A Position Analysis of Coupled Spherical Mechanisms in Action Origami

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Dear Prof. A. Kecskeméthy,

We are pleased to submit the following paper for publication in *Machine and Mechanism Theory*:


This is the first time the paper has been submitted, and no part of its contents have been submitted to any other journal or conference. It fits within the general areas of spherical mechanisms and overconstrained mechanisms.

If there is any additional information that would be helpful to you I would be happy to provide it.

Thank you for your consideration,

Landen A. Bowen
Highlights

An origami vertex is equivalent to a spherical change-point mechanism.
Most action origami achieves motion through coupled systems of spherical mechanisms.
The position analysis of an origami vertex includes output angle and coupler path.
A method for analyzing coupled systems of spherical mechanisms is proposed.
The results can help create compact, deployable mechanisms for application.
A Position Analysis of Coupled Spherical Mechanisms Found in Action Origami

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Abstract

Origami has been previously utilized in design to create deployable systems. Action origami, origami designed to move, has the ability to deploy to a larger state and have motion in the deployed state. The majority of action origami achieves motion through coupled systems of spherical mechanisms. An origami vertex, the point at which folds converge, is shown to be equivalent to a spherical change-point mechanism. A position analysis of an origami vertex is presented, resulting in a relationship between input and output angles as well as the path of the coupler link. A method for analyzing coupled systems of repeated spherical mechanisms is proposed and demonstrated using two examples. Better understanding the kinematics of action origami increases the ability of designers to create compact, deployable mechanisms for use in packaging, space, and medical industries.

Keywords: Spherical Mechanisms, Origami

1. Introduction

The ancient Japanese art of origami has intrigued people for centuries with a beautiful complexity arising from the single fabrication process of folding. The quantity and complexity of origami models has been increasing partly due to the introduction of mathematical tools which model and characterize the design parameters found in the art [1, 2]. Engineers and designers have looked to origami for inspiration due to its potential in deployable systems. Origami, for example, has been used as a source of inspiration for space applications [3, 4] and automobile safety [5, 6].

Figure 1: Shafer’s “Monster Mouth” [7] is an example of action origami. This model can be stored in a flat state (1a). Once deployed (1b), it can then open its mouth (1c).

Action origami refers to the subset of origami models designed to move. Simple examples include flapping birds and opening mouths (Figure 1). Action origami has received relatively little attention in the literature but has great potential to inspire new and useful mechanisms. This is in part due to the ability of many action origami models to...
deploy from a compact (and even flat) state to a larger state. In addition, these models are designed to exhibit motion in the deployed state. There are several areas of application for such mechanisms, including solar pointing arrays and minimally invasive surgical tools.

Action origami has been shown to achieve motion through the use of spherical mechanisms [8]. Action origami that achieves motion through the use of spherical mechanisms is termed “kinematic origami.” All possible kinematic origami configurations have been classified based on the number and arrangement of spherical mechanisms [9]. Some kinematic origami models contain dozens of coupled spherical mechanisms, but can be fabricated by simple folding in just a few minutes. As kinematic origami often utilizes symmetry for visual appeal, many models consisting of several spherical mechanisms use the same mechanism (equivalent link lengths) repeated throughout the model.

In order to utilize the unique properties of kinematic origami in engineering design, the kinematics of coupled spherical mechanism systems must first be understood. The current literature dealing with coupled spherical systems is sparse, possibly because there has not previously been motivation to investigate them. Recent studies show that spherical mechanisms (both single and coupled) demonstrate several advantages over planar mechanisms including lower values of inertial forces and better pressure angle values [10, 11].

Making explicit the commonality between spherical mechanisms and kinematic origami makes possible mathematical models that can greatly enhance the analysis, optimization, and synthesis of new mechanisms with motion inspired by kinematic origami.

The purpose of this paper is to take a first step toward the advanced analysis and synthesis of kinematic origami-inspired mechanisms by describing and demonstrating a method for the position analysis of coupled systems of repeated spherical mechanisms. Such repeated systems are common in kinematic origami models and tessellations. This method has the potential to make kinematic origami-based design more effective and structured, moving much of the design process to computer-based tools.

2. Literature Review

The growing relationship between origami artists and technical designers has resulted in increased literature on the subject of connections between origami and mechanism design. Greenberg et al. [12] point out that origami is a form of compliant mechanisms where the creases act as pin joints to allow movement, and goes on to show that graph theory can act as a step between kinematics and origami. Liu and Dai similarly identify carton panels as mechanism links and creases as joints when investigating carton mobility [13].

The origami community has long identified and labeled characteristics of origami models and has particularly noted movement about a single vertex [14], the point at which fold lines converge. It has been noted that an origami vertex (including the surrounding folds and panels) is equivalent to a traditional spherical mechanism [15, 8, 16].

A recent examination of action origami models verifies that much of action origami is composed of configurations of spherical mechanisms [9], where action origami which achieves motion from spherical mechanisms is called “kinematic origami”. A classification scheme based on the spherical mechanism structure of kinematic origami was presented that encompasses all possible configurations. Several of these categories, “Single”, “Coupled”, and “Loop”, are investigated in this paper.

Lamina emergent mechanisms (LEM$s$) are a subset of compliant mechanisms that are monolithic, planar, and have motion emerging from the plane of fabrication [17]. Action origami shares all of these characteristics. All possible LEM spherical mechanisms have been classified [18], resulting in 21 possible LEM spherical 4R types being sorted into 3 categories.

The motion of origami has been investigated previously using several different methods. Screw theory has been used to calculate the mobility of a carton during the folding process [13]. Spherical trigonometry has been utilized to perform some motion studies of a packaging carton [19]. Quaternions and dual quaternions have been used to analyze the motion of an origami model known as the square twist [20]. Other methods have also been used [21, 22, 1].

The motion of traditional spherical mechanisms is well-documented [23, 24, 25], and the modeling and visualization of spherical mechanisms has been demonstrated [26].
3. Method

This section describes the position analysis of a single origami vertex, which is the basic building block of the more complex systems described later. The method applies to systems that are rigidly foldable, meaning that motion is possible without warping when origami creases are considered to be joints and facets are considered to be rigid links. Many multi-vertex kinematic origami models are composed of the same mechanism repeated one or more times, thus the results of one vertex can be translated and rotated to the other vertices. This is demonstrated with two examples: Shafer’s “Chomper” [27] and the traditional “Square Twist”. Taking advantage of symmetry in this manner can result in a significant reduction of computation time for larger multi-vertex models.

3.1. Origami vertex

As mentioned above, an origami vertex is equivalent to a spherical mechanism when folds are treated as joints and panels as links. A byproduct of the flat-foldability of origami is the fact that all origami vertices’ link lengths sum to 360°. This means that each origami vertex is actually a spherical change point mechanism. Traditional spherical mechanisms reside on a portion of a sphere (less than a hemisphere) whereas an origami vertex resides on an entire hemisphere. Although other types are possible, the majority of vertices found in action origami have 4 links, and thus only spherical 4R mechanisms are discussed below.

![Figure 2: Single origami vertex and corresponding spherical mechanism to be analyzed.](image)

3.2. Relationship between input and output angle

Let us consider the origami vertex shown in Figure 2a and its corresponding spherical mechanism in Figure 2b. Note that we decide upon a link that will be grounded. Also, a radius is selected for the mechanism because this will determine joint locations along the folds. We will use a radius of 1, or the unit sphere. The input angle, $\phi$, is defined as the internal angle between ground and the input link. The output angle, $\psi$, is defined as the external angle between ground and the output link.

Let ground be labelled $f$, the input link as $a$, the coupler link as $c$, and the output link as $b$.

From [23], an explicit expression of $\psi$ in terms of $f$, $a$, $c$, $b$, and $\phi$ is as follows:

$$a + c + b + f = 360^\circ.$$
Figure 3: Position analysis of a single origami vertex where \( a = c = 45^\circ, b = f = 135^\circ, \theta_0 = 30^\circ, \theta_P = 0^\circ, \) and \( R_P = 1.5. \)

\[
k_1 = \cos(a) \cdot \cos(b) \cdot \cos(f) \tag{1}
\]
\[
k_2 = \cos(c) \tag{2}
\]
\[
k_3 = \sin(a) \cdot \cos(b) \cdot \sin(f) \tag{3}
\]
\[
k_4 = \cos(a) \cdot \sin(b) \cdot \sin(f) \tag{4}
\]
\[
k_5 = \sin(a) \cdot \sin(b) \cdot \cos(f) \tag{5}
\]
\[
k_6 = \sin(a) \cdot \sin(b) \tag{6}
\]

and let

\[
h_1 = k_1 - k_2 + k_3 \cdot \cos(\phi) \tag{7}
\]
\[
h_2 = -k_4 + k_5 \cdot \cos(\phi) \tag{8}
\]
\[
h_3 = k_6 \cdot \sin(\phi) \tag{9}
\]

Using the above parameters, we obtain the following expression for \( \psi \) obtained from spherical trigonometry:

\[
\tan(\psi) = \frac{h_2 \cdot h_3 \pm [h_2^2 \cdot h_3^2 - (h_2^2 - h_3^2) \cdot (h_1^2 - h_3^2)]^{1/2}}{h_1^2 - h_3^2} \tag{10}
\]

Note that the above equation has two solutions. Taking the solution that most closely resembles the expected relationship and accounting for a change of quadrant in the output link results in a relationship between input and output angles (Figure 3a). The location at which the change in quadrant of the output link occurs must be accounted for.

### 3.3. Angle between input and coupler

The internal angle \( \beta \) between the input link and the coupler link is useful in determining the location of the coupler point. An equation for \( \beta \) in terms of \( f, a, c, b, \) and \( \psi \) is as follows [24]:

\[
\cos(\beta) = \frac{\sin(b) \cdot \sin(f) \cdot \cos(\psi) + \cos(b) \cdot \cos(f) - \cos(a) \cdot \cos(c)}{\sin(a) \cdot \sin(c)} \tag{11}
\]
3.4. Equation for a coupler point

We now have all of the information necessary to determine the location of a coupler point for all input angles. Let \( \theta_0 \) be defined as the length of the link to the desired coupler point \( P \), \( \theta_P \) as the offset angle from the coupler link to the point \( P \), and \( R_P \) as the distance from the center to the desired coupler point (Figure 4a). Three equations for the cartesian coordinates of a coupler point \( P \) are given as [25]:

\[
\begin{align*}
    r_{Px} &= \left[ \cos(\theta_0) \cdot \cos(a) + \sin(\theta_0) \cdot \cos(\beta + \theta_P) \cdot \sin(a) \right] \cdot R_P \\
    r_{Py} &= \left[ \cos(\theta_0) \cdot \cos(\phi) \cdot \sin(a) + \sin(\theta_0) \cdot \sin(\phi) \cdot \sin(\beta + \theta_P) - \sin(\theta_0) \cdot \cos(\beta + \theta_P) \cdot \cos(a) \cdot \cos(\phi) \right] \cdot R_P \\
    r_{Pz} &= \left[ \cos(\theta_0) \cdot \sin(\phi) \cdot \sin(a) - \sin(\theta_0) \cdot \cos(\phi) \cdot \sin(\beta + \theta_P) - \sin(\theta_0) \cdot \cos(\beta + \theta_P) \cdot \cos(a) \cdot \sin(\phi) \right] \cdot R_P
\end{align*}
\]

Note that \( \theta_P \) causes the coupler point to rest on the sphere at radius \( R_P \). As origami deals with flat panels for links rather than the traditional spherical links, a coupler point for origami can be defined with \( \theta_0 \) as described above, \( \theta_P = 0 \), and a radius \( R_P \) to the desired coupler point (Figure 4b).

The path of the coupler point as a function of the input angle is shown in Figure 3b and the path of the coupler link for several input angles is shown in Figure 5.

4. Examples

With the equations for the basic building block defined, it is possible to extend the model to systems with multiple vertices. This section describes two examples of the position analysis of coupled spherical systems. The first is Shafer’s “Chomper”, which is composed of two identical spherical mechanisms coupled to one another (Figure 6a). This configuration is commonly used in action origami as a mouth for various creatures.

The second example is a model known as the “Square Twist” in which 4 identical spherical mechanisms are coupled to one another in a loop (Figure 6b).

It should be pointed out that in both examples the spherical centers (two in the first example and four in the second) are fixed to ground. This simplifies the analysis, allowing the angles and coupler position of one spherical mechanism to be solved for all input angles, the results of which are simply translated and rotated to the other spherical centers (as they are identical) to understand the bulk motion of the mechanism.
4.1. Coupled

The “Chomper” model is composed of two identical spherical mechanisms which are coupled as shown in Figure 7a. A more traditional representation the spherical mechanism located at the origin is shown in Figure 7b. By analyzing one mechanism and applying the results to the other mechanism the motion of the entire model can be discovered in an efficient manner. Note that the second mechanism is translated 1 unit (as we are using the unit sphere) in the x-direction and rotated 180 degrees from the first mechanism.

Solving the first spherical mechanism yields the input(\(\phi\))-output(\(\psi\)) relationship found in Figure 8a. Again, the input-output relationship of the second mechanism is similar to that of the first. The approach for translating the coupler curve from the first mechanism to the second and rotating by 180 degrees is described next.

The second center is located a distance of \(R\) (which for our example is 1) from the second mechanism along the x-axis. Thus, \(R\) must be added to each x coordinate in the first coupler curve. The y and z coordinates remain the same. Note that \(x_i\) represents the initial x position, \(x_t\) the translated x position, and \(x_f\) the final x position. Numbers in the subscript designate specific vertices. The same subscripts are used for y and z.

\[
\begin{align*}
x_t &= x_i + R \\
y_t &= y_i \\
z_t &= z_i
\end{align*}
\]
Next the translated coupler curve must be rotated 180 degrees. This is accomplished using the following equations for rotations of cartesian coordinates about the z-axis (because the mechanism centers are fixed, the z-axis is also fixed thus the rotation are fairly simple):

\[
x_f = x_t \cos(\theta) - y_t \sin(\theta) \quad (18)
\]
\[
y_f = x_t \sin(\theta) + y_t \cos(\theta) \quad (19)
\]
\[
z_f = z_t \quad (20)
\]

For this example, \( \theta = 180^\circ \), thus the above simplifies to:

\[
x_{2f} = -x_{1t} \quad (21)
\]
\[
y_{2f} = -y_{1t} \quad (22)
\]
\[
z_{2f} = z_{1t} \quad (23)
\]

The path of the coupler points of both mechanisms is presented in Figure 8b. From this we see the expected motion of the two sides converging. The motion of the two couplers at several points throughout their motion is shown in Figure 9.

4.2. Loop

The “Square Twist” model is composed of four identical spherical mechanisms coupled in such a manner that they form a loop (Figure 10a). A traditional kinematic representation of the spherical mechanism located at the origin is found in Figure 10b. Note that one center is located a distance of 1 from the origin (again using the unit sphere) in the y-direction and is rotated by -90°. The next center (moving clockwise around the loop) is located a distance of 1 in both the x- and y-directions and is rotated 180°. The last mechanism is located a distance of 1 in the x-direction from the origin and is rotated by 90°. By leveraging symmetry, one of these mechanisms can be solved and that solution can be translated and rotated to the other three spherical centers.

Solving the first spherical mechanism yields the input-output relationship found in Figure 11a. The input-output relationship of the remaining three mechanisms is similar to that of the first. Translating the coupler curve from the first mechanism to the second and rotating by -90 degrees is accomplished using the following translations:
Figure 7: “Chomper” and corresponding coupled spherical mechanisms to be analyzed.

Figure 8: Position analysis of the “Chomper” [27] with dimensions as specified in Figure 7.

\[ x_{2i} = x_{1i} \]  \hspace{1cm} (24)
\[ y_{2i} = y_{1i} + R \]  \hspace{1cm} (25)
\[ z_{2i} = z_{1i} \]  \hspace{1cm} (26)

and the following rotations:

\[ x_{2f} = y_{2i} \]  \hspace{1cm} (27)
\[ y_{2f} = -x_{2i} \]  \hspace{1cm} (28)
\[ z_{2f} = z_{2i} \]  \hspace{1cm} (29)

Moving the coupler curve to the third spherical center is accomplished by the following translations:
and the following rotations.

\[
\begin{align*}
\phi_{3t} &= \phi_{1t} + R \\
\phi_{3f} &= -\phi_{3t} \\
\phi_{3t} &= \phi_{1t} \\
\phi_{3f} &= \phi_{3t} 
\end{align*}
\]

Moving the coupler curve to the fourth spherical center is accomplished by first translating as follows:

\[
\begin{align*}
x_{3t} &= x_{1t} + R \\
y_{3t} &= y_{1t} + R \\
z_{3t} &= z_{1t} \\
x_{3f} &= x_{3t} \\
y_{3f} &= y_{3t} \\
z_{3f} &= z_{3t} \\
x_{4t} &= x_{1i} + R \\
y_{4t} &= y_{1i} \\
z_{4t} &= z_{1i} \\
\end{align*}
\]
(a) “Square Twist” fold pattern and geometry. $V_1$, $V_2$, $V_3$, and $V_4$ are the spherical centers.

(b) Equivalent spherical mechanism of the first mechanism in the square twist.

Figure 10: “Square Twist” and corresponding loop of spherical mechanisms to be analyzed.

\begin{align*}
    x_{4f} &= -y_{4t} & (39) \\
    y_{4f} &= x_{4t} & (40) \\
    z_{4f} &= z_{4t} & (41)
\end{align*}

The paths of the overall mechanism is presented in Figure 11b. The motion of the four couplers at several angles is shown in Figure 12. When actuated from the edges, the center square of this model twists. When the center is fixed, the twist is imparted to the coupler links.

5. Future work

The method that has been described and demonstrated above simplifies the solution of systems of identical spherical mechanisms whose centers are attached to a ground link. Not all action origami fits within these restrictions. An example exception is Shafer’s “Frog’s Tongue” model [7] shown in Figure 13a. When a panel is chosen as ground and
the model is actuated, spherical centers along the center of the model move through space. In the examples performed above the spherical centers were always fixed. When the centers are allowed to move, the analysis becomes more complex.

2D periodic models such as the “Miura-Ori” [28] fold (Figure 13b) also have moving centers that require an extension of the method outlined here.

6. Conclusion

Utilizing the knowledge that much of action origami is composed of spherical mechanisms allows the use of traditional kinematic equations for understanding the behavior of these systems. Every origami vertex is a spherical change point mechanism whose links sum to 360°. A position analysis of a single origami vertex was performed, resulting in a relationship between input and output angles as well as the path of the coupler link.

In coupled systems of spherical mechanisms, if the mechanisms are identical (i.e. have the same link lengths) and the spherical centers are grounded, one of the mechanisms can be solved and the results applied to each vertex. In particular, the coupler curve can be moved from the solved mechanism to every other using translations and rotations to understand the overall motion of the system. Taking advantage of symmetry in this manner results in a significant reduction of computation time for large, multi-vertex models.

Future work can be done to perform velocity and acceleration analyses, in addition to motion, path, and function generation. The capabilities of the presented method could also be expanded to address systems in which the spherical
centers are not grounded. This would extend the application of the presented model significantly.

Successfully modeling the motion of action origami contributes meaningful insight to the kinematics of coupled systems of spherical mechanisms. The position analyses presented are a first step to the design of products based on coupled spherical systems with applications in the packaging, medical device, and space industries.

7. Acknowledgements

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References


Figure 13: Future work includes the analysis of models with moving spherical centers, such as these two models.