The Weak Cayley Table and the Relative Weak Cayley Table

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The Weak Cayley Table and the Relative Weak Cayley Table

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A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of
Master of Science

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Abstract

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In 1896, Frobenius began the study of character theory while factoring the group determinant. Later in 1963, Brauer pointed out that the relationship between characters and their groups was still not fully understood. He published a series of questions that he felt would be important to resolve. In response to these questions, Johnson, Mattarei, and Sehgal developed the idea of a weak Cayley table map between groups. The set of all weak Cayley table maps from one group to itself also has a group structure, which we will call the weak Cayley table group.

We will examine the weak Cayley table group of $AGL(1,p)$ and the dicyclic groups, find a normal subgroup of the weak Cayley table group for a special case with Camina pairs and Semi-Direct products with a normal Hall-$\pi$ subgroup, and look at some nontrivial weak Cayley table elements for certain $p$-groups.

We also define a relative weak Cayley table and a relative weak Cayley table map. We will examine the relationship between relative weak Cayley table maps and weak Cayley table maps, automorphisms and anti-automorphisms, characters and spherical functions.

Keywords: Finite Group, Weak Cayley Table, Relative Weak Cayley Table, Camina Pair
I would like thank my advisor, Dr. Stephen Humphries, for his patient guidance throughout the research and writing process.
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In 1896 Frobenius and Dedekind corresponded through a series of letters on the problem of factoring the group determinant [Cu, p.50-53]. Previously, characters had been defined for abelian groups, and it was during this time that Frobenius defined characters in a general sense. He chose to do so in such a way that each irreducible factor of the group determinant corresponded to an irreducible character of the group. As part of producing an algorithm to take a character to its corresponding irreducible factor, Frobenius defined $k$-characters recursively as follows [JS]:

**Definition 1.1.** Let $\chi$ be a character of a finite group $G$. Let the 1-character, $\chi^{(1)}$, be equal to $\chi$. Then define the $k$-character $\chi^{(k)} : G^k \to \mathbb{C}$ to be the map

$$\chi^{(k)}(g_1, g_2, \ldots, g_k) = \chi(g_1)\chi^{(k-1)}(g_2, \ldots, g_k) - \chi^{(k-1)}(g_1g_2, \ldots, g_k) - \chi^{(k-1)}(g_2, g_1g_3, \ldots, g_k) - \cdots - \chi^{(k-1)}(g_2, \ldots, g_1g_k).$$

In particular, the 2-character is defined to be

$$\chi^{(2)}(g_1, g_2) = \chi(g_1)\chi(g_2) - \chi(g_1g_2).$$

This was the beginning of character and representation theory. Mathematicians began to implement these new ideas to prove powerful results about groups, such as Burnside’s $pq$-theorem.

In 1963, Brauer wrote a paper where he proposed several open questions about the relationship between characters and their groups [Br]. Some of his questions were:
• In addition to the character table, what information is necessary to determine a finite group?

• Given a group $G$, how much information about the automorphism group $Aut(G)$ of a group can be obtained from the characters of $G$?

• Given a set of conjugacy classes that form a normal subgroup, is there enough information in the character table to determine if the normal subgroup is abelian?

In response to these questions, Johnson, Mattarei, and Sehgal published a paper in 2000 developing the concept of a weak Cayley table. They were interested in the question, “What properties of a group can be determined by the 1- and 2- characters which cannot be determined by the 1-characters alone?” [JMS]. In this paper, they define a weak Cayley table and proved that knowing the weak Cayley table of a group is equivalent to knowing the 1- and 2-characters of a group.

1.1 Weak Cayley Tables

A weak Cayley table is similar to a multiplication table for a group, only instead of the table containing the product of the two indices, the entries of a weak Cayley table contain the conjugacy class of their product. More specifically, given a finite group $G$ of order $n$, order the elements of $G$, index the rows and columns of an $n \times n$ table with the ordered elements; then in the $i^{th}$ row and $j^{th}$ column enter the conjugacy class of $ij$ in $G$. The resulting table is a weak Cayley table for the group. Any two weak Cayley tables for a group $G$ differ only by an ordering of the rows and columns.

Example 1.2. As an example, consider $S_3$. The conjugacy classes of $S_3$ are

$$C_1 = \{1\}, C_2 = \{(12), (13), (23)\} \text{ and } C_3 = \{(123), (132)\}.$$ 

Then a weak Cayley table for $S_3$ is
When Johnson, Mattarei, and Sehgal defined the weak Cayley table, they proved:

**Theorem 1.3** (JMS, Proposition 2.4). If the irreducible 1- and 2-characters of a group are given, its weak Cayley table can be constructed. Conversely, if the weak Cayley table is given, the irreducible 1- and 2- characters can be calculated.

Thus by examining weak Cayley tables, we can further understand the relationship between groups and their characters.

### 1.2 Weak Cayley Table Maps

Weak Cayley tables are not unique to a specific group. For example the two non-abelian non-isomorphic groups of order $p^3$, where $p$ is an odd prime, have the same weak Cayley table [JMS]. The authors of [JMS] defined a weak Cayley table map to be a bijection between two groups that preserves the weak Cayley table structure. If $G_1$ and $G_2$ are two groups, then a *weak Cayley table map* $\phi : G_1 \to G_2$ is a bijection that satisfies two conditions:

(i) $\phi(g^{G_1}) = \phi(g)^{G_2}$

(ii) for every $g$ and $h$ in $G_1$, $\phi(gh) \sim \phi(g)\phi(h)$.

Where $\sim$ denotes the equivalence relation of conjugacy. We say $G_1$ and $G_2$ have *the same weak Cayley table* if there exists such a map.
**Example 1.4.** Let $G_1$ and $G_2$ be the non-isomorphic groups of order $p^3$ ($p$ an odd prime) with the following presentations:

$$G_1 = \langle a, b, c : a^p = b^p = c^p = 1, b^p = bc \rangle$$

where $\langle c \rangle$ generates the center $Z(G_1)$, and

$$G_2 = \langle x, y, z : x^p = z, x^{p^2} = y^p = z^p = 1, x^p = x^{p+1} \rangle$$

where the center $Z(G_2)$ is $\langle z \rangle$. See [DF, p. 183].

One of the nice properties of $G_1$ and $G_2$ is that both groups form a Camina pair over their center.

**Definition 1.5.** A *Camina pair* is a group $G$ with a normal subgroup $H$ such that the conjugacy classes of $G$ not intersecting $H$ are unions of cosets of $H$.

For example in $G_1$ the conjugacy classes are

$$\{1\}, \{c\}, \{c^2\}, \ldots, \{c^{p-1}\},$$

along with classes of the form

$$a^i b^j \{1, c, c^2, \ldots, c^{p-1}\} \text{ for } 0 \leq i \leq p - 1, 0 \leq j \leq p - 1,$$

where $i$ and $j$ are not both equivalent to 0 mod $p$.

The conjugacy classes for $G_2$ are

$$\{1\}, \{z\}, \{z^2\}, \ldots, \{z^{p-1}\},$$

together with

$$x^i y^j \{1, z, z^2, \ldots, z^{p-1}\} \text{ for } 0 \leq i \leq p - 1, 0 \leq j \leq p - 1.$$
where \(i\) and \(j\) are not both equivalent to \(0\) mod \(p\).

Let the map \(\phi : G_1 \to G_2\) be defined by

\[
\phi(1) = 1, \quad \phi(a^r) = y^r, \quad \phi(b^s) = x^s, \quad \phi(c^t) = z^{-t}, \quad \text{and} \quad \phi(a^r b^s c^t) = x^s y^r z^{rs-t}.
\]

From the above description of the conjugacy classes it is easy to see that \(\phi\) maps \(G_1\) conjugacy classes to \(G_2\) conjugacy classes, thus satisfying condition (i) of a weak Cayley table map.

To meet condition (ii), \(\phi(gh)\) must be conjugate to \(\phi(g)\phi(h)\) for all \(g\) and \(h\) in \(G_1\). Let \(g = a^r b^s c^t\) and let \(h = a^i b^j c^k\) in \(G_1\). Then

\[
gh = (a^r b^s c^t)(a^i b^j c^k) = (a^r b^s)(a^i b^j)c^{t+k} = a^{r+i}b^{s+j}c^{s+t+k}.
\]

First assume that either \(r + i \neq p\) or \(s + j \neq p\). Then

\[
\phi(gh) = \phi(a^{r+i}b^{s+j}c^{s+t+k}) = x^{s+j} y^{r+i} z^{(s+j)(r+i-t-k)},
\]

and

\[
\phi(g)\phi(h) = \phi(a^r b^s c^t)\phi(a^i b^j c^k) = (x^s y^r z^{rs-t})(x^i y^j z^{ij-k}) = x^{s+j} y^{r+i} z^{rs-t+ij-k+rj}.
\]

Both \(\phi(gh)\) and \(\phi(g)\phi(h)\) are in the same coset of the center, and therefore must be conjugate to each other since \(r + i \neq p\) or \(s + j \neq p\).

Next assume that \(r + i = p\) and \(s + j = p\). Then since \(gh\) is a central element, in order for \(\phi(gh) \sim \phi(g)\phi(h)\), we need for \(\phi(gh) = \phi(g)\phi(h)\). Doing a similar computation, we get that

\[
\phi(gh) = \phi(c^{-rs+t+k}) = z^{rs-t-k},
\]

and
φ(g)φ(h) = \phi(a^rb^sc^t)\phi(a^{-r}b^{-s}c^k)
= (x^sy^rz^{rs-1})(x^{-s}y^{-r}z^{rs-k}).
= x^s y^r x^{-s} y^{-r} z^{2rs-t-k}
= x^s y^r y^{-r} z^{-rs} z^{2rs-t-k}
= z^{rs-t-k}.

Which gives us \( \phi(gh) = \phi(g)\phi(h) \), as required. Therefore \( \phi \) is a weak Cayley table map.

In their paper, Johnson, Mattarei, and Sehgal also identified other convenient facts about weak Cayley table maps, several of which are summarized in the following theorem [JMS].

**Theorem 1.6** (Johnson, Mattarei, and Sehgal). Let \( \phi : G_1 \to G_2 \) be a weak Cayley table map. Then

- (i) \( \phi(1_{G_1}) = 1_{G_2} \)
- (ii) \( \phi(x^{-1}) = \phi(x)^{-1} \)
- (iii) \( \phi \) sends normal subgroups of \( G_1 \) to normal subgroups of \( G_2 \)
- (iv) \( \phi \) preserves the cosets of any normal subgroup \( N \) of \( G_1 \)
- (v) Any automorphism (or anti-automorphism) of a group \( G \) is a weak Cayley table map
- (vi) The composition of two weak Cayley table maps is also a weak Cayley table map
- (vii) If \( g \in G \) is an involution, then \( \phi(g) \) is also an involution.

Another interesting fact is that for two non-isomorphic groups, having the same weak Cayley table is a stronger condition than of having the same character table. For example, consider \( D_8 \), the dihedral group of order 8 and the Quaterions \( Q_8 \). This is a classic example of two non-isomorphic groups possessing the same character table [DF, p. 882]. However, since the number of involutions in both groups are not the same, there is not a bijection.
between $D_8$ and $Q_8$ that preserves inverses, and so no weak Cayley table map between $D_8$ and $Q_8$ can exist [JMS].

**Proposition 1.7.** The set of weak Cayley table maps from a group $G$ to itself is a group, denoted $WCT(G)$.

**Proof.** As stated above, the composition of two weak Cayley table maps is still a weak Cayley table map. Then, since $\phi$ is a bijection, $\phi^{-1}$ exists and is also a weak Cayley table map. So $WCT(G)$ is a group. $\square$

Any automorphism or anti-automorphism is called a trivial weak Cayley table map. If for some group $G$, the group $WCT(G)$ consists only of automorphisms or anti-automorphisms then $WCT(G)$ is said to be trivial. [Hu] proved that for all $n \geq 1$, $WCT(S_n)$ is trivial and that for all dihedral groups $D_{2n}$, $WCT(D_{2n})$ is also trivial.

Since any weak Cayley table map $\phi : G_1 \rightarrow G_2$ preserves cosets of normal subgroups $N$ of $G$, we can let $\overline{\phi} : G/N \rightarrow G/\phi(N)$ be the map where $\overline{\phi}(gN) = \phi(g)\phi(N)$. Johnson proved that $\overline{\phi}$ is a weak Cayley table map.

Other results from Johnson, Mattarei, and Sehgal’s work on weak Cayley table maps are the following:

**Theorem 1.8** (JMS, Theorem 3.1). Let $G_1$ and $G_2$ be finite groups and $N$ a normal subgroup in both $G_1$ and $G_2$. Suppose further that $G_1/N \cong G_2/N$ and the order of $G_i/N$ is odd. If $(G_i, N)$ forms a Camina pair, then $G_1$ and $G_2$ have the same weak Cayley table.

An example of this theorem would be the two non-isomorphic groups of order $p^3$ where $p$ is odd, as examined above. Their centers are isomorphic, both groups form Camina pairs with their center, and their quotients are isomorphic and odd ordered, so they meet the criteria of the hypotheses.

The next theorem in [JMS] uses the structure properties of extensions and Camina pairs to create a weak Cayley table map between two non-isomorphic groups.
Definition 1.9. If $H$, $G$ are groups and $N$ is an abelian group such that

$$1 \to N \to H \to G \to 1$$

is a short exact sequence, we say that $H$ is an extension of $G$ by $N$.

Theorem 1.10 (JMS, Theorem 4.1). Suppose that $G_1$ and $G_2$ have the same weak Cayley table via $\alpha : G_1 \to G_2$. Let $H_i$ be an extension of $G_i$ by the module $N$, for $i = 1, 2$ where $n^g = n^{\alpha(g)}$ for all $g \in G$ and suppose that $(H_1, N)$ and $(H_2, N)$ are Camina pairs. Finally, having written each $H_i$ as an extension of $N$ by $G_i$, suppose that for all involutions $x \in G_1$ we have

$$(e, x)^2 = (e, \alpha(x))^2.$$ 

Then $H_1$ and $H_2$ have the same weak Cayley tables.

The strong conditions on the theorem above might often force an isomorphism between $H_1$ and $H_2$. An interesting case of this is when $H_i$ is a Frobenius group.

Definition 1.11. Let $G$ and $N$ be finite groups, and let $G$ act on $N$. Then the action of $G$ on $N$ is said to be Frobenius if $n^g \neq n$ for all nonidentity elements $n \in N$ and $g \in G$. The group $H = N \rtimes G$ is called a Frobenius group if the action of $G$ on $N$ is Frobenius [Is, p.177].

To understand why two Frobenius groups with the same weak Cayley table must be isomorphic, suppose that the action of $G_1$ and $G_2$ on $N$ was Frobenius. Then, with the condition $n^g = n^{\alpha(g)}$ for all $g \in G_1$, we would have

$$n^{gh} = n^{\alpha(gh)} = (n^g)^h = (n^\alpha)^{\alpha(h)}$$

Therefore the Frobenius property shows that $\alpha(gh) = \alpha(g)\alpha(h)$. Thus $G_1 \cong G_2$ and they both act identically on $N$, so in the case that the action is Frobenius, $H_1$ and $H_2$ in the
previous theorem are isomorphic. Since Frobenius groups have interesting properties, the authors of [JMS] also published the following result which eliminates the condition that $G_1$ and $G_2$ act the same way on $N$, thus allowing for the case when there are two non-isomorphic, Frobenius groups.

**Theorem 1.12** (JMS, Theorem 4.3). *Suppose that $G_1$ and $G_2$ have the same weak Cayley table via $\alpha : G_1 \rightarrow G_2$. Let $H_i$ be an extension of $G_i$ by the abelian normal subgroup $N$, such that the conjugacy classes of $H_1$ which lie in $N$ are the same as the conjugacy classes of $H_2$ in $N$. Suppose that $(H_1, N)$ and $(H_2, N)$ are Camina pairs. Finally, having fixed a representation for each $H_i$ as an extension of $G_i$ by $N$, suppose that for every involution $x \in G_1$ we have

\[
(e, x)^2 = (e, \alpha(x))^2, \\
n^x = n^{\alpha(x)} \quad \text{for all } n \in N.
\]

Then $H_1$ and $H_2$ have the same weak Cayley table.*
Chapter 2. Weak Cayley group of $AGL(1,p)$

In this chapter, let $p$ be an odd prime and let $G_p = AGL(1,p)$, the group of affine transformations of $A'(\mathbb{F}_p)$. There are two different ways of referring to the elements of the group $G_p$. One way is where we use the presentation:

$$\langle a, b | a^{p-1} = b^p = 1, a^b = b^r \rangle,$$

where $r$ is a generator for $\mathbb{F}_p^\times$. The other is to represent elements of $G_p$ as a set of matrices of the form \((\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix})\) such that $x \in \mathbb{F}_p^\times$, and $y \in \mathbb{F}_p$. (Notice that $AGL(1,3) \cong S_3$ and $AGL(1,5) \cong F_{20}$.) There is an isomorphism determined by the map $a \mapsto (\begin{smallmatrix} r & 0 \\ 0 & 1 \end{smallmatrix})$ and $b \mapsto (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$. Then $a^i \mapsto (\begin{smallmatrix} r^i & 0 \\ 0 & 1 \end{smallmatrix})$ and $b^j \mapsto (\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix})$.

Any element in the group can be written as $a^ib^j$ where $0 \leq i \leq p-2$ and $0 \leq j \leq p-1$, by using the conjugation $a^{-1}ba = b^r$. So we can simplify any expression using the following identities:

$$a^{-1}ba^k = b^kr,$$
$$a^{-1}ba^2 = a^{-1}b^ra = b^{r^2},$$
$$a^{-s}ba^s = b^{r^s},$$
$$a^{-s}ba^s = b^{kr^s}.$$

Next let $B$ be the subgroup generated by $\langle b \rangle$ (in the matrix notation $B = \{ (\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}) \mid y \in \mathbb{F}_p \}$).

Then if we take an element $(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}) \in B$ and conjugate it by any element $(\begin{smallmatrix} w & \tilde{y} \\ 0 & 1 \end{smallmatrix}) \in AGL(1,p)$, we get
\[
\begin{pmatrix}
  w^{-1} & -w^{-1}z \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  1 & y \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  w & z \\
  0 & 1
\end{pmatrix} =
\begin{pmatrix}
  w^{-1} & -w^{-1}z \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  w & z + y \\
  0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 & w^{-1}y \\
  0 & 1
\end{pmatrix}.
\]

Thus \(B\) is a normal subgroup and \(B - \{(1,0)\}\) is a conjugacy class of \(G_p\) since we can take \(w^{-1}\) to be anything in \(\mathbb{F}_p^\times\).

Now consider the element \((\frac{x}{0}, 0)\) where \(x \neq 1\). If we conjugate by any element \((\frac{w}{0}, \frac{z}{1})\), observe that
\[
\begin{pmatrix}
  w^{-1} & -w^{-1}z \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  x & 0 \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  w & z \\
  0 & 1
\end{pmatrix} =
\begin{pmatrix}
  w^{-1} & -w^{-1}z \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  xw & xz \\
  0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
  x & w^{-1}z(x - 1) \\
  0 & 1
\end{pmatrix}.
\]

By conjugating \((\frac{x}{0}, 0)\) over all the elements of \(G_p\), the entry \(w^{-1}z(x - 1)\) will range over all of \(\mathbb{F}_p\). Thus the conjugacy class of \((\frac{x}{0}, 0)\) is the coset \((\frac{x}{0}, 0)B\), which implies that \((G_p, B)\) is a Camina pair. In the terms of the generators \(a\) and \(b\), the conjugacy classes of \(G_p\) are \(\{1\}, B - \{1\}\), and the cosets \(a'B\) for every \(1 \leq i \leq p - 2\).

**Lemma 2.1.** Given any element \((\frac{x}{0}, \frac{y}{1}) \notin B\), then
\[
\begin{pmatrix}
  x & y \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  x & y \\
  0 & 1
\end{pmatrix}^{-1} =
\begin{pmatrix}
  1 & 0 \\
  0 & 1
\end{pmatrix}.
\]

**Proof.** If \((\frac{x}{0}, \frac{y}{1}) \notin B\), then \(x \neq 1\). If \(y = 0\), then the result follows from \(x \in \mathbb{F}_p^\times\). If \(y \neq 0\) then,
\[
\begin{pmatrix}
  x & y \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  x & y \\
  0 & 1
\end{pmatrix} =
\begin{pmatrix}
  x^2 & (x + 1)y \\
  0 & 1
\end{pmatrix}.
\]
and
\[
\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} x^3 (x^2 + x + 1)y \\ 0 & 1 \end{pmatrix}.
\]

An inductive argument shows that for any positive integer \( k \),
\[
\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} x^k (x^{k-1} + x^{k-2} + \cdots + x + 1)y \\ 0 & 1 \end{pmatrix}.
\]

In particular, let \( k = p - 1 \), then
\[
\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^{p-1} = \begin{pmatrix} x^{p-1} (x^{p-2} + x^{p-3} + \cdots + x + 1)y \\ 0 & 1 \end{pmatrix}.
\]

Using the identity \( x^{p-1} = 1 \), this becomes
\[
\begin{pmatrix} x^{p-1} (x^{p-2} + x^{p-3} + \cdots + x + 1)y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 (x^{p-2} + x^{p-3} + \cdots + x + x^{p-1})y \\ 0 & 1 \end{pmatrix}.
\]

If we consider the upper right entry on both sides of the equation, we notice that
\[
(x^{p-2} + x^{p-3} + \cdots + x + 1)y = (x^{p-2} + x^{p-3} + \cdots + x + x^{p-1})y
\]
\[
= x(x^{p-2} + x^{p-3} + \cdots + x + 1)y.
\]

However \( x \neq 1 \) and \( y \neq 0 \), so we must have that \((x^{p-2} + x^{p-3} + \cdots + x + 1) = 0\). Therefore
\[
\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^{p-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

\[ \square \]

Considering our conjugacy classes, since \( B \) has \( p \) elements, the conjugacy classes of the form \( a^iB \) have size \( p \) (\( i \neq 0 \)), and the class \( B - \{1\} \) is the unique class of size \( p - 1 \). If
\( \varphi \in WCT(G_p) \), then \( \varphi \) is a bijection that preserves conjugacy classes. Thus \( \varphi(B) = B \).

Also, \( \varphi \) maps the unique class of involutions (the coset \( a^{\frac{p-1}{2}}B \)) to itself because \( \varphi \) also preserves inverses.

Let \( \Phi : WCT(G_p) \to WCT(G_p/B) \) be the map that sends \( \varphi \in WCT(G_p) \) to \( \overline{\varphi} \in WCT(G_p/B) \).

This map is well-defined, since we have shown that \( B \) is fixed by any \( \phi \in WCT(G_p) \) and that the other cosets \( a^iB \) are permuted by \( \phi \).

Also \( G_p/B \cong \mathbb{F}_p^\times \) is an abelian group and any weak Cayley table map of \( G_p/B \) would have to be an automorphism to satisfy condition (ii) in the definition. Thus \( WCT(G_p/B) = Aut(G_p/B) \), and \( \Phi \) is a map from \( WCT(G_p) \) to \( Aut(G_p/B) \).

Let \( K \) be the kernel of \( \Phi \). Then \( K \) is not trivial, since it contains the automorphism \( \rho \) which sends \( a \to a, b \to b^s \), for any \( s \) that generates \( \mathbb{F}_p^\times \). For any \( \phi \in K \), \( \phi(a^iB) = a^iB \). Therefore \( \phi \) permutes elements inside cosets. Then we can express \( \phi(a^ib^j) \) as \( a^{i\alpha_{\phi}(i,j)} \) for some function \( \alpha_{\phi} \), where for each \( i \), \( \alpha_{\phi}(i, \_ : \mathbb{F}_p \to \mathbb{F}_p \) is an injective function with \( \alpha_{\phi}(0, 0) = 0 \) (since \( \phi(1) = 1 \)).

Also \( \phi \) preserves inverses, so \( (\phi(a^ib^j))^{-1} = \phi((a^ib^j)^{-1}) \).

Now

\[
(\phi(a^ib^j))^{-1} = (a^{i\alpha_{\phi}(i,j)})^{-1} = b^{-\alpha_{\phi}(i,j)}a^{-i} = a^{-i}b^{-r^{-i}\alpha_{\phi}(i,j)},
\]

and

\[
\phi((a^ib^j)^{-1}) = \phi(b^{-j}a^{-i}) = \phi(a^{-ib^{-r^{-j}}}) = a^{-i}b^{\alpha_{\phi}(-i, -r^{-j})}.
\]

These two expressions are equal since \( \alpha_{\phi} \) preserves inverses, thus for every \( \alpha_{\phi} \)

\[
-r^{-i}\alpha_{\phi}(i, j) = \alpha_{\phi}(-i, -r^{-j}).
\]
where \(0 \leq i \leq p - 2\) and \(0 \leq j \leq p - 1\).

**Lemma 2.2.** The kernel \(K\) in \(WCT(G_p)\) is the set of all bijections \(\phi\) such that \(\phi(a^i b^j) = a^i b^{\alpha(\phi)(i,j)}\) where \(\alpha(\phi)(i,j)\) is an injective function on \(\mathbb{F}_p\) to \(\mathbb{F}_p\) such that \(\alpha(\phi)(0,0) = 0\), and

\[
-r^{-i}\alpha(\phi)(i,j) = \alpha(\phi(-i,-r^{-i}j))
\]

for every \(0 \leq i \leq p - 2, 0 \leq j \leq p - 1\).

**Proof.** We have already shown that any map in the kernel must satisfy these conditions. All that is left is to show that any map of this form is a weak Cayley table map in \(K\).

So let \(\psi\) be a map such that \(\psi(a^i b^j) = a^i b^{\alpha(\psi)(i,j)}\) where \(\alpha(\psi)(i,j)\) is an injective function on \(\mathbb{F}_p\) to \(\mathbb{F}_p\) such that \(\alpha(\psi)(0,0) = 0\) and \(-r^{-i}\alpha(\psi)(i,j) = \alpha(\psi(-i,-r^{-i}j))\) for every \(0 \leq i \leq p - 2, 0 \leq j \leq p - 1\). Since \(\psi(a^i b^j) = a^i b^{\alpha(\psi)(i,j)}\), we have \(\psi(a^i B) = a^i B, \psi(B - \{1\}) = B - \{1\},\) and \(\psi(1) = \psi(a^0 b^0) = a^0 b^{\alpha(\psi)(0,0)} = a^0 b^0 = 1\). So \(\psi\) takes conjugacy classes to the same conjugacy class, which means that it also fixes the cosets of \(B\). Thus if \(\psi \in WCT(G_p)\), then \(\psi \in K\).

Then given two elements \(a^i b^j\) and \(a^s b^t\) in \(G_p\), to satisfy condition (ii), we require \(\psi(a^i b^j a^s b^t) \sim \psi(a^i b^j) \psi(a^s b^t)\). Now

\[
\psi(a^i b^j a^s b^t) = \psi(a^{i+s} b^{r^i j + t})
\]

and

\[
\psi(a^i b^j) \psi(a^s b^t) = a^{i+s} b^{\alpha(\psi)(i,j)} a^{s} b^{\alpha(\psi)(s,t)}
\]

If \(a^s \neq a^{-i}\), then \(a^{i+s} b^{\alpha(\psi)(i+s,r^i j + t)} \sim a^{i+s} b^{\alpha(\psi)(i,j) + \alpha(\psi)(s,t)}\) because they belong to the same coset.

If \(a^s = a^{-i}\), then

\[
a^{i+s} b^{\alpha(\psi)(i+s,r^i j + t)} = b^{\alpha(\psi)(0,r^{-i}j + t)},
\]
\[ a^i b^s a_\psi(i, j) + a_\psi(s, t) = b^{r^{-i}a_\psi(i, j)} + a_\psi(-i, t). \]

When \( b^{r^{-i}a_\psi(i, j)} + a_\psi(-i, t) = 1 \), then \( r^{-i}a_\psi(i, j) + a_\psi(-i, t) = 0 \), which implies

\[-r^{-i}a_\psi(i, j) = a_\psi(-i, t).\]

Above we found \(-r^{-i}a_\psi(i, j) = a_\psi(-i, -r^{-i}j)\), which implies \( t = -jr^{-i} \mod p \). Then

\[ b^{a_\psi(0, r^{-i}j + t)} = b^{a_\psi(0, r^{-i}j - jr^{-i})} = b^{a_\psi(0, 0)} = 1. \]

By a similar argument, if \( b^{a_\psi(0, r^{-i}j + t)} = 1 \), then \( b^{r^{-i}a_\psi(i, j)} + a_\psi(-i, t) = 1 \).

If \( t \neq -jr^{-i} \), then both \( b^{a_\psi(0, r^{-i}j + t)} \) and \( b^{r^{-i}a_\psi(i, j)} + a_\psi(-i, t) \) are in the conjugacy class \( B - \{1\} \). Therefore, \( \psi(a^ib^ja^sb^t) \sim \psi(a^ib^t)\psi(a^sb^t) \), and \( \psi \) is a weak Cayley table map in \( K \).

Thus we can construct elements of \( K \) by considering permutations of each of the cosets \( a^iB \) (\( i \neq 0 \)) that preserve inverses. For the class \( a^mB \), where \( 1 \leq m \leq \frac{p-3}{2} \), its inverse class is equal to \( a^{p-m-1}B \). So by choosing any permutation on the elements of \( a^mB \), the corresponding permutation of \( a^{p-m-1}B \) is determined. So we can conclude that the kernel of \( \Phi \) contains \( \frac{p-3}{2} \) copies of \( S_p \).

There is also one coset of involutions, \( a^{\frac{p-1}{2}}B \), which is sent to itself by any weak Cayley table map, and any permutation on these elements will respect inverses, so this contributes another copy of \( S_p \) to the kernel.

Finally the subgroup \( B \) contains its own inverses, and so the allowable permutations on \( B \) are those that respect the inverses. The Coxeter group of type B on \( \frac{p-1}{2} \) elements is the set of permutations which respect those inverses. It is denoted here as \( Cox_B(\frac{p-1}{2}) \).
The permutations referred to above are all independent of each other and by Lemma 2.2 they can be composed together to give all of $K$. So we have shown

**Lemma 2.3.** The kernel of $\Phi$ is isomorphic to $Cox_B(\frac{p-1}{2}) \times S_p \times S_{p-3}^{p-3}$. □

Next consider any weak Cayley table map $\phi$. We can view it as a permutation on the cosets of $B$ composed with an element $\psi$ of the kernel $K$. Since we know the structure of the kernel, to finish identifying all weak Cayley table maps we need to determine what effect $\phi$ can have on the cosets of $B$.

As seen above, the map $\phi$ will always satisfy $\phi(B) = B$. Further $\phi$ will take involutions to involutions, so $\phi(a^{\frac{p-1}{2}}B) = a^{\frac{p-1}{2}}B$ since $a^{\frac{p-1}{2}}B$ is the unique class of involutions in $G_p$. Therefore $\phi$ only (possibly) permutes the remaining $p - 3$ conjugacy classes amongst themselves while preserving inverses.

Condition (ii) of a weak Cayley table map guarantees that $\phi$ must preserve inverses. Conveniently all the inverses of elements in $a^iB$ lie in the conjugacy class $a^{-i}B$. Therefore $\phi$ must permute the remaining $p - 3$ cosets in such a way that preserves the coset containing the inverses. This gives us another Coxeter group acting on these $p - 3$ classes.

However these coset permutations are not completely independent of the permutation of the elements inside of the $p - 3$ classes found in the kernel, since the order of these actions matters. So the possibilities for the action of $\phi$ on the $p - 3$ cosets are $S_{p-3}^{p-3} \rtimes Cox_B(\frac{p-3}{2})$.

Then, how $\phi$ permutes the elements of $B$ and $a^{\frac{p-1}{2}}B$ are completely independent of the permutations of the other cosets, thus we have:

**Theorem 2.4.** $WCT(G_p) \cong Cox_B(\frac{p-1}{2}) \times S_p \times (S_{p-3}^{p-3} \rtimes Cox_B(\frac{p-3}{2}))$. □
Chapter 3. Camina pairs \((G,N)\) with \(G/N\) Abelian

In Chapter 2, the kernel of the map \(\Phi\) played a key role in helping us understand the group \(WCT(AGL(1,p))\). The goal of this chapter is to find similar results about a well defined map \(\Phi : WCT(G) \to WCT(G/N)\), where \((G,N)\) form a Camina pair. First we start with a lemma.

**Lemma 3.1.** If \((G,N)\) is a Camina pair, then \(G/N\) is abelian if and only if the conjugacy classes of \(G - N\) are cosets \(Ng\).

*Proof.* First assume that the conjugacy classes of \(G - N\) are of the form \(Ng\). Then for \(g, h \in G\),

\[(Nh)^{-1}(Ng)(Nh) = Nh^{-1}gh.\]

Let \(g \in G - N\). Then, since \(h^{-1}gh \sim g\) and \((G,N)\) is a Camina pair, \(h^{-1}gh = ng\) for some \(n \in G\). Then

\[(Nh)^{-1}(Ng)(Nh) = Nng = Ng.\]

Thus \(G/N\) is abelian.

Next, assume that \(G/N\) is abelian. If \(g \in G - N\) and \(n \in N\), we have that \(g\) and \(ng\) are in the same coset \(Ng\), and therefor are conjugate, since \((G,N)\) is a Camina pair. Then note that the conjugacy classes off of \(N\) are unions of cosets of \(N\). We also observe that the commutator subgroup \(G'\) is contained in \(N\) since \(G/N\) is abelian. However, no class can have size greater than \(|G'|\), and so each class is a coset of \(N\). \(\Box\)

**Lemma 3.2.** Let \((G,H)\) be a Camina pair. Then for every weak Cayley table map \(\phi\), \(\phi(H) = H\). Thus the map \(\Phi : WCT(G) \to WCT(G/H)\) that sends \(\phi\) to \(\bar{\phi}\) is a well-defined map.
Proof. Now \( \phi \) preserves the set of conjugacy classes and sends cosets of \( H \) to cosets of \( \phi(H) \). Thus if \( (G, H) \) is a Camina pair then so is \( (G, \phi(H)) \). Now if \( \phi(H) \neq H \), then there would exist two subgroups \( H_1 \) and \( H_2 \) of the same order in \( G \) such that \( (G, H_1) \) and \( (G, H_2) \) are both Camina pairs.

Suppose by way of contradiction that there exist two subgroups \( H_1 \) and \( H_2 \) such that \( H_1 \neq H_2 \), \( (G, H_1) \) and \( (G, H_2) \) are Camina pairs, and the order of \( H_1 \) equals the order of \( H_2 \). Next pick \( h \in H_1 - H_2 \).

Then \( h^G \) is not contained in \( H_2 \), which implies \( h^G \) is the union of cosets of \( H_2 \), i.e. \( h^G = \cup H_2 b_i \) for some elements \( b_i \). Then \( |h^G| \geq |H_2| \). However \( h^G \subseteq H_1 - H_2 \) which implies \( |h^G| < |H_1| \), which gives

\[
|h^G| \geq |H_2| = |H_1| > |h^G|.
\]

This is a contradiction, so \( H_1 = H_2 \).

Thus there can be only one subgroup that forms a Camina pair of size \( |H| \). This gives \( \phi(H) = H \) for all weak Cayley table maps \( \phi \).

\[\square\]

We note that what the above result really proves is

**Corollary 3.3.** Let \( (G, H_1) \) and \( (G, H_2) \) be Camina pairs. Then either \( H_1 \subseteq H_2 \) or \( H_2 \subseteq H_1 \). \[\square\]

**Theorem 3.4.** Let \( (G, N) \) be a Camina pair such that \( G/N \) is abelian and \( N = \{1\} \) is a conjugacy class. Let \( \Phi : WCT(G) \rightarrow WCT(G/N) \) be the map that sends \( \phi \) to \( \overline{\phi} \). Then the kernel of \( \Phi : WCT(G) \rightarrow WCT(G/N) \) is the set of all bijections from \( G \) to \( G \) that take inverses to inverses and preserve conjugacy classes (i.e. maps a conjugacy class to itself).

**Proof.** Note by Lemma 3.2 that \( \Phi \) is a well defined map, and by Lemma 3.1 the conjugacy classes of \( G \) off of \( N \) are the cosets of \( N \). Let \( K \) be the kernel of \( \Phi \) and let \( L \) be the set of bijections \( \psi \) that satisfy the hypothesis that \( \psi(g^G) = g^G \), and \( \psi(g^{-1}) = \psi(g)^{-1} \) for all \( g \in G \).
First let $\phi \in K$. Then $\phi$ is a bijection and $\phi(x^{-1}) = \phi(x)^{-1}$ for all $x \in G$. Then $\phi \in K$ implies $\phi(N - \{1\}) = N - \{1\}$ and for $g \notin N,$

\[
\begin{align*}
\phi(g^G) &= \phi(N g) \\
&= \phi(N) \phi(g) \\
&= N \phi(g) \\
&= Ng \\
&= g^G.
\end{align*}
\]

Thus any map $\phi \in K$ is also in $L$.

Next let $\psi$ be in $L$. We need to show that $\psi$ is a weak Cayley table map. Any map in $L$ will take conjugacy classes to conjugacy classes, thus satisfying condition (i) of a weak Cayley table map. To show that any map $\psi \in L$ also satisfies condition (ii) of a weak Cayley table map, we need to consider three cases.

**Case 1:** For $g, h \in G$ and $gh \notin N$ : Then there exists $n_1, n_2 \in N$ such that $\psi(g) = n_1g$, $\psi(h) = n_2h$ and so

\[
\psi(g)\psi(h) = n_1gn_2h.
\]

Then

\[
\psi(g)\psi(h) \in NgNh = Ngh.
\]

We also have $\psi(gh) \sim \psi(g)\psi(h) = n_1gn_2h = n_3gh$, so that

\[
N\psi(gh) = Ngh
\]

Note that $Ngh$ is a conjugacy class for $gh \notin N$, so

\[
\psi(g)\phi(h) \sim \psi(gh).
\]

**Case 2:** $g, h \in G$, and $gh = 1$. Then $h = g^{-1}$. Moreover since $\psi \in L$, $\psi(g^{-1}) = \psi(g)^{-1}$,
so

\[ \psi(gh) = \psi(gg^{-1}) = 1 = \psi(g)\psi(g)^{-1} = \psi(g)\psi(h). \]

**Case 3:** For \( g, h \in G \) and \( gh \in N - \{1\} \): Then \( Ng = Nh^{-1} \) and for some \( n \in N \), \( g = nh^{-1} \). Then

\[
\psi(gh) = \psi(nh^{-1}h) \\
= \psi(n) \\
\in N - \{1\}.
\]

Also

\[
\psi(g)\psi(h) = \psi(nh^{-1})\psi(h) \\
\in Nh^{-1}Nh \\
= N - \{1\}.
\]

Thus, for all \( g, h \in G \), \( \psi(gh) \sim \psi(g)\psi(h) \). So \( \psi \) is in \( K \).

\[\Box\]
In this chapter we generalize the results that we obtained above in the situation where we have a Camina pair (see Theorem 3.4).

**Definition 4.1.** Given a finite group $G$ and a set $\pi$ of prime numbers, a Hall $\pi$-subgroup is a subgroup $H$ such that all primes which divide the order of $H$ are in $\pi$ and no prime in $\pi$ divides the index $[G : H]$. [Is, p. 86]

The following is from [Is, p. 87]

**Theorem 4.2** (Hall). Suppose that $G$ is a finite solvable group, and let $\pi$ be an arbitrary set of primes. Then all Hall $\pi$-subgroups of $G$ are conjugate.

In this section, we will consider groups $G = A \rtimes B$ with $Z = Z(G)$ such that $A$ is a normal Hall $\pi$-subgroup and $Ab - Z$ is a conjugacy class for every $b \in B$. We call such a group a Camina-Z group.

**Example 4.3.** An example of a group with these properties is the group

$$G = \langle a, b | a^5 = b^8 = 1, a^b = a^2 \rangle \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_8$$

with $A = \langle a \rangle$ and $B = \langle b \rangle$. One way to examine the conjugacy classes of this group is to consider a part of the weak Cayley table, where the columns are indexed by the elements of $A$, and the rows are indexed by elements of $B$. We can then see that the number of conjugacy classes is 10, and that $\langle b^4 \rangle$ is the center. (In this table below, the numbers 1-10 represent different conjugacy classes of $G$).
Here $A$ is a normal Sylow-5 subgroup, and so is a normal Hall $\{5\}$-subgroup, and where $A \cap Z = \{1\}$. Thus for example $b^4 A = \{b^4\} \cup \{b^4 a, b^4 a^2, b^4 a^3, b^4 a^4\}$ is a union of two conjugacy classes.

**Theorem 4.4.** Let $G = A \rtimes B$ with $Z = Z(G)$ such that $G$ is a Camina-Z group and $A$ is the normal Hall-$\pi$ subgroup. Let $K$ be the kernel of

$$\Phi : WCT(G) \to WCT(G/A).$$

Then $K$ is the set of functions $\phi$ such that

1. $\phi$ is a bijection,
2. $\phi$ preserves conjugacy classes (i.e. maps a conjugacy class to itself),
3. $\phi$ satisfies $\phi(x^{-1}) = \phi(x)^{-1}$,
4. $\phi(xz) = \phi(x)z$ for all $x \in G$ and $z \in Z$.

**Proof.** The map $\Phi : WCT(G) \to WCT(G/A)$ is well defined, since $A$ is a normal Hall-$\pi$ subgroup, and so it is the unique normal subgroup of its order by Theorem 4.2. Since any
weak Cayley table map sends normal subgroups to normal subgroups and is a bijection, the set $A$ must be fixed by every weak Cayley table map. So if $\phi$ is a weak Cayley table map, 
$\overline{\phi} \in WCT(G/\phi(A)) = WCT(G/A)$.

Let $L$ be the set of all bijections $\phi : G \to G$ satisfying (ii)-(iv).

If $\phi \in K$, then $\phi$ is a bijection that preserves classes and respects inverses. Also for $z \in Z(G)$, we have $\phi(z) = \phi(x^{-1}xz) \sim \phi(x)^{-1}\phi(xz)$. Since $z$ is central, $\phi(z)$ is central, and we have

$$\phi(z) = \phi(x)^{-1}\phi(xz)$$

and so

$$\phi(x)\phi(z) = \phi(xz).$$

Because $\phi \in K$, $\phi(z) = z$ and $\phi(x)z = \phi(xz)$. So $\phi \in L$.

Next let $\phi \in L$. Every map in $L$ preserves conjugacy classes, which implies $\phi(Ab) = Ab$ for all $b \in B$. Therefore we can think of $\phi$ as a permutation on the elements in each $Ab$. Let $\phi(ab) = \phi_b(a)b$, where $\phi_b \in Sym(A)$. So in order to show that $\phi$ is a weak Cayley table map, it is sufficient to show that for every $g, h \in G$, $\phi(gh) \sim \phi(g)\phi(h)$. Let $g = a_1b_1$ and $h = a_2b_2$. Then

$$\phi(gh) = \phi(a_1b_1a_2b_2)$$

$$= \phi(a_1a_2^{-1}b_1b_2)$$

$$= \phi_{b_1b_2}(a_1a_2^{-1})b_1b_2,$$

and

$$\phi(g)\phi(h) = \phi(a_1b_1)\phi(a_2b_2)$$

$$= \phi_{b_1}(a_1)b_1\phi_{b_2}(a_2)b_2$$

$$= \phi_{b_1}(a_1)\phi_{b_2}(a_2)b_1b_2^{-1}.$$

Here we have two cases:
Case 1: $b_1b_2 \notin \mathbb{Z}$. Then

$$\phi_{b_1b_2}(a_1a_2^{b_1^{-1}})b_1b_2 \sim \phi_{b_1}(a_1)\phi_{b_2}(a_2)^{b_1^{-1}}b_1b_2,$$

since they are in the same conjugacy class $Ab_1b_2$.

Case 2: $b_1b_2 = z \in \mathbb{Z}$. If

$$\phi(a_1b_1a_2b_2) = y$$

for some $y \in \mathbb{Z}$, then

$$a_1b_1a_2b_2 = y,$$

since $\phi$ satisfies (iv). Then

$$a_2b_2 = (a_1b_1)^{-1}y,$$

so

$$\phi(a_1b_1)\phi(a_2b_2) = \phi(a_1b_1)\phi((a_1b_1)^{-1}y) = \phi(a_1b_1)\phi(a_1b_1)^{-1}y = y.$$

On the other hand, if

$$\phi(a_1b_1)\phi(a_2b_2) = y$$

for some $y \in \mathbb{Z}$, then

$$\phi(a_2b_2) = \phi(a_1b_1)^{-1}y = \phi((a_1b_1)^{-1}y).$$

Since $\phi$ is a bijection, we have

$$a_2b_2 = (a_1b_1)^{-1}y,$$

so

$$a_1b_1a_2b_2 = y.$$
which implies
\[ \phi(a_1b_1a_2b_2) = y. \]

So if either \( \phi(a_1b_1a_2b_2) \) or \( \phi(a_1b_1)\phi(a_2b_2) \) is central, then both are central and they are equal. If they are both not central, then
\[
\phi(a_1b_1a_2b_2) = \phi_{b_1b_2}(a_1a_2^{b_2^{-1}})b_1b_2 \\
= \phi_{b_1b_2}(a_1a_2^{b_2^{-1}})z,
\]
which is a non-central element in the coset \( A \). Also
\[
\phi(a_1b_1)\phi(a_2b_2) = \phi_{b_1}(a_1)\phi_{b_2}(a_2^{b_2^{-1}})b_1b_2 \\
= \phi_{b_1}(a_1)\phi_{b_2}(a_2^{b_2^{-1}})z,
\]
which is again is a non-central element of \( A \). Thus we have
\[ \phi(a_1b_1a_2b_2) \sim \phi(a_1b_1)\phi(a_2b_2). \]
Chapter 5. Dicyclic Groups

Humphries proved a similar result to the following theorem in his paper on Weak Cayley table Groups in 1997 [Hu]. In that paper, he proved that the \( WCT(G) \) is trivial for all dihedral groups \( G \). We will prove this is also true for dicyclic groups.

**Definition 5.1.** A group with the presentation \(<a, x| x^{2n} = 1, x^n = a^2, xa = x^{-1}>\) is called a dicyclic group.

**Theorem 5.2.** If \( G_{4n} \) is a dicyclic group, then \( WCT(G_{4n}) \) is trivial.

**Proof.** Let \( G_{4n} \). If \( g \in G_{4n} \) we can write \( g \) in the form \( x^k \) or \( ax^k \) for some \( k \) with \( 0 \leq k \leq n \). Note that the conjugacy classes of \( G_{4n} \) are

\[
\{1\}, \{a^2 = x^n\}, \{x^i, x^{-i}\} \text{ for } 1 \leq i \leq n - 1, \{ax^i|i \text{ is even}\} \text{ and } \{ax^j|j \text{ is odd}\},
\]

and each element that has the form \( ax^k \) is of order two if \( 0 < k < n \) [JL, p.420]. Observe that \( Aut(G_{4n}) \) acts transitively on noncentral involutions since the map

\[a \rightarrow ax^k, \quad x \rightarrow x\]

determines an automorphism of \( G_{4n} \). So if we are given \( f \in WCT(G_{4n}) \), we can assume that \( f(a) = a \) by composing with an automorphism.

Given \( f \in WCT(G_{4n}) \) such that \( f(a) = a \), we know that \( f \) must send conjugacy classes to classes and so by considering the classes of \( G_{4n} \), we note that \( f(x^k) = x^{\alpha(k)} \) for some bijection \( \alpha : \mathbb{Z}/(n\mathbb{Z}) \rightarrow \mathbb{Z}/(n\mathbb{Z}) \), and \( f(ax^k) = ax^{\beta(k)} \) for some bijection \( \beta : \mathbb{Z}/(n\mathbb{Z}) \rightarrow \mathbb{Z}/(n\mathbb{Z}) \).

Then the following relations are a result of \( f \in WCT(G_{4n}) \):
\[ x^{\alpha(k+m)} = f(x^{k+m}) = f(x^k x^m) \sim f(x^k) f(x^m) = x^{\alpha(k)} x^{\alpha(m)}, \]

\[ x^{\alpha(k)} = f(x^k) = f(a^3 ax^k) \sim a^3 f(ax^k) = a^3 ax^{\beta(k)} = x^{\beta(k)}, \]

\[ x^{\alpha(m-k)} = f(x^{-k} x^m) = f(a^4 x^{-k} x^m) = f(a^3 x^k ax^m) \sim f(a^3 x^k) f(ax^m) = a^3 x^{\beta(k)} ax^{\beta(m)} = x^{\beta(m-k)} \sim \beta(k). \]

Using these equations in conjunction with the structure of the conjugacy classes, we find

\[ \alpha(k+m) = \pm (\alpha(k) + \alpha(m)), \]

\[ \alpha(k) = \pm \beta(k), \]

\[ \alpha(m-k) = \pm (\beta(m) - \beta(k)) \]

for all \( k, m \in \mathbb{Z}/(n\mathbb{Z}) \). Since \( f \in WCT(G_{4n}) \), we know \( \alpha(0) = 0, \beta(0) = 0, \alpha(-k) = -\alpha(k) \) and \( \beta(-k) = -\beta(k) \) for all \( k \in \mathbb{Z}/(n\mathbb{Z}) \).

Now suppose that \( \alpha(1) = r \) for some \( r \in \mathbb{Z}/(n\mathbb{Z}) \). Since \( f \) is a bijection, we know that \( \gcd(r, n) = 1 \). Then \( \alpha(-1) = -r \), since \( \alpha(-1 + 1) = \pm (\alpha(-1) + \alpha(1)) \). By the same equation, we also know that \( \alpha(2) = \pm 2r \). Then, if we consider \( \alpha(3) = \alpha(2 + 1) = \pm (\alpha(2) + \alpha(1)) = \pm (\pm 2r + r) \). Then since \( \alpha \) is a bijection, \( \alpha(3) \neq \alpha(-1) = -r \), so \( \alpha(2) = +2r \). Then \( \alpha(-2) = -2r \). So \( \alpha(3) = \pm 3r \). By similar reasoning, since \( \alpha(4) = \alpha(3 + 1) = \pm (\alpha(3) + \alpha(1)) = \pm (\pm 3r + r) \), we see that \( \alpha(4) = 4r \), and we can continue this to show that \( \alpha(k) = kr \) for all \( k \). So \( \alpha \) is an automorphism.

Then, since \( \alpha(k) = \pm \beta(k) \), we have \( \beta(k + m) = \pm (\beta(k) + \beta(m)) \) for all \( k, m \in \mathbb{Z}/(n\mathbb{Z}) \).
Then similar logic as shown above will show that $\beta$ is also an automorphism, and that $\beta(1) = \pm r$. Now if $\beta(1) = -r$, we can compose $\beta$ with the inverse map to get $\beta(1) = r$.

Then we have:

$$f(x^k x^m) = x^{\alpha(k+m)} = x^{\alpha(k)} x^{\alpha(m)} = f(x^k)f(x^m),$$

$$f(ax^k x^m) = ax^{\alpha(k+m)} = ax^{\alpha(k)} x^{\alpha(m)} = f(ax^k)f(x^m),$$

$$f(x^k ax^m) = ax^{\alpha(-k+m)} = x^{\alpha(k)} ax^{\alpha(m)} = f(x^k)f(ax^m);$$

$$f(ax^k ax^m) = a^2 x^{\alpha(-k+m)} = ax^{\alpha(k)} ax^{\alpha(m)} = f(ax^k)f(ax^m).$$

Therefore $f$ is then an automorphism, so the $WCT(G_{4n})$ is trivial.
Chapter 6. Some Non-trivial Weak Cayley Table Maps

Often it is difficult to find weak Cayley table maps that are not trivial. In this chapter, we are going to define some nontrivial maps for particular groups that are Camina pairs over their center.

In this chapter, given a finite group $G$ and elements $g_1, \ldots, g_r \in G$, we will write $(g_1, g_2, \ldots, g_r)$ to denote the permutation of $G$ which sends $g_1$ to $g_2$, $g_2$ to $g_3$, and so on. Thus $(g_1, g_2, \ldots, g_r) \in \text{Sym}(G)$.

6.1 Groups with a Camina Pair Structure Over a Center of Order 2

Theorem 6.1. Let $G$ be a group with center $Z = Z(G) = \langle z \rangle$ of order 2 and $(G, Z)$ is a Camina pair. For $g \in G - Z$ of order 2, let

$$\phi_g = (g, gz)$$

and for $g$ of order greater than 2 let

$$\phi_g = (g, gz)(g^{-1}, g^{-1}z).$$

Then $\phi_g$ is a weak Cayley table map for any $g \in G - Z$.

Proof. Let $< z > \subseteq Z$. Since $(G, Z)$ is a Camina pair, the conjugacy classes of $G$ are $\{1\}, \{z\}$, and then unions of sets of the form $\{g, gz\}$ for $g \notin G$. By interchanging $g$ and $gz$, the conjugacy classes of $G$ are preserved, so $\phi_g$ satisfies condition (i) for the definition of a weak Cayley table map.
To check that $\phi_g$ satisfies condition (ii), let $x, y \in G$ and consider the cases below. We may also assume that $x, y \neq 1$. Further the cases $x = z$ or the cases $y = z$ are easily checked, so we assume $x, y, \neq z$.

**Case 1:** $x \notin \{g, gz, g^{-1}, g^{-1}z\}$ and $y \notin \{g, gz, g^{-1}, g^{-1}z, x^{-1}g, x^{-1}gz, x^{-1}g^{-1}, x^{-1}g^{-1}z\}$.

Then $\phi_g$ fixes $xy, x$ and $y$. Thus $\phi_g(x) = xy = \phi_g(x)\phi_g(y)$.

**Case 2:** $x \notin \{g, gz, g^{-1}, g^{-1}z\}$ and $y = g$. Then $xy = xg \neq 1, z$,

\[
\phi_g(xy) = \phi_g(xg) = xg \text{ or } xgz, \quad \phi_g(x)\phi_g(y) = \phi_g(x)\phi_g(g) = xgz \text{ or } xg.
\]

Then note that $xg \sim xgz$, since $G$ is a Camina pair over $Z$ and $xg \notin Z$. So $\phi_g(xy) \sim \phi_g(x)\phi_g(y)$. Similar reasoning shows this for $y \in \{g, gz, g^{-1}, g^{-1}z\}$.

**Case 3:** $x \notin \{g, gz, g^{-1}, g^{-1}z\}$ and $y = x^{-1}g$. Then

\[
\phi_g(xy) = \phi_g(xx^{-1}g) = \phi_g(g) = gz, \quad \phi_g(x)\phi_g(y) = \phi_g(x)\phi_g(x^{-1}g) = xx^{-1}g = g.
\]

Since $g \sim gz$, we have $\phi_g(xy) \sim \phi_g(x)\phi_g(y)$. Similar reasoning shows this for $y$ an element of $\{x^{-1}g, x^{-1}g^{-1}, x^{-1}g^{-1}z\}$.

**Case 4:** $x = g$ and $y \notin \{g, gz, g^{-1}, g^{-1}z\}$. Then
\[ \phi_g(xy) = \phi_g(gy) \]
\[ = gy, \]
\[ \phi_g(x)\phi_g(y) = \phi_g(g)\phi_g(y) \]
\[ = gzy \]
\[ = gyz. \]

Whereas \( gy \sim gyz \), we observe that \( \phi_g(xy) \sim \phi_g(x)\phi_g(y) \). The same argument also works for \( x \in \{gz, g^{-1}, g^{-1}z\} \).

**Case 5:** \( x = g \) and \( y = g \). Then
\[ \phi_g(xy) = \phi_g(g^2) \]
\[ = g^2, \]
\[ \phi_g(x)\phi_g(y) = \phi_g(g)\phi_g(g) \]
\[ = ggz \]
\[ = g^2. \]

So \( \phi(xy) = \phi(x)\phi(y) \). This also works for the case when \( x \in \{g, gz\} \) and \( y \in \{g, gz\} \) or the case when \( x \in \{g^{-1}, g^{-1}z\} \) and \( y \in \{g^{-1}, g^{-1}z\} \).

**Case 6:** \( x = g \) and \( y = g^{-1} \). Then
\[ \phi_g(xy) = \phi_g(1) \]
\[ = 1 \]
\[ \phi_g(x)\phi_g(y) = \phi_g(g)\phi_g(g^{-1}) \]
\[ = ggz^{-1}z \]
\[ = 1. \]

So \( \phi(xy) = \phi(x)\phi(y) \). This final argument also works for \( x \in \{g, gz\} \) and \( y \in \{g^{-1}, g^{-1}z\} \) or \( x \in \{g^{-1}, g^{-1}z\} \) and \( y \in \{g, gz\} \).

**Example 6.2.** A quick example of such a group is the dihedral group \( D_8 \) of order 8, with presentation
\[ D_8 = \langle a, b \mid a^4 = b^2 = 1, ab = a^{-1} \rangle. \]
Then the center of $D_8$ is $Z = \langle a^2 \rangle$, and $(D_8, Z)$ is a Camina pair. Then $|b| = 2$ and $b \notin Z$, so

$$\phi_b = (b, ba^2)$$

is an element of $WCT(D_8)$. Since $WCT(D_8)$ is trivial [Hu], we know that $\phi_b$ is either an automorphism or anti-automorphism. With some simple computations, one can show that $\phi_b$ is an anti-automorphism for $D_8$.

### 6.2 $p$-Groups with a Camina Pair Structure

**Theorem 6.3.** Let $G$ be a group with cyclic center $\langle z \rangle = Z$, $|Z| = p$, such that $(G, Z)$ is a Camina pair, $G/Z$ is elementary $p$-abelian, and let $g \in G$ be noncentral element. Then the map

$$\phi_g = (g, gz, gz^2, \ldots, gz^{p-1})(g^{-1}, (gz)^{-1}, (gz^2)^{-1}, \ldots, (gz^{p-1})^{-1})$$

is a weak Cayley table map.

**Proof.** Since $G/Z$ is abelian and $|Z| = p$, we see that $Z = G'$, the commutator subgroup. If $p = 2$ we can use theorem 6.1, so assume $p$ is odd prime, and let

$$C = \{g, gz, gz^2, \ldots, gz^{p-1}\}$$

and

$$K = \{g^{-1}, (gz)^{-1}, (gz^2)^{-1}, \ldots, (gz^{p-1})^{-1}\}.$$

Then $C$ and $K$ are conjugacy classes in $G$ and $\phi_g$ fixes $C$, $K$ and all other conjugacy classes. So $\phi_g$ satisfies condition (i) of the definition of a weak Cayley table map.

The following are some cases to consider to prove the condition (ii) of a weak Cayley table map $\phi_g(xy) \sim \phi_g(x)\phi_g(y)$.

Let $x, y \in G$. The cases where $x \in \langle z \rangle$ or $y \in \langle z \rangle$ are easily checked, we we assume
$x, y \notin \langle z \rangle$.

**Case 1:** $x \notin C \cup K, y \notin C \cup K$, and $xy \notin C \cup K$. Then $\phi_g$ fixes $x, y$ and $xy$, so

$$\phi_g(xy) = xy = \phi_g(x)\phi_g(y).$$

**Case 2:** $x \notin C \cup K$, and $y = x^{-1}gz^i$. Then $y \notin C \cup K$ and so

$$\phi_g(xy) = \phi_g(xx^{-1}gz^i) = \phi_g(gz^i) = gz^{i+1},$$

$$\phi_g(x)\phi_g(y) = \phi_g(x)\phi_g(x^{-1}gz^i) = xx^{-1}gz^i = gz^i.$$  

Then $gz^{i+1} \sim gz^i$ and so $\phi_g(xy) \sim \phi_g(x)\phi_g(y)$.

**Case 3:** $x \notin C \cup K$, and $y = x^{-1}g^{-1}z^i$ is a similar argument as above.

**Case 4:** Then the cases where $y \notin C \cup K$ and $x = gz^iy^{-1}$ or $x = g^{-1}z^iy^{-1}$ are the same as the above, since $g, g^{-1}$ and $z$ all commute.

**Case 5:** $x = gz^i$, $y \notin C \cup K$, and $y \neq z^k$ or $g^{-2}z^k$. Then $xy \notin C \cup K$, so

$$\phi_g(xy) = \phi_g(gz^i y) = gyz^i,$$

$$\phi_g(x)\phi_g(y) = \phi_g(gz^i)\phi_g(y) = gz^{i+1}y.$$  

Since $gyz^i \sim gyz^{i+1}$, we have $\phi_g(xy) \sim \phi_g(x)\phi_g(y)$.

**Case 6:** $x = gz^i$, $y = g^{-2}z^k$. Then $xy = g^{-1}z^{i+k}$, so
\[
\phi_g(xy) = \phi_g(g^{-1}z^{i+k}) = g^{-1}z^{i+k-1},
\]
\[
\phi_g(x)\phi_g(y) = \phi_g(gz^i)\phi_g(g^{-2}z^k) = gz^{i+1}g^{-2}z^k = g^{-1}z^{i+k+1}.
\]

Since \(g^{-1}z^{i+k-1} \sim g^{-1}z^{i+k+1}\), we have \(\phi_g(xy) \sim \phi_g(x)\phi_g(y)\).

**Case 7:** \(x = g^iz, y = g^zk\).

\[
\phi_g(xy) = \phi_g(g^2z^{i+k}) = g^2z^{i+k},
\]
\[
\phi_g(x)\phi_g(y) = \phi_g(gz^i)\phi_g(gz^k) = gz^{i+1}gz^{k+1} = g^2z^{i+k+2}.
\]

Then since \(g^2z^{i+k} \sim gz^{i+k+2}\), so \(\phi_g(xy) \sim \phi_g(x)\phi_g(y)\).

**Case 8:** \(x = g^iz, y = g^{-1}z^k\). Then

\[
\phi_g(xy) = \phi_g(gg^{-1}z^{i+k}) = \phi_g(z^{i+k}) = z^{i+k},
\]
\[
\phi_g(x)\phi_g(y) = \phi_g(gz^i)\phi_g(g^{-1}z^k) = gz^{i+1}g^{-1}z^{k-1} = z^{i+k}.
\]

So in this case, \(\phi_g(xy) = \phi_g(x)\phi_g(y)\).

**Case 9:** Then the cases where \(x = g^{-1}z^i\) are the same as previous ones.

\[
\Box
\]

**Example 6.4.** A group that satisfies these hypotheses is the extraspecial 3-group of order
27 with exponent 3. Its presentation is given by

\[ G_{27} = \langle a, b, c : a^3 = b^3 = c^3 = 1, b^a = bc, a^c = a, b^c = b \rangle. \]

Note that \(|G_{27}| = 27\), it has center \(Z = \langle c \rangle\), the order of \(c\) is 3, \((G_{27}, Z)\) is a Camina pair, and \(G_{27}/Z\) is elementary abelian. Then consider the element \(a \in G_{27} - Z\). By Theorem 6.2, the map

\[ \phi_a = (a, ac, ac^2) (a^{-1}, (ac)^{-1}, (ac^2)^{-1}) \]

\[ = (a, ac, ac^2) (a^2, a^2c^2, a^2c) \]

is an element of \(WCT(G_{27})\). This map is non-trivial, since if we consider the element \(ab \in G_{27}\), we notice

\[ \phi_a(ab) = ab, \]

\[ \phi_a(a)\phi_a(b) = acb = abc, \]

and

\[ \phi_a(b)\phi_a(a) = bac = abc^2 \]

which are conjugate but not equal. Thus \(\phi_a\) is a nontrivial element of \(WCT(G_{27})\).

6.3 Another map for \(p\)-Groups with a Camina Pair Structure

**Theorem 6.5.** Let \(p\) be an odd prime, and let \(G\) be a \(p\)-group with cyclic center \(< z > = Z, |Z| = p\) such that \((G, Z)\) is a Camina pair, \(G/Z\) is elementary abelian, and let \(g \in G\) be a noncentral element of order \(q\). Then the map

\[ \phi_g = (g, gz, gz^2, \ldots, gz^{p-1}) (g^{-1}, (gz)^{-1}, (gz^2)^{-1}, \ldots, (gz^{p-1})^{-1}) \]

\[ (g^2, g^2z, g^2z^2, \ldots, g^2z^{p-1}) (g^{-2}, (g^2z)^{-1}, (g^2z^2)^{-1}, \ldots, (g^2z^{p-1})^{-1}) \]

\[ \vdots \]

\[ (g^{q-1}, g^{q-1}z, g^{q-1}z^2, \ldots, g^{q-1}z^{p-1}) (g^{-q+1}, (g^{q-1}z)^{-1}, (g^{q-1}z^2)^{-1}, \ldots, (g^{q-1}z^{p-1})^{-1}) \]

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is a weak Cayley table map.

Proof. Let

\[ C = \{ g^i, g^i z, g^i z^2, \ldots, g^i z^{p-1} | 1 \leq i \leq \frac{q-1}{2} \}, \]

and

\[ K = \{ g^{-i}, (g^i z)^{-1}, (g^i z^2)^{-1}, \ldots, (g^i z^{p-1})^{-1} | \frac{q-1}{2} \leq i \leq p-1 \}. \]

Also note that since \((G, Z)\) is a Camina pair and \(G/Z\) is elementary \(p\)-abelian, conjugacy classes are of the form \(xZ\) off of the center.

Note that \(\phi_g\) just permutes the elements of each conjugacy class, so the condition that \(\phi_g(x^G) = \phi_g(x)^G\) is met. All that is left is to check to see if \(\phi_g(xy) \sim \phi_g(x)\phi_g(y)\) for every \(x, y\) in \(G\). There are several cases to check:

**Case 1:** \(x \notin C \cup K\), \(y \notin C \cup K\), and \(xy \notin C \cup K\). Then \(\phi_g\) fixes \(x, y\) and \(xy\), so

\[ \phi_g(xy) = xy = \phi_g(x)\phi_g(y). \]

**Case 2:** \(x \notin C \cup K\) and \(y = x^{-1} g^i z^j\) where \(g^i z^j \in C\). Then

\[ \phi_g(xy) = \phi_g(xx^{-1} g^i z^j) = \phi_g(g^i z^j) = g^i z^{j+1}. \]

\[ \phi_g(x)\phi_g(y) = xx^{-1} g^i z^{j+1} = g^i z^{j+1}. \]

So \(\phi_g(xy) = xy = \phi_g(x)\phi_g(y)\).

**Case 3:** \(x \notin C \cup K\) and \(y = x^{-1} g^i z^j\) where \(g^i z^j \in K\). Then
\[
\phi_g(xy) = \phi_g(x^{-1}g^i z^j)
\]
\[
= \phi_g(g^i z^j)
\]
\[
= \phi_g(g^i z^j)
\]
\[
= g^i z^{j-1},
\]
\[
\phi_g(x)\phi_g(y) = xx^{-1}g^i z^{j-1}
\]
\[
= g^i z^{j-1}.
\]

So \(\phi_g(xy) = xy = \phi_g(x)\phi_g(y)\).

Similarly, for \(y \notin C \cup K\) and \(x = g^i z^j y^{-1}\), \(\phi_g(xy)\) is still conjugate to \(\phi_g(x)\phi_g(y)\).

**Case 4:** \(x = g^i z^j\) and \(y \notin Z\), \(y \notin C \cup K\), and so \(xy \notin C \cup K\). Then
\[
\phi_g(xy) = xy
\]
\[
= g^i y z^j,
\]
\[
\phi_g(x)\phi_g(y) = \phi_g(g^i z^j)\phi_g(y)
\]
\[
= g^i z^{j+1} y.
\]

Then \(g^i y z^j\) and \(g^i z^{j+1} y\) are elements of \(g^i y Z\). Thus \(\phi_g(xy) \sim \phi_g(x)\phi_g(y)\).

**Case 5:** \(x = g^i z^j\) and \(y = z^k\), which means \(xy = g^i z^{j+k}\). So we have that
\[
\phi_g(xy) = \phi_g(g^i z^{j+k})
\]
\[
= g^i z^{j+k+1},
\]
\[
\phi_g(x)\phi_g(y) = \phi_g(g^i z^j)\phi_g(z^k)
\]
\[
= g^i z^{k+1} z^k
\]
\[
= g^i z^{j+k+1}.
\]

Therefore \(\phi_g(xy) = \phi_g(x)\phi_g(y)\). The cases where \(y = g^i z^j\) and \(x \notin C \cup K\) are the same as above.

**Case 6:** If \(x = g^i z^j\) and \(y = g^l z^k\), where \(l \neq -i\), \(\phi_g\) will keep \(x, y,\) and \(xy\) in the conjugacy class of \(g^{i+l} Z\), so \(\phi_g(xy) \sim \phi_g(x)\phi_g(y)\).

**Case 7:** \(x = g^i z^j\) and \(y = g^{-i} z^k\). Then \(xy = z^{j+k}\). So then we have
\[
\begin{align*}
\phi_g(xy) &= \phi_g(z^{j+k}) \\
&= z^{j+k},
\end{align*}
\]
\[
\begin{align*}
\phi_g(x)\phi_g(y) &= \phi_g(g^i z^j)\phi_g(g^{-i} z^k) \\
&= g^i z^{j \pm 1} g^{-i} z^{k \mp 1} \\
&= z^{j+k}.
\end{align*}
\]

Therefore, \(\phi_g(xy) \sim \phi_g(x)\phi_g(y)\).

\[\Box\]

**Example 6.6.** An example of a group that satisfies the hypothesis of Theorem 6.5 is the extraspecial 5-group of order 125 with exponent 25. Its presentation is given by

\[G_{125} = \langle x, y, z | x^{25} = y^5 = z^5 = 1, x^5 = z, x^y = xz, z \ \text{central} \rangle.\]

The center is \(Z = \langle z \rangle, |z| = 5\), and \((G_{125}, Z)\) is a Camina pair. Then note that since \(y \notin Z\), by Theorem 6.3, the map

\[
\phi_y = (y, yz, yz^2, yz^3, yz^4)(y^{-1}, (yz)^{-1}, (yz^2)^{-1}, (yz^3)^{-1}, (yz^4)^{-1})
\]
\[
(y^2, y^2 z, y^2 z^2, y^2 z^3, y^2 z^4)(y^{-2}, (y^2 z)^{-1}, (y^2 z^2)^{-1}, (y^2 z^3)^{-1}, (y^2 z^4)^{-1})
\]

\[
= (y, yz, yz^2, yz^3, yz^4)(y^4, y^4 z^4, y^4 z^3, y^4 z^2, y^4 z)
\]
\[
(y^2, y^2 z^2, y^2 z^3, y^2 z^4)(y^3, y^3 z^4, y^3 z^3, y^3 z^2, y^3 z)
\]

is a weak Cayley table map. Note that

\[
\begin{align*}
\phi_y(xy) &= xy \\
\phi_y(x)\phi_y(y) &= xyz \\
\phi_y(y)\phi_y(x) &= yzx \\
&= xyz^2
\end{align*}
\]

are not equal, thus \(\phi_y\) is not an anti-automorphism or automorphism of \(G_{125}\).
Chapter 7. Relative Conjugacy Classes and Relative Weak Cayley Tables

We define a conjugacy classes for an element $x$ of a group $G$ to be the set $\{g^{-1}xg | g \in G\}$. Relative conjugacy classes are similar, only instead of conjugating an element of $x \in G$ over the entire group, we conjugate $x$ only by the elements of a particular subgroup of $G$. So if $H$ is a subgroup of $G$, the relative conjugacy class of $x$ with respect to $H$ (or the $H$-conjugacy class of $x$) is the set $\{h^{-1}xh | h \in H\}$. We will use the notation $x \sim_H y$ to mean that $x$ is conjugate to $y$ by an element in $H$.

This essentially splits some of the conjugacy classes into distinct parts. In particular, the set of relative conjugacy classes will have at least as many elements as the set of conjugacy classes of the group. For example, consider the dihedral group of order 8, $D_8$, with the presentation $\langle a, b | a^4 = b^2 = 1, a^b = a^3 \rangle$. Then the conjugacy classes are $\{1\}, \{a^2\}, \{a, a^3\}, \{b, ba^2\}$, and $\{ba, ba^3\}$.

If we let $H = \langle a \rangle$, then the relative conjugacy classes for $D_8$ with respect to $H$ would be $\{1\}, \{a\}, \{a^2\}, \{a^3\}, \{b, ba^2\}$, and $\{ba, ba^3\}$.

We can use these relative conjugacy classes to define a relative weak Cayley table. This is similar to a weak Cayley table except the entries of the table contain relative conjugacy classes. For example, if we consider the group $S_3$ with the subgroup $H = \langle (123) \rangle$, then the relative conjugacy classes of $S_3$ with respect to $H$ are $B_1 = \{1\}, B_2 = \{(12), (13), (23)\}, B_3 = \{(123)\}$ and $B_4 = \{(132)\}$. Then the relative weak Cayley table for $S_3$ with respect to $H$ is:
### 7.1 Relative Weak Cayley Table Maps

As with weak Cayley tables, it is convenient to know when two groups with given subgroups have the same relative weak Cayley table. To do so, we will define a map that preserves the weak Cayley table structure. Given two groups $G_1, G_2$ with subgroups $H_1, H_2$ respectively, a relative weak Cayley table map is a bijection $\phi : G_1 \to G_2$ such that

1. $\phi(H_1) = H_2$;
2. $\phi(x^{H_1}) = (\phi(x))^{H_2}$, for all $x \in G_1$;
3. $\phi(xy) \sim_{H_2} \phi(x)\phi(y)$ for all $x, y \in G_1$.

Since this map preserves the structure of the relative $H$-conjugacy classes, we say two groups with two specified subgroups have the same relative weak Cayley tables if there exists a relative weak Cayley table map between the two groups.

**Theorem 7.1.** There exists a relative weak Cayley table map between two non-isomorphic groups.

**Proof.** Consider the two non-isomorphic non-abelian groups of order $p^3$. They have presentations

$$G_1 = \langle a, b, c : a^p = b^p = c^p = 1, b^a = bc, a^c = a, b^c = b \rangle;$$

$$G_2 = \langle x, y, z : x^p = z, x^{p^2} = y^p = z^p = 1, xy = x^{p+1}, x^2 = x, y^2 = y \rangle;$$
with

\[ Z(G_1) = \langle c \rangle; \ Z(G_2) = \langle z \rangle. \]

Let \( H_1 = \langle a \rangle \) and \( H_2 = \langle y \rangle. \)

Then define the map \( \phi : (G_1, H_1) \to (G_2, H_2) \) by

\[ a^r b^s c^t \to x^s y^r z^{r s - t} \]

where \( rs \) is not congruent to 0 mod \( p. \)

We will show that \( \phi \) is a relative weak Cayley table map by finding the relative conjugacy classes and comparing several cases that arise.

Notice these two groups are Camina pairs over their centers, so this makes the relative \( H_i \)-conjugacy classes easy to compute: for \( (G_1, H_1) \):

\[
\begin{align*}
\{1\}, & \quad \{c\}, & \quad \{c^2\}, & \ldots, & \quad \{c^{p-1}\}, \\
\{a\}, & \quad \{ac\}, & \quad \{ac^2\}, & \ldots, & \quad \{ac^{p-1}\}, \\
\vdots & & & & \\
\{a^i\}, & \quad \{a^i c\}, & \quad \{a^i c^2\}, & \ldots, & \quad \{a^i c^{p-1}\}, \\
\vdots & & & & \\
\{a^{p-1}\}, & \quad \{a^{p-1} c\}, & \quad \{a^{p-1} c^2\}, & \ldots, & \quad \{a^{p-1} c^{p-1}\}
\end{align*}
\]

and then

\[ a^i b^j \{1, c, c^2, \ldots, c^{p-1}\} \]

for \( 0 \leq i \leq p - 1, \ 0 < j \leq p - 1; \)

and for \( (G_2, H_2) \):
\[
\{1\}, \quad \{z\}, \quad \{z^2\}, \quad \ldots, \quad \{z^{p-1}\}, \\
\{y\}, \quad \{yz\}, \quad \{yz^2\}, \quad \ldots, \quad \{yz^{p-1}\}, \\
\quad \vdots \\
\{y^i\}, \quad \{y^i z\}, \quad \{y^i z^2\}, \quad \ldots, \quad \{y^i z^{p-1}\}, \\
\quad \vdots \\
\{y^{p-1}\}, \quad \{y^{p-1} z\}, \quad \{y^{p-1} z^2\}, \quad \ldots, \quad \{y^{p-1} z^{p-1}\},
\]
and then
\[
x^i y^j \{1, z, z^2, \ldots, z^{p-1}\}
\]
for \(0 < i \leq p - 1, 0 \leq j \leq p - 1\).

By inspection, it is easy to see that \(\phi\) will send \(H_1\) to \(H_2\), and that \(\phi\) will send \(H_1\)-conjugacy classes in \(G_1\) to \(H_2\)-conjugacy classes in \(G_2\), which are the first two conditions of a relative weak Cayley table map. The last thing to check is to see if \(\phi(\alpha \beta) \sim_{H_2} \phi(\alpha)\phi(\beta)\) for all \(\alpha, \beta \in G_1\).

Let \(\alpha = a^i b^j c^k\) and \(\beta = a^r b^s c^t\). Then there are three cases that can happen which we need to compare to determine if \(\phi\) is a relative weak Cayley table map: when \(j + s < p\), \(j + s > p\), and \(j + s = p\).

Case 1: Let \(j + s < p\). Then \(\alpha \beta = a^i b^j a^r b^s c^{k+t} = a^{i+r} b^{i+s} c^{k+t+rj}\), so
\[
\phi(\alpha \beta) = x^{j+s} y^{i+r} z^{-k-t-rj}.
\]
On the other hand \(\phi(\alpha) = x^i y^j z^{ij-k}\) and \(\phi(\beta) = x^s y^r z^{rs-t}\). Therefore we have
\[
\phi(\alpha)\phi(\beta) = x^{j+y} y^{ij-k} x^s y^r z^{rs-t} = x^{j+s} y^{i+r} z^{ij-k+rs-t+si}.
\]
This gives \(\phi(\alpha \beta)\) is \(H_2\)-conjugate to \(\phi(\alpha)\phi(\beta)\).
**Case 2:** Let \( j + s > p \), then \( j + s = w + p \) for some \( 0 < w < p \). Thus

\[
\phi(\alpha\beta) = x^w y^{j+r} z^{k-t-rj+1},
\]

and

\[
\phi(\alpha)\phi(\beta) = x^w y^{j+r} z^{ij-k+rs-t+si+1},
\]

both of which are also in the same \( H_2 \)-conjugacy class.

**Case 3:** Let \( j + s = p \). Then \( \phi(\alpha\beta) \) needs to be equal to \( \phi(\alpha)\phi(\beta) \) in order to be \( H_2 \)-conjugate. Note that \( j = -s \mod(p) \).

In this case we have

\[
\phi(\alpha\beta) = y^{j+r} z^{-k-t-rj+1} = y^{j+r} z^{-k-t+rs+1},
\]

and

\[
\phi(\alpha)\phi(\beta) = y^{j+r} z^{ij-k+rs-t+si+1} = y^{j+r} z^{-is-k+rs-t+is+1} = y^{j+r} z^{-k-t+rs+1}.
\]

Thus \( \phi(\alpha\beta) = \phi(\alpha)\phi(\beta) \).

\[\square\]

**Proposition 7.2.** Let \( G_1, G_2 \) and \( G_3 \) be groups with subgroups \( H_1, H_2 \) and \( H_3 \) respectively such that there exist relative weak Cayley table maps \( \phi : (G_1, H_1) \to (G_2, H_2) \) and \( \psi : (G_2, H_2) \to (G_3, H_3) \). Then the map \( \psi \circ \phi : (G_1, H_1) \to (G_3, H_3) \) is a relative weak Cayley table map.

**Proof.** By the definition of \( \phi \) and \( \psi \), we have

\[
\phi(H_1) = H_2,
\]
\[ \psi(H_2) = H_3 \]

and so

\[ \psi \circ \phi(H_1) = H_3. \]

Let \( a, b \) be elements of \( G_1 \). Then if we consider the relative conjugacy class \( a^{H_1} \), we note that

\[ \phi(a^{H_1}) = \phi(a)^{H_2}, \]

and

\[ \psi(\phi(a)^{H_2}) = (\psi \circ \phi(a))^{H_3}. \]

Lastly, consider the relationships

\[ \phi(ab) \sim_{H_2} \phi(a)\phi(b), \]

which implies

\[ \psi(\phi(ab)) \sim_{H_3} \psi(\phi(a))\psi(\phi(b)). \]

\[ \square \]

**Corollary 7.3.** Let \( G_1 \) and \( G_2 \) have subgroups \( H_1 \) and \( H_2 \) respectively such that there exists a relative weak Cayley table map \( \phi : (G_1, H_1) \rightarrow (G_2, H_2) \). Also let \( \alpha \in \text{Aut}(G_1) \) and \( \beta \in \text{Aut}(G_2) \). Then \( \phi \circ \alpha : (G_1, \alpha^{-1}(H_1)) \rightarrow (G_2, H_2) \) and \( \beta \circ \phi : (G_1, H_1) \rightarrow (G_2, \beta(H_2)) \) are also relative weak Cayley table maps.

**Proof.** Automorphisms are relative weak Cayley table maps, so by using Proposition 7.2, we can easily see these compositions are relative weak Cayley table maps. \[ \square \]
Chapter 8. Relative Weak Cayley Table Groups

We define the set of relative weak Cayley table maps from $G$ with a subgroup $H$ to itself by $RWCT(G, H)$.

**Theorem 8.1.** Let $\phi \in RWCT(G, H)$, then $\phi \in WCT(G)$.

*Proof.* By definition if $\phi \in RWCT(G, H)$, then $\phi(xy) \sim_H \phi(x)\phi(y)$ which also implies $\phi(xy) \sim_G \phi(x)\phi(y)$. This implies the second condition of a weak Cayley table map.

Then, let $x' = \phi^{-1}(x)$ and $y' = \phi^{-1}(y)$. Then we know that

$$\phi(x'y') \sim_H \phi(x')\phi(y') = xy$$

also

$$\phi(y'x') \sim_H \phi(y')\phi(x') = yx$$

and $xy \sim_G yx$, which means that $\phi(x'y') \sim_G \phi(y'x')$.

So given any $g \in G$, $\phi(g) = \phi(gaa^{-1})$ for any $a \in G$. Then $\phi(gaa^{-1}) \sim_G \phi(a^{-1}ga)$ by the above argument. This implies that $\phi(g^G) = \phi(g)^G$, which is the first condition of a weak Cayley table map. Thus $\phi \in WCT(G)$. 

**Theorem 8.2.** $RWCT(G, H)$ is a subgroup of $WCT(G)$.

*Proof.* Given $\phi, \psi \in RWCT(G, H)$, then $\phi, \psi \in WCT(G)$. Clearly, $\phi \circ \psi$ takes $H$-conjugacy classes to $H$-classes. Then given two maps $\phi$ and $\psi$ in $RWCT(G, H)$, we know from Proposition 7.2 that $\psi \circ \phi$ is also in $RWCT(G, H)$.

Since $WCT(G)$ is a group, for every $\phi \in RWCT(G, H)$, there is a $\phi^{-1} \in WCT(G)$. Since $\phi(H) = H, \phi^{-1}(H) = H$, and since $\phi$ sends $H$-conjugacy classes to $H$-conjugacy classes, $\phi^{-1}$ does the same.
Then let $x' = \phi^{-1}(x)$ and $y' = \phi^{-1}(y)$. Then, since $\phi \in RWCT(G)$, we have that $\phi(x'y') \sim_H \phi(x')\phi(y') = xy$. Since we know that $\phi^{-1}$ takes $H$-conjugacy classes to $H$-conjugacy classes, we know that $\phi^{-1}\phi(x'y') \sim_H \phi^{-1}(xy)$. So $x'y' \sim_H \phi^{-1}(xy)$, which implies that $\phi^{-1}(x)\phi^{-1}(y) \sim_H \phi^{-1}(xy)$. Therefore, $\phi^{-1} \in RWCT(G, H)$, and $RWCT(G, H)$ is a subgroup of $WCT(G)$.

$\square$
**Chapter 9. Extensions of Results from [JMS]**

**Theorem 9.1.** Let $G$ be a group with $|G|$ odd. Let $N$ be an abelian group that is also a $G$-module and assume the $N$-conjugacy classes outside of $N$ are $Ng$. Let $G_1$ and $G_2$ be two non-isomorphic groups which are extension of $N$ by $G$ such that $(G_1, N)$ and $(G_2, N)$ are Camina pairs. Then $(G_1, N)$ and $(G_2, N)$ have the same relative weak Cayley tables.

**Proof.** This result was proven for weak Cayley tables in [JMS] without the assumption that the $N$-classes outside of $N$ are $Ng$. We will show that the map that he defined is also a relative weak Cayley table map between $(G_1, N)$ and $(G_2, N)$ with this additional condition.

Some notation that [JMS] used was to use the extension structure of $G_1$ and $G_2$ to write

$$(n_1, g_1) \circ_i (n_2, g_2) = (n_1 n_2^{g_1^{-1}} f_i(g_1, g_2), g_1 g_2),$$

where $\circ_i$ represents the multiplication in the particular group $G_1$ or $G_2$ and $f_i$ is a 2-cocycle in $H^2(G, N)$. One can assume that $f_i(g, e) = f_i(e, g) = e$ for all $g \in G_i$.

They then partitioned $G - \{e\}$ into two subsets, $S_1$ and $S_2$, where if $g \in S_1$, then $g^{-1} \in S_2$, and $S_1 \cup S_2 = G$. This is possible since $|G|$ is odd. Using these subsets, they defined the map $\phi : G_1 \to G_2$ as follows:

$$\phi(n, e) = (n, e) \quad \text{for } n \in N,$$

$$\phi(n, g) = (n, g) \quad \text{for } g \in S_1,$$

$$\phi((n, g)^{inv(1)}) = (n, g)^{inv(2)} \quad \text{for } g \in S_1,$$

where $inv(i)$ represents the inverse in $G_i$.

In [JMS], they then went on to prove that this is a weak Cayley table map.

To see that this map is also a relative weak Cayley table map, note that since we have Camina pairs over the abelian subgroup $N$ and that the $N$-classes outside of $N$ are $Ng$, the
$N$-conjugacy classes of $G_i$ ($i = 1, 2$) are the singleton sets $\{1\}, \{(n, e)\}$ for every $n \in N$, and the cosets $Ng$ for every $g \in G - \{1\}$. It is clear that $\phi$ preserves the $N$-conjugacy classes, thus satisfying condition (ii) of a relative weak Cayley table map.

For condition (iii), let $A = \phi((n_1, g_1) \circ_1 (n_2, g_2))$ and let $B = \phi(n_1, g_1) \circ_2 \phi(n_2, g_2)$, and we want to show that $A$ and $B$ are in the same $N$-conjugacy class, which would imply that $\phi$ is a relative weak Cayley table map between $G_1$ and $G_2$.

We then need to consider three cases:

**Case 1:** $g_1 = g_2 = e$. Then $A = (n_1 n_2, e) = B$.

**Case 2:** $g_1 \neq g_2^{-1}$. Then $A = (m_1, g_1 g_2)$ for some $m_1 \in N$, and $B = (m_2, g_1 g_2)$ for some $m_2 \in N$. Then, since $g_1 g_2 \notin N$, $A$ and $B$ are in the same coset $Ng_1 g_2$, so $A$ and $B$ are conjugate by an element of $N$.

**Case 3:** $g_2 = g_1^{-1} \neq e$. Assume without loss of generality that $g_1 \in S_1$. Then $(n, g) = (n, e) \circ (e, g)$ for all $n \in N$ and $g \in G$. So then

$$(n, g) \circ (m, g)^{-1} = (n, e) \circ (e, g) \circ (e, g)^{-1} \circ (m, e)^{-1} = (nm^{-1}, e).$$

Which means

$$A = \phi((n_1, g_1) \circ_1 (m, g_1)^{\text{inv}(1)}) = \phi((n_1 m^{-1}, e)) = (n_1 m^{-1}, e),$$

and

$$B = \phi(n_1, g_1) \circ_2 \phi((m, g_1)^{\text{inv}(1)}) = (n_1, g_1) \circ_2 (m, g_1)^{\text{inv}(2)} = (n_1 m^{-1}, e).$$

So $A = B$, which is what was needed for $A$ and $B$ to be $N$-conjugate.

The following lemma shows that the conditions required in Theorem 4.1 from [JMS] (referenced in this paper as Theorem 1.8) force any weak Cayley table to be an automorphism...
when the action of $G_i$ is Frobenius on $N$.

**Lemma 9.2.** Suppose that $G_1$ and $G_2$ have the same weak Cayley table, and $\alpha : G_1 \rightarrow G_2$ is a weak Cayley table map. Suppose that $H_i$ is a Frobenius extension of $G_i$ by the module $N$ in such a way that $n^g = n^{\alpha(g)}$ for all $g$ in $G_1$. Then $G_1 \cong G_2$.

**Proof.** Note that all $g, h \in G_1$, we have

$$n^{gh} = n^{\alpha(gh)}.$$ 

From the group actions we also have:

$$n^{\alpha(gh)} = n^{gh} = (n^g)^h = (n^{\alpha(g)})^{\alpha(h)} = n^{\alpha(g)\alpha(h)}.$$ 

This means that

$$n^{\alpha(gh)} = n^{\alpha(g)\alpha(h)}.$$ 

So

$$n^{\alpha(gh)(\alpha(g)\alpha(h))^{-1}} = n.$$ 

Since the action is Frobenius, this implies that

$$\alpha(gh)(\alpha(g)\alpha(h))^{-1} = 1,$$

and so

$$\alpha(gh) = \alpha(g)\alpha(h),$$

which shows that $\alpha$ is a homomorphism, which means that it is also an isomorphism. □

The following theorem allows us to remove the condition that $N$ be abelian from the statement of Theorem 4.1 of [JMS] if we require that the action of $G_i$ is Frobenius on $N$. 

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Theorem 9.3. [Is, p.179] Let $H_i = N \rtimes G_i$ be a Frobenius group with kernel $N$. If the order of $G_i$ is even, then $G_i$ has at most one involution and $N$ must be abelian.

When the order of $G_i$ is odd, the authors of [JMS] comment that in the proof of Theorem 4.1, the fact that $N$ is abelian is not necessary.

Theorem 9.4. Suppose there exists a weak Cayley table map $\alpha : G_1 \to G_2$. Let $H_i$ be a Frobenius extension of $G_i$ by the normal subgroup $N$, such that in $H_1$ and $H_2$, the relative $N$-conjugacy classes outside of $N$ are unions of cosets of $N$. Lastly, for any involution $x \in G_1$, we require

$$ (e, x)^2 = (e, \alpha(x))^2. $$

Then $H_1$ and $H_2$ have the same relative weak Cayley table.

Proof. First we note that $(H_1, N)$ and $(H_2, N)$ are Camina pairs, since $H_i$ is a Frobenius extension [Is, pg. 185]. As in the proof of 9.1, write $H_1$ and $H_2$ as group extensions. Let $I$ denote the set of involutions of $G_1$ (if it exists). Next, partition $G_1 - \{e\} - I$ into two subsets $S$ and $S^{-1}$ (where $S^{-1} = \{s^{-1} | s \in S\}$). Then we have that $G_1 = \{e\} \cup I \cup S \cup S^{-1}$ and $G_2 = \{e\} \cup \alpha(I) \cup \alpha(S) \cup \alpha(S^{-1})$.

The map that the authors of [JMS] prove is a weak Cayley table map is

$$ \phi(n, g) = (n, \alpha(g)), \quad \text{for } g \in \{e\} \cup \{x\} \cup S, $$

$$ \phi((n, g)^{-1}) = (n, \alpha(g))^{-1}, \quad \text{for } g \in S, $$

$$ \phi((e, x)) = (e, \alpha(x)), \quad \text{for } x \in I. $$

To see this is a relative weak Cayley table map, we need to know the $N$-conjugacy classes of $H_i$. Note that by the hypothesis, the $N$-conjugacy classes contained in $H_1$ lying in $N$ are the same as those lying in the copy of $N$ in $H_2$, so $\phi$ automatically preserves those $N$-classes. Then the rest of the conjugacy classes are unions of cosets $\{(n, g) | n \in N\}$ for a $g \in G_i$. Knowing the $N$-conjugacy classes, it is clear that $\phi$ is a bijection that sends $N$-conjugacy classes to $N$-conjugacy classes.
Then to show that \( \phi \) is a relative weak Cayley table map, we need to show that \( \phi((n, g)(m, h)) \sim_N \phi(n, g)\phi(m, g) \) for all \( n, m \in N \) and \( g, h \in G \). Let \( A = \phi((n, g)(m, h)) \) and let \( B = \phi(n, g)\phi(m, g) \).

**Case 1:** \( g = h = e \). Then \( A = (nm, e) = B \).

**Case 2:** \( g \neq h^{-1} \). Then \( A = (m_1, gh) \) for some \( m_1 \in N \), and \( B = (m_2, gh) \) for some \( m_2 \in N \). Then \( A \) and \( B \) are in the same coset \( Ng_1g_2 \), so \( A \) and \( B \) are conjugate by an element of \( N \).

**Case 3:** \( h = g^{-1} \neq e \). The authors of [JMS] show that for all \( g \in G \), \( \phi((n, g))^{-1} = \phi((n, g)^{-1}) \), and this is the only spot in his proof where he used the action of \( G \) on \( N \). If the order of \( G \) is even, then in order for \( \phi((n, g))^{-1} = \phi((n, g)^{-1}) \) to hold, \( N \) must be abelian. However, since \( N \) is a Frobenius kernel of \( G \), the fact that \( |G| \) is even forces \( N \) to be abelian. If the order of \( G \) is odd, then no assumptions on \( N \) are needed to obtain \( \phi((n, g))^{-1} = \phi((n, g)^{-1}) \).

Therefore, an equivalent statement to \( A \sim_N B \) is to show

\[
\phi((n, g)(m, h)^{-1}) \sim_N \phi((n, g))\phi((m, h)^{-1}).
\]

Consider the two computations:

\[
\phi((n, g)(m, g)^{-1}) = \phi((nm^{-1}, e)) = (nm^{-1}, e)
\]

and

\[
\phi((n, g))\phi((m, g)^{-1}) = \phi(n, g)(\phi(m, g))^{-1} = (n, \alpha(g))(m, \alpha(g))^{-1} = (n, e)(\alpha(g)((m, e)(e, \alpha(g)))^{-1} = (n, e)(e, \alpha(g))^{-1}(m^{-1}, e) = (nm^{-1}, e).
\]

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These show that $A = B$, and hence conjugate, which means $\phi$ is a relative weak Cayley table map. □

The authors of [KR] describe a construction of groups that meet the criteria for Theorem 9.4. They start with a semi-direct product of semi-linear maps acting on a finite vector space, and then choose specific subgroups of this semi-direct product.

**Example 9.5.** [KR, Definition 3.1 and 3.2, pg. 278-279] Let $G_n$ be a group with a cyclic normal subgroup $D_n = \langle d_n \rangle$ of order $n$ with complement $C_{\mu(n)} = \langle c \rangle$, where $\mu(n)$ is the Euler function, i.e., the number of primitive $n^{th}$ roots of unity. The group $D_n$ should be interpreted as the group of $n^{th}$ roots of unity, on which $C_{\mu(n)}$ acts as the Galois group.

By $G_{n,m}$ we denote the subgroup of $G_n$, where the complement $C_m = \langle c_m \rangle$ is generated by an element of order $m$ dividing $\mu(n)$. In addition, we require that $m^2$ divides $n$, and that $m$ and $n/m^2$ are relatively prime.

A further assumption is that $p$ is a rational prime, such that $\mathbb{F}_{p^m}$ is the smallest field of characteristic $p$ containing all $n^{th}$ roots of unity. Then $M = \mathbb{F}_{p^m}$ is an irreducible $G_{n,m}$-module, on which $d_n$ acts by multiplication with a primitive $n^{th}$ root of unity, and $c_m$ acts as the Frobenius automorphism, i.e., raising to the $p^{th}$ power.

By $G_{n,m,p}$ we denote the semi-direct product $M \rtimes G_{n,m}$. Let $b_m = d_n^{m/n^2}$ be an element of order $m^2$ in $D_n$. There is no loss of generality if we assume that $b_m^{c_m} = b_m^{m+1}$. We note that $b_m^m$ lies in the center of $G_{n,m}$.

The authors of [KR] then show that the element

$$(c_m b_m^i)^m = b_m^{im},$$

is central, and then proceed to define the subgroups which meet the criteria of Theorem 9.4.

Let $1 \leq i \leq m - 1$ be relatively prime to $m$, and define the group $H_{n,m,i}$ as a subgroup
of $G_{n,m}$ by

\[ H_{n,m,i} = \langle d_n^{m^2}, b_m^m, c_m b_m^i \rangle, \]

and put

\[ H_{n,m,i,p} = M \rtimes H_{n,m,i}, \]

the semi-direct product with the module $M$.

An example of such groups are the subgroups of $\mathbb{F}_{7^3} \rtimes (\mathbb{F}_{7^3}^* \times C_3)$. Then $d^9 \in \mathbb{F}_{7^3}$ has order 19, $b \in \mathbb{F}_{7^3}$ has order 9, $c$ generates $C_3$, and

\[ H_{n,m,i} = \langle d^9, b^3, cb^i \rangle, \]

for $i = 1, 2$. Then

\[ H_{n,m,i,p} = \mathbb{F}_{7^3} \rtimes \langle d^9, b^3, cb^i \rangle. \]

These $H_{n,m,i,p}$ are Frobenius groups with kernel $M$, and the $H_{n,m,i}$ are isomorphic for all $i$ such that $1 \leq i \leq m - 1$ and $i$ is relatively prime to $m$. Also, for each $1 \leq i \leq m - 1$, the orbits of $H_{n,m,i}$ are the same on $M$ [KR, pg. 279]. This is the same as stating that the relative $M$-classes of $H_{n,m,i,p}$ inside $M$ are the same for all $i$ relatively prime to $m$, $(1 \leq i \leq m - 1)$. Since $H_{n,m,i,p}$ is a Frobenius group with kernel $M$, $(H_{n,m,i,p}, M)$ is a Camina pair. Thus the groups $H_{n,m,i,p}$ satisfy all the hypotheses of Theorem 9.4. □
Chapter 10. Relative Weak Cayley Table Map Group of $AGL(1, p)$

As in Chapter 2 let $G = AGL(1, p)$ have the presentation

$$G = \langle a, b | a^{p-1} = b^p = 1, a^b = b^r \rangle,$$

and let $B$ be the subgroup $\langle b \rangle$.

Then the $B$-conjugacy classes of $G$ are the singletons $\{1\}, \{b\}, \{b^2\}, \cdots \{b^{p-1}\}$, and the cosets of the form $a^iB$ where $i \neq 0$.

Since every weak Cayley table map sends $B$ to itself, we can define a map

$$\Psi : RWCT(G, B) \rightarrow WCT(G/B)$$

to be the restriction of the map $\Phi : WCT(G, B) \rightarrow WCT(G/B)$ (as defined in chapter 2) to the subgroup $RWCT(G, B)$. Then let $K = Ker(\Psi)$.

If $\psi \in K$ and is also a weak Cayley table map, then from Lemma 2.2 in Chapter 2, we know that $\psi(a^i b^j) = a^i b^{\alpha(i,j)}$ where $\alpha(i, j)$ is an injective function on $(F)_p$ to itself such that $\alpha(0, 0) = 0$, and $-r^{-i} \alpha(i, j) = \alpha(-i, -r^{-i}j)$ for every $0 \leq i \leq p - 2$ and $0 \leq j \leq p - 1$.

Also $\psi|_B$ has to be the identity map on $B$, since the $B$-conjugacy classes on $B$ are singletons. Also, since $B$ is abelian, $\psi|_B$ must be an automorphism.

**Lemma 10.1.** Let $\beta$ be an automorphism of $B$. Then map $\beta^* : G \rightarrow G$ which sends $b \rightarrow \beta(b)$ and $a^i b \rightarrow a^i \beta(b)$ is an automorphism.

**Proof.** Let $a^i b^j, a^k b^l$ be in $G$, then using the relations established in Chapter 2, we have:

$$\beta^*(a^i b^j a^k b^l) = \beta^*(a^{i+k} b^{r^j+l}) = a^{i+k} \beta(b^{r^j+l}),$$

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and
\[ \beta^*(a^i b^j) \beta^*(a^k b^l) = a^i \beta(b^j) a^k \beta(b^l) = a^{i+k} \beta(b^j) \beta(b^l) = a^{i+k} \beta(b^l j k). \]

Similar to chapter 2, we can construct a relative weak Cayley table map in the kernel $K$ by permuting elements of cosets and respecting their inverse cosets. For $a^i B$, for $1 < i \leq \frac{p-3}{2}$ pick any permutation on the elements of the coset. This determines a permutation of its inverse class. Also there is one coset of involutions, $a^{\frac{p-1}{2}} B$ and any permutation of the elements of this coset will preserve inverses. So from these permutations, we have a subgroups of $RWCT(G, B)$ isomorphic to $S_p \times S_{\frac{p-3}{2}}$.

Then, given one of the above maps, we can compose it with an automorphism like those in Lemma 10.1 to get permissable permutations of the elements in $B$. These give you all of the maps $\psi$ in the kernel $K$. This means

\[ K \cong (S_p \times S_{\frac{p-3}{2}}) \rtimes \text{Aut}(B). \]

Then for any relative weak Cayley table map, we can view it as a composition of permutations on the nontrivial cosets of $B$ composed with an element of the kernel $K$. As above in chapter 2, this gives a subgroup of $WCT(G)$ isomorphic to

\[ ((S_p \times S_{\frac{p-3}{2}}) \rtimes \text{Aut}(B)) \rtimes \text{Cox}_B(\frac{p-3}{2}). \]
Chapter 11. Automorphisms, Anti-Automorphisms and $RWCT(G, H)$

Note that automorphisms always satisfy the requirements for a relative Weak Cayley table map for $G_1 = G_2$, since they are isomorphisms. However, while anti-automorphisms are weak Cayley table maps, they are not always relative weak Cayley table maps.

Example 11.1. Consider the group $S_3$. The relative-$S_2$ conjugacy classes are

$$B_1 = \{1\}, \quad B_2 = \{(12)\}, \quad B_3 = \{(13), (23)\}, \quad B_4 = \{(123), (132)\}.$$ 

Since we can write any anti-automorphism as an automorphism composed with the inverse map, it is sufficient to check if the inverse map is a relative weak Cayley table map.

However, note that the inverse map $\alpha : S_3 \to S_3$ given by $\alpha(g) = g^{-1}$ fails to be a relative weak Cayley table map, since

$$\alpha((132)(13)) = \alpha((12)) = (12).$$

However,

$$\alpha((132))\alpha((13)) = (123)(13) = (23).$$

Note that $(12)$ is not $S_2$-conjugate to $(23)$, so the inverse map $\alpha$ fails to be a relative weak Cayley table map, which implies that no anti-automorphisms of $S_3$ are in $RWCT(S_3, S_2)$.

Theorem 11.2. Given a group $G$ with a subgroup $H$, $RWCT(G, H)$ contains the anti-automorphisms if and only if for every $a \not\in H, Ha \cap C_G(g) \neq \emptyset$ for all $g \in G$.

Proof. Since any anti-automorphism can be expressed as an automorphism composed with the inverse map, it is sufficient to find when the inverse map $\alpha : G \to G$ is in $RWCT(G, H)$.

Note that since $\alpha$ permutes $H$-conjugacy classes, $\alpha \in RWCT(G, H)$ is equivalent to $\alpha(ab) \sim_H \alpha(a)\alpha(b)$ for all $a, b \in G$. Then since
\[ \alpha(ab) = (ab)^{-1} = b^{-1}a^{-1}, \]

and

\[ \alpha(a)\alpha(b) = a^{-1}b^{-1}, \]

the statement \( \alpha(ab) \sim_H \alpha(a)\alpha(b) \) for all \( a, b \in G \) is equivalent to \( ab \sim_H ba \) for all \( a, b \in G \). This is the same as \( (ab)^h = ba \) for some \( h \in H \) or \( abh = hba \).

If either \( a \) or \( b \) are in \( H \), then we can find \( h \in H \) such that \( abh = hba \). The reason for this is if \( b \in H \), then take \( h = b^{-1} \). Then

\[ abh = abb^{-1} = a \]

and

\[ hba = b^{-1}ba = a. \]

If \( a \in H \), take \( h = a \), then

\[ abh = aba = hba. \]

So suppose \( a \) and \( b \) are not in \( H \). Then we note that since \( ba = (ab)^a \),

\[
(ab)^h = ba = (ab)^a \iff (ab)^ha^{-1} = ab
\]

\[
\iff ha^{-1} \in C_G(ab)
\]

\[
\iff Ha^{-1} \cap C_G(g) \neq \emptyset \text{ for all } g \in G, \text{ and all } a \in G - H.
\]

\[ \square \]

An example of a group and a subgroup that satisfies the hypotheses in Theorem 11.2 is the group \( S_3 \times C_2 \), with the subgroup \( S_3 \). Let \( C_2 = \langle t \rangle \), then we can write the two cosets of \( S_3 \) are itself and \( S_3t \).

Note that \( S_3t \) contains the element \( t \), which is central. Certainly \( t \in C_{S_3 \times C_2}(g) \) for every \( g \in S_3 \times C_2 \), so the pair \( (S_3 \times C_2, S_3) \) fits the criteria of Theorem 11.2, and so anti-automorphisms of \( S_3 \times C_2 \) are elements of \( RWCT(S_3 \times C_2, S_3) \).
Chapter 12. Relative Weak Cayley Table Maps and Characters

Let \( \phi : (G_1, H_1) \to (G_2, H_2) \) be a relative weak Cayley table map from a group \( G_1 \) with subgroup \( H_1 \) to a group \( G_2 \) with subgroup \( H_2 \). Further, let \( \chi_{1,i} \) be the irreducible characters of \( G_1 \), and let \( \chi_{2,i} \) be the irreducible characters of \( G_2 \). Let \( \psi_{1,i} \) be the irreducible characters of \( H_1 \) and let \( \psi_{2,i} \) be the irreducible characters of \( H_2 \). Then define an action by \( \phi \) on the characters \( \chi_{2,i} \) of \( G_2 \) by \( \phi \cdot \chi_{2,i} = \chi_{2,i}(\phi(g_1)) \) where \( g_1 \in G_1 \). Thus for every character \( \chi_{2,i} \) of \( G_2 \) we obtain a function \( \phi \cdot \chi_{2,i} : G_1 \to \mathbb{C} \). We prove that

**Proposition 12.1.** \( \phi \cdot \chi_{2,i} \) is a character of \( G_1 \).

**Proof.** Let \( g_1 \) be conjugate to \( k_1 \) in \( G_1 \). Since \( \phi \) is a relative weak Cayley table map we have

\[
\phi(g_1) \sim \phi(k_1).
\]

Then considering that \( \chi_{2,i} \) is a character of \( G_2 \), we know that

\[
\chi_{2,i}(\phi(g_1)) = \chi_{2,i}(\phi(k_1)).
\]

This can be rewritten as \( \phi \cdot \chi_{2,i}(g_1) = \phi \cdot \chi_{2,i}(k_1) \), which means that \( \phi \cdot \chi_{2,i} \) is constant on the conjugacy classes of \( G_1 \), so \( \phi \cdot \chi_{2,i} \) is a character of \( G_1 \). \( \square \)

**Proposition 12.2.** For a relative weak Cayley table map \( \phi : (G_1, H_1) \to (G_2, H_2) \), and a character \( \psi_{2,i} \) of \( G_2 \) as in the above, we have that \( \phi \cdot \psi_{2,i} \) is an \( H_1 \)-class function.

**Proof.** The proof is very similar to the above proof for \( \phi \cdot \chi_{2,i} \). Let \( g_1 \sim_{H_1} k_1 \) in \( G_1 \). Then since \( \phi \) is a relative weak Cayley table map,

\[
\phi(g_1) \sim_{H_2} \phi(k_1),
\]

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\[
\psi(\phi(g_1)) = \psi(\phi(k_1)),
\]
which shows \(\phi \cdot \psi_2(g_1) = \phi \cdot \psi_2(k_1)\). Therefore \(\phi \cdot \psi_2\) is an \(H_1\)-class function.

**Theorem 12.3.** The character \(\phi \cdot \chi_{2, i}\) is irreducible if and only if \(\chi_{2, i}\) is irreducible.

**Proof.** Since \(\chi_{2, i}\) is irreducible, we know that the inner product

\[
(\chi_{2, i}, \chi_{2, i}) = \frac{1}{|G_2|} \sum_{g_2 \in G_2} \chi_{2, i}(g_2) \bar{\chi}_{2, i}(g_2) = 1.
\]

Then the inner product of \(\phi \cdot \chi_{2, i}\) with itself is

\[
(\phi \cdot \chi_{2, i}, \phi \cdot \chi_{2, i}) = \frac{1}{|G_1|} \sum_{g_1 \in G_1} \phi \cdot \chi_{2, i}(g_1) \bar{\phi} \cdot \chi_{2, i}(g_1)
= \frac{1}{|G_1|} \sum_{g_1 \in G_1} \chi_{2, i}(\phi(g_1)) \bar{\chi}_{2, i}(\phi(g_1)).
\]

Since \(\phi\) is a relative weak Cayley table map, we can rewrite this in terms of \(G_2\):

\[
(\phi \cdot \chi_{2, i}, \phi \cdot \chi_{2, i}) = \frac{1}{|G_2|} \sum_{g_2 \in G_2} \chi_{2, i}(g_2) \bar{\chi}_{2, i}(g_2)
= (\chi_{2, i}, \chi_{2, i}) = 1.
\]

So \(\phi \cdot \chi_{2, i}\) is an irreducible character of \(G_1\). The other implication is similar.

**Theorem 12.4.** The \(H\)-class function \(\phi \cdot \psi_{2, i}\) is irreducible if and only if \(\psi_{2, i}\) is irreducible.
Proof. Assume that $\psi_{2,1}$ is an irreducible $H$-class function. Then the inner product of $\phi \cdot \psi_{2,i}$ with itself is

$$
(\phi \cdot \psi_{2,i}, \phi \cdot \psi_{2,i}) = \frac{1}{|H_1|} \sum_{h_1 \in H_1} \phi \cdot \psi_{2,i}(h_1) \overline{\phi \cdot \psi_{2,i}(h_1)}
$$

$$
= \frac{1}{|H_1|} \sum_{h_1 \in H_1} \psi_{2,i}(\phi(h_1)) \overline{\psi_{2,i}(\phi(h_1))}.
$$

Then, since $\phi$ maps $H_1$ into $H_2$ bijectively, we can rewrite this expression as

$$
(\phi \cdot \psi_{2,i}, \phi \cdot \psi_{2,i}) = \frac{1}{|H_2|} \sum_{h_2 \in H_2} \psi_{2,i}(h_2) \overline{\psi_{2,i}(h_2)}
$$

$$
= (\psi_{2,i}, \psi_{2,i}) = 1.
$$

Thus $\phi \cdot \psi_{2,i}$ is an irreducible character of $H_1$. \qed

Definition 12.5. Given a group $G$ with a subgroup $H$, we call the map $\psi : G \to \mathbb{C}$ an $H$-class function of $G$ if $\psi$ is constant on the $H$-classes.

Theorem 12.6. A relative weak Cayley table map $\phi : (G_1, H_1) \to (G_2, H_2)$ determines a correspondence between the irreducible characters $\chi_{1,1}, \chi_{1,2}, \ldots, \chi_{1,s}$ of $G_1$ and $\chi_{2,1}, \chi_{2,2}, \ldots, \chi_{2,s}$ of $G_2$, and a correspondence between the irreducible $H_1$-characters $\psi_{1,1}, \psi_{1,2}, \ldots, \psi_{1,s}$ on $G_1$ and the $H_2$-characters $\psi_{2,1}, \psi_{2,2}, \ldots, \psi_{2,r}$ on $G_2$.

Proof. Above we showed that $\phi \cdot \chi_{2,i}$ is an irreducible character of $G_1$ obtained from an irreducible character $\chi_{2,i}$ of $G_2$. Note that $G_1$ and $G_2$ have the same number of irreducible characters. Thus it is sufficient to show that if $\chi_{2,i}$ and $\chi_{2,j}$ are two distinct irreducible characters of $G_2$, then $\phi \cdot \chi_{2,i} \neq \phi \cdot \chi_{2,j}$. This will complete the correspondence required.
If $H_2$-characters $\chi_{2,i}$ and $\chi_{2,j}$ of $G_2$ are distinct, then for some element $g_2 \in G_2$, $\chi_{2,i}(g_2) \neq \chi_{2,j}(g_2)$. Further since $\phi$ is a bijection and $\phi^{-1}(g_2)$ is an element in $G_1$, we find
\[
\phi \cdot \chi_{2,i}(\phi^{-1}(g_2)) = \chi_{2,i}(\phi(\phi^{-1}(g_2))) = \chi_{2,i}(g_2),
\]
and
\[
\phi \cdot \chi_{2,j}(\phi^{-1}(g_2)) = \chi_{2,j}(\phi(\phi^{-1}(g_2))) = \chi_{2,j}(g_2).
\]
So $\phi \cdot \chi_{2,i}$ and $\phi \cdot \chi_{2,j}$ must be distinct, irreducible characters of $G_1$.

The same argument on the irreducible $H_1$-characters and $H_2$-characters show the correspondence for the subgroups’ characters. □
Chapter 13. Overview of Spherical functions

Most of the information in this section came from [Tr]. Spherical functions are very similar to characters. They are functions that are constant on the relative conjugacy classes for a particular subgroup, and they have many of the same properties that characters possess.

Definition 13.1. Let $G$ be a finite group with subgroup $H$, let $\chi$ be a character of $G$, and let $\psi$ be a character of $H$. Then the spherical function $Y_{\chi \psi} : G \to \mathbb{C}$ is defined as

$$Y_{\chi \psi}(g) = \frac{1}{|H|} \sum_{\sigma \in H} \chi(g\sigma)\psi(\sigma^{-1}).$$

The following properties of spherical functions can be found in [Tr]:

(i) $Y_{\chi \psi}(1) = (\chi|_H, \psi)$,

(ii) $Y_{\chi \psi}(g^{-1}) = \overline{Y_{\chi \psi}(g)}$,

(iii) For $h$ in $H$, $Y_{\chi \psi}(hgh^{-1}) = Y_{\chi \psi}(g)$,

(iv) Let $f_\chi$ be the degree of $\chi$ and let $\xi$ be another character of $G$. Then

$$Y_{\chi \psi} \ast \xi = \delta_{\chi \xi} \frac{1}{f_\chi} Y_{\chi \psi},$$

where $\ast$ denotes the convolution. In other words if $\phi_1$ and $\phi_2$ are functions from $G$ into the complex numbers, then

$$(\phi_1 \ast \phi_2)(g) = \frac{1}{|G|} \sum_{h \in G} \phi_1(gh^{-1})\phi_2(h). \ [Ga]$$

(v) Let $\phi$ be another character of $H$, and let $\overline{\phi}$ represent the function of $G$ that vanishes
off $H$, and is equal to $|G : H| \cdot \phi$ on $H$, then

$$Y_{\chi \psi} \ast \bar{\phi} = \delta_{\phi \psi} \frac{1}{f_{\phi}} Y_{\chi \psi},$$

and

$$Y_{\chi \psi} \ast Y_{\xi \phi} = \delta_{\chi \xi} \delta_{\phi \psi} \frac{1}{f_{\chi} f_{\psi}} Y_{\chi \psi}.$$

(vi) The regular representation can be written as:

$$R(g) = \sum_{\chi, \psi} f_{\chi} f_{\psi} Y_{\chi \psi}(g).$$

(vii)

**Theorem 13.2.** [Tr, Theorem 1] The following are equivalent

(a) $Y_{\chi \psi}$ is a $G$-class function;

(b) $Y_{\chi \psi}$ is proportional to $\chi$;

(c) $\chi|_H = c_{\chi \psi} \cdot \psi$.

(viii)

**Theorem 13.3.** [Tr, Theorem 1′] The following are equivalent

(a) $Y_{\chi \psi}$ vanishes off of $H$;

(b) $Y_{\chi \psi}$ is proportional to $\bar{\psi}$;

(c) $\psi^G = c_{\chi \psi} \cdot \chi$. 
Chapter 14. Relative Weak Cayley Tables and Spherical Functions

For notation in this chapter let $G_1, G_2$ be groups with $H_1, H_2$ as subgroups respectively, such that $\phi : (G_1, H_1) \to (G_2, H_2)$ is a relative weak Cayley table map. Also let $\chi_{1,1}, \chi_{1,2}, \ldots, \chi_{1,s}$ be the irreducible characters of $G_1$, and let $\chi_{2,1}, \chi_{2,2}, \ldots, \chi_{2,s}$ be those of $G_2$. Further let $\psi_{1,1}, \psi_{1,2}, \ldots, \psi_{1,s}$ be the irreducible $H_1$-characters and $\psi_{2,1}, \psi_{2,2}, \ldots, \psi_{2,s}$ be those of $H_2$.

Theorem 14.1. Let $\chi_2$ be an irreducible character for $G_2$, and let $\chi_1 = \phi \cdot \chi_2$ be the corresponding irreducible character in $G_1$. Also let $\psi_2$ be an irreducible character for $H_2$, and let $\psi_1 = \phi \cdot \psi_2$ be the corresponding irreducible character in $H_1$. Then

$$Y_{\chi_2 \psi_2}(g_2) = Y_{\chi_1 \psi_1}(\phi^{-1}(g_2))$$

for all $g_2 \in G_2$.

Proof. By the definition of a spherical function

$$Y_{\chi_1 \psi_1}(\phi^{-1}(g_2)) = \frac{1}{|H_1|} \sum_{\sigma \in H_1} \chi_1(\phi^{-1}(g_2)\sigma) \psi_1(\sigma^{-1})$$

$$= \frac{1}{|H_1|} \sum_{\sigma \in H_1} \chi_1(\phi^{-1}(g_2)\phi^{-1}(\phi(\sigma))) \psi_1(\sigma^{-1}).$$

Then because $\phi$ is a relative weak Cayley table map, $\phi^{-1}(g_2)\phi^{-1}(\phi(\sigma)) \sim_{H_1} \phi^{-1}(g_2\phi(\sigma))$ and since $Y_{\chi_1 \psi_1}$ is constant on conjugacy classes, we can rewrite the above equation as:
\[ Y_{\chi_1\psi_1}(\phi^{-1}(g_2)) = \frac{1}{|H_1|} \sum_{\sigma \in H_1} \chi_1(\phi^{-1}(g_2\phi(\sigma)))\psi_1(\sigma^{-1}) \]
\[ = \frac{1}{|H_1|} \sum_{\sigma \in H_1} \chi_2(\phi(\phi^{-1}(g_2\phi(\sigma))))\psi_2(\phi(\sigma^{-1})) \]
\[ = \frac{1}{|H_1|} \sum_{\sigma \in H_1} \chi_2(g_2\phi(\sigma))\psi_2(\phi(\sigma^{-1})). \]

Since \( \phi \) is a bijection between \( H_1 \) and \( H_2 \), this becomes

\[ = \frac{1}{|H_2|} \sum_{\sigma \in H_2} \chi_2(g_2\sigma)\psi_2(\sigma^{-1}) \]
\[ = Y_{\chi_2\psi_2}(g_2). \]
Chapter 15. An Example of Relative Weak Cayley Table Maps With Spherical Functions

The goal of this Chapter is to illustrate Theorem 14.1 with the non-isomorphic, non-abelian groups of order \(p^3\).

Example 15.1. For a prime \(p\), let \(G_1\) and \(G_2\) be the two non-isomorphic, non-abelian groups of order \(p^3\). As in Chapter 7, let them have the following presentations:

\[
G_1 = \langle a, b, c : a^p = b^p = c^p = 1, b^a = bc, a^c = a, b^c = b \rangle;
\]

\[
G_2 = \langle x, y, z : x^p = z, x^{p^2} = y^p = z^p = 1, x^y = x^{p+1}, x^z = x, y^z = y \rangle.
\]

Let \(H_1 = \langle a \rangle\) and \(H_2 = \langle y \rangle\). In Chapter 7 we defined a bijection \(\phi : G_1 \rightarrow G_2\) which is a relative weak Cayley table map from \((G_1, H_1)\) to \((G_2, H_2)\). Recall that \(\phi\) was defined to be the map

\[
a^r b^s c^t \mapsto x^s y^r z^{r^s-t}.
\]

Note that \(G_1\) and \(G_2\) have the same character table. The irreducible characters of \(G_1\) are

\[
\chi_{1:u,v}(a^r b^s c^t) = e^{ru+sv},
\]

and

\[
\xi_{1:u}(a^r b^s c^t) = \begin{cases} 
0 & \text{if } a \neq 0 \text{ or } b \neq 0; \\
pe^{ut} & \text{otherwise}; 
\end{cases}
\]
where $0 \leq u, v \leq p - 1$ and $\epsilon$ is a primitive $p^{th}$ root of unity [JL, p.301-304]. For $G_2$ we have

$$\chi_{2;u,v}(x^s y^r z^t) = \epsilon^{ru+sv},$$

and

$$\xi_{1;u}(x^s y^r z^t) = \begin{cases} 0 & \text{if } x \neq 0 \text{ or } y \neq 0; \\ pe^{ut} & \text{otherwise.} \end{cases}$$

As shown above, we can act on the character $\chi_{2;u,v}$ of $G_2$ by $\phi$ to get an irreducible character of $G_1$:

$$\phi \cdot \chi_{2;u,v}(a^r b^s c^t) = \chi_{2;u,v}(\phi(a^r b^s c^t)) = \chi_{2;u,v}(x^s y^r z^{-t}) = \epsilon^{ru+sv} = \chi_{1;u,v}(a^r b^s c^t).$$

And if we look at the irreducible characters of $H_1$ these are just $\psi_{1;j}(a^i) = \epsilon^{ij}$ since $H_1$ is cyclic of order $p$. (For $H_2$ the irreducible characters are $\psi_{2;j}(y^i) = \epsilon^{ij}$.)

Then to examine the spherical functions of $G_1$ with $H_1$, we notice

$$Y_{\chi_{1;u,v},\psi_{1;j}}(a^r b^s c^t) = \frac{1}{|H|} \sum_{y' \in H} \chi_{1;u,v}(a^r b^s c^t a^i) \psi_{1;j}(a^{-i})$$

$$= \frac{1}{p} \sum_{y' \in H} \chi_{1;u,v}(a^r b^s c^t) \psi_{1;j}(a^{-i}),$$

since $a^r b^s c^t a^i \sim_{G_1} a^r a^r b^s c^t$, so
\[
Y_{\chi_{1:u,v},\psi_{1:j}}(a^rb^sc^t) = \frac{1}{p} \sum_{y^i \in H} \chi_{1:u,v}(a^{r+i}b^sc^t) \psi_{1:j}(a^{-i})
\]

\[
= \frac{1}{p} \sum_{i=1}^{p} \epsilon^{(r+i)u+sv} \epsilon^{-ij}
\]

\[
= \frac{1}{p} \epsilon^{ru+sv} \sum_{i=1}^{p} \epsilon^{ij(u-j)}
\]

If \( u = j \) this becomes

\[
Y_{\chi_{1:u,v},\psi_{1:j}}(a^rb^sc^t) = \frac{1}{p} \epsilon^{ru+sv} \sum_{i=1}^{p} \epsilon^{i(0)}
\]

\[
= \epsilon^{ru+sv},
\]

and if \( u \neq j \) then \( \epsilon^{ij(u-j)} \) runs over all the roots of unity, so we get

\[
Y_{\chi_{1:u,v},\psi_{1:j}}(a^rb^sc^t) = \frac{1}{p} \epsilon^{ru+sv} \sum_{i=1}^{p} \epsilon^{i(u-j)}
\]

\[
= \frac{1}{p} \epsilon^{ru+sv}(0)
\]

which implies

\[
Y_{\chi_{1:u,v},\psi_{1:j}}(a^rb^sc^t) = 0.
\]

To summarize,

\[
Y_{\chi_{1:u,v},\psi_{1:j}}(a^rb^sc^t) = \begin{cases} 
\epsilon^{ru+sv} & \text{if } u = j; \\
0 & \text{otherwise.}
\end{cases}
\]

Next if we consider the other characters, \( \xi_{1:u} \), on \( G_1 \), we notice
\[
Y_{\xi_1, u, \psi_{1,j}}(a^r b^s c^t) = \frac{1}{p} \sum_{y_i \in H} \xi_{1:u}(a^r b^s c^t a^i) \psi_{1:j}(a^{-i})
\]

\[
= \frac{1}{p} \sum_{y_i \in H} \xi_{1:u}(a^{r+i} b^s c^t) \psi_{1:j}(a^{-i}).
\]

Because of how \(\xi_{1:u}\) is defined, when \(r + i \neq 0\) we have that \(\xi_{1:u}(a^{r+i} b^s c^t) = 0\). Thus

\[
\frac{1}{p} \sum_{y_i \in H} \xi_{1:u}(a^{r+i} b^s c^t) \psi_{1:j}(a^{-i}) = \frac{1}{p} \xi_{1:u}(a^{r-r} b^s c^t) \psi_{1:j}(a^r)
\]

\[
= \frac{1}{p} \xi_{1:u}(b^s c^t) \psi_{1:j}(a^r).
\]

Further \(\xi_{1:u}(b^s c^t) = 0\) if \(s \neq 0\). Therefore if \(s \neq 0\),

\[
Y_{\xi_1, u, \psi_{1,j}}(a^r b^s c^t) = 0,
\]

and if \(s = 0\),

\[
Y_{\xi_1, u, \psi_{1,j}}(a^r c^t) = \frac{1}{p} \xi_{1:u}(c^t) \psi_{1:j}(a^r)
\]

\[
= \frac{1}{p} (p \epsilon^{ut})(\epsilon^{rj}) = \epsilon^{ut+rj}.
\]

To summarize, these spherical functions are of the form

\[
Y_{\xi_1, u, v, \psi_{1,u}}(a^r b^s c^t) = \begin{cases} 
0 & \text{if } s \neq 0; \\
\epsilon^{ut+rj} & \text{if } s = 0.
\end{cases}
\]

The same sort of calculations will result in similar spherical functions for \(G_2\) with \(H_2\).
Lastly we note the inverse of $x^r y^s z^t$ is

$$\phi^{-1}(x^r y^s z^t) = a^s b^r c^{-t}$$

and the action of $\phi$ on $\chi_{2;u,v}$ and $\psi_{2;j}$ gives

$$\phi \cdot \chi_{2;u,v} = \chi_{1;u,v} \text{ and } \phi \cdot \psi_{2;j} = \psi_{1;j}.$$  

Thus $\chi_{1;u,v}, \chi_{2;u,v}, \psi_{1;j},$ and $\psi_{2;j}$ satisfy the hypothesis to Theorem 14.1. Then notice

$$Y_{\chi_{2;u,v}, \psi_{2;u}}(x^r y^s z^t) = \frac{1}{p} e^{ru+sv}$$

and

$$Y_{\chi_{1;u,v}, \psi_{1;u}}(a^s b^r c^{-t}) = \frac{1}{p} e^{ru+sv}.$$  

So we have

$$Y_{\chi_{2;u,v}, \psi_{2;u}}(x^r y^s z^t) = Y_{\chi_{1;u,v}, \psi_{1;u}}(\phi^{-1}(x^r y^s z^t)),$$

which is the conclusion to Theorem 14.1.
Chapter 16. Questions for Further Research

Here we list questions for future work.

(i) If $H$ is normal in $G$ what can one say about the subgroup $RWCT(G, H)$ of $WCT(G)$?

(ii) For any subgroup, when is $RWCT(G, H)$ a normal subgroup of $WCT(G)$?

**Example 16.1.** $RWCT(S_3)$ does not contain anti-automorphisms, then $RWCT(G)$ is strictly contained in $WCT(G)$. Also $WCT(S_3)$ is trivial, meaning it is only composed of automorphisms and anti-automorphisms, which means that $RWCT(G, H)$ has index 2, and therefore is normal.

(iii) Are there any non-trivial groups $G$ such that $WCT(G) = RWCT(G, H)$?

(iv) Are the anti-automorphisms of $S_4$ in $RWCT(S_4, S_3)$?

(v) Given a subgroup $H$ of $G$, if $\phi \in RWCT(G, H)$, then we have an induced map $\phi^* : \mathbb{C}G^H \to \mathbb{C}G^H$. What conditions do we need to go the other direction?

(vi) Do there exist non-isomorphic groups $G_1, G_2$ and $\phi : G_1 \to G_2$ a weak Cayley table map such that for every subgroup $H$ contained in $G_1$, $\phi(g^H) = \phi(g)^{\phi(H)}$ and $\phi^*(RWCT(G_1, H)) = RWCT(G_2, \phi(H))$?

(vii) What other applications do $RWCT(G, H)$ maps have to spherical functions?

(viii) What other conditions are equivalent to the condition: given a group $G$ and a subgroup $H$, for every $a \notin H, Ha \cap C_G(a)$ is non-empty?

(ix) What conditions are needed for a weak Cayley table map to be a relative weak Cayley table map for some nontrivial subgroup?
(x) If $WCT(G)$ is not trivial, is there always a nontrivial relatively weak Cayley table map for some nontrivial subgroup?
Bibliography


