2016-07-01

Insight into Student Conceptions of Proof

Steven Daniel Lauzon
Brigham Young University

Insight into Student Conceptions of Proof

Steven Daniel Lauzon

A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Arts

Steve Williams, Chair
Doug Corey
Steven Jones
Denise Halverson

Department of Mathematics Education

Brigham Young University

July 2016

ABSTRACT

Insight into Student Conceptions of Proof

Steven Daniel Lauzon
Department of Mathematics Education, BYU
Master of Arts

The emphasis of undergraduate mathematics content is centered around abstract reasoning and proof, whereas students' pre-college mathematical experiences typically give them limited exposure to these concepts. Not surprisingly, many students struggle to make the transition to undergraduate mathematics in their first course on mathematical proof, known as a bridge course. In the process of this study, eight students of varied backgrounds were interviewed before during and after their bridge course at BYU. By combining the proof scheme frameworks of Harel and Sowder (1998) and Ko and Knuth (2009), I analyzed and categorized students' initial proof schemes, observed their development throughout the semester, and their proof schemes upon completing the bridge course. It was found that the proof schemes used by the students improved only in avoiding empirical proofs after the initial interviews. Several instances of ritual proof schemes used to generate adequate proofs were found, calling into question the goals of the bridge course. Additionally, it was found that students' proof understanding, production, and appreciation may not necessarily coincide with one another, calling into question this hypothesis from Harel and Sowder (1998).

Keywords: Proof, proof schemes, bridge course, undergraduate mathematics

ACKNOWLEDGEMENTS

I'd like to thank the many people who helped and supported me through this process. First, I'd like to thank my advisor, Steve Williams, for all the many hours he spent, for the efforts he made to help me with this study, and for being an excellent mentor to me during my time at BYU. I'd like to thank my wife, Natascha, who was patient, loving, and encouraging in spite of all of the late nights spent working on this thesis. I'd also like to thank my family and friends who supported me, and gave me advice throughout this process.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1: INTRODUCTION

Rationale

As students progress through K-12 mathematics, proof takes an increasingly prominent role in the curriculum. However, students are given few opportunity to prove in these early grades and it is not until their early undergraduate studies in mathematics that proof becomes a fundamental skill indispensable to their development as competent mathematics students (Knuth, 2000; Weber & Alcock, 2004). The word *proof* alone is often enough to make many students nervous because of prior experiences they have had. Regardless of their background, all students who study mathematics as an undergraduate must learn to become competent in proof. As a result, it is worthwhile to investigate how students make the transition to formal proof from their K-12 experiences to their undergraduate studies.

One important justification for the study of proof is that although teachers are being instructed to introduce students to proof in the early elementary grades (American Mathematical Society [AMS], 2001; Common Core State Standards Initiative [CCSSI], 2010; Mathematical Association of America [MAA], 2004; National Council of Teachers of Mathematics [NCTM], 2000), students' experiences as a result of these introductions are few and rarely complete, leaving students with a difficult transition into undergraduate mathematics (Healy & Hoyles, 1998; Knuth, 2000; Moore, 1994; J. Selden, Benkhalti, & Selden, 2014; Tall, 1992). Knuth (2002; 2015) and Stylianides (2009) have suggested that students who are engaged in discovery and inquiry-based activities throughout their mathematics education are better prepared to be introduced to formal proof. In addition, Moore (1994) and J. Selden, Benkhalti and Selden (2014) have found that students that cannot perform well in proof-related tasks (e.g. analyzing examples or making intuitive arguments) struggle to produce formal proofs as early undergraduate students. However, no attempt has been made to follow students with varying backgrounds with proof-related tasks from their K-12 experience into their undergraduate studies (Fawcett, 1938; Harel & Sowder, 1998; Knuth, 2000; A. Stylianides, 2007).

A second justification is that students' difficulties with proof in their early undergraduate studies is well documented. Difficulties arise at many levels, from mathematical discovery in the early grades, to the understanding and production of all types of proofs including direct, indirect and inductive proofs as well as proofs by example/counterexample (Alcock & Weber, 2005; Harel & Sowder, 1998; Knuth, 2000, 2002; Peled & Zaslavsky, 1997; Reid & Knipping, 2010; A. Selden & Selden, 2003; A. Stylianides, Stylianides, & Philippou, 2004; Weber, 2002b; Weber & Alcock, 2004; Wilson, 2014). As a result, Harel, A. Selden and Selden (2006) have called for further research in novice and expert mathematical behavior as it relates to proof. Since proof is the defining characteristic of pure mathematics at the undergraduate level and beyond and informs and influences applied mathematics and related disciplines (Knuth, 2002; Schoenfeld, 1994), it is important to know what affects students' success in proving.

It is worth acknowledging that the word *proof* in mathematics connected to a variety of meanings and connotations. In general, proof is seen as the use of deductive reasoning to establish new results from previously established or accepted results (Mariotti, 2006; A. Stylianides, 2007). A proof is thus a collection of mathematical arguments, constructed from axioms and previously known results for the purpose of supporting or refuting a mathematical claim. Some scholars see proof as much more than a systematic way of presenting arguments in order to arrive at a particular conclusion; it is a way that students can externally communicate both the characteristics and the existence of mathematical objects (Hanna, 1989; Knipping, 2010). Proof is also a vital part of the axiomatic structure of mathematics, which provides a way of systematizing known results and organizing them in a hierarchical deductive sequence involving the choice of suitable starting points as axioms (Bell, 1979).

Considering that proof is a vital part of doing mathematics, it is important to understand what it means to be successful at proving. A proof can only be considered correct within a *reference theory,* that is, it is context-dependent (Mariotti, 2006; A. Stylianides, 2007). For mathematics students, this context includes their level of education, age and classroom norms and the reference theory dictates three things (Mariotti, 2006; A. Stylianides, 2007). First, a proof must be constructed from results known within the classroom community. Second, it must employ forms of communication that are known to or within the conceptual reach of students. Third, it must be communicated using forms of argumentation and representation that are accepted or within the conceptual reach of the students. With this perspective, proof-related activities can be made available to all students regardless of their age or mathematical conceptions. As a community, mathematical learners, educators and researchers must come to understand that the formality of mathematical proof the way it is typically introduced to early undergraduates is not what characterizes proof; instead, proof is deeply connected to mathematical exploration, discovery, making conjectures and logical connections (Knuth, 2015).

According to Harel and Sowder (1998), there are three aspects that must be present in order to have a well-rounded conception of proof: Proof Understanding, Production and Appreciation (PUPA). Proof understanding concerns a student's ability to analyze a proof to determine whether or not it is complete. Proof production concerns a student's ability to write a valid proof or disproof. Proof appreciation concerns the capacity of a student to see the need for and the role of proof within mathematics both generally and in specific situations. In this study, I will focus on each of these three aspects of proof.

Harel and Sowder (1998) also introduce what they call a proof schemes framework to describe students' understanding of proof. They suggest that their framework could be used to map the development of college mathematics students' PUPA over a period of time. A proof scheme is "what constitutes ascertaining and persuading for that person" (p. 244). The proof schemes framework provides a way of categorizing and analyzing the lens through which a student looks at proof and it encompasses ways of understanding (evaluating the correctness of) proofs, producing proofs, and appreciating proof (seeing its significance or necessity within a context or in general). By better understanding how a students' initial proof schemes affects their ability to understand, produce and appreciate proofs, we will be able to understand what initial proof schemes are the most constructive conceptions to build upon and which ones are more difficult to build upon. Little research has been done to consider the impact that the students' initial proof schemes have on their ability to form fully developed, mature proof schemes as undergraduate mathematics students.

The ideal setting in which to investigate the development of students' proof conceptions is in a student's first course on proofs, typically known as a *bridge course*. A bridge course is intended to help students to learn to communicate using the language unique to mathematics, reason logically, and construct proofs. Although such a course is not offered by every mathematics department, such courses have been established by many colleges and universities for several years to facilitate the transition to advanced mathematics courses (Moore, 1994). Students' experiences with formal proof prior to the bridge course are typically limited to two-column proofs in geometry and proofs involving the precise definition of the limit. Knuth (2000) and Selden et al. (2014) have suggested that this type of introduction to proof may not be the best foundation for students to build upon. Because I plan to study proof development in a bridge course, the results from this study should shed some light on how to prepare students to have a constructive conception of proof and reasoning as they make the transition to undergraduate mathematics.

In order to investigate the effect a students' initial conceptions have on his or her ability to understand and perform proof tasks in their bridge course, we must establish what skills are expected of them in the learning objectives of the course. As mentioned previously, a fully developed conception of proof includes the ability to determine the validity of a proof, construct proofs and learn to appreciate the role and necessity of proof within mathematics (Harel & Sowder, 1998). The proof schemes framework outlines several different conceptions students have when they are given proof tasks. Some of these proof schemes are productive, in that they have the potential to lead to formal proof. Other proof schemes are deficient in general. It is expected that students will use different proof schemes as they build upon their initial conceptions of proof in the bridge course. However, categorizing their thinking, and in particular their proof schemes, will allow us to see how previous proof schemes affect the proof schemes that they eventually develop. By tracing back from the more mature final (end of course) conceptions of proof, we will better see what initial proof schemes were most constructive and how those initial proof schemes influenced the students throughout the course.

My intent in this study is to identify the ways that undergraduate mathematics students understand, produce and appreciate (see the value of) proofs upon beginning their bridge course and how these initial conceptions affect their learning of proof throughout that course. In exploring this, it is beneficial to consider what factors affect the development of a students' PUPA. Although it is not possible to trace their understanding of proof and argumentation from their K-12 experience to the bridge course without conducting a longitudinal study, insights into this problem were able to be addressed during the students' bridge course. In order to accomplish this, I investigated the connections between a student's initial proof schemes and their more developed proof schemes throughout the course and upon completion of the course. Thus, by determining a student's proof schemes and proficiency at proving before, during and after the course, I was able to investigate the connections between their initial proof schemes and their more mature proof schemes within the context of the bridge course.

CHAPTER 2: LITERATURE REVIEW AND THEORETICAL FRAMEWORK

Literature Review

In this section, I discuss what we know about undergraduate students' experiences with proof, both as they make the initial transition to proof and afterwards. This will be followed by a more thorough description of PUPA as it relates to Harel and Sowder's (1998) proof schemes framework.

Making the Transition to Formal Proof

Once students begin to study mathematics in college, they suddenly must make a transition away from mathematics that can be learned through mastering procedures and memorizing equations to a subject that is characterized by abstraction, proof and argumentation. Harel & Sowder (1998) outline some of the disconnections between the way proofs are used in advanced mathematics and the way they are presented in grades K-12. For example, mathematicians do not very often prove theorems of well-known, elegantly stated propositions, while this is quite common in most geometry curriculums. A key component of constructing proofs for mathematicians is making and testing conjectures, with which students often have little experience in K-12, in spite of the recommendations in the *Principles and Standards* (Ellis, Lockwood, Dogan, Williams, & Knuth, 2011). In addition, students struggle making connections with many different types of proofs. However, K-12 students deal primarily with proofs that are geometric and in two columns, which is a practice that is not adopted within undergraduate mathematics. Although these proofs may serve as a mathematical foundation for learning to write proofs formally, its use as a means for understanding proof has been criticized by several authors (Harel & Sowder, 1998; Manaster, 1998; Shaughnessy & Burger, 1985).

As discussed in chapter 1, students studying mathematics are required to take a bridge course in their first or second year of studies in mathematics where they are introduced to forms of argumentation and mathematical proof. When they reach this point, students may or may not have begun to develop their capacity to comprehend proof or mathematical arguments. As a result, they often struggle to understand their importance in the development of their mathematical understanding of the topics they study and, in essence, they see them as being a pointless "hoop" to jump through (Alcock, Hodds, Roy, & Inglis, 2015; Harel, 1998; Harel & Sowder, 1998). Harel argues that one of the reasons students feel this way is that some of the results are considered by the students to be obvious or meaningless. For example, in a course on real analysis, students struggle to see the meaning of such rigorous proofs to show such elementary properties as $\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$. One other reason that is offered by Weber (2002a) is that some proofs merely convince that a result is true and shed absolutely no greater light on the theorem itself. According to Tall (1992), students are better able to understand constructive proofs and arguments than results that are obtained indirectly. It would seem clear that without being exposed to the power and enrichment of knowledge and understanding that can come from mathematical proof, we cannot expect our students to come to value the role of proof in their mathematical learning and thus have a complete understanding of proof (Harel & Sowder, 1998).

Students' difficulties with proof are evident in all forms of argumentation, such as direct proof, proof by contradiction, contrapositive and proof by example/counterexample. In addition, students are not taught how to reason in general and, as a result, they develop a variety of deficient methods of reasoning (Reid & Knipping, 2010). However, students that use deficient forms of reasoning often know these forms of reasoning to be deficient, but they use them because they have no other means of arguing mathematically (Healy & Hoyles, 1998). Thus, due to their limited exposure to proof and forms of reasoning they do not have a mathematical foundation to build upon when they begin to formally learn about proofs in the bridge course.

Undergraduate mathematics students retain the same difficulties K-12 students have with understanding the purpose of proof and constructing valid arguments; however, in their deeper study of mathematics, additional challenges with proof become apparent. Undergraduate students struggle to acquire optimal strategies and acquire their own deficient coping mechanisms, such as using the converse of a statement as proof, accepting empirical arguments as general proof and refuting a single counterexample of a universal statement as acceptable for refutation of the universal statement (Galbraith, 1981; Healy & Hoyles, 2000; Knuth, 2002; Schoenfeld, 1978). Knuth (2002) investigated how advanced mathematics students accept or refute arguments and found that many students recognized that beginning a proof by assuming the conclusion to be true was unusual. Yet even then, they would be convinced to a greater extent by the form of the remainder of the argument than by the suspicion they felt by the initial statement. Moore (1994) argued that undergraduate students are rarely able to produce examples to support a claim, do not know how to begin proofs or communicate using proper mathematical discourse and are often unable to transfer between intuitive arguments and mathematical proof. Galbraith (1981) as well as Barkai, Tsamir, Tirosh and Dreyfus (2002) investigated how students deal with counterexamples and examples in proofs. Both authors found that most students would provide at least one example or counterexample. Barkai et al. further found that a large number of students felt that using more examples would strengthen their argument or that any number of counterexamples or examples are sufficient to prove or disprove. In addition, many students thought that a single counterexample is invalid if the statement is "usually true". All in all, students struggle with the deductive aspect of proof as well as the consequences of universal and existential quantifiers.

Proof Production, Understanding and Appreciation

As discussed previously, according to Harel and Sowder's (1998) proof schemes framework, students have a working conception of proof if they are able to understand, produce and appreciate proofs. In order for students to be successful at proving, they must first know what a correct argument should look like. Proof understanding (evaluating the correctness of a proof) is fundamental to a student's proof scheme. Several authors have found that students struggle to understand and evaluate proofs and argue that there is a correlation between students' abilities to understand proofs and produce proofs (Healy & Hoyles, 2000; Knuth, 2002; Recio & Godino, 2001; Sowder & Harel, 2003). Teachers can give their students the opportunity to engage in the derivations and constructions of mathematical ideas, yet proof in K-12 and in early undergraduate courses prior to the bridge course (such as linear algebra, calculus and differential equations), even when proof is part of the curriculum, it is typically presented as an obstacle that needs to be overcome before the class can apply a given result (Harel, 1998). Evaluation of the teacher's proof rarely, if ever, takes place. Students need to have a way to determine which arguments are correct and which are incorrect in order to begin to critically evaluate logical arguments and construct an intuitive understanding of techniques of proof. Thus, students are usually led to become convinced by the form of the argument more than the statements contained within them and struggle to understand the logical quantifiers and their role in the proof, especially as they first begin to prove.

Proof production is the most intuitive aspect of mastering proof since it consists of the student performing the act of proving. Weber and Alcock (2004) outline three major issues related to students' being unable to produce proofs. First, students struggle to understand and identify relevant definitions. Without understanding formal definitions, it is impossible for students to be able to construct arguments from them. Second, students can often accurately state a definition or a concept without actually understanding the definition. This may happen when students' misconceptions are not challenged (Erlwanger, 1973). Finally, students often produce intuitive arguments based on the mental images they construct for the proof, ignoring the formal construction of an argument (Moore, 1994; Tall & Mejia-Ramos, 2006). This can occur whether or not the student's understanding of the concept is correct. These obstacles can make it extremely difficult for students to be able to produce valid proofs.

Students who have difficulty recognizing the role and purpose of proof within mathematics have been found to struggle to prove. According to the PUPA framework, each aspect of proof is deeply connected accordingly, weak proof appreciation often coexists with weak proof understanding and proof production (Harel & Sowder, 1998). Several studies have shown that early undergraduate students often see proof as an arbitrary, disconnected string of statements that are only important because of the implications they lead to (Healy & Hoyles, 2000; Knipping, 2010; Knuth, 2002; Martin & Harel, 1989). These studies considered primary, secondary and early undergraduate students and found relatively small differences in the students in each category's ability to identify correct proofs and incorrect proofs. Some difficulties were rooted in the students' inability to understand relevant definitions, differentiate between trivial statements and key components to proofs, restate arguments in their own words and identify the specific data supporting a given claim (J. Selden et al., 2014). In addition, according to Mejia-Ramos, Fuller, Weber, Rhoads and Samkroff (2012), undergraduate students have difficulty transferring proof techniques between arguments since they struggle to break down proofs into modules and intuitive arguments. That being said, there is little work that is done, if any, to help students evaluate mathematical arguments in K-12 or in their early undergraduate studies (Doerr & English, 2003; Knuth, 2015). Thus early undergraduate students are often over-exposed to proof production and under-exposed to proof appreciation and proof understanding.

In summary, according to the research, undergraduate students struggle with all three aspects of PUPA, both in K-12 and in their undergraduate studies. Although students have limited experiences with proof understanding, it informs proof production and is an important skill of its own right in mathematics. Proof appreciation is deeply connected to both proof understanding and production, revealing its nature in both the production of proof and its evaluation.

Theoretical Framework

Defining Proof

In order to provide clarity for the use of the notion of proof in this study, I will revisit the definition of proof given in chapter one, where proof was defined as the use of deductive reasoning used to establish new results from previously established known results (Mariotti, 2006; A. Stylianides, 2007). While researchers have not reached a common definition of proof due to its wide range of applications and subjectivity, it is nonetheless important to discuss the meaning of proof to clarify what constitutes a proof for the sake of this study. According to Bell (1979) proof is a way of establishing new results from known axioms and previously known results. Additionally, proof is a way of situating results within a deductive or axiomatic system. In addition, a proof can only be considered correct within a *reference theory,* or in other words, it is context-dependent (Mariotti, 2006; A. Stylianides, 2007). That is, the sophistication of an argument is dependent on the knowledge that is accessible to the student within the classroom community. In the context of a mathematics course, the validity of a proof must reflect the level of the students' understanding of and instruction in mathematics. In Reid & Knipping (2010) the fluency of the student in proof discourse is compared to a tool-box. If a result or definition is known to the students, they are able to use those things to construct arguments. In other words, those things are in the student's tool box. Any analysis of a student's work cannot be separated from the accepted knowledge that the student has access to within the classroom community. In fact, Inglis and Mejia-Ramos (2009) and Knuth (2002) argue that the process of students being convinced by an argument or proof is usually connected to an appeal to what an authority sees as being an acceptable argument, whether the authority is a teacher, a textbook or otherwise. Even students that have a mastery of the axiomatic systems in question and are able to differentiate between the arguments in the proof and the structure of the proof often accept the arguments because they are similar to the arguments used by the authority in question.

For the purposes of this study, we will consider three conditions that have to be met in order for a proof to be considered mathematically rigorous in the given reference theory (Mariotti, 2006; A. Stylianides, 2007). A reference theory in the context of this study always refers to a classroom of students, and never to an individual. First, the proof must use statements that are accepted by the classroom community. This refers specifically to the facts contained within the student's communal (classroom) tool-box. No reference is made here to the structure of the arguments. Second, it must employ forms of reasoning that are valid and known to the classroom community. For example, students cannot be expected to use proofs by contradiction without first being taught the form of such proofs. Third, it must be communicated in a form that is known to or within the conceptual reach of the students. In my study these three criteria will be used to determine whether or not a proof can be considered rigorous or complete within the reference theory in question.

The definition of proof provided above is used because it is consistent with the frameworks used in this study, particularly those of Harel and Sowder (1998) and Ko and Knuth (2009). One reason for this is that it allows for proofs to be categorized according to the level of mathematical knowledge appropriate for students at a given level in their education. In addition, this definition allows the researcher to not only categorize a student's proof scheme, but also assess whether a proof is considered to be complete for a student in the bridge course. Finally, since the bridge course has the students mostly use basic definitions and theorems from set theory and elementary number theory, the axiomatic structure of proof is emphasized throughout the course and the study.

Proof Schemes

In order to gain insight into students' development in their understanding of proof, it is beneficial to be able to categorize their methods of proof and modes of argumentation. Harel & Sowder (1998) constructed a framework to be able to categorize a student's proof understanding, production and appreciation (PUPA). The benefit of such a framework in this study is four-fold. First, it is a method of mapping the students' cognitive schemes of mathematical proof. This will facilitate the process of coming to understand a student's thoughts and beliefs about proof at the time of the interview. Second, it allows the researchers to be able to document the progress college mathematics students make in building their conceptions of mathematical proof in the bridge course. Since we can construct the students' proof scheme at the time of an interview, we can see how it develops throughout the semester. Third, it gives a context to better understand how to help students using a given proof scheme reach a level where they can understand, construct and appreciate proofs. Fourth, understanding students' proof schemes can lead to improved pedagogical practices. By better understanding the students' strengths and weaknesses in proof understanding, production and appreciation, teachers can either diversify these aspects in their teaching or focus on particular aspects of PUPA where the students' understanding is deficient.

Proof understanding and production come as two natural indicators of one's capacity to prove, as discussed previously in the literature review, yet proof appreciation is deeply connected to and informed by both of these aspects of proof. For example, if a student thinks that mathematics is simply a list of facts that are accumulated over time, then they may be far more likely to resort to an appeal to a teacher or textbook to convince themselves that a result is true. (Harel & Sowder, 1998). If a student believes that all mathematics can be constructed from elementary notions (axioms) they are more likely to use a type of axiomatic proof scheme.

According to Harel & Sowder (1998), "a person's proof scheme consists of what constitutes ascertaining and persuading for that person" (p. 244). A student's proof scheme tells a lot about what a student believes is the purpose of doing mathematics. There are three main categories into which proof schemes are categorized. First, *external conviction proof schemes* are proof schemes that are often based in a premature over-emphasis on formality and not on the logical construction of arguments. Students may develop these types of proof schemes when the teacher is the main source of knowledge and students are focused on the formal presentation of the arguments rather than their logical construction. Within this category of proof schemes, there are three subcategories. First, *ritual proof schemes* rely more heavily on the structure of the argument than on anything else. For example, a student using a ritual proof scheme is likely to reject a proof by counterexample because the arguments are void of any symbolic manipulation or deductive reasoning (Martin & Harel, 1989). Students using such a proof scheme want proofs to "look like" other proofs. The second subcategory of external proof schemes is *authoritarian proof schemes.* Students using such a proof scheme become convinced when they are given a proof by some form of authority, usually a teacher or a textbook. The third subcategory of external proof schemes is the *symbolic proof scheme*. Students using this proof scheme do not necessarily associate the symbols used in a proof with any specific meaning, nonetheless they see the symbols as being an essential element of proofs and that any proof given without a manipulation of symbols is invalid. In sum, external conviction proof schemes can be characterized by proofs that only convince the student by the form, appearance, or source of an argument, regardless of whether the arguments follow a logical flow. It is of interest to this study to understand how and why a student relies on an external conviction proof scheme and how a student builds this conception throughout the bridge course to gain a fuller understanding of proof and logical forms of reasoning.

The second main category of proof schemes is *empirical proof schemes* (Harel & Sowder, 1998). Students who use this proof scheme make conclusions and arguments based on physical facts and sensory experiences. Empirical proof schemes can either be inductive or perceptual. Inductive proof schemes are those proofs that rely exclusively on specific examples, which are meant to represent a general class of mathematical objects. Although not all aspects of inductive proof schemes are altogether wrong, students must eventually develop more complex ideas of proof and reasoning as they develop in their understanding of mathematics. Another type of empirical proof schemes is the perceptual proof scheme. Perceptual proof schemes are those proof schemes that are based on static physical or mental images. The major distinction here from the generic example proof scheme is that no changes are made to the mental/physical image of the mathematical object in an effort to represent an arbitrary object(Knuth & Elliott, 1998).

The third and final category of proof schemes is the *analytical proof scheme* (Harel & Sowder, 1998). Simply put, these are the proof schemes that are produced through logical deduction. There are two main subcategories of analytical proof schemes: *transformational proof schemes* and *axiomatic proof schemes*. Transformational proof schemes may be present in several forms, therefore; there are three subcategories of the transformational proof schemes. First, *internalized proof schemes* are those that make a connection between their experiences from working with the mathematical objects in question in order to use their informal conjectures as facts in order to logically deduce a given result. For example, consider a student given the problem: "Let $\overrightarrow{v_1}, \ldots, \vec{v}_m$ be vectors in $\mathbf{R}^n$. If m > n, are these vectors linearly independent?" The student using an internalized proof scheme would notice that there are more equations than unknowns (likely reifying from their experiences using operations for row reduction), and make a conclusion that it is necessary that the set of vectors be linearly dependent. The second subcategory of transformational proof schemes is the *interiorized proof scheme*. Implicit within this proof scheme is the act of recognizing a connection between their experiences from applied problems, as with the internalized proof scheme. The difference between these two schemes is that students using the interiorized proof scheme take the connection they intuitively made between their experiences and the problem at hand to the next level by formalizing it. That is, they will relate their experience from practice to the problem at hand and then use the accepted forms of reasoning within their reference theory to argue those things. For example, a student using this proof scheme would make the same observations as the student using the internalized proof scheme, except, he or she might formalize this by using a proof by contradiction, for example, by noting that the $(n+1)^{st}$ vector would have to depend on the previous ones if all of the previous vectors were independent. The third subcategory of transformational proof schemes is the *restrictive proof scheme*. This proof scheme is used by those who see a proof as being transformational, however with a restriction on the context of the conjecture, the generality of the justification, or the mode of justification. Therefore, there are three types of restrictive proof schemes: *contextual, generic* and *constructive.* A contextual proof scheme is a proof scheme where the student imagines a proof spatially (imagined or graphically), yet they place restrictions on the proof based on spatial restrictions within their concept image of the proof. For example, a person using such a proof scheme may struggle to imagine a geometric proof generalizing beyond the picture that they had drawn to represent the situation. Students who interpret conjectures in general terms but express their proofs in a particular context use a generic proof scheme. The student using this proof scheme may discuss their argument in general terms but their argument is based on a specific example. The constructive proof scheme is used by students that can only interpret their conjectures by an appeal to a construction of objects. Mere existence of objects is insufficient for them.

Axiomatic proof schemes, the second subcategory of analytical proof schemes, are used by those who recognize that mathematical justification is based on a set of definitions and axioms (Harel & Sowder, 1998). There are however, three different types of axiomatic proof schemes: *intuitive, structural and axiomatizing*. Intuitive proof schemes are used by students who are only able to use and understand axioms that correspond to their intuition or that are self-evident (such as $a = b \Rightarrow b = a$). Those using a structural proof scheme are able to see how conjectures relate to one another and to the axiomatic system. What distinguishes this proof scheme from the axiomatic-intuitive proof scheme is that here, the focus is on the structure of the statements and not on the axiomatic system. Therefore, students using this proof scheme build upon previously known axioms and previously proven results to obtain new results. Finally, the student using the axiomatizing proof scheme is able to see the implications of a given set of axioms. For example, they can see how certain results may differ as we consider different sets of axioms. Students using this proof scheme can recognize the need to prove further results after being introduced to a set of axioms and results.

Ultimately, a proof scheme refers to what convinces a person, and to what the person offers to convince others (Harel & Sowder, 1998). As we considered above, there are several different ways that a person can be convinced of a result and can provide justification to others. Harel and Sowder (1998) have suggested that monitoring the development of the proof schemes of several students as they progress through their bridge course is a worthwhile investigation to gain further insights into students' understanding of proof. However, this endeavor has not yet been attempted. By assessing students' proof schemes throughout the bridge course, we should be able to better understand the influence one's initial proof scheme has on them as they enrich and develop their understanding of mathematical proof in their bridge course.

Classifying Proofs

The proof schemes described above accomplish their purpose of allowing the researcher to understand and describe what students believe about proof and its role within mathematics as well as what convinces the student and what the student offers as a mathematical argument to convince others. However, it does not strictly tell the researcher if the student's proof is correct or not. Students' use of a particular proof scheme may be of some use in evaluating the correctness of a proof. Indeed, some always or almost always produce deficient proofs. But this is not always the case. For example, a student that uses an external conviction proof scheme may memorize a proof that they are expected to need to produce on a test and accurately reproduce it. A student may also use a proof scheme that is more developed and still be unsuccessful at proving. Thus, a student using a structural-axiomatic proof scheme may be unsuccessful at proving, simply because they do not fully understand the definitions and previous results connected to the proof they are attempting to prove (J. Selden et al., 2014; Weber & Alcock, 2004). Thus, the proofs produced by the students need to be separately evaluated (beyond the proof schemes involved) to assess the validity of the proof.

Ko and Knuth's (2009) framework provides a way of categorizing students' proof types and counterexample productions. There are seven types of proof production. First, "no response" refers to a proof that was left blank or that the student believed that the statement was true or false but provided no explanation to support the claim at all. Second, "restatement" refers to the student interpreting the statement in their own words, yet again providing no explanation as to why the statement would be true or false. Third, "counterexample" refers to a student attempting to disprove a statement that is true. Fourth, "empirical" encompasses any type of proof produced using an empirical proof scheme, as described in Harel and Sowder's (1998) framework. Fifth, "non-referential symbolic" refers to proofs that involve some relevant symbolic manipulations, but making logical errors and ultimately resulting in an incomplete proof. Sixth, "structural" refers to proofs where the student recognizes all relevant definitions and results are needed to construct a proof, but makes logical errors and does not provide a complete proof. Lastly, "complete" refers to a proof that is considered to be valid within the context of the bridge course.

Since counterexample production is a major part of proving in mathematics, Ko and Knuth (2009) also provide a framework with six categories of counterexample productions.

First, “no response” refers to a disproof that is left blank or that has no justification to support whether the statement is true or not. Second, “proof” refers to an attempted proof of a false statement. Third, “inadequate” refers to an attempted disproof where the student provided an example that either did not exist, was irrelevant or did not disprove the statement. Fourth, “justification” refers to a student providing a narrative to explain why they believe that the statement is false, but ultimately fail to provide a counterexample. Fifth, “incomplete” refers to a disproof where the student provided a correct example, but failing to justify why the example provided is sufficient to refute the claim. Sixth, “adequate” refers to a complete, well-justified counterexample.

By combining both of these frameworks, I will be able to assess a student’s proof scheme as well as how successful they are at proof production. If they are unsuccessful at proving, we will also be able to categorize their methods used that led them to providing an incomplete argument. This may help us see whether certain proof schemes often result in complete proof or in particular errors, which would inform math educators as to which conceptions of proof lead to success in proving.

Research Questions

The literature review and the theoretical framework provide the context necessary to state the research questions for this study:

1. What influence do students’ initial proof schemes have on the proof schemes they eventually develop in their bridge course?
2. What connections can be made between success in proving and students’ proof schemes?

CHAPTER 3: METHODOLOGY

Setting and Participants

The students in the study were selected from the bridge course offered at Brigham Young University, Math 290, *Fundamentals of Mathematics.* This course is designed as an introductory course in proof and mathematical logic. The topics covered in the course include set theory (set builder notation, Venn diagrams, DeMorgan's Laws and set operations such as union and intersection), logic (truth tables, quantifiers, negations and implications), proof techniques (direct proof, proof by contradiction and proof by contrapositive), relations (reflexive, symmetric and transitive relations and equivalence classes), functions (one-to-one and onto, inverse functions, bijective functions, permutations), mathematical induction (well-ordering principle, mathematical induction and the method of descent), cardinal numbers (numerical equivalence, countable and uncountable sets and the Schröder-Bernstein theorem) and elementary number theory (division algorithm, Euclid's algorithm, infinitude of primes and unique factorization). This course was chosen because, as an introductory course on proof, it proved to be a good opportunity to see how students are making the transition from their previous calculation-based mathematical background into undergraduate pure mathematics with an emphasis on proof.

All of the participants were chosen from the same section of Math 290 taught by a Mathematics faculty member who I have given the pseudonym Dr. Smith. This section was chosen because Dr. Smith has expressed interest in having this study carried out in her class, and as a result of her support of this project, I proved to have an opportunity for rich data collection that may not have been possible if I were to be a passive observer in the classroom. All students in Dr. Smith's class were given a pre-test and eight students were selected to be interviewed based on their varied use of proof schemes in the pre-test (only six of the eight students selected completed the course). The pre-test had six questions and the students were given twenty-five minutes to work on them. In both the pre-test and the interviews students were given one proof appreciation question, three proofs (including some by counterexample) and two proof analyses.

I collected each of the pre-tests completed by the students and categorized their uses of proof schemes throughout the tests. My goal was to find two students who relied heavily on external conviction proof schemes, two students who relied heavily on empirical proof schemes, two students who relied heavily on transformational proof schemes and two students who relied heavily on axiomatic proof schemes. When I found multiple students who were similar in their use of proof schemes, I selected the students who were the most descriptive in their explanations. Although the lines between each proof scheme were often not perfectly clear, I was able to select a group of students who thought in varied ways coming into the class and were comfortable sharing their thinking with me as the researcher.

Data Collection

In order to address the research questions outlined, there are two major characteristics of the students' proof understanding, production and appreciation that I focused on: One was the actual proof scheme and the amount of thought the student put into that proof scheme, and the other was the overall effectiveness of the student's proof scheme (which may be a hybrid of many different proof schemes as categorized by Harel and Sowder, 1998) in producing correct proofs. This former characteristic allowed me to characterize the student's proof schemes as being productive or deficient, with analytical proof schemes being considered productive and empirical and external conviction proof schemes as being deficient proof schemes. The latter characteristic listed above allowed me to characterize the actual proofs written down to be characterized as being productive deficient, with complete/adequate proofs being considered productive and all other proof types considered deficient. According to Harel and Sowder (1998), more productive proof schemes do not always produce superior proofs, however, some proof schemes have been found to typically create more powerful arguments than others. Thus, each proof will be categorized according to its type of proof construction or counterexample production (as categorized by Ko and Knuth, 2009), as well as by its proof scheme. This dual framework allowed me to compare and contrast performance on proof production problems with the proof schemes used.

The students selected were interviewed at the beginning of the semester, twice during the semester and after the final exam. Students' submitted class work was analyzed in order to gain greater clarity on the students' proof schemes. Students were given proofs to analyze and they were asked about their decision-making process on proofs that they previously submitted in class. They were also given opportunities in the interviews to produce proofs and counterexamples for problems that were similar to those dealt with by the students in Math 290 as well as new proofs that were less familiar to the students. Interviews varied from 20 to 40 minutes long, which allowed the students enough time to answer a question about the purpose of proof, construct three proofs and analyze one of their own proofs and another's proof. The questions that were selected had the purpose of challenging students and moving them beyond the routine proofs that are often encountered in the course. The questions were chosen to be of this type since it was not the purpose of this study to evaluate students' abilities for memorization or emulation. Problems such as these that forced them to reason mathematically were thought to highlight their proof schemes and show diversity between the students. Some routine questions were asked as well with the purpose of balancing the difficulty of the interview questions in order to prevent the students from being discouraged in the process of the interviews. Proof appreciation questions helped me see the connections between the students' beliefs about the role of proof and the connections of these beliefs to their ways of understanding and producing proofs.

Semi-structured interviews were selected as the main way for obtaining information about the students' proof schemes for several reasons. First, Sowder & Harel (2003) suggest themselves that written work is insufficient to fully categorize a student's proof scheme. Second, semi-structured interviews are a good way of observing what decisions, false starts and intuitive arguments a student makes in a problem solving context (Ginsberg, Kossan, Schwartz, & Swanson, 1981; Goldin, 1997; McGivney & DeFranco, 1995). Third, it also allows for greater consistency of results and analysis with other authors who have done research in proof, since semi-structured interviews are the most common way used in the literature regarding early proof (Harel & Sowder, 1998; Peled & Zaslavsky, 1997; A. Stylianides et al., 2004; Weber & Alcock, 2004).

The eight students were selected for interviewing based on their performance on the pre-test as discussed above. Because it was necessary to answering the research question that the incoming students used a variety of proof schemes, after hypothesizing their proof schemes from the pre-test, the hypothesis was confirmed through interviewing. It was never the case that a students' proof scheme in the interview was significantly different than the hypothesized proof scheme from the pre-test, thus it was not necessary to re-select some students for the study. The eight students selected were Albert, Ashley, Adam, Jason, Joe, Miles, Melanie and Trevor. Trevor and Ashley were the only students in the study that had taken a class which involved proof production beyond high school geometry. In both cases they took an introductory linear algebra course at BYU. The linear algebra course did not focus on mathematical logic or general proof techniques and as a result, these students were still considered to be beginners in proof, as far as this study in concerned and did not have a clear advantage on the other students.

Additional data for the study were collected from informal interactions with students and from classroom discussions. I attended the lectures and was an active participant in the classroom environment, being willing to talk to students and help them outside of class if requested. I took field notes from these interactions with students in order to gain a clearer image of the development of their mathematical thinking throughout the semester. I was willing to answer students' questions when approached and on occasion, I allowed students to discuss content with me outside of class. In doing this, I realized that the students do not usually have such a service in a regular Math 290 class at Brigham Young University, but my intent is to understand the development of students' conceptions of proof, not to study the effectiveness of the course delivery.

The final type of data that I collected is student's written work. In the Math 290 course, the students typically submitted three assignments per week. I copied these assignments before they were graded and reviewed them with the intent of conjecturing which proof schemes are typical for the students to use. By doing this, my intention was to understand when deficient proof types appear in their work, what proof schemes they use in analyzing the proof and whether they are able to identify the error in their proof without the problem being graded. I wanted to know if the associated conceptions were deeply held or if they were simply appearing because of a lack of effort or focus on the task. One of these proofs were selected for analysis in each subsequent interview.

Interview Questions

The problems presented to the students in the pre-test are found in Appendix A. Since it was not of interest to this study to assess the students' ability to recall relevant definitions, but rather to assess their proof schemes and effectiveness in proving, students were instructed to ask for relevant definitions, as they are required to use them. They were only given the definitions they asked for and they were written down on cards ahead of time according to the definitions given in their textbook. On occasion, some irrelevant definitions were requested and these were still provided to the student.

Data Analysis

Each interview was recorded, coded and the statements and actions made by the student were transcribed. I coded their statements according to the proof schemes being used during each point in the interview. If a proof scheme appears only once in the interview, I had to determine if that idea had a significant part in building the proof or counterexample. If, for example, it was only a temporary slip-up and the student quickly resorted to another strategy and never resorts back to another method of proving, the statement is considered not to be significant enough to be coded, even as a secondary (open) proof scheme. The proof scheme that is coded the most and statements that contributed most significantly to their final result will be coded as a primary (closed) proof scheme. The proof was then categorized according to the final written proof into the table according to its proof type (Ko & Knuth, 2009).

There are two major ways that student's proof production and analysis (understanding) were categorized. The first is by categorizing the proof schemes (Harel & Sowder, 1998) used by the student in each problem using the table in Figure 1 (this table is specifically for Jason during the third interview). In Figure 1, PA indicates the task evaluated the proof appreciation of the student, problems P1 – P3 involved proof production and problems P4 - P5 involved proof understanding:

Proof Scheme:	PA	P1	P2	P3	P4	P5
External Conviction:						
Ritual			○		●	
Authoritarian	○			○		●
Symbolic		●			○	
Empirical:						
Inductive						
Perceptual						
Analytical-Transformational:						
Internalized		○				
Interiorized			●			
Analytical-Transformational						
Restrictive:						
Contextual/ Spatial						
Generic						
Constructive						
Axiomatic:						
Intuitive-Axiomatic						
Structural		○	○	●		
Axiomatizing	●					

Figure 1. Jason's Distribution of Proof Schemes During Interview 3

Students received an open circle in the box under a given proof activity if they showed evidence of using the particular proof scheme. Methods of categorizing students proof schemes are described extensively in the theoretical framework section. Thus, a thorough description of how students' proof schemes were categorized is omitted from this section. A student's most dominant proof scheme, that is, the proof scheme that is shown to be evident in most if not all sections of the proof or disproof and was specifically used to formulate the main part of their argument was given a closed circle to indicate a strong reliance on that particular proof scheme.

It is important to recall once again that a student's proof scheme does not necessarily determine whether a proof is correct or incorrect. Thus, it is important to also categorize how successful the student is at proving, using Ko and Knuth's (2009) framework. As opposed to the proof schemes categorization, the type of proof production categorization has only one possible categorization per proof. By coordinating both of these frameworks, we can observe both the student's proof scheme and the effectiveness of that proof scheme in solving each problem. Consider Figure 2 used to categorize Jason's proof type during interview 3.

Types of Counterexample Productions:	P3	Types of Proof Productions:	P1	P2
No response		No response		
Proof		Restatement		
Inadequate		Counterexample		
Justification		Empirical		
Incomplete		Non-referential symbolic	●	
Adequate	●	Structural		
		Complete		●

Figure 2. Jason's Distribution of Proof Types During Interview 3

In order to better understand the process by which these codes were produced, consider Jason's responses on the first problem of the third interview and the process by which he was categorized:

Problem (I3): *True or false: Every even integer is the sum of two odd integers.*

Jason: *Seems true… It can be but it doesn't have to be.*

Interviewer: *What do you mean by that?*

Jason: *Well, look at six. Six is 2 + 4 but it's also 3 + 3. So it can be written as a sum of odd numbers, but it doesn't have to. But I'm assuming that the statement is saying that even numbers* ***can*** *be written as the sum of two odd numbers, is that right?*

Interviewer: *Yes.*

Jason: *Okay, then it's true.*

Interviewer: *Okay, can you prove it?*

Jason: *Alright, so every even integer can be written as 2k, for some integer k. I'll try a contradiction proof [thinks about it for a few seconds] ... or maybe not. I'll take two odd numbers, 2m+1 and 2n+1 for integers m and n and add them together. [Jason then spends about a minute algebraically showing that the sum of these two odd numbers is an even number and then stares at his proof for several seconds].*

Interviewer: *What did you accomplish?*

Jason: *I showed that two odd integers add to make an even integer.*

Interviewer: *Okay.*

Jason: *Should I move on to the next problem?*

Interviewer: *Absolutely, you're done with this one, then?*

Jason: *Yes.*

In this problem, Jason's proof type was coded as non-referential symbolic since his proof included relevant definitions, but ultimately only proved the converse of the desired result, thereby leading his proof to be logically flawed and therefore deficient. His primary proof scheme on this problem was symbolic since in deriving this proof, he was so focused on the algebraic manipulations that he neglected to notice that he did not prove the desired statement. The proof does have relevant symbolic manipulations but they only prove the converse of the desired statement. His written proof and verbal argumentation show that he was aware of what his symbolic manipulations accomplished, that is, that he showed that the sum of odd numbers is even, however, he failed to acknowledge that this was insufficient for proving the desired statement. He was also coded as structural because he recognized exactly what his algebraic manipulations allowed him to conclude (i.e. the converse of the statement), however, this was considered as a secondary proof scheme since he appeared to be so lost in the symbolic manipulations that he was unable to interpret his conclusion in the context of the original problem. Additionally, he was coded as internalized since he used examples to generate a general theory. However, this was also a secondary proof scheme since his empiricism began with an even number and generated two odd numbers, however, since he made a converse error in his proof, there was not a very big connection between his proof and his empiricism.

In order to answer the research questions, students' proof schemes and proof types in each interview were recorded and categorized using the tables described above and I observed changes that took place from interview-to-interview and from student-to-student. Additionally, I compared their proof scheme to their proof type in order to see what proof schemes were most successful in helping students produce complete proofs. In order to answer research question one, I compared students' final proof schemes with their initial proof schemes. I wanted to observe patterns such as whether or not all students arrived at the same proof scheme by the end of the bridge course or if they were influenced by their initial proof schemes. In order to answer research question two, each interview was coded according to proof scheme and proof type and contrasts were considered within and between students during the process of interviews.

During the interviews I took field notes with the time noted so that I could review the video in order to justify my coding of the student's proof scheme. Thus, for each interview I made a heading for each problem dealt with by each student in the interview with a brief explanation of why he was coded as such. Beneath this, I supported the initial coding with citations from the interview using direct quotes from the student.

Early Interviews and Results from Pilot Studies

A student was given the pre-test as well as the first interview questions to work through. This student was a physics and mathematics major who was enrolled to take Math 290 four months after this interview took place. Therefore, he was likely to be a typical student that I could expect in the Math 290 course. Among the results that were important in making edits to the problems was the unfamiliarity with basic mathematical definitions and notation. For example, although the student was familiar with the fact that "∈" was the mathematical symbol for "is an element of" and he knew that **R** meant "the set of real numbers", he was unable to coordinate the symbols well though to understand that $x \in \mathbf{R}$ meant that x was a real number. Since the student's understanding of mathematical notation and relevant definitions was unrelated to my research questions, more explanation was added to the problems used during interviews and I chose to prepare relevant definitions and theorems on a piece of paper to give to the student only when they asked for them. They were made aware that this was available to them, but I discouraged them from asking for it before necessary. Also, since the notion of proof was completely perplexing to the student, it was beneficial to ask the student to give an informal mathematical argument to the statements that the student believed to be true. This modification was a product of the student providing true or false answers with minimal explanation because he thought that the main purpose of the activity was to determine if the statement was true or false. It is possible that phenomenon was a product of the conditioning from the multiple-choice nature of true-false problems in mathematics classes. This student's relative naïveté with respect to proofs indicated that it is likely that some students will enter the bridge course with little to no training in proof. I also instructed the students to write and tell as much as they possibly can for each problem. The student had seen the proof that 0.99999… = 1 before in high school. In weighing the pros and cons of this fact, I decided that this could help to categorize student proof schemes, since the student saw his teacher prove it but still felt that the teacher was making an approximation, meaning that the student was motivated by his internal conviction more than the authority of a teacher or textbook. It took him seventeen minutes to complete the pre-test, which was estimated to take twenty minutes. It took him twenty-three minutes to complete the first interview. This is significantly shorter than the time I expected it to take, but within this shorter timeframe, I was still able to understand his motivations for the decisions he was making and classify his proof type and proof scheme.

In the process of collecting the data for this study, the classroom anecdotal notes did not prove to be directly helpful in answering the research questions since without asking follow-up questions, it was impossible to classify the proof schemes used by the students. Additionally, only students who did not participate in the study requested out-of-class assistance. However, these interactions did make me more aware of common misconceptions among the students in the class and as a result, it helped me choose follow-up questions during interviews. Similarly, the homework collected helped me to choose good problems to discuss during interviews. However, without further questioning, conclusions about proof schemes could not be made. As a result, these homework questions were not analyzed, instead the discussion about the proof was analyzed in the proof understanding section of interviews two, three and four.

CHAPTER 4: RESULTS

There are two parts to this chapter: the results section and the discussion section. The results section is presented in three parts. The first part presents results of the study relevant to the first research question from chapter 3: What influence do students' initial proof schemes have on the proof schemes they eventually develop in their bridge course? The second part presents results relevant to the second research question: What connections can be made between success in proving and students' proof schemes? The third part contains additional insights about students in their early stages of proof development and how this relates to the way proof schemes are currently conceptualized by researchers in the field of proof. More specifically, I show that in the process of the interviews, students' proof understanding, production and appreciation were not aligned with one another as predicted by Harel and Sowder (1998) and other authors. In the discussion section, I explain how the results of this study can be situated within previous research on proof and provide some insights this study has given the mathematics education community about the development of students in the early stages of learning about proof. It is important to consider that these results are subject to my interpretation of the statements made by the students during the process of the interviews.

It is important to understand that the data I gathered are not sufficient to make general statements about the success of either the bridge course or the teacher that were the setting for the study. Although results may yield some general insights and possible directions for teaching and research, the goal is to more fully understand students' thinking about proof

Results

Evaluating the Relationships Between Initial and Final Proof Schemes

In order to obtain an overview of the proof schemes used by the students in the process of the interviews, first consider the distribution among the students of the proof schemes used. Table 1 gives for each interview the number of questions (out of six) where a proof of each main type occurs. Note that Albert and Jason completed only one and three interviews, respectively.

Table 1

Primary and Secondary Occurrences of Each Major Classification of Proof Scheme

Interview Number	External Conviction Proof Scheme (Primary/Secondary)	Empirical Proof Scheme (Primary/Secondary)	Analytical Proof Scheme (Primary/Secondary)
		Albert	
1	2/3	3/1	1/2
		Ashley	
1	0/1	0/1	6/0
2	4/1	0/0	2/1
3	6/0	0/1	0/4
4	1/3	2/0	3/1
		Adam	
1	2/0	1/0	3/2
2	5/0	0/0	1/3
3	2/3	0/0	4/0
4	1/2	1/0	4/0
		Jason	
1	1/1	0/2	5/1
2	1/0	0/0	5/1
3	3/3	0/0	3/1
		Joe	
1	0/1	0/1	6/0
2	2/1	0/0	4/1
3	1/0	0/0	5/0
4	1/1	1/0	4/0
		Miles	
1	1/2	1/0	4/2
2	0/1	0/0	6/0
3	1/1	0/1	5/1
4	0/1	1/0	5/1
		Melanie	
1	3/2	2/1	1/1
2	3/1	1/0	2/1

3	4/0	0/0	2/0
4	3/2	1/0	2/1
	Trevor		
1	0/0	0/1	6/0
2	0/2	0/0	6/0
3	1/1	0/0	5/0
4	1/0	1/0	4/2
	Totals (Not Including Albert and Jason)		
1	6/6	4/4	26/5
2	14/6	1/0	21/6
3	15/5	0/2	21/5
4	7/9	7/0	22/5

There are several important observations to be made from Table 1. First, we notice that the sophistication of the students' proof schemes does not appear to show a steady improvement throughout the course relative to the increase in sophistication of the problems considered during interviews. This can be observed in Figure 3. There, we observe that analytical proof schemes (which are the goal of instruction) are most evident in the first interview. We should add a stipulation that Albert and Jason are excluded from these figures since we do not have data from every interview from them. However, Albert used an analytical proof scheme only on the problem about proof appreciation during interview 1. In addition, with the exception of interview four, we note that empirical proof schemes are almost completely absent after interview one. Six out of the seven empirical proof schemes from interview four came from one problem that was thought to be challenging by all of the six interviewees. However, external conviction proof schemes appear to be relatively consistent throughout the course, notably being rarest during interviews one and four.

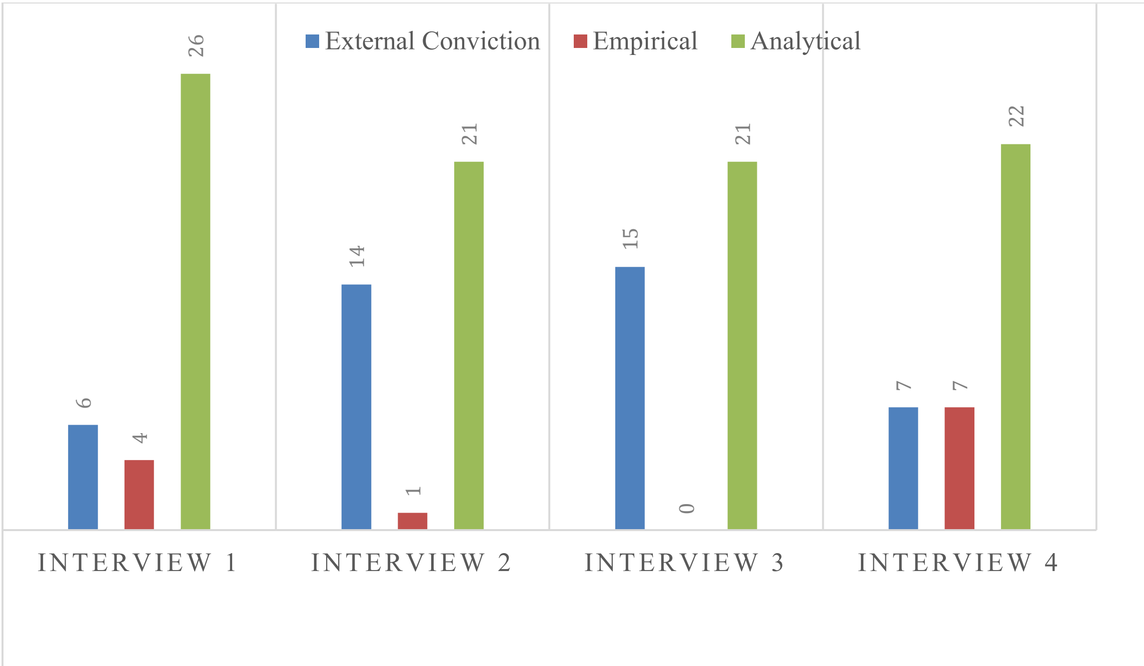


Figure 3. Primary Proof Scheme Totals

It is important to consider that the phenomenon found in Table 1 and in Figure 3 may have other explanations that those considered here. For example, the majority of instances of external proof schemes occurred during interviews two and three when the focus of the course was heavily on the structure of proofs. By further categorizing the students' proof schemes according to proof understanding, proof production and proof appreciation, the same consistency from interview-to-interview is found as in Table 1. However, there were only two instances of students using a proof scheme other than an analytical proof scheme on the proof appreciation section: Ashley and Melanie, both during interview three. However, this likely came to light since the question "How do you know whether you can accept a statement as true (like an axiom) or if a statement requires a proof?" stimulated a discussion of whether the teacher or the mathematics was the ultimate authority of truth. Such reflection did not occur with other proof appreciation questions, showing evidence that the form of the question had an impact on how a student's proof scheme was categorized. However, it is important to take note that the students were barely distinguishable for one another in terms of their performance on the proof appreciation questions. Results from proof production and proof understanding questions were consistent with the pattern from Table 1, being generally distributed between external conviction proof schemes and analytical proof schemes.

These data suggest that empiricism initially declines throughout the course and then increases towards the end, but external conviction initially increases and decreases towards the end. Therefore, there must be other factors that affect a student's proof scheme. Thus, we will investigate particular problems that caused students to use deficient proof schemes throughout the interviewing process to identify possible explanations for these patterns in students' proof schemes.

We will begin with the problem mentioned previously in interview four where all six students interviewed used an external conviction proof scheme. The problem was as follows: "True or false: Let a and b be real numbers such that b > a. There is a rational number, r such that a < r < b." On this problem, Ashley argued that since the real numbers were more numerous than the rational numbers, it didn't feel true to her. Adam argued in a similar way but also argued that the statement seemed too hard to prove true, so it had to be false, and only made efforts to find a counterexample, which is why he was also coded as using an external conviction proof scheme (ritual, to be precise). Joe and Trevor described themselves as imagining two real numbers beside each other somewhere. Miles attempted to argue the cases of rational, irrational and mixed endpoints, yet ultimately concluded that he could only deal with the rational endpoints case and resolved to use an argument similar to the previous students. Melanie did not understand the question well enough to provide a clear answer. Although no other problems provided such an extreme black hole of deficient proof schemes, the common theme was that students frequently used deficient proof schemes on problems that required a novel (for them) approach. For example, the second question from the third interview was as follows: "*Given that for all real numbers x and y,* $|x + y| \leq |x| + |y|$, *show that* $|x| - |y| \leq |x - y|$". Consider below some excerpts from the students' responses to this question:

Ashley: *I think there are some other definitions you could use. There's some identity out there that you could just plug into.*

Melanie: *There is some clever trick for this one. You can change all of the signs on both sides to negative.*

It is interesting to consider that the students in this study are all math majors or minors and were successful in math to that point in their schooling, yet still struggled with misconceptions about writing proofs. In Ashley's case, she continued to believe that proofs consisted of formulaic procedures that we can "plug into" to prove other theorems. In Melanie's case, she was rarely internally convinced of any proof throughout the interview process, and used algebraic manipulations that were not valid in order to construct proofs. In later other interviews, she seemed to pursue a proof path that likely contradicted her own mathematical knowledge:

Problem (I4): *As far as this problem is concerned,* π *is the ratio of the circumference of a circle to its diameter or the pi from the equation* $A = \pi r^2$. *Imagine that you don't know what pi is and that you want to discover it.*

True or false: $\pi > 4$.

Melanie: *I'm not sure… It could be… A couple of interviews ago you said that 0.999… = 1, so maybe pi could be bigger than 4.*

Obviously, since Melanie knew the decimal expansion of pi, she knew that this statement is false. This problem highlights the fact that she continues to allow herself to be satisfied with external conviction in proof problems. This pattern continued in the following instance:

Problem (I4): *Analyze one of your own proofs from class.*

Interviewer: *Do you remember doing this problem?*

Melanie: *Yes*

Interviewer: *How did you come up with this solution?*

Melanie: *I based it off other proofs I saw in the book.*

Interviewer: *How did you decide that your proof was similar enough to the one you were basing it off of?*

Melanie: *I don't know, it just seemed close. I know it doesn't make any sense, I just copied the proof from the textbook.*

In this last case, Melanie was unable to understand her own proof and wanted to match up her proof with the one in the textbook. Melanie and Albert were the only students who used analytical proof schemes in less than half of the cases. While Albert left the class after two weeks, the same issue of using deficient proof schemes continued through the entire process for Melanie.

Another example of a new proof problem that induced deficient proof schemes is problem 4 from interview four where students were asked to evaluate the argument of the proposed solution of the Calvin and Phoebe problem (see Appendix A to review the particular problem). Consider the responses of Joe, Melanie and Trevor below:

Joe: *It takes a case and shows why it is invalid. And there is nothing wrong with the argument. The problem that I'm having is that the logic is sound but the* *statement they are proving is false. I can't find the misstep that allowed that error.*

Melanie: *I can't say that I am convinced because I know that the answer they came up with is not correct… I don't know where they went wrong because everything they say checks out.*

Trevor: *So it proves that Calvin is a liar. Then the proof evaluated every possibility and the statement is convincing.*

With the exception of this problem, Trevor and Joe performed very well in this interview. However, the new context may have caused them to be absorbed in the details of the argument and were unable to see the major problem in the big picture (see problem 4 from interview 4 in Appendix A to see to review the problem and the argument presented to the students). Thus, it becomes imperative to take a closer look at proof schemes on proof production tasks as they ranged in difficulty.

In order to analyze proof schemes according to difficulty, each proof production task was categorized into three categories. The first category is *standard*, which were problems that were similar to tasks considered in class. The second category is *extension*, which were problems that were accessible but somewhat beyond the tasks that were considered in class. The third category is *new context*, which were problems that required the students to apply their knowledge of proofs to new contexts (either geometry or calculus). The problems from the first interview were not considered in this categorization since some students had more experience than others with proof-related tasks and the categorization would have to be individualized to the student which may affect the accuracy of the results. The proof production tasks and their categorizations are found in Table 2. See Appendix A to review the problems.

Table 2

Categorization of Proof Production Tasks for Interviews 2-4

Interview	Problem	Categorization
2	1	Extension
2	2	New Context
2	3	Standard
3	1	Extension
3	2	Extension
3	3	Standard
4	1	Standard
4	2	New Context
4	3	New Context

In Figures 4-6, consider the primary proof schemes used for each category of proof production task.

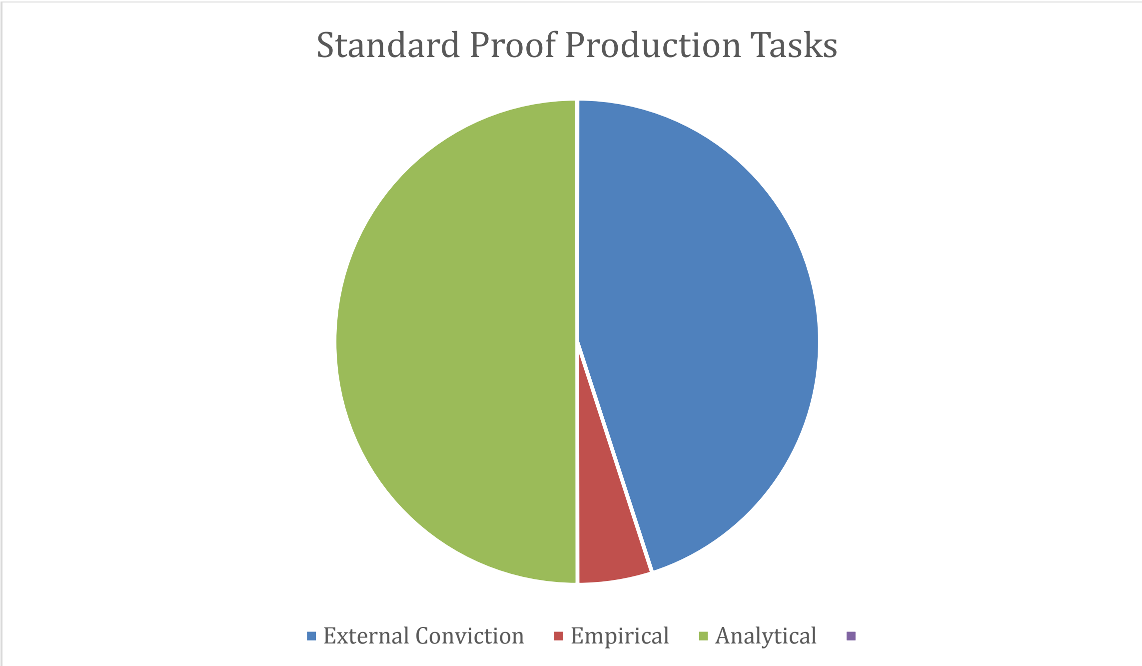


Figure 4. Distribution of Proof Schemes among Standard Proof Production Tasks

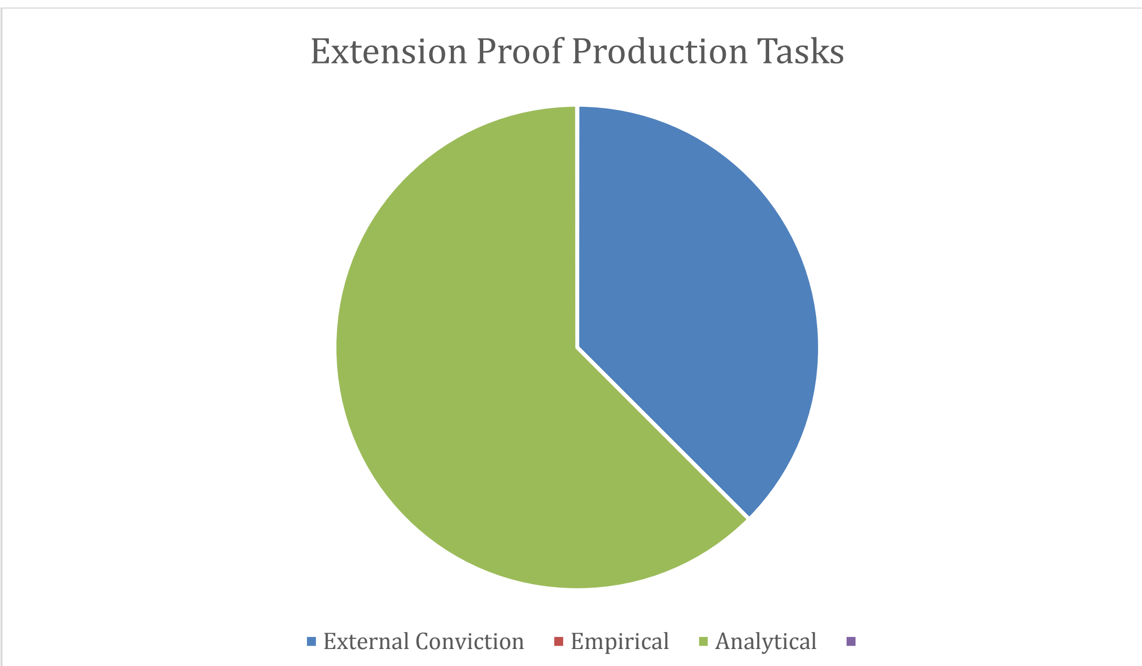


Figure 5. Distribution of Proof Schemes among Extension Proof Production Tasks

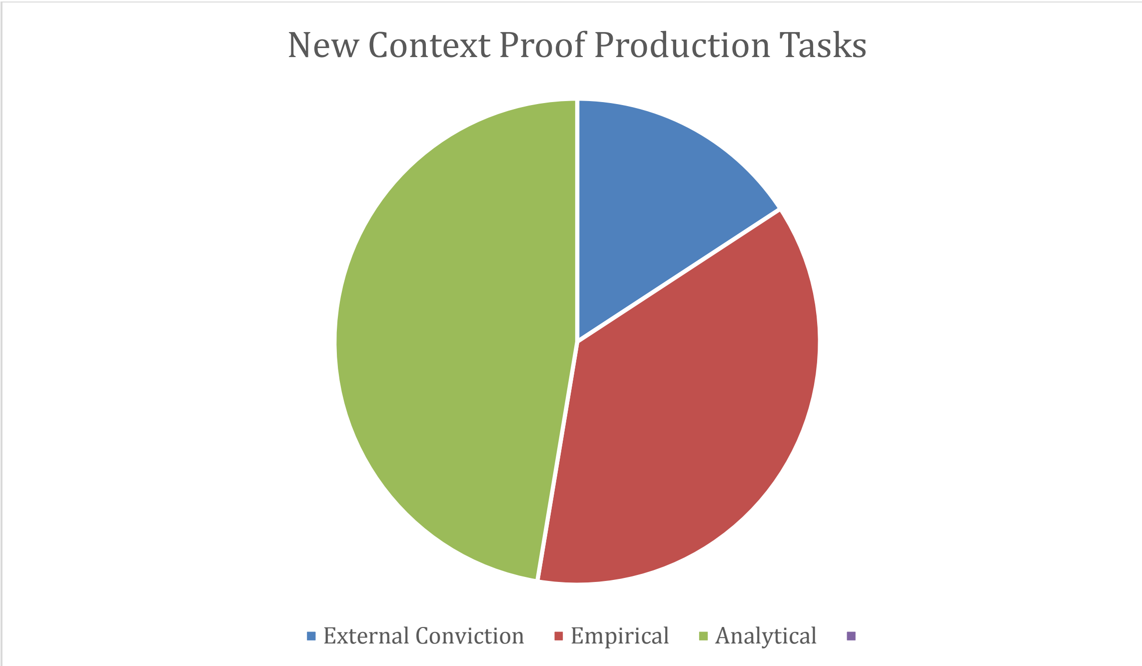


Figure 6. Distribution of Proof Schemes among New Context Proof Production Tasks

In Figures 4-6,three key observations may be made. First, analytical proof schemes were used most frequently on proof tasks which were extensions of the knowledge gained in class. That is, tasks that were similar, but somewhat beyond anything they had previously considered. Second, external conviction proof schemes were used less by far in problems with a new context than they were in any other type of proof production task. That is, as the familiarity with a problem increased, the likelihood that the students would deal with the problem using the lens of an external conviction proof scheme increased. Third, analytical proof schemes appeared almost just as much in standard problems as they did in problems given in a new context. Students' success in proving, however, was best in standard proof production problems, where 60% were done correctly as opposed to 40 % and 35% in extension and new context proof production problems, respectively. This suggests that students' performance on proof-related tasks do not directly correlate with the maturity of the proof schemes employed. This will be discussed further in the following section.

These data suggest that while some students showed some moderate progress in the maturity of their proof schemes, standard proof production problems and those in new contexts are often dealt with by the use of deficient proof schemes. In most cases, external conviction proof schemes were used particularly often for familiar proofs, regardless of the time in the semester that the students were interviewed. Additionally, when students were asked to prove a statement where they must argue using results which they do not have a full knowledge or understanding of, students did well at avoiding symbolic, authoritarian or ritualistic proof schemes. Also, instruction in general methods of proof seems to have done well at helping the students avoid using empiricism in the process of proving. While external conviction proof schemes are most common in proof production tasks that students are familiar with, these were also the tasks where students performed the best.

Evaluating the Connections Between Success in Proof and Proof Schemes

In this part we will examine what the data tells us about the relationship between success with proving and the proof schemes employed by the students in the process. In Figure 7, we examine the primary proof schemes used by students according to proof type of the final answer.

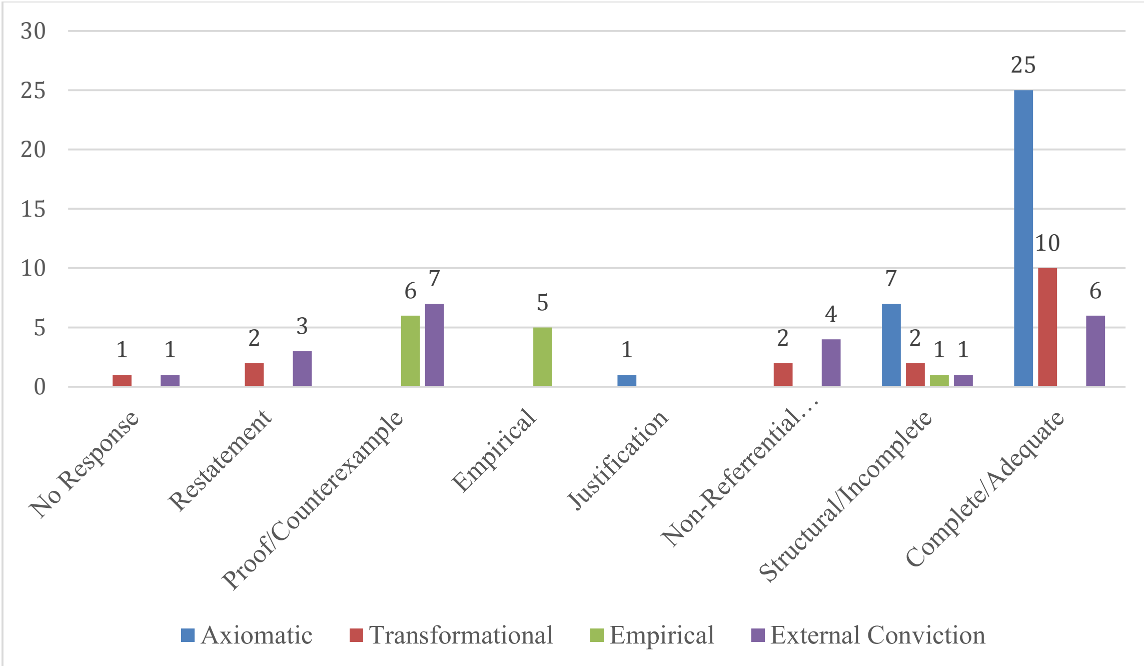


Figure 7. Proof Schemes According to Proof Types

As we can see in Figure 7, the overwhelming majority of students who are successful at proving used analytical proof schemes (either transformational or axiomatic). There are instances of several other proof types in the interviews. However, as discussed in the previous sections with respect to proof scheme, it appeared as though there was some relationship between the proof type and the problem posed to the students. Therefore, we will highlight two major findings from Figure 7. First, in all but one case, axiomatic proof schemes led to proofs that were either coded as structural or complete/adequate in type. This suggests that if a student's proof production scheme is axiomatic, a student will produce a proof that will either be adequate, or at least will be based on a mathematically-based argument. By contrast, proofs that were based on an external conviction proof scheme had a wide variety of forms, several of which had issues far beyond simply being incomplete.

Second, complete/adequate proofs were either based on an axiomatic, transformational or external conviction proof schemes. None of the complete/adequate proofs were based on empirical proof schemes however, cause for concern are the six proofs that were complete/adequate but whose primary proof scheme was coded as external conviction (ritual in every case). Below we will consider several instances where this occurred in the process of the interviews:

Problem (I1): *True or false: If m is an integer such that m > 2, then* $m^2 - 4$ *is composite.*

Jason: *… Well, if I try m = 3, I get 5 which is prime, so it's false. Let me just double check 4… true… 5… true… 6… true. It seems like all of the other ones are true.*

Interviewer: *So is the statement true or false?*

Jason: *It's false.*

Interviewer: *If you're convinced by the m = 3 case, then why did you try all of the other cases?*

Jason: *I would have felt better if I could find more examples, but even if I can't it's still false.*

Here we see that Jason knows that only one counterexample is necessary, however he believes that further counterexamples will strengthen his conviction. This pattern can be observed in several other instances:

Problem (I1): *True or false: If n is an integer and* $|n+1| < 1, then\ |n^2 - 1| < 4$

Joe: *… I mean, it only works if n = -1, 0 and 1 [note: n = -1 is the only number that makes the hypothesis true and Joe later recognizes this]. I think this should be simple, I have no clue what I'm supposed to write. I want to use the intermediate value theorem, but I don't know how… [Long pause] It [the hypothesis] only applies for n = -1. It [the conclusion] holds there. There is no other case to consider, so it has to be true… [pause] That can't be it?*

Interviewer: *So is the statement true or false?*

Joe: *It's true.*

Interviewer: *Have you proven it?*

Joe: *I don't know how to prove it. I have never dealt with anything like this before. I feel like I'm just spinning my wheels. That's the best that I can come up with.*

Here we see again an instance of the student completing the proof and not being completely confident in their argument. In this case, Joe struggled to recognize his proof as being correct because he has never seen a proof like this previously. In this case, he was not convinced by his proof to the extent that he never bothered to write it down. The theme of students struggling in new contexts and with new proof structures from the previous section appears here as well. I will carry on with other similar instances of students providing proofs using a ritual proof scheme:

Problem (I3): *[Homework problem] For a function f: A → B and subsets C and D of A and E and F of B, prove the following:*$f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$

$f^{1}(EUF$

Interviewer: *... I don't want you to feel bad about making a mistake in your homework, but if you were to start the problem over, what would you do?*

Melanie: *[Tries several false starts and recognizes them as such]*

Interviewer: *So you have a challenging homework problem. What do you do?*

Melanie: *I try to find a similar theorem, look at the proof and plug it in.*

Interviewer: *And if that doesn't work?*

Melanie: *If that doesn't work, I usually just ask Dr. Smith in class or I ask my math teacher friend, she's pretty good at this stuff.*

Interviewer: *And that usually turns out well for you?*

Melanie: *Yeah, I always get good grades, so I must be doing something right.*

In this instance, Melanie admitted that her correct answers were sometimes merely a matching game where she used previously solved problems to act as a template for solving new problems. In this instance she was coded as using an authoritarian proof scheme and although she claimed that she was successful on her homework and tests throughout the bridge course, she struggled to solve problems without her textbook providing a template for her. Again, we see that Melanie felt successful, even though she admitted that she was unable to produce a proof without assistance from her teacher or others.

Adam was questioned about the same problem and below is an excerpt from his solution:

Problem (I3): *[Homework problem] For a function f: A → B and subsets C and D of A and E and F of B, prove the following:* $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$

Adam: *[Considers his original response] I think it follows logically. I think the f(x) should change to a y, but other than that it all makes sense.*

Interviewer: *How would the argument change if you had y instead of f(x)?*

Adam: *It wouldn't change at all, it's just that with these [types of problems] you are supposed to use y and use element chasing.*

Interviewer: *How did you know that?*

Adam: *That's how Dr. Smith wants it done. With these types of proofs, you are just supposed to chase the elements from each side of the equation into the other set.*

Interviewer: *Why do you think Dr. Smith wants it done that way?*

Adam: *For lots of reasons… [long pause]… It probably wouldn't work quite right the other way. This way you're going to be sure that you're chasing your elements from one set to the other one.*

In this instance, Adam is able to correctly explain how a correct proof is to be written for this problem with the stipulation that the choice of variable differed from what he was accustomed to. Although he flagged this as a necessary correction to the proof, he was unable to explain why this change was necessary beyond the fact that it differed from the way his instructor wrote it. Thus in this case, he was coded as authoritarian despite correctly analyzing the structure of the proof. This example clearly illustrates that Adam is unable to differentiate between what his instructor does that is necessary in a proof for logic to prevail and what is convention or habit.

After considering the number of ritual proofs that were adequate/complete, I will now focus on the other proofs based on ritual proof schemes. Among all proofs that relied primarily on a ritual proof scheme, in other words, proofs that were focused more on the appearance of the proof than on the structure of the statements, one of these proofs was coded as "no response", two were coded as being a proof when the statement was false and one instance was coded as non-referential symbolic. Six such proofs were coded as adequate or complete. Thus 60% of primarily ritual proof schemes were complete proofs. Among analytical proof schemes, one proof was coded as "justification", meaning that a reason was given for a statement being false without a counterexample provided. Nine proofs based on an analytical proof scheme were coded as incomplete or structural. Two proofs based on an analytical proof scheme were coded as restatement. Two proofs based on an analytical proof scheme were coded as inadequate or non-referential symbolic. Thirty-five proofs based on an analytical proof scheme were coded as complete or adequate. Thus 70% of analytical proofs were coded as complete or adequate. Since all complete/adequate proofs were coded as either analytical or ritual, these two categories of primary proof schemes fully characterize all proof schemes that led to correctly produced proofs. The distributions of proof type according to proof scheme are summarized in Figures 8 and 9.

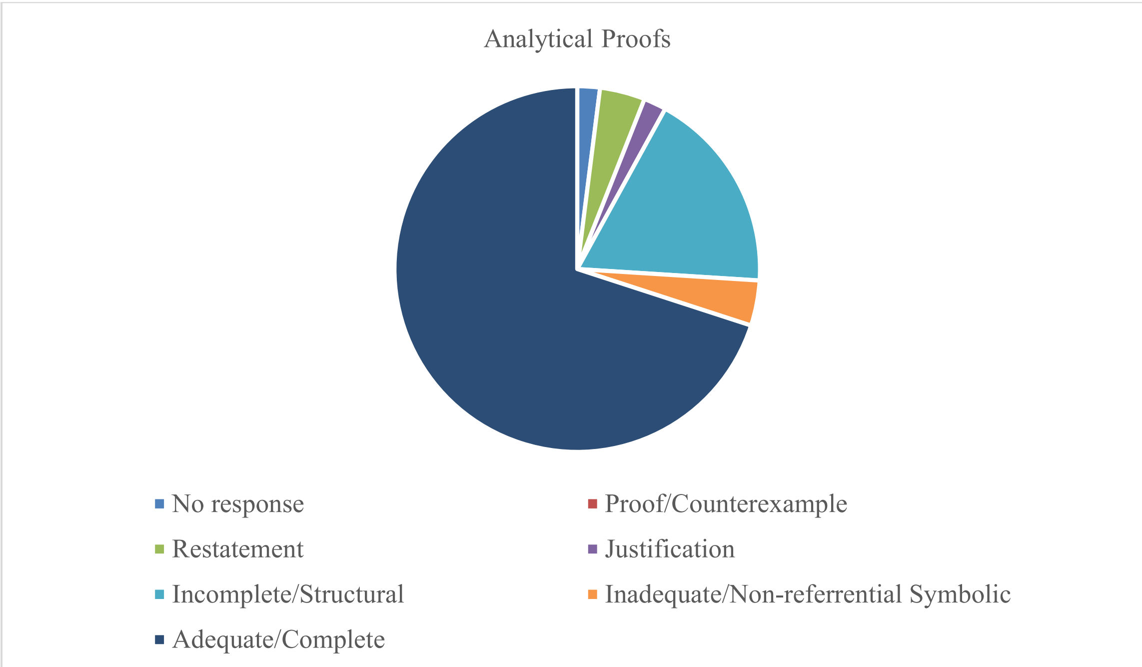


Figure 8. Analytical Proof Schemes and their Proof Types

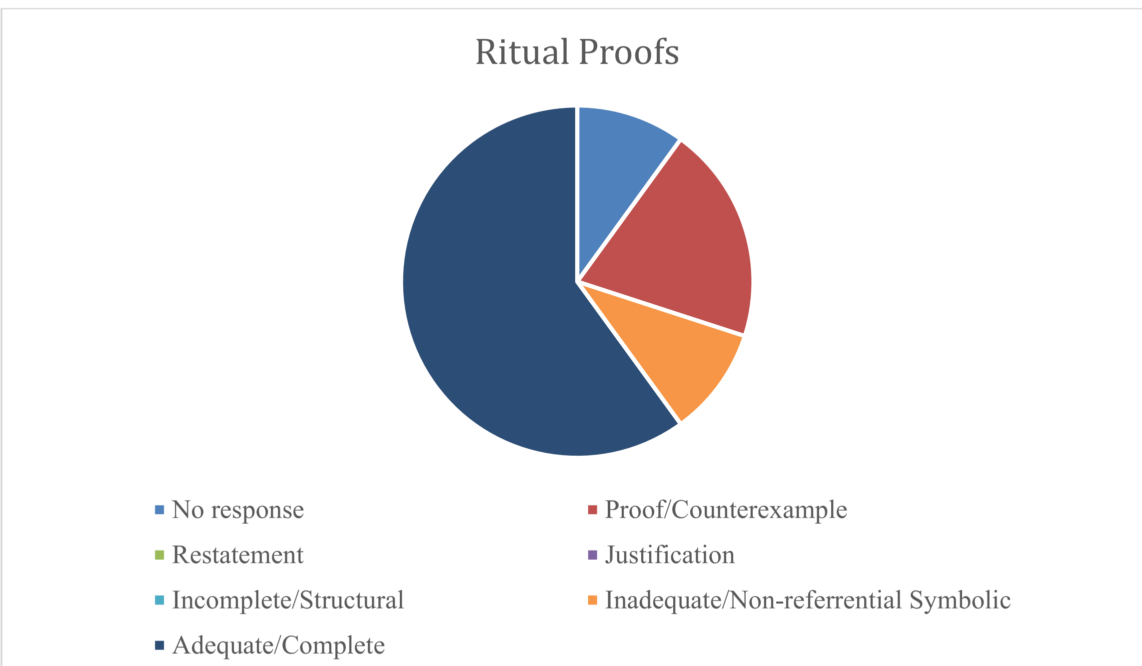


Figure 9. Ritual Proof Schemes and their Proof Types

Thus, in considering the connections between the proof schemes and the proof types used, the data suggest that correct answers came most often when students used an analytical proof of some type. However, a close percentage of students using a ritualistic proof scheme were able to be successful in proving in the sense of often having correct answers. This is consistent with the results from many studies in mathematics education that show students being successful on tasks by ritually following rules or mimicking examples without deep understanding of the underlying principles.

Just as in other contexts, where correct answers to algebra or arithmetic problems do not imply understanding, correct proofs imply understanding. Thus, a major result from this section is that students with a high degree of competence in a proof setting may or may not have a correct understanding of the meaning and purpose of proof. Nevertheless, the strong majority of proofs showed significant understanding (e.g. incomplete/structural proof types). By contrast, proofs that were ritual resulted in a greater number of deficient proof types including several proofs without a response or clear objective, attempted proofs of false statements and attempted counterexamples of true statements.

Re-Examining the Differences Within PUPA

In this part, we will examine a phenomenon that arose in the interview process: that the students' proof understanding, proof production and proof appreciation were, on several occasions, categorized quite differently. In order to become convinced that this is the case, several individual cases must be considered.

In Albert's case, for example, he was very competent at explaining the purpose of proof within mathematics, exhibiting mostly axiomatic proof schemes on this part of the interview. However, when asked to prove or when asked about someone else's proof, Albert did not continue to use the same proof schemes and resorted almost exclusively to external conviction and empirical proof schemes. Although Albert did not complete any other interviews, his case was a very strong example of the misalignment of different aspects of PUPA with respect to proof schemes. Consider below excerpts from the interview with Albert, which exemplify some of the inconsistencies among his proof understanding, proof production and proof appreciation:

Problem (I1): *You are taking a class that is totally devoted to teaching you to prove. Why do you think mathematicians emphasize proof so much?*

Albert: *From what I understand, proofs are a way of helping us understand why we do the things we do in math, why things work, why we get the answers that we do. I don't think it's enough to be able to crunch numbers but more so that you can explain why and how things work because of this, this and that.*

Here we see that Albert's conceptions are well characterized by an analytical proof scheme. He does very well at identifying the purpose and gives a glimpse of the process of proving in his answer. Next, we will consider some of his responses from the proof production and proof understanding sections:

Problem (I1): *True or false: If n is an integer then* $n^2 + n$ *is an even number. Justify your reasoning.*

Albert: *I think the first thing I would do is start plugging in numbers so that I can kind of get an idea of whether it is true or false. The first number that comes to mind is 2. I'm trying an even number first. That works. I will try an odd number. Three works. So based on the pattern I see, they should all come out even. So it's true.*

Interviewer: *How would you go about arguing that?*

Albert: *I don't know how I would prove that. I just know that all of the numbers that I can think of always come out that way, so it should be true.*

Interviewer: *So those two examples are good enough to make a general statement about all integers?*

Albert: *Well, normally, I would use a lot more than two examples. More like five or six. If those all worked out, I would just assume that they all would work.*

Here we see that Albert's statement that "*I don't think it's enough to be able to crunch numbers*" doesn't apply to this problem. This pattern continues with the following problem in the interview:

Problem (I1): *True or false: If m is an integer such that $m > 2$, then $m^2 - 4$ is composite. Justify your reasoning.*

Albert: *[re-states the problem several times]. Let's try $m = 3$… That's a prime number. Let's try $m = 4$… That's composite. So based on that first example, it's false.*

Interviewer: *Obviously, you made that decision based on the first example, so why did you test $m = 4$ if you didn't need it?*

Albert: *I always like to be sure because I second guess myself a lot, so I always use more than one example.*

Interviewer: *So what would you do if you got composite, composite when testing numbers?*

Albert: *Then I would just assume that is was always composite and say that it's true then.*

Here, we see that in the first two proof production problems, Albert's conception from the proof appreciation question do not seem to align with his schemes for proving. In the third proof production problem, he complained about an inability to plug in numbers and explained "*my go-to is to plug in numbers*", directly contradicting his statement from the proof appreciation section that "*I don't think it's enough to be able to crunch numbers*". And although he did not have the knowledge at this point in the course to prove thoroughly, he seems to have no issue being convinced by a very limited number of examples. Additionally, the ritual of using an odd and an even example for conviction appears to be strongly held. Below we will consider his responses on the two proof understanding problems:

Problem (I1): *See problem 4 in Appendix A.*

Albert: *It was really hard to follow…* But *I'm more inclined to believe this kind of thing than not to believe it, so if my teacher wrote this kind of thing on the board, I would assume that it's probably true.*

Interviewer: *What is challenging about understanding this proof?*

Albert: *What I think is tough is that I can't plug in numbers here. I wish they had done it for n = 5 instead of just n. That's confusing and I have no idea why they are doing what they are doing.*

As was the case with Albert's proof production, the pattern of empiricism continued to manifest itself in the following problem:

Problem (I1): *See problem 5 in Appendix A.*

Albert: *I feel like that's pretty close to one.*

Interviewer: *What is the author claiming here?*

Albert: *I would say that they are saying that zero point nine repeating is equivalent to one. I don't know if that is true or not. [Re-reads problem]. Okay, I think this is convincing, I like that I can plug in 0.999… in there. That's what I like to do. This way of proving works well with my thought process.*

Here we see that Albert's schemes for proving and analyzing proofs line up with one another in that they are mainly empirical and at least partially external. However, they are quite different from his schemes for proof appreciation. This occurrence manifested itself several times in the study, especially in the case of proof appreciation when compared to proof production and understanding. In fact, in almost every case, students had a very good understanding of what a proof is and a general idea of its philosophical meaning and significance within mathematics, however, it often did not manifest into their doing of mathematics. Note also that Albert did not participate in the class long enough to complete any other interviews.

In some cases, their schemes for proof understanding and appreciation did not align with one another. In fact, there were twelve instances where the proof schemes of the student's PUPA did not align with one another. Notably, Trevor was the only student who did not have a clear instance of this in any of the interviews. In the following paragraphs we will demonstrate an instance where the student's proof understanding and proof production were each misaligned. We will begin by analyzing the third interview with Jason:

Problem (I3): *How do you know whether to accept a statement as true (like an axiom) or if a statement requires proof?*

Jason: *For me, I'm told in class what is an axiom.*

Interviewer: *What's an axiom?*

Jason: *It's a truth that the whole mathematical community agrees on. So basically in class, I learn what truths that everyone has agreed on and I use those results to build new results and theorems.*

Here we see that Jason's proof appreciation conception is well characterized by an analytical proof scheme, an axiomatizing proof scheme, to be more precise. Identifying the origin of his knowledge being the instructor did not appear to be part of his primary conception. Instead, he sees axioms as being building blocks that can be used to derive further results. Next, consider some of his responses from the proof production sections:

Problem (I3): *Use the following result: for all real numbers x and y,* $|x+y| \leq |x| + |y|$ *Show that* $|x| - |y| \leq |x-y|$.

Jason: *I'm just going to try some numbers?*

Interviewer: *How is that helpful?*

Jason: *It just helps me to better understand what's going on [tries three examples and find them to be valid]. Do I need to use that result?*

Interviewer: *You don't have to, that's a result that you can use if you choose to.*

Jason: *Okay, I'll just try substituting positive and negatives then and see what happens [long delay while he attempts to work through this]. I don't think this proof is going to work as well as the other one did [referring to the triangle inequality].*

Interviewer: *What do you mean by that?*

Jason: *There are going to be tons and tons of cases here because I don't know whether x – y will be positive or negative [begins to list all of the possible cases].*

Interviewer: *What happens next?*

Jason: *I would just have to go through each of these cases, I could do a WLOG on two of them at least. It'll take a while but I can do it. [Jason then outlined all of the cases he had to consider, proved one of them and left out the other cases to expedite the process of the interview].*

Here we see that Jason's deductive organization of the argument led to a successful proof. This pattern continues with the following problem in the interview:

Problem (I3): *True or false: Let a and n be integers. If a divides* n^2 *then a divides n*

Jason: *I've seen this before. We used it to prove that the square root of 2 is irrational. [Lists relevant theorems and uses them to try to construct a deductive argument, but has several false starts in the derivation of the proof]. Oh wait. I don't know if it is [true]. I was thinking of the opposite thing. The converse might not be true [thinks for several seconds]. If a is 4 and n is 2 it fails.*

Here, we see that in two proof production problems, Jason is able to interpret the problem in the context of what he knows and consider how he can use these results in order to prove the desired results. In the proof by counterexample, he recognizes the problem as being similar to one considered in class, however, in the process of organizing the relevant results into a proof, notices that the statement is false. Thus, Jason is focused on organizing a deductive argument, rather than trying to replicate similar proofs. These proof schemes are deeply related to his conception from the proof appreciation question since, although his teacher shares the axioms with the student, the results and axioms stand by themselves to prove necessary results and theorems. He was ultimately able to use analytical proof schemes to outline the related results and adapt similar proofs to fit the context of what he was proving. Below we will consider his responses on the two proof understanding problems:

Problem (I3): *See problem 4 in Appendix A.*

Jason: *I find this proof to be convincing*

Interviewer: *What do you like about it?*

Jason: *I like geometry, it's visual and I can see it. I like the Pythagorean theorem because I understand it well. I find that with these ones [having to analyze a* *proof], I look more for sins of commission than sins of omission, if you know what I mean.*

Interviewer: *Why don't you tell me exactly what you mean, so I am sure.*

Jason: *When I look at a proof, I want to find something blatantly wrong with it, if I don't, I just assume it's right. If it's missing something, I'll probably miss that part… What is the point of this [question] anyway?*

Interviewer: *The author is trying to prove that the forward direction of the Pythagorean theorem is true. I just want to know what comes to mind for you as we are going through their argument.*

Jason: *I think it's convincing. I don't know if it's missing something what would be required to prove what it wants to prove. But I'm not a master of proving or anything.*

Here we see that Jason defaulted to believing that a proof is correct and only adjusted this belief if he saw a blatant error in the proof. Thus, the deductive rigor he subjected himself to in the previous proof production problems did not apply. This pattern continued with the following problem in the interview:

Problem (I3): *[Homework problem] Prove that* $f(C \cup D) = f(C) \cup f(D)$

Note that for this problem, in Jason's solution from his homework, it was clear that he was unaware of whether he was working in the image or the pre-image or in the original sets.

Interviewer: *Why did you choose* $y \in C \cup D$ *and* $x = C \cup D$*?*

Jason: *Because you use x for your inputs and y for your outputs.*

Interviewer: *Do you remember solving this problem?*

Jason: *Yeah*

Interviewer: *How did you come up with this method of proving?*

Jason: *That's how the book did it. I figured it must be right so I basically did the same thing.*

Interviewer: *Was this problem difficult?*

Jason: *I don't think it was difficult. I'm not entirely sure if it's right, but writing it wasn't hard.*

Interviewer: *What makes you think it's not right?*

Jason: *Because I get bad grades on set proofs. I mean, I can't really ever say that I'm 100 % convinced by set proofs, but I do my best to write the proofs and I always end up losing a bunch of points.*

It is clear from the past two prompts in the interview that Jason's understanding of proofs seems to depend on schemes different from those used in the proof appreciation and proof production problems. In problem four, he admitted that he was unable to follow the arguments in the proof closely enough to find an omission, and he was only able to identify blatant falsehoods. The proof was missing part of the section where the rhombus is proven to be a square. Thus, the proof has both an omission and a part of the proof (labelling of the angles) that are never used. Jason is unable to identify these things and is clearly looking at the argument globally, being satisfied that it looks like other proofs and that it is proving a result he was familiar with. This is why his primary proof scheme was coded as external (ritual) on this problem. On the following problem, where he analyzed his proof about sets, he explained that the conviction that was important was the conviction of his teacher, and not his personal conviction. Even though he was not able to interpret the meaning of the proof, he argued that the proof was correct because it was modeled after the textbook's proof. For these reasons, he was again coded as using an external (authoritarian) proof scheme in working through this problem. As a result, we clearly see that Jason's schemes for proof production and proof appreciation were shown to be consistently analytical during this interview, whereas his proof understanding is based on external conviction.

This pattern continues as we consider interview four with Ashley, in which her proof production schemes appear to be inconsistent with the other aspects of her PUPA.

Problem (I4): *What is the role of proof within mathematics? How do you think your opinion has changed throughout your Math 290 course?*

Ashley: *[Explains that she used to think that a proof is the first thing that you do when you're onto a new topic in math]. Now I see it more as a demonstration. You know, this is what we have found to be true and here [are] the rigorous steps that you can follow in order to arrive at that conclusion.*

Interviewer: *So how do you personally benefit from having a proof of a theorem?*

Ashley: *It helps me see the techniques used in proving and also the concepts involved in proving that particular result. I mean, it helps me see how a theorem fits within that particular branch of mathematics and not just skipping to the plug and chug part of it.*

Here we see that Ashley's conceptions are well characterized by an analytical proof scheme (structural). Ashley emphasized here the importance of situating a result in the context of other results from any given branch of mathematics. Because she emphasized the deductive, connected nature of mathematics, she was coded as using a structural proof scheme. Next, we will consider some of her responses from the proof understanding sections:

Problem (I4): *See problem 4 in Appendix A.*

Ashley: *[reads the problem and then pauses for about twenty seconds] It doesn't work.*

Interviewer: *What do you mean?*

Ashley: *Calvin lying and Phoebe telling the truth doesn't work. If she is telling the truth then he is telling the truth but if he is telling the truth, she is lying according to what he says. So obviously the proof isn't right. You could do it by cases, I think.*

Interviewer: *What do you mean?*

Ashley: *Well, they kind of missed part of it. They show that Calvin is lying because it doesn't make any sense for him to be telling the truth. But just because he's lying doesn't mean Phoebe is automatically telling the truth. You would need to say something like: "Okay so now that we know that Calvin is lying then 'Exactly one of us is lying' is false, so they must both be telling the truth or both lying, but Calvin is lying so Phoebe's lying" and go from there.*

We see above that Ashley does very well at dissecting the statement and analyzing the steps the author took to arrive at their conclusion, which is consistent with what she answered on the proof appreciation problem. This pattern continued in the following proof understanding problem:

Problem (I4): *[Homework problem] For positive integers a, b and c, the greatest common divisor of a, b and c is the largest positive integer that divides all of a, b and c. Let d = gcd(a, b, c), e = gcd(a, b) and f = gcd(e, c). Prove that d = f.*

Ashley: *[Explains what theorems were used to get from one step to the other]. To show that d equals f, I would probably want to show that d is less than or equal to f and d is less than or equal to f… But I don't think that's what I did in my homework so that proof is wrong.*

Interviewer: *What's wrong with it?*

Ashley: *It kind of jumps the gun. Like everything I wrote leading up to it is right, but that stuff doesn't mean that $d = f$. I just showed that d and f are common divisors of c.*

Interviewer: *How did you end up having this error? It seems to me that you can perfectly explain to me what you're doing, why it's incorrect and what needs to happen for it all to get fixed. But in spite of all of that, why did you write what you wrote in the first place?*

Ashley: *I think I just got caught up in listing all of the information and didn't have my eye on what the major goal of the proof was. Sometimes it's helpful to think backwards; like, I know where I want to end up, so what happens right before that, and before that, and so on.*

Here, we see that in both proof understanding problems that Ashley's conception from the proof appreciation question is deeply related to her schemes for understanding proofs. She clearly made connections about the process of proving and needing to succinctly move from one point of deduction to the next. This is described in the proof appreciation section and enacted in her analysis of the proofs that she was presented with. Her responses on the proof production problems, however did not align with these conceptions. Below I will consider her responses on two of the proof production problems:

Problem (I4): *True or false: Let n be an odd integer. Then $n^2 = 1 \pmod 8$*

Ashley: *Okay… So if n^2 is congruent to one mod eight, then eight divides $n^2 - 1$, which means that $n^2 - 1 = 8x$, where x is an integer. Okay now n being odd means that $n = 2y + 1$ for an integer y. So $n^2 - 1 = 4(y^2 + y)$ [omitting the algebraic steps]. Thus $4(y^2 + y) = 4(2x)$. So it looks like it's true because you can factor out a four out of both.*

Interviewer: *So it's true, then? What exactly are you basing this on?*

Ashley: *Because x and y are integers, then it must be true.*

In this problem, Ashley was quick to "jump the gun" by using the converse to prove the statement. She was thus coded as external (symbolic) on this problem since the solution was focused on an algebraic manipulation, rather than the structure of the statements. However, she was critical about jumping to conclusions in the proof understanding section. In the following problem, she seemed to make an attempt at organizing her argument logically, but quickly abandoned this strategy:

Problem (I4): *As far as this problem is concerned, π is the ratio of the circumference of a circle to its diameter or the pi from the equation $A = \pi r^2$. Imagine that you don't know what pi is and that you want to discover it.*

True or false: $\pi > 4$.

Ashley: *So pi is 3.14-whatever. So it's false.*

Interviewer: *Can you show me why? Imagine that you don't have Google to tell you what the decimal representation of π is.*

Ashley: *Okay, so pi is irrational, then, which means that it cannot be written in the p/q manner… So if we considered area, for example, we could consider radii that are positive, negative and zero. Oh wait… It's never going to be negative and if [the radius] is zero the area is zero. I just don't know what I should plug in because I want to solve for pi [has several false starts] ... I bet you would need to do like a squeeze theorem kind of thing, I just don't know how I am supposed to do that.*

Here, we see that in two proof production problems Ashley's conception from the proof appreciation and proof understanding questions are quite different. The first proof seems to be based on a major converse error with a conclusion that cannot be convincing to anyone who understands the symbols being used and their meaning. She was coded as using an external (symbolic) proof scheme on this problem since the only part of her argument that was coherent was the part where she listed the relevant definitions. The rest appeared to simply be symbol pushing. On the second proof, Ashley was looking for a one-step previous theorem that she would be able to "plug into" in order to obtain the desired result. Notably, she proposed to use positive, negative and zero radii. This was clearly ritual. Thus, she was again coded as using an external conviction proof scheme. The proof schemes used in these two responses are quite different from those used in the proof appreciation and proof understanding questions. This once again calls into question whether a student's PUPA categories are consistent.

It is clear from the past two prompts in the interview that Ashley's production of proofs did not follow the same schemes as those used in the proof appreciation and proof understanding sections. In problem five, she criticized her own proof for being a product of her getting distracted from the overarching goal of the proof and focusing on details that were irrelevant. However, in considering the two proofs production problems, she repeated the same type of behavior. Furthermore, it is curious that she was able to identify so well exactly what was missing from problem five in order to complete the proof and then made a blatant converse error on the first proof production problem. As a result, we clearly see that Ashley's schemes for proof production and proof understanding were shown to be consistently analytical during this interview, whereas her schemes for proof production are based on external conviction.

After considering the cases of Albert, Jason and Ashley during three separate interviews, we can see that the schemes for proof appreciation, proof production and proof appreciation can be shown to be quite different. The other cases are similar to these, with two other instances of a misaligned proof understanding scheme and six other cases of a misaligned proof appreciation scheme. Multiple such occurrences were present in all four interviews as we can see in Table 3.

Table 3

Instances of Misalignment of Proof Schemes

Student	Interview	Misaligned aspect of PUPA
Albert	1	Proof appreciation
Ashley	2	Proof appreciation
Ashley	4	Proof production
Adam	1	Proof understanding
Adam	2	Proof appreciation
Jason	3	Proof understanding
Joe	2	Proof understanding
Joe	3	Proof understanding
Miles	3	Proof appreciation
Melanie	1	Proof appreciation
Melanie	2	Proof appreciation
Melanie	4	Proof appreciation

Discussion

There are several significant findings that have occurred from this study. The findings related to the first research question can be summarized as follows: the proof schemes in the study only improved in the area of empiricism. External conviction proof schemes appeared not to decrease throughout the class and that the proof scheme appeared in many cases to be induced by the nature of the problem and often times deficient proof schemes appeared in cases where problems in new contexts were posed. A similar improvement in the area of empiricism has been identified by G. Stylianides and Stylianides as a natural stepping stone as students develop their understanding of proofs, even if they are introduced to proofs in the early grades (2009). The problem of the persistent use of external conviction has clearly been identified in this study as well; however, no analogous method is proposed in the research on proof schemes that address this issue through instruction (Harel & Sowder, 2007). The issue of the proof scheme being induced by the problem at hand is an obstruction to students succeeding in proof since they struggle to use the strategies for proofs learned in the bridge course in new contexts. It is a significant finding that students do not use the same proof schemes for proofs that they can construct as the schemes used for problems that they cannot solve. This has not been explored in the literature and will be discussed further in chapter five. The observation from this study that students have difficulty applying their knowledge of proofs to new contexts (like geometry or calculus) has been suggested by Ko and Knuth (2009) and aligns well with the findings of Shoenfeld (1989) where he observed that students see math as largely memorization, which makes the process of mathematical discovery a challenge to students. It is nevertheless a process which is indispensable in being able to prove a wide range of results.

The findings related to the second research question are that although most adequate/complete proofs were associated with analytical proof schemes, external conviction proof schemes resulted six times in adequate/complete proofs. Harel and Sowder (1998, 2007) argue that external conviction proof schemes cannot be used in increasingly complex proving situations. However, memorization and ritual presentation of arguments appeared to be surprisingly effective in this study at producing correct proofs. This seemed to occur on relatively simple proofs and when attempting proofs that were similar to others that students had seen. It is not clear whether this pattern would persist with more novel or difficult proofs that might appear later in the course or in a real analysis or abstract algebra class. This behavior is reminiscent of Benny's being able to produce correct answers without understanding (Erlwanger, 1973). It is important that it is clear to both teachers and researchers that a major implication with respect to this research question is that correct proofs do not imply internal conviction and that some issues in proving may only arise in a verbal interaction with the student. Several instances in the process of the interviews have showed that students were able to succeed with little to no understanding of the results or forms of argumentation by becoming familiar with the structure of a class of proofs. As proof problems move further from canonical examples, however, it is not clear that this strategy would continue to be effective for students.

Although not explicitly part of either research question for this study, a significant finding that arose from the six students who completed the course is that the students' proof understanding, proof production, and proof appreciation may be very different than one another, as occurred in almost half of the interviews for this study. This finding challenges the claims of Harel and Sowder (1998) who argue that proof appreciation goes hand-in-hand with proof production and understanding. Other authors such as Healy and Hoyles (2000), Knuth (2002), Recio and Godino (2001) and Sowder and Harel (2003) have argued that proof production and proof understanding occur simultaneously. Further investigation into these studies show that Harel and Sowder's (1998) argument for which all aspects of PUPA are related to one another is based on an appeal to what appears to be reasonable. That is, that a person's beliefs about the purpose and meaning of proof will certainly manifest itself in the way that they write and interpret proofs. Although that seems rational and reasonable at the surface, that is not investigated by the authors and never called into question. This study shows that such an assumption must not be made in every case. One possible reason for this is that proof understanding and proof production require a deeper level of understanding of proof than proof appreciation since there are so many technical details involved with these cognitive processes. As a result, it is possible that proof appreciation may not distinguish students in their competence in proof-related activities as do proof understanding and proof production. Additionally, the lack of diversity in students' primary proof scheme on the proof appreciation tasks suggests that a competent basis in understanding the purpose of proof may be a pre-requisite for success in proof production and proof appreciation. Healy and Hoyles (2000) do however, show that strength in proof understanding is correlated with strength in proof production, which is supported by this study, however, caution must be taken in not over-stating these results. That is, that while there may be a relationship between the aspects of a student's PUPA, it cannot be assumed that all of these aspects are identical to one another and each must be nurtured in order for a student to have a comprehensive understanding of proof.

CHAPTER 5: CONCLUSION

Summary of the Study

This study was designed to add to our knowledge about the development of students' understanding of proof during their first undergraduate course on proofs, known as the bridge course. The main motivation for such a study comes from the increased emphasis in American schools to introduce proofs in the early elementary school mathematics classrooms (AMS, 2001; CCSSI, 2010; MAA, 2004; NCTM, 2000). However, the difficult transition from the elementary grades to abstract mathematics continues to be a major challenge for many students (J. Selden et al., 2014). Thus, this study is directed at understanding both the impact of the students' initial conceptions of proof on their more mature conceptions upon completion of the bridge course and better understanding the process of this development. As a result, this study was directed towards answering the following research questions:

1. What influence do students' initial proof schemes have on the proof schemes they eventually develop in their bridge course?
2. What connections can be made between success in proving and students' proof schemes?

In order to address these research questions, a section of Brigham Young University's bridge course, known as Math 290: *Fundamentals of Mathematics* was selected and eight students were selected for interviews based on their varied conceptions of proof. Using the dual frameworks of Harel and Sowder (1998) and Ko and Knuth (2009) and by interviewing each student at the beginning of the semester, twice during the semester and upon completion of the course, I was able to categorize their proof schemes and proof types. Each interview included problems to address their schemes for proof understanding, proof production and proof appreciation. Thus, in addition to the research questions discussed above, I was able to analyze the relationship between these aspects of proof.

During the course of the study, an analysis of the proof schemes used by the students showed that the only significant change in the proof schemes used by the students was that after the first interview, empirical proof schemes were used very rarely in comparison to the first interview. However, external conviction proof schemes continued to be used throughout the course, with many students citing “convincing the teacher” as their goal for proofs and not striving towards internal conviction. Additionally, in many cases, the type of problem was a better predictor of the proof scheme used by the student than the student’s performance in other proof-related problems. Thus, many students in the study employed a wide variety of proof schemes.

By comparing and contrasting proof schemes and adequacy of produced proofs, it appeared that there were some instances of correct proofs in spite of a primary use of external conviction proof schemes, meaning that students with very little understanding about proof techniques and logic were nonetheless successful at producing some proofs. While most correct proofs were produced while a student used an analytical proof scheme, it remains a cause for concern that ritual proofs led to correct answers in the process of the interviews.

Additionally, this study called into question the reliability of the claims of Harel and Sowder (1998) that a student’s schemes for proof understanding, proof production and proof appreciation were necessarily related to one another. In several cases, the skills, insights, and schemes of the students in one aspect of proof was inconsistent with their skills, insights, and schemes that I would observe when I was assessing another aspect of proof.

Contributions to the Mathematics Education Research Community

My study has contributed to the Mathematics Education research community in a number of ways. First, my thesis shows that it may be difficult to change a student's beliefs about and orientations towards proving in a single semester bridge course once a student is in his or her undergraduate studies. A single semester bridge course alone should not stand as the solve way students learn about proofs. This means that efforts to introduce students to proof thinking in their early years may be able to help students later in their capacities to prove. This study also suggests that the initial conceptions of the students at the beginning of the study strongly influenced their proof scheme upon completing the course. Thus, this study suggests that further initiatives to increase mathematical curiosity, reasoning and sense making and productive use of examples in proof-related activities (following Knuth, 2015) may be helpful. As a result, this study further motivates the already numerous initiatives to increase the influence of proof in the early grades (AMS, 2001; CCSSI, 2010; MAA, 2004; NCTM, 2000).

A second contribution this study makes to the Mathematics Education research community is that it raises concerns about students' use of external conviction proof schemes, even as they begin to mature in their ability to prove. Harel and Sowder's (1998) proof schemes framework can map a student's progress as a mathematics student, however, since success in proving is defined by many students from their grade achievement and not actual understanding, students should experience a variety of proof-related activities and avoid the learning activities which allow them to succeed by using memorization and thereby avoiding true understanding (Middleton & Spanias, 1999). Thus, this study identifies the problem that external conviction remains a part of many students' proof schemes. It is important that bridge courses give students experiences in proving that cause them to rethink and readjust their own beliefs with respect to the role of proof within mathematics and exactly what impact that meaning has for them as mathematics students.

A third contribution that this study makes is that it demonstrates that while there may be a correlation between a student's proof understanding, proof production and proof appreciation, it cannot be assumed that each of these aspects of proof come hand-in-hand. Several authors have identified the connection between proof understanding and proof production (Healy & Hoyles, 2000; Knuth, 2002; Recio & Godino, 2001; Sowder & Harel, 2003) however, their findings may be overstated. In the case of the connection between all aspects of PUPA, no attempt was made by Harel and Sowder (1998) to justify the claims for the relationship between all three aspects of PUPA and this study, in part, addresses that claim. Thus, as a result, this study shows that further care needs to be taken in order to assess a students' complete understanding of proof. Thus, studies such as Sowder and Harel (2003) and Ko and Knuth (2009) that attempt to understand students' PUPA only by having the students produce proofs may not be collecting accurate information and while it may be worthwhile to assess proof production in isolation, care must be taken not to extrapolate these findings to conclusions about the student's entire PUPA.

Implications for Practice

It should be reiterated here that my analysis cannot evaluate the success of either the bridge course or the teacher that were the setting of the study. In this section, however, I suggest general insights and some possible directions for teaching proof. First, given the assumption that Knuth's (2015) suggestions for K-12 mathematics instruction build a more advanced and adaptable PUPA, these activities should be increased in K-12 and early undergraduate mathematics classes. The three major learning activities noted by Knuth (2015), fostering mathematical curiosity, reasoning and sense making and productive use of examples in proof-related activities, provide a solid basis for students to form productive conceptions about proof, which should be nurtured and strengthened in the bridge course. Undoubtedly the bridge course could be more effective to the extent that students can come to it having built an understanding about proofs from instruction in the early grades or from major adaptations to the bridge course.

A second implication for instruction which arises as a result of this study is that teachers of proof should be aware of the prevalence of the external conviction proof scheme among their students. Some of the students in this study did not see internal conviction as being important since getting good grades and making sure that the teacher is satisfied with the form and appearance of their proofs were often at the forefront of their minds. Teachers could help students move beyond this by encouraging flexibility in the form for proving, that is, by avoiding the use of repetitive sentence structure which is motivated by habit rather than necessitated by logic. Teachers may also be able to accomplish this emphasizing internal conviction during instruction. This can be accomplished through classroom discussions or individual introspection. Additionally, teachers can de-center themselves as the ultimate authorities on truth in mathematics and help students understand that the ultimate authority on truth in mathematics is the logic and the axioms (Weber, 2004). This may be accomplished by allowing students to discover proof methods as new axiomatic structures are considered and incorporating mathematical investigation and discovery into the classroom (Jones, 1977).

A third implication for instruction is that the curriculum in the bridge course should move beyond repetitive, similar proofs and include a wider variety of results and types of proofs. Several students in the study considered themselves to be successful at proving since they were able to master the *format* of proofs and thereby create other similar proofs, while not necessarily being completely able to apply the learning objectives of the bridge course beyond these examples.

A fourth implication for instruction which resulted from this study is that students must be engaged in all aspects of PUPA throughout their bridge course. This study showed that it cannot be assumed that a student's capacity to prove implied that they understood results presented to them or that they understood the significance of the consequences of the proof. As a result, students must continue to analyze and criticize their own, their teachers', and even their peers' proofs in order to contribute to a full understanding of proof. Additionally, the purpose and the significance of proofs in particular cases (as opposed to an overarching philosophical understanding of why mathematicians prove) is also critical to students' competence in proof and may not receive the attention it needs in the course of instruction during the bridge course. An emphasis on proof production alone is insufficient to help students understand and appreciate proofs.

Implications for Future Research

One of the major issues that arose during the study is the issue of external conviction not decreasing in the process of the bridge course. Strategies for helping students decrease their use of empiricism in proof-related problems have been well researched, however the same insights must be reached for external conviction proof schemes (G. Stylianides & Stylianides, 2009). However, although some suggestions were proposed in the previous section, strategies for helping students decrease their reliance on external conviction proof schemes should be verified and tested in order to evaluate their effectiveness. Additionally, further methods of helping students decrease their reliance on external conviction proof schemes should be investigated and verified as well as those discussed here.

Another issue related to external conviction proof schemes is that there were several instances of successful proofs stemming from external conviction proof schemes. Although this issue was discussed in the previous section, the implementation of a curriculum to address this problem is nontrivial and further research should be conducted into refining teaching methods and curriculum that allow students to prove without relying upon memorization and other external conviction-based methods of proving.

One major result from the study showed that students' proof schemes were not always consistent for proof understanding, proof production and proof appreciation. Although this study was not focused on understanding that relationship, because I ensured that that at least one problem of each type was considered during each interview, I was able to notice that these aspects of proof were not clearly identical as is suggested by Harel and Sowder (1998) was able to come to my attention. Thus by restructuring a study that gives an equal treatment of all three aspects of PUPA, researchers may be able to better understand the reasons why these aspects are not always consistent with one another. Additionally, in light of these findings, researchers such as Ko and Knuth (2009), Sowder and Harel (2003) and Harel and Sowder (2007) and should be cautious about making claims about a student's PUPA while only assessing their competence in proof production. Thus similar studies can be conducted where one or all aspects of proof may become the focus of inquiry.

Another result that stemmed from this study is that the nature of the problem posed to the student had a major impact on the scheme used by the student to solve the problem. That is, students often did not use the same proof schemes to approach the problems they were familiar with as they did in less familiar contexts. Thus, further investigation may help researchers come to a better understanding of what causes students to abandon the knowledge gained from proof in one context in a new or unfamiliar context. By better understanding the causes for this phenomenon and by later investigating how teachers may be able to deal with this concern during instruction, students may become more competent upon completion of the bridge course in other courses where proof is emphasized.

Limitations of this Study and Further Research Implications

There are several limitations for this study that must be considered. First, as discussed in the methods section, students were chosen based on their most likely proof schemes used on the pre-test, however, two of the eight students did not complete the course. Additionally, it was not possible to find two students who heavily favored external conviction proof schemes on the initial interview, thus it was difficult to choose a variety of students since there were so few students relying heavily on empirical and external conviction proof schemes initially. Also, some students who expressed interest in the study declined to participate when they were invited for interviews. Although the six remaining students provided some variety, there were more students which relied primarily on analytical proof schemes from the initial interview and only Ashley and Melanie relied heavily on deficient proof schemes at the start of the course and the study. In a future study, perhaps a different university may be considered since Brigham Young University is a competitive school which accepts less than half of its applicants. A school with a more open enrollment would allow for easier access to students who have weak conceptions of proof upon entering the bridge course.

Second, all students selected from the study were sampled from one section of the bridge course at Brigham Young University. Thus, it is possible that some of these findings are specific to this particular class, curriculum, and teacher. However, there is no reason to believe that her class is significantly different from any other section of the bridge course at Brigham Young

University and her class was selected to maximize my access to the students' work and to the classroom atmosphere. While using other sections, perhaps at other universities may be a good way to make more generalizable conclusions, it may be challenging to get the same amount of student access, for example, to administer a pre-test and have access to students' written work. Thus, the study would likely have to modify or omit certain components in order to be implemented in a variety of classrooms.

Third, this study was not quantitatively-based. While no attempt is made in this thesis to make conclusions about the mathematical behavior of all students, it may be possible to generalize these findings by verifying them quantitatively and conducting appropriate statistical tests for verifications. This study was qualitatively-based since my intention was to obtain as many details as possible about the eight students chosen from this study and I was not restricted to a categorical/quantitative answer to my research questions. Now that the groundwork has been laid out in this study, it would be worthwhile to discover exactly how often one aspect of a student's PUPA differs from his or her other aspects of PUPA, or to investigate whether external conviction proofs truly do increase throughout the bridge course. In this study, it was found that external conviction proofs schemes rose after the first interview, but the reason for it was not clear, thus we simply concluded that it did not improve. However, by collecting more data and analyzing it quantitatively, it would be possible to determine whether the bridge course actually has a way of nurturing the development of external proof schemes within students.

Lastly, all of the coding in this study was done by myself and was not verified by an additional researcher. For a thesis like this one, it was not possible to have someone else review and code over fifteen hours of interview data. While the distinction between some proof schemes were often subtle and occasionally subjective, particularly in terms of which proof scheme was the primary proof scheme, this study generally looked at proof schemes according to their main categorizations of external conviction proof schemes, empirical proof schemes and analytical proof schemes, thereby lessening the effect of this limitation. Harel and Sowder (1998) and Sowder and Harel (2003) explain that the result of coding a student according to proof scheme is not unique since it is not possible to fully understand what a student is thinking. Nonetheless this framework is the most commonly accepted framework to understand students' schemes with respect to proof and its use in this study was imperative to answering the research questions (Knipping, 2010).

Conclusion

The purpose of this study was to better understand the impact that a student's initial conceptions of proof has on their more matured conceptions upon completion of the bridge course, as well as to better understanding the development of their proof schemes and competence in proving throughout the course. By interviewing six students once before, twice during and once after the bridge course, I found that the only improvement in students' proof schemes came from them having less of a tendency to construct empirical proofs after only a few weeks of instruction. Additionally, students often used different proof schemes for proofs of a very familiar form and used deficient proof schemes to attempt to prove results that were less familiar to them. This study also identified several instances of students constructing coherent, complete proofs, while primarily relying on external conviction proof schemes. The production of complete proofs while relying on external conviction schemes means students were able to memorize patterns in proofs without having very much understanding. Lastly, this study challenged the claim of Harel and Sowder (1998) that a student's schemes for proof understanding, proof production and proof appreciation were connected since I identified several instances in the course of this study where there was a misalignment of at least one of these aspects of PUPA. With these results, we can begin to work to help teachers learn to help students develop appropriate conceptions of proof prior to entering the bridge course as well as to make curriculum and instructional adaptations in order to better help students improve upon their external conviction proof schemes.

References

Alcock, L., Hodds, M., Roy, S., & Inglis, M. (2015). Investigating and Improving Undergraduate Proof Comprehension. *Notices of the American Mathematical Society, 62*(7), 741-752.

Alcock, L., & Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. *The Journal of Mathematical Behavior, 24*(2), 125-134.

Barkai, R., Tsamir, P., Tirosh, D., & Dreyfus, T. (2002). *Proving or Refuting Arithmetic Claims: The Case of Elementary School Teachers.* Paper presented at the Annual Meeting of the International Group for the Psychology of Mathematics Education, Norwich, England.

Bell, A. W. (1979). The learning of process aspects of mathematics. *Educational Studies in Mathematics, 10*(3), 361-387.

Doerr, H. M., & English, L. D. (2003). A modeling perspective on students' mathematical reasoning about data. *Journal for Research in Mathematics Education, 34*(2), 110-136.

Ellis, A. B., Lockwood, E., Dogan, M. F., Williams, C. C., & Knuth, E. (2011). Choosing and using examples: How example activity can support proof insight. *Proceedings of 37th Conference of the International Group for the Psychology of Mathematics Education, 2*(1), 265-272.

Erlwanger, S. H. (1973). Benny's conception of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior, 1*(2), 7-26.

Fawcett, H. P. (1938). *The nature of proof*. New York: Teachers College, Columbia University.

Galbraith, P. L. (1981). Aspects of proving: A clinical investigation of process. *Educational Studies in Mathematics, 12*(1), 1-28.

Ginsberg, H. P., Kossan, N., Schwartz, R., & Swanson, D. (1981). Protocol methods in research on mathematical thinking. In H. P. Ginsberg (Ed.), *Development of mathematical thinking* (pp. 4-49). New York: Academic Press.

Goldin, G. A. (1997). Observing mathematical problem solving: Perspectives on structured, taskbased interviews. In A. Teppo (Ed.), *Qualitative Research Methods in Mathematics* (pp. 40-62). Reston, VA: National Council of Teachers of Mathematics.

Hanna, G. (1989). Proofs and that prove and proofs that explain. *Proceedings of the International Group for Psychology of Mathematics Education, 2*, 45-51.

Harel, G. (1998). Two dual assertions: The first on learning and the second on teaching (or vice versa). *The American Mathematical Monthly, 105*(6), 497-507.

Harel, G., Selden, A., & Selden, J. (2006). Advanced mathematical thinking. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 147-172). The Netherlands: Sense Publishers.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In T. Dick (Ed.), *Research in Collegiate Mathematics Education. III* (Vol. 7, pp. 234-282). USA: American Mathematical Society.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (Vol. 2, pp. 805-842). USA: National Council of Teachers of Mathematics

Healy, L., & Hoyles, C. (1998). *Justifying and proving in school mathematics: Technical report on the nationwide survey,* Institute of Education, University of London.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 31*(4), 396-428.

Inglis, M., & Mejia-Ramos, J. P. (2009). The effect of authority on the persuasiveness of mathematical arguments. *Cognition and Instruction, 27*(1), 25-50.

Jones, F. B. (1977). The Moore Method. *American Mathematical Monthly, 84*(4), 273-278.

Knipping, C. R., David A. (2010). *Proof in Mathematics Education*. Boston, Massachusetts: Sense Publishers.

Knuth, E. J. (2000). The rebirth of proof in school mathematics in the United States. *International Newsletter on the Teaching and learning of Mathematical Proof.* Retrieved from http://www.didactique.imag.fr/preuve/Newslatter/0.../000506ThemeUK.htm

Knuth, E. J. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education, 33*(5), 379-405. doi:10.2307/4149959

Knuth, E. J. (2015). *Laying a Foundation for Proving: Fostering Mathematical Curiosity, Sense Making and Reasoning.* Paper presented at the Utah Association of Mathematics Teacher Educators, Brigham Young University, Provo, Utah.

Knuth, E. J., & Elliott, R. L. (1998). Characterizing students' understandings of mathematical proof. *The Mathematics Teacher, 91*(8), 714-717.

Ko, Y.-Y., & Knuth, E. (2009). Undergraduate mathematics majors' writing performance producing proofs and counterexamples about continuous functions. *The Journal of Mathematical Behavior, 28*(1), 68-77.

Manaster, A. B. (1998). Some characteristics of eighth grade mathematics classes in the TIMSS videotape study. *American Mathematical Monthly, 105*(9), 793-805. doi:10.2307/2589208

Mariotti, M. A. (2006). Proof and proving in mathematics education. In A. Gutiérrez & P. Boero (Eds.), *Handbook of Research on Mathematics Teaching and Learning: Past Present and Future* (pp. 173-204). The Netherlands: Sense Publishers.

Martin, W. G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education, 20*(1), 41-51.

McGivney, J. M., & DeFranco, T. C. (1995). Geometry proof writing: A problem-solving approach a la Polya. *The Mathematics Teacher, 88*(7), 552-555.

Mejia-Ramos, J. P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics, 79*(1), 3-18.

Middleton, J. A., & Spanias, P. A. (1999). Motivation for achievement in mathematics: Findings, generalizations, and criticisms of the research. *Journal for Research in Mathematics Education, 30*(1), 65-88. doi:10.2307/749630

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics, 27*(3), 249-266.

Peled, I., & Zaslavsky, O. (1997). Counter-Examples That (Only) Prove and Counter-Examples That (Also) Explain. *FOCUS on Learning Problems in mathematics, 19*(3), 49-61.

Recio, A. M., & Godino, J. D. (2001). Institutional and personal meanings of mathematical proof. *Educational Studies in Mathematics, 48*(1), 83-99.

Reid, D. A., & Knipping, C. (2010). Proof in mathematics education. *Research, learning and teaching. Rotterdam: Sense Publisher*.

Schoenfeld, A. H. (1978). Presenting a strategy for indefinite integration. *American Mathematical Monthly, 85*(8), 673-678.

Schoenfeld, A. H. (1989). Explorations of Students' Mathematical Beliefs and Behavior. *Journal for Research in Mathematics Education, 20*(4), 338-355.

Schoenfeld, A. H. (1994). What do we know about mathematics curricula? *The Journal of Mathematical Behavior, 13*(1), 55-80.

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education, 34*(1), 4-36. doi:10.2307/30034698

Selden, J., Benkhalti, A., & Selden, A. (2014). *An analysis of transition-to-proof course students' proof constructions with a view towards course redesign.* Paper presented at the 17 th Annual Conference on Research in Undergraduate Mathematics Education.

Shaughnessy, J. M., & Burger, W. F. (1985). Spadework prior to deduction in geometry. *The Mathematics Teacher, 78*(6), 419-428.

Sowder, L., & Harel, G. (2003). Case studies of mathematics majors' proof understanding, production, and appreciation. *Canadian Journal of Math, Science & Technology Education, 3*(2), 251-267.

Stylianides, A. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*(3), 289-321.

Stylianides, A., Stylianides, G., & Philippou, G. (2004). Undergraduate students' understanding of the contraposition equivalence rule in symbolic and verbal contexts. *Educational Studies in Mathematics, 55*(1-3), 133-162.

Stylianides, G. (2009). Reasoning-and-proving in school mathematics textbooks. *Mathematical thinking and learning, 11*(4), 258-288.

Stylianides, G., & Stylianides, A. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education, 40*(3), 314-352.

Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity and proof. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 495-511). New York, NY: Macmillan Publishing.

Tall, D., & Mejia-Ramos, J. P. (2006). *The long-term cognitive development of different types of reasoning and proof.* Paper presented at the Conference on Explanation and Proof in Mathematics: Philosophical and Educational Perspectives.

Weber, K. (2002a). Beyond proving and explaining: Proofs that justify the use of definitions and axiomatic structures and proofs that illustrate technique. *For the learning of mathematics, 22*(3), 14-17.

Weber, K. (2002b). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48*(1), 101-119.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *The Journal of Mathematical Behavior, 23*(2), 115-133.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56*(2-3), 209-234.

Wilson, K. (2014). *Moving Students from Conjecture to Proof*. Unpublished manuscript, University of Delaware. Newark, DE.

Appendix A: Pre-test and Interview Protocol

Pre-Test Questions

Proof Appreciation Question:
Recalling some of the major results from your first calculus classes, such as the Intermediate Value Theorem, the Mean Value Theorem and the Fundamental Theorem of Calculus, how did you come to understand and become convinced of the theorem's validity?

Problem 1 (Proof Production – Counterexample):
Let m and n be integers that are both greater than zero. Is $m^2 - n^2$ prime, composite or impossible to tell?

Problem 2 (Proof Production – Direct Proof):
Let f(x) be a function defined on the real numbers. Assume $f(x)$ is continuous over the real numbers, is $|f(x)|$ continuous as well? Why or why not?

Problem 3 (Proof Production – Direct Proof):
Third, "Is there a number right after zero? That is, is there a smallest number, a such that $a > 0$ and there is no smaller number, b such that $0 < b < a$? Why or why not?".

Note that the last portion of the test was given after the students have submitted their responses to the first three problems because of the similarities between the first proof analysis in problem 4 and problem 3.

Problem 4 (Proof Analysis):
Consider the following two proposed proofs. Which one has a more convincing argument to you? Why do you prefer this proof to the other?

First, consider the following proof given by a student that there is no greatest number:

> *Proof:* *Let's imagine that there was a greatest number. Let's call it N. Now consider the number N. Since N+1 is one greater than what N is, N+1 is bigger than N, which is the biggest number. That's impossible! We can't have a number that's bigger than the biggest number, N. Therefore, it must be the case that there is no such thing as a biggest number, because we can always find a number that's bigger than it by adding one. Therefore, there is no biggest number.*

Consider the following proof of the same proposition.

> *Proof:* *Let $N \in R$. Assume for all $n \in R$, $n \leq N$.*
> Note: $(n + 1) > n$ and $(n + 1) \in R$
> Which is a contradiction.
> Thus, there is no biggest number.

Interview One Questions

Proof Appreciation Question:
You are taking a class that is totally devoted to teaching you to prove. Why do you think mathematicians emphasize proof so much?

Problem 1 (Proof Production – Direct Proof):
True or false: If n is an integer then $n^2 + n$ is an even number. Justify your reasoning.

Problem 2 (Proof Production – Counterexample):
True or false: If m is an integer such that m > 2, then $m^2 – 4$ is composite. Justify your reasoning.

Problem 3 (Proof Production – Direct Proof):
True or false: Assume f(x) and g(x) are odd functions. If we define (f + g)(x) as (f + g)(x) = f(x) + g(x). Then (f + g)(x) is an odd function.

Problem 4 (Proof Analysis):
Consider the following proposed proof. Do you find the proof to be convincing? Why or why not?

<u>Claim</u>: Let N be a positive integer. Then $1 + 2 + \ldots. + \mathrm{N} = \frac{N(N+1)}{2}$

<u>Proof</u>: Let's imagine ourselves adding the following numbers:

$$1 + 2 + 3 + \cdots + (\mathrm{N} - 2) + (\mathrm{N} - 1) + \mathrm{N}$$

Now let's imagine ourselves lining the same sum up backwards directly below the previous sum:

$$\begin{array}{ccccccccc} 1 + & 2 & + & 3 & + \cdots + & (\mathrm{N}-2) + & (\mathrm{N}-1) + & \mathrm{N} \\ N + & (N-1) & + & (N-2) & + \ldots + & 3 & + \quad 2 & +1 \end{array}$$

Notice the following by adding the numbers that are lined up:
1 + N = N + 1
2 + (N – 1) = N + 1
3 + (N – 2) = N + 1
……………………
(N – 2) + 3 = N + 1
(N – 1) + 2 = N + 1
N + 1 = N + 1
By combining the sums we notice that we have N occurrences of (N + 1) in the combined sum which means that the combined sum is equal to N(N + 1). However, in the combined sum, we added all of the numbers twice, thus we must half our result, which yields:

$$1 + 2 + \ldots. + \mathrm{N} = \frac{N(N+1)}{2}$$

Problem 5 (Proof Analysis):
Consider the following proposed proof. Do you find the proof to be convincing? Why or why not?

Claim: 0.99999… = 1

Proof: Let a = 0.99999…

Then 10a = 9.99999…

Then we have the following:

(10a – a) = 9.99999… – 0 .99999

(10 – 1)a = 9

9a = 9

a = 1

We began with a = 0.99999… and we obtained that it is necessary that a = 1. Thus, it must be the case that 0.99999… = 1.

Interview Two Questions

The second interview took place after the students had taken their first midterm test. This was about a third of the way through the semester.

Proof Appreciation Question:
What is gained by having a proof of a conjecture as opposed to having countless examples to support the conjecture? Other than being 100% sure of the truth of the conjecture are there any other benefits?

Problem 1 (Proof Production – Direct Proof):
True or false: If n is an integer and $|n+1| < 1, then\ |n^2 - 1| < 4.$

Problem 2 (Proof Production – Direct Proof):
Prove the following result: If f(x) and g(x) are differentiable on R, then (f + g)(x) is differentiable on R and (f + g)'(x) = f'(x) + g'(x).

Problem 3 (Proof Production – Counterexample):
True or false: If n is an integer, then $n^2 - n + 11$ is a prime number.

Problem 4 (Proof Analysis):
Consider the following proposed proof. Do you find the proof to be convincing? Why or why not?

Claim: Let x and y be integers and let a and b be odd integers. If ax + by is even then x and y are of the same parity.

Proof: Assume the contrary! That is, assume x and y have opposite parity. Assume that x is the even number and y is the odd number, thus we assign:

a = 2r + 1, for some integer, r;

b = 2s + 1, for some integer, s;

x = 2p, for some integer, p;

y = 2q + 1, for some integer, q.

Therefore, we have:

ax+by = (2r + 1)(2p) + (2s + 1)(2q + 1)
= 4pr + 2p + 4qs + 2s + 2q + 1
= 2(2pr + p + 2qs + s + q) + 1

Thus, ax + by is an odd integer since (2pr + p + 2qs + s + q) is an integer, being that it is a product and sum of integers.

This is a contradiction, since we were to assume that ax + by was even. Since the assumption that x and y were of opposite parity logically led to a contradiction, it must be the case that x and y are of the same parity.

Problem 5 (Proof Analysis):
The student was given one of his or her own proofs to evaluate.

Interview Three Questions

The third interview took place after the students had taken their second midterm test. This was about two thirds of the way through the semester.

Proof Appreciation Question:
How do you know whether you can accept a statement as true (like an axiom) or if a statement requires a proof?

Problem 1 (Proof Production – Direct Proof):
True or false: Every even integer is the sum of two odd integers.

Problem 2 (Proof Production – Direct Proof):
Use the following result: for all real numbers x and y,

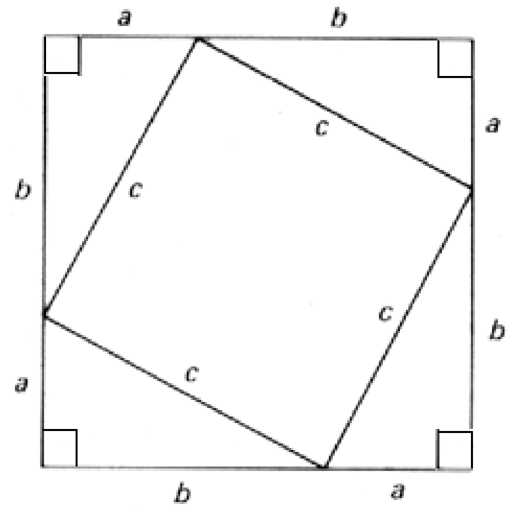
$$|x + y| \leq |x| + |y|$$

Show that $|x| - |y| \leq |x - y|$.

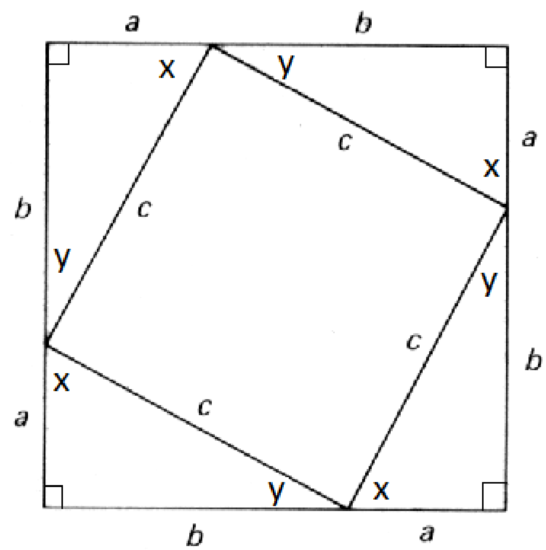
Problem 3 (Proof Production – Counterexample)
True or false: Let a and n be integers. If a divides n^2 then a divides n.

Problem 4 (Proof Analysis):
Consider the following proposed proof. Do you find the proof to be convincing? Why or why not?

Claim: Let a and b the length of the legs of a right triangle and let c be the length of the hypotenuse. Then $a^2 + b^2 = c^2$.
Proof: Consider the following figure:



The figure is constructed by repeating the right triangle described above four times and arranging them in such a way that a square of side length (a+b) is constructed. We note that there is a rhombus of side length c in the middle of the four triangles. We also note that since the four triangles are identical, then we can assign values, x and y to their angles, as we see below:



Now we can calculate the area of the square with side lengths (a+b) in two different ways, first by calculating the area directly and second by adding up the areas of the square and four triangles:
First: Area = $(a + b)^2 = a^2 + 2ab + b^2$
Second: $Area = 4\frac{ab}{2} + c^2 = 2ab + c^2$ *(since there are four triangles and the area of a triangle is given by* $\frac{(base)(height)}{2}$*)*
Now by combining both calculations of the area we have:
$a^2 + 2ab + b^2 = 2ab + c^2$*, thus* $a^2 + b^2 = c^2$.

Problem 5 (Proof Analysis):
The student was given one of his or her own proofs to evaluate.

Interview Four Questions

The final interview took place after the students had taken their final exam and completed the course.

Proof Appreciation Question:
What is the role of proof within mathematics? How do you think your opinion of the purpose of proof has changed throughout your Math 290 course?

Problem 1 (Proof Production – Direct Proof):
True or false: Let n be an odd integer. Then $n^2 = 1 \pmod 8$.

Problem 2 (Proof Production – Direct Proof):
True or false: Let a and b be real numbers such that $b > a$. There is a rational number, r such that $a < r < b$.

Problem 3 (Proof Production – Direct Proof):
As far as this problem is concerned, π is the ratio of the circumference of a circle to its diameter or the pi from the equation $A = \pi r^2$. *Imagine that you don't know what pi is and that you want to discover it.*

True or false: $\pi > 4$.

Problem 4 (Proof Analysis):
Consider the following argument. Do you find the argument to be convincing? Why or why not?

> On a certain island, each inhabitant always lies or always tells the truth. Calvin and Phoebe live on the island.
> Calvin says: "Exactly one of us is lying."
> Phoebe says: "Calvin is telling the truth"
> Determine who is telling the truth and who is lying.
>
> *Answer: Suppose Calvin is telling the truth. Then we know that "Exactly one of us is lying" is true thus Phoebe must be the liar since we assumed that Calvin was telling the truth. Thus we know that "Calvin is telling the truth" is false, thus Calvin is lying. Which contradicts the assumption that Calvin is telling the truth. Thus it must be the case that Calvin is lying. Therefore, we know that Phoebe is the truth teller. Thus Calvin is lying and Phoebe is telling the truth.*

Problem 5 (Proof Analysis):
The student was given one of his or her own proofs to evaluate.