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ABSTRACT

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In this thesis, we will explore the structure of Terwilliger algebras over several different types of finite groups. We will begin by discussing what a Schur ring is, as well as providing many different results and examples of them. Following our discussion on Schur rings, we will move onto discussing association schemes as well as their properties. In particular, we will show every Schur ring gives rise to an association scheme. We will then define a Terwilliger algebra for any finite set, as well as discuss basic properties that hold for all Terwilliger algebras. After specializing to the case of Terwilliger algebras resulting from the orbits of a group, we will explore bounds of the dimension of such a Terwilliger algebra. We will also discuss the Wedderburn decomposition of a Terwilliger algebra resulting from the conjugacy classes of a group for any finite abelian group and any dihedral group.

Keywords: Terwilliger algebra, Wedderburn decomposition, association scheme, Bose-Mesner algebra, Schur ring, representation theory, dihedral groups

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Chapter 1. Introduction

Association schemes (Definition 3.1) are a useful tool in studying many combinatorial problems. Commutative association schemes give rise to subalgebras of a matrix algebra with complex entries, which are called Terwilliger algebras (Definition 4.1). By studying Terwilliger algebras we can better understand the association schemes that they result from. Doing so also allows further study into those combinatorial problems in which association schemes are useful.

Terwilliger algebras were originally developed in the 1990's by Paul Terwilliger to provide a method for studying commutative association schemes. In particular he looked at P-Polynomial (Definition 3.11) and Q-Polynomial (Definition 3.14) association schemes. Over the course of the three papers [41, 42, 43] in which Terwilliger algebras were originally introduced, Terwilliger manages to find a combinatorial characterization of thin (Definition 4.7) P-Polynomial and Q-Polynomial association schemes.

Since Terwilliger algebras were defined by Paul Terwilliger, others have studied them with relation to various commutative association schemes. Terwilliger algebras have been studied as they relate to Johnson schemes in [22, 25, 26, 40]. Extending this study of Johnson schemes, the Johnson geometry was studied using Terwilliger algebras in [17, 27]. The Terwilliger algebra for Hamming schemes was studied in [23] and the incidence matrix of the Hamming graph was studied in [47]. Another common place where Terwilliger algebras are used is in looking at distance regular graphs, which also give rise to association schemes. Papers that have studied this include [31, 38, 39, 44, 45]. In addition to distance regular graphs, Terwilliger algebras have also been applied to bipartite distance-regular graphs in [8, 29, 30], almost-bipartite distance-regular graphs in [7], and distance-biregular graphs in [12]. Terwilliger algebras have also been studied for group association schemes in [1, 3, 14]. In addition to these commonly studied areas, Terwilliger algebras have also been studied with relation to wreath products of association schemes in [34, 37], the hypercube in [13], the latin square in [9], odd graphs in [18], a strongly regular graph in [48], quantum adjacency algebras in [46], almost-bipartite P- and Q-polynomial association schemes in [6], Lee association schemes over $\mathbb{Z}_4$ in [33], cyclotomic association schemes in [15], semidefinite programming in [5], and generalized quadrangles in [24]. Generalizations of Terwilliger algebras can be found in [11, 51].

The goal of this thesis is to study Terwilliger algebras that are created using finite groups. In particular, we find the Wedderburn decomposition for several Terwilliger algebras created from groups, which were previously unknown. This gives us the explicit structure of the Terwilliger algebra, which is a very important tool in studying the Terwilliger algebra further. Our primary focus of Terwilliger algebras created from groups is centered around those that are created using the conjugacy classes of the group, however other Terwilliger algebras are also considered.

In Chapter 2 of this thesis we will discuss an algebraic objected called a Schur ring (Definition 2.9). We shall give many examples of these algebraic objects along with several properties that they posses. Our motivation for this is that Schur rings give rise to (not necessarily commutative) association schemes over a group (Theorem 3.10). Thus, by using them we can immediately determine some association schemes that result from them. By using the fact that the Terwilliger algebra comes from a Schur ring we are also able to place our proofs in a more group theoretic setting.

In Chapter 3 we introduce association schemes. In that chapter we will provide three different definitions of an association scheme (Definitions 3.1, 3.7, and 3.8) and show they are equivalent. Many results and examples of association schemes will also be provided. The definitions as well as examples of P-Polynomial and Q-Polynomial association schemes will be provided in this section as well. For more complete information on association schemes we recommend the reader see [2].

In Chapter 4 we will introduce Terwilliger algebras, as well as some basic results and definitions that are related to them. We also prove the existence of an irreducible two-sided ideal possessed by all Terwilliger algebras with a specific base point created using orbit Schur rings (Definition 2.14). See (Theorem 4.12) for this result. The existence of this ideal allows us to easily find the Wedderburn decomposition of the Terwilliger algebra resulting from several different orbit Schur rings (Theorem 4.15, and Theorem 4.16). The irreducible two-sided ideal is later used when we find the complete Wedderburn decomposition of the Terwilliger algebra of any dihedral group D_{2n} where n is odd. (See Corollary 5.13, which is the focus of Chapter 5).

This thesis also includes several appendices. The first of these covers results related to semisimple algebras, which are used in showing that Terwilliger algebras are semi-simple (Proposition 4.4). We also include tables in an appendix which provide the dimension of the Terwilliger algebra created from a group algebra as well as the dimension of a subalgebra of the Terwilliger algebra that is of interest, and the character degrees of the Terwilliger algebra for all groups of orders 1 through 63. These calculations were done using Magma and the code used for the calculations is provided in an appendix as well.

We now close with some notation that will be used throughout. Unless otherwise specified G will be any group, F will be a field of characteristic 0, and $F[G]$ will the group ring over G with coefficients in F. Any other notation that is used, is provided over the course of the thesis.

Chapter 2. Schur Rings

In every group ring there is a collection of subrings that are of particular interest, which are known as Schur rings (see Definition 2.9). Schur rings were developed by Schur [36] and Wielandt [49] in the early part of the 20th century. They were created as an alternative way to study permutations groups instead of character theory. Schur rings were later studied in connection with algebraic combinatorics [16, 28]. In particular they have connections with association schemes, strongly-regular graphs, and difference sets. The original definition was for finite groups but was later extended to infinite groups in [4].

Two operations that are of particular interest when looking at a group ring $F[G]$ are the star operator and Hadamard product. We define these operations as follows. For $\alpha = \sum_{g\in G}\alpha_g g$ and $\beta = \sum_{g\in G}\beta_g g$ in $F[G]$ the *star operator* applied to α gives

$$\alpha^* = \sum_{g\in G}\alpha_g g^{-1}$$

and the *Hadamard product* is given by

$$\alpha \circ \beta = \sum_{g\in G}\alpha_g\beta_g g.$$

If C is a finite subset of G, then we let $\overline{C} = \sum_{g\in C} g \in F[G]$ and call this a *simple quantity*. These are one of the basic structures involved in defining Schur rings. Another natural thing to look at when considering any element of a group ring is the set of elements of the group that actually show up in the summation. For $\alpha \in F[G]$, we call this the *support* of α, denoted $\operatorname{supp}(\alpha)$. That is, if $\alpha = \sum_{g\in G}\alpha_g g$ then

$$\operatorname{supp}(\alpha) = \{g \in G \colon \alpha_g \neq 0\}.$$

For example, $\operatorname{supp}(\overline{C}) = C$.

The underlying structure from which we create a Schur ring in a group ring $F[G]$ is that of a Schur module. For this reason we will now begin looking at Schur modules and discuss results relating to them.

Definition 2.1 (Schur Module)**.** Let G be a group, R be a ring, and let $\mathfrak{S}$ be an R-submodule of $R[G]$. We say that $\mathfrak{S}$ is a *Schur module* (or an *S-module*) if there exists a partition $\mathcal{D}(\mathfrak{S})$ of G, with each set in $\mathcal{D}(\mathfrak{S})$ having finite support, such that

$$\mathfrak{S} = \operatorname{Span}_R\{\overline{C} \colon C \in \mathcal{D}(\mathfrak{S})\}.$$

We call the sets C that appear in $\mathcal{D}(\mathfrak{S})$ the *primitive sets*. We also call a simple quantity $\overline{C}$, where $C \in \mathcal{D}(\mathfrak{S})$, a *primitive quantity*. Primitive quantities are the building blocks of Schur modules.

If $D \subseteq G$ happens to be such that $D = \bigcup_{E\in\mathcal{E}} E$ for some $\mathcal{E} \subseteq \mathcal{D}(\mathfrak{S})$ then we say that D is an $\mathfrak{S}-$*set*. If in addition to being just a subset, D is a subgroup of G, then D is called an $\mathfrak{S}$-*subgroup*.

Wielandt proved the following basic properties of Schur modules over finite groups in [49, 50]. Most of them extend to infinite groups using the same proof that Wielandt provided.

Proposition 2.2 ([50], Proposition 22.2)**.** *Let $\mathfrak{S}$ be a Schur module over $F[G]$. Let $\alpha \in \mathfrak{S}$. Then* $\operatorname{supp}(\alpha)$ *is an $\mathfrak{S}$-set.*

Proposition 2.3 ([50], Proposition 22.3)**.** *Let $\mathfrak{S}$ be a Schur module over a group G and let $\alpha = \sum_{g\in G} \alpha_g g \in \mathfrak{S}$. Let $f : F \to F$ be a function such that $f(0) = 0$. We define*

$$f[\alpha] = \sum_{g\in G} f(\alpha_g)g.$$

Then $f[\alpha] \in \mathfrak{S}$.

Proposition 2.4 ([50], Proposition 22.4)**.** *Schur modules are closed under the Hadamard product.*

Proof. Suppose that we have a Schur module $\mathfrak{S}$. Let $\alpha, \beta \in \mathfrak{S}$. Then $\alpha = \sum_{i=1}^{n} \alpha_i \overline{C_i}$ and $\beta = \sum_{j=1}^{m} \beta_j \overline{D_j}$ where every $C_i, D_j \in \mathcal{D}(\mathfrak{S})$. Let $\mathcal{E} = \{C_1, C_2, \ldots C_n, D_1, D_2, \ldots D_m\}$. We relabel all the distinct elements in $\mathcal{E}$ as $E_1, E_2, \ldots, E_r$. Then we can write $\alpha = \sum_{i=1}^{r} \alpha_i \overline{E_i}$ and $\beta = \sum_{i=1}^{r} \beta_i \overline{E_i}$ using 0 coefficient for terms that didn't originally appear. We then have

$$\alpha \circ \beta = \left(\sum_{i=1}^{r} \alpha_i \overline{E_i} \right) \circ \left(\sum_{i=1}^{r} \beta_i \overline{E_i} \right) = \sum_{i=1}^{r} \alpha_i \beta_i (\overline{E_i} \circ \overline{E_i}) = \sum_{i=1}^{r} \alpha_i \beta_i \overline{E_i}.$$

Notice $\sum_{i=1}^{r} \alpha_i \beta_i \overline{E_i} \in \mathfrak{S}$. Thus, $\mathfrak{S}$ is closed under the Hadamard product. $\square$

We note that the above are true in the case when G is infinite.

Theorem 2.5 (Wielandt, [49]). *Let $F \subseteq \mathbb{R}$ be a subfield and let G be a finite group. Every F-subalgebra of $F[G]$ which is closed under $*$ (called a $*$-subalgebra) is semisimple.*

Proof. Suppose that S is a $*$-subalgebra of $F[G]$, but not semisimple. Let $\mathcal{J}(S)$ denote the Jacobson radical of S as given in Definition A.5. By Lemma A.11, $\mathcal{J}(S) \neq 0$ and contains a simple left ideal $S\alpha$, since S is Artinian. But $\alpha \in \mathcal{J}(S)$, so $\alpha(S\alpha) = 0$, and hence $\alpha\alpha^*\alpha = 0$. Then $\alpha\alpha^*\alpha\alpha^* = (\alpha\alpha^*)(\alpha\alpha^*)^* = 0$.

We now claim that the only solution $\beta \in F[G]$ to the equation $\beta\beta^* = 0$ is 0 itself. Suppose $\beta = \sum_{g \in G} \beta_g g$. Then the coefficient of e in $\beta\beta^*$ is $\sum_{g \in G} \beta_g^2$. Now, a sum of squares is 0 in F if and only if $\beta_g = 0$ for every $g \in G$. This means that $\beta = 0$.

By the above claim, $\alpha\alpha^* = 0$. Again using the claim, we conclude that $\alpha = 0$, which contradicts $\mathcal{J}(S) \neq 0$. Therefore, S is semisimple. $\square$

Proposition 2.6 ([50], Proposition 22.1). *Let $\mathfrak{S}$ be a Schur module over $F[G]$. Let $\alpha \in \mathfrak{S}$ and write $\alpha = \sum_{g \in G} \alpha_g g$. Then $K(\alpha, c) = \{g \in G \mid \alpha_g = c\}$ is an $\mathfrak{S}$-set for each $c \in F$, called the* coefficient complex *associated to α with respect to c.*

Two further results about Schur modules that are of some interest come from [4] where they are proven for the infinite case.

Lemma 2.7 ([4], Lemma 2.6)**.** *If $\mathfrak{S}$ is an F-subspace of $F[G]$ closed under $\circ$ and $f(x) \in F[x]$, then $f[\alpha] \in \mathfrak{S}$ for all $\alpha \in \mathfrak{S}$.*

Lemma 2.8 ([4], Lemma 2.7)**.** *If $\mathfrak{S}$ is an F-subspace of $F[G]$ closed under $\circ$, then $\overline{K(\alpha, c)} \in \mathfrak{S}$ for all $\alpha \in \mathfrak{S}$ and $c \in F \setminus \{0\}$.*

Having now covered the preliminary information we can give the formal definition for a Schur ring.

Definition 2.9 (Schur Ring)**.** A *Schur ring*, $\mathfrak{S}$, over a group G is a Schur module over $F[G]$ such that

(i) $\{1\} \in \mathcal{D}(\mathfrak{S})$

(ii) if $C \in \mathcal{D}(\mathfrak{S})$, then $C^* \in \mathcal{D}(\mathfrak{S})$

(iii) for all $C, D \in \mathcal{D}(\mathfrak{S})$,

$$\overline{C} \cdot \overline{D} = \sum_{E \in \mathcal{D}(\mathfrak{S})} \lambda_{CDE} \overline{E},$$

where all but finitely many λ_{CDE} are equal to 0. The coefficients λ_{CDE} are called the *structure constants* of $\mathfrak{S}$.

We use the same terminology of primitive sets, primitive quantities, $\mathfrak{S}$-sets, and $\mathfrak{S}$-subgroups for Schur rings that we used for Schur modules. We say that a Schur ring is *symmetric* if for all $C \in \mathcal{D}(\mathfrak{S})$, we have $C = C^*$.

As by definition every Schur ring is a Schur module, all of the above results regarding Schur modules also hold for Schur rings. One result related to Schur rings that is particularly useful comes from Muzychuck [35].

Theorem 2.10 ([35], Lemma 1.3)**.** *Suppose that $\mathfrak{S}$ is a F-subalgebra of $F[G]$ for a finite group G. Then $\mathfrak{S}$ is a Schur ring if and only if $\mathfrak{S}$ is closed under both $*$ and $\circ$ and contains both 1 and $\overline{G}$.*

This follows as a result of the semisimplicity of $\mathfrak{S}$. We know that $\mathfrak{S}$ is semisimple as a result of $\mathfrak{S}$ being closed under $*$ and Theorem 2.5. Theorem 2.10 provides an alternative definition of a Schur ring over a finite group. This definition does not extend to infinite groups. Modifying it to some degree we do get an alternative definition for a Schur module over infinite groups. This alternative definition is provided below.

Theorem 2.11 ([4], Theorem 2.8)**.** *Let G be a group and let $\mathfrak{S}$ be an F-subspace of $F[G]$. Then $\mathfrak{S}$ is a Schur module if and only if $\mathfrak{S}$ is closed under $\circ$ and for all $g \in G$ there exists some $\alpha \in \mathfrak{S}$ such that $g \in \operatorname{supp}(\alpha)$.*

Proof. Sufficiency is immediate from Proposition 2.4 and the existence of the partition $\mathcal{D}(\mathfrak{S})$. For necessity, let $\mathcal{D}$ be the set of supports of all primitive quantities of $\mathfrak{S}$. We claim that $\mathcal{D}$ is a partition of G. Let $g \in G$. By assumption, there exists some $\alpha \in \mathfrak{S}$ such that $g \in \operatorname{supp}(\alpha)$. By Lemma 2.8, we may assume that α is simple. If α is not primitive, then there exists some simple quantity β such that $\alpha \circ \beta \neq 0$ and $\operatorname{supp}(\alpha) \not\subseteq \operatorname{supp}(\beta)$. If $g \in \operatorname{supp}(\beta)$, then $g \in \operatorname{supp}(\alpha \circ \beta)$. If $g \notin \operatorname{supp}(\beta)$, then $g \in \operatorname{supp}(\alpha - \alpha \circ \beta)$. In either case, g is contained in a simple quantity whose support is strictly smaller than the support of α. Repeating this process recursively, we see that there exists some primitive quantity containing g in its support. Suppose next that α, β are primitive quantities of $\mathfrak{S}$ such that $g \in \operatorname{supp}(\alpha) \cap \operatorname{supp}(\beta)$. But $\operatorname{supp}(\alpha) \cap \operatorname{supp}(\beta) = \operatorname{supp}(\alpha \circ \beta)$. If $\alpha \neq \beta$, then $\operatorname{supp}(\alpha \circ \beta) = \emptyset$, a contradiction. Therefore, $\mathcal{D}$ is a partition of G. As $\mathcal{D}$ consists of all the supports of the primitive quantities of $\mathfrak{S}$, we have $\mathfrak{S} = \operatorname{Span}_F\{\overline{C} \mid C \in \mathcal{D}\}$. $\square$

We now give several basic examples of Schur rings.

Example 2.12 (Trivial Schur Ring)**.** Suppose that G is a finite group. We can then partition G into the sets $\{1\}, G \setminus \{1\}$. The fact that this partition produces sets that satisfy the first two conditions of a Schur ring is obvious. All that remains to check is that we have closure under multiplication. Note that $\overline{\{1\}} \cdot \overline{\{1\}} = \overline{\{1\}}$, $\overline{G \setminus \{1\}} \cdot \overline{\{1\}} = \overline{G \setminus \{1\}}$, and $\overline{\{1\}} \cdot \overline{G \setminus \{1\}} = \overline{G \setminus \{1\}}$. When doing $\overline{G \setminus \{1\}} \cdot \overline{G \setminus \{1\}}$ we note that we have $\overline{G \setminus \{1\}} \cdot \overline{G \setminus \{1\}} = (\overline{G} - 1)(\overline{G} - 1) =$

$\overline{G}\overline{G} - 2\overline{G} + 1 = (|G| - 2)\overline{G} + 1$. This then confirms that we have closure of multiplication. Then picking $\mathcal{D}(\mathfrak{S}) = \{\{1\}, G \setminus \{1\}\}$ produces a Schur ring. We call this Schur ring the *trivial Schur ring*. This is often denoted $F[G]^0$.

Example 2.13. Consider the group $D_8 = \langle r, s \colon r^4 = s^2 = 1, rs = s^{-1}r \rangle$. We partition this group into classes based on conjugation by $\langle s \rangle$. That is we shall let

$$C_1 = \{1\},\ C_2 = \{r, r^3\},\ C_3 = \{r^2\},\ C_4 = \{s\},\ C_5 = \{rs, r^3s\},\ C_6 = \{r^2s\}.$$

The fact that this is a partition of D_8 which satisfies the first two conditions of a Schur ring - namely $\{1\}$ is a primitive set, and that we have closure under the star operator - are clear. All that remains to be verified is that these sets are in fact closed under multiplication. By direct computation we find that

	$\overline{C_1}$	$\overline{C_2}$	$\overline{C_3}$	$\overline{C_4}$	$\overline{C_5}$	$\overline{C_6}$
$\overline{C_1}$	$\overline{C_1}$	$\overline{C_2}$	$\overline{C_3}$	$\overline{C_4}$	$\overline{C_5}$	$\overline{C_6}$
$\overline{C_2}$	$\overline{C_2}$	$2\overline{C_1} + 2\overline{C_3}$	$\overline{C_2}$	$\overline{C_5}$	$\overline{C_3} + 2\overline{C_4} + \overline{C_6}$	$\overline{C_5}$
$\overline{C_3}$	$\overline{C_3}$	$\overline{C_2}$	$\overline{C_1}$	$\overline{C_6}$	$\overline{C_5}$	$\overline{C_4}$
$\overline{C_4}$	$\overline{C_4}$	$\overline{C_5}$	$\overline{C_6}$	$\overline{C_1}$	$\overline{C_2}$	$\overline{C_3}$
$\overline{C_5}$	$\overline{C_5}$	$\overline{C_3} + 2\overline{C_4} + \overline{C_6}$	$\overline{C_5}$	$\overline{C_2}$	$2\overline{C_1} + 2\overline{C_3}$	$\overline{C_2}$
$\overline{C_6}$	$\overline{C_6}$	$\overline{C_5}$	$\overline{C_4}$	$\overline{C_3}$	$\overline{C_2}$	$\overline{C_1}$

With this we see that the sets are indeed closed under multiplication. This then verifies that taking $\mathcal{D}(\mathfrak{S}) = \{C_1, C_2, \dots C_6\}$ gives a Schur ring over the group D_8.

The previous example is one of a more general type of Schur ring that can be constructed over any group.

Definition 2.14 (Orbit Schur ring)**.** Let G be a group and $H \leq \mathrm{Aut}(G)$. Then we define

$$F[G]^H = \{\alpha \in F[G] \colon \sigma(\alpha) = \alpha \text{ for all } \sigma \in H\}.$$

Then $F[G]^H$ is Schur ring over G called the *Orbit Schur ring* with respect to H, whose primitive sets are the orbits of H over G.

Example 2.15 (Direct Product)**.** Let $\mathfrak{S}$ and $\mathfrak{T}$ be Schur rings over groups G and H respectively. We can naturally view G and H are subgroups of $G \times H$; namely, as $G \times \{1\}$ and $\{1\} \times H$. Let

$$\mathcal{P} = \{CD \colon C \in \mathcal{D}(\mathfrak{S}), D \in \mathcal{D}(\mathfrak{T})\},$$

that is, $\mathcal{P}$ is the partition of $G \times H$ generated by all the possible products of a primitive set of $\mathfrak{S}$ by a primitive set of $\mathfrak{T}$. We let $\mathfrak{S} \times \mathfrak{T} = \mathrm{Span}_F\{\overline{C} \cdot \overline{D} \colon C \in \mathcal{D}(\mathfrak{S}), D \in \mathcal{D}(\mathfrak{T})\}$, the subalgebra of $F[G \times H]$ afforded by $\mathcal{P}$. The fact that $\mathfrak{S} \times \mathfrak{T}$ is closed under the Hadamard product follows directly from the fact that $\mathfrak{S}$ and $\mathfrak{T}$ are closed under the Hadamard product. For every $(g, h) \in G \times H$ we have that $g \in P$ for some $P \in \mathcal{D}(\mathfrak{S})$ and $h \in Q$ for some $Q \in \mathcal{D}(\mathfrak{T})$. Then $(g, h) \in PQ$ as $(g, 1)(1, h) = (g, h)$. Then $\overline{P} \cdot \overline{Q} \in \mathfrak{S} \times \mathfrak{T}$ where $(g, h) \in \mathrm{supp}(\overline{P} \cdot \overline{Q})$. Then we have by Theorem 2.11 that $\mathfrak{S} \times \mathfrak{T}$ is a Schur module. As both $\mathcal{D}(\mathfrak{S})$ and $\mathcal{D}(\mathfrak{T})$ contain $\{1\}$ we have $\{1\} \in \mathcal{P}$. Let $C \in \mathcal{D}(\mathfrak{S})$ and $D \in \mathcal{D}(\mathfrak{T})$. Next we show for any $C \in \mathcal{D}(\mathfrak{S})$ and $D \in \mathcal{D}(\mathfrak{T})$ that $(CD)^* = C^*D^*$. Notice for any $(g, h) \in (CD)^*$, we have $(g^{-1}, h^{-1}) = (g, h)^{-1} \in CD$. So $g^{-1} \in C$ and $h^{-1} \in D$. Then $g \in C^*$ and $h \in D^*$. So $(g, h) \in C^*D^*$. Thus, $(CD)^* \subseteq C^*D^*$. If $(a, b) \in C^*D^*$, then $a \in C^*$ and $b \in D^*$ so $a^{-1} \in C$ and $b^{-1} \in D$. Thus, $(a^{-1}, b^{-1}) = (a, b)^{-1} \in CD$. This then gives that $(a, b) \in (CD)^*$. We thus have $(CD)^* = C^*D^*$. As $C^* \in \mathcal{D}(\mathfrak{S})$ and $D^* \in \mathcal{D}(\mathfrak{T})$ we conclude that $(CD)^* \in \mathcal{P}$ for any $CD \in \mathcal{P}$. We lastly need to check for any $C_1, C_2 \in \mathcal{D}(\mathfrak{S})$ and $D_1, D_2 \in \mathcal{D}(\mathfrak{T})$ that $(\overline{C_1} \cdot \overline{D_1}) \cdot (\overline{C_2} \cdot \overline{D_2}) \in \mathfrak{S} \times \mathfrak{T}$. We know that $\overline{C_1} \cdot \overline{C_2} \in \mathfrak{S}$ and that $\overline{D_1} \cdot \overline{D_2} \in \mathfrak{T}$ since these are Schur rings. As multiplication is done component-wise in $G \times H$ we have that

$$(\overline{C_1} \cdot \overline{D_1}) \cdot (\overline{C_2} \cdot \overline{D_2}) = (\overline{C_1} \cdot \overline{C_2})(\overline{D_1} \cdot \overline{D_2}) \in \mathfrak{S} \times \mathfrak{T}.$$

This then confirms that we have closure of multiplication. Thus, $\mathfrak{S} \times \mathfrak{T}$ is indeed a Schur ring. We call $\mathfrak{S} \times \mathfrak{T}$ the *direct product Schur Ring* of $\mathfrak{S}$ and $\mathfrak{T}$.

Like Schur modules, many of the properties of Schur rings remain true when G is allowed to be infinite. One property of all Schur rings is that as they are closed under $*$ by definition. So we have by Theorem 2.5 that for a finite group they are semisimple. Some of these results are included below, and are slight extensions of those given by Wielandt in [50], which only dealt with finite groups.

Proposition 2.16 (cf. [50], Proposition 23.5)**.** *Let $\mathfrak{S}$ be a Schur ring over a group G. Let $\alpha \in \mathfrak{S}$ and $\mathrm{Stab}(\alpha) = \{g \in G \mid \alpha g = \alpha\}$. Then $\mathrm{Stab}(\alpha)$ is an $\mathfrak{S}$-subgroup of G.*

Lemma 2.17 ([4], Lemma 2.15)**.** *Let $\mathfrak{S}$ be a Schur ring over a group G. Let $g \in G$ such that $\{g\} \in \mathcal{D}(\mathfrak{S})$. Then $gC, Cg \in \mathcal{D}(\mathfrak{S})$ for all $C \in \mathcal{D}(\mathfrak{S})$.*

Theorem 2.18 (cf. [19], Lemma 1.2)**.** *Suppose G is a finite group. Let $\varphi : G \to H$ be a group homomorphism with $\ker \varphi = K$ and let $\mathfrak{S}$ be a Schur ring over G. Suppose that K is an $\mathfrak{S}$-subgroup. Then*

(i) If $C \in \mathcal{D}(\mathfrak{S})$ such that

$$C = g_1A_1 \cup g_2A_2 \cup \ldots \cup g_kA_k$$

where $A_1, A_2, \ldots, A_k \subseteq K$ are non-empty, and $g_1K, g_2K, \ldots, g_kK$ are distinct cosets, then $|A_1| = |A_2| = \ldots = |A_k|$.

(ii) If $C, D \in \mathcal{D}(\mathfrak{S})$, then either $\varphi(C) \cap \varphi(D) = \emptyset$ or $\varphi(C) = \varphi(D)$. In other words, if two $\mathfrak{S}$-classes both intersect some coset of K, then they intersect all the same cosets of K.

(iii) The image $\varphi(\mathfrak{S})$ is a Schur ring over $\varphi(G)$ where $\mathcal{D}(\varphi(\mathfrak{S})) = \{\varphi(C) \mid C \in \mathcal{D}(\mathfrak{S})\}$.

With Theorem 2.18 we can discuss an additional type of Schur ring. This is the wedge product of Leung and Man [21]. Let $H, K \leq G$ be two nontrivial, proper subgroups such that $K \leq H$ and $K \trianglelefteq G$. Let $\mathfrak{S}$ be a Schur ring over H with K as an $\mathfrak{S}$-subgroup. Suppose $\varphi : H \to H/K$ is the natural quotient map. Then $\varphi(\mathfrak{S})$ is a Schur ring over H/K by Theorem 2.18. Let $\mathfrak{T}$ be a Schur ring over G/K such that $\mathfrak{T}_{H/K} = \varphi(\mathfrak{S})$. Then we define the *wedge product* $\mathfrak{S} \wedge \mathfrak{T}$ by the partition

$$\mathcal{D}(\mathfrak{S} \wedge \mathfrak{T}) = \mathcal{D}(\mathfrak{S}) \cup \{\varphi^{-1}(D) \mid D \in \mathcal{D}(\mathfrak{T}) \setminus \mathcal{D}(\mathfrak{T}_{H/K})\}.$$

Under these conditions, $\mathfrak{S} \wedge \mathfrak{T}$ is a Schur ring over G (see [21] for details). Alternatively, a Schur ring $\mathfrak{S}$ over G is a wedge product if there exist nontrivial, proper $\mathfrak{S}$-subgroups $H, K \leq G$ such $K \leq H$, $K \trianglelefteq G$, and every $\mathfrak{S}$-class outside of H is a union of K-cosets. In this case, we say that $1 < K \leq H < G$ is a *wedge-decomposition* of $\mathfrak{S}$. The wedge product construction is valid for arbitrary groups so long as K is finite.

Example 2.19 (Wedge Product Example)**.** Let $H \leq G$ and let $\mathfrak{S}$ be the subspace of $F[G]$ given by the partition $\mathcal{D}(\mathfrak{S}) = \{\{1\}, H \setminus 1, G \setminus H\}$. Then $\mathfrak{S} = \text{Span}_F\langle 1, \overline{H} - 1, \overline{G} - \overline{H}\rangle = \text{Span}_F\langle 1, \overline{H}, \overline{G}\rangle$. We see that $\overline{H} \cdot \overline{H} = |H|\overline{H}, \overline{H} \cdot \overline{G} = |H|\overline{G}$, and $\overline{G}^2 = |G|\overline{G}$. Also $\overline{H}^* = \overline{H}$. Then by Theorem 2.10 $\mathfrak{S}$ is a Schur ring. In fact, this is the wedge product Schur ring $F[H]^0 \wedge F[G]^0$.

Having gone over some examples of Schur rings along with some basic results about them, we can now look at an additional operation on Schur rings that is of particular importance. This operation is the Frobenius map. For $\alpha = \sum_{g \in G} \alpha_g g \in F[G]$, we define the *nth Frobenius map* of α for $n \in \mathbb{Z}$ as $\alpha \mapsto \alpha^{(n)} = \sum_{g \in G} \alpha_g g^n$. This was first studied by Wielandt [50] in the context of Schur rings. The following identities are easily proven:

i) $(a\alpha + b\beta)^{(m)} = a\alpha^{(m)} + b\beta^{(m)}$,

ii) $\alpha^{(mn)} = (\alpha^{(m)})^{(n)}$,

iii) $\alpha^{(-1)} = \alpha^*$,

iv) $\overline{C}^{(m)} = \overline{C^{(m)}}$,

for all $\alpha, \beta \in F[G]$, $a, b \in F$, $m, n \in \mathbb{Z}$, and $C \subseteq G$ (for any arbitrary group G).

Theorem 2.20 ([50], Theorem 23.9)**.** *Suppose $\mathfrak{S}$ is a Schur ring over G where G is an abelian group. Suppose m is an integer coprime to the orders of all torsion elements of G. Then for all $\alpha \in \mathfrak{S}$ we have that $\alpha^{(m)} \in \mathfrak{S}$. In particular, if G is torsion-free, then all Schur rings over G are closed under all Frobenius maps.*

Proposition 2.21 ([50], Proposition 23.6). *Let $\mathfrak{S}$ be a Schur ring over G. Let $\alpha \in \mathfrak{S}$, and let H be the subgroup generated by* $\text{supp}(\alpha)$. *Then H is an $\mathfrak{S}$-subgroup of $\mathfrak{S}$.*

Proof. Suppose $\mathfrak{S}$ is a Schur ring over G. Let $\alpha \in \mathfrak{S}$. Consider $L = \text{supp}(\alpha)$. So $H = \langle L \rangle$. Letting $K = L \cup L^*$, then also $H = \langle K \rangle$. This means that

$$H = \bigcup_{i=0}^{\infty} K^i.$$

Since L is the support of an element of $\mathfrak{S}$, it is a union of primitive sets. Thus, the same is true for K. It follows, by induction, the same is true for K^i. That is, K^i is an $\mathfrak{S}$-set. As H is a union of $\mathfrak{S}$-sets, we conclude that H is an $\mathfrak{S}$-subgroup. $\square$

The three major types of Schur rings that we have defined, namely orbit Schur rings, direct product Schur rings, and wedge product Schur rings are called *traditional Schur rings*. This name is inspired by the following classification theorem of Leung and Man.

Theorem 2.22 (Leung and Man, [20]). *Every Schur ring over a finite cyclic group is either a trivial Schur ring, orbit Schur ring, direct product Schur ring, or wedge product Schur ring. That is every Schur ring over a finite cyclic group is a traditional Schur ring.*

Chapter 3. Association Schemes

Schur rings are examples of association schemes (Theorem 3.10). As mentioned before, association schemes have applications in graph theory as well as combinatorics.

Consider a finite set Ω. An association scheme is created by looking at subsets of $\Omega \times \Omega$. If we let $C \subseteq \Omega \times \Omega$ then we say that the *dual subset* of C, denoted C', is

$$C' = \{(b, a) \colon (a, b) \in C\}.$$

If we have $C = C'$ then we say that C is *symmetric*. There is one special symmetric subset of a set $\Omega \times \Omega$ that is of particular interest for an association scheme. This set is called the *diagonal* subset and is denoted $\mathrm{diag}(\Omega)$. It is defined by

$$\mathrm{diag}(\Omega) = \{(\omega, \omega) \colon \omega \in \Omega\}.$$

With these ideas we can now give the first definition of an association scheme.

Definition 3.1 (Association Scheme)**.** An *association scheme* of class d on a finite set Ω is a partition of $\Omega \times \Omega$ into sets $R_0, R_1, \ldots, R_d$ such that

(i) $R_0 = \mathrm{diag}(\Omega)$;

(ii) for all $i = 1, 2, \ldots, d$, $R_i' \in \{R_1, R_2, \ldots, R_d\}$; i.e., $\{R_0, R_1, \ldots R_d\}$ is closed under duals;

(iii) for all $i, j, k \in \{0, 1, 2, \ldots, d\}$ there is an integer p_{ij}^k such that, for all (α, β) in R_k,

$$|\{\gamma \in \Omega \colon (\alpha, \gamma) \in R_i \text{ and } (\gamma, \beta) \in R_j\}| = p_{ij}^k.$$

A short hand notation that we will use to denote such an association scheme $\mathcal{A}$ is

$$\mathcal{A} = (\Omega, \{R_i\}_{0 \le i \le d}).$$

In an association scheme, $\mathcal{A}$, we call the sets $R_0, R_1, \ldots, R_d$ the *associate classes* of $\mathcal{A}$. An association scheme is said to be *commutative* if for all i, j, k we have that $p_{ij}^k = p_{ji}^k$. We further have that an association scheme is said to be *symmetric* if each associate class is symmetric. That is, $R_i' = R_i$ for each i. For a symmetric association scheme two elements $\alpha, \beta \in \Omega$ are called *i-associates* if $(\alpha, \beta) \in R_i$. Following immediately from the symmetric property is that the number of i-associates, for any given element of Ω, is given by p_{ii}^0. For this reason we often let $a_i = p_{ii}^0$ and call this the *valency* of the i^{th} associate class. For $\alpha \in \Omega$, we let

$$R_i(\alpha) = |\{\beta \in \Omega \colon (\alpha, \beta) \in R_i\}|.$$

The following is an obvious lemma, as the R_i partition all of $\Omega \times \Omega$.

Lemma 3.2. *For a symmetric association scheme $\mathcal{A}$ over a set Ω we have that*

$$\sum_{i=0}^{d} a_i = |\Omega|.$$

Example 3.3. Let $|\Omega| = n \geq 2$. We let R_0 be the diagonal subset of $\Omega \times \Omega$. Let $R_1 = (\Omega \times \Omega) \setminus R_0$. Taking classes R_0, R_1 we get an association scheme called the *trivial association scheme* on Ω. In this scheme it is easily checked that $p_{11}^0 = n - 1$ and $p_{11}^1 = n - 2$.

Example 3.4 (Triangular Association Scheme)**.** Let Ω consist of all size 2 subsets of a set of size $n \geq 2$. For $i = 0, 1, 2$ let

$$R_i = \{(\alpha, \beta) \in \Omega \times \Omega \colon |\alpha \cap \beta| = 2 - i\}.$$

Using these sets R_0, R_1, R_2 we get an association scheme that is symmetric. For this association scheme we have $a_1 = 2(n-2)$, since we have two choices for which element a 1-associate β, of α shares with α, and we have $n-2$ choices for the other element of β (as it cannot be something in α). To compute a_2 for a given $\alpha \in \Omega$ we need to count the elements of Ω that are disjoint from α. Such elements are constructed by picking two elements of our set of size n that are not in α. There are $n-2$ such elements, so $a_2 = \binom{n-2}{2}$. This association scheme is called the *triangular association scheme*.

The triangular association scheme is just a special case of the following type of association scheme.

Example 3.5 (Johnson Scheme)**.** Let Ω consist of all subsets of size m of a set with size n. For $i = 0, 1, \ldots, m$ define α and β to be i^{th} *associates if*

$$|\alpha \cap \beta| = m - i.$$

Doing this creates an association scheme. In this association scheme we can compute a_i by counting all elements β that contain exactly $m - i$ of the elements in any fixed α. There are m elements in α and we must pick $m - i$ of them. So we have $\binom{m}{m-i}$ choices for which elements are intersecting. For the other i elements of β, the only restriction is that they must not be in α. We have $n - m$ remaining elements to chose from and there are $\binom{n-m}{i}$ ways to pick our remaining i elements. So this means that $a_i = \binom{m}{m-i}\binom{n-m}{i}$. We call these association schemes *Johnson Schemes*. We denote a particular Johnson Scheme by $J(n, m)$.

Example 3.6. (Petersen Graph) Let Ω be the set of vertices of the Petersen graph in Figure 3.1.

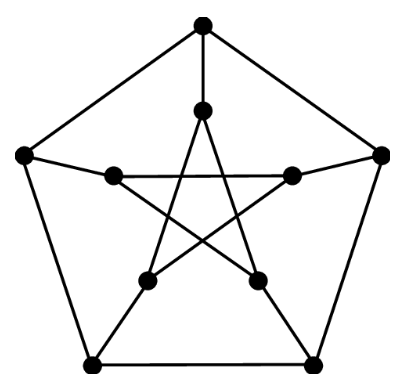

Figure 3.1: Petersen Graph

Let $C_0 = \text{diag}(\Omega)$. We will let $C_1 \subseteq \Omega \times \Omega$ be the set of all pairs of vertices of the graph, joined by an edge. Let $C_2 = (\Omega \times \Omega) \setminus C_1 \setminus C_0$. Clearly each C_i is symmetric.

By directly inspecting the graph we can see that each vertex is joined by 3 edges to other vertices. Thus, $a_1 = 3$. Then this means that $a_2 = 10 - 3 - 1 = 6$. Next we will let $C_i(\alpha) = \{\beta \in \Omega \colon (\alpha, \beta) \in C_i\}$. For any pair $(\alpha, \beta) \in C_1$, the number of $\gamma \in C_i(\alpha) \cap C_j(\beta)$ is given in the following table.

	$C_0(\beta)$	$C_1(\beta)$	$C_2(\beta)$	
$C_0(\alpha)$	0	1	0	1
$C_1(\alpha)$	1	0		3
$C_2(\alpha)$	0			6
	1	3	6	10

Table 3.1: Partial table for number of i-associates of α and j-associates of β for a pair $(\alpha, \beta) \in C_1$ for the Petersen Graph

We know the row and column totals, since they are the number of i-associates of α and β respectively. Then we can easily get the complete table, which follows:

	$C_0(\beta)$	$C_1(\beta)$	$C_2(\beta)$	
$C_0(\alpha)$	0	1	0	1
$C_1(\alpha)$	1	0	2	3
$C_2(\alpha)$	0	2	4	6
	1	3	6	10

Table 3.2: Table for number of i-associates of α and j-associates of β for a pair $(\alpha, \beta) \in C_1$ for the Petersen Graph

We next discuss the table for $(\alpha, \beta) \in C_2$ of the Petersen graph. Looking at the Petersen graph, one can quickly tell that any two vertices which are not joined by an edge have exactly one vertex in common. Thus, $p_{11}^2 = 1$. With this and using subtraction with the row and column totals we can complete the same type of table as above for $(\alpha, \beta) \in C_2$ of the Petersen graph. (See Table 3.3).

With these two completed tables (Table 3.2 and Table 3.3), we can now easily see that taking classes C_0, C_1, C_2 we obtain an association scheme.

This previous example motivates another definition of a symmetric association scheme using graphs.

	$C_0(\beta)$	$C_1(\beta)$	$C_2(\beta)$	
$C_0(\alpha)$	0	0	1	1
$C_1(\alpha)$	0	1	2	3
$C_2(\alpha)$	1	2	3	6
	1	3	6	10

Table 3.3: Table for number of i-associates of α and j-associates of β for a pair $(\alpha, \beta) \in C_2$ for the Petersen Graph

Definition 3.7 (Symmetric Association Scheme (Graph Version))**.** A *symmetric association scheme* of class d on a finite set Ω is a coloring of the edges of the complete undirected graph with vertex set Ω by d colors such that:

(i) for all $i, j, k \in \{1, 2, \ldots, d\}$ there is an integer p_{ij}^k such that whenever $\{\alpha, \beta\}$ is an edge of color k, then

$$p_{ij}^k = |\{\gamma \in \Omega\colon \{\alpha, \gamma\} \text{ has color } i \text{ and } \{\gamma, \beta\} \text{ has color } j\}|,$$

(ii) every color is used at least once, and

(iii) there are integers a_i for $i \in \{1, 2, \ldots d\}$ such that every vertex is contained in exactly a_i edges of color i.

This definition is not quite the same as Definition 3.1. Instead this gives an alternative definition for a symmetric association scheme. Our association classes from the original definition are obtained by letting R_i be the set of all points (α, β) and (β, α) such that the edge $\{\alpha, \beta\}$ has color i. To construct $R_0 = \mathrm{diag}(\Omega)$ we simply add this as another class, which is not given in the construction in Definition 3.7. Condition 3.1(ii) is satisfied since in the graph version every edge is an undirected set of two points. So $R_i' = R_i$ for all i. Conditions 3.7(i) and 3.7(iii) together give us that condition 3.1(iii) in the original definition holds. Note 3.7(i) tells us what happens for all cases except the case of p_{ii}^0 that we had for the original definition. The p_{ii}^0 terms are covered by 3.7(iii). We need condition 3.7(ii) as we had a partition in the original definition, which requires each set to be non-empty.

A third definition of an association scheme can be obtained by thinking in terms of matrices. To do so we let F be any field and let F^{Ω} be the set of all functions from Ω to F. This is a vector space with addition defined by $(f+g)(\omega) = f(\omega) + g(\omega)$ for all $f, g \in F^{\Omega}$, $\omega \in \Omega$ and scalar multiplication defined for all $\lambda \in F$ and $f \in F^{\Omega}$ by $(\lambda f)(\omega) = \lambda(f(\omega))$. For any subset $\Lambda \subseteq \Omega$ we define *the characteristic function* χ_{Λ} as follows

$$\chi_{\Lambda}(\omega) = \begin{cases} 1 & \text{if } \omega \in \Lambda \\ 0 & \text{if } \omega \notin \Lambda \end{cases}.$$

Fix an ordering on Ω. Doing this we can think of χ_{Λ} as a vector where the rows are indexed by the elements of Ω, and get a 1 in position ω if that $\omega \in \Lambda$, otherwise we get a 0. We now extend this same idea to $\Omega \times \Omega$ by having for any $\Lambda \subseteq \Omega \times \Omega$ a function χ_{Λ} defined by

$$\chi_{\Lambda}(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \Lambda \\ 0 & \text{otherwise} \end{cases}.$$

We can now think of this function χ_{Λ} as a matrix where we index the rows and columns by the first and second entries in $\Omega \times \Omega$, respectively. We then have a 1 in entry (x, y) if $(x, y) \in \Lambda$, and 0 otherwise. If we have an association scheme $\mathcal{A}$ with classes $R_0, R_1, \ldots, R_d$ we call χ_{R_i} the i^{th} *adjacency matrix* of $\mathcal{A}$ and denote it by A_i.

Let $\mathcal{A} = (\Omega, \{R_i\}_{0 \leq i \leq d})$ be an association scheme with adjacency matrices $A_0, A_1, \ldots, A_d$. Let

$\mathfrak{A} = \mathrm{Span}_{\mathbb{C}}(A_0, A_1, \ldots, A_d)$. The fact the A matrices are all linearly independent is easy to see. We then have that $\mathfrak{A}$ is a $d+1$ dimensional subalgebra of the matrix algebra $M_{|\Omega|}(\mathbb{C})$. We call $\mathfrak{A}$ the *Bose-Mesner Algebra*.

Before continuing, we discuss entry-wise multiplication of matrices on $M_{|\Omega|}(\mathbb{C})$, denoted by $\circ$, and called the *Hadamard product*. It is easy to see that the Bose-Mesner Algebra $\mathfrak{A}$ is closed under Hadamard product. Further notice that $\mathfrak{A}$ is commutative under the Hadamard product.

Now we let $E_0, E_1, \ldots, E_d$ be the primitive idempotents of $\mathfrak{A}$. As $\mathfrak{A}$ is closed under component-wise multiplication, we have $E_i \circ E_j \in \mathfrak{A}$. Then there are complex numbers q_{ij}^k, called the *Krein parameters*, such that

$$E_i \circ E_j = \frac{1}{|\Omega|} \sum_{k=0}^{d} q_{ij}^k E_k$$

There are matrices related to the A_i and E_i matrices that are defined as follows. We define the diagonal matrices E_i^* and A_i^* *with base point* $x \in \Omega$ by

$$(E_i^*(x))_{yy} = \begin{cases} 1 & (x, y) \in R_i; \\ 0 & \text{otherwise}; \end{cases}$$

$$(A_i^*(x))_{yy} = |\Omega|(E_i)_{xy}.$$

Using these matrices we define the *dual Bose-Mesner algebra* with respect to x as

$$\mathfrak{A}^*(x) = \mathrm{Span}_{\mathbb{C}}(A_0^*(x), A_1^*(x), \ldots, A_d^*(x)) = \mathrm{Span}_{\mathbb{C}}(E_0^*(x), E_1^*(x), \ldots, E_d^*(x)).$$

Both the Bose-Mesner algebra and the dual Bose-Mesner algebra are of interest in their own right. However, our interest will be in the subalgebra of $M_{|\Omega|}(\mathbb{C})$ that they generate together, which is a Terwilliger algebra. We shall study these in more detail in Chapters 4, 5.

We can define an association scheme using matrices that are similar to the A_i defined above. In this case the matrices we used to create the association scheme are its adjacency matrices. Doing this gives us our third definition for an association scheme.

Definition 3.8 (Association Scheme (Matrix Version))**.** An *association scheme* of class d on a finite set Ω is a set of matrices $A_0, A_1, \ldots, A_d \in M_{|\Omega|}(\mathbb{C})$, all of whose entries are equal to 0 or 1 such that

(i) $A_0 = I_{|\Omega|}$;

(ii) for all $i = 1, 2, \ldots, d$ we have $A_i^t = A_j$ for some $j = 1, 2, \ldots, d$;

(iii) for all $i, j \in \{1, 2, \ldots, d\}$ we have $A_i A_j = \sum_{k=1}^{d} p_{ij}^k A_k$;

(iv) none of the A_i is equal to $0_{|\Omega|}$ and $\sum_{i=0}^{d} A_i$ is the all 1 matrix.

Note that this definition matches closely with Definition 3.1 where 3.1(i) corresponds to 3.8(i), 3.1(ii) corresponds to 3.8(ii) and 3.1(iii) corresponds to 3.8(iii) between these two definitions. Note that the p_{ij}^k in this definition are the same as in Definition 3.1. We have to add 3.8(iv) to this definition as we need a partition in the original definition and this condition (iv) in the matrix definition provides us the equivalent condition.

To illustrate the three definitions of an association scheme we now look at an example using all three definitions.

Example 3.9. Let $G = D_6 = \langle r, s \colon r^3 = s^2 = 1, rs = sr^{-1} \rangle$. Consider the conjugacy classes of D_6, which are

$$C_0 = \{1\},\ C_1 = \{r, r^2\},\ C_2 = \{s, rs, r^2 s\}.$$

We shall see in Theorem 3.10 that defining

$$R_i = \{(x, y) \colon yx^{-1} \in C_i\}$$

gives us an association scheme. This association scheme is called the *group association scheme* for D_6. We then use this to get associate classes

$$R_0 = \{(1,1), (r,r), (r^2,r^2), (s,s), (rs,rs), (r^2 s, r^2 s)\}$$

$$R_1 = \{(1,r), (r,1), (r^2,1), (1,r^2), (r,r^2), (r^2,r), (rs,s), (s,rs), (rs,r^2 s), (r^2 s,rs), (r^2 s,s), (s,r^2 s)\}$$

$$R_2 = \{(1,s), (s,1), (rs,1), (1,rs), (r^2 s,1), (1,r^2 s), (s,r), (r,s), (s,r^2),$$

$$(r^2,s), (r,rs), (rs,r), (r^2 s,r), (r,r^2 s), (rs,r^2), (r^2,rs), (r^2 s,r^2), (r^2,r^2 s)\}.$$

The colored graph corresponding to this association scheme is the following graph

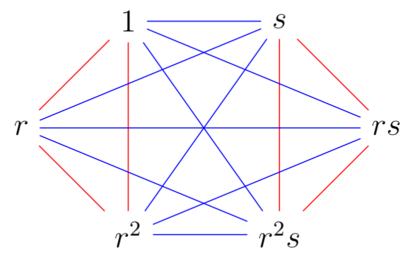

Figure 3.2: Colored Graph of the Group Association Scheme of D_6

The adjacency matrices for this association scheme are

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

These together give us the three different ways we can represent this same association scheme for D_6.

Something of particular interest to us is how Schur rings and association schemes are related. From the similarity in the definitions it is reasonable to guess that many Schur rings give rise to association schemes for the group they are defined over. This we will now show to be the case.

Theorem 3.10. *Every Schur ring over a finite group produces an association scheme for the group. Further, commutative Schur rings give commutative association schemes, and symmetric Schur rings give symmetric association schemes.*

Proof. Let G be a finite group and let $P_0 = \{1\}, P_1, P_2, \ldots, P_d$ be primitive sets for a Schur ring $\mathfrak{S}$ over G. For $i = 0, 1, 2, \ldots, d$ we define the relation R_i as follows:

$$R_i = \{(x, y) \colon yx^{-1} \in P_i\}.$$

We note that as the P_i form a partition of G, defining the R_i as above provides us a partition of $G \times G$. Next we note that if $(x, y) \in R_0$, then $yx^{-1} = 1$, so $y = x$. Furthermore $xx^{-1} = 1$ so $(x, x) \in R_0$ for all x. Thus, $R_0 = \text{diag}(G)$. For any R_i with $i = 1, 2, \ldots, d$ suppose that $(x, y) \in R_i$. Then $yx^{-1} \in P_i$. This tells us that $xy^{-1} \in P_i^*$. We know that $P_i^* = P_j$ for some j. Thus, $(y, x) \in R_j$. So $R_i' \subseteq R_j$. Similar reasoning gives that $R_j \subseteq R_i'$. So $R_i' = R_j$. Thus, for any $i = 1, 2, \ldots, d$ there exists a $j = 1, 2, \ldots, d$ so that $R_i' = R_j$. Note that if $P_i = P_i^*$, we see that $R_i = R_i'$, so once we have shown we do indeed get an association scheme this tells us that a symmetric Schur ring gives a symmetric association scheme. With this we have completed the proof that conditions $(i), (ii)$ of an association scheme are satisfied by these R_i.

For any $i, j, k \in \{0, 1, 2, \ldots, d\}$ consider $(\alpha, \beta) \in R_k$. We will compute

$$|\{\gamma \in G \colon (\alpha, \gamma) \in R_i \text{ and } (\gamma, \beta) \in R_j\}|.$$

By hypothesis $\beta\alpha^{-1} \in P_k$. Similarly if we have $(\alpha, \gamma) \in R_i$ and $(\gamma, \beta) \in R_j$ then $\gamma\alpha^{-1} \in P_i$ and $\beta\gamma^{-1} \in P_j$. From the third condition of a Schur ring we know that in the support of $\overline{P_j} \cdot \overline{P_i}$ each element of P_k shows up exactly the same number of times. We shall denote this number as p_{ij}^k. Having $\gamma\alpha^{-1} \in P_i$ and $\beta\gamma^{-1} \in P_j$ tell us that $\beta\alpha^{-1}$ appears in the product of $\overline{P_j} \cdot \overline{P_i}$ exactly p_{ij}^k times for each (α, β) that we pick. Thus the number of $\gamma \in G$ so that $\gamma\alpha^{-1} \in P_i$ and $\beta\gamma^{-1} \in P_j$ is the same for every $(\alpha, \beta) \in R_k$, namely it is p_{ij}^k. Hence, we have

$$|\{\gamma \in G \colon (\alpha, \gamma) \in R_i \text{ and } (\gamma, \beta) \in C_j\}| = p_{ij}^k$$

for all $(\alpha, \beta) \in R_k$. This confirms the third condition of an association scheme. Thus if $\mathfrak{S}$ is a Schur ring, then taking classes $R_i = \{(x, y) \colon yx^{-1} \in P_i\}$ we get an association scheme.

If $\mathfrak{S}$ is a commutative Schur ring then for all i and j, $\overline{P_i} \cdot \overline{P_j} = \overline{P_j} \cdot \overline{P_i}$. Then using the same argument as above to find p_{ij}^k and p_{ji}^k from the Schur ring, we have that $p_{ij}^k = p_{ji}^k$ for all i, j, k. Thus, the association scheme is commutative. $\square$

This theorem gives rise to the following way we can construct a symmetric association scheme for any finite group $G = \Omega$. Take P_i to be the conjugacy classes of G. These give us an orbit Schur ring. We just proved that this Schur ring produces a related association scheme $\mathcal{A}$. An association scheme created in this way is called the *group association scheme* of G. Note that when constructing such an association scheme we can partially order the group elements using the conjugacy classes. The partial order is defined as follows: all of the elements in the conjugacy class P_0 come first, then all those in P_1, and so on. Then the adjacency matrices can be thought of as having blocks corresponding to conjugacy classes.

Two special types of association schemes that are of interest to us are called $P-$polynomial and $Q-$polynomial schemes.

Definition 3.11 (P-polynomial association scheme)**.** Let $\mathcal{A}$ be a symmetric association scheme with associate classes $R_0, R_1, \ldots, R_d$. Let $A_0, A_1, \ldots, A_d$ be its related adjacency matrices. Then $\mathcal{A}$ is called a *$P-$polynomial association scheme* with respect to this ordering of the A_i, if there exist complex coefficient polynomials $v_i(x)$ of degree i for each $0 \leq i \leq d$ such that $A_i = v_i(A_1)$ for all i, where multiplication is done using standard matrix multiplication.

Example 3.12. Let $G = C_5 = \langle t \rangle$, the cyclic group of order 5 with generator t. We partition $G \times G$ as

$$P_0 = \{0\},\ P_1 = \{t\},\ P_2 = \{t^2\},\ P_3 = \{t^3\},\ \text{and}\ P_4 = \{t^4\}.$$

This partition can quickly be verified to form a Schur ring over G. It is created using the conjugacy classes of G to form the partition sets. Let $\mathcal{A} = (G, \{R_i\}_{0 \leq i \leq 4})$ be the resulting association scheme from Theorem 3.10. Let A_0, A_1, A_2, A_3, A_4 be the adjacency matrices for associate classes R_0, R_1, R_2, R_3, R_4 respectively. We define $v_i(x) = x^i$ for all $0 \leq i \leq 4$. We now consider $v_i(A_1) = A_1^i$. As a result of Theorem 3.10 we know the coefficients from multiplying two adjacency matrices together are the same as those we get when multiplying two primitive quantities from the related Schur ring together. We thus look at $\overline{P_1}^i$ for $0 \leq i \leq 4$. It is obvious by looking at the primitive sets P_i that $\overline{P_1}^i = \overline{P_i}$ for all i. Thus, we have $A_1^i = A_i$ for all i. Thus, $v_i(x)$ are polynomials of degree i with $0 \leq i \leq d$, such that $A_i = v_i(A_1)$ for all i. Thus, our association scheme is a $P-$polynomial association scheme.

Remark: There is nothing particularly special about C_5 in Example 3.12. By the same reasoning we get that for any finite cyclic group G, the conjugacy classes of G give rise to a $P-$polynomial association scheme.

Example 3.13. Now consider a cube labeled as shown in Figure 3.3.

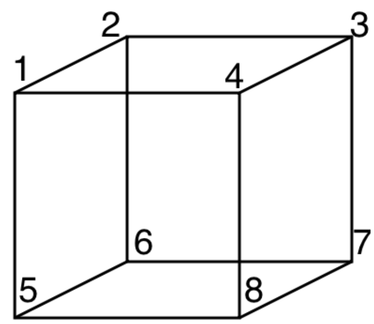

Figure 3.3: Labelling of Cube for Association Scheme

We create an association scheme on the set $\Omega = \{1, 2, \ldots, 8\}$ using this cube. Let $d(x, y)$ be the minimum number of edges one must move along to get from vertex x to vertex y. Then we define

$$R_i = \{(x, y) \in \Omega \times \Omega \colon d(x, y) = i\}.$$

Note that there are only four R_i classes, as three is the largest value $d(x, y)$ takes for the cube. We can easily see that R_0, R_1, R_2, R_3 form a partition of $\Omega \times \Omega$. Now $R_0 = \operatorname{diag}(\Omega)$, as the only point distance 0 from any given point is the point itself. Each of R_0, R_1, R_2, R_3 are symmetric since $d(x, y) = d(y, x)$ so condition (ii) of an association scheme is satisfied. Through direct inspection we also find that the third condition of an association scheme is satisfied. Then $\mathcal{A} = (\Omega, \{R_i\}_{0 \leq i \leq 3})$ is an association scheme. We let A_0, A_1, A_2, A_3 be the related adjacency matrices for R_0, R_1, R_2, R_3. We find that

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \ A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

With these matrices we now define the polynomials

$$v_0(x) = x^0, \ v_1(x) = x, \ v_2(x) = \frac{1}{2}x^2 - \frac{3}{2}, \text{ and } v_3(x) = \frac{1}{6}x^3 - \frac{7}{6}x.$$

Notice that clearly $v_0(A_1) = A_0$, and $v_1(A_1) = A_1$. By direct computation one can find that $v_2(A_1) = A_2$ and $v_3(A_1) = A_3$. We then have that $\mathcal{A}$ is a P-polynomial scheme.

Definition 3.14 (Q-polynomial association scheme)**.** Let $\mathcal{A}$ be a symmetric association scheme and let E_i be the primitive idempotents of the Bose-Mesner algebra, created from the adjacency matrices of $\mathcal{A}$. Then $\mathcal{A}$ is called a *Q-polynomial association scheme* with respect to the ordering $E_0, E_1, \ldots, E_d$ if there exists some complex coefficient polynomials

$v_i^*(x)$ with degree i such that $E_i = v_i^*(E_1)$, $0 \leq i \leq d$, where multiplication is done using the Hadamard product of matrices.

Example 3.15 (P-polynomial and Q-polynomial association scheme)**.** Let us consider the triangular association scheme for a set Ω with $|\Omega| = 5$. Using sets R_1, R_2 as defined in Example 3.4 along with R_0 as the diagonal of $\Omega \times \Omega$ we get an association scheme; call it $\mathcal{A}$. The related adjacency matrices are

$$A_0 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0&1&1&1&1&1&1&0&0&0 \\ 1&0&1&1&1&0&0&1&1&0 \\ 1&1&0&1&0&1&0&1&0&1 \\ 1&1&1&0&0&0&1&0&1&1 \\ 1&1&0&0&0&1&1&1&1&0 \\ 1&0&1&0&1&0&1&1&0&1 \\ 1&0&0&1&1&1&0&0&1&1 \\ 0&1&1&0&1&1&0&0&1&1 \\ 0&1&0&1&1&0&1&1&0&1 \\ 0&0&1&1&0&1&1&1&1&0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0&0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&1&1&0&0&1 \\ 0&0&0&0&1&0&1&0&1&0 \\ 0&0&0&0&1&1&0&1&0&0 \\ 0&0&1&1&0&0&0&0&0&1 \\ 0&1&0&1&0&0&0&0&1&0 \\ 0&1&1&0&0&0&0&1&0&0 \\ 1&0&0&1&0&0&1&0&0&0 \\ 1&0&1&0&0&1&0&0&0&0 \\ 1&1&0&0&1&0&0&0&0&0 \end{bmatrix}.$$

With these matrices in this same ordering we take

$$v_0(x) = x^0,\ v_1(x) = x,\ \text{and } v_2(x) = \frac{1}{4}x^2 - \frac{3}{4}x - \frac{3}{2}.$$

It is obvious that $v_0(A_1) = A_0$ and $v_1(A_1) = A_1$. We can also find through direct computation that $v_2(A_1) = \frac{1}{4}A_1^2 - \frac{3}{4}A_1 - \frac{3}{2}I = A_2$. Thus the triangular association scheme $\mathcal{A}$ is a P-polynomial association scheme.

Next we find the primitive idempotents of the Bose-Mesner algebra generated by the adjacency matrices for $\mathcal{A}$. These are the E_i matrices for the association scheme.

$$E_0 = \frac{1}{10} \cdot \text{(the all 1's matrix)},$$

$$E_1 = \begin{bmatrix} 2/5 & 1/15 & 1/15 & 1/15 & 1/15 & 1/15 & 1/15 & -4/15 & -4/15 & -4/15 \\ 1/15 & 2/5 & 1/15 & 1/15 & 1/15 & -4/15 & -4/15 & 1/15 & 1/15 & -4/15 \\ 1/15 & 1/15 & 2/5 & 1/15 & -4/15 & 1/15 & -4/15 & 1/15 & -4/15 & 1/15 \\ 1/15 & 1/15 & 1/15 & 2/5 & -4/15 & -4/15 & 1/15 & -4/15 & 1/15 & 1/15 \\ 1/15 & 1/15 & -4/15 & -4/15 & 2/5 & 1/15 & 1/15 & 1/15 & 1/15 & -4/15 \\ 1/15 & -4/15 & 1/15 & -4/15 & 1/15 & 2/5 & 1/15 & 1/15 & -4/15 & 1/15 \\ 1/15 & -4/15 & -4/15 & 1/15 & 1/15 & 1/15 & 2/5 & -4/15 & 1/15 & 1/15 \\ -4/15 & 1/15 & 1/15 & -4/15 & 1/15 & 1/15 & -4/15 & 2/5 & 1/15 & 1/15 \\ -4/15 & 1/15 & -4/15 & 1/15 & 1/15 & -4/15 & 1/15 & 1/15 & 2/5 & 1/15 \\ -4/15 & -4/15 & 1/15 & 1/15 & -4/15 & 1/15 & 1/15 & 1/15 & 1/15 & 2/5 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 1/2 & -1/6 & -1/6 & -1/6 & -1/6 & -1/6 & -1/6 & 1/6 & 1/6 & 1/6 \\ -1/6 & 1/2 & -1/6 & -1/6 & -1/6 & 1/6 & 1/6 & -1/6 & -1/6 & 1/6 \\ -1/6 & -1/6 & 1/2 & -1/6 & 1/6 & -1/6 & 1/6 & -1/6 & 1/6 & -1/6 \\ -1/6 & -1/6 & -1/6 & 1/2 & 1/6 & 1/6 & -1/6 & 1/6 & -1/6 & -1/6 \\ -1/6 & -1/6 & 1/6 & 1/6 & 1/2 & -1/6 & -1/6 & -1/6 & -1/6 & 1/6 \\ -1/6 & 1/6 & -1/6 & 1/6 & -1/6 & 1/2 & -1/6 & -1/6 & 1/6 & -1/6 \\ -1/6 & 1/6 & 1/6 & -1/6 & -1/6 & -1/6 & 1/2 & 1/6 & -1/6 & -1/6 \\ 1/6 & -1/6 & -1/6 & 1/6 & -1/6 & -1/6 & 1/6 & 1/2 & -1/6 & -1/6 \\ 1/6 & -1/6 & 1/6 & -1/6 & -1/6 & 1/6 & -1/6 & -1/6 & 1/2 & -1/6 \\ 1/6 & 1/6 & -1/6 & -1/6 & 1/6 & -1/6 & -1/6 & -1/6 & -1/6 & 1/2 \end{bmatrix}.$$

Now with these matrices we consider the following polynomials

$$v_0^*(x) = \frac{1}{10}x^0, \; v_1^*(x) = x, \text{ and } v_2^*(x) = \frac{9}{2}x^2 - \frac{1}{10}x - \frac{9}{50}.$$

Plugging E_1 into each of these where multiplication is done by the Hadamard product, and the identity is the all 1 matrix, we get $v_0^*(E_1) = E_0$ and $v_1^*(E_1) = E_1$. By direct computation we find that $v_2^*(E_1) = \frac{9}{2}E_1^2 - \frac{1}{10}E_1 - \frac{9}{50}E_1^0 = E_2$. This then means that using these polynomials we have that $\mathcal{A}$ is a Q-polynomial scheme. This then means that the triangular association scheme for a set of size 5 is both a P-polynomial scheme and a Q-polynomial scheme.

We now turn our attention to looking at Terwilliger algebras. In particular, we shall discuss many basic results about them in this chapter as well as provide the Wedderburn decomposition of several Terwilliger algebras. Terwilliger algebras are a subalgebra of the matrix algebra $M_n(\mathbb{C})$ that are defined using the Bose-Mesner algebra.

Definition 4.1 (Terwilliger Algebra)**.** Let $\mathcal{A}$ be an association scheme over a finite set Ω. We define the *Terwilliger algebra* of $\mathcal{A}$ with base point x, denoted $T(x)$, to be the subalgebra of $M_{|\Omega|}(\mathbb{C})$ under matrix multiplication, generated by the Bose-Mesner algebra $\mathfrak{A}$ and the related dual Bose-Mesner algebra $\mathfrak{A}^*(x)$ obtained from $\mathcal{A}$.

The choice of the base point x determines $\mathfrak{A}^*(x)$. Given different base points we can get different Terwilliger algebras for the same association scheme. These different Terwilliger algebras still have many of the same properties as can be seen in [42] where the results are proven for any chose of base point. Examples of some of these properties follow. The first of these is that Terwilliger algebras are almost always non-commutative. This is stated without proof in [41].

Proposition 4.2. *Suppose that $|\Omega| > 1$. Then for any association scheme $\mathcal{A} = (\Omega\colon \{R_i\}_{0\leq i\leq d})$ and any $x \in \Omega$ the Terwilliger algebra $T(x)$ is non-commutative.*

Proof. Let $A_0, A_1, A_2, \ldots, A_d$ be the adjacency matrices for our association scheme $\mathcal{A}$. Let $y \in \Omega$ such that $y \neq x$. As the sum of the A_i matrices is the all 1 matrix, there exists a unique A_r such that the entry of A_r indexed by (x, y) is 1. Consider $E_0^*(x)$. There is a 1 in the entry of $E_0^*(x)$ indexed by (z, z) for each $z \in \Omega$ such that $(x, z) \in R_0$. As R_0 is the diagonal of $\Omega \times \Omega$ the only term of the form (x, z) in R_0 is $(x, x) \in R_0$. So the entry of $E_0^*(x)$ indexed by (x, x) is 1 and all other entries are 0. Thus $E_0^*(x)A_r$, has the same x^{th} row as A_r and all other rows are zero. Note that the entry of A_r indexed by (x, y) is 1, so the entry of $E_0^*(x)A_r$ indexed by (x, y) is 1. Looking at $A_rE_0^*(x)$ we see that the column indexed by x of $A_r E_0^*(x)$ equals the column indexed by x of A_r and all other entries are 0. In particular, the entry indexed by (x, y) is 0 as it is not in the column indexed by x. Therefore, $E_0^*(x) A_r \neq A_r E_0^*(x)$. Hence, the Terwilliger algebra is non-commutative. $\square$

We next will show that $T(x)$ is semisimple for any base point ([41], Lemma 3.4(i)). To that end we start by proving the first part of Lemma 3.4(i) of [41] and then prove semisimplicity.

Lemma 4.3 ([41], Lemma 3.4(i))**.** *A subalgebra of $M_n(\mathbb{C})$ that is closed under the conjugate-transpose is semi-simple.*

Proof. Suppose $\mathcal{B}$ is a subalgebra of $M_n(\mathbb{C})$ that is closed under conjugate-transpose, but is not semisimple. Let $\mathcal{J}(\mathcal{B})$ denote the Jacobson radical of $\mathcal{B}$. Since $\mathcal{B}$ is a left Artinian ring, we have by Theorem A.11 that $\mathcal{J}(\mathcal{B}) \neq 0$. So $\mathcal{J}(\mathcal{B})$ contains a simple left ideal, say $\mathcal{B}A$. Note $A \in \mathcal{J}(\mathcal{B})$ since if it isn't, then $\mathcal{B}A$ being simple is contradicted. So $A(\mathcal{B}A) = 0$. As $\mathcal{B}$ is closed under conjugate-transpose we have that $\overline{A^t} \in \mathcal{B}$. As $\mathcal{B}$ is an algebra that means $\overline{A^t}^2 \in \mathcal{B}$. Then $A(\overline{A^t}^2 A) = 0$, which means $(A\overline{A^t})(\overline{A^t}A) = (A\overline{A^t})\overline{(A\overline{A^t})^t} = 0$.

We now claim that the only solution $B \in M_n(\mathbb{C})$ to the equation $B\overline{B^t} = 0$ is the zero matrix. Say $B = (b_{ij})$. Then the (i, i) entry of $B\overline{B^t}$ is $\sum_{j=1}^n b_{ij}\overline{b_{ij}}$. This must be 0. Note that $b_{ij}\overline{b_{ij}} = |b_{ij}|^2$, the square of the modulus of b_{ij}. This is a non-negative real number. As $\sum_{j=1}^n b_{ij}\overline{b_{ij}} = 0$ we must have that $|b_{ij}|^2 = 0$ for all j. This is only true if $b_{ij} = 0$ for all j. We chose i arbitrarily. Thus, $b_{ij} = 0$ for all i and j. So, $B\overline{B^t} = 0$ implies $B = 0$.

As $(A\overline{A^t})\overline{(A\overline{A^t})^t} = 0$, we have $A\overline{A^t} = 0$. This then means that $A = 0$, which contradicts the fact $\mathcal{B}A$ is simple. Therefore, $\mathcal{B}$ is semi-simple. $\square$

Proposition 4.4 ([41], Lemma 3.4)**.** *Let $T(x)$ be a Terwilliger algebra for $\mathcal{A} = (\Omega, \{R_i\}_{0\leq i\leq d})$ with base point $x \in \Omega$. Then $T(x)$ is semi-simple.*

Proof. Every matrix in $T(x)$ is a linear combination of products of adjacency matrices A_i and matrices $E_i^*(x)$. Both of these types of matrices only consist of 1's and 0's, so they are unchanged by complex conjugation. Also the set of A_k's is closed under transposes because of condition 3.8(ii). Note that the $E_k^*(x)$ are diagonal, so they equal their transpose. Thus, the conjugate-transpose of A_k and $E_k^*(x)$ for any k is still in $T(x)$. So the conjugate-transpose of any sum of products of A_i's and $E_i^*(x)$'s is in $T(x)$. So by Lemma 4.3 $T(x)$ is semi-simple. □

One result of some interest is that we can write each adjacency matrix as a sum of blocks as follows.

Lemma 4.5. *Let $T(x)$ be a Terwilliger algebra with base point x for an association scheme $\mathcal{A} = (\Omega\colon \{R_i\}_{0\leq i\leq d})$. Let A_k be any adjacency matrix of $\mathcal{A}$. Then $A_k = \sum_{i,j} E_i^*(x) A_k E_j^*(x)$.*

Proof. We know that $\sum_{i=1}^{d} E_i^*(x) = I = I_{|\Omega|}$. Then $IA_kI = \sum_{i,j} E_i^*(x) A_k E_j^*(x)$. □

The above lemma motivates us to look at products $E_i^*(x)A_kE_j^*(x)$ as they form building blocks for the adjacency matrix. To that end the following is a lemma proven in [3] that will be of use in discussing a subspace of a Terwilliger algebra $T(x)$ that has the matrices $E_i^*(x)A_kE_j^*(x)$ as a basis. The following lemma also lets us look at a subspace of the Terwilliger algebra with basis matrices $E_iA_j^*(x)E_k$.

Lemma 4.6 ([2],Lemma 1)**.** *For an association scheme $\mathcal{A} = (\Omega, \{R_i\}_{0\leq i\leq d})$ with matrices $A_i, E_i, A_i^*(x)$, and $E_i^*(x)$ we have*

- $tr(E_i^*(x)A_jE_k^*(x)(\overline{E_\ell^*(x)A_mE_n^*(x)})^T) = \delta_{i\ell}\delta_{jm}\delta_{kn}p_{ij}^k|R_k|$
- $tr(E_iA_j^*(x)E_k(\overline{E_\ell A_m^*(x)E_n})^T) = \delta_{i\ell}\delta_{jm}\delta_{kn}q_{ij}^k \operatorname{rank} E_k$

where δ_{ij} represents the Kronecker-delta function, p_{ij}^k are the structure constants of the association scheme $\mathcal{A}$, and the q_{ij}^k are the Krein parameters.

A Terwilliger algebra $T(x)$ with base point x over the association scheme $\mathcal{A} = (\Omega, \{R_i\}_{0\leq i\leq d})$ has two subspaces of particular interest that have dimensions that are closely related to that of the Terwilliger algebra itself. They are defined as follows:

$$T_0(x) = \mathrm{Span}_{\mathbb{C}}(E_i^*(x)A_jE_k^*(x)\colon 0 \leq i,j,k \leq d),$$

$$T_0^*(x) = \mathrm{Span}_{\mathbb{C}}(E_iA_j^*(x)E_k\colon 0 \leq i,j,k \leq d).$$

Notice that $E_i^*(x)A_jE_k^*(x)$ will give the rows of A_j indexed by elements R_i and the columns of A_j indexed by elements of R_k. Then the entry of $E_i^*(x)A_jE_k^*(x)$ indexed by (y,z) is nonzero if and only if $(x,y) \in R_i$, $(y,z) \in R_j$ and $(x,z) \in R_k$. By definition of p_{ij}^k, $(x,y) \in R_i$, $(y,z) \in R_j$ and $(x,z) \in R_k$ if and only if $p_{ij}^k \neq 0$. We thus have $E_i^*(x)A_jE_k^*(x) \neq 0$ if and only if $p_{ij}^k \neq 0$. This means that

$$\dim T_0(x) = |\{(i,j,k) \colon p_{ij}^k \neq 0\}|, \tag{4.1}$$

It is proven in ([41], Lemma 3.2) using Lemma 4.6 that $E_iA_j^*(x)E_k \neq 0$ if and only if $q_{ij}^k \neq 0$. We thus have

$$\dim T_0^*(x) = |\{(i,j,k) \colon q_{ij}^k \neq 0\}|. \tag{4.2}$$

Definition 4.7. Let $T(x)$ be a Terwilliger algebra, where x is the base point, for an association scheme over a finite set Ω. If W is an irreducible $T(x)-$module such that $\dim E_i^*(x)W \leq 1$ for all i, then we say that W is *thin*. We say that an association scheme is *thin with respect to* x if each irreducible $T(x)-$module is thin. An association scheme is *thin* if it is thin with respect to every $x \in \Omega$.

As it would happen, every Terwilliger algebra has at least one irreducible module that is thin ([41]), which we now prove. To do so we first let $W = \mathbb{C}^n$ and let $\langle , \rangle$ denote the Hermitean form $\langle u, v\rangle = u^t\overline{v}$. We then call $W, \langle , \rangle$ the *standard module* of $M_n(\mathbb{C})$. Since a Terwilliger algebra T over a set of size n is a subalgebra of $M_n(\mathbb{C})$ we have that W is also a $T-$module for any Terwilliger algebra T over a set of size n ([41]).

Proposition 4.8 (Lemma 3.6, [41])**.** *Let Ω be a finite set and $\mathcal{A} = (\Omega, \{R_i\}_{0\leq i\leq d})$. Let $x \in \Omega$ be any element of Ω. Suppose that $\mathfrak{A}$ is the Bose-Mesner algebra of $\mathcal{A}$ and $\mathfrak{A}^*(x)$ is the dual Bose-Mesner algebra of $\mathcal{A}$. We write $E_i^* = E_i^*(x)$, $A_i^* = A_i^*(x)$, $\mathfrak{A}^* = \mathfrak{A}^*(x)$, and $T = T(x)$. Then*

$$A_i\hat{x} = E_j^*v \ (0 \leq i \leq d), \ \textit{where } A_i^t = A_j,$$

$$A_i^*v = |\Omega|E_i^t\hat{x} \ (0 \leq i \leq d),$$

where v is the all 1's vector in the standard module and $\hat{x}$ is the vector $(0, 0, \dots, 1, 0, \dots 0)^t$ *where the entry indexed by x is the only nonzero entry. In particular, $\mathfrak{A}\hat{x} = \mathfrak{A}^*v$ is a thin irreducible $T-$module of dimension $d+1$.*

Proof. As $\hat{x}$ is a vector with 1 in the entry indexed by x and 0 elsewhere we have that $A_i\hat{x}$ is just the column of A_i indexed by x. Thus, an entry of $A_i\hat{x}$ indexed by y is 1 if $(x, y) \in R_i$ and is 0 otherwise. We know that $R_j = \{(b, a) \colon (a, b) \in R_i\}$, so E_j^* is a diagonal matrix where an entry indexed by (y, y) is 1 if $(x, y) \in R_i$ and 0 otherwise. Then as v is the all 1 vector we have that E_j^*v is a vector that corresponds exactly to the main diagonal of E_j^*. Thus, an entry of E_j^*v indexed by y is 1 if $(x, y) \in R_i$ and is 0 otherwise. We thus have that $A_i\hat{x} = E_j^*v$ for any $(0 \leq i \leq d)$ where $A_i^t = A_j$. The proof that $A_i^*v = |\Omega|E_i^t\hat{x}$ for any $0 \leq i \leq d$ is similar using the fact that $(A_i^*)_{yy} = |\Omega|(E_i)_{xy}$ for all $y \in \Omega$.

From these two facts above we have by the definition of $\mathfrak{A}$ and $\mathfrak{A}^*$ that $\mathfrak{A}\hat{x} = \mathfrak{A}^*v$. Since $\mathfrak{A}\hat{x}$ is $\mathfrak{A}-$invariant and $\mathfrak{A}^*v$ is $\mathfrak{A}^*-$invariant, we have that $\mathfrak{A}\hat{x}$ is a $T-$module. As the standard module can be decomposed into an orthogonal direct sum of irreducible $T-$modules, there exists an irreducible $T-$module U that is not orthogonal to $\hat{x}$. Then $\hat{x} \in E_0^*U \subseteq U$. Then as U is a $T-$module containing $\hat{x}$ and $\mathfrak{A} = \mathrm{Span}_{\mathbb{C}}(A_0, A_1, \dots A_d)$ we have $\mathfrak{A}\hat{x} \subseteq U$. Since U is irreducible we must have $\mathfrak{A}\hat{x} = U$. So $\mathfrak{A}\hat{x}$ is a irreducible $T-$module. As $A_i\hat{x} = E_j^*v$ where $A_i^t = A_j$ for all i we know that $\dim(E_i^*(\mathfrak{A}\hat{x})) = 1$ for all i. Thus, $\mathfrak{A}\hat{x}$ is a thin $T-$module, which clearly has dimension $d+1$. □

Now that we have covered some of the basic information about Terwilliger algebras we turn our attention to proving the existence of a certain irreducible ideal of any Terwilliger algebra created using an orbit Schur ring over any finite group G. Note we can do this as, by Theorem 3.10, every Schur ring gives rise to an association scheme. We shall let $P_0, P_1, \dots, P_d$ be the primitive classes of G and $T(G)$ be the Terwilliger algebra created with base point e. We begin by constructing the elements of $T(G)$ that will serve as a basis for the ideal.

Notice that by construction E_0^* is a matrix with 1 in position $(1,1)$ and all other entries are 0. It follows immediately that $E_0^* A_i$ is a matrix with the same first row as A_i and all other rows are zero. Similarly we have that $A_i E_0^*$ is a matrix with the same first column as A_i and all other columns are zero. Let $X_i = E_0^* A_i$ and $Y_i = A_i E_0^*$.

As X_i consists only of the first row of A_i and the first row of A_i has 1 in the entry indexed by (e, g) for those $g \in G$ such that $ge = g \in P_i$, we have that X_i's first row consists exactly of 1's in positions indexed by (e, g) for the $g \in P_i$. Looking at Y_i, since its nonzero entries are just the first column of A_i we have that its first column consists of $1'$s in entries indexed by (g, e) for those $g \in G$ such that $g^{-1}e \in P_i$. This is the same as saying $g \in P_i^*$. So Y_i has 1's exactly in the entries indexed by (g, e) corresponding to elements $g \in P_i^*$. In fact $Y_j = X_i^t$, where $A_j = A_i^t$.

Let X_{b^*} and Y_{b^*} denote respectively the X_i, and Y_i matrices with their only nonzero entries corresponding to the elements of P_b^*. Note that $X_{b^*} = E_0^* A_b^t$ and $Y_{b^*} = A_b^t E_0^*$. As we have ordered G by the primitive sets, rows and columns corresponding to elements of primitive sets P_i and P_j are all grouped together. We will then say that the P_i, P_j block of a matrix D is the block of D with entries indexed by (x, y) corresponding to $x \in P_i$ and $y \in P_j$.

Lemma 4.9. *Let X_i, Y_j be defined as above. Then $Y_j X_i$ is a matrix whose P_j^*, P_i block is all* 1*'s and all other entries are* 0*.*

Proof. The matrix Y_j has 1's in the entries indexed by $(u, 1)$ with $u \in P_j^*$ and 0's elsewhere. X_i has 1's in the entries indexed by $(1, v)$ with $v \in P_i$ and 0's elsewhere. We then see that $Y_j X_i$ is a matrix with 1's in every position indexed by (u, v) where $u \in P_j^*$ and $v \in P_i$, and all other entries are 0. Thus, the P_j^*, P_i block of $Y_j X_i$ is all 1's and all other entries are 0. □

We now let v_{ij} be the the matrix with the all 1's P_i, P_j block and all other entries 0. Lemma 4.9 tells us $v_{ij} = Y_{i^*} X_j$. As an additional note we know that the sum of all the adjacency matrices is the all 1's matrix. Then $Y_{i^*} X_j$ is just the P_i, P_j block of the sum of all the adjacency matrices. So if $A_0, A_1, \dots, A_d$ are all the adjacency matrices we have $Y_{i^*}X_j = E_i^*(A_0 + A_1 + \cdots + A_d)E_j^* = \sum_{k=0}^{d} E_i^* A_k E_j^*$.

We next shall prove a lemma that will be helpful in demonstrating that $\mathrm{Span}_{\mathbb{C}}(v_{ij})$ is closed under multiplication by A_k for any k.

Lemma 4.10. *Given an adjacency matrix A_k for an orbit Schur ring over a finite group G of order n with orbits $P_0, P_1, \dots, P_d$, the number of 1's in every row of the P_i, P_j block of A_k does not depend on the row. The number of 1's in every column of the P_i, P_j block of A_k does not depend on the column.*

Proof. Let $H \leq \mathrm{Aut}(G)$ be the subgroup of $\mathrm{Aut}(G)$ that gives the orbits used for the orbit Schur ring over G. Suppose that the P_i, P_j block of A_k has rows corresponding to group elements $h_1, h_2, \dots, h_m$ and columns corresponding to group elements $f_1, f_2, \dots, f_n$. Note every $h_u \in P_i$ and every $f_v \in P_j$ by construction of the P_i, P_j block. As P_i is an orbit of G, for any $h_u \in P_i$ there exists some $\alpha \in H$ such that $h_u = \alpha(h_1)$. For any $f_v \in P_j$ we have that the P_i, P_j block of A_k has a 1 in the entry indexed by (h_1, f_v) if and only if $f_v^{-1}h_1 \in P_k$. Applying α to $f_v^{-1}h_1$ we get

$$\alpha(f_v^{-1}h_1) = \alpha(f_v^{-1})\alpha(h_1) = \alpha(f_v)^{-1}h_u.$$

Since P_j is an orbit resulting from H, $\alpha(f_v) \in P_j$. Thus, $\alpha(f_v)^{-1}h_u$ corresponds to some nonzero entry in the h_u row of the P_i, P_j block of A_k. Since P_k is an orbit, $\alpha(f_v^{-1}h_1) \in P_k$ if and only if $f_v^{-1}h_1 \in P_k$. Thus the entries of A_k indexed by (h_1, f_v) and $(h_u, \alpha(f_v))$ will have the same value. Doing this for every entry indexed by (h_1, f_v), $(1 \leq v \leq n)$ we see that the entries indexed by (h_1, f_v) and $(h_u, \alpha(f_v))$ will have the same value. This is because elements of H map elements of P_j to P_j. Thus, the row indexed by h_1 and the row indexed by h_u must have the same number of 1's as the entries are in one-to-one correspondence. We chose h_u arbitrarily so every row of the P_i, P_j block of A_k must have the same number of 1's. By similar reasoning all the columns of this block also have the same number of 1's. $\square$

We now prove a lemma that will be very helpful in proving that a certain subspace of a Terwilliger algebra is in fact a two-sided ideal of the Terwilliger algebra.

Lemma 4.11. *Let $\mathcal{A} = (\Omega, \{R_i\}_{0 \le i \le d})$ be an association scheme over a nonempty set Ω. For any $x \in \Omega$, let $T(x)$ be the Terwilliger algebra with base point x for $\mathcal{A}$. Suppose that W is a subspace of $T(x)$ that is closed under transposition, and is a left ideal of $T(x)$. Then W is a two-sided ideal of $T(x)$.*

Proof. Let $A_0, A_1, \ldots, A_d$ be the adjacency matrices of $\mathcal{A}$. Let A_k be any adjacency matrix of $\mathcal{A}$. Then $A_k^t = A_b$ for some b, by definition of an association scheme. Now let $w \in W$. Then $w^t \in W$ as well. By assumption we know that $A_b w^t \in W$ as W is a left ideal of $T(x)$. Since V is closed under transposition we have that $(A_b w^t)^t = wA_b^t = wA_k \in W$. Thus, for any $w \in W$ we have that $wA_k \in W$. For any $E_k^*(x)$ we know that it is diagonal and thus symmetric. As W is a left ideal of $T(x)$ we have that $E_k^*(x)w^t \in W$. Then as W is closed under transposition we have that $(E_k^*(x)w^t)^t = wE_k^*(x) \in W$ for any $w \in W$. Hence, W is closed under multiplication by any A_k and $E_k^*(x)$ matrix on both the right and the left. Therefore, as $T(x)$ is generated by the matrices A_k, E_k^* for $0 \le k \le d$, we have that W is a two-sided ideal of $T(x)$. □

At this point we can now show that $\text{Span}_{\mathbb{C}}(v_{ij} : 0 \le i, j \le d)$ is a two-sided ideal of $T(G)$.

Theorem 4.12. *Let $T(G)$ be the Terwilliger algebra related to an orbit Schur ring for a finite group G. Then $V = \text{Span}_{\mathbb{C}}(v_{ij} : 0 \le i, j \le d)$ is a two-sided ideal of $T(G)$.*

Proof. Note that $v_{ij} = Y_{i^*} X_j = \sum_{k=0}^{d} E_i^* A_k E_j^*$ and thus is in $T(G)$. Then $V \subseteq T(G)$. The fact that we are taking the span of a set gives that we have a nonempty set, closure under addition, and closure under additive inverses. Thus, all that remains to be checked is closure under multiplication by any element of $T(G)$. As $T(G)$ is generated by $\mathfrak{A} = \text{Span}_{\mathbb{C}}(A_0, A_1, \ldots, A_d)$ and $\mathfrak{A}^* = \text{Span}_{\mathbb{C}}(E_0^*, E_1^*, \ldots, E_d^*)$ it suffices for us to check that V is closed under multiplication by each A_i and E_i^*. Let v_{rs} be an arbitrary spanning element of V.

We first consider $E_i^* v_{rs}$. We know that E_i^* is a diagonal matrix with 1's in the P_i, P_i block and 0's in all other diagonal entries. By definition v_{rs} has a block of all 1's in the P_r, P_s block with all other entries being 0. We then have

$$E_i^* v_{rs} = \begin{cases} v_{rs} & \text{if } P_i = P_r \\ 0 & \text{if } P_i \neq P_r \end{cases} .$$

We now turn our attention to $A_k v_{rs}$. Using block notation and letting $a_{t,i}$ be the P_t, P_i block of A_k, with $*$ symbolizing an unknown value, we have

$$A_k v_{rs} = \begin{bmatrix} * & \cdots & a_{0,r} & * & \cdots \\ * & \cdots & a_{1,r} & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & a_{t,r} & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & a_{n,r} & * & \cdots \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & d_{rs} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots \end{bmatrix} = \begin{bmatrix} 0 & \cdots & a_{0,r}d_{rs} & 0 & \cdots \\ 0 & \cdots & a_{1,r}d_{rs} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & a_{t,r}d_{rs} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & a_{n,r}d_{rs} & 0 & \cdots \end{bmatrix} \tag{4.3}$$

where d_{rs} is the all 1 matrix in the P_r, P_s block. We now consider $a_{t,s}d_{rs}$ for any $a_{t,s}$.

As d_{rs} is the all 1's matrix and since by Lemma 4.10 every row and column in the P_t, P_r block of A_k has the same number of 1's, the product $a_{t,r}d_{rs}$ is just the number of 1's in a row of $a_{t,r}$ multiplied by the all 1's matrix of appropriate dimensions, namely $|P_t| \times |P_s|$. Let $q_{t,r}$ be the number of 1's in a row of $a_{t,r}$. We then have that since v_{ts} has d_{ts} as its P_t, P_s block with all other entries being 0, that $A_k v_{rs} = \sum_{t=0}^{n} q_{t,r} v_{ts} \in V$.

With the fact that $E_i^* v_{rs} \in V$ and $A_k v_{rs} \in V$ for all i, k, r, s we see that V is a left ideal of our algebra. We now proceed show that $(v_{rs})^t \in V$. Notice that $(v_{rs})^t = (Y_{r^*} X_s)^t = X_s^t Y_{r^*}^t = Y_{s^*} X_r = v_{sr}$. Thus, we have that $(v_{rs})^t = v_{sr}$. From this and the fact that V is the span of all v_{rs} we have that V is closed under transposition. Then as V is a left ideal of $T(G)$ that is closed under transposition we have by Lemma 4.11 that V is a two-sided ideal of $T(G)$. $\square$

We have shown that V is a two-sided ideal. In fact, it is irreducible. To prove this we will need the following lemma.

Lemma 4.13. *Let $A_a \in V$ be any of the A matrices with a nonzero P_k, P_i block. Then $E_k^* A_a v_{ij} = \lambda v_{kj}$ for some nonzero constant λ and $v_{jk} A_a E_i^* = \mu v_{ji}$ for some nonzero constant μ.*

Proof. We begin by noting that we found in the proof of Theorem 4.12 that $A_a v_{ij} = \sum_b q_{b,i} v_{bj}$, where $q_{b,i}$ is the number of 1's in each row of the P_b, P_i block of A_a. This results in a matrix where the only nonzero columns correspond to the P_j block. We note that by assumption the P_k, P_i block of A_a is nonzero, thus $q_{k,i} \neq 0$. Then $q_{k,i} v_{kj} \neq 0$, so the P_k, P_j block of $A_a v_{ij}$ is a nonzero multiple of the all 1 matrix. Now we multiply by E_k^*. As E_k^* is a diagonal matrix with 1's in its P_k, P_k block, and 0's elsewhere we have that $E_k^* A_a v_{ij}$ will just have nonzero elements in the P_k row block of $A_a v_{ij}$ and those entries will be the entries of $A_a v_{ij}$. The only nonzero column block of $A_a v_{ij}$ is the P_j column block. Thus, the only nonzero block in $E_k^* A_a v_{ij}$ is the P_k, P_j block, which we found to be $q_{k,i}$ multiplied by the all 1 matrix. This means that $E_k^* A_a v_{ij} = q_{k,i} v_{kj}$. Thus, as claimed, $E_k^* A_a v_{ij}$ is a nonzero multiple of v_{kj}.

The second claim follows by essentially the same reasoning. □

Now using this lemma we are ready to show that V is not only a two-sided ideal of $T(G)$, but is in fact irreducible as well.

Corollary 4.14. *Let $\mathfrak{S}$ be an orbit Schur ring over a finite group G with primitive sets $P_0, P_1, \ldots, P_d$. Let $T(G)$ be the related Terwilliger algebra with base point e. Then the two-sided ideal V is an irreducible two-sided ideal of $T(G)$ whose dimension is $(d+1)^2$.*

Proof. Let $\alpha \in V$ be any nonzero element of V. We show that $I = \langle \alpha \rangle = V$. We can write $\alpha = \sum_{i,j} c_{i,j} v_{ij}$ as it is in V. Since $\alpha \neq 0$, there is a $c_{i,j} \neq 0$. Then the P_i, P_j block of α is nonzero. We note that $E_i^* \alpha$ has the same P_i row block as α and all other row blocks are 0. We now multiply this by E_j^* on the right which will zero out all the column blocks of $E_i^* \alpha$

except for the P_j column block. Thus, $E_i^*\alpha E_j^*$ has the same P_i, P_j block as α and all other entries are 0. As we know $\alpha = \sum_{i,j} c_{i,j}v_{ij}$ we have that every entry in the P_i, P_j block of α is $c_{i,j} \neq 0$. Then this means that $E_i^*\alpha E_j^* = c_{i,j}v_{ij}$. So $v_{ij} \in I$.

Since $v_{ij} \in I$ and we have by Lemma 4.13 for all k that $E_k^* A_a v_{ij} = \lambda v_{kj}$ and $\lambda \neq 0$. Since $v_{ij} \in I$ this then means that $\lambda v_{kj} \in I$ and thus $v_{kj} \in I$ for all k. Now for each k we fix v_{kj} and use Lemma 4.13 once again to get that $v_{kj}A_a E_m^* = \mu v_{km}$ for all m with $\mu \neq 0$. This then tells us that $v_{km} \in I$ for all m. Doing this for all k we have $v_{km} \in I$ for all k and all m. This means the spanning set for V is in I. Thus, $V \subseteq I$. Then as $I \subseteq V$ we have $I = V$. So V has no proper nonzero subideals and thus is irreducible.

We now find the dimension of V. We know $P_0, P_1, P_2, \ldots, P_d$ are our partition sets for our Schur ring. Then there are $(d+1)^2$ different P_i, P_j blocks in any matrix in V. So there are $(d+1)^2$ different v_{ij}. As each v_{ij} is only nonzero in its P_i, P_j block the set of all the v_{ij} must be linearly independent. As $V = \mathrm{Span}_{\mathbb{C}}(v_{ij} \colon 0 \leq i, j \leq d)$ we have that the v_{ij} form a basis for V. There are $(d+1)^2$ of them so $\dim(V) = (d+1)^2$. □

With the existence of this irreducible component of the Terwilliger algebra created using any orbit Schur ring over a finite group now proven, we turn our attention to several specific types of Schur rings and the Terwilliger algebras resulting from them.

Theorem 4.15. *Let G be a finite group. Let $\mathfrak{S}$ be the orbit Schur ring created by acting by the identity. Let $T(G)$ be the resulting Terwilliger algebra. Then $T(G)$ has dimension $|G|^2$ and V is the only component in the Wedderburn decomposition of $T(G)$.*

Proof. As we act by the identity to get our primitive sets for $\mathfrak{S}$ each element of G is in its own primitive set, so there are $|G|$ primitive sets. We know that $V \subseteq T(G)$. By Corollary 4.14 we have that $\dim(V) = |G|^2$. We also know that $T(G) \subseteq M_{|G|}(\mathbb{C})$ and that $\dim(M_2(\mathbb{C})) = |G|^2$. Then we have that $V = T(G) = M_{|G|}(\mathbb{C})$ in this case and $T(G)$ has dimension $|G|^2$, with V clearly being the only component in the Wedderburn decomposition of $T(G)$. □

Theorem 4.16. *Let G be a finite group of order n with $n \geq 3$. Let $\mathfrak{S}$ be the trivial Schur ring over G. Let $T(G)$ be the resulting Terwilliger algebra. Then $T(G)$ has dimension 5.*

Proof. As $\mathfrak{S}$ is the trivial Schur ring over G, the primitive classes of $\mathfrak{S}$ are $\{e\}, G \setminus \{e\}$. Then the resulting adjacency matrices for $T(G)$ are the identity matrix I, call it A_0, and $A_1 = J - I$, where J is the all 1 matrix. Let $B_1 = E_0^* A_1$, $B_2 = A_1 E_0^*$, $B_3 = E_0^*$, $B_4 = E_1^*$, and $B_5 = E_1^*(A_0 + A_1)E_1^*$. It is easy to express each of these matrices in block form using blocks based on our two primitive sets. We shall use I_m for the $m \times m$ identity matrix and $J_{m,k}$ for the $m \times k$ all 1's matrix in expressing these matrices. We then have

$$B_1 = \begin{bmatrix} 0 & J_{1,n-1} \\ 0 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 & 0 \\ J_{n-1,1} & 0 \end{bmatrix},\ B_3 = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix},\ B_4 = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-1} \end{bmatrix},\ B_5 = \begin{bmatrix} 0 & 0 \\ 0 & J_{n-1,n-1} \end{bmatrix}.$$

We claim $\mathcal{B} = \{B_1, B_2, B_3, B_4, B_5\}$ is a basis for $T(G)$. The fact that $\mathcal{B}$ is linearly independent is clear. Thus, we need only verify that $\mathcal{B}$ spans $T(G)$. Notice that $A_0 = B_3 + B_4$, $A_1 = B_5 - B_4$, $E_0^* = B_3$, $E_1^* = B_4$. Thus, all of the A_i, E_i^* matrices are in Span($\mathcal{B}$). Since $T(G)$ is generated by the A_i, E_i^* matrices as long as Span($\mathcal{B}$) is closed under multiplication we have that every element of $T(G)$ is in Span($\mathcal{B}$). To this end we provide the multiplication table (Table 4.1) for the elements of $\mathcal{B}$ below.

	B_1	B_2	B_3	B_4	B_5
B_1	0	$(n-1)B_3$	0	B_1	$(n-1)B_1$
B_2	B_5	0	B_2	0	0
B_3	B_1	0	B_3	0	0
B_4	0	B_2	0	B_4	B_5
B_5	0	$(n-1)B_2$	0	B_5	$(n-1)B_5$

Table 4.1: Multiplication table of basis for Terwilliger algebra resulting from Trivial Schur ring

We can clearly see from Table 4.1 that Span($\mathcal{B}$) is closed under multiplication. Thus, we then have that $T(G)$ is spanned by $\mathcal{B}$. Thus, $\mathcal{B}$ is a basis for $T(G)$. Hence, $T(G)$ has dimension 5 as claimed. $\square$

Before continuing with our next proof we note that the map defined by $\langle A, B\rangle = \sum_{i,j} a_{ij}\overline{b_{ij}}$ for all $A = (a_{ij}), B = (b_{ij}) \in M_n(\mathbb{C})$ is an inner product. We call this the *Hadamard inner product on* $M_n(\mathbb{C})$.

Corollary 4.17. *Let G be a finite group of order n with $n \geq 3$. Let $\mathfrak{S}$ be the trivial Schur ring over G. Let $T(G)$ be the resulting Terwilliger algebra. Then the Wedderburn decomposition for $T(G)$ is $T(G) = V \oplus I$ where $I = Span_{\mathbb{C}}(B_5 - (n-1)B_4)$ and V is defined as in Theorem 4.12.*

Proof. We start by showing that V as defined in Theorem 4.12 is an ideal of $T(G)$. Using the notation from Theorem 4.12 we see that $v_{00} = B_3$, $v_{10} = B_2$, $v_{01} = B_1$, and $v_{11} = B_5$. From this and the fact that $T(G) = \text{Span}(\mathcal{B})$, we immediately see that V is closed under multiplication by elements of $T(G)$ on the left and right. It obviously has dimension 4. We now show that it is irreducible. Let $\alpha \in V$ be any nonzero element of V. We will show that $K = \langle \alpha \rangle = V$. As $\alpha \in V$ we have that $\alpha = a_0 v_{00} + a_1 v_{01} + a_2 v_{10} + a_3 v_{11}$, for some $a_0, a_1, a_2, a_3 \in \mathbb{C}$. We know at least one of these $a_i \neq 0$. Without loss of generality say it is a_0. As K is a two-sided ideal of $T(G)$ and $J_n \in T(G)$ we have that $J_n\alpha \in K$ as well as $\alpha J_n \in K$. Simple calculations show that $\alpha J_n = (a_0+a_1)v_{00}+(a_0+a_1)v_{01}+(a_2+a_3)v_{10}+(a_2+a_3)v_{11}$ and $J_n\alpha = (a_0+a_2)v_{00}+(a_1+a_3)v_{01}+(a_0+a_2)v_{10}+(a_1+a_3)v_{11}$. If $a_1 = 0$, then αJ_n is an element of K that has a nonzero v_{00} and v_{01} component, namely both having coefficient a_0. We then can proceed with this matrix calling it β. If $a_1 \neq 0$, but $a_2 = 0$ we let $\beta = J_n\alpha$. Notice that β has a nonzero v_{00}, v_{10} component as both have coefficient a_0. If $a_1 \neq 0$, $a_2 \neq 0$, but $a_3 = 0$ we let $\beta = J_n\alpha$. This β has nonzero v_{01} and v_{11} components both having coefficient a_1. We can then repeat this process using β. Doing so we can produce elements of K, where between all of them we can find an element with nonzero v_{ij} component for $i, j \in \{0, 1\}$. Now as K is two-sided ideal we know $E_i^*\gamma E_j^* \in K$ for $i, j \in \{0, 1\}$. Doing so isolates the v_{ij} component of γ. Applying this to the set of matrices we created above we see that each of $v_{00}, v_{10}, v_{01}, v_{11} \in K$. Thus, $V \subseteq K$. So $K = V$. Any nonzero subideal of V must contain some $\alpha \neq 0$ and we just found that such an ideal is V itself. Thus, V has no proper nonzero subideals and is thus irreducible.

As $T(G)$ has dimension 5 and V has dimension 4 we have that $T(G) = V \oplus J$ for some one-dimensional ideal J. We now show that I is this ideal. It is clear that $I \subseteq T(G)$. We need now check only closure under multiplication. To do so we need only check that I is closed under multiplication by each of the E_i^* and A_i matrices. As $A_0 = I_n = B_3 + B_4$, $A_1 = B_5 - B_4$, $E_0^* = B_3$, $E_1^* = B_4$ we need only check closure under multiplication by the B_i's. We do this now using Table 4.1.

$$B_1(B_5 - (n-1)B_4) = (n-1)B_1 - (n-1)B_1 = 0 = (B_5 - (n-1)B_4)B_1,$$

$$B_2(B_5 - (n-1)B_4) = 0 + 0 = 0 = (n-1)B_2 - (n-1)B_2 = (B_5 - (n-1)B_4)B_2,$$

$$B_3(B_5 - (n-1)B_4) = 0 + 0 = 0 = 0 + 0 = (B_5 - (n-1)B_4)B_3,$$

$$B_4(B_5 - (n-1)B_4) = B_5 - (n-1)B_4 = (B_5 - (n-1)B_4)B_4,$$

$$B_5(B_5 - (n-1)B_4) = (n-1)B_5 - (n-1)B_5 = (B_5 - (n-1)B_4)B_5.$$

From this we see that I is indeed closed under multiplication. Thus, it is a two-sided ideal of $T(G)$. It is clearly dimension one and thus irreducible. We can also see that I is orthogonal to v_{00}, v_{10}, v_{01} under the Hadamard inner product on $M_n(\mathbb{C})$. Thus, for it to be orthogonal to V we need only check $\langle B_5 - (n-1)B_4, v_{11}\rangle$. Since the row sum of each row of $B_5 - (n-1)B_4$ is 0 in the P_1, P_1 block, and v_{11} is the all 1 matrix in the P_1, P_1 block and 0 elsewhere, we have that $\langle B_5 - (n-1)B_4, v_{11}\rangle = 0$. Hence, I is orthogonal to V. Therefore, we have that $T(G) = V \oplus I$ as claimed. □

Our above proof only dealt with the trivial Schur ring for groups of order $n \geq 3$. We now consider when $|G| = 2$. The Terwilliger algebra $T(G)$ using the trivial Schur ring in this case is the same Schur ring that we get when we act on the group by the identity, so by Theorem 4.15 we have $\dim(T(G)) = 4$ and $T(G) = V = M_2(\mathbb{C})$.

We next look at $G = \mathbb{Z}_n \times \mathbb{Z}_m$, and partition the group using the cosets of $\{1\} \times \mathbb{Z}_m$. Splitting the coset $\{1\} \times \mathbb{Z}_m$ into $\{(1,1)\}, \{1\} \times (\mathbb{Z}_m \setminus \{1\})$ we get a partition of G that results in a Schur ring. Notice that when $n = 1$ this is just the group ring (Theorem 4.15) and when $m = 1$ this gives the trivial S-ring (Theorem 4.16 and Corollary 4.17).

Proposition 4.18. *Let $G = \mathbb{Z}_n \times \mathbb{Z}_m = \langle a \rangle \times \langle b \rangle$, $n, m \geq 2$. Let $\mathfrak{S}$ be the Schur ring resulting from the partition of G as $P_0 = \{(1,1)\}, P_1 = \{1\} \times (\mathbb{Z}_m \setminus \{1\}), P_2 = a \times \mathbb{Z}_m, P_3 = a^2 \times \mathbb{Z}_m, \ldots, P_n = a^{n-1} \times \mathbb{Z}_m$. Let $T(G)$ be the resulting Terwilliger algebra from this partition. Then we have the following:*

(a) $T(G)$ has, for each i with $2 \leq i \leq n$, an irreducible one dimensional ideal $I_i = Span_{\mathbb{C}}(D_i)$ where the P_i, P_i block of D_i is

$$\frac{-1}{m-1}J + \frac{m}{m-1}I = \begin{bmatrix} 1 & \frac{-1}{m-1} & \cdots & \frac{-1}{m-1} \\ \frac{-1}{m-1} & 1 & \ddots & \frac{-1}{m-1} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{-1}{m-1} & \frac{-1}{m-1} & \cdots & 1 \end{bmatrix} \tag{4.4}$$

and D_i only has 0's outside this block.

(b) When $m > 2$, $T(G)$ has an irreducible one dimensional ideal $I_1 = Span_{\mathbb{C}}(D_1)$ where the P_1, P_1 block of D_1 is

$$\frac{-1}{m-2}J + \frac{m-1}{m-2}I = \begin{bmatrix} 1 & \frac{-1}{m-2} & \cdots & \frac{-1}{m-2} \\ \frac{-1}{m-2} & 1 & \ddots & \frac{-1}{m-2} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{-1}{m-2} & \frac{-1}{m-2} & \cdots & 1 \end{bmatrix} \tag{4.5}$$

and D_1 only has 0's outside this block.

Proof. Notice that $(y, b^r)(x, b^s)^{-1} \in P_1$ if and only if $y \times \mathbb{Z}_m = x \times \mathbb{Z}_m$ and $(y, b^r) \neq (x, b^s)$. So for $(y, b^r)(x, b^s)^{-1} \in P_1$ we must have $(x, b^s) \neq (y, b^r)$ but they are in the same partition class. This tells us that the only nonzero entries of A_1 are in its P_j, P_j blocks for each j. Further in the P_j, P_j block of A_1 all non-diagonal entries must be 1 as these are entries indexed by (u, v) where $u \neq v$ and $u, v \in P_j$. The diagonal entries must all be 0 as these are entries indexed by (u, v) with $u = v$, so $vu^{-1} = 1 \notin P_1$. For all other A_i with $2 \leq i \leq n$ the entry indexed by $((a^r, b^s), (a^j, b^k))$ is 1 in A_i if and only if $(a^r, b^s)(a^{-j}, b^{-k}) = (a^{r-j}, b^{s-k}) \in P_i$. This is true if and only if $a^{r-j} = a^{i-1}$ (it does not depend on the second coordinate at all). From this we then have that the P_j, P_r block of A_i is all 1's if and only if $a^{r-j} = a^{i-1}$, and is the all zero matrix otherwise.

We start by proving (a). We show that for any i with $2 \leq i \leq n$ we have that $D_i \in T(G)$. As we know that $A_0 = I_{nm}$ and we just found what A_1 looks like we can easily see that $D_i = E_i^*(A_0 - \frac{1}{m-1}A_1)E_i^*$. Thus, $D_i \in T(G)$. So $I_i = \text{Span}_{\mathbb{C}}(D_i) \subseteq T(G)$.

Since I_i is clearly closed under scaling and addition, we need only check multiplication by elements of $T(G)$. To do this we need only check that $A_kD_i, D_iA_k, E_k^*D_i, D_iE_i^* \in I_i$ for all i and k. Using the fact that $D_i = E_i^*(A_0 - \frac{1}{m-1}A_1)E_i^*$ we can immediately see that

$$\begin{aligned} D_iE_k^* = D_i = E_k^*D_i \quad & \text{if } k = i \\ D_iE_k^* = 0 = E_k^*D_i \quad & \text{if } k \neq i. \end{aligned}$$

Thus, $D_iE_k^*, E_k^*D_i \in I_i$ for any choice of i and k. Now consider A_kD_i for any $0 \leq k \leq n$. Using block notation and letting $a_{t,i}$ be the P_t, P_i block of A_k with $*$ symbolizing an unknown value we have essentially the same multiplication (with different letters for the indices, and a different size matrix) as in equation (4.3) except that D_i replaces v_{rs} and d_{rs} is the matrix in (4.4). We now consider three cases based on what k is.

Case 1: $k = 0$. In this case $A_k = I_{mn}$, so $A_kD_i = D_i \in I_i$ and we are done.

Case 2: $k = 1$. In this case the only $a_{t,i} \neq 0$ is $a_{i,i}$. Here the P_i, P_i block of A_1D_i equals

$$\begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{-1}{m-1} & \cdots & \frac{-1}{m-1} \\ \frac{-1}{m-1} & 1 & \ddots & \frac{-1}{m-1} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{-1}{m-1} & \frac{-1}{m-1} & \cdots & 1 \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{m-1} & \cdots & \frac{1}{m-1} \\ \frac{1}{m-1} & -1 & \ddots & \frac{1}{m-1} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{m-1} & \frac{1}{m-1} & \cdots & -1 \end{bmatrix}.$$

This is just the negative of the P_i, P_i block of D_i. Since that is the only nonzero block of D_i, we have $A_1D_i = -D_i \in I_i$.

Case 3: $2 \leq k \leq n$. In this case each $a_{t,i}$ is either the all 0 or all 1 matrix. Clearly $a_{t,i}d_i = 0$ if $a_{t,i}$ is the all 0 matrix. Since the row and column sum of d_i is 0 we have that if $a_{t,i}$ is the all 1 matrix then $a_{t,i}d_i = 0$. So for all t we have $a_{t,i}d_i = 0$. Thus, $A_k D_i = 0 \in I_i$ in this case.

From these three cases we see that for all k we have $A_k D_i \in I_i$. Thus, I_i is a left ideal of $T(G)$. Notice that clearly $D_i^t = D_i$ since the only nonzero block is a diagonal block, which is symmetric. Thus, I_i is closed under transposition. As I_i is a left ideal of $T(G)$ that is closed under transposition we have by Lemma 4.11 that I_i is a two-sided ideal of $T(G)$ for all $2 \leq i \leq n$. The fact that it is one dimensional is clear. Since it is one dimensional it must be irreducible. This completes the proof of (a). For part (b) notice that $D_1 = E_1^*(A_0 - \frac{1}{m-2}A_1)E_1^*$, so it is in $T(G)$. The proof that it is an irreducible one dimensional ideal of $T(G)$ is essentially the same as the proof for part (a). □

Chapter 5. Terwilliger Algebras for Group Association Schemes

Among Terwilliger algebras resulting from orbit Schur rings, there is one particular type of said Terwilliger algebra that has been more thoroughly researched than others. These are the Terwilliger algebras resulting form using the conjugacy classes a group G, to create our Schur ring. Recall the related association scheme created in this way is called the group association scheme. (See [1, 3, 14] additional research done on these types of Terwilliger algebras.)

Using the group association scheme for a group G we can generate a Terwilliger algebra $T(x)$ where $x \in G$. From this point on, unless otherwise specified, $\mathcal{A}$ is the group association scheme for a group G and $T(x)$ is a Terwilliger algebra related to it with base point $x \in G$. We will take $C_0, C_1, \ldots C_d$ to be the conjugacy classes of G, which also serve as our partition sets. Additionally, if no base point x is specified we assume it to be the identity, e, of the group. We shall denote a Terwilliger algebra created in this way as $T(G)$. We shall also let $T_0(G)$ and $T_0^*(G)$ denote the subspaces of $T(G)$ defined after Lemma 4.6.

Let $\tilde{T} = \mathrm{End}_G(\mathbb{C}[G])$ be the centralizer algebra of the permutation representation of G acting on G by conjugation. This is the centralizer algebra of the Bose-Mesner algebra for $\mathcal{A}$. The dimension of $\tilde{T}$ is the number of orbits of G acting on $G \times G$ by simultaneous conjugation. This is just equal to the average of the number of fixed points, which means

$$\dim \tilde{T} = \frac{1}{|G|} \sum_{a \in G} |C_G(a)|^2 = \sum_{i=0}^{d} \frac{|G|}{|C_i|}. \tag{5.1}$$

Theorem 5.1 (Theorem 2, [3]). *Let G be a finite group. Given the Terwilliger algebra $T(G)$ resulting from the group association scheme on G we have the following bounds on its dimension:*

(i) $|\{(i,j,k) \colon p_{ij}^k \neq 0\}| \leq \dim T(G)$;

(ii) $|\{(i,j,k)\colon q_{ij}^k \neq 0\}| \leq \dim T(G)$;

(iii) $\dim T(G) \leq \sum_{i=0}^{d} |G|/|C_i|$.

Proof. Clearly $T_0(G) \subseteq T(G)$ and $T_0^*(G) \subseteq T(G)$ so $\dim T_0(G) \leq \dim T(G)$ and $\dim T_0^*(G) \leq \dim T(G)$. Thus we have by equations (4.1) and (4.2) that $|\{(i,j,k)\colon p_{ij}^k \neq 0\}| \leq \dim T(G)$ and $|\{(i,j,k)\colon q_{ij}^k \neq 0\}| \leq \dim T(G)$.

To prove (iii) we will show that $T(G) \subseteq \tilde{T}$. To do so we define the matrices $B(g)$ with $g \in G$ where the entries are indexed by the elements of G with $B(g)_{x,y} = 1$ if $gxg^{-1} = y$ and is 0 otherwise. Then $\tilde{T} = \mathrm{End}_G(\mathbb{C}[G])$ is just the set of all matrices that commute with $B(g)$ for all $g \in G$ ([2], Chapter 2, Theorem 1.3).

Now we note that for any matrix D where the entries correspond to elements of G, that $(B(g)DB(g)^{-1})_{x,y} = D_{gxg^{-1},gyg^{-1}}$. Thus, D commutes with $B(g)$ if and only if $D_{x,y} = D_{gxg^{-1},gyg^{-1}}$ for all entries (x,y). Now we look at A_i for any i. For A_i we have 1 in the entry indexed by (x,y) if and only if $yx^{-1} \in C_i$. As C_i is a conjugacy class $yx^{-1} \in C_i$ means for all $g \in G$, $gyx^{-1}g^{-1} \in C_i$. So $gyg^{-1}(gxg^{-1})^{-1} \in C_i$, and the entry of A_i indexed by (gxg^{-1}, gyg^{-1}) is also 1. Thus the entry of A_i indexed by (x,y) is 1 if and only if the entry of A_i indexed by (gxg^{-1}, gyg^{-1}) is 1. Therefore, $(A_i)_{x,y} = (A_i)_{gxg^{-1},gyg^{-1}}$ for all x, y. As g is arbitrary this is true for all $g \in G$ as well. Therefore, A_i commutes with $B(g)$ for all $g \in G$. Thus, $A_i \in \tilde{T}$. As A_i was chosen arbitrarily we have that every $A_i \in \tilde{T}$.

Next we show every $E_i^* \in \tilde{T}$. Notice that $(E_i^*)_{x,y}$ is 1 if and only if $x = y$ and $y \in C_i$. Then for any $g \in G$ we have that $(E_i^*)_{gxg^{-1},gyg^{-1}} = 1$ if and only if $gxg^{-1} = gyg^{-1}$ and $gyg^{-1} \in C_i$. This is true if and only if $x = y$ and $y \in C_i$. Therefore, $(E_i^*)_{x,y} = (E_i^*)_{gxg^{-1},gyg^{-1}}$ for all $g \in G$ and for all entries indexed by (x,y). Therefore, $E_i^* \in \tilde{T}$.

We have shown that every A_i and every E_j^* is in $\tilde{T}$. Since $T(G)$ is generated by the A_i and E_j^* this means that $T \subseteq \tilde{T}$. Thus, $\dim T \leq \dim \tilde{T} = \sum_{i=0}^{d} |G|/|C_i|$. This proves (iii). □

Proposition 5.2. *If G is an abelian group and $T(G)$ is a Terwilliger algebra for the group association scheme of G then* $\dim T(G) = |G|^2$.

Proof. As G is abelian, conjugating by any element is the same as acting by the identity. Thus, the group association scheme in this case corresponds to the orbit Schur ring created by acting by the identity. Then by Theorem 4.15, we have $\dim T(G) = |G|^2$. $\square$

There are two additional properties that Terwilliger algebras of group association schemes can have which are of interest.

Definition 5.3. Let G be a finite group with conjugacy classes $C_0, C_1, \ldots, C_d$. If G acts transitively by conjugation on the set

$$S_{ijk} = \{(g, h) \in C_i \times C_j \colon gh \in C_k\}$$

for any $i, j, k \in \{0, 1, 2, \ldots, d\}$ where $S_{ijk} \neq \emptyset$, then G is said to be *triply transitive.*

Definition 5.4. We say that a finite group G is *triply regular* if $T(G) = T_0(G)$. Also, G is *dually triply regular* if $T_0^*(G) = T(G)$.

Lemma 5.5. *Let G be a finite group. Then G is triply transitive if and only if* $\dim T_0(G) = \dim \tilde{T}$. *In this case* $T_0(G) = T(G) = \tilde{T}$.

Proof. Let $C_0, C_1, \ldots, C_d$ be the conjugacy classes of G. These are also the partition sets for the Schur ring, that we create an association scheme from. Now $S_{ijk} \neq \emptyset$ if and only if there is some $(g, h) \in C_i \times C_j$ such that $gh \in C_k$. This is true if and only if $\overline{C_k}$ has a nonzero coefficient in the product $\overline{C_i} \cdot \overline{C_j}$. This coefficient is p_{ij}^k. So $S_{ijk} \neq \emptyset$ if and only if $p_{ij}^k \neq 0$. This fact and equation (4.1) tell us that $\dim T_0(G) = |\{(i, j, k) \colon S_{ijk} \neq \emptyset\}|$.

If G is triply transitive then G acts transitively by conjugation on S_{ijk} for all $S_{ijk} \neq \emptyset$. This fact and the fact that $G \times G = \bigcup_{i,j,k} S_{ijk}$ tell us that each of the orbits of G acting on $G \times G$ by simultaneous conjugation come from a union of nonempty S_{ijk}. Then the number of orbits is less than or equal to the number of nonempty S_{ijk}. So $\dim \tilde{T} \leq |\{(i, j, k) \colon S_{ijk} \neq \emptyset\}| = \dim T_0(G)$. By Theorem 5.1 we have $\dim T_0(G) \leq \dim \tilde{T}$. Thus, in this case $\dim \tilde{T} = \dim T_0(G)$.

If $\dim \tilde{T} = \dim T_0(G)$, then the number of orbits of G acting on $G \times G$ by simultaneous conjugation equals the number of nonempty S_{ijk}. Clearly the S_{ijk} are closed under conjugation, so each one must lie in some orbit of G acting on $G \times G$ by simultaneous conjugation. As there are the same number of orbits as there are nonempty S_{ijk} we see that the S_{ijk} are the orbits of G acting by conjugation. A group acts transitively on each of its orbits, so G acts transitively by conjugation on S_{ijk} for all $i, j, k \in \{0, 1, 2 \ldots, d\}$ where $S_{ijk} \neq \emptyset$. So G is triply transitive.

As $T_0(G) \subseteq T(G) \subseteq \tilde{T}$ and each is finite dimensional, $\dim T_0(G) = \dim \tilde{T}$ implies that $T_0(G) = T(G) = \tilde{T}$. $\square$

Note that Lemma 5.5 immediately shows that if a group is triply transitive, then it is triply regular.

Example 5.6. One group of interest that is not triply regular is S_4. To show this we order our conjugacy classes as follows:

$$C_0 = \{(1)\},$$

$$C_1 = \{(1\ 3)(2\ 4), (1\ 2)(3\ 4), (1\ 4)(2\ 3)\},$$

$$C_2 = \{(1\ 2), (1\ 4), (3\ 4), (2\ 3), (1\ 3), (2\ 4)\},$$

$$C_3 = \{1\ 2\ 3), (1\ 3\ 4), (2\ 4\ 3), (1\ 2\ 4), (2\ 3\ 4), (1\ 4\ 3), (1\ 4\ 2), (1\ 3\ 2)\},$$

$$C_4 = \{1\ 4\ 3\ 2), (1\ 2\ 3\ 4), (1\ 3\ 4\ 2), (1\ 4\ 2\ 3), (1\ 3\ 2\ 4), (1\ 2\ 4\ 3)\}.$$

Using this ordering of elements and classes we can create the matrices A_k, E_k^*. We shall show that

$$(E_3^* A_4 E_4^*)(E_4^* A_4 E_3^*) = E_3^*(A_4 E_4^* A_4) E_3^*$$

is not in $T_0(S_4)$. It is clear the only nonzero block of this matrix is the C_3, C_3 block. So if it were in $T_0(S_4)$ it would be a linear combination of matrices of the form $E_3^* A_k E_3^*$ where $0 \leq k \leq 4$. Direct computation shows that $E_3^* A_3 E_3^*$ has 1's in entries $(1\ 2\ 3), (1\ 4\ 3)$ and

$(1\ 2\ 3), (1\ 3\ 2)$. All other $E_3^* A_k E_3^*$ has 0 in both these positions. Therefore any linear combination of these matrices must have the same value in the $(1\ 2\ 3), (1\ 4\ 3)$ and $(1\ 2\ 3), (1\ 3\ 2)$ entries. Computing $(E_3^* A_4 E_4^*)(E_4^* A_4 E_3^*)$ shows it has 2 in entry $(1\ 2\ 3), (1\ 4\ 3)$ and 0 in entry $(1\ 2\ 3), (1\ 3\ 2)$. Therefore it is not a linear combination of the $E_3^* A_k E_3^*$. As it is not a linear combination of these matrices it is not in $T_0(S_4)$. Clearly it is in $T(S_4)$. Therefore $T_0(S_4) \neq T(S_4)$. Hence, S_4 is not triply regular.

Using Terwilliger algebras we can now show the existence of a class of association schemes related to the group association scheme for a triply regular group.

Proposition 5.7. *Let G be a finite, triply regular group with conjugacy classes $C_0, C_1, \ldots, C_d$. Let $\mathcal{A}$ be the group association scheme for G with adjacency matrices $A_0, A_1, \ldots, A_d$. Let $E_i^* = E_i^*(e)$ where e is the identity of the group. Let $T(G)$ be the Terwilliger algebra resulting from these matrices. Then for any $0 \leq i, k \leq d$, we let $\alpha_{i,k}$ be the C_i, C_i block of A_k. For any $0 \leq i \leq d$ let*

$$\mathcal{AS}(C_i) = \{\alpha_{i,k} \colon 0 \leq k \leq d, \alpha_{i,k} \neq 0\}.$$

Then $\mathcal{AS}(C_i)$ is an association scheme over $C_i \times C_i$.

Proof. We shall show that $\mathcal{AS}(C_i)$ satisfies the conditions in definition 3.8. First note that the C_i, C_i block of A_0 is diagonal with 1's along the diagonal as A_0 is the identity matrix of size $|G|$. So $\alpha_{i,0}$ is the identity matrix of size $|C_i|$. Then condition (i) of an association scheme is met. For any $1 \leq k \leq d$ as $\mathcal{A}$ is an association scheme we know $A_k^t = A_j$ for some $j = 1, 2, \ldots, d$. The C_i, C_i block of A_k^t, is $\alpha_{i,k}^t$ as $\alpha_{i,k}$ is a block along the diagonal. So $\alpha_{i,k}^t$ is the C_i, C_i block of A_j. Thus, $\alpha_{i,k}^t = \alpha_{i,j}$. This means condition (ii) of an association scheme is met by $\mathcal{AS}(C_i)$. By construction none of the $\alpha_{i,k}$ are 0 matrices. The sum of all the A_k matrices is the all 1 matrix. So the sum of all the C_i, C_i blocks of the A_k matrices must be the all 1 matrix of size $|C_i| \times |C_i|$. That is $\sum_{k=1}^{d} \alpha_{i,k} = J_{|C_i|}$ where $J_{|C_i|}$ is the all 1 matrix of size $|C_i| \times |C_i|$. Then

$$\sum_{\{\alpha_{i,k} \colon \alpha_{i,k} \neq 0\}} \alpha_{i,k} = J_{|C_i|}.$$

This proves that condition (iv) of an association scheme is met.

To prove that condition (iii) is met we will look at $(E_i^* A_m E_i^*)(E_i^* A_n E_i^*)$. Multiplying by E_i^* on the left and right just isolates the C_i, C_i block of A_m and A_n respectively. That is

$$E_i^* A_m E_i^* = \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \alpha_{i,m} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

where the only nonzero block is the C_i, C_i block which we have been calling $\alpha_{i,m}$. Then

$$(E_i^* A_m E_i^*)(E_i^* A_n E_i^*) = \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \alpha_{i,m}\alpha_{i,n} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}.$$

Recall that G is triply regular, so $T(G) = T_0(G)$. As $T(G) = T_0(G)$ has $\{E_j^* A_k E_t^* : 0 \leq j, k, t \leq d\}$ as a basis, we know that the only matrices in $T(G)$ that are nonzero in the C_i, C_i block and zero outside of this block are of the form $\sum_{k=0}^{d} \beta_k E_i^* A_k E_i^*$. So $(E_i^* A_m E_i^*)(E_i^* A_n E_i^*) = \sum_{k=0}^{d} \beta_{mn}^k E_i^* A_k E_i^*$. That is $\alpha_{i,m}\alpha_{i,n} = \sum_{k=0}^{d} \beta_{mn}^k \alpha_{i,k}$. Therefore condition (iii) of an association scheme is met. We therefore have that $\mathcal{AS}(C_i)$ is an association scheme. $\square$

We now will begin studying the Terwilliger algebra for the group association scheme for dihedral groups. We begin by showing that all dihedral groups are triply transitive.

Theorem 5.8 (Theorem 5, [3])**.** *Let*

$$D_{2n} = \langle r, s\colon r^n = s^2 = 1\colon rs = sr^{-1}\rangle$$

be the dihedral group of order $2n$*. Then* D_{2n} *is triply transitive.*

Proof. Let T be the Terwilliger algebra over D_{2n} resulting from the group association scheme. We consider first if n is odd. Say that $n = 2m+1$. In this case the conjugacy classes of D_{2n} are

$$C_0 = \{e\},\ C_i = \{r^i, r^{-i}\} \text{ (for } 1 \le i \le m),\ \text{and } C_{m+1} = s\langle r\rangle.$$

Using equation (5.1) we compute the dimension of $\tilde{T}$ to be

$$\dim \tilde{T} = \frac{2(2m+1)}{1} + \frac{2m(2m+1)}{2} + \frac{2(2m+1)}{2m+1} = 2m^2 + 5m + 4.$$

To compute the dimension of T_0 we will look at the product C_iC_j. If $1 \le i \le m$, then

$$C_iC_j = \begin{cases} C_i & \text{if } j = 0, \\ C_0 \cup C_{2i} & \text{if } j = i, \\ C_{i+j} \cup C_{i-j} & \text{if } 1 \le j \le m \text{ and } j \ne i, \\ C_{m+1} & \text{if } j = m+1, \end{cases}$$

subscripts are taken (mod m) as needed. Also we have

$$C_{m+1}C_j = \begin{cases} C_{m+1} & \text{if } 0 \le j \le m, \\ C_0 \cup C_1 \cup \cdots \cup C_m & \text{if } j = m+1. \end{cases}$$

With this we now count the number of triples (i, j, k) such that $(C_iC_j) \cap C_k \ne \emptyset$. Note this is exactly when $p_{ij}^k \ne 0$, which will give $\dim T_0$. If $i = 0$, then any choice of C_j will work as long as we pick $C_j = C_k$. This gives $m+2$ different triples (i, j, k) with nonzero p_{ij}^k. For each $1 \le i \le m$ we get from the above that if $j = 0$ we have 1 choice of C_k, for $j = i$ we have 2 choices for C_k, and if $j = m+1$ we have 1 choice of C_k. For $1 \leq j \leq m$ and $j \neq i$ we have 2 choices for C_k based on what C_j is. There are $m-1$ possible choices of C_j here. All together we get $1+1+2+2(m-1) = 2m+2$ triples (i,j,k) such that $p_{ij}^k \neq 0$ for each i. As there are m choices of i this gives a total of $m(m+2)$ triples.

We now consider when $i = m+1$. If $0 \leq j \leq m$ then as long as $C_k = C_{m+1}$ any choice of C_j will work. This gives $m+1$ triples. If $j = m+1$ we can pick C_k to be any of $C_0, C_1, \cdots, C_m$. This gives $m+1$ choices. Thus, when $i = m+1$ we have $2m+2$ triples (i,j,k) such that $p_{ij}^k \neq 0$. Putting all of this together we get $\dim T_0 = (m+2) + m(2m+2) + 2m+2 = 2m^2+5m+4 = \dim \tilde{T}$. As $\dim T_0 = \dim \tilde{T}$ we have by Lemma 5.5 that T is triply transitive.

We now assume that n is even, with $n = 2m$. Then the conjugacy classes of D_{2n} are

$$C_0 = \{e\},\ C_i = \{r^i, r^{-i}\}(\text{ for } 1 \leq i \leq m-1),\ C_m = \{r^m\},\ C_{m+1} = s\langle r^2\rangle, \text{ and } C_{m+2} = sr\langle r^2\rangle.$$

We compute that

$$\dim \tilde{T} = \frac{4m}{1} + \frac{4m}{2}(m-1) + \frac{4m}{1} + \frac{4m}{m} + \frac{4m}{m} = 2m^2 + 6m + 8.$$

We now compute C_iC_j as we did in the previous case to determine $\dim T_0$. For $1 \leq i \leq m-1$

$$C_iC_j = \begin{cases} C_i & \text{if } j = 0, \\ C_0 \cup C_{2i} & \text{if } j = i, \\ C_{i+j} \cup C_{i-j} & \text{if } 1 \leq j \leq m-1 \text{ and } j \neq i, \\ C_{m+i} & \text{if } j = m, \\ C_{m+1} & \text{if } j = m+1 \text{ and } i \text{ is even}, \\ C_{m+2} & \text{if } j = m+1 \text{ and } i \text{ is odd}, \\ C_{m+2} & \text{if } j = m+2 \text{ and } i \text{ is even}, \\ C_{m+1} & \text{if } j = m+2 \text{ and } i \text{ is odd}, \end{cases}$$

with subscripts taken $(mod\ m-1)$ as needed. For $i=m$ with m even we get

$$C_mC_j = \begin{cases} C_m & \text{if } j=0, \\ C_{m+i} & \text{if } 1 \le j \le m-1, \\ C_0 & \text{if } j=m, \\ C_{m+1} & \text{if } j=m+1, \\ C_{m+2} & \text{if } j=m+2. \end{cases}$$

Note if m is odd we get the same things other than C_{m+1} and C_{m+2} switch places on the last two lines.

Next we look at $i=m+1$ and $i=m+2$. As with C_m depending on if m is even or odd determines what $C_{m+1}C_m$ and $C_{m+2}C_m$ look like. We only write it for m even as the only difference comes in what $C_{m+1}C_m$, $C_{m+2}C_m$ equal with one being C_{m+1} and the other C_{m+2}.

$$C_{m+1}C_j = \begin{cases} C_{m+1} & \text{if } j=0, \\ C_{m+1} & \text{if } 1 \le j \le m-1 \text{ and } j \text{ is even}, \\ C_{m+2} & \text{if } 1 \le j \le m-1 \text{ and } j \text{ is odd}, \\ C_{m+1} & \text{if } j=m, \\ \bigcup_{k\ even} C_k & \text{if } j=m+1, \\ \bigcup_{k\ odd} C_k & \text{if } j=m+2. \end{cases}$$

$$C_{m+2}C_j = \begin{cases} C_{m+2} & \text{if } j=0, \\ C_{m+1} & \text{if } 1 \le j \le m-1 \text{ and } j \text{ is odd}, \\ C_{m+2} & \text{if } 1 \le j \le m-1 \text{ and } j \text{ is even}, \\ C_{m+2} & \text{if } j=m, \\ \bigcup_{k\ odd} C_k & \text{if } j=m+1, \\ \bigcup_{k\ even} C_k & \text{if } j=m+2. \end{cases}$$

With these we now count the number of triples (i,j,k) such that $(C_iC_j) \cap C_k \neq \emptyset$ as we did before. If $i = 0$, then $C_iC_j = C_j$ so we need to pick $k = j$. This is for any j. Thus, we get $m + 3$ possible triples here. We now assume $1 \leq i \leq m - 1$ is fixed. Going through the possible j's as we did in the odd case we find there are $2m + 2$ possible nonzero triples for each i. There are $m - 1$ possible i's we could choose so we get $(m - 1)(2m - 2)$ different triples in this case. Next consider $i = m$. Running through the possible j's we have $m + 3$ nonzero triples with $i = m$. Next we consider $m = i + 1$. There are $2m + 2$ possible nonzero triples when $i = m + 1$. When $i = m + 2$ the counting is identical to that of when $i = m + 1$ so we get $2m + 2$ nonzero triples in this case as well.

We now put all of this together to find that

$$\dim T_0 = m + 3 + (m - 1)(2m + 2) + m + 2 + 2m + 2 + 2m + 2 = 2m^2 + 6m + 8 = \dim \tilde{T}.$$

Thus, we have that $\dim T_0 = \dim \tilde{T}$ in this case. Then by Lemma 5.5 this means T is triply transitive. So regardless of whether n is even or odd we have that D_{2n} is triply transitive. $\square$

As a direct consequence of Theorem 5.8 and Lemma 5.5 we have the following corollary.

Corollary 5.9. *If $T(G)$ is a Terwilliger algebra for the group association scheme over D_{2n} we have*

$$\dim T(G) = \begin{cases} 2m^2 + 5m + 4 & \text{if } n = 2m + 1, \\ 2m^2 + 6m + 8 & \text{if } n = 2m. \end{cases}$$

We now turn our attention to finding the Wedderburn decomposition for group association schemes over dihedral groups. Let $D_{2n} = \langle r, s \colon r^n = s^2 = 1,\ r^{-1}s = sr \rangle$ be a dihedral group. The two-sided ideal V (Theorem 4.12) is one of the irreducible Wedderburn components for the Terwilliger algebra of the group association scheme of D_{2n}. To determine the others we will consider cases based on if n is odd or even.

Let us first consider the case when n is odd, $n = 2m + 1$, $m \in \mathbb{Z}$. As a reminder, the conjugacy classes for D_{2n} are $C_0 = \{e\}$, $C_i = \{r^i, r^{-i}\}$ for $1 \leq i \leq m$ and $C_{m+1} = s\langle r \rangle$. For any C_i, C_j with $1 \leq i \leq m$ and $1 \leq j \leq m$ we construct matrices Z_{ij} whose entries corresponding to (r^i, r^j) and (r^{-i}, r^{-j}) are 1 and whose entries corresponding to (r^{-i}, r^j) and (r^i, r^{-j}) are -1. All other entries 0.

Theorem 5.10. *Let $T(G)$ be the Terwilliger algebra created from the group association scheme for $G = D_{2n}$ where $n = 2m+1$, for any $m \in \mathbb{N}$. Then $Z = \mathrm{Span}_{\mathbb{C}}\{Z_{ij} : 1 \leq i, j \leq m\}$ is a two-sided ideal of $T(G)$.*

Proof. We start by showing that Z is a subset of $T(G)$. Notice for any $x, y \in G$ the (x, y) entry of A_i is 1 if and only if $yx^{-1} \in C_i$. All of the conjugacy classes of D_{2n} are closed under inverses, so $(yx^{-1})^{-1} = xy^{-1}$ is in C_i. Then the (y, x) entry of A_i is also 1. As the sum of all the A_k matrices is the all 1's matrix we know for any choice of $i, j \in \{1, 2, \ldots, m\}$ there exists a unique A_k whose (r^i, r^j) entry is 1 and a unique A_s whose (r^{-i}, r^j) entry is 1. Note that $r^{j-i} \in C_k$ as (r^i, r^j) is a nonzero entry of A_k. Then $C_k = \{r^{j-i}, r^{i-j}\}$. Similarly we know $C_s = \{r^{-i-j}, r^{i+j}\}$. Clearly C_k and C_s are not the same, so $A_k \neq A_s$. The (r^{-i}, r^j) and (r^i, r^{-j}) entries of A_k must be 0 as A_s is the unique adjacency matrix with 1's in these entries. Similarly A_s has 0's in entries (r^i, r^j) and (r^{-i}, r^{-j}). Then $E_i^* A_k E_j^*$ has 1's in entries (r^i, r^j) and (r^{-i}, r^{-j}) with 0's elsewhere and $E_i^* A_s E_j^*$ has 1's in entries (r^{-i}, r^j) and (r^i, r^{-j}) with all other entries being 0. We then notice that $Z_{ij} = E_i^* A_k E_j^* - E_i^* A_s E_j^* \in T(G)$ for all i, j. So $Z \subseteq T(G)$.

The fact that Z is closed under addition and scaling is clear. We now are left with showing that Z is closed under multiplication by an element of $T(G)$. As $T(G)$ is generated by the set of all A_i, E_i^* for $0 \leq i \leq m+1$ we need only check that Z is closed under multiplication by the A_k and E_k^* for any k. Starting with E_k^* we have as E_k^* is a diagonal matrix with 1's along the diagonal in the entries corresponding to C_k that

$$E_k^* Z_{ij} = \begin{cases} Z_{ij} & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases} \qquad \text{and} \qquad Z_{ij} E_k^* = \begin{cases} Z_{ij} & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

So Z is closed under multiplication by E_k^*. We now must check that Z is closed under multiplication on the left by A_k for any $0 \leq k \leq m+1$. Using block notation and letting

$a_{t,i}$ be the C_t, C_i block of A_k, with $*$ symbolizing an unknown value, we have the same multiplication as seen in Equation (4.3) replacing v_{rs} with Z_{ij} where in this case

$$d_{ij} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

We now consider the possible $a_{t,i}$ we could have for A_k. Doing so we will determine what the C_j column of $A_k Z_{ij}$ looks like. In the C_0, C_i block of A_k we have a 1×2 matrix. If either entry is 1, that means either r^i or r^{-i} is in C_k. Either way $C_k = C_i$ in this case and both entries are nonzero in A_k. So the C_0, C_i block of A_k is either $(0,0)$ or $(1,1)$. Notice $(0,0)d_{ij} = 0$ and $(1,1)d_{ij} = 0$. So $a_{0,i}d_{ij} = 0$.

Next we consider $a_{m+1,i}d_{ij}$. Note $a_{m+1,i}$ is a $m \times 2$ matrix. Every entry of C_{m+1} is of the form $r^b s$ for some $0 \leq b \leq m-1$. So if $a_{m+1,i}$ has 1 in an entry this means $r^i(r^b s)^{-1} \in C_k$ or $r^{-i}(r^b s)^{-1} \in C_k$. Either way $C_k = C_{m+1}$. If $C_k = C_{m+1}$ then $r^i(r^b s)^{-1}, r^{-i}(r^b s)^{-1} \in C_k$ for all b. Thus, $a_{m+1,i}$ is the all 1's matrix in this case. So if $a_{m+1,i}$ has a nonzero entry it is the all 1's matrix. Thus, $a_{m+1,i}$ is either the 0 matrix or the all 1's matrix. Either way $a_{m+1,i}d_{ij} = 0$.

We now look at $a_{t,i}d_{ij}$ for $1 \leq t \leq m$. Now $a_{t,i}$ has a 1 in the entry corresponding to (r^t, r^i) if and only if $r^{i-t} \in C_k$, which is true if and only if $r^{-i+t} \in C_k$. This is true if and only if the (r^{-t}, r^{-i}) entry of $a_{t,i}$ is 1. Note in this case $r^{i+t}, r^{-i-t} \notin C_k$ so the (r^{-t}, r^i) and (r^t, r^{-i}) entries of $a_{t,i}$ are 0. Similarly $a_{t,i}$ has 1 in the (r^t, r^{-i}) entry if and only if the (r^{-t}, r^i) entry is 1. In this case both the entries of $a_{t,i}$ indexed by (r^t, r^i) and (r^{-t}, r^{-i}) must be 0. This tells us that $a_{t,i}$ is one of three matrices

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then $a_{t,i}d_{ij}$ is one of the following

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

or

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

We now know that $A_k Z_{ij}$ is a matrix with 0's in all entries not in the C_j column block. Within this block the only potentially nonzero blocks are the C_t, C_j where $1 \leq t \leq m$. In the case where the C_t, C_j block is nonzero it is either d_{ij} or $-d_{ij}$. We then have $A_k Z_{ij} = \sum_{t=1}^{m} c_{ijt} Z_{tj}$ where

$$c_{ijt} = \begin{cases} 0 & \text{if } a_{t,i}d_{ij} = 0, \\ 1 & \text{if } a_{t,i}d_{ij} = d_{ij}, \\ -1 & \text{if } a_{t,i}d_{ij} = -d_{ij}. \end{cases}$$

So $A_k Z_{ij} \in Z$ for all k. Thus, we have that Z is a left ideal of $T(G)$.

We now show Z is closed under transposition. For any $1 \leq i, j \leq m$ we have that Z_{ij} has 1 in entries corresponding to (r^i, r^j) and (r^{-i}, r^{-j}), -1 in entries corresponding to (r^{-i}, r^j) and (r^i, r^{-j}), and all other entries are 0. Then Z_{ij}^t has 1 in entries corresponding to (r^j, r^i) and (r^{-j}, r^{-i}), -1 in entries (r^{-j}, r^i) and (r^j, r^{-i}), and all other entries are 0. Thus $Z_{ij}^t = Z_{ji}$. As Z is the span of all the Z_{ij} we have that Z is then closed under transposition. Then by Lemma 4.11 we have that Z is a two-sided ideal of $T(G)$. ☐

Corollary 5.11. *Consider D_{2n} where $n = 2m+1$. The two-sided ideal Z is an irreducible two-sided ideal of $T(G)$ whose dimension is $m^2 = (\frac{n-1}{2})^2$.*

Proof. Note each Z_{ij} is nonzero only in the C_i, C_j block. The set of all Z_{ij} is then linearly independent as they are each nonzero in different blocks. Thus, the set of all Z_{ij} is a basis for Z. There are m different choices for both i and j. We get an element of the basis for Z

for any choice of $1 \leq i,j \leq m$. So there are a total of m^2 elements of the basis for Z. Note $m = \frac{n-1}{2}$, so $\dim Z = (\frac{n-1}{2})^2$.

Now we show Z is irreducible. Let $\alpha \in Z$ be any nonzero element of Z. We show that $I = \langle \alpha \rangle = Z$. We can write $\alpha = \sum_{i,j} c_{i,j} Z_{ij}$ as it is in Z. Since $\alpha \neq 0$, we can chose i,j so that $c_{i,j}$ is nonzero. Then the C_i, C_j block of α is nonzero. As E_i^* is diagonal with 1's only in the C_i, C_i block and E_j^* is diagonal with 1's only in the C_j, C_j block we have $E_i^* \alpha E_j^*$ has the same C_i, C_j block as α and all other entries are 0. Then this means that $E_i^* \alpha E_j^* = c_{i,j} Z_{ij}$. So $Z_{ij} \in I$. We now show for any $1 \leq x, y \leq m$ that $Z_{xy} \in I$. As the sum of the A matrices is the all 1's matrix there exists an A_t such that the C_x, C_i block of A_t is nonzero. Using the notation from the proof of Theorem 5.10 this means

$$a_{x,i} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We then know from Theorem 5.10 that $A_t Z_{ij} = \sum_{k=1}^{m} c_{ijk} Z_{kj}$ where $c_{ijk} = 0, 1, -1$. We also know that $c_{ijx} \neq 0$ as $a_{x,i} \neq 0$. So Z_{xj} has a nonzero coefficient in this sum. Computing $E_x^*(A_t Z_{ij})$ we note this gives the C_x row block of $A_t Z_{ij}$ and everything else is 0. The only entry in the C_x row block of $A_t Z_{ij}$ is $c_{ijx} Z_{xj}$ as all of the other Z_{kj} are 0 in this row block for all k. Thus, $E_x^*(A_t Z_{ij}) = c_{ijx} Z_{xj}$, so $Z_{xj} \in I$. We know there is an A_w that has a nonzero C_y, C_j block. We consider $Z_{xj} A_w$. Note that $A_w Z_{jx} = \sum_{k=1}^{m} c_{jxk} Z_{kx}$. As $a_{y,j} \neq 0$ we know that $c_{jxy} \neq 0$. Note from the proof of Theorem 5.10 that

$$Z_{xj} A_w = (A_w Z_{jx})^t = \left(\sum_{k=1}^{m} c_{jxk} Z_{kx} \right)^t = \sum_{k=1}^{m} c_{jxk} Z_{xk}.$$

So $Z_{xj} A_w = \sum_{k=1}^{m} c_{jxk} Z_{xk}$. Since $c_{jxy} \neq 0$ we see that Z_{xy} has a nonzero coefficient in this sum. Computing $(Z_{xj} A_w) E_y^*$ we get only the C_y column block of $Z_{xj} A_w$ which is $c_{jxy} Z_{xy}$. This implies that $Z_{xy} \in I$. As we chose $1 \leq x, y \leq m$ arbitrary we have for all $1 \leq x, y \leq m$ that $Z_{xy} \in I$. Thus, $Z \subseteq I$ and we have $Z = I$. Any nonzero ideal of Z must contain some

$\alpha \neq 0$ and the ideal generated by such an α is Z itself. Hence, Z has no proper nonzero subideals and is thus irreducible. $\square$

From Corollaries 4.14 and 5.11 we have found two irreducible components of D_{2n} where n is odd. Using the same notation as before these components are V and Z. We now consider the Hadamard inner product $\langle v_{ij}, Z_{rs}\rangle$ for any basis element v_{ij}, $(0 \leq i, j \leq m+1)$ of V and Z_{rs}, $(1 \leq r, s \leq m)$ of Z. Recall v_{ij} is the all 1's matrix in the C_i, C_j block and all 0 outside the C_i, C_j block. Similarly Z_{rs} is all 0 outside the C_r, C_s block, where C_r and C_s are size two conjugacy classes. The C_r, C_s block of Z_{rs} is of the form

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

If v_{ij} and Z_{rs} have different nonzero blocks, that is $(i, j) \neq (r, s)$, then we have that $\langle v_{ij}, Z_{rs}\rangle$ is a sum of zeros, which is 0. This is because v_{ij} is all 0's in its C_r, C_s block and Z_{rs} is all 0's in its C_i, C_j block. If v_{ij} and Z_{rs} have the same nonzero block, that is $(i, j) = (r, s)$, we have $\langle v_{ij}, Z_{rs}\rangle = 1(1) + 1(-1) + 1(-1) + 1(1) = 0$. Hence for all i, j, r, s we have that $\langle v_{ij}, Z_{ij}\rangle = 0$. Extending this we get that V and Z are orthogonal. Now we can write the Terwilliger algebra of D_{2n} as $T(D_{2n}) = V \oplus Z \oplus R$ where R is the orthogonal complement of $V \oplus Z$ in $T(D_{2n})$. We shall show in Theorem 5.12 that R is commutative and thus is a direct sum of irreducible one dimensional components.

Theorem 5.12. *Let $D_{2n} = \langle a, b \colon a^n = b^2 = 1, ab = ba^{-1}\rangle$ be a dihedral group with n odd. Let $T(D_{2n})$ be the Terwilliger algebra of the group association scheme of D_{2n}. Then writing $T(D_{2n}) = V \oplus Z \oplus R$ we have that R is commutative.*

Proof. Let $r \in R$. We begin by determining what r looks like. As $T(D_{2n})$ is triply regular by Theorem 5.8 and $r \in T(D_{2n})$ we know that $r = \sum_{i,k,j} \lambda_{i,k,j} E_i^* A_k E_j^*$ where each $\lambda_{i,k,j} \in \mathbb{C}$. We begin determining the form of r by looking at the $\lambda_{i,k,j}$ terms. We do this in cases. Before beginning we note that because of the way the A_k matrices are created for each entry

(x, y) there exists a unique A_k matrix that has a nonzero value in that entry. This implies that for a linear combination $\sum_k \lambda_{i,k,j} E_i^* A_k E_j^*$ for fixed i, j to be the 0 matrix all of the coefficients $\lambda_{i,k,j}$ where $E_i^* A_k E_j^* \neq 0$ must be 0. If $E_i^* A_k E_j^*$ is the 0 matrix we may, and will, assume $\lambda_{i,k,j} = 0$. So if the C_i, C_j block of r is the 0 matrix we have that $\lambda_{i,k,j} = 0$ for all k. We shall use this in all of the following cases.

Case 1: Suppose C_i and C_j are both conjugacy classes of size 2. Notice that in this case the (a^i, a^j) entry of $E_i^* A_k E_j^*$ is 1 if and only if $a^{j-i} \in C_k$, which is true if and only if $a^{i-j} \in C_k$. This is true if and only if the (a^{-i}, a^{-j}) entry of A_k is 1. So the (a^i, a^j) and (a^{-i}, a^{-j}) entries of the C_i, C_j block of r must be the same. Similar reasoning gives that the (a^i, a^{-j}) and (a^{-i}, a^j) entries of r must also be the same. Thus, the C_i, C_j block of r looks like

$$\begin{bmatrix} c & d \\ d & c \end{bmatrix}$$

for some $c, d \in \mathbb{C}$. We consider $\langle v_{ij}, r \rangle = 1c + 1d + 1d + 1c = 2c + 2d$. This must be 0 as $r \in R$, so $2c + 2d = 0$. Then $c = -d$. Similarly $\langle Z_{ij}, r \rangle = 0$, so $c - d - d + c = 0$. That is $2c - 2d = 0$, so $c = d$. We thus have $d = -d$. Therefore, $d = 0 = c$. Hence, the C_i, C_j block of r is all 0's. So $\lambda_{i,k,j} = 0$ for all k in this case.

Case 2: $C_i = C_0 = \{1\}$ and C_j is a conjugacy class of size 2. The $(1, a^j)$ entry of $E_0^* A_k E_j^*$ is 1 if and only if $a^j \in C_k$, which is true if and only if $a^{-j} \in C_k$. In this case the entry of r indexed by $(1, a^{-j})$ is also 1. Thus, in the C_0, C_j block of r both entries have the same value. Call it c. Taking $\langle v_{0,j}, r \rangle$ we get $2c$. As this must be 0, we have $c = 0$. Thus, the C_0, C_j block of r is the 0 matrix, so $\lambda_{0,k,j} = 0$ for all k.

Case 3: C_i is a conjugacy class of size 2 and $C_j = C_0$. This is similar to case 2 and we get $\lambda_{i,k,0} = 0$ for all k.

Case 4: $C_i = C_0$ and $C_j = C_{n+1}$, where C_{n+1} is the conjugacy class of size n. For any $0 \leq \ell \leq n-1$, the $(1, a^\ell b)$ entry of $E_0^* A_k E_{n+1}^*$ is 1 if and only if $a^\ell b \in C_k$, which is true if and only if $a^t b \in C_k$ for all $0 \leq t \leq n-1$ as these elements are all conjugate. So every element of the C_0, C_{n+1} block of r must have the same value. Call this value c. Then as $\langle v_{ij}, r\rangle = nc$ and must be 0, so $c = 0$. Thus, the C_0, C_{n+1} block of r is all 0's. So $\lambda_{0,k,n+1} = 0$ for all k.

Case 5: $C_i = C_{n+1}$ and $C_j = C_0$. The same argument as in Case 4 can be used here to get $\lambda_{n+1,k,0} = 0$ for all k.

Case 6: C_i be a conjugacy class of size 2 and $C_j = C_{n+1}$. For any $0 \leq \ell \leq n-1$ the $(a^i, a^\ell b)$ entry of $E_i^* A_k E_{n+1}^*$ is 1 if and only if $a^\ell b a^{-i} \in C_k$. This is just saying that $a^{\ell+i} b \in C_k$. This is true if and only if $C_k = C_{n+1}$. In this case for all $0 \leq t \leq n-1$ we have $a^{t+i} b \in C_k$ and $a^{t-i} b \in C_k$. Then the entry $(a^i, a^\ell b)$, for some $0 \leq \ell \leq n-1$, is 1 if and only if we have the $(a^i, a^t b)$ and $(a^{-i}, a^t b)$ entries are also 1 for all $0 \leq t \leq n-1$. This tells us that every entry in the C_i, C_{n+1} block of r must be the same value. Say that value is c. Then we consider the fact that $\langle v_{i,n+1}, r\rangle = 0$. We also know it equals the sum of the entries of the C_i, C_{n+1} block of r. So it equals $2nc$. Thus, $2nc = 0$. Then $c = 0$. So the C_i, C_{n+1} block of r is all zeroes. Then $\lambda_{i,k,n+1} = 0$ for all k.

Case 7: $C_i = C_{n+1}$ and C_j is a conjugacy class of size 2. This case can be done the same way as Case 6 to get $\lambda_{n+1,k,j} = 0$ for all k.

Case 8: $C_i = C_0$ and $C_j = C_0$. In this case the C_0, C_0 block of r has a single entry. Call it c. Since $\langle v_{00}, r\rangle = 0$ and also equals c we have that $c = 0$. So $\lambda_{0,k,0} = 0$ for all k.

Having done all of these cases we have shown that the only potentially nonzero $\lambda_{i,k,j}$ are of the form $\lambda_{n+1,k,n+1}$. This means we can write $r = \sum_k \lambda_k E_{n+1}^* A_k E_{n+1}^*$. This was done for an arbitrary $r \in R$.

Now let $r, s \in R$. We have $r = \sum_i \lambda_i E_{n+1}^* A_i E_{n+1}^*$, and $s = \sum_j \mu_j E_{n+1}^* A_j E_{n+1}^*$. Then to show $rs = sr$ it suffices to show $(E_{n+1}^* A_i E_{n+1}^*)(E_{n+1}^* A_j E_{n+1}^*) = (E_{n+1}^* A_j E_{n+1}^*)(E_{n+1}^* A_i E_{n+1}^*)$ for all i, j.

First we show that $E_{n+1}^* A_j E_{n+1}^*$ is symmetric for all j. As the E_{n+1}^* matrices are diagonal clearly they are symmetric. In A_j the entry (x, y) is 1 if and only if $yx^{-1} \in C_j$. As in D_{2n} every element and its inverse are in the same conjugacy class we have $yx^{-1} \in C_j$ if and only if $xy^{-1} \in C_j$ which is true if and only if the (y, x) entry of A_j is 1. Thus, the (x, y)

and (y,x) entries of A_j must the be the same for all x,y. Therefore, $A_j = A_j^t$. Then $(E^*_{n+1}A_jE^*_{n+1})^t = (E^*_{n+1})^tA_j^t(E^*_{n+1})^t = E^*_{n+1}A_jE^*_{n+1}$. So as claimed it is symmetric.

Next we note that $(A_iE^*_{n+1})(E^*_{n+1}A_j) \in T(D_{2n})$. Recall by Lemma 5.5 and Theorem 5.8 we have $T(D_{2n}) = T_0(D_{2n})$. Then $(A_iE^*_{n+1})(E^*_{n+1}A_j) \in T_0(D_{2n})$ and we can write it as a sum of basis elements. That is

$$(A_iE^*_{n+1})(E^*_{n+1}A_j) = \sum_{i,j,k} \beta_{i,j,k}E^*_iA_jE^*_k.$$

Then multiplying on each side of this by E^*_{n+1} and using the fact that $E^*_iE^*_j = \delta_{ij}E^*_i$ (where δ_{ij} is 1 if $i=j$ and 0 if $i \neq j$) we have

$$(E^*_{n+1}A_iE^*_{n+1})(E^*_{n+1}A_jE^*_{n+1}) = \sum_{i,j,k} \beta_{i,j,k}E^*_{n+1}E^*_iA_jE^*_kE^*_{n+1} = \sum_j \beta_{n+1,j,n+1}E^*_{n+1}A_jE^*_{n+1}.$$

Therefore we have

$$[(E^*_{n+1}A_iE^*_{n+1})(E^*_{n+1}A_jE^*_{n+1})]^t = \left(\sum_k \beta_{n+1,k,n+1}E^*_{n+1}A_kE^*_{n+1}\right)^t$$

$$= \sum_k \beta_{n+1,k,n+1}(E^*_{n+1}A_kE^*_{n+1})^t = \sum_k \beta_{n+1,k,n+1}E^*_{n+1}A_kE^*_{n+1} = (E^*_{n+1}A_iE^*_{n+1})(E^*_{n+1}A_jE^*_{n+1}).$$

However, we also know that

$$\left[(E^*_{n+1}A_iE^*_{n+1})(E^*_{n+1}A_jE^*_{n+1})\right]^t = (E^*_{n+1}A_jE^*_{n+1})^t(E^*_{n+1}A_iE^*_{n+1})^t = (E^*_{n+1}A_jE^*_{n+1})(E^*_{n+1}A_iE^*_{n+1}).$$

We thus have $(E^*_{n+1}A_iE^*_{n+1})(E^*_{n+1}A_jE^*_{n+1}) = (E^*_{n+1}A_jE^*_{n+1})(E^*_{n+1}A_iE^*_{n+1})$ for all i,j. As noted before this is sufficient for us to show that $rs = sr$ for all $r,s \in R$. Therefore R is commutative. □

Corollary 5.13. *Let D_{2n} be a dihedral group with $n = 2m+1$. Let $T(D_{2n})$ be the Terwilliger algebra of D_{2n}. Then $T(D_{2n}) = V \oplus Z \oplus I_1 \oplus \cdots \oplus I_m$ where each I_i is an irreducible ideal of $T(D_{2n})$ of dimension* 1.

Proof. We have already seen that we can write $T(D_{2n}) = V \oplus Z \oplus R$. By Corollary 5.9 we have that $\dim(T(D_{2n}) = 2m^2 + 5m + 4$. We further know from Corollaries 4.14 and 5.11 that $\dim(V) = (m+2)^2$ and $\dim(Z) = m^2$. As the sum of the dimensions of V, Z, R must be $\dim(T(D_{2n}))$ we have $\dim(R) = 2m^2 + 5m + 4 - (m+2)^2 - m^2 = m$. By Theorem 5.12 we know that R is commutative. Then R is a direct sum of irreducible ideals each with dimension 1. That is $R = \bigoplus_{i=1}^{m} I_i$ where each I_i has dimension 1. Therefore, we have that $T(D_{2n}) = V \oplus Z \oplus I_1 \oplus I_2 \oplus \cdots \oplus I_m$. $\square$

We now move onto looking at D_{2n} when n is even. Let $n = 2m$ for some $m \in \mathbb{N}$. In this case the conjugacy classes are

$$C_0 = \{e\},\ C_i = \{r^i, r^{-i}\} \text{ (for } 1 \leq i \leq m-1),\ C_m = \{r^m\},\ C_{m+1} = s\langle r^2\rangle, \text{ and } C_{m+2} = sr\langle r^2\rangle.$$

For any C_i, C_j, where $1 \leq i, j \leq m-1$ we shall create the Z_{ij} matrices the same way as we did in the n odd case.

Theorem 5.14. *Let $T(G)$ be the Terwilliger algebra created from the group association scheme for D_{2n} where $n = 2m$, for any $m \in \mathbb{N}$. Then $Z = \mathrm{Span}_{\mathbb{C}}\{Z_{ij} \colon 1 \leq i, j \leq m-1\}$ is a two-sided ideal of $T(G)$.*

Proof. The proof that Z is a subset of $T(G)$ is nearly identical to the proof of this fact for Theorem 5.10. It is also clear that Z is closed under addition and scaling. We just need to show closure under multiplication by elements of $T(G)$. As $T(G)$ is generated by the set of all A_k, E_k^* for $0 \leq k \leq m+2$ we need to only check that Z is closed under multiplication by the A_k and E_k^* for any k. Starting with E_k^* we have

$$E_k^* Z_{ij} = \begin{cases} Z_{ij} & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases} \quad \text{and} \quad Z_{ij} E_k^* = \begin{cases} Z_{ij} & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

So Z is closed under multiplication by E_k^*. We now must check that Z is closed under multiplication by A_k for any $0 \leq k \leq m+2$. We shall do this in much the same as we did for Theorem 5.10. Using block notation and letting $a_{t,i}$ be the C_t, C_i block of A_k, with $*$ symbolizing an unknown value, we have the same multiplication as in Equation (4.3) replacing v_{rs} with Z_{ij} and having

$$d_{ij} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

We now consider the possible $a_{t,i}$ we could have for A_k. Doing so we will determine what the C_j column of $A_k Z_{ij}$ looks like. In the C_0, C_i block of A_k we have a 1×2 matrix. If either entry is 1, that means either r^i or r^{-i} is in C_k. Either way $C_k = C_i$ in this case both entries are nonzero in A_k. So the C_0, C_i block of A_k is either $(0,0)$ or $(1,1)$. $(0,0)d_{ij} = 0$ and $(1,1)d_{ij} = 0$, so $a_{0,i}d_{ij} = 0$.

Next we consider $a_{m,i}d_{ij}$. Note that $a_{m,i}$ is a 1×2 matrix. If either entry is 1, that means that r^{m-i} or r^{m+i} is in C_k. We note that r^m is its own inverse, so $r^{i+m} = r^{i-m} = (r^{m-i})^{-1}$. Thus, if either entry of $a_{m,i}$ is 1 then $C_k = C_{m-i}$. In this case both entries are nonzero. So the C_m, C_i block of A_k is either $(0,0)$ or $(1,1)$. Note $(0,0)d_{ij} = 0$ and $(1,1)d_{ij} = 0$, so $a_{m,i}d_{ij} = 0$.

We now look at $a_{t,i}d_{ij}$ for $1 \leq t \leq m-1$. Using the same argument as in the proof of Theorem 5.10 we have $a_{t,i}d_{ij} = 0, d_{ij}$ or $-d_{ij}$.

Next we consider $a_{m+1,i}d_{ij}$ which is a $m \times 2$ matrix. Every entry of C_{m+1} is of the form $r^{2b}s$ for some $0 \leq b \leq m-1$. So if $a_{m+1,i}$ has 1 in an entry this means $r^i(r^{2b}s)^{-1} = r^{2b+i}s \in C_k$, or $r^{-i}(r^{2b}s)^{-1} = r^{2b-i}s \in C_k$. If i is even both of these would imply $C_k = C_{m+1}$. If i is odd both of these statements imply $C_k = C_{m+1}$. If $C_k = C_{m+1}$, then $r^i(r^{2b}s)^{-1}, r^{-i}(r^{2b}s)^{-1} \in C_k$ for all b. Similarly if $C_k = C_{m+2}$, then $r^i(r^{2b}s)^{-1}, r^{-i}(r^{2b}s)^{-1} \in C_k$ for all b. Thus, $a_{m+1,i}$ is the all 1's matrix in this case. So if $a_{m+1,i}$ has a nonzero entry it is the all 1's matrix. Thus, $a_{m+1,i}$ is either the 0 matrix, or the all 1's matrix. Either way $a_{m+1,i}d_{ij} = 0$. Nearly identical reasoning can be applied to get $a_{m+2,i}d_{ij} = 0$.

From these computations we have $A_k Z_{ij} = \sum_{t=1}^{m-1} c_{ijt} Z_{tj}$ where

$$c_{ijt} = \begin{cases} 0 & \text{if } a_{t,i} d_{ij} = 0, \\ 1 & \text{if } a_{t,i} d_{ij} = d_{ij}, \\ -1 & \text{if } a_{t,i} d_{ij} = -d_{ij}. \end{cases}$$

So $A_k Z_{ij} \in Z$ for all k. Thus, Z is a left ideal of $T(G)$. The proof that Z is closed under transposition is the same as in the proof of Theorem 5.10. Then by Lemma 4.11 we see that Z is a two-sided ideal of $T(G)$. □

Corollary 5.15. *Consider D_{2n} where $n = 2m$. The two-sided ideal Z is an irreducible two-sided ideal of $T(G)$ whose dimension is $(m-1)^2$.*

Proof. The proof that Z is irreducible is the same as in the proof of Corollary 5.11. The proof that the Z_{ij} are linearly independent is also the same as in the proof of Corollary 5.11. We have a basis element of Z for any choice of $1 \leq i, j \leq m-1$. So in total there are $(m-1)^2$ basis element for Z. Thus, $\dim(Z) = (m-1)^2 = (\frac{n}{2} - 1)^2$. □

We shall continue to order the conjugacy classes of D_{2n} as in the proof of Theorem 5.8. Now we specify the order of the elements in C_{m+1} and C_{m+2}. We order them by increasing powers of r. That is $C_{m+1} = \{s, r^2 s, r^4 s, \cdots\}, C_{m+2} = \{rs, r^3 s, r^5 s, \cdots\}$. Doing this gives rise to the following result.

Lemma 5.16. *Let $G = D_{2n} = \langle r, s \colon r^n = 1 = s^2, r^{-1}s = sr \rangle$, where $n = 2m$. Let $T(G)$ be the Terwilliger algebra of the group association scheme of G created using the conjugacy classes ordered as discussed above. Then for all $0 \leq i \leq m+2$ we have that the C_{m+1}, C_{m+1} and C_{m+2}, C_{m+2} blocks of A_i are equal.*

Proof. Let $a_{i,m+1}$ be the C_{m+1}, C_{m+1} block of A_i and $a_{i,m+2}$ be the C_{m+2}, C_{m+2} block of A_i for any $0 \leq i \leq m+2$. Both of $a_{i,m+1}$ and $a_{i,m+2}$ are $m \times m$ matrices. Due to our fixed ordering of C_{m+1} and C_{m+2} the $(r^p s, r^q s)$ entry of $a_{i,m+1}$ corresponds to the $(r^{p+1}s, r^{q+1}s)$ entry of $a_{i,m+2}$. Note the entries of $a_{i,m+1}$ and $a_{i,m+2}$ are all either 0 or 1, being part of

A_i. The $(r^p s, r^q s)$ entry of $a_{i,m+1}$ is 1 if and only if $(r^q s)(r^p s)^{-1} = r^{q-p} \in C_i$. Notice that $r^{q-p} = (r^{q+1}s)(r^{p+1}s)^{-1}$. Therefore, $r^{q-p} \in C_i$ if and only if $(r^{q+1}s)(r^{p+1}s)^{-1} \in C_i$. Therefore, the $(r^p s, r^q s)$ entry of $a_{i,m+1}$ is 1 if and only if $(r^{q+1}s)(r^{p+1}s)^{-1} \in C_i$, which is true if and only if the $(r^{p+1}s, r^{q+1}s)$ entry of $a_{i,m+2}$ is 1. As we chose the entry $(r^p s, r^q s)$ of $a_{i,m+1}$ arbitrarily we have that each of the corresponding entries of $a_{i,m+1}$ and $a_{i,m+2}$ are equal. Therefore, $a_{i,m+1} = a_{i,m+2}$. As we chose i arbitrarily this is true for all i. □

This preceding lemma simplifies the study of what the Wedderburn decomposition of the Terwilliger algebra resulting from the group association scheme of D_{2n} when n is even. From Table C.11 in Appendix C it is clear that the decomposition for such a Terwilliger algebra depends on if $4 \mid n$ or not. This thesis has described what two of the components will be. Namely V from Theorem 4.12 and Z from Theorem 5.14. We now provide the structure of two irreducible dimension 1 components when $4 \mid n$.

Lemma 5.17. *Let $G = D_{2n} = \langle r, s \colon r^n = 1 = s^2, r^{-1}s = sr \rangle$, where $n = 2m$ and $4 \mid n$. Let $T(G)$ be the Terwilliger algebra of the group association scheme of G created using the conjugacy classes ordered as discussed above. Let D be the following $m \times m$ matrix,*

$$D = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{5.2}$$

Then the C_{m+1}, C_{m+1} and C_{m+2}, C_{m+2} blocks of A_{2i} $(0 < i < \frac{m}{2})$ equal $D^i + D^{-i}$. Also the C_{m+1}, C_{m+1} and C_{m+2}, C_{m+2} blocks of A_k with $k = 0$ and $k = m$ are D^0 and $D^{m/2}$ respectively.

Proof. First notice that D is just the image under the permutation representation of the generator of the cyclic group of order m.

We now consider what the C_{m+1}, C_{m+1} block of an A_i matrix can look like. The definition of how we create the A_i matrices tells us that the entry indexed by (a, b) is 1 if and only if $b^{-1}a \in C_i$. Using this we can create a grid of what value must be in C_i in order for A_i to have a 1 in the entry indexed by (x, y) where x is the row term and y is the column term in the grid. Note we create this grid so that the elements are ordered as we specified before. See Table 5.1 for this grid.

	s	$r^2 s$	$r^4 s$	$\cdots$	$r^{-2} s$
s	1	r^2	r^4	$\cdots$	r^{-2}
$r^2 s$	r^{-2}	1	r^2	$\cdots$	r^{-4}
$r^4 s$	r^{-4}	r^{-2}	1	$\vdots$	r^{-6}
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\vdots$
$r^{-2} s$	r^2	r^4	$\cdots$	$\cdots$	1

Table 5.1: Grid for C_{m+1}, C_{m+1} block of an A_i matrix in D_{2n} where $4 \mid n$

Using this grid we can immediately determine the value of the entry of the C_{m+1}, C_{m+1} block of A_i indexed by $(r^j s, r^k s)$ for any i. By way of quick observation it is easy to see that the 1's in D^k line up with those entries in this grid that are r^{-2k}. Then $D^i + D^{-i}$ has 1's in the entries corresponding to the grid entries that are in the set $\{r^{2i}, r^{-2i}\}$. So for A_{2i} for any $0 < i < \frac{m}{2}$ the entries of the C_{m+1}, C_{m+1} block that are 1 correspond exactly to the entries that are 1 in $D^i + D^{-i}$. That is as claimed the C_{m+1}, C_{m+1} block of A_{2i} for $0 < i < \frac{m}{2}$ equals $D^i + D^{-i}$. We know that D^0 is the $m \times m$ identity matrix which is the C_{m+1}, C_{m+1} block of A_0. We also know that $D^{m/2}$ has 1's in exactly the same entries that A_m has 1's since from the grid we already know that $D^{m/2}$ has 1's exactly where the grid has an entry $r^{-m} = r^m$, which is the only element of A_m. Therefore, the claim is true for the C_{m+1}, C_{m+1} block of A_{2i} for $0 \le i \le \frac{m}{2}$. By Lemma 5.16 we then have it is true also for the C_{m+2}, C_{m+2} blocks. $\square$

It immediately follows from the previous lemma that the C_{m+1}, C_{m+1} block of $\sum_{i=0}^{n/4}(-1)^i A_{2i}$ and the C_{m+2}, C_{m+2} block of $\sum_{i=0}^{n/4}(-1)^i A_{2i}$ both equal

$$\begin{bmatrix} 1 & -1 & 1 & \cdots & -1 \\ -1 & 1 & -1 & \cdots & 1 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ -1 & 1 & -1 & \cdots & 1 \end{bmatrix}.$$

We will now show that isolating the C_{m+1}, C_{m+1} block of $\sum_{i=0}^{n/4}(-1)^i A_{2i}$ and taking the $\mathbb{C}-$span of this matrix gives an irreducible ideal of $T(G)$. The same is true if we instead isolate the C_{m+2}, C_{m+2} block.

Proposition 5.18. *Let $G = D_{2n} = \langle r, s\colon r^n = 1 = s^2, r^{-1}s = sr\rangle$, where $n = 2m$ and $4 \mid n$. Let $T(G)$ be the Terwilliger algebra of the group association scheme of G. Then $J_{m+1} = \mathrm{Span}_{\mathbb{C}}\left(\sum_{i=0}^{n/4}(-1)^i E^*_{m+1}A_{2i}E^*_{m+1}\right)$ and $J_{m+2} = \mathrm{Span}_{\mathbb{C}}\left(\sum_{i=0}^{n/4}(-1)^i E^*_{m+2}A_{2i}E^*_{m+2}\right)$ are one dimensional irreducible ideals of $T(G)$.*

Proof. The fact that J_{m+1} and J_{m+2} are one dimensional is clear. It is also obvious that these sets are closed under addition, scaling, and are subsets of $T(G)$. All that is left to show is closure under multiplication. We start with J_{m+1} and note that J_{m+2} being closed under multiplication will follow by similar reasoning. Since $T(G)$ is generated by the set of all A_i, E^*_i for $0 \le i \le m+2$, we need only check that J_{m+1} is closed under multiplication by the A_k and E^*_k. Starting with E^*_k we note that

$$E^*_k \sum_{i=0}^{n/4}(-1)^i E^*_{m+1}A_{2i}E^*_{m+1} = \begin{cases} \sum_{i=0}^{n/4}(-1)^i E^*_{m+1}A_{2i}E^*_{m+1} & \text{if } k = m+1, \\ 0 & \text{if } k \neq m+1. \end{cases}$$

So J_{m+1} is clearly closed under multiplication by E^*_k on the left. We now turn our attention to multiplying by A_k on the left. Using block notation and letting $a_{t,i}$ be the C_t, C_i block of A_k we have the same multiplication structure as in Equation (4.3) where v_{rs} is replaced by $\sum_{i=0}^{n/4}(-1)^i E^*_{m+1}A_{2i}E^*_{m+1}$ and

$$d_{m+1,m+1} = \begin{bmatrix} 1 & -1 & 1 & \cdots & -1 \\ -1 & 1 & -1 & \cdots & 1 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ -1 & 1 & -1 & \cdots & 1 \end{bmatrix}.$$

We now consider the possible $a_{t,m+1}$ we could have for A_k. Then we look at $a_{t,i}d_{m+1,m+1}$.

In the C_0, C_{m+1} block of A_k we have a $1 \times m/2$ matrix. If any of the entries is nonzero then that means $r^t s \in C_k$ for some even t. For this to be the case we must have that $C_k = C_{m+1}$. In this case every entry in the C_0, C_{m+1} block of A_k is 1. So the C_0, C_{m+1} block of A_k is either all 0's or all 1's. As the row sum of $d_{m+1,m+1}$ is 0, we have in either case that $a_{0,m+1}d_{m+1,m+1} = 0$.

Next we consider $a_{t,m+1}d_{m+1,m+1}$ for $1 \leq t \leq m-1$. As $a_{t,m+1}$ is the C_t, C_{m+1} block of A_k we have that its entries are indexed by elements of the form $(r^t, r^a s)$ and $(r^{-t}, r^a s)$ where a is even. An entry of A_k indexed by $(r^t, r^a s)$ is 1 if and only if $r^a s r^{-t} = r^{t+a}s \in C_k$. An entry of A_k indexed by $r^{-t}, r^a s$ is 1 if and only if $r^a s r^t = r^{a-t}s \in C_k$. Since a is even, $a+t$ and $a-t$ have the same parity, so $r^{a+t}s$ and $r^{a-t}s$ are in the same conjugacy class of G. Then for any even a we have that the entries of A_k indexed by $(r^t, r^a s)$ and $(r^{-t}, r^a s)$ have the same value. We next see that if t is even then $a+t$ is even for every even a. In this case $r^{a+t}s \in C_{m+1}$ for every a, so the entry of A_k indexed by $(r^t, r^a s)$ for any even a is 1 if and only if $C_k = C_{m+1}$. This tells us that if t is even then every entry in the r^t row of $a_{m+1,m+1}$ is the same. Since for any even a we have that the entries of A_k indexed by $(r^t, r^a s)$ and $(r^{-t}, r^a s)$ have the same value, we have that if $C_k \neq C_{m+1}$ then $a_{m+1,m+1}$ is the 0 matrix and if $C_k = C_{m+1}$ then $a_{m+1,m+1}$ is the all 1's matrix. If t is odd similar reasoning can be applied to show that if $C_k \neq C_{m+2}$ then $a_{m+1,m+1}$ is the 0 matrix and if $C_k = C_{m+2}$ then $a_{m+1,m+1}$ is the all 1's matrix. Hence, $a_{t,m+1}$ is the 0 matrix, or the all 1's matrix. Direct computation shows that if $a_{t,m+1}$ is the all 1's matrix then $a_{t,m+1}d_{m+1,m+1} = 0$. Clearly if $a_{t,m+1}$ is the 0 matrix then, $a_{t,m+1}d_{m+1,m+1} = 0$. Hence, $a_{t,m+1}d_{m+1,m+1} = 0$ for all $1 \leq t \leq m-1$.

Looking at $a_{m,m+1}d_{m+1,m+1}$, we have that $a_{m,m+1}$ is a $1 \times m/2$ matrix. Then entries of A_k in the C_m, C_{m+1} block are indexed by elements of the form $(r^m, r^t s)$ for even t. If any of the entries in $a_{m,m+1}$ is nonzero then that means $r^t s r^{-m} = r^{t+m} s \in C_k$ for some even t. As m is even, $t + m$ is even. Then for $r^{t+m} s \in C_k$ we must have that $C_k = C_{m+1}$. In this case every entry in the C_0, C_{m+1} block of A_k is 1. So the C_m, C_{m+1} block of A_k is either all 0's or all 1's. As the row sum of $d_{m+1,m+1}$ is 0, we have in either case that $a_{m,m+1}d_{m+1,m+1} = 0$.

Now we consider $a_{m+2,m+1}d_{m+1,m+1}$. As $a_{m+2,m+1}$ is the $C_{m+2,m+1}$ block of A_k we have that its entries are indexed by elements of the form $(r^a s, r^b s)$ for odd $1 \leq a < n$ and even $0 \leq b < n$. An entry indexed by $(r^a s, r^b s)$ of A_k is 1 if and only if $r^b s(r^a s)^{-1} = r^{b-a} \in C_k$. As b is even and a is odd $b - a$ is odd. Then since m is even we know that $b - a \neq m$. Thus, if $r^{b-a} \in C_k$, then $C_m \neq C_k$. So for A to have a nonzero entry in its C_{m+2}, C_{m+1} block we must have that $C_k = \{r^k, r^{-k}\}$. If A has no nonzero entry in its C_{m+2}, C_{m+1} block then $a_{m+2,m+1}$ is the all 0's matrix and $a_{m+2,m+1}d_{m+1,m+1} = 0$.

Now assume that $C_k = \{r^k, r^{-k}\}$ for some $1 \leq k \leq m - 1$. For any fixed odd value $1 \leq a < n$, we know there exists a unique value $0 \leq b < n$ such that $b - a \equiv k(mod\ n)$, namely $a + k(mod\ n)$. As a, k are both odd and n is even, we have that b is even. Similar reasoning gives that there exists a unique even integer $0 \leq c < n$ such that $c - a \equiv -k(mod\ n)$. Then we have that $r^{b-a} \in C_k$ and $r^{c-a} \in C_k$. We therefore have that the entries indexed by $(r^a s, r^b s)$ and $(r^a s, r^c s)$ are both 1 in A_k. By the uniqueness of b and c these are the only nonzero entries in the row of A_k indexed by $r^a s$. Notice that $b \equiv a + k(mod\ n)$ and $c \equiv a - k(mod\ n)$, so $b - c \equiv 2k(mod\ n)$. As k is odd and n is divisible by 4 we then have that b and c must differ by a value not divisible by 4. Namely the difference between b and c is 2ℓ for some odd integer ℓ. This tells us in the $r^a s$ row of A_k, there is an even number of 0's between the 1 in the entry indexed by $r^a s, r^b s$ and the entry indexed by $r^a s, r^c s$. All of these computations have been done for an arbitrary odd $1 \leq a < n$. Therefore, we have that every row of A_k in the C_{m+2}, C_{m+1} block has exactly two entries that are 1, which have an even number of 0's between them.

We now consider the product $a_{m+2,m+1}d_{m+1,m+1}$. To get the (x, y) entry of $a_{m+2,m+1}d_{m+1,m+1}$ we take the dot product of the x^{th} row of $a_{m+2,m+1}$ with the transpose of the y^{th} column of $d_{m+1,m+1}$. We found that the x^{th} row of $a_{m+2,m+1}$ has exactly two nonzero entries, which are both 1 and they have an even number of 0's between them. The y^{th} column of $d_{m+1,m+1}$ is a vector with entries that alternate between 1 and -1. Direct observation gives that entries in the y^{th} column of $d_{m+1,m+1}$ that have an even number of entries between them have different values. Then in the dot product of the x^{th} row of $a_{m+2,m+1}$ and the y^{th} column of $d_{m+1,m+1}$, one of the nonzero entries of the x^{th} row of $a_{m+2,m+1}$ will be multiplied by 1 and the other will be multiplied by -1. Then the dot product of the x^{th} row of $a_{m+2,m+1}$ and the transpose of the y^{th} column of $d_{m+1,m+1}$ is $(1)(1) + (1)(-1) = 0$. Hence, the (x, y) entry of $a_{m+2,m+1}d_{m+1,m+1}$ is 0. Since we chose this entry arbitrary we have that every entry of $a_{m+2,m+1}d_{m+1,m+1}$ is 0. Hence, $a_{m+2,m+1}d_{m+1,m+1} = 0$.

To this point we have considered $a_{t,m+1}d_{m+1,m+1}$ for every t except $t = m+1$ and found in all these cases that $a_{t,m+1}d_{m+1,m+1} = 0$. We now look at $a_{m+1,m+1}d_{m+1,m+1}$. This is the product of the C_{m+1}, C_{m+1} block of A_k with the C_{m+1}, C_{m+1} block of $\sum_{i=0}^{n/4}(-1)^i E^*_{m+1}A_{2i}E^*_{m+1}$. Let D be the matrix in Equation (5.2). We shall show that $D^k d_{m+1,m+1} = d_{m+1,m+1}$ for even k and $D^k d_{m+1,m+1} = -d_{m+1,m+1}$ for odd k. Recall that

$$d_{m+1,m+1} = \begin{bmatrix} 1 & -1 & 1 & \cdots & -1 \\ -1 & 1 & -1 & \cdots & 1 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ -1 & 1 & -1 & \cdots & 1 \end{bmatrix}.$$

Each row is the negative of the previous one. Now then looking at $D^k d_{m+1,m+1}$ as we know that $D^k = (1\ 2\ \cdots\ m)^k \cdot I_m$, where the action is permuting the rows of the matrix, we have that $D^k d_{m+1,m+1}$ is the same as $(1\ 2\ \cdots m)^k d_{m+1,m+1}$. If we act by $(1\ 2\ \cdots m)$ an even number of times we will get $d_{m+1,m+1}$ back since every other row of $d_{m+1,m+1}$ is equal. If we act by $(1\ 2\ \cdots m)$ an odd number of times we get $-d_{m+1,m+1}$ as each row is the negative of the row below it. So as claimed $D^k d_{m+1,m+1} = d_{m+1,m+1}$ if k is even and $D^k d_{m+1,m+1} = -d_{m+1,m+1}$ if k is odd.

By Lemma 5.17 we know that the C_{m+1}, C_{m+1} block of A_k is D^0 if $k = 0$, $D^{m/2}$ if $k = m$ and $D^{k/2} + D^{-k/2}$ if $0 < k < m$. Thus we have

$$a_{m+1,m+1}d_{m+1,m+1} = \begin{cases} d_{m+1,m+1} & \text{if } k = 0, \\ d_{m+1,m+1} & \text{if m/2 is even and } k = m/2, \\ -d_{m+1,m+1} & \text{if m/2 is odd and } k = m/2, \\ 2d_{m+1,m+1} & \text{if k is even and } 0 < k < m/2, \\ -2d_{m+1,m+1} & \text{if k is odd and } 0 < k < m/2. \end{cases}$$

As we have shown this is the only nonzero block in $A^k \sum_{i=0}^{n/4}(-1)^i E^*_{m+1}A_{2i}E^*_{m+1}$ and $\sum_{i=0}^{n/4}(-1)^i E^*_{m+1}A_{2i}E^*_{m+1}$ equals $d_{m+1,m+1}$ in this block and has all 0's outside of this block we have

$$A^k \sum_{i=0}^{n/4}(-1)^i E^*_{m+1}A_{2i}E^*_{m+1} = \begin{cases} \sum_{i=0}^{n/4}(-1)^i E^*_{m+1}A_{2i}E^*_{m+1} & \text{if k=0, or k=m/2 is even,} \\ -\sum_{i=0}^{n/4}(-1)^i E^*_{m+1}A_{2i}E^*_{m+1} & \text{if m/2 is odd and k=m/2,} \\ 2\sum_{i=0}^{n/4}(-1)^i E^*_{m+1}A_{2i}E^*_{m+1} & \text{if k is even and } 0 < k < m/2, \\ -2\sum_{i=0}^{n/4}(-1)^i E^*_{m+1}A_{2i}E^*_{m+1} & \text{if k is odd and } 0 < k < m/2. \end{cases}$$

In all cases we have that this is in J_{m+1}. Thus, J_{m+1} is closed under multiplication on the left. Notice that in D_{2n} the conjugacy classes are closed under inversion of elements. That is $x \in C_i$ if and only if $x^{-1} \in C_i$. The (x, y) entry of A_i is 1 if and only if $yx^{-1} \in C_i$, which is true if and only if $(yx^{-1})^{-1} \in C_i$. This is true if and only if $xy^{-1} \in C_i$, which is true if and only if the (y, x) entry of A_i is 1. This tells us that the A_i matrices are symmetric. Clearly the E^*_i matrices are symmetric. We then have that

$$\left(\sum_{i=0}^{n/4}(-1)^i E^*_{m+1}A_{2i}E^*_{m+1}\right)^t = \sum_{i=0}^{n/4}(-1)^i (E^*_{m+1}A_{2i}E^*_{m+1})^t = \sum_{i=0}^{n/4}(-1)^i (E^*_{m+1})^T (A_{2i})^t (E^*_{m+1})^t$$

$$= \sum_{i=0}^{n/4} (-1)^i E^*_{m+1} A_{2i} E^*_{m+1}.$$

As every element of J_{m+1} is a scalar multiply of $\sum_{i=0}^{n/4}(-1)^i E^*_{m+1} A_{2i} E^*_{m+1}$ we thus conclude that J_{m+1} is closed under transposition. Then by Lemma 4.11 we have that J_{m+1} is a two-sided ideal of $T(G)$. The fact that it is one dimensional means it must be irreducible.

The proof that J_{m+2} is a one dimensional irreducible two-sided ideal of $T(G)$ is essentially the same as that for J_{m+1} replacing E^*_{m+1} by E^*_{m+2} everywhere. This is largely because of Lemma 5.16. This then concludes the proof. $\square$

Appendix A. Semisimple Algebras

Definition A.1. Let R be a ring with unity. Let $x \in R$. We say that x is *nilpotent* if there exists some positive integer n such that $x^n = 0$. If I is a subset of R, we say that I is *nilpotent* if there exists some positive integer n such that $I^n = 0$,

Proposition A.2. *Suppose that x is a central element in a ring R such that $x^n = 0$ for some positive integer n. Then Rx is a nilpotent 2-sided ideal of R with $(Rx)^n = 0$.*

Proof. Let $s_i \in R$ for some $1 \leq i \leq n$. Then we have that $(s_1x)(s_2x)\cdots(s_nx) = (s_1s_2\cdots s_n)x^n = 0$, where the first equality holds as x is central. Therefore, $(xR)^n = 0$. □

Definition A.3. A commutative ring R is said to be *Artinian* if there is no infinite decreasing chain of ideals in R. That is whenever

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

is a decreasing chain of ideals of R, then there is a positive integer m such that $I_k = I_m$ for all $k \geq m$.

Definition A.4. Let R be a ring and M be a left $R-$module. Then $M \neq 0$ is *simple* if it has no nontrivial, proper submodules.

Definition A.5. Let R be a ring. We define the *Jacobson Radical* of R, denoted by $\mathcal{J}(R)$, to be the intersection of all maximal left (or equivalently right) ideals of R.

Note that a simple left R-module M is of the form $M \cong R/I$ for a maximal left ideal I of R. That is I is the annihilator of M. So $\mathcal{J}(R)$ is the intersection of all the annihilators of simple left $R-$modules. So we have that $\mathcal{J}(R)$ is a $2-$sided ideal of R since the annihilators of left modules are 2-sided ideals.

Proposition A.6. *If I is a nilpotent left ideal of R, then $I \subseteq \mathcal{J}(R)$.*

Proof. Since I is nilpotent, there exists some positive integer m such that $I^m = 0$. Let U be a simple left $R-$module. If $IU \neq 0$, then by the irreducibility of U, we have $U = IU$. Repeating this process to we get

$$U = IU = I^2U = \cdots = I^mU = 0.$$

This contradicts U being simple. Thus, $IU = 0$. Since U was an arbitrary simple module, $I \subseteq \mathcal{J}(R)$. □

Lemma A.7 (Nakayama's Lemma)**.** *Let V be a finitely generated nonzero $R-$module. Then*

$$\mathcal{J}(R)V \subsetneq V.$$

Proof. Assume V' is a maximal submodule of V. Such a maximal submodule exists by the fact that V is a nonzero, finitely generated module and a Zorn's Lemma argument. So V/V' is a simple module. So $\mathcal{J}(R)(V/V') = 0$, which implies that $\mathcal{J}(V) \subseteq V' \subsetneq V$. □

Theorem A.8. *Let A be a left Artinian ring. Then $\mathcal{J}(A)$ is the maximal nilpotent left ideal of A.*

Proof. Let $\mathcal{J} = \mathcal{J}(A)$. Consider the descending chain of left ideals of A

$$\mathcal{J} \supseteq \mathcal{J}^2 \supseteq \mathcal{J}^3 \supseteq \cdots$$

Since A is Artinian, there exists some n such that $\mathcal{J}^n = \mathcal{J}^m$ for all $m \geq n$. Since A is Artinian, A is Noetherian. This implies that $\mathcal{J}(A)$ is finitely generated. Then by Nakayama's Lemma, $\mathcal{J}^n = 0$. So $\mathcal{J}$ is nilpotent. Then by Proposition A.6, $\mathcal{J}$ is the maximal nilpotent left ideal. □

Definition A.9. Let R be a ring and M be a left $R-$module. Then we say M is *semisimple* if it is the direct sum of simple left modules. A ring R is *semisimple* if it is semisimple as a module over itself.

Theorem A.10 ([10], Theorem 15.3)**.** *The following are equivalent for a left $R-$module M.*

1. *M is semisimple*

2. *M is a sum of simple modules.*

3. *Every submodule of M is a direct summand of M.*

Theorem A.11 ([32], Theorem A.12)**.** *Let A be a left (or right) Artinian ring. Then A is semisimple if and only if $\mathcal{J}(A) = 0$.*

Proof. Suppose that A is semisimple. Then there exists a left ideal I of A such that $A = I \oplus \mathcal{J}(A)$. If $\mathcal{J}(A) \neq 0$, then I is contained in a maximal left ideal M. However, as $I \subseteq M$ and $\mathcal{J}(A) \subseteq M$, we have that $M = R$ by maximality of $\mathcal{J}(A)$. This contradicts M being simple. Thus, $\mathcal{J}(A) = 0$.

Conversely, suppose that $\mathcal{J}(A) = 0$. Let L_1 be a minimal left ideal of A. As $\mathcal{J}(A) = 0$, there exists a maximal left ideal $M_1 \neq 0$ which does not contain L_1. Note, $L_1 + M_1 = A$, by the maximality of M_1. By the minimality of L_1, we have that $L_1 \cap M_1 = 0$. Thus, $A = L_1 \oplus M_1$. Since $M_1 \neq 0$, it must contain a minimal left idea L_2. We then have, that there exists a maximal left ideal M_2 which does not contain L_2 and

$$A = L_1 \oplus L_2 \oplus (M_1 \cap M_2).$$

Repeating this process, we may define L_k and M_k recursively as long as $\cap_{j=1}^{k-1} M_j \neq 0$. If $\cap_{j=1}^{k-1} M_j \neq 0$ for all k. Doing this we get an infinite descending chain of left ideals of A

$$M_1 \supsetneq M_1 \cap M_2 \supsetneq M_1 \cap M_2 \cap M_3 \supsetneq .$$

This contradicts A being Artinian. Thus, $A = L_1 \oplus L_2 \oplus \cdots \oplus L_k$ for some k. Thus, A is semisimple. $\square$

Corollary A.12. *Let F be a field. Let A be a semisimple F-algebra and let B be a finite-dimensional central F-subalgebra of A. Then B is semisimple.*

Proof. Suppose that B is not semisimple. Since B is a finite-dimensional F-algebra, B is Artinian. Then it must be that $\mathcal{J}(B) \neq 0$. So there exists some $x \in \mathcal{J}(B)$ that is nonzero. Thus, x is a nilpotent element of A. By Proposition A.2, Ax is a nonzero nilpotent ideal of A. So by Proposition A.6, $Ax \subseteq \mathcal{J}(A) \neq 0$. This however contradicts Theorem A.11. $\square$

Definition A.13. An element ϵ of a ring R is *idempotent* if $\epsilon^2 = \epsilon$. A pair of central idempotents δ, ϵ is *orthogonal* if $\delta\epsilon = 0$. A central idempotent is *primitive* if it cannot be expressed as a sum of two nonzero orthogonal central idempotents. If the sum of a set of orthogonal idempotents is 1, we say that set of idempotents is *complete*.

Appendix B. Unanswered Questions

- What is the dimension of $T(S_n)$ and $T_0(S_n)$ for any n where we use the group association scheme to create $T(S_n)$?
- What is the complete Wedderburn decomposition for the Terwilliger algebra for the group association scheme of D_{2n} for even n?
- Where do the one dimensional Wedderburn components of the Terwilliger algebra for the group association scheme of D_{2n} for odd n come from?
- What impact does the base point have on the Terwilliger algebra for a group when looking at the group association scheme? What about for other association schemes?
- Can the Wedderburn component V(Theorem 4.12) for a Terwilliger algebra resulting from an orbit Schur ring be generalized beyond orbit Schur rings?
- Can the Wedderburn component Z(Theorems 5.10,5.14) for a Terwilliger algebra resulting from the group association scheme for a dihedral group be generalized beyond dihedral groups.
- Is the association scheme in Proposition 5.7 always commutative?
- For which commutative Schur rings over a group G is the association scheme created thin?
- For which commutative Schur rings over a group G is the association scheme created a $P-$polynomial scheme?
- For which commutative Schur rings over a group G is the association scheme created a $Q-$polynomial scheme?
- Let G be a group and H be a subgroup. When partitioning G be the orbits under conjugation by elements of H, which resulting association schemes are $P-$polynomial?

Appendix C. Terwilliger Algebra Dimension Tables for Group Association Schemes

Group	Dimension T	Dimension T_0	Dimension T_0^*	character degrees of T
$G_{1,1}$	1	1	1	1
$G_{2,1}$	4	4	4	2
$G_{3,1}$	9	9	9	3
$G_{4,1}$	16	16	16	4
$G_{4,2}$	16	16	16	4
$G_{5,1}$	25	25	25	5
$G_{6,1}$	11	11	11	1, 1, 3
$G_{6,2}$	36	36	36	6
$G_{7,1}$	49	49	49	7
$G_{8,1}$	64	64	64	8
$G_{8,2}$	64	64	64	8
$G_{8,3}$	28	28	28	1, 1, 1, 5
$G_{8,4}$	28	28	28	1, 1, 1, 5
$G_{8,5}$	64	64	64	8
$G_{9,1}$	81	81	81	9
$G_{9,2}$	81	81	81	9
$G_{10,1}$	22	22	22	1, 1, 2, 4
$G_{10,2}$	100	100	100	10
$G_{11,1}$	121	121	121	11
$G_{12,1}$	44	44	44	2, 2, 6
$G_{12,2}$	144	144	144	12
$G_{12,3}$	19	19	19	1, 1, 1, 4
$G_{12,4}$	44	44	44	2, 2, 6
$G_{12,5}$	144	144	144	12
$G_{13,1}$	169	169	169	13
$G_{14,1}$	37	37	37	1, 1, 1, 3, 5
$G_{14,2}$	196	196	196	14
$G_{15,1}$	225	225	225	15

Table C.1: Terwilliger algebra dimension table orders 1-15

Group	Dimension T	Dimension T_0	Dimension T_0^*	character degrees of T
$G_{16,1}$	256	256	256	16
$G_{16,2}$	256	256	256	16
$G_{16,3}$	112	112	112	2, 2, 2, 10
$G_{16,4}$	112	112	112	2, 2, 2, 10
$G_{16,5}$	256	256	256	16
$G_{16,6}$	112	112	112	2, 2, 2, 10
$G_{16,7}$	64	64	64	1, 1, 2, 3, 7
$G_{16,8}$	64	64	64	1, 1, 2, 3, 7
$G_{16,9}$	64	64	64	1, 1, 2, 3, 7
$G_{16,10}$	256	256	256	16
$G_{16,11}$	112	112	112	2, 2, 2, 10
$G_{16,12}$	112	112	112	2, 2, 2, 10
$G_{16,13}$	112	112	112	2, 2, 2, 10
$G_{16,14}$	256	256	256	16
$G_{17,1}$	289	289	289	17
$G_{18,1}$	56	56	56	1, 1, 1, 1, 4, 6
$G_{18,2}$	324	324	324	18
$G_{18,3}$	99	99	99	3, 3, 9
$G_{18,4}$	56	56	56	1, 1, 1, 1, 4, 6
$G_{18,5}$	324	324	324	18
$G_{19,1}$	361	361	361	19
$G_{20,1}$	88	88	88	2, 2, 4, 8
$G_{20,2}$	400	400	400	20
$G_{20,3}$	29	29	29	1, 1, 1, 1, 5
$G_{20,4}$	88	88	88	2, 2, 4, 8
$G_{20,5}$	400	400	400	20
$G_{21,1}$	37	35	35	1, 1, 1, 1, 2, 2, 5
$G_{21,2}$	441	441	441	21
$G_{22,1}$	79	79	79	1, 1, 1, 1, 1, 5, 7
$G_{22,2}$	484	484	484	22
$G_{23,1}$	529	529	529	23
$G_{24,1}$	176	176	176	4, 4, 12
$G_{24,2}$	576	576	576	24
$G_{24,3}$	75	75	73	1, 5, 7
$G_{24,4}$	116	116	116	1, 1, 2, 2, 5, 9
$G_{24,5}$	176	176	176	4, 4, 12
$G_{24,6}$	116	116	116	1, 1, 2, 2, 5, 9
$G_{24,7}$	176	176	176	4, 4, 12
$G_{24,8}$	116	116	116	1, 1, 2, 2, 5, 9
$G_{24,9}$	576	576	576	24
$G_{24,10}$	252	252	252	3, 3, 3, 15

Table C.2: Terwilliger algebra dimension table orders 16-24

Group	Dimension T	Dimension T_0	character degrees of T
$G_{24,11}$	252	252	3, 3, 3, 15
$G_{24,12}$	43	42	1, 2, 2, 3, 5
$G_{24,13}$	76	76	2, 2, 2, 8
$G_{24,14}$	176	176	4, 4, 12
$G_{24,15}$	576	576	24
$G_{25,1}$	625	625	25
$G_{25,2}$	625	625	25
$G_{26,1}$	106	106	1, 1, 1, 1, 1, 1, 6, 8
$G_{26,2}$	676	676	26
$G_{27,1}$	729	729	27
$G_{27,2}$	729	729	27
$G_{27,3}$	137	137	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 11
$G_{27,4}$	137	137	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 11
$G_{27,5}$	729	729	27
$G_{28,1}$	148	148	2, 2, 2, 6, 10
$G_{28,2}$	784	784	28
$G_{28,3}$	148	148	2, 2, 2, 6, 10
$G_{28,4}$	784	784	28
$G_{29,1}$	841	841	29
$G_{30,1}$	275	275	5, 5, 15
$G_{30,2}$	198	198	3, 3, 6, 12
$G_{30,3}$	137	137	1, 1, 1, 1, 1, 1, 1, 7, 9
$G_{30,4}$	900	900	30
$G_{31,1}$	961	961	31
$G_{32,1}$	1024	1024	32
$G_{32,2}$	448	448	4, 4, 4, 20
$G_{32,3}$	1024	1024	32
$G_{32,4}$	448	448	4, 4, 4, 20
$G_{32,5}$	448	448	4, 4, 4, 20
$G_{32,6}$	142	142	1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 11
$G_{32,7}$	142	142	1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 11
$G_{32,8}$	142	142	1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 11
$G_{32,9}$	256	256	2, 2, 4, 6 14
$G_{32,10}$	256	256	2, 2, 4, 6 14
$G_{32,11}$	256	256	2, 2, 4, 6 14
$G_{32,12}$	448	448	4, 4, 4, 20
$G_{32,13}$	256	256	2, 2, 4, 6 14
$G_{32,14}$	256	256	2, 2, 4, 6 14
$G_{32,15}$	256	256	2, 2, 4, 6 14
$G_{32,16}$	1024	1024	32

Table C.3: Terwilliger algebra dimension table order 24-32

Group	Dimension T	Dimension T_0	character degrees of T
$G_{32,17}$	448	448	4, 4, 4, 20
$G_{32,18}$	184	184	1, 1, 2, 2, 2, 7, 11
$G_{32,19}$	184	184	1, 1, 2, 2, 2, 7, 11
$G_{32,20}$	184	184	1, 1, 2, 2, 2, 7, 11
$G_{32,21}$	1024	1024	32
$G_{32,22}$	448	448	4, 4, 4, 20
$G_{32,23}$	448	448	4, 4, 4, 20
$G_{32,24}$	448	448	4, 4, 4, 20
$G_{32,25}$	448	448	4, 4, 4, 20
$G_{32,26}$	448	448	4, 4, 4, 20
$G_{32,27}$	256	256	2, 2, 2, 2, 2, 2, 6, 14
$G_{32,28}$	256	256	2, 2, 2, 2, 2, 2, 6, 14
$G_{32,29}$	256	256	2, 2, 2, 2, 2, 2, 6, 14
$G_{32,30}$	256	256	2, 2, 2, 2, 2, 2, 6, 14
$G_{32,31}$	256	256	2, 2, 2, 2, 2, 2, 6, 14
$G_{32,32}$	256	256	2, 2, 2, 2, 2, 2, 6, 14
$G_{32,33}$	256	256	2, 2, 2, 2, 2, 2, 6, 14
$G_{32,34}$	256	256	2, 2, 2, 2, 2, 2, 6, 14
$G_{32,35}$	256	256	2, 2, 2, 2, 2, 2, 6, 14
$G_{32,36}$	1024	1024	32
$G_{32,37}$	448	448	4, 4, 4, 20
$G_{32,38}$	448	448	4, 4, 4, 20
$G_{32,39}$	256	256	2, 2, 4, 6, 14
$G_{32,40}$	256	256	2, 2, 4, 6, 14
$G_{32,41}$	256	256	2, 2, 4, 6, 14
$G_{32,42}$	256	256	2, 2, 4, 6, 14
$G_{32,43}$	142	142	1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 11
$G_{32,44}$	142	142	1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 11
$G_{32,45}$	1024	1024	32
$G_{32,46}$	448	448	4, 4, 4, 20
$G_{32,47}$	448	448	4, 4, 4, 20
$G_{32,48}$	448	448	4, 4, 4, 20
$G_{32,49}$	304	304	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 17
$G_{32,50}$	304	304	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 17
$G_{32,51}$	1024	1024	32
$G_{33,1}$	1089	1089	33
$G_{34,1}$	172	172	1, 1, 1, 1, 1, 1, 1, 1, 8, 10
$G_{34,2}$	1156	1156	34
$G_{35,1}$	1225	1225	35

Table C.4: Terwilliger algebra dimension table 32-35

Group	Dimension T	Dimension T_0	character degrees of T
$G_{36,1}$	224	224	2, 2, 2, 2, 8, 12
$G_{36,2}$	1296	1296	36
$G_{36,3}$	171	171	3, 3, 3, 12
$G_{36,4}$	224	224	2, 2, 2, 2, 8, 12
$G_{36,5}$	1296	1296	36
$G_{36,6}$	396	396	6, 6, 18
$G_{36,7}$	224	224	2, 2, 2, 2, 8, 12
$G_{36,8}$	1296	1296	36
$G_{36,9}$	48	48	1, 1, 1, 1, 1, 1, 1, 1, 2, 6
$G_{36,10}$	121	121	1, 1, 1, 1, 3, 3, 3, 3, 9
$G_{36,11}$	171	171	3, 3, 3, 12
$G_{36,12}$	396	396	6, 6, 18
$G_{36,13}$	224	224	2, 2, 2, 2, 8, 12
$G_{36,14}$	1296	1296	36
$G_{37,1}$	1369	1369	37
$G_{38,1}$	211	211	1, 1, 1, 1, 1, 1, 1, 1, 1, 9, 11
$G_{38,2}$	1444	1444	38
$G_{39,1}$	89	85	1, 1, 1, 1, 1, 1, 1, 1, 4, 4, 7
$G_{39,2}$	1521	1521	39
$G_{40,1}$	352	352	4, 4, 8, 16
$G_{40,2}$	1600	1600	40
$G_{40,3}$	116	116	2, 2, 2, 2, 10
$G_{40,4}$	268	268	1, 1, 2, 2, 2, 2, 9, 13
$G_{40,5}$	352	352	4, 4, 8, 16
$G_{40,6}$	268	268	1, 1, 2, 2, 2, 2, 9, 13
$G_{40,7}$	352	352	4, 4, 8, 16
$G_{40,8}$	268	268	1, 1, 2, 2, 2, 2, 9, 13
$G_{40,9}$	1600	1600	40
$G_{40,10}$	700	700	5, 5, 5, 25
$G_{40,11}$	700	700	5, 5, 5, 25
$G_{40,12}$	116	116	2, 2, 2, 2, 10
$G_{40,13}$	352	352	4, 4, 8, 16
$G_{40,14}$	1600	1600	40
$G_{41,1}$	1681	1681	41
$G_{42,1}$	55	55	1, 1, 1, 1, 1, 1, 7
$G_{42,2}$	148	140	2, 2, 2, 2, 4, 4, 10
$G_{42,3}$	539	539	7, 7, 21
$G_{42,4}$	333	333	3, 3, 3, 9, 15
$G_{42,5}$	254	254	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 10, 12
$G_{42,6}$	1764	1764	42
$G_{43,1}$	1849	1849	43

Table C.5: Terwilliger algebra dimension table orders 36-43

Group	Dimension T	Dimension T_0	character degrees of T
$G_{44,1}$	316	316	2, 2, 2, 2, 2, 10, 14
$G_{44,2}$	1936	1936	44
$G_{44,3}$	316	316	2, 2, 2, 2, 2, 10, 14
$G_{44,4}$	1936	1926	44
$G_{45,1}$	2025	2025	45
$G_{45,2}$	2025	2025	45
$G_{46,1}$	301	301	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 11, 13
$G_{46,2}$	2116	2116	48
$G_{47,1}$	2209	2209	47
$G_{48,1}$	704	704	8, 8, 24
$G_{48,2}$	2304	2304	48
$G_{48,3}$	124	119	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 5, 5, 8
$G_{48,4}$	704	704	8, 8, 24
$G_{48,5}$	464	464	2, 2, 4, 4, 10, 18
$G_{48,6}$	368	368	1, 1, 2, 2, 2, 2, 2, 11, 15
$G_{48,7}$	368	368	1, 1, 2, 2, 2, 2, 2, 11, 15
$G_{48,8}$	368	368	1, 1, 2, 2, 2, 2, 2, 11, 15
$G_{48,9}$	704	704	8, 8, 24
$G_{48,10}$	464	464	2, 2, 4, 4, 10, 18
$G_{48,11}$	704	704	8, 8, 24
$G_{48,12}$	464	464	2, 2, 4, 4, 10, 18
$G_{48,13}$	464	464	2, 2, 4, 4, 10, 18
$G_{48,14}$	464	464	2, 2, 4, 4, 10, 18
$G_{48,15}$	233	233	1, 1, 1, 3, 3, 4, 4, 6, 12
$G_{48,16}$	233	233	1, 1, 1, 3, 3, 4, 4, 6, 12
$G_{48,17}$	233	233	1, 1, 1, 3, 3, 4, 4, 6, 12
$G_{48,18}$	233	233	1, 1, 1, 3, 3, 4, 4, 6, 12
$G_{48,19}$	464	464	2, 2, 4, 4, 10, 18
$G_{48,20}$	2304	2304	48
$G_{48,21}$	1008	1008	6, 6, 6, 30
$G_{48,22}$	1008	1008	6, 6, 6, 30
$G_{48,23}$	2304	2304	48
$G_{48,24}$	1008	1008	6, 6, 6, 30
$G_{48,25}$	576	576	3, 3, 6, 9, 21
$G_{48,26}$	576	576	3, 3, 6, 9, 21
$G_{48,27}$	576	576	3, 3, 6, 9, 21
$G_{48,28}$	130	130	1, 2, 3, 4, 6, 8
$G_{48,29}$	130	130	1, 2, 3, 4, 6, 8
$G_{48,30}$	172	168	2, 4, 4, 6, 10
$G_{48,31}$	304	304	4, 4, 4, 16
$G_{48,32}$	300	300	2, 10, 14

Table C.6: Terwilliger algebra dimension table orders 44-48

Group	Dimension T	Dimension T_0	character degrees of T
$G_{48,33}$	300	300	2, 10, 14
$G_{48,34}$	464	464	2, 2, 4, 4, 10, 18
$G_{48,35}$	704	704	8, 8, 24
$G_{48,36}$	464	464	2, 2, 4, 4, 10, 18
$G_{48,37}$	464	464	2, 2, 4, 4, 10, 18
$G_{48,38}$	308	308	1, 1, 1, 1, 1, 1, 3, 3, 3, 5, 5, 15
$G_{48,39}$	308	308	1, 1, 1, 1, 1, 1, 3, 3, 3, 5, 5, 15
$G_{48,40}$	308	308	1, 1, 1, 1, 1, 1, 3, 3, 3, 5, 5, 15
$G_{48,41}$	308	308	1, 1, 1, 1, 1, 1, 3, 3, 3, 5, 5, 15
$G_{48,42}$	704	704	8, 8, 24
$G_{48,43}$	464	464	2, 2, 4, 4, 10, 18
$G_{48,44}$	2304	2304	48
$G_{48,45}$	1008	1008	6, 6, 6, 30
$G_{48,46}$	1008	1008	6, 6, 6, 30
$G_{48,47}$	1008	1008	6, 6, 6, 30
$G_{48,48}$	172	168	2, 4, 4, 6, 10
$G_{48,49}$	304	304	4, 4, 4, 16
$G_{48,50}$	124	119	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 5, 5, 8
$G_{48,51}$	704	704	8, 8, 24
$G_{48,52}$	2304	2304	48
$G_{49,1}$	2401	2401	49
$G_{49,2}$	2401	2401	49
$G_{50,1}$	352	352	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 12, 14
$G_{50,2}$	2500	2500	50
$G_{50,3}$	550	550	5, 5, 10, 20
$G_{50,4}$	352	352	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 12, 14
$G_{50,5}$	2500	2500	50
$G_{51,1}$	2601	2601	51
$G_{52,1}$	424	424	2, 2, 2, 2, 2, 2, 12, 16
$G_{52,2}$	2704	2704	52
$G_{52,3}$	85	76	1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3, 7
$G_{52,4}$	424	424	2, 2, 2, 2, 2, 2, 12, 16
$G_{52,5}$	2704	2704	52
$G_{53,1}$	2809	2809	53

Table C.7: Terwilliger algebra dimension table order 48-53

Group	Dimension T	Dimension T_0	character degrees of T
$G_{54,1}$	407	407	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 13, 15
$G_{54,2}$	2916	2916	54
$G_{54,3}$	504	504	3, 3, 3, 3, 12, 18
$G_{54,4}$	891	891	9, 9, 27
$G_{54,5}$	127	127	1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 10
$G_{54,6}$	127	127	1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 10
$G_{54,7}$	407	407	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 13, 15
$G_{54,8}$	216	208	4, 5, 5, 5, 5, 10
$G_{54,9}$	2916	2916	54
$G_{54,10}$	548	548	2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 22
$G_{54,11}$	548	548	2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 22
$G_{54,12}$	891	891	9, 9, 27
$G_{54,13}$	504	504	3, 3, 3, 3, 12, 18
$G_{54,14}$	407	407	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 13, 15
$G_{54,15}$	2916	2916	54
$G_{55,1}$	73	63	1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 7
$G_{55,2}$	3025	3025	55
$G_{56,1}$	592	592	4, 4, 4, 12, 20
$G_{56,2}$	3136	3136	56
$G_{56,3}$	484	484	1, 1, 2, 2, 2, 2, 2, 2, 13, 17
$G_{56,4}$	592	592	4, 4, 4, 12, 20
$G_{56,5}$	484	484	1, 1, 2, 2, 2, 2, 2, 2, 13, 17
$G_{56,6}$	592	592	4, 4, 4, 12, 20
$G_{56,7}$	484	484	1, 1, 2, 2, 2, 2, 2, 2, 13, 17
$G_{56,8}$	3136	3136	56
$G_{56,9}$	1372	1372	7, 7, 7, 35
$G_{56,10}$	1372	1372	7, 7, 7, 35
$G_{56,11}$	71	71	1, 1, 1, 1, 1, 1, 1, 8
$G_{56,12}$	592	592	4, 4, 4, 12, 20
$G_{56,13}$	3136	3136	56
$G_{57,1}$	165	159	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 6, 6, 9
$G_{57,2}$	3249	3249	57
$G_{58,1}$	466	466	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 14, 16
$G_{58,2}$	3364	3364	58
$G_{59,1}$	3481	3481	59

Table C.8: Terwilliger algebra dimension table order 54-59

Group	Dimension T	Dimension T_0	character degrees of T
$G_{60,1}$	1100	1100	10, 10, 30
$G_{60,2}$	792	792	6, 6, 12, 24
$G_{60,3}$	548	548	2, 2, 2, 2, 2, 2, 2, 14, 18
$G_{60,4}$	3600	3600	60
$G_{60,5}$	73	71	1, 2, 3, 3, 5, 5
$G_{60,6}$	261	261	3, 3, 3, 3, 15
$G_{60,7}$	143	132	1, 1, 1, 1, 1, 1, 2, 3, 3, 3, 5, 9
$G_{60,8}$	242	242	1, 1, 1, 1, 2, 2, 3, 3, 4, 4, 6, 12
$G_{60,9}$	475	475	5, 5, 5, 20
$G_{60,10}$	792	792	6, 6, 12, 24
$G_{60,11}$	1100	1100	10, 10, 30
$G_{60,12}$	548	548	2, 2, 2, 2, 2, 2, 2, 14, 18
$G_{60,13}$	3600	3600	60
$G_{61,1}$	3721	3721	61
$G_{62,1}$	529	529	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 15, 17
$G_{62,2}$	3844	3844	62
$G_{63,1}$	333	315	3, 3, 3, 3, 6, 6, 15
$G_{63,2}$	3969	3969	63
$G_{63,3}$	333	315	3, 3, 3, 3, 6, 6, 15
$G_{63,4}$	3969	3969	63

Table C.9: Terwilliger algebra dimension table order 60-64

Group	Dimension T	Dimension T_0
$G_{21,1}$	37	35
$G_{24,12}$	43	42
$G_{39,1}$	89	85
$G_{42,2}$	148	140
$G_{48,3}$	124	119
$G_{48,30}$	172	168
$G_{48,48}$	172	168
$G_{48,50}$	124	119
$G_{52,3}$	85	76
$G_{54,8}$	216	208
$G_{55,1}$	73	63
$G_{57,1}$	165	159
$G_{60,5}$	73	71
$G_{60,7}$	143	132
$G_{63,1}$	333	315
$G_{63,3}$	333	315
$G_{64,32}$	226	223
$G_{64,33}$	226	223
$G_{64,34}$	226	223
$G_{64,35}$	226	223
$G_{64,36}$	226	223
$G_{64,37}$	226	223
$G_{64,41}$	376	361
$G_{64,42}$	376	361
$G_{64,43}$	376	361
$G_{64,46}$	376	361
$G_{64,134}$	376	367
$G_{64,135}$	376	367
$G_{64,136}$	376	367
$G_{64,137}$	376	367
$G_{64,138}$	376	367
$G_{64,139}$	376	367
$G_{64,152}$	376	361
$G_{64,153}$	376	361
$G_{64,154}$	376	361
$G_{64,190}$	376	361
$G_{64,191}$	376	361
$G_{64,257}$	592	583
$G_{64,258}$	592	583
$G_{64,260}$	592	583

Table C.10: Non triply regular groups of order 1-64

Group	Dimension T	Dimension of Components
D_6	11	1, 1, 9
D_8	28	1, 1, 1, 25
D_{10}	22	1, 1, 4, 16
D_{12}	44	4, 4, 36
D_{14}	37	1, 1, 1, 9, 25
D_{16}	64	1, 1, 4, 9, 49
D_{18}	56	1, 1, 1, 1, 16, 36
D_{20}	88	4, 4, 16, 64
D_{22}	79	1, 1, 1, 1, 1, 25, 49
D_{24}	116	1, 1, 4, 4, 25, 81
D_{26}	106	1, 1, 1, 1, 1, 1, 36, 64
D_{28}	148	4, 4, 4, 36, 100
D_{30}	137	1, 1, 1, 1, 1, 1, 1, 49, 81
D_{32}	184	1, 1, 4, 4, 4, 49, 121
D_{34}	172	1, 1, 1, 1, 1, 1, 1, 1, 64, 100
D_{36}	224	4, 4, 4, 4, 64, 144
D_{38}	211	1, 1, 1, 1, 1, 1, 1, 1, 1, 81, 121
D_{40}	268	1, 1, 4, 4, 4, 4, 81, 169
D_{42}	254	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 100, 144
D_{44}	216	4, 4, 4, 4, 4, 100, 196
D_{46}	301	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 121, 169
D_{48}	368	1, 1, 4, 4, 4, 4, 4, 121, 225
D_{50}	352	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 144, 196
D_{52}	424	4, 4, 4, 4, 4, 4, 144, 256
D_{54}	407	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 169, 225
D_{56}	484	1, 1, 4, 4, 4, 4, 4, 4, 169, 289
D_{58}	466	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 196, 256
D_{60}	548	4, 4, 4, 4, 4, 4, 4, 196, 324
D_{62}	529	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 225, 289
D_{64}	616	1, 1, 4, 4, 4, 4, 4, 4, 4, 225, 361

Table C.11: Terwilliger algebra dimension table of Dihedral Groups of orders 6 to 64

Appendix D. Magma Code

The following is the magma code that was used in computing the tables in Appendix C. It is broken into several parts with each part being explained to some degree. One thing worth noting is that in each separate section of the code, the same thing may be done. For example the $A-$matrices for the association scheme are created in many of these pieces of code. In using several pieces of the provided code at the same time, redundant parts like this can be removed, so they are only done once.

The first piece of code is magma code that creates the conjugacy classes for a given group. This code creates them and stores them as a sequence of sets.

```
orbsc:=function(g);
cc:=Classes(g);
sc:=[Class(g,x[3]):x in cc];
ga:=GroupAlgebra(RationalField(),g);
psga:=PowerSet(ga);
UU:=[&+(psga!x):x in sc ];
return UU;
end function;
```

The next piece of code is magma code that creates the Terwilliger algebra with base point e and finds the dimension for the group association for a given group that is called g in the code. It also creates the Wedderburn components for the Terwilliger algebra, which are given in a sequence called I and gives the dimension of each of them. Changing the first line of the code to the group that you want to look at will allow this code to be used to find the explained information for any group that magma is capable of handling.

```
//Create Terwilliger Algebra
//Forms the associate classes from the conjugacy classes.
g:=SmallGroup(6,1);
o:=orbsc(g);
q:=#g/2;
ssc:=o;
ga:=Parent(ssc[1]);g:=Group(ga);
F:=CyclotomicField(Floor(#g));
FC:=QuadraticField(Floor(q));
F1:=Compositum(F,FC);
r:=CosetImage(g,sub< g| >);
m:=GModule(r,F1);
ag:=ActionGenerators(m);
mg:=MatrixGroup< #g, F1|ag >;
mr:=MatrixRing(F1,#g);
so:=[Support(x):x in o];
eg:= [ ];
for i:=1 to #so do
    so2:=SetToSequence(so[i]);
    eg:= eg cat so2;
end for;
    //Creates the A-matrices using the ordering of the associate classes in eg.
A:=[ ];
for i:=1 to #so do
    D:=ZeroMatrix(F1,#g,#g);
    for j:=1 to #eg do
        for k:=1 to #eg do
```

```
            if eg[k]*eg[j]∧-1 in so[i] then
                D[j][k] :=1;
            end if;
        end for;
    end for;
    A:= A cat [D];
end for;
    //Creates the E* matrices using the A-matrices
ES:=[ ];
for i:=1 to #o do
    m:=mr!0;
    for j:=1 to #g do
        m[j,j]:= A[i][1,j];
    end for;
    ES:=ES cat [m];
end for;
    //Creates the Terwilliger algebra and finds its dimension.
TC:=sub<mr|A,ES>;
Dimension(TC);
    //Creates the E-matrices and uses them to find the Wedderburn components.
sumc:=sub<mr|TC>;
EC:=CentralIdempotents(sumc);
I:=[ideal<sumc|EC[i]> : i in [1.. #EC]];
for i:=1 to #EC do
    i,Dimension(I[i]);
end for;
```

The next section of magma code is code that will create $T_0(g)$ for a given group g. It also computes the dimension of $T_0(g)$.

```
//Create T0 basis
//Forms the associate classes from the conjugacy classes.
g:=SmallGroup(6,1);
o:=orbsc(g);
q:=#g/2;
ssc:=o;
ga:=Parent(ssc[1]);g:=Group(ga);
F:=CyclotomicField(Floor(#g));
FC:=QuadraticField(Floor(q));
F1:=Compositum(F,FC);
r:=CosetImage(g,sub< g| >);
m:=GModule(r,F1);
ag:=ActionGenerators(m);
mg:=MatrixGroup< #g, F1|ag >;
mr:=MatrixRing(F1,#g);
so:=[Support(x):x in o];
eg:= [ ];
for i:=1 to #so do
    so2:=SetToSequence(so[i]);
    eg:= eg cat so2;
end for;
    //Creates the A-matrices using the ordering of the associate classes in eg.
A:=[ ];
for i:=1 to #so do
    D:=ZeroMatrix(F1,#g,#g);
```

```
    for j:=1 to #eg do
        for k:=1 to #eg do
            if eg[k]*eg[j]∧-1 in so[i] then
                D[j][k] :=1;
            end if;
        end for;
    end for;
    A:= A cat [D];
end for;
    //Creates the E* matrices using the A-matrices
ES:=[ ];
for i:=1 to #o do
    m:=mr!0;
    for j:=1 to #g do
        m[j,j]:= A[i][1,j];
    end for;
    ES:=ES cat [m];
end for;
    //Computes a basis for T0. Then T0 is created and its dimension is found.
t0:=[ ];
for i:= 1 to #ES do;
    for j:= 1 to #A do;
        for k:= 1 to #ES do;
            t1:=ES[i]*A[j]*ES[k];
            if t1 ne 0 then
                t0:= t0 cat [t1];
            end if;
```

```
        end for;
    end for;
end for;
MR:=KMatrixSpace(F1,#g,#g);
T0:=sub< MR|t0 >;
Dimension(T0);
```

This next piece of code creates $T_0^*(g)$. It then computes its dimension.

```
//Create T0* basis //Forms the associate classes from the conjugacy classes.
//Forms the associate classes from the conjugacy classes. g:=SmallGroup(6,1);
o:=orbsc(g);
q:=#g/2;
ssc:=o;
ga:=Parent(ssc[1]);g:=Group(ga);
F:=CyclotomicField(Floor(#g));
FC:=QuadraticField(Floor(q));
F1:=Compositum(F,FC);
r:=CosetImage(g,sub< g| >);
m:=GModule(r,F1);
ag:=ActionGenerators(m);
mg:=MatrixGroup< #g, F1|ag >;
mr:=MatrixRing(F1,#g);
so:=[Support(x):x in o];
eg:= [ ];
for i:=1 to #so do
    so2:=SetToSequence(so[i]);
    eg:= eg cat so2;
```

```
end for;
    //Creates the A-matrices using the ordering of the associate classes in eg.
A:=[ ];
for i:=1 to #so do
    D:=ZeroMatrix(F1,#g,#g);
    for j:=1 to #eg do
        for k:=1 to #eg do
            if eg[k]*eg[j]∧-1 in so[i] then
                D[j][k] :=1;
            end if;
        end for;
    end for;
    A:= A cat [D];
end for;
    //Creates the Bose-Mesner Algebra from the A-matrices. It then finds the E-matrices.
BM:=sub< mr|A >;
E:=CentralIdempotents(BM);
    //Creates the A* matrices.
AS:=[ ];
for i:=1 to #o do
    m:=mr!0;
    for j:=1 to #g do
        m[j,j]:= #g*E[i][1,j];
    end for;
    AS:=AS cat [m];
end for;
    //Creates a basis for T_0^*(g) as well as creating T_0^*(g). Its dimension is then calculated.
```

```
t0S:=[ ];
for i:= 1 to #AS do;
    for j:= 1 to #E do;
        for k:= 1 to #AS do;
            t1:=E[i]*AS[j]*E[k];
            t0S:= t0S cat [t1];
        end for;
    end for;
end for;
MR:=KMatrixSpace(F1,#g,#g);
T0S:=sub< MR|t0S >;
Dimension(T0S);
```

By using the three pieces of code above, one can find all of the values that are included in the first 7 tables in Appendix C.

The next section of code is code that has been specifically written for the group association scheme for dihedral groups. This code orders the conjugacy classes as well as the elements that appear in the conjugacy classes.

```
//Dihedral Group code
//Create the conjugacy classes with the desired ordering of elements.
n:=8;
g:=DihedralGroup(n);
q:=#g/2;
F:=CyclotomicField(Floor(#g));
FC:=QuadraticField(Floor(q));
F1:=Compositum(F,FC);
r:=CosetImage(g,sub< g| >);
m:=GModule(r,F1);
```

```
ag:=ActionGenerators(m);
mg:=MatrixGroup< #g, F1|ag >;
mr:=MatrixRing(F1,#g);
so:=[Id(g)];
eg:=[Id(g)];
so2:={ };
so3:={ };
if IsOdd(n) then
    m:=Floor((n-1)/2);
    for i:=1 to m do
        so:=so cat [g.1^i,g.1^-i];
        eg:=eg cat [g.1^i];
        eg:=eg cat [g.1^-i];
    end for;
    for i:= 0 to n-1 do
        so2:= so2 join g.1^i*g.2;
        eg:= eg cat [g.1^i*g.2];
    end for;
    so:= so cat [so2];
else
    m:=Floor(n/2);
    for i:=1 to m-1 do
        so:=so cat [g.1^i,g.1^-i];
        eg:=eg cat [g.1^i];
        eg:=eg cat [g.1^-i];
    end for;
    so:=so cat [g.1^m];
```

```
    eg:=eg cat [g.1∧m];
    for i:=0 to m-1 do
        so2:= so2 join g.1∧(2*i)*g.2;
        eg:= eg cat [g.1∧(2*i)*g.2];
    end for;
    so:= so cat [so2];
    for i:=0 to m-1 do
        so3:= so3 join g.1∧(2*i+1)*g.2;
        eg:= eg cat [g.1∧(2*i+1)*g.2];
    end for;
    so:= so cat [so3];
end if;
A:=[ ];
for i:=1 to #so do
    D:=ZeroMatrix(F1,#g,#g);
    for j:=1 to #eg do
        for k:=1 to #eg do
            if eg[k]*eg[j]∧-1 in so[i] then
                D[j][k] :=1;
            end if;
        end for;
    end for;
    A:= A cat [D];
end for;
ES:=[ ];
for i:=1 to #so do
    m:=mr!0;
```

```
    for j:=1 to #g do
        m[j,j]:= A[i][1,j];
    end for;
    ES:=ES cat [m];
end for;
TC:=sub< mr|A, ES >;
Dimension(TC);
sumc:=sub< mr|TC >;
EC:=CentralIdempotents(sumc);
I:=[ideal< sumc|EC[i] > : i in [1.. #EC]];
for i:=1 to #EC do
    i,Dimension(I[i]);
end for;
```

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