

A topic related to F-c/s that we discussed both in class and in interviews is that of distortion. What I mean by that is, given an x-interval of a specified size, the distribution of those x's to their corresponding y-values approaches linearity (less distorted) as you move to the right on the graph, and becomes more distorted the more you move to the left. The students I interviewed tended to believe the graph would cause distortion most around the x-intercept and least around either end of the graph, probably because the graph looks linear on either end and curvy towards the middle. However, the case is in fact that as you move to the left, the distortion is always greater (see figure 2 for an illustration of this).

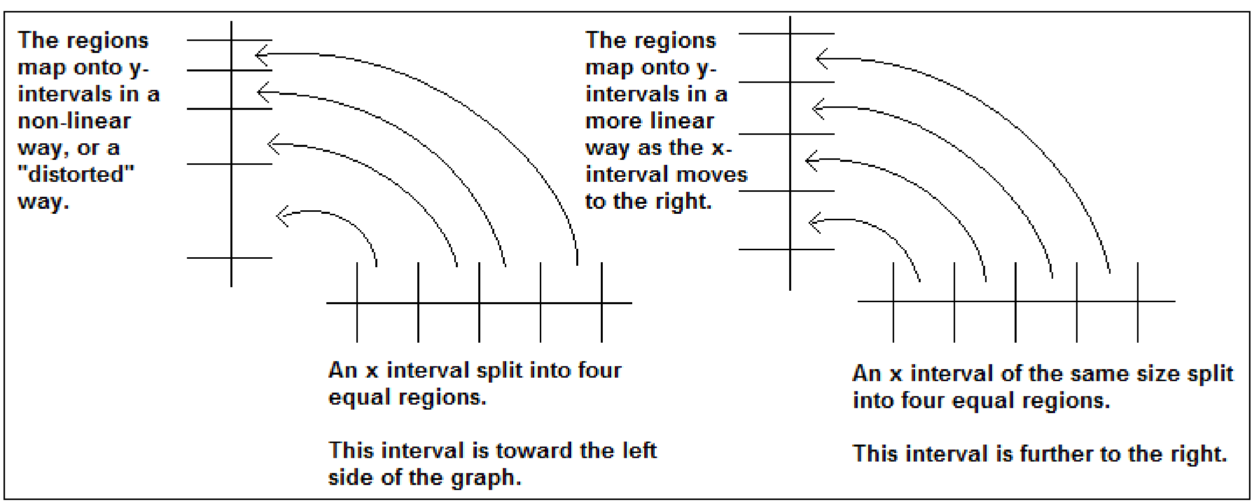

Figure 1. Logarithmic functions cause more distortion towards the left side of the graph.

Mark did guess this was the case, but he could not explain why. I have not added this concept of distortion to my framework because I believe it is not applicable in very many situations.

F-d/r and F-asym were grouped together in the revised version because they were essentially numerical and graphical interpretations of the same concept. I also added that the student ought to be able to explain why this is the case using object or process definition because often the students would say something like "I don't know why, it just looks that way" or "I remember my teacher told me that once" as justification. I do not feel these explanations suffice for understanding why the logarithm has a finite domain but an infinite range, and thought rather that the students ought to be able to explain this through use of a definition.

F-inv had a minor change, that is, I added the words "graphically and symbolically" to the description of understanding. I added these because I think the idea that one function's input is the other's output and vice versa is a very important idea, and so is the idea that the graphs are reflections about the line $y = x$. Julie provided an example of how confusing logarithms can be without understanding that exponentials and logarithms are graphical inverses. She attempted to draw a picture of $y = 2^x$ and $y = log_2 x$ on the same axes, and she did not realize her graphs were wrong because she did not understand what graphical inverses should look like. For a longer discussion of this instance, see p. 54. On the other hand, Mark provided a good instance of when it might be useful to think of logarithms and exponentials as inverses symbolically. In his simple but elegant proof of the product/sum log rule, you can see between step one and step two he utilized the idea of inverses to rewrite a as $10^{\log(a)}$ and likewise for b. His proof reads as follows:

$$\log(ab) = \log\left(10^{\log(a)} \cdot 10^{\log(b)}\right) = \log\left(10^{\log(a)+\log(b)}\right) = \log(a) + \log(b)$$

I believe both orientations (numerical and symbolic) are useful for many types of problems, and I believe a student would be unsuccessful at some problem types if they only understood one orientation.

In relation to P-elim and P-exp, I mentioned I would revisit the idea of taking the logarithm of both sides of an equation. F-ineq was adjusted and F-1to1 was created to address issues that were not addressed in my initial framework. First, there are two kinds of equations (or inequalities) that involve logarithms. Either they are exponential equations (as in P-exp), which can be solved using logarithms, or they are logarithmic equations such as $2lnx = \ln(2-x)$. F-ineq was redesigned to address the idea that students can solve this sort of equation and interpret their solutions in terms of the original equation, which here would include discarding one solution as extraneous (if solved the traditional algebraic way). On a basic level this involves the notion that exponentials and logarithms are inverse operations, much like multiplication and division are inverse operations. On a deeper level, however, it is important to realize that there is a reason we can take the logarithm of both sides, or raise e to each side of the equation: both the logarithmic function and the exponential function are one-to-one. When solving an equation such as $x^2 = 4$, you may take the square root of both sides and obtain 2 as your answer. However, we are taught to always write ± 2 as our answer. This is because $y = x^2$ is not a one-to-one function, so an adjustment must be used when taking its "inverse." No adjustments like this must be made when using logarithms and exponentials as inverses because both functions are one-to-one. To capture this deeper understanding of why we can do this, I added F-1to1 to the framework.

Contextual understanding revised.

Table 8 shows the changes for the contextual understanding category between the original framework and the revised framework. Three of the four sub-categories in the contextual understanding category remained unchanged. I believe this category (contextual understanding) is qualitatively different from the other three. In the other three categories, I am comfortable referring to each sub-category as a concept. However, in this category, I do not believe the sub-categories could be considered concepts, particularly not the first three. Rather, I might explain them as essential abilities students must acquire in order to have a good understanding of logarithms. As I explain on p. 55 of this document, category four was difficult for me to analyze. However, I found no evidence to suggest that I ought to change the first three sub-categories in this category, and I have justified keeping the second sub-category (C-rec) unchanged already on p. 39.

Table 8
Evidence of Contextual Understanding - Revised

Label	Description of Understanding (Original)	Revised Description
C-reas	The student is able to reason about (and solve) real-world problems involving logarithmic properties, whether or not the problem contains typical characteristics of symbolic logarithm problems such as “logarithm” or “ln.”	Same
C-rec	The student is able to recognize when a real-world problem is most easily solved using a logarithm, whether or not the problem contains typical characteristics of symbolic logarithm problems such as “logarithm” or “ln.”	Same
C-real	The student can explain *why* logarithms are useful in the real world, as opposed to the mathematics classroom (e.g. to change a scale; to solve for exponent variables in exponential relationships).	Same
C-sym	The student is able to relate information given in a real world problem involving logarithms to symbolic notation (for example, they might write an exponential or logarithmic equation to represent information in a problem that could be solved using such an equation).	The student can change flexibly between representations of logarithms (such as from verbal to graphical or from graphical to symbolic).

Only one piece of understanding was changed; C-sym was changed to become broader in what it covered. Originally, I had in mind a kind of “translation” from word problems to symbolic notation. However, I realized as I was analyzing data that this was not the only kind of important "translation." I believe it is very important for a student to be able to switch back and forth between different representations of logarithms, because sometimes a single problem will be understood best if looked at through multiple representations. Sometimes, the representation in which the problem is posed is not the best representation for exploring the problem or presenting a solution. For example, I noticed that sometimes the students solved equations or inequalities involving logarithms by graphing rather than by symbolic manipulation. My only hesitation in the change I made to C-sym was that I was unsure whether understanding C-sym still belonged in category four, since it no longer applies only to contextual problems. In the end, I decided to keep it in category four because that is where it originated from, and also because it didn't seem to fit better anywhere else.

Using the Framework to Assess Student Understanding

In general, the framework was useful for identifying areas in which the students displayed good understanding as well as for identifying gaps in their understanding. Throughout data analysis, I reworded concept descriptions and revised the framework in some places so it would be more helpful in assessing whether the students understood the concept or not. Since one major purpose of creating this framework was to be able to better assess what students do and do not understand about logarithms, I believe it is important to show how the framework can help to assess such understanding. My intent in this section is to present several examples of where the framework helped me identify good understanding and also where it helped me discover holes in understanding. The examples are presented in order by the concept they address, and are organized according to the order in which these concepts appear in my framework. I chose examples of students displaying good understanding as well as displaying a lack of understanding; I also chose examples across the first three categories for variety. Since the examples chosen for this section were used only to illustrate how the framework could be used to assess whether or how a student understands a particular concept, I only included a few examples. The examples I chose were ones I believe illustrate particularly well how the framework aided me in assessing whether and how students understood certain concepts of logarithms. The fourth category is addressed immediately following these examples.

The first example is taken from my first interview with Sarah and shows an area in the framework (O-num) which Sarah did not understand, and reveals a hole in her understanding of logarithms. This example shows that Sarah did not really believe logarithmic expressions were numbers:

> Sarah: So x would equal $\frac{\frac{ln1}{lne}+1}{3}$, and that would give you some decimal, but I'd have to plug it into a calculator to figure out what it was. (After some prodding, she realized she did know the values for $ln1$ and lne.)
> Sarah: (after several minutes of discussion) But if it had different logs, then I wouldn't know how to get terribly specific.
> Interviewer: Would you be satisfied with an answer that looks like $\frac{\frac{ln5}{ln12}+1}{3}$, would you be satisfied with that as an answer?
> Sarah: Depending on why I was doing it. If it was just for homework, then yeah, I would say, close enough.

In this example, Sarah believed the answer to a problem involving logarithms should be a decimal, indicated by the fact that after she had come up with her answer she went on to explain how she would then use a calculator to get a decimal to "figure out what it was." Further, she said the answer, which involved a logarithmic expression, was "close enough," despite it being the *exact* answer to the problem. On the other hand, Holly displayed a very good understanding that logarithmic expressions are numbers in the following example:

> Holly: I would just leave that [$ln5$] and say that $x = \frac{ln5+1}{3}$.
> Interviewer: Why would you leave it as $ln5$?

> Holly: Because that's the most precise answer. I mean I could try to estimate it, the natural log of 5 would be somewhere between 1 and 2, but my estimation I don't think would be very good.
> Interviewer: Is this answer [referring to her answer for x above] a number, as it's written?
> Holly: Yeah, it is a number.

In this next example, Mark used P-def (the process definition of logarithms) quite effectively to find the value of $log_2 4096$.

> Mark: Well I know that 2^{10} is 1024 and 1024 times 4 is that number. So that's 2^{10} times 4 which is 2^2, when we multiply we add so that should be 2^{12} so the answer would be 12.

Mark converted 4096 to 2 raised to some power and gave the power as the answer. This proved to be very effective for finding the value of a logarithm with such a large argument, and shows one instance in which the process definition for logarithms is quite useful.

In the following example, Julie displayed an understanding of the relationship between roots, logarithms, and exponents (P-root) when I asked her if taking the fourth root would help you find log base four of a number:

> Julie: Taking the fourth root, that means that you'd have y^4, but that's not right because you have 4 to the something, so it wouldn't help at all.

Although this explanation is quite short, it shows that Julie has a grasp on P-root from my framework because she realized that taking the fourth root would help her solve for the base if the exponent was four, but would not help her solve for the exponent if the base was four.

In this next example, Sarah displays an understanding of why the domain is restricted for logarithms, (F-d/r in the framework).

> Sarah: [The graph of $y = log_2 x$] won't ever hit x=0, because that would mean that, so if x is 0, then $2^y = 0$, and there's no value of y where that's true. So it's never gonna hit 0 and it's never gonna hit negatives.

It is difficult to tell whether Sarah's thought process involved using the process definition of logarithms, the idea of switching forms (which is addressed later), or the idea of exponentials as inverses of the logarithm. Despite the fact that I could not tell which of these ideas she used to come up with her equation $2^y = 0$, she did have a reasonable explanation for the restriction on the domain in logarithms. Other students sometimes justified the domain restriction with "it looks like that" or "my teacher told me that," so Sarah's understanding of domain restriction is certainly a step above that.

Another example of a gap in understanding comes from when Julie was attempting to graph $y = log_2 x$ and $y = 2^x$ on the same axes. Her first thought was that the graphs were exactly the same (a misconception which I believe stemmed from her replacing the process definition with switching forms, which is explained later). When I asked her to plot by points, she made several graphing mistakes and ended up with a drawing I have reproduced in Figure 2:

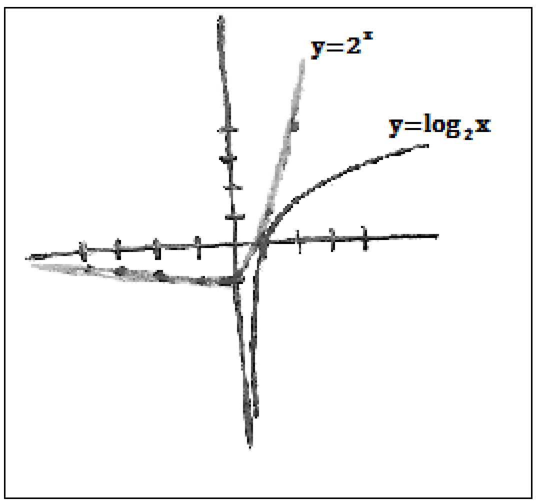

Figure 2. Julie's graphs of $y = log_2 x$ and $y = 2^x$ on the same axes.

When I asked how the two graphs were related, she said that $y = 2^x$ was a reflection of $y = log_2 x$. I asked what she meant by that and she said it was flipped over (indicating diagonally with her hand). Unlike the other three students who answered this question, she did not mention the graphs were inverses. She did not seem to realize that her graph of $y = 2^x$ was wrong. Eventually, with a lot of prompting, she did correct her graph, but she wasn't sure why she had graphed it incorrectly to begin with. To me, this graphing instance shows Julie did not understand F-inv, or what it meant for the two functions to be inverses. It is possible she did not even recall that the two functions *were* inverses. She never mentioned the line $y = x$ or explained the numerical relation of inverse functions, and thus did not see her graph was wrong.

From the examples I have presented I believe it is clear my framework does help in assessing for student understanding of logarithms. I believe that because the tasks in my interviews were designed to get at the different concepts in the framework, the students' solutions revealed many different instances of either good understanding or gaps in understanding. I believe my framework addresses many things that students commonly misunderstand about logarithms and provides a good basis for what students must understand in order to have a good understanding of logarithms. However, I do believe my framework is weak in category four, which I explain in the next section.

The Weakness of Category Four

I mentioned before that I had some problems with category four, which was qualitatively different from the first three categories in that it did not consist of a list of concepts like the other categories did. Although I believe category four (contextual problems) is an important one to include in a framework that describes good understanding of logarithms because I believe the skills it describes are necessary for working with logarithms in a meaningful way, it was difficult to assess whether the way I had written category four was useful. I attribute the problem in assessing the usefulness of category four mainly to the scarcity of applicable problems; the four interviews combined only had three problems that really addressed category four. When designing the interview protocols, I anticipated these three problems would take a significant amount of time, which is why I didn't include more of them. However, two of the problems (interview 1 question 5 and interview 2 question 6) took the students very little time and were not dwelt on for long.

Interview 1, question 5, reads in part, “given the following sequence: 3, 9, 27, 81, … where 3 is the first term, 9 is the second term, etc., how could you find out which term of the sequence is 1594323?” I did discover on this question that Holly, who showed a method for solving the problem by repeated division, could not recognize that the problem could be solved with logarithms even when prompted for other solution methods. The other three students almost immediately recognized that you should just take log base 3 of 1594323. Interview 2, question 6, reads “a google is 1 with 100 zeros after it. What is the logarithm (base 10) of a google? Explain your thinking. Would the natural log of a google be bigger or smaller than that? Explain your thinking.” All four students solved this problem correctly and extremely quickly. The question from interview 1 was the only contextual, or story problem, which did not include any hints that we were using logarithms. Because it was the only problem in my study without a verbal or symbolic clue that you should use logarithms, and only one student did not use logarithms to solve it, I am hesitant to say the problem provides sufficient evidence to justify keeping C-rec in my framework. However, I likewise did not find any reason to remove it from my framework, and I still find the data from Kastberg (2002) to be a compelling reason to leave it in. Also, the question from interview two seemed unhelpful because when I noticed the ease with which the students solved it, I realized the problem was really not very contextualized and might actually be taken as confirming evidence for P-def (the process definition of logarithms) rather than for C-reas, the ability to reason about and solve contextual problems involving logarithms.

Thus, the only really meaningful discussion on category four in the interviews focused on problem 9 from interview 4, and focused almost completely on logarithmic scales. All four students seemed to grasp the idea that using a logarithmic scale means that for every increase by one on the scale the original measurement is increased by a factor of the base of the logarithm (that is, going from 2 to 3 on the Richter scale means the original measurements are 10 times larger). Their reasons for why you might use a logarithmic scale varied, but mainly focused around the argument "it's easier" or the argument "the data just looks logarithmic."

The main reason I created category four was to address the issue of being able to determine when a logarithm would be helpful in solving a problem, as well as the issue of being able to successfully solve contextual problems involving logarithms. I believe that to adequately test whether category four of my framework is helpful in determining this, a study would have to be done where more time is spent on contextual problems involving logarithms. It might also be helpful to not inform the students that the study is about logarithms or give clues by asking them to solve many logarithm problems before giving them a contextual problem without the word logarithm in it, but that logarithms would be helpful in solving.

Overall, I believe category four is an important one, but might need to be revised after further study. Because of my lack of data, I am unsure whether the category needs revising, or if it just needs to be tested with different tasks. Although I only presented a few examples of how the other three categories could be used to assess for students understanding of logarithms, there was a vast number of examples from the first three categories which displayed either good understanding or a lack of understanding. There were very few examples from category four, which leads me to believe category four requires further study.

The Process Meaning vs. Switching Forms

In this section I first explain what I mean by switching forms and how it differs from the process meaning of logarithms. After that, I describe how a student can become dependent on switching forms and thus fail to grasp the process meaning of logarithms; the student may not see a difference between the two ideas. Last, I explain why relying on switching forms rather than a process understanding can hinder students understanding of logarithms.

The Difference between Process Meaning and Switching Forms

The process meaning for logarithms is that a logarithm takes the argument, converts it to the base raised to some power, and gives the power as the answer. For example, if you were to use the process meaning of logarithms to simplify $log_2 16$, you would rewrite 16 as 2 to some power and take the power as the answer. In symbolic notation, you might write $log_2 2^4$ and then realize that 4 is your answer.

Switching forms can look a lot like the process meaning for logarithms. A typical diagram for how to switch forms between a logarithm and an exponential is pictured in Figure 3.

$$log_2 x = y \quad \textbf{means} \quad 2^y = x$$

Figure 3. A diagram of how to switch from log form to exponential form.

If we were to take the same example as before, a student using the switching forms method would write $log_2 16 = x$, followed by $2^x = 16$, and then come up with an answer of 4 because 16 is 2^4.

It may be difficult at first to differentiate the process meaning of logarithms from the switching forms method. The main difference between these two ideas is that the process meaning for logarithms is a meaning of logarithms, while the switching forms essentially takes logarithms out of the problem. Instead, using switching forms allows the student to change a logarithmic problem into an exponential one, and thus allows them to avoid developing meaning for logarithms. When teaching a unit on logarithms, it is typical for the idea of switching forms to be a main idea that is taught near the beginning of the unit. I believe the practice of switching forms promotes far less understanding of logarithms than the process meaning of logarithms, particularly because it often becomes little more than a pattern to follow (as I tried to illustrate with the arrows in Figure 3). I believe that if switching forms is the main practice for students, the word logarithm becomes little more than a command to change the problem into an exponential one.

A Student May Become Too Reliant on Switching Forms

In the class from which I drew my four interview subjects, the professor was trying to teach the object meaning for logarithms and the process meaning for logarithms. In my interviews, I could see that for two of the four students, the process meaning never really fully developed because the students consistently used switching forms instead. Although both would occasionally parrot word for word what they had been taught in class as the process meaning for logarithms, in practice they would often convert logarithmic expressions into an exponential expression and avoid logarithms altogether. For example, in interview, Sarah simplified $log_2 16$ by writing $2^x = 16$, followed by $2^2 = 4$, $2^3 = 8$, and $2^4 = 16$, then gave four as her answer. When I asked her how she had thought about it, she said "I took this number [2], raised it to this [x], and set it equal to that [16]. [As she talked, she drew the arrows as in Figure 3.] So it's not really thinking about anything." Essentially, she showed me that she had learned a pattern which allowed her not to think about logarithms at all. However, Sarah often compensated for her lack of understanding the process meaning by using graphs in creative ways to answer questions. Julie did not appear to have this flexibility, and so her understanding of logarithms suffered more than Sarah's, as I explain in the next section of this document.

In addition to Sarah's assertion that she isn't really thinking about anything (when switching forms), here are a few examples to show that Julie was heavily dependent on switching forms. One telling remark was her response to my question in interview three, which asked what she thought about the idea that taking a logarithm eliminates the base of an exponential expression. She said, "I don't think it's eliminating it, I think it's just putting it in a different form so you can find what you're looking for." In her explanation, she circled the different parts of an equation and used hand gestures (rather than the arrows in Figure 3) to indicate the pattern for switching forms.

Another example comes from Julie's fourth interview. When I asked her to show me how she would find out what $log_2 16$ is, she had to rewrite it as an equation to show me what she would do. She wrote $log_2 16 = N$, then switched forms to exponential by writing $2^N = 16$, figured out that N was 4, revised her original equation to read $log_2 16 = N = 4$. I prompted her for alternate ways to find it, but she did not come up with the process meaning, so I presented it to her as I did earlier in this section, with the key step being to write $log_2 2^4$ and take 4 as the answer. She said that although it was written differently, it was really the same as her process meaning (which was really switching forms). She could not distinguish a difference between switching forms and process meaning for logarithms. I believe she had used the method of switching forms so many times that it was ingrained too deeply in her for her to replace it with the process meaning of logarithms.

In another example, from interview three with Julie, I asked her a series of inequalities problems, such as $log_2 x < 0$ (solve for x). When she had previously been presented with equations, she did just fine by using switching forms, but inequalities did not translate as well using her switching forms method and she became confused. For $log_2 x < 0$, she initially wrote

$2^0 <$, but erased it before she went any further. Once she realized she could not simply switch forms because there was no equals sign, she tried to use the object meaning to answer the question. She said "this [$log_2 x$] is the exponent you have to raise something to, to get something less than 0" and wrote $2^{log_2 x} < 0$. She asked if it was okay for there to be no solution. Finally, I asked guiding questions until she arrived at the answer by thinking about x as a fraction.

Replacing the Process Meaning of Logarithms with Switching Forms May be Harmful

I believe it may be harmful for students to become dependent on switching forms. While the object meaning for logarithms is important for understanding logarithms and is useful in many situations, the process meaning for logarithms is also vital and is often useful in situations where the object meaning for logarithms is not helpful. However, I believe if a student becomes too dependent on switching forms, they may use switching forms instead of the process meaning for logarithms and never fully grasp the process meaning for logarithms. This would mean that switching forms fills the hole in understanding logarithms that is created by not understanding the process meaning, but does not serve the student as well as the process meaning does. My assumption here is based on what happened with Julie. Even though she was *explicitly taught* the process meaning for logarithms both in class and in our interviews, Julie continued to rely on switching forms instead of the process meaning for logarithms. I believe this dependence on switching forms may have caused some major gaps in Julie's understanding.

Recall the example from figure 2 of Julie's graphs of $y = log_2 x$ and $y = 2^x$ on the same axes. I wrote that Julie first believed the two graphs were identical. At first I was astounded that she could think these two graphs would look identical, until I considered the fact that she relied so heavily on switching forms. If you consider that every time she was confronted with something like $log_2 x$, she converted it to an *equivalent* form by raising 2 to some power, the idea that $y = log_2 x$ and $y = 2^x$ might be equivalent does not seem so far-fetched. When she did realize the graphs were different, she graphed $y = 2^x$ incorrectly, and that didn't bother her because although by that point she had remembered the graphs were inverses, she didn't seem to actually know what that meant. I can see how knowing that logarithms and exponentials are inverses might be confusing if every time you are presented with a logarithm, you change it into an exponential expression. It seems rather confusing that something that is considered equivalent to a logarithm can also be considered its inverse. Of course, there really *are* exponential equations that are equivalent to logarithmic ones (like $y = 2^x$ is equivalent to $log_2 y = x$), and yet exponentials are inverses of logarithms, but it does make sense that this could be confusing.

One more example of Julie's problems that I attribute to her switching forms is an example from my last interview with Julie. Her task was to simplify the following expression: $log_2 3 \cdot log_3 4 \cdot log_4 5 \cdot log_5 6 \cdots log_{63} 64$. After some time and a few nudges in the right direction, Julie gave as her answer $log_2 64$. I asked her if she could find out what that was, and she said "sure!" It didn't take her very long to discover that the answer was 6, but I think she did not initially do so because it was not an equation. In order to come to the conclusion that the answer was 6, she wrote an equation and switched forms. I think because there was no equal sign in the problem, it did not occur to her to switch forms immediately, and so it did not occur to her that the answer might simply be a number. Also, in another simplification task, she did get a numerical answer and commented, "I don't know if I was supposed to solve it, though." The idea of a logarithmic expression without an equal sign becoming a number seemed to be somewhat confusing to her because she could not switch forms.

Although I did not begin my study with the difference between switching forms and the process meaning for logarithms in mind, I believe it is an important finding. I had not initially included switching forms in my framework because it did not occur to me that students might consider that to be a way of understanding logarithms. I don't believe switching forms (especially when it becomes just a pattern) really helps students understand logarithms; it just helps them to avoid logarithms. Further, I believe it may prevent them from understanding the process meaning of logarithms and lead to confusion when they cannot rely on switching forms.

Beyond the Framework: Exceptional Understanding

In this section, I discuss the capacity of my framework. That is, my framework serves to separate good understanding from a lack of understanding, but does not necessarily help separate good understanding from exceptionally good understanding. In this section, I first discuss the one subject (Mark) of my study who led me to discover that the difference between good understanding and exceptional understanding cannot be captured by my framework. Next, I explain my theory of why Mark's understanding of logarithms was so exceptionally good. Finally, I revisit the intent of my framework and explain why it is acceptable that my framework does not capture the difference between good understanding of logarithms and exceptional understanding (like Mark's).

Mark's Exceptional Understanding

I first became aware of Mark's exceptional understanding during my first interview with him. In class, the students had discussed the idea of "pulling out" the base. Mark explained this idea using $log_4 72$. He began factoring out fours, ending with $(4)(4)(4)(1.125)$ or $4^3(1.125)$. He said he liked how this method showed the exponent, and that it would get you pretty close to the actual answer. I then brought up another suggestion from a class member, that is, to divide the remainder by the base for the "part after the decimal", so the estimate of $log_4 72$ would be $3 + \frac{1.125}{4}$. He said he liked that method of estimating, but was the only student to explain why it provided a reasonable estimate. He drew the graphs of $y = log_4 x$ and $y = x/4$ on the same axis, as pictured in figure 4. He then explained that the two functions are really close between 1 and the base of the logarithm (in this case 4), but the closer the remainder gets to 1, the worse the approximation gets.

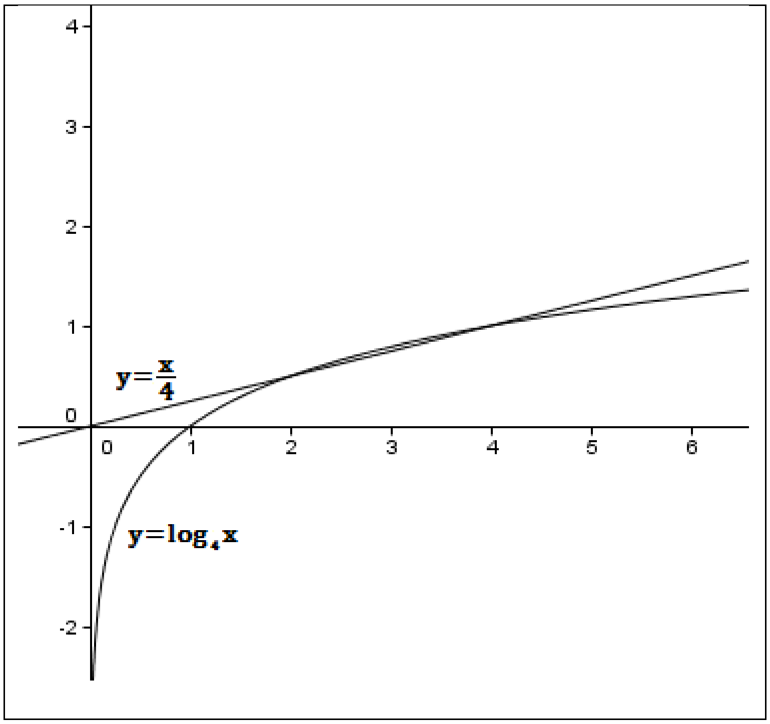

Figure 4. Mark's graph of why dividing the remainder by four gives a good estimate.

Mark's classmates did not understand why dividing the remainder by four provided a good estimate. His spontaneous use of functions and graphs to explain why this process provided a good estimate showed a deeper understanding than my framework describes. Although I did adjust the C-sym cell of my framework to read "the student can change flexibly between representations of logarithms (such as from verbal to graphical or from graphical to symbolic)" to better capture what I intended in the original framework, I believe this example goes beyond even this rewording of C-sym. My reasoning is that he did not simply translate a word problem into an algebraic manipulation problem or an algebraic manipulation problem into a graph, but rather he thought about the problem in a completely different way. Instead of focusing on the action of dividing by four, he thought about how the two functions were related, particularly around a specific range of values.

Another case where Mark explained something that showed his understanding was above and beyond what my framework encompassed was in interview 3 when he was analyzing the inequality $2^x < 3^x$. He said you could take the log of both sides if you wanted because the log function is increasing and monotonic. I, myself, didn't understand what he meant by that remark until I was analyzing data. To get an idea of what would happen if you were to apply a non-monotonic function to both sides of the inequality, I plotted $y = \sin(2^x)$ and $y = \sin(3^x)$ to compare to the original inequality graph. The two graphs are pictured in figure 5.

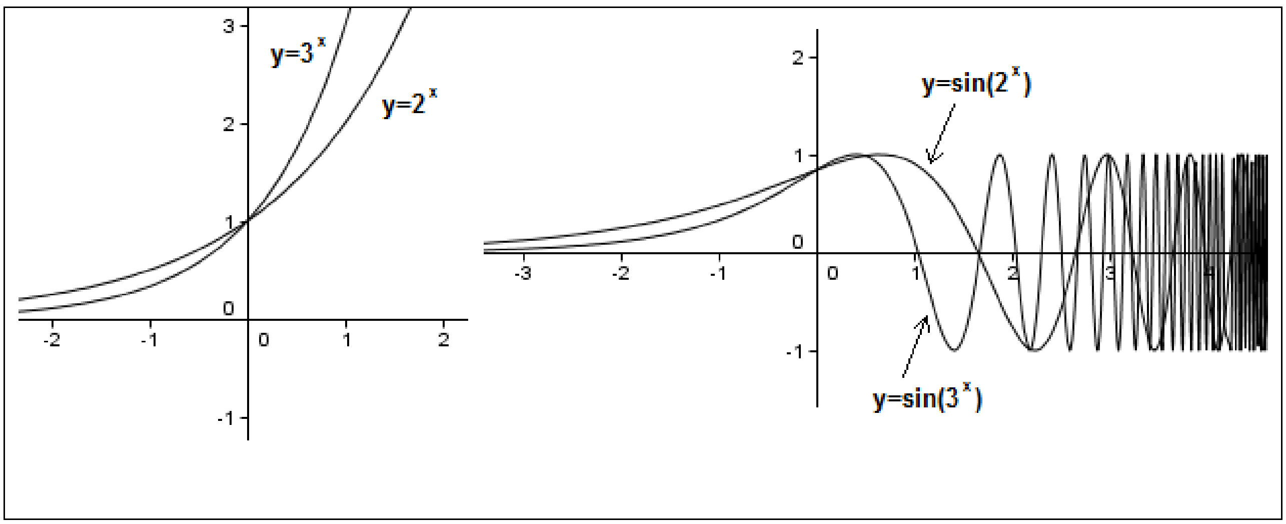

Figure 5. A graphical comparison of $2^x < 3^x$ with $\sin(2^x) < \sin(3^x)$.

As you can see, the solution to the inequality does not remain the same if you apply the sine function to both sides. Further, applying a decreasing function to both sides would have reversed the inequality. Although this kind of an exploration might be interesting for a class, I would not consider it part of the standard knowledge that anyone should have if they are to have a good understanding of logarithms. I did add into my framework the idea that the logarithm graph is one-to-one as an important concept because of the way it relates for solving quadratic equations using plus and minus, and why we don't have to do that with logarithms. This idea, however, seemed somewhat more tangential and interesting rather than a core concept of logarithms.

Besides the two instances provided, Mark displayed exception understanding of logarithms and how logarithms relate to other mathematics in many instances. In general, he solved most problems faster than any of the other students, and could solve the problems in more ways than the other students. He was also exceptionally good at providing justifications for his solutions. He frequently checked his answers without being prompted, both to see if the answer he gave answered the question that was asked, and to see if his answer seemed reasonable. For example, when solving the equation $2lnx = \ln(2-x)$ he factored the equation $x^2 + x - 2 = 0$ to obtain the answers $x = -2$ and $x = 1$. Before I could ask him what those answers meant in terms of the original problem, however, he eliminated $x = -2$ as not a solution. Watching Mark solve problems was an interesting experience, because I sometimes felt he knew more even than I did about logarithms, despite the fact that I was studying them in depth.

A Theory of Why Mark's Understanding Was Exceptional

Based on my observations of Mark both while I was interviewing him and then again as I watched and re-watched the videos of our interviews, I believe I have identified some reasons Mark's understanding of logarithms was so exceptionally good. I believe his exceptional understanding stemmed from his own internal interest in mathematics, his strong problem solving abilities, his ability to connect other topics in mathematics to logarithms, and his experience as a tutor.

There were times during our interviews that it became clear Mark thought about math as a hobby, not just for his classes. When he was explaining how he solved a problem, he mentioned he knew that $2^{10} = 1024$. I asked his if he knew this because of computers. He said no, he knew it because he liked to estimate things, and for estimating it is good to know that $10^3 \approx 2^{10}$. He said this was just a hobby of his. There were also other times during our interviews when he would talk about some sort of math concept he had been thinking about during his free time. It sounded like he frequently thought about math concepts outside of homework and class – just for his own enjoyment. I believe that the fact that he found math interesting and enjoyable is part of what allowed him to develop such an extremely good understanding of logarithms. I am sure his good understanding does not relate only to logarithms, but probably to most areas of mathematics that he has studied.

I think another reason Mark was so extraordinarily capable of solving any problem I presented to him was because he had good problem solving skills in general. Although problem solving skills are important to have in order to understand any area of mathematics, it would not make sense to include "general problem solving skills" in my framework because my framework focuses specifically on logarithms, and does not extend to other abilities or prerequisite knowledge.

Mark also displayed a tendency to make connections between different areas of mathematics. He would frequently connect whatever problem we were dealing with to several areas of mathematics. Often, these areas would be obviously related to logarithms, like division, exponents, etc. but sometimes he would bring up things that connected more obscurely, like limits or monotonicity. I believe different aspects of mathematics are so connected in his mind that it allows him to understand concepts like logarithms in a way that relates to mathematics as a whole and to other mathematical topics.

I also believe one thing that set Mark apart as having exceptional understanding of logarithms was his experience with them. Although all the students in my interviews had fairly extensive experience with logarithms, having seen them at least in precalculus and calculus, Mark revealed in his first interview that he had been a math tutor for several years, starting in high school. I know he sometimes helped students understand logarithms in his tutoring experience, which meant he had most likely spent a lot more time thinking about logarithms than any of the other students I interviewed. Further, I believe people gain a greater understanding for a concept if they teach it to someone else. Since he had taught logarithms before through his tutoring experience, I imagine that helped him develop a deeper understanding of logarithms.

Revisiting the Intent of My Framework

Although my framework was useful to distinguish that Mark had a good understanding of logarithms, it does not show that Mark's understanding is actually beyond what is considered "good understanding" according to my framework. As you have just read, I had to create my own theory as to why Mark's understanding of logarithms was so exceptionally good. However, the intent of my framework was to identify many of the essential ideas and understandings associated with logarithms, and it may not identify the many and varied possible connections between these concepts and other mathematical concepts. Of course, while this would be valuable to explore, it is beyond the scope of this study. Also, as I have already mentioned, there are certain skills (like problem solving skills) that help a person to better understand logarithms, but do not belong in my framework because they are too general. My framework is restricted to understanding logarithms, so general skills that help in all areas of mathematics do not belong in the framework. Thus, although my framework did not help distinguish that Mark's understanding was exceptional, it did what it was designed to do, that is, identified that he had good understanding (as opposed to inadequate understanding).

CHAPTER VI: DISCUSSION AND CONCLUSIONS

In this concluding chapter of my thesis, I summarize the findings of my study, explain the pedagogical and theoretical implications of my study, discuss the limitations of my study, and lay out some suggestions for further research in the area of what it means to understand logarithms.

Summary of Contributions

Recall that prior to this study, limited research had been done about what it means to understand logarithms. The little research that had been done about students understanding logarithms explored some concepts students clearly did not understand about logarithms, but did not offer a research-based description of what understanding logarithms ought to look like. The main findings of my study relate to the creation and refinement of a framework to describe what understanding logarithms should look like, the idea of switching forms, and the usefulness of my framework.

The main purpose of my study was to create a research-based framework of what it means to understand logarithms. As a result of analyzing the data I collected during my study, I was able to refine the initial framework I had outlined prior to data collection. I reworded a few of the concepts in category one (logarithms as objects) and category two (logarithms as processes), for clarity and conciseness. I also made a few minor additions to the first two categories, such as adding repeated multiplication to P-div (the concept of logarithms as repeated division). Category three (understanding logarithms as functions) underwent a great deal of changes, not only for clarity but also for content. Within the category, some concepts were reworded, two concepts were combined, and two concepts were added. One noteworthy change was the addition of the concept that the logarithm function is one-to-one, which is the reason we can take the logarithm of both sides of an equation and have the answer remain the same. Category four (logarithms in contextual problems) underwent only one change and was under-researched in my study as previously described. The only change in category four was to address the idea that simply translating a word problem into a symbolic representation was too narrow, so C-sym was changed to include changing representations of logarithms in many ways (verbal to symbolic, symbolic to graphical, etc.). Given that my framework was partially based on a framework by Berezovski and Zazkis (2006) where category one and two were briefly explained, category three was only mentioned, and category four was not yet thought of, my study represents a substantial contribution to the literature.

One unintentional finding from my study was the difference between the practice of switching forms (logarithms to exponentials) and the process meaning for logarithms. While switching forms is a practice frequently taught when logarithms are introduced to students, I believe it can become a crutch students depend on instead of developing a process meaning for logarithms. Further, students can use switching forms to avoid thinking about logarithms by changing logarithm problems to exponential ones whenever possible. My study has shown a few holes in understanding which are likely related to students relying too heavily on switching forms. Thus, I conclude that it may not be helpful for students to depend on the practice of switching forms as a primary way to deal with logarithms. Instead, I would suggest that students depend primarily upon the object and process definitions for logarithms as their primary ways to deal with logarithms.

Another finding from my study is that my framework was useful for assessing which concepts within logarithms students do not understand well and which concepts they do understand well. I believe the breakdown of the categories (particularly the first three) allows us to pinpoint where a student's understanding breaks down. The framework also allows us to find out where students have strong understanding.

On a related topic, the framework is not particularly useful for distinguishing between good understanding of logarithms and exceptionally good understanding of logarithms (like Mark's). However, I believe Mark's exceptional understanding stemmed primarily from the mathematical connections he had made between the concepts within logarithms and also between logarithms and other mathematics. His genuine interest in mathematics, his problem solving skills, and his experience as a mathematics tutor may also have enhanced his ability to work with logarithms so well. Although my framework is not suitable to differentiate between good understanding of logarithms and exceptional understanding (like Mark's), altering the framework to be able to make this distinction would require that the framework extends beyond logarithms in scope. Although having a framework that explores connections within logarithms as well as between logarithms and other mathematics could be useful, it is beyond the scope of this study.

Implications

I have divided this implications section into two subsections. The first subsection describes the pedagogical implications of my study for those who intend to teach logarithms to students. The second subsection describes the theoretical implications for mathematics education research as a whole and particularly research pertaining to understanding logarithms.

Pedagogical Implications

One way my study can have a positive impact for those teaching logarithms is that it provides a framework that could serve as a planning guide and formative/summative assessment guide. Depending on the level of students and time to teach logarithms, a teacher may decide not to attempt to teach all of the aspects of my framework. However, my framework does provide a detailed list of concepts for understanding logarithms that a teacher may consider before planning to teach a unit on logarithms. When using formative assessment methods, a teacher might refer to my framework to find out which concepts her students are struggling with and which concepts they understand well. Also, my framework could be used to ensure summative assessments are balanced and include a wide variety of problems to test for understanding of many different concepts of logarithms. Using my framework as a guide for instruction may help to reduce emphasis on memorizing rules and place more emphasis on being flexible by thinking about logarithms in many different ways.

Another important implication for teaching logarithms which resulted from my study is that a focus on switching forms from logarithmic expression/equations to exponential may not be the most helpful way to teach students about logarithms. Although the practice of switching forms can be helpful for solving many problems, other ways of thinking about logarithms (like the object and process meanings for logarithms) may promote deeper understanding. If students rely too heavily on switching forms, they may not be flexible in the ways they think about logarithms. In fact, they may use switching forms to avoid thinking about logarithms, since switching forms allows them to change logarithm problems into exponential ones.

Theoretical Implications

The main contribution of this research is a research-based framework for what it means to have a good understanding of logarithms. Recall from chapter one of this document (p. 1) that there has been work by researchers such as Carpenter, Fennema, Franke, Levi, & Empson (1999) to explain what it means for students to understand more elementary mathematical concepts like addition, subtraction, multiplication, and division. There is less work generally in what it means for students to understand secondary level mathematics concepts. Also, previous to my study, there was no research-based theory of what it means for students to understand logarithms. Berezovski and Zazskis (2006) did use a framework to assess students understanding of logarithms, but there was no indication that their framework was the result of research and they did not refine it according to their data (since that was not their purpose).

As noted, many secondary level mathematics concepts lack research in the area of what it means for students to understand them. Another theoretical implication of my research is that this study could be used as a template for future research to focus on other areas of secondary or higher level mathematical concepts. The pattern I used for researching what it means to understand logarithms could be implemented for research on topics such as mathematical induction, combinations and permutations, or trigonometric functions.

Limitations

I have already discussed that there was insufficient time spent examining category four of my framework (logarithms in contextual problems); this was one of the main limitations of my study. Other limitations were that my framework was tested on only a very small group of students, and also that the group was fairly homogenous. In the following paragraphs, I address each of these limitations.

As I have already described in the results chapter, I believe my study was insufficient to fully analyze category four of my framework, because I overestimated the length of time the students would spend on each of the few contextualized problems I included in my interviews, and did not include enough contextualized problems, and very little time was spent examining category four in this study. As a result, category four was not revised much and may not be the best indicator of what it means to understand logarithms in contextual or “real-world” situations. While I believe the other three categories of my framework describe well what it means to understand logarithms within their respective categories, I believe category four is still in a preliminary stage and requires further research.

It should also be noted that my study included only a small group of students. Choosing only four interview subjects allowed me to study each person's understanding in depth, but it also limited my findings to four (albeit bright and insightful) students' conceptions of logarithms.

Similarly, the group of students I chose for my study was fairly homogenous. All four students were attending the same university, had the same major, were enrolled in the same class, and were traditional-aged Caucasian college students. It is possible this may have limited my findings, and having a more varied group of interview subjects might provide additional information on what it means to understand logarithms. However, having a fairly homogeneous group of interview subject was also helpful for my study. Attending class with all four subjects of my study allowed me to bring insights from class into the interviews that they were already familiar with and had thought about previous to the interviews.

Future Research

I would like to conclude my thesis with some suggestions for future research related to understanding logarithms. A study focusing more or entirely on category four (contextual problems) would be helpful in explicating what it means to understand logarithms as they relate to contextual problems. I believe the first three categories of my framework have been refined fairly well, and a study focusing on category four would add significantly to the theory of what it means to understand logarithms.

Since my study revolved around a fairly homogeneous group of students, another possibility for future research is to test the framework with different subjects. Studies with different groups of interview subjects would help test the usefulness of the framework with people who are not preservice mathematics teachers. Some examples of possible interview subjects might be students who are first learning about logarithms, chemistry students, mathematics majors, mathematicians, engineers, or mathematics teachers. This could be particularly interesting because my framework is useful only for differentiating between inadequate and good understanding, but could conceivably be extended to also differentiate between good and exceptional understanding (like Mark's). In order to extend the framework in this way, it would be necessary to use expert subjects.

Research on what it means to understand logarithms in particular and secondary mathematical topics more generally, is far from exhausted. While mathematics education in many places is trying to form standards based upon understanding mathematics, it is important to keep researching what it means to understand the various topics in mathematics that we teach in schools. There is a continual need for studies like this one to help establish what students ought to be learning in their mathematics classes. The main purpose of my study was to establish what students should understand about logarithms, which I fulfilled by creating a research-based framework that describes what understanding logarithms should look like.

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APPENDIX A: SURVEY

Logarithms Survey
Directions: Please answer all of the questions in the order they are given. Directions are given throughout the questionnaire in **bold** face font, questions are given in *italics*, and options are given in regular font. If you need more room, write on the back of the survey. Please write thorough descriptions, where applicable.

1. *What is a logarithm?*
   (Please write in)
   ____
   ____
   ____
   ____

2. *Are you willing to be interviewed (if chosen) for a research project which explores how students think about logarithms?*
   (Please circle one)

   No Yes

   2b. If yes, please provide your email address so that you may be contacted
   Email:____________________________________

3. *How would you know if a logarithm would be useful in a real life situation?*
   (Please write in)
   ____
   ____
   ____

4. *Do you know any real life situations in which logarithms are used?*
   (Please write in)
   ____
   ____

   5b. *Why do you think people use logarithms in these situations?*
   (Please write in)
   ____
   ____
   ____

5. *How are logarithms related to exponents?*
   (Please write in)
   ____
   ____
   ____

6. *What logarithm rules do you remember?*
(Please write in)

7b. *Choose one of the above and justify why it works.*
(Please write in)

7. *What does it mean to "take the logarithm"?*
(Please write in)

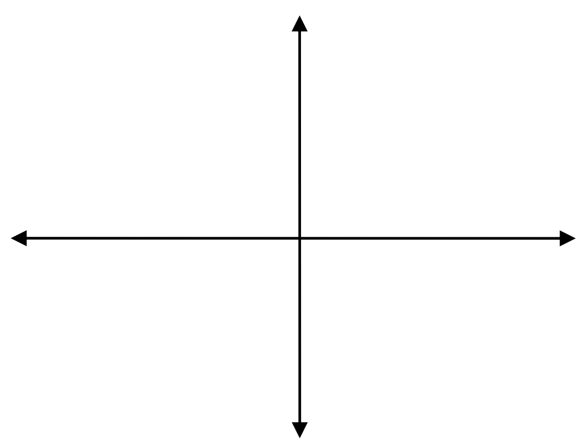
8. *What is the domain of the function* $y = log_3(x)$*?*
(Please write in)

6b. *Explain how you know that is the domain.*

9. *What is the range of the function* $y = log_3(x)$*?*
(Please write in)

6b. *Explain how you know that is the range.*

10. *Please graph the function* $y = log_2(x)$.
(Please show all work and provide labels as appropriate)



APPENDIX B: INTERVIEW PROTOCOLS

Interview 1
Introduce yourself, briefly explain the interviews, ask for "thinking out loud" and explanations, explain that wrong answers are okay.

(1) Introductory questions: What experience do you have with logarithms? When did you first see them? Next see them? Do you think you are good at logarithms? Why or why not? When you think of logarithms, what comes to mind? What topics in mathematics are logarithms related to?

(2) What did you think of the five possible process meanings for logarithms we discussed in class? (This question added as a result of in-class discussion)
(Convert to exponent, repeated division, repeated rooting, finding xth root, repeated multiplication, what were they?)
(P-def, P-div, P-root)

(3) What is $log_2 16$? How do you know? What does the 2 mean? The 16? Is $\frac{1}{2} log_2 16$ equal to $log_2 8$? Why or why not? What is it equal to? Why do you think that works? What does the expression $log_2 17$ mean?
(O-def, O-num, O-rule, O-part, P-def)

(4) Solve for x: $e^{3x-1} = 1$. Explain all your steps. Ok, now what if instead of 1 it was a 5, as in $e^{2x} = 5$? Can you find the exact answer? An approximate decimal answer? Can you interpret what your answer means in terms of the original equation? Could you write an equation that produces an x-value that is half of the one you found?
(P-elim, P-exp, O-def, P-def, O-num, O-not)

(5) Given the following sequence: 3, 9, 27, 81, …
Where 3 is the first term, 9 is the second term, etc., how could you find out which term of the sequence is 1594323? Describe the process you would use and explain why you would do that. Another way? Another? Follow up (if they use logs to solve it): how did you know to use a log?
(Adapted from Kastberg, 2002)
(O-def/Pdef?, P-div, C-reas, C-rec, C-symb)

(6) What is the value of $log_2 4096$, why, and what does it mean? How about $log_2 400$? Is there any way I could use addition to find this value? Subtraction? Multiplication? Division? Square roots, or other roots? What else?
(O-def or P-def, O-num, P-est, P-div, P-root)

(7) e, π, and $\sqrt{2}$ are some irrational numbers. Are the outputs of logarithmic functions irrational numbers? Always, sometimes, or never? Can you write a logarithmic expression that is equal to a whole number, a fraction, an irrational number. Why did you use this base? Is there another expression you could write that would be exactly equal to this one (irrational one)?
(O-def, O-num, O-not, O-part)

Interview 2

Let the interview subject know that there is a graphing calculator available for use but that they must ask permission to use it and I may ask them to try without it.

(1) Introductory questions: What have you learned about logarithms lately? Can you think of more than one way to define logarithms? How many can you think of? What are they? Explain.
(O-def, P-def)

(2) Can you graph for me $y = log_2 x$? What labels can you put on there? Can you graph for me $y = \ln(x)$? How about $y = 2^x$? How are these graphs related? (If they initially aren't sure what the graphs look like, ask them if there is any way they could find out.)
(F-graph, F-inv)

(3) How can you tell if a logarithm is a function or not? So is it a function?
(F-def)

(4) Consider the graph of $y = log_2 x$. How many times does this graph intersect with…
$y = c$ where c is any constant? How do you know?
$x = k$ where k is any constant? How do you know?
$y = mx + b$ where m and b are constants? If I gave you a slope, could you make the line intersect 0 times, once, or twice? (If they don't consider this) what if the slope was negative?
$y = 2^x$? How do you know?
$y = \sqrt{x}$? How do you know? (Or how could you find out?)
$y = \log_4 x$? How do you know? Would this one be higher or lower than $y = log_2 x$?
(F-def, F-graph, F-d/r, F-inv, P-root)

(5) Can you talk to me about the domain and range of the log functions? Do you know why that is the domain and why that is the range? Are there any holes or asymptotes (vertical, horizontal, slant)? How do you know?
(F-d/r, F-asym)

(6) A google is 1 with 100 zeros after it. What is the logarithm (base 10) of a google? Explain your thinking. Would the natural log of a google be bigger or smaller than that? Explain your thinking.
(C-reas, C-rec, C-symb)

(7) Do you know what a logarithmic scale is? (If no, bring up Richter scale). How does a logarithmic scale work? Can you think of a reason why we would use a logarithmic scale in real life? Can you think of anything in real life that logarithms are useful for? How about besides a logarithmic scale?
(C-real)

(8) Concluding question: did you think this interview was very different from the last one? (If so, how?)

Interview 3

(1) What are all the rules you remember about logarithms? How would you know to use these rules? Can you justify why they work? (If they don't come up with all the rules) What about the change of base rule? Etc.
(O-rule, O-def, O-not, O-part, P-def)

(2) If a problem didn't have the word "logarithm" in it, how would you know if you need to use logarithms to solve the problem? Are there any other ways you would know to use a logarithm?
(C-rec)

(3) Some people think of taking a logarithm as eliminating the base of an expression. What do you think of that? (Demonstrate on board what this means)
(P-elim)

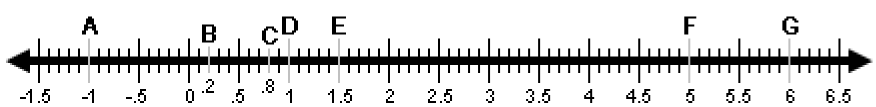
Find the solution set for each of the following inequalities. Explain as you go.

(a) $log_2 x < 0$
(b) $log_{10} x \geq -1$
(c) $2 < log_3 x < 3$
(d) $-2 \leq log_{10} x \leq -1$
(e) $2^x > 10$
(f) $2^x < 3^x$

(O-def, P-elim, P-exp, F-ineq, others?)

(4) Can you solve this for x, explaining as you go?
$2\ln(x) = \ln(3x - 4)$
Explain your answer. What does it mean in terms of the original problem? *Note: This question was written wrong. It was rewritten for some of the interviews to give an extraneous solution, which I initially intended it to have (one solution, one extraneous).
(O-rule, F-ineq)

(5) Using only whole number bases, write logarithmic expressions (try to use just one logarithm) exactly equal to the values on the number line below corresponding to A-G. Could you do more than one for each? Can you use rules of logarithms to write them?



(O-def, P-def, O-not, O-part)

(6) Explain the difference between solving $x^y = a$ for x and for y.
(P-root)

Interview 4

(1) How could you find out what $log_2 16$ is? Could you do it another way? Another way? Can you do it using an object conception of logs? Can you do it using a process conception of logs? Someone did it this way, what meaning do you think they were using? They said they were using the process conception. Were they?
(Write this as the "way": $log_2 16 = log_2 2^4 = 4$)

(O-def, P-def, P-div)

(2) If $log_9 7 = A, log_9 4 = B, log_9 10 = C$, find $log_9 \frac{1}{16}$ and $log_9 810$.
(O-rule, O-def, P-def)

(3) Simplify: $2log_3 7 - 4log_3 49 + log 10 + 5log_3 6$
(O-rule, O-def, O-part, P-def)

(4) Simplify: $log \frac{1}{2} + log \frac{2}{3} + log \frac{3}{4} + \cdots + log \frac{98}{99} + log \frac{99}{100}$

$log_2 3 \cdot log_3 4 \cdot log_4 5 \cdot log_5 6 \cdots log_{63} 64$
(O-rule, O-def, P-def)

(5) In class, we've been talking about compacting and stretching. What does that mean to you?

(6) In class, we've also been talking about distortion and intervals. What does that mean to you?
*Questions 5 and 6 were added as a result of in-class discussion. The ideas contained in these questions were not in my original framework but were inserted later.

(7) Suppose we were looking at a graph of a logarithm. If I zoomed in really close onto one part of the graph, would it look the same as if I zoomed in on another portion? Would it look the same as the original graph? Justify your response.
(F-Graph)

(8) If they got the wrong version of the equation question in interview 3, give them the right version: $2lnx = \ln(2 - x)$. Ask them to interpret their solutions.
If they got the right version last time, ask them to try to write an equation using logarithms that produces at least one extraneous solution.
(O-rule, F-ineq)

(9) What is this map about? How are logarithms related to this map? Why do you think they used a logarithmic scale? Why wouldn't they use a linear scale, like 1-19,999 and 20,000-39,999 etc.?
(C-rec, C-real)

http://upload.wikimedia.org/wikipedia/commons/e/e0/COB_data_Tsunami_deaths.PNG