PAM Representation of Ternary Continuous Phase Modulation

Michael D. Rice
mdr@byu.edu

Erik Perrins

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It is well-known that binary continuous phase modulation (CPM) waveforms can be exactly represented as a pulse amplitude modulation (PAM) waveforms. This has also been shown for nonbinary CPM waveforms with even-sized symbol alphabets. In this paper we show how this is also accomplished for ternary CPM waveforms. We focus the development on signals with shorter frequency pulses, since these cases have the most practical value in designing reduced-complexity detectors. However, we also show the pattern by which these results can be extended to arbitrary frequency pulse durations and also to arbitrary odd-sized symbol alphabets, although little practical value is perceived in either of these extensions. We conclude with a thorough application of the results using the ternary CPM waveform SOQPSK-TG, which is an aeronautical telemetry standard.

I. Introduction

In this report, we develop the PAM representation of CPM for \( M = 3 \), where \( M \) is the size of the symbol alphabet. This is a continuation of previously published work for binary \( (M = 2) \) CPM in [1] and for nonbinary CPM with \( M \)-even in [2].

We concentrate on the special case of \( L \leq 2 \), where \( L \) is the duration (in symbol times) of the frequency pulse. There is not much merit in considering \( L > 2 \) since the primary motivation of the PAM representation is in building reduced-complexity detectors. When \( L > 2 \) the number of terms in the exact PAM representation grows impractically large. In such cases, the detector is not built using the exact PAM representation. Instead, the typical approach is to truncate the number of PAM terms to the equivalent of the \( L = 1 \) or \( L = 2 \) case [3]. Using this common approach, a detector for a signal with arbitrary \( L \) can be constructed with the relatively low complexity of the \( L \leq 2 \) cases. Therefore, our results include those cases of practical value. Furthermore, we demonstrate that these results have a pattern that can easily be extended to yield an exact PAM representation for arbitrary pulse durations, if so desired. We also show how these results generalize for odd-sized symbol alphabets other than ternary; although the use of such alphabets is not known to the authors.

We conclude with a thorough application of the PAM representation using the aeronautical telemetry standard SOQPSK-TG, which is a partial-response \( (L = 8) \) ternary CPM. We design a simple detector based on a two-pulse PAM approximation. We characterize the performance of this detector both analytically and with computer simulations, where it is shown that the PAM-based detector has negligible performance losses relative to the optimal detector.

II. Signal Model

The CPM signal may be represented as [4]

\[
s(t; \alpha) = \exp \{ j \phi(t; \alpha) \}
\]

where the phase is a pulse train of the form

\[
\phi(t; \alpha) = 2\pi h \sum_i \alpha_i q(t - iT)
\]
and $\alpha_i$ is an $M$-ary symbol, $T$ is the duration of each $\alpha_i$, and $h$ is the digital modulation index. For $M$-even, we have $\alpha_i \in \{\pm 1, \pm 3, \ldots, \pm (M - 1)\}$, and for $M$-odd we have $\alpha_i \in \{0, \pm 2, \ldots, \pm (M - 1)\}$. The phase pulse $q(t)$ is usually thought of as the time-integral of a frequency pulse $f(t)$ with area $1/2$ and duration $LT$. When $L = 1$ the signal is full-response and when $L > 1$ it is partial-response. Due to the constraints on $f(t)$ and $q(t)$, and assuming a rational modulation index $h = k/p$, the phase may be expressed as

$$\phi(t; \alpha) = 2\pi h \sum_{i=n-L+1}^{n} \alpha_i q(t - iT) + \pi h \sum_{i=0}^{n-L} \alpha_i$$

where $nT \leq t < (n + 1)T$. The phase state $\theta_{n-L}$ can only assume $p$ distinct values given by

$$\theta_{n-L} = \theta[I_{n-L}] = \frac{2\pi I_{n-L}}{p}$$

where $\theta[\cdot]$ is a $p$-entry look-up table indexed by

$$I_{n-L} = \left( k \sum_{i=0}^{n-L} \alpha_i \right) \mod p.$$

**III. The Equivalent PAM Representation for CPM**

In this section we show that the right-hand side of (1) can be expressed in an entirely different way using a PAM format. The development of this alternate representation has progressed in stages. We review the existing work first before developing the PAM representation for ternary CPM.

**A. Review of the PAM Representation for Binary CPM**

It is well known that binary CPM signals can be represented as a linear combination of PAM waveforms [1], as in

$$s_{\text{bin}}(t; \gamma) = \sum_{k=0}^{Q-1} \sum_{i} b_{k,i} c_k(t - iT), \quad Q = 2^{L-1}$$

(2)

where $\gamma_i \in \{\pm 1\}$ are binary antipodal data. The pulses $c_k(t)$ are given by

$$c_k(t) = \prod_{j=0}^{L-1} u(t + jT + LT \beta_{k,j}), \quad 0 \leq k \leq Q - 1$$

(3)

where

$$u(t) = \begin{cases} 
\sin (2\pi h q(t)) / \sin(\pi h), & 0 \leq t < LT \\
\sin (\pi h - 2\pi h q(t - LT)) / \sin(\pi h), & LT \leq t < 2LT \\
0, & \text{otherwise}.
\end{cases}$$

The parameter $\beta_{k,j} \in \{0, 1\}$ is the $j$-th bit in the radix-2 decomposition of the integer $k$, or

$$k = \sum_{j=1}^{L-1} 2^{j-1} \beta_{k,j}, \quad 0 \leq k \leq Q - 1$$

and $\beta_{k,0}$ is defined as zero for all $k$.

The pseudo-symbols $b_{k,i}$ are derived from the binary data $\gamma_i$ by the nonlinear mapping

$$b_{k,i} = \exp \left\{ j \pi h \left[ \sum_{m=0}^{i} \gamma_m - \sum_{l=0}^{L-1} \gamma_{i-l} \beta_{k,l} \right] \right\}.$$  

(4)

Table 1 gives an example of (3) and (4) for the binary $L = 2$ case.
Table 1. Pseudo-symbols and pulses for the binary $L = 2$ case.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$b_{k,i}$</th>
<th>$c_k(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\exp\left{ j\pi h \sum_{m=0}^{\gamma_m} \right}$</td>
<td>$u(t)u(t+T)$</td>
</tr>
<tr>
<td>1</td>
<td>$\exp\left{ j\pi h \sum_{m=0}^{\gamma_m - \alpha_i-1} \right}$</td>
<td>$u(t)u(t+T+2T)$</td>
</tr>
</tbody>
</table>

B. Review of the PAM Representation for Nonbinary CPM with $M$-even

In [2] it was shown that $M$-ary CPM signals with $M$-even can be represented as

$$s_{M\text{-even}}(t; \alpha) = \prod_{l=0}^{P-1} s_{\text{bin}}(t; 2^l \gamma_l)$$  \hspace{1cm} (5)

where $P$ is the positive integer that satisfies

$$2^{P-1} < M \leq 2^P.$$  

The key idea is that the original $M$-ary symbols $\alpha_i \in \{\pm 1, \pm 3, \ldots, \pm (M-1)\}$ can be represented by

$$\alpha_i = \sum_{l=0}^{P-1} 2^l \gamma_{i,l}$$  \hspace{1cm} (6)

which is a sum of binary symbols. Inserting (6) into (1) leads to the product in (5). Therefore, the $M$-ary CPM signal is viewed as the product of $P$ binary sub-signals, where the $l$-th sub-signal has the modulation index $h_l = 2^l h$. Each binary sub-signal has a PAM representation given by (2). Inserting (2) into (5) and expanding the product gives

$$s_{M\text{-even}}(t; \alpha) = \sum_{k=0}^{N-1} \sum_{i=1}^{P-1} a_{k,i} g_k(t - iT), \quad N = Q^P (2^P - 1)$$  \hspace{1cm} (7)

where formulas for $a_{k,i}$ and $g_k(t)$ can be found in [2] which are simply products of their binary counterparts $b_{k,i}$ and $c_k(t)$. Table 2 gives an example of the pseudo-symbols and pulses in (7) for the $M = 4, L = 2$ case. A superscript is added to $b_{k,i}^{(l)}$ and $c_k^{(l)}(t)$ to index the two different binary sub-signals with modulation indexes $h_l = 2^l h, 0 \leq l \leq 1$.

C. Derivation of the PAM Representation for Ternary CPM

The development of the PAM representation of ternary CPM is patterned after the approach in Section III-B. Here, the 3-ary symbol $\alpha_i \in \{0, \pm 2\}$ is represented as

$$\alpha_i = \gamma_{i,1} + \gamma_{i,0}.$$  \hspace{1cm} (8)

Inserting (8) into (1) yields

$$s_{\text{tern}}(t; \alpha) = s_{\text{bin}}(t; \gamma_1) s_{\text{bin}}(t; \gamma_0).$$

The ternary CPM signal is thus viewed as the product of two binary sub-signals with identical modulation indexes $h_l = h$. This has implications for the superscripts in Table 2. In the case of the pseudo-symbols $b_{k,1}^{(l)}$, the superscript indexes the data $\gamma_{i,1}$ and also indicates the modulation index $h_l$. For the pulses $c_k^{(l)}(t)$, which are data-independent, the superscript indicates only the modulation index $h_l$, which is always constant ($h_l = h$) in the present ternary case. Therefore, the superscripts can be removed from the pulses in Table 2 for the ternary case and several of the table entries can be combined. (Combining the entries in the table is
we give a few insights regarding the form of the PAM representation for the general case of the authors, this is the only odd-sized alphabet that is used in practice. However, for the sake of completeness, in Section III-C we discussed the PAM representation for the special case where $M = 3$. To the knowledge of the authors, this is the only odd-sized alphabet that is used in practice. However, for the sake of completeness, we give a few insights regarding the form of the PAM representation for the general case of $M$-odd.

### Table 2. Pseudo-symbols and pulses for the quaternary $L = 2$ case. The expressions for $b_{k,i}$ and $c_k(t)$ are as shown in Table 1.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$a_{k,i}$</th>
<th>$g_k(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$h_{0,i}^{(0)}b_{0,i}^{(1)}$</td>
<td>$c_0^{(0)}(t)c_0^{(1)}(t)$</td>
</tr>
<tr>
<td>1</td>
<td>$b_{0,i}^{(0)}k_{0,i}^{(1)}$</td>
<td>$c_0^{(0)}(t + T)c_0^{(1)}(t)$</td>
</tr>
<tr>
<td>2</td>
<td>$b_{0,i}^{(0)}k_{0,i}^{(1)}$</td>
<td>$c_0^{(0)}(t + 2T)c_0^{(1)}(t)$</td>
</tr>
<tr>
<td>3</td>
<td>$b_{0,i}^{(0)}k_{0,i}^{(1)}$</td>
<td>$c_1^{(0)}(t)c_1^{(1)}(t)$</td>
</tr>
<tr>
<td>4</td>
<td>$b_{0,i}^{(0)}k_{0,i}^{(1)}$</td>
<td>$c_1^{(0)}(t)c_1^{(1)}(t)$</td>
</tr>
<tr>
<td>5</td>
<td>$b_{0,i}^{(0)}k_{0,i}^{(1)}$</td>
<td>$c_2^{(0)}(t)c_2^{(1)}(t)$</td>
</tr>
<tr>
<td>6</td>
<td>$b_{0,i}^{(0)}k_{0,i}^{(1)}$</td>
<td>$c_2^{(0)}(t)c_2^{(1)}(t)$</td>
</tr>
<tr>
<td>7</td>
<td>$b_{0,i}^{(0)}k_{0,i}^{(1)}$</td>
<td>$c_3^{(0)}(t)c_3^{(1)}(t)$</td>
</tr>
<tr>
<td>8</td>
<td>$b_{0,i}^{(0)}k_{0,i}^{(1)}$</td>
<td>$c_3^{(0)}(t)c_3^{(1)}(t)$</td>
</tr>
<tr>
<td>9</td>
<td>$b_{0,i}^{(0)}k_{0,i}^{(1)}$</td>
<td>$c_4^{(0)}(t)c_4^{(1)}(t)$</td>
</tr>
<tr>
<td>10</td>
<td>$b_{0,i}^{(0)}k_{0,i}^{(1)}$</td>
<td>$c_4^{(0)}(t)c_4^{(1)}(t)$</td>
</tr>
<tr>
<td>11</td>
<td>$b_{0,i}^{(0)}k_{0,i}^{(1)}$</td>
<td>$c_5^{(0)}(t)c_5^{(1)}(t)$</td>
</tr>
</tbody>
</table>

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<td>$c_0^{(0)}(t)c_0^{(1)}(t)$</td>
</tr>
<tr>
<td>1</td>
<td>$b_{0,i}^{(0)}k_{0,i}^{(1)}$</td>
<td>$c_0^{(0)}(t + T)c_0^{(1)}(t)$</td>
</tr>
<tr>
<td>2</td>
<td>$b_{0,i}^{(0)}k_{0,i}^{(1)}$</td>
<td>$c_0^{(0)}(t + 2T)c_0^{(1)}(t)$</td>
</tr>
<tr>
<td>3</td>
<td>$b_{0,i}^{(0)}k_{0,i}^{(1)}$</td>
<td>$c_1^{(0)}(t)c_1^{(1)}(t)$</td>
</tr>
<tr>
<td>4</td>
<td>$b_{0,i}^{(0)}k_{0,i}^{(1)}$</td>
<td>$c_1^{(0)}(t)c_1^{(1)}(t)$</td>
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<td>5</td>
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</tr>
<tr>
<td>11</td>
<td>$b_{0,i}^{(0)}k_{0,i}^{(1)}$</td>
<td>$c_5^{(0)}(t)c_5^{(1)}(t)$</td>
</tr>
</tbody>
</table>

Table 2. Pseudo-symbols and pulses for the quaternary $L = 2$ case. The expressions for $b_{k,i}$ and $c_k(t)$ are as shown in Table 1.

Since the pseudo symbols in Tables 1 and 2 have unity magnitude, we normalize the new pseudo-symbol $d_{1,i}$ in (9) by 1/2 to remain consistent with the binary and $M$-even cases. This in turn leads to the amplification of the pulse $p_i(t)$ by a factor of 2.

With the above simplifications in mind, the ternary CPM signal for the $L = 2$ case is

$$ s_{tern,L=2}(t; \alpha) = \sum_{k=0}^{6} \sum_{i=1} d_{k,i} p_k(t - iT) $$

and for the $L = 1$ case is

$$ s_{tern,L=1}(t; \alpha) = \sum_{k=0}^{1} \sum_{i=1} d_{k,i} p_k(t - iT) $$

where $d_{k,i}$ and $p_k(t)$ are given in Table 3.

From the pattern established by Tables 2 and 3, it is easy to extend the results to cases other than $L \leq 2$. For the purposes of building reduced-complexity detectors, it is reasonable to approximate ternary CPM signals with arbitrary $L$ by

$$ s_{tern}(t; \alpha) \approx \sum_{k=0}^{K-1} \sum_{i=1} d_{k,i} p_k(t - iT) $$

where $K \in \{2, 7\}$. An example of this is given in Section IV.

### D. Notes on the PAM Representation for Nonbinary CPM for the General Case with $M$-odd

In Section III-C we discussed the PAM representation for the special case where $M = 3$. To the knowledge of the authors, this is the only odd-sized alphabet that is used in practice. However, for the sake of completeness, we give a few insights regarding the form of the PAM representation for the general case of $M$-odd.
Table 3. Pseudo-symbols and pulses for the ternary $L = 2$ case. The expressions for $b_{k,j}$ and $c_k(t)$ are as shown in Table 1.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$d_{k,i}$</th>
<th>$p_k(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$b_{0,0}^{(0)}b_{0,i}^{(1)}$</td>
<td>$c_0^0(t)$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{2} (b_{0,i-1}^{(0)} + b_{0,i}^{(0)}b_{0,i-1}^{(1)})$</td>
<td>$2c_0(t)c_0(t + T)$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{2} (b_{0,i-2}^{(0)} + b_{0,i}^{(0)}b_{0,i-2}^{(1)})$</td>
<td>$2c_0(t)c_0(t + 2T)$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{2} (b_{0,i}^{(0)}b_{0,1}^{(1)} + b_{0,i}^{(0)}b_{0,1}^{(1)})$</td>
<td>$2c_0(t)c_1(t)$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1}{2} (b_{0,i}^{(0)}b_{0,i-1}^{(1)} + b_{0,i}^{(0)}b_{0,i-1}^{(1)})$</td>
<td>$2c_0(t + T)c_1(t)$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{2} (b_{0,i}^{(0)}b_{0,i-2}^{(1)} + b_{0,i}^{(0)}b_{0,i-2}^{(1)})$</td>
<td>$2c_0(t + 2T)c_1(t)$</td>
</tr>
<tr>
<td>6</td>
<td>$b_{0,i}^{(0)}b_{0,i}^{(1)}b_{0,i}^{(1)}$</td>
<td>$c_2^1(t)$</td>
</tr>
</tbody>
</table>

In general, the $M$-ary symbol $\alpha_i \in \{0, \pm 2, \pm 4, \ldots, \pm (M - 1)\}$ is represented as

$$\alpha_i = \sum_{l=2}^{P'-1} 2^{l-1} \gamma_{i,l} + \gamma_{i,1} + \gamma_{i,0} \quad (10)$$

where the positive integer $P'$ satisfies

$$2^{P'-2} < M - 1 \leq 2^{P'-1}.$$ 

Inserting (10) into (1) yields

$$s_{M\text{-odd}}(t; \alpha) = \left( \prod_{l=2}^{P'-1} s_{\text{bin}}(t; 2^{l-1} \gamma_{i,l}) \right) s_{\text{bin}}(t; \gamma_{i,1}) s_{\text{bin}}(t; \gamma_{i,0}). \quad (11)$$

The $l$-th binary sub-signal in (11) has the modulation index

$$h_l = \begin{cases} 2^{l-1} h, & l \geq 2 \\ h, & 0 \leq l \leq 1. \end{cases}$$

The product in (11) can be expanded using the formulas in [2] which yield a set of $N = Q^{P'} (2^{P'} - 1)$ terms. Any pulses in this set can be combined, as exemplified in Tables 2 and 3, when they are identical upon removal of the superscript on $c_k^{(0)}(t)$ and $c_k^{(1)}(t)$.

IV. Application: SOQPSK-TG

We now give an example of the ternary PAM representation that was derived in Section III-C. We use a variant of shaped offset-QPSK known as SOQPSK-TG [5]. This is a ternary partial-response CPM with $L = 8$ and a frequency pulse defined by

$$f_{\text{TG}}(t) = A \frac{\cos \left( \frac{\pi Bt}{2T} \right)}{1 - 4 \left( \frac{\rho Bt}{2T} \right)^2} \times \sin \left( \frac{\pi Bt}{2T} \right) \times w(t) \quad (12)$$

where the window is

$$w(t) = \begin{cases} 1, & 0 \leq \left| \frac{t}{2T} \right| < T_1 \\ \frac{1}{2} + \frac{1}{2} \cos \left( \frac{\pi}{T_1} \left( \frac{t}{2T} - T_1 \right) \right), & T_1 \leq \left| \frac{t}{2T} \right| \leq T_1 + T_2 \\ 0, & T_1 + T_2 < \left| \frac{t}{2T} \right|. \end{cases}$$
Fig. 1. The length-8T frequency and phase pulses for SOQPSK-TG.

The constant $A$ is chosen to give the pulse an area of $1/2$ and $T_1 = 1.5$, $T_2 = 0.5$, $\rho = 0.7$, and $B = 1.25$. Figure 1 shows the frequency pulse in (12) and the corresponding phase pulse.

Although it makes no difference to the PAM representation, the ternary symbols $\alpha_i \in \{0, \pm 2\}$ are derived from binary information bits $a_i \in \{\pm 1\}$ according to the precoder

$$\alpha_i = (-1)^{i+1} a_{i-1} (a_i - a_{i-2}).$$

(13)

The modulation index for SOQPSK is $h = 1/4$.1

The received signal model is

$$r(t) = s(t; \alpha) + n(t)$$

where $n(t)$ is complex-valued additive white Gaussian noise with single-sided power spectral density $N_0$.

A. Optimal Performance of SOQPSK-TG

The error performance of CPM is governed by the normalized squared Euclidian distance [4, ch.2]

$$d^2 = \frac{\log_2 M_{\text{info}}}{2T} \int |s(t; \alpha_{\text{Tx}}) - s(t; \alpha_{\text{Rx}})|^2 dt$$

(14)

where $\log_2 M_{\text{info}}$ is the number of information bits per symbol (for SOQPSK we have $M_{\text{info}} = 2$). We consider the single-error bit sequences

$$\alpha_{\text{Tx}} = \ldots, a_{e-1}, a_e, a_{e+1}, \ldots$$

$$\alpha_{\text{Rx}} = \ldots, a_{e-1}, \bar{a}_e, a_{e+1}, \ldots$$

(15)

where the error location is arbitrarily chosen as $a_e$. For SOQPSK, we must compute (14) for all 4 possible values of $a_{e-1}$ and $a_{e+1}$. These sequences are first passed through the precoder (13) and then the signals $s(t; \alpha_{\text{Tx}})$ and $s(t; \alpha_{\text{Rx}})$ are generated. The minimum squared distance $d^2_{\text{min}} = 1.60$ corresponds to the two sequences where $a_{e-1} = a_{e+1}$. The two sequences where $a_{e-1} = \bar{a}_{e+1}$ have a squared distance of $d^2 = 2.59$. With these distances, the probability of bit error for SOQPSK-TG is

$$P_b \leq \frac{1}{2} \left( \sqrt{1.60 \frac{E_b}{N_0}} \right) + \frac{1}{2} \left( \sqrt{2.59 \frac{E_b}{N_0}} \right)$$

(16)

where $E_b$ is the energy-per-bit and

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2} \, du.$$
B. Reduced-Complexity Detector for SOQPSK-TG

We propose a reduced-complexity PAM-based detector using the two-pulse approximation

\[ \tilde{s}_{\text{tern}}(t; \alpha) = \sum_{k=0}^{1} \sum_{i} d_{k,i} p_{k}(t - iT) \]  

(17)

where the pulses \( p_{0}(t) \) and \( p_{1}(t) \) are shown in Figure 2. This approximation results in a normalized mean-squared error of

\[ \sigma^{2} = \frac{1}{T} \int_{0}^{T} E \{|\tilde{s}_{\text{tern}}(t; \alpha) - s_{\text{tern}}(t; \alpha)|^{2}\} \, dt = 4.26 \times 10^{-3}. \]

The detector based on (17) is patterned after the one in [3]. The metric increment for the \( l \)-branch in the trellis is

\[ z_{l}(n) = \text{Re} \left[ e^{-j\theta[S_{l}]} \sum_{k=0}^{1} y_{k}(n) \delta_{k,l}^{*} \right] \]  

(18)

where \((\cdot)^{*}\) is the complex conjugate, \( S_{l} \in \{0, 1, 2, 3\} \), and the sampled matched filter output is

\[ y_{k}(n) = \int_{nT}^{(n+L+1-k)T} r(t) p_{k}(t - nT) \, dt. \]  

(19)

In (18), we have explicitly factored the phase state \( \theta[S_{l}] \) out of the pseudo-symbols \( d_{k,l} = e^{-j\theta[S_{l}]\delta_{k,l}} \). The part that remains, \( \delta_{k,l}^{*} \), is a function only of the branch symbol \( \alpha_{l} \), the three possible values of which are listed in Table 4. Therefore, it is clear that the branch metric increment (18) depends only on the phase state index \( S_{l} \) and the branch symbol \( \alpha_{l} \). Thus, the PAM approximation has given SOQPSK-TG a full-response interpretation and the 8-state trellis in [6]\(^2\) can be used in the detector. The \( l \)-th branch in the trellis, where \( 0 \leq l \leq 15 \), is associated with an input bit \( a_{l} \), an output symbol \( \alpha_{l} \), and a phase state index \( S_{l} \). These quantities are used to compute (18).

C. Performance of the Reduced-Complexity Detector

The performance of PAM-based CPM detectors has been studied in [7]. The squared distance measure from [7], \( d'^{2} \), must be computed over a larger interval of bits than (15) due to 1) the non-constant envelope of (17), and 2) the partial-response of SOQPSK-TG [7]. It is sufficient to consider the values of \( d'^{2} \) from the

---

1The modulation index for SOQPSK is usually defined as \( h = 1/2 \), and the ternary data are usually defined as \( \alpha_{n} \in \{0, \pm1\} \). This is due to the historical roots of SOQPSK. In this paper we use strict CPM notation to define these quantities.

2The 8-state trellis in [6] was developed for the full-response MIL-STD variant of SOQPSK.
Table 4. The relationship between the branch $\alpha_l$, the binary $\gamma_{l,1}$ and $\gamma_{l,0}$, and the pseudo-symbols $\delta_{k,l}$ for SOQPSK.

<table>
<thead>
<tr>
<th>$\alpha_l$</th>
<th>$\gamma_{l,1}$</th>
<th>$\gamma_{l,0}$</th>
<th>$\delta_{0,l}$</th>
<th>$\delta_{1,l}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>$\exp{-j\pi/2}$</td>
<td>$\exp{-j\pi/4}$</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
<td>1, or 1</td>
<td>1</td>
<td>$\cos(\pi/4)$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\exp{j\pi/2}$</td>
<td>$\exp{j\pi/4}$</td>
</tr>
</tbody>
</table>

Fig. 3. Distance spectrum of the PAM-based detector for SOQPSK-TG.

64 possible values of \(\{a_{e-3}, a_{e-2}, a_{e-1}, a_{e+1}, a_{e+2}, a_{e+3}\}\). The resulting values are plotted in Figure 3. These 64 values form the probability of bit error

\[
P_b \leq \frac{1}{64} \sum_{i=0}^{63} Q\left(\sqrt{\frac{d_i^2}{E_b/N_0}}\right).
\]  

(20)

Figure 4 shows (16) and (20) along with data points from computer simulations of the PAM-based detector. From the figure we conclude 1) that the simulation results show strong agreement with the analysis, and 2) that the two pulse PAM approximation has near-optimal performance, with a loss of only 0.10 dB relative to the optimal detector at $P_b = 10^{-5}$. The minimum distance from Figure 3 suggests an eventual loss of $10\log_{10}(1.60/1.43) = 0.48$ dB, although this occurs for values of $E_b/N_0$ outside the range of practical interest.

We point out that the two pulses in Figure 2 have a combined duration of $9 + 8 = 17$ symbol times. Since they are real-valued and $r(t)$ is complex-valued, implementing (19) requires 34 length-$T$ filtering operations. However, it is obvious from Figure 2 that the two pulses are near zero for much of their combined 17 symbol times. By truncating the pulses to a combined duration of 3.5 symbol times (7 length-$T$ filtering operations), the overall loss at $P_b = 10^{-5}$ is 0.12 dB, and the eventual loss is $10\log_{10}(1.60/1.38) = 0.64$ dB.

V. Conclusions

We have developed the PAM representation for ternary CPM with $L \leq 2$. We have also shown that these results can be easily extended to longer pulse lengths and to odd-length symbol alphabets other than ternary; although little practical value is perceived in pursuing either of these extensions. We have presented a thorough application of the PAM representation by designing a reduced-complexity detector for the partial-response ternary CPM waveform known as SOQPSK-TG, which is an aeronautical telemetry standard. We have assessed the performance of the reduced-complexity detector both analytically and with computer simulations, and have demonstrated that the PAM-based approach results in very minor performance losses.
Fig. 4. Performance of the PAM-based detector for SOQPSK-TG.

References


