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Observability Analysis of Bearing-only Cooperative Localization

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Observability Analysis of Bearing-only Cooperative Localization

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0.1 Observability of an edge between two robots

To determine the observability matrix of $G_2^0$, we first find the Lie derivatives of $\eta_{ij}$. The zeroth order Lie derivatives is:

$$L^0_g h(X) = \eta_{ij}. \quad (1)$$

Differentiating $L^0_g h(X)$, we obtain

$$\nabla L^0_g h(X) = \begin{bmatrix} H^T_{11ij} & -1 & -H^T_{12ij} & 0 \end{bmatrix}, \text{ where } H^T_{11ij} = \begin{bmatrix} -\frac{y_i - y_j}{R_{ij}} & \frac{x_i - x_j}{R_{ij}} \end{bmatrix}. \quad (2)$$

This leads to the first Lie derivative

$$L_1^g h(X) = \frac{\partial}{\partial X} L^0_g h(X) X = H^T_{11ij} (f_i - f_j) - \omega_1. \quad (3)$$

Differentiating $L^1_f(h)$ we obtain

$$\nabla L^1_f (h) = \begin{bmatrix} H^T_{21ij} & H^T_{11ij} F_i & -H^T_{22ij} & -H^T_{11ij} F_j \end{bmatrix} \quad (4)$$

where $H^T_{21ij} = (f_i - f_j)^T J_{1ij}$, $F_i = V_i [\cos \psi_i \sin \psi_i]^T$, $F_i = \frac{\partial \psi_i}{\partial \theta_i}$, and

$$J_{1ij} = \frac{\partial H_{ij}}{\partial X} = \begin{bmatrix} 2a_{ij}b_{ij} & (a_{ij}^2 - b_{ij}^2) & (a_{ij}^2 - b_{ij}^2) \end{bmatrix} \quad (5)$$

Following a similar derivation, we obtain the second Lie derivative as

$$L^2_g (h) = (f_i - f_j)^T J_{1ij} (f_i - f_j) + H^T_{11ij} (F_i^0 \omega_i - F_j \omega_j). \quad (6)$$

Differentiating $L^2_g(h)$ we obtain

$$\nabla L^2_g (h) = \begin{bmatrix} H^T_{31ij} & H^T_{41ij} & -H^T_{32ij} & H^T_{51ij} \end{bmatrix} \quad (7)$$

where $H^T_{31ij} = (f_i - f_j)^T J_{21ij}$, $J_{21ij} = 2(a^2 + b^2)|J_{21ij}|J_{31ij}$, $J_{22ij}$ = $\begin{bmatrix} a & b & b & -a \end{bmatrix}$, $J_{31ij} = (f_i - f_j)^T J_{1ij} F_i + H^T_{12ij} F_i - \omega_i H^T_{11ij} f_i$, and $H^T_{51ij} = -(f_i - f_j)^T J_{11ij} F_j - H^T_{22ij} F_j + \omega_i H^T_{11ij} f_j$.

Higher order Lie derivatives are linear combinations of $(f_i - f_j)$ and $(F_i \omega_i - F_j \omega_j)$ and therefore do not contribute to the rank of the observability matrix. The observability matrix for $G_2^0$ can now be written as

$$O^{ij} = \frac{\partial}{\partial X} \Omega = \begin{bmatrix} H^T_{11ij} & -1 & -H^T_{12ij} & 0 \\
H^T_{21ij} & H^T_{11ij} F_i & -H^T_{22ij} & -H^T_{11ij} F_j \\
H^T_{31ij} & H^T_{11ij} F_i & -H^T_{32ij} & H^T_{51ij} \end{bmatrix}. \quad (8)$$

Conditions for the maximum rank of (8) are stated in the following lemma.
Lemma 1. The observability matrix given in (8) has rank three if and only if 
$V_j > 0, V_i \neq V_j$ or $\psi_i \neq \psi_j$ , and $\psi_i \neq H_{ij}(f_i - f_j)$.

Proof. We first prove the sufficiency of the listed conditions. The rank of ob-

servability matrix in (8) is three iff all the three rows are linearly independent.
To show the linear independence we perform gaussian elimination on the rows 
of (8), and show that the reduced-row echelon form (RREF) of the observability

matrix $O_{ij}$ is, 
\[
\bar{O}_{ij} \triangleq \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & y_i - y_j \\
0 & 1 & 0 & 0 & -1 & x_j - x_i \\
0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}.
\] (9)

We can expand the observability matrix and write, 
\[
O = \begin{bmatrix} -a & b & -1 & a & -b \ & 0 \\
H_{21} & H_{22} & H_{23} & -H_{21} & -H_{22} & H_{26} \\
H_{31} & H_{32} & H_{33} & -H_{31} & -H_{32} & H_{36} \end{bmatrix} \] (10)

If all the conditions of lemma 1 are satisfied we can perform gaussian elimination

on $O$ to find the basis which spans the observability space for two robots.

We first perform these operation on $R_2$ and $R_3$. (1) $R_2 = R_2 - \frac{R_3}{\alpha} H_{21}$ (2)

$R_3 = R_3 - \frac{R_3}{\alpha} H_{31}$, and obtain

\[
O = \begin{bmatrix} -a & b & -1 & a & -b \ & 0 \\
0 & H'_{22} & H'_{23} & 0 & -H'_{22} & H_{26} \\
0 & H'_{32} & H'_{33} & 0 & -H'_{32} & H_{36} \end{bmatrix} \] (11)

$R_3 = R_3 - \frac{R_3}{\alpha} H'_{32}$

\[
O = \begin{bmatrix} -a & b & -1 & a & -b \ & 0 \\
0 & H'_{22} & H'_{23} & 0 & -H'_{22} & H_{26} \\
0 & 0 & H'_{33} & 0 & 0 & H'_{36} \end{bmatrix} \] (12)

Plugging the values it can shown that $H''_{33} = -H'_{36}$.

$R_3 = \frac{R_3}{\alpha} H''_{33}$

\[
O = \begin{bmatrix} -a & b & -1 & a & -b \ & 0 \\
0 & H''_{22} & H''_{23} & 0 & -H''_{22} & H_{26} \\
0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \] (13)

$R_2 = R_2 - R_3 H'_{23}$ $R_1 = R_1 + R_3$

\[
O = \begin{bmatrix} -a & b & 0 & a & -b \ & -1 \\
0 & H'_{22} & 0 & 0 & -H'_{22} & H'_{26} \\
0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \] (14)
\( R_2 = \frac{R_3}{H_{zz}} \) and using \( \frac{H_{zz}}{H_{zz}} = x_j - x_i, \)

\[
O = \begin{bmatrix}
-a & b & 0 & a & -b & -1 \\
0 & 1 & 0 & 0 & -1 & x_j - x_i \\
0 & 0 & 1 & 0 & 0 & -1 \\
\end{bmatrix} \tag{15}
\]

\( R_1 = R_3 - R_2 b \) and \( R_1 = \frac{R_3}{a} \)

\[
O = \begin{bmatrix}
1 & 0 & 0 & -1 & 0 & \frac{-1-b(x_j-x_i)}{-a} \\
0 & 1 & 0 & 0 & -1 & x_j - x_i \\
0 & 0 & 1 & 0 & 0 & -1 \\
\end{bmatrix} \tag{16}
\]

Plugging in values of \( a \) and \( b \) we can show that \( \frac{-1-b(x_j-x_i)}{-a} = y_i - y_j \) and then we can write

\[
O = \begin{bmatrix}
1 & 0 & 0 & -1 & 0 & y_i - y_j \\
0 & 1 & 0 & 0 & -1 & x_j - x_i \\
0 & 0 & 1 & 0 & 0 & -1 \\
\end{bmatrix} \tag{17}
\]

It can be easily verified that all the rows in (9) are linearly independent and therefore, \( \text{rank}(O_{ij}) = 3 \). To prove the necessity of the lemma.

(a) Assume that \( V_i > 0 \) and \( V_j = 0 \). Plugging these into (8) we obtain,

\[
O^{ij} = \begin{bmatrix}
H^0_{1ij} & -1 & -H^0_{1ij} & 0 \\
f_i^T J_{1ij} & H^T_{1ij} F_i & -f_i^T J_{1ij} & 0 \\
J_{2ij}^T f_i & 2f_i^T J_{1ij} F_i & -f_i^T J_{2ij} & 0 \\
\end{bmatrix} \]

It can be seen that second and third rows are not linearly independent and therefore, \( \text{rank}(O^{ij}) = 2 < 3 \).

(b) Note that if \( V_i = V_j \) and \( \psi_i = \psi_j \) (i.e., the robots are moving in formation) then \( \dot{\psi}_{ij} = 0 \), leading to insufficient measurements for observability. Similarly, when \( \omega_i = H^T_{1ij} (f_i - f_j) \), \( \dot{\psi}_{ij} = 0 \) leading to a rank 1 observability matrix. \( \square \)

Here we derive the observability conditions for an edge \( \eta_{il} \) between a robot and landmark.

The zeroth order Lie derivative is \( L^0_{f(h)} = \eta_{il} \). Differentiating \( L^0_{f(h)} \) we obtain,

\[
\frac{\partial}{\partial X} L^0_{f(h)} = \begin{bmatrix}
H^T_{1il} & -1 \\
\end{bmatrix}, \]

where \( H_{1il} \) is as defined in Equation (2), leading to a first order Lie derivative of \( L^1_{f(h)} = H^T_{1il} f_i - \omega_i \). Differentiating \( L^1_{f(h)} \) we obtain

\[
\nabla L^1_{f(h)} = \begin{bmatrix}
H^T_{2il} & H^T_{1il} F_i \\
\end{bmatrix} \]

where, \( H^T_{2il} = f_i^T J_{1il} \) and \( J_{1il} \) is similar to \( J_{1ij} \) defined in (5). Second and higher order Lie derivatives will be multiples of \( f_i \) and \( F_i \). We can write the observability matrix entries for the observing robot as

\[
O_{il} = \begin{bmatrix}
H^T_{1il} & -1 \\
H^T_{2il} & H^T_{1il} F_i \\
\end{bmatrix}. \tag{18}
\]

Conditions for the maximum rank of (18) are stated in following lemma.

**Lemma 2.** The number of linearly independent rows contributed by edge \( \varepsilon_{il} \) between a robot and a landmark is two if and only if \( V_i > 0 \) and \( \omega_i \neq H^T_{1il} f_i \).
Figure 1: The observability conditions between these four possible configurations of a connected, 3-node RPMG are identical.

Proof. If all the conditions in above lemma are satisfied we can perform gaussian elimination (similarly to lemma 1) on the rows of (18), and can write its RREF as,

$$\tilde{O}_d = \begin{bmatrix} 1 & 0 & y_i - y_l \\ 0 & 1 & x_l - x_i \end{bmatrix}.$$  \hspace{1cm} (19)

which has rank two. If $V_i = 0$ or $\omega_i = H_{il}^T f_l$, then $\dot{\eta}_p = 0$, leading to an observability matrix rank of 1. \hfill \Box

Definition 1. A RPMG $G_n^l$ is called a proper RPMG if all the edges between robot nodes satisfies the conditions of lemma 1 and all the edges between robots and landmarks satisfies lemma 2.

From (9) we can write the basis for the row space of an observability matrix of an edge between two robots in the proper graph $G_n^l$ as,

$$\begin{bmatrix} 0_{3 \times 3(i-1)} & O_i^{ij} & 0_{3 \times (3(j-1) - 3i)} & O_j^{ij} & 0_{3 \times (n - 3j)} \end{bmatrix},$$  \hspace{1cm} (20)

where $\tilde{O}_i^{ij} = I_3$, and $\tilde{O}_j^{ij} = \begin{bmatrix} -1 & 0 & y_i - y_j \\ 0 & -1 & x_j - x_i \\ 0 & 0 & -1 \end{bmatrix}$.

Similarly, from (19) we can write the basis for the row space of an observability matrix of an edge between a robot and a landmark in the proper $G_n^l$ as,

$$[0_{2 \times 3(i-1)} \tilde{O}_d 0_{2 \times 3(n-i)}].$$  \hspace{1cm} (21)

0.2 Observability of three nodes

Lemma 3. If a three node proper RPMG $G_3^l$ is connected then the rank of observability matrix is six.

Proof. There are four possible configurations of a connected graph $G_3^l$, shown as sub-figures (a) through (d) in Fig. 1. We can write the observability matrix
for these configurations using (20) as

\[
O_a = \begin{bmatrix}
O_{12}^{12} & O_{12}^{12} & 0 \\
O_{13}^{12} & 0 & O_{33}^{13} \\
O_{12}^{12} & 0 & 0 \\
0 & O_{23}^{23} & O_{33}^{23} \\
O_{13}^{13} & 0 & O_{33}^{13}
\end{bmatrix}
\]

\[
O_b = \begin{bmatrix}
O_{12}^{12} & O_{12}^{12} & 0 \\
0 & O_{22}^{12} & O_{33}^{22} \\
0 & 0 & O_{33}^{23} \\
0 & 0 & 0 \\
O_{13}^{13} & 0 & O_{33}^{13}
\end{bmatrix}
\]

\[
O_c = \begin{bmatrix}
1 & 0 & 0 & -1 & 0 & y_1 - y_2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & x_2 - x_1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & y_1 - y_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & x_3 - x_1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
\]

\[
O_d = \begin{bmatrix}
O_{13}^{12} & 0 & O_{33}^{13} \\
0 & O_{23}^{23} & O_{33}^{23} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

First we perform gaussian elimination on \(O_a\). Plugging values of matrices \(O_{12}^{12}\), \(O_{13}^{12}\), and \(O_{13}^{13}\) we get,

\[
O_a = \begin{bmatrix}
1 & 0 & 0 & -1 & 0 & y_1 - y_2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & x_2 - x_1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & y_1 - y_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & x_3 - x_1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
\]

By performing following elementary operations \(R_4 = R_4 - R_1\), \(R_5 = R_5 - R_2\), and \(R_6 = R_6 - R_3\) we get,

\[
R_1 = R_1 + R_4, \quad R_2 = R_2 + R_5, \quad \text{and} \quad R_3 = R_3 + R_1
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 0 & y_1 - y_3 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & x_3 - x_1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & y_1 - y_3 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & x_3 - x_1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1
\end{bmatrix}
\]
\[ R4 = R4 + R6(y_1 - y_2) \text{ and } R5 = R5 + R6(x_2 - x_1) \]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 0 & y_1 - y_3 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & x_3 - x_1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & y_1 - y_3 - (y_1 - y_2) \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & x_3 - x_1 - (x_2 - x_1) \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
\]

\begin{equation}
(25)
\end{equation}

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 0 & y_1 - y_3 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & x_3 - x_1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & y_2 - y_3 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & x_3 - x_2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
\]

\begin{equation}
(26)
\end{equation}

Therefore,

\[ \bar{O}_a = \begin{bmatrix} I_3 & 0 & O_{13}^{13} \\ 0 & I_3 & O_{23}^{13} \end{bmatrix} \]

\begin{equation}
(27)
\end{equation}

Similarly, we can perform Gaussian elimination and elementary row operations on \( O_b \) and \( O_c \) and show that \( \bar{O}_a = \bar{O}_b = \bar{O}_c = O_d \) and there rank is six.

\subsection*{0.3 Observability with landmarks}

Consider RPMG (\( G_2^1 \)) with two robot nodes and one landmark. We can write observability matrix of \( G_2^1 \) where landmark \( k \) is connected to node \( i \),

\[ O = \begin{bmatrix} O_{ij}^{ij} & O_{ik}^{ij} \\ O_{ik} & 0_{2 \times 3} \end{bmatrix}. \]

\begin{equation}
(28)
\end{equation}

Plugging values of matrices \( O_{ij}^{ij} \), and \( O_{ik} \) we get,

\[ O = \begin{bmatrix}
1 & 0 & 0 & -1 & 0 & y_i - y_j \\
0 & 1 & 0 & 0 & -1 & x_j - x_i \\
0 & 0 & 1 & 0 & 0 & -1 \\
1 & y_i - y_l & 0 & 0 & 0 \\
0 & 1 & x_l - x_i & 0 & 0 & 0
\end{bmatrix}
\]

\begin{equation}
(29)
\end{equation}

First \( R4 = R4 - R1 \) and \( R5 = R5 - R2 \)

\[
\begin{bmatrix}
1 & 0 & 0 & -1 & 0 & y_i - y_j \\
0 & 1 & 0 & 0 & -1 & x_j - x_i \\
0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & y_i - y_l & 1 & 0 & -(y_l - y_j) \\
0 & 0 & x_l - x_i & 0 & 1 & -(x_j - x_i)
\end{bmatrix}
\]

\begin{equation}
(30)
\end{equation}
\[ R_4 = R_4 - R_3(y_i - y_l) \] and \[ R_5 = R_5 - R_3(x_l - x_i) \]

\[
\begin{bmatrix}
1 & 0 & 0 & -1 & 0 & y_i - y_j \\
0 & 1 & 0 & 0 & -1 & x_j - x_i \\
0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & -(y_i - y_j) + y_l - y_i \\
0 & 0 & 0 & 0 & 1 & -(x_j - x_i) + x_l - x_i
\end{bmatrix}
\]

(31)

\[
\begin{bmatrix}
1 & 0 & 0 & -1 & 0 & y_i - y_j \\
0 & 1 & 0 & 0 & -1 & x_j - x_i \\
0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & y_j - y_i \\
0 & 0 & 0 & 0 & 1 & x_l - x_j
\end{bmatrix}
\]

(32)

\[
\begin{bmatrix}
O_{ij}^i & O_{ij}^j \\
0_{2 \times 3} & O_{jl}^j
\end{bmatrix}
\]

(33)

This is the observability matrix for landmark \( l \) connected to node \( i \) is equal to landmark \( p \) connected to node \( j \).