2020-12-08

Eigenvalues of Differential Operators and Nontrivial Zeros of L-functions

Dongsheng Wu
Brigham Young University

Eigenvalues of Differential Operators and Nontrivial Zeros of L-functions

Dongsheng Wu

A dissertation submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Xian-Jin Li, Chair
David Cardon
Todd Fisher
Michael Griffin
Paul Jenkins

Department of Mathematics

Brigham Young University

ABSTRACT

Eigenvalues of Differential Operators and Nontrivial Zeros of L-functions

Dongsheng Wu
Department of Mathematics, BYU
Doctor of Philosophy

The Hilbert-Pólya conjecture asserts that the non-trivial zeros of the Riemann zeta function $\zeta(s)$ correspond (in a certain canonical way) to the eigenvalues of some positive operator. R. Meyer constructed a differential operator D_- acting on a function space $\mathcal{H}$ and showed that the eigenvalues of the adjoint of D_- are exactly the nontrivial zeros of $\zeta(s)$ with multiplicity correspondence.

We follow Meyer's construction with a slight modification. Specifically, we define two function spaces $\mathcal{H}_\cap$ and $\mathcal{H}_-$ on $(0,\infty)$ and characterize them via the Mellin transform. This allows us to show that $Z\mathcal{H}_\cap \subseteq \mathcal{H}_-$ where $Zf(x) = \sum_{n=1}^{\infty} f(nx)$. Also, the differential operator D given by $Df(x) = -xf'(x)$ induces an operator D_- on the quotient space $\mathcal{H} = \mathcal{H}_-/Z\mathcal{H}_\cap$. We show that the eigenvalues of D_- on $\mathcal{H}$ are exactly the nontrivial zeros of $\zeta(s)$. Moreover, the geometric multiplicity of each eigenvalue is one and the algebraic multiplicity of each eigenvalue is its vanishing order as a nontrivial zero of $\zeta(s)$.

We generalize our construction on the Riemann zeta function to some L-functions, including the Dirichlet L-functions and L-functions associated with newforms in $\mathcal{S}_k(\Gamma_0(M))$ with $M \geq 1$ and k being a positive even integer. We give spectral interpretations for these L-functions in a similar fashion.

Keywords: Hilbert Pólya conjecture, Riemann zeta function, L-functions, spectral interpretation, Poisson summation formulas.

Acknowledgements

I would like to express my sincere gratitude to my advisor Prof. Xian-Jin Li for the continuous support of my PhD study and related research, for his patience, motivation, and immense knowledge. His guidance helped me in all the time of research and writing of this dissertation.

Besides my advisor, I would like to thank the rest of my dissertation committee: Prof. David Cardon, Prof. Todd Fisher, Prof. Michael Griffin and Prof. Paul Jenkins, for their insightful comments and encouragement.

Contents

Chapter 1. Introduction

1.1 Relevant History

The distribution of prime numbers does not follow any regular pattern. However, B. Riemann [18] observed that the frequency of prime numbers is very closely related to the behavior of zeros of the Riemann zeta function, which is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\mathrm{Re}(s) > 1$. It admits the Euler product

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}}, \quad \mathrm{Re}(s) > 1$$

where p runs over all prime numbers. Riemann noticed that $\zeta(s)$ can be extended meromorphically to the whole complex plane except a simple pole at $s = 1$ and satisfies the functional equation

$$\xi(1 - s) = \xi(s), \tag{1.1}$$

where $\xi(s) = s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$; see [24, (2.1.13), p. 16]. Moreover, the function ξ is an entire of order 1. Since $\Gamma(s)$ has simple poles at $s = 0, -1, -2, \cdots$ and $\zeta(s)$ vanishes nowhere when $\mathrm{Re}(s) > 1$ by its Euler product, we see that $\zeta(s)$ has simple zeros at $s = -2, -4, -6, \cdots$. These zeros are called the *trivial zeros* of $\zeta(s)$. The remaining zeros of $\zeta(s)$ are called the *nontrivial zeros*. The celebrated *Riemann Hypothesis* asserts that all nontrivial zeros of the Riemann zeta function lie on the vertical line $\mathrm{Re}(s) = 1/2$, called the *critical line*. This has been checked for the first 100 billion nontrivial zeros of the Riemann zeta function, and they all lie on the critical line.

One possible approach to the Riemann Hypothesis is the *Hilbert-Pólya conjecture*, which roughly says that the Riemann Hypothesis is true because non-trivial zeros of the zeta function correspond (in a certain canonical way) to the eigenvalues of some positive operator.

In 1973, H. L. Montgomery [15] conjectured that the pair correlation function between nontrivial zeros of $\zeta(s)$ (normalized to have unit average spacing) is

$$1 - \left(\frac{\sin \pi u}{\pi u}\right)^2,$$

as F. J. Dyson pointed out to him that this is precisely the pair correlation function of the eigenvalues of a random complex Hermitian or unitary matrix of large order. Montgomery also noted in [15] that a positive proportion of zeros have a spacing between them strictly smaller than the mean spacing of the nontrivial zeros of $\zeta(s)$. More precisely,

Theorem (Montgomery, [15], 1973)**.** *Under the assumption of the Riemann Hypothesis, we have*

$$\liminf_n (\gamma_{n+1} - \gamma_n)(\log \gamma_n / 2\pi) \leq \lambda < 1$$

where $\gamma_n > 0$ is the imaginary part of the n-th nontrivial zero of $\zeta(s)$ in the upper half plane.

Using the above theorem, Brad Rodgers and Terence Tao [19] proved that the *De Bruijn-Newman constant* is non-negative. To be precise, for any real number $t \in \mathbb{R}$, define the entire function

$$H_t(z) := \int_0^\infty e^{tu^2} \Phi(u) cos(zu) du$$

where

$$\Phi(u) := \sum_{n=1}^{\infty} (2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}) \exp(-\pi n^2 e^{4u}).$$

Then the Riemann Hypothesis holds if and only if all the zeros of H_0 are real. This assertion follows from the classical fact (see [24, p. 255])

$$\frac{1}{16} \xi \left(\frac{1}{2} + \frac{iz}{2}\right) = H_0(z).$$

Newman [17] showed that there exists a finite constant Λ (the *De Bruijn-Newman constant*) such that the zeros of H_t are all real precisely when $t \geq \Lambda$. The Riemann Hypothesis is then equivalent to the assertion that $\Lambda \leq 0$. Assume $\Lambda < 0$ (which implies the Riemann Hypothesis), B. Rodgers and T. Tao showed that

$$\lim_{n\to\infty} (\gamma_{n+1} - \gamma_n)(\log \gamma_n/2\pi) = 1,$$

in contradiction to Montgomery's Theorem.

In 2019, M. Griffin, K. Ono, L. Rolen and D. Zagier [7] proved a derivative aspect Gaussian Unitary Ensemble (GUE) for the Riemann zeta function, which is "a rigorously proved result of this type and not just conjectured or made plausible by numerical evidence;" see E. Bombieri [2]. More precisely, let $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ and

$$(-1 + 4s^2)\Lambda\left(\frac{1}{2} + s\right) = \sum_{n=0}^{\infty} \frac{\gamma(n)}{n!} \cdot s^{2n}.$$

They showed that the *Jensen Polynomials*

$$J_\gamma^{d,n}(X) := \sum_{j=0}^{d} \binom{d}{j} \gamma(n+j) X^j$$

can be nicely approximated by *Hermite polynomials*. As a consequence, they proved that for any $d \geq 1$, the Jensen Polynomials $J_\gamma^{d,n}$ are *hyperbolic* for all sufficiently large n.

In the direction of the Hilbert-Pólya conjecture, a spectral interpretation for critical zeros (zeros on the critical line $\mathrm{Re}(s) = 1/2$) is given by A. Connes [4], which is modelled on Selberg's trace formula [21]. He constructed a closed, densely defined, unbounded differential operator D_χ and a Hilbert-Pólya space $\mathcal{H}_\chi$. His operator D_χ has discrete spectrum, which is the set of imaginary parts of critical zeros of the L-function with Grössencharacter χ [4, Theorem 1, p. 40]. Note that $\mathcal{H}_\chi$ is indeed a Hilbert space, but D_χ is not selfadjoint. For a nice reference of Grössencharacter, see [9].

R. Meyer [13] introduced another kind of Hilbert Pólya space. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space on $\mathbb{R}$ (see [22, p. 134]). Functions in $\mathcal{S}(\mathbb{R})$ are smooth (i.e., infinitely differentiable) and rapidly decreasing when $|x| \to \infty$; more precisely,

$$\mathcal{S}(\mathbb{R}) = \{f | \sup_{x \in \mathbb{R}} |x^m f^{(n)}(x)| < \infty \text{ for } m, n \in \mathbb{N}\},$$

where $\mathbb{N} = \{0, 1, 2, \cdots\}$. Denote by $\mathcal{H}_\cap$ the subspace of all even functions f in $\mathcal{S}(\mathbb{R})$ with $f(0) = \mathfrak{F}f(0) = 0$. Here

$$\mathfrak{F}f(x) = \int_{-\infty}^{\infty} f(y) e^{-2\pi i x y} dy$$

is the Fourier transform of f.

Let $\mathcal{H}_-$ be the set of all smooth functions on $(0, \infty)$ that decrease rapidly when $x \to 0+$ and ∞, that is,

$$\mathcal{H}_- = \{f : (0, \infty) \to \mathbb{C} | \sup_{x > 0} |x^m f^{(n)}(x)| < \infty \text{ for } |m|, n \in \mathbb{N}\}.$$

For $f \in \mathcal{H}_\cap$, define the operator Z by $Zf(x) = \sum_{n=1}^{\infty} f(nx)$. By the Poisson summation formula (see [22, Theorem 3.1, p. 154]), we have $Z\mathcal{H}_\cap \subseteq \mathcal{H}_-$. The quotient space $\mathcal{H} := \mathcal{H}_-/Z\mathcal{H}_\cap$ can be regarded as a Hilbert-Pólya space.

The fundamental differential operator D on $\mathcal{H}_-$ is given by

$$Df(x) = -xf'(x), \quad f \in \mathcal{H}_-.$$

Notice that $D\mathcal{H}_- \subseteq \mathcal{H}_-$ and $D(Z\mathcal{H}_\cap) \subseteq Z\mathcal{H}_\cap$. Then D induces an operator D_- on $\mathcal{H}$.

In [6], Liming Ge studied the operation of the differential operator D on the space $\mathcal{H}_-/Z(\mathcal{H}_- \cap \mathcal{H}_\cap)$, and obtained that the point spectrum of D coincides with the zeros of $\zeta(s)$ (see [6]).

In [13], Meyer proved in [13, Corollary 4.2, p. 8] that the eigenvalues of the transpose D_-^t of D_- (acting on the space of continuous linear functionals on $\mathcal{H}$) are exactly the non-trivial zeors of $\zeta(s)$, and the algebraic multiplicity of an eigenvalue is equal to the vanishing order of the corresponding zero of $\zeta(s)$.

Li [12] proved by an explicit construction that the nontrivial zeros of $\zeta(s)$ are the eigenvalues of D_- acting on $\mathcal{H}$ (instead of D_-^t in [13]), which we state as follows (Theorem 1.1 and Theorem 1.4 in [12]).

Theorem 1.1. *Let ρ be a nontrivial zero of $\zeta(s)$. Let F_ρ be the function given by*

$$F_\rho(x) = \int_1^\infty Z\eta(tx)t^{\rho-1}dt, \; x > 0,$$

where

$$\eta(x) = 8\pi x^2 \left(\pi x^2 - \frac{3}{2}\right) e^{-\pi x^2}.$$

Then F_ρ is an eigenfunction of D_- on $\mathcal{H}$ associated with eigenvalue ρ. Moreover, the eigenvalue ρ has geometric multiplicity one.

In the above, we describe the dimension of the corresponding eigen space as the *geometric multiplicity* of the eigenvalue.

1.2 Zeta operators

1.2.1 Zeta operator for Riemann zeta function. First we restate Meyer's construction as follows.

Let $\mathcal{H}_0$ be the set of smooth functions f on $\mathbb{R}_+^\times = (0, \infty)$ such that

$$\lim_{x\to\infty} x^m f^{(n)}(x) = 0$$

and

$$f^{(n)}(0) := \lim_{x\to 0+} f^{(n)}(x)$$

exists for any $m, n \in \mathbb{N}$.

Let

$$\mathcal{H}_{\cap} = \{f \in \mathcal{H}_0 \big| \int_0^{\infty} f(x)dx = 0, f(0) = 0 \text{ and } f^{(2n+1)}(0) = 0 \text{ for } n \in \mathbb{N}\}$$

and

$$\mathcal{H}_{-} = \{f \in \mathcal{H}_0 \big| f^{(n)}(0) = 0 \text{ for } n \in \mathbb{N}\}.$$

It is easy to see that the above definitions of $\mathcal{H}_{\cap}$ and $\mathcal{H}_{-}$ coincide with Meyer's original construction.

For $f \in \mathcal{H}_{\cap}$, denote by Z the operator given by $Zf(x) = \sum_{n=1}^{\infty} f(nx)$. We will show that $Z\mathcal{H}_{\cap} \subseteq \mathcal{H}_{-}$ (Theorem 2.9). We regard the quotient space $\mathcal{H}_{-}/Z\mathcal{H}_{\cap}$, denoted by $\mathcal{H}$, as a Hilbert-Pólya space. The operator D defined as $Df(x) = -xf'(x)$ induces an operator on $\mathcal{H}$, which is denoted by D_{-}.

Let f be a function on $(0, \infty)$. The *Mellin tranform* of f is

$$\widehat{f}(s) = \int_0^{\infty} f(x)x^{s-1}dx.$$

For $f \in \mathcal{H}_{-}$, it is easy to check that $\widehat{f}(s)$ is entire on $\mathbb{C}$. By partial integration,

$$\widehat{Df}(s) = -\int_0^{\infty} xf'(x)x^{s-1}dx = -\int_0^{\infty} x^s df(x) = s\int_0^{\infty} f(x)x^{s-1}dx = s\widehat{f}(s).$$

Let g be a function in $\mathcal{H}_{\cap}$. We have

$$\widehat{Zg}(s) = \int_0^{\infty} \left(\sum_{n=1}^{\infty} g(nx)\right) x^{s-1}dx = \sum_{n=1}^{\infty} \int_0^{\infty} g(y) \left(\frac{y}{n}\right)^{s-1} d\left(\frac{y}{n}\right) = \zeta(s)\widehat{g}(s)$$

when $\mathrm{Re}(s) > 1$.

Recall that the *algebraic multiplicity* of an eigenvalue λ of an operator T is the dimension of the space $\bigcup_{n\geq 1} \ker(T-\lambda)^n$. Our *main theorem* asserts that the eigenvalues of D_{-} acting on $\mathcal{H}$ are precisely the nontrivial zeros of $\zeta(s)$, which we state as follows.

Theorem 1.2. *A complex number ρ is an eigenvalue of D_- on $\mathcal{H}$ if and only if ρ is a nontrivial zero of $\zeta(s)$. Moreover, the geometric multiplicity of such an eigenvalue ρ is 1, and its algebraic multiplicity is equal to its vanishing order as a zero of $\zeta(s)$.*

1.2.2 Zeta operators for Dirichlet L-functions.

Let χ be a primitive Dirichlet character modulo $N > 1$. Recall that the Dirichlet L-function is defined as

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for s with $\mathrm{Re}(s) > 0$ (see [15, p. 120]). It admits the Euler product

$$L(s,\chi) = \prod_{p} \frac{\chi(p)}{1-p^{-s}}, \quad \mathrm{Re}(s) > 1$$

where p is taken over all prime numbers.

Let $\tau(\chi) = \sum_{n \bmod N} \chi(n) e^{2\pi i n/N}$ be the *Gauss Sum* of χ and let $\epsilon = \frac{1-\chi(-1)}{2}$. The function $L(s,\chi)$ satisfies the functional equation

$$\Lambda(s,\chi) = (-i)^{\epsilon} \tau(\chi) N^{-s} \Lambda(1-s,\overline{\chi}), \tag{1.2}$$

where $\Lambda(s,\chi) = \pi^{-\frac{s+\epsilon}{2}} \Gamma\left(\frac{s+\epsilon}{2}\right) L(s,\chi)$ is an entire function of order 1. Since $\Gamma(s)$ has simple poles at $s = 0, -1, -2, \cdots$ and $L(s,\chi) \neq 0$ when $\mathrm{Re}(s) > 1$ by its Euler product, we deduce that $L(s,\chi)$ has simple zeros at $s = -\epsilon, -2-\epsilon, -4-\epsilon, \cdots$ (see [15, p. 333]). These zeros are called the trivial zeros of $L(s,\chi)$. The remaining zeros are called the nontrivial zeros of $L(s,\chi)$. The *Generalized Riemann Hypothesis* for $L(s,\chi)$ asserts that all nontrivial zeros of $L(s,\chi)$ lie on the vertical line $\mathrm{Re}(s) = 1/2$.

Similar to the construction of the Hilbert-Pólya space for $\zeta(s)$, we can define a Hilbert-Pólya space for $L(s,\chi)$. Let $\mathcal{H}_{\cap}^{\chi}$ be the subspace of all functions in $\mathcal{H}_0$ satisfying $f^{(2n+1)}(0) = 0$ for $n \in \mathbb{N}$ if $\chi(-1) = 1$ and the subspace of all functions in $\mathcal{H}_0$ such that $f^{(2n)}(0) = 0$ for

$n \in \mathbb{N}$ if $\chi(-1) = -1$.

For $f \in \mathcal{H}_{\cap}^{\chi}$, denote by Z_χ the operator given by $(Z_\chi f)(x) = \sum_{n=1}^{\infty} \chi(n) f(nx)$. We will see that $Z_\chi \mathcal{H}_{\cap}^{\chi} \subseteq \mathcal{H}_-$ (Theorem 2.9). We regard the quotient space $\mathcal{H}_-/Z_\chi \mathcal{H}_{\cap}^{\chi}$, denoted by $\mathcal{H}^\chi$, as the Hilbert-Pólya space for $L(s,\chi)$. Notice that the differential operator D satisfies $DZ_\chi \mathcal{H}_{\cap}^{\chi} \subseteq Z_\chi \mathcal{H}_{\cap}^{\chi}$. It induces an operator D_-^χ on $\mathcal{H}^\chi$.

Let $g \in \mathcal{H}_{\cap}^{\chi}$. Then

$$\widehat{Z_\chi g}(s) = \int_0^\infty \left(\sum_{n=1}^{\infty} \chi(n) g(nx) \right) x^{s-1} dx = \sum_{n=1}^{\infty} \chi(n) \int_0^\infty g(y) \left(\frac{y}{n} \right)^{s-1} d\left(\frac{y}{n} \right) = L(s,\chi) \widehat{g}(s)$$

when $\mathrm{Re}(s) > 1$.

In [14, Theorem 5.11], Meyer constructed a summable representation π whose spectrum consists of the poles and zeros (with multiplicity) of L-functions of an arbitrary algebraic number field. We will show the set of eigenvalues of D_-^χ acting on $\mathcal{H}^\chi$ is equal to the set of nontrivial zeros of $L(s,\chi)$ with multiplicities correspondence as follows.

Theorem 1.3. *A complex number ρ is an eigenvalue of D_-^χ on $\mathcal{H}^\chi$ if and only if ρ is a nontrivial zero of $L(s,\chi)$. Moreover, the geometric multiplicity of every eigenvalue ρ is one and its algebraic multiplicity is equal to the vanishing order of $L(s,\chi)$ at ρ.*

1.2.3 Zeta operators for L-functions associated with newforms.

We first introduce some basic theory of modular forms. For detailed discussion on this topic, we refer to [5].

Let $\mathbb{H} = \{z = x + iy : x \in \mathbb{R}, y > 0\}$ be the upper half-plane. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}^+)$, denote

$$\gamma z = \frac{az+b}{cz+d}, \quad z \in \mathbb{H}.$$

One can check that this defines a transitive action of the group $GL_2(\mathbb{R}^+)$ on $\mathbb{H}$.

Let k be an integer. For any function $f : \mathbb{H} \to \mathbb{C}$ and real matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with positive determinant, the function $f[\gamma]_k$ is defined as

$$(f[\gamma]_k)(z) = (ad - bc)^{k/2}(cz + d)^{-k} f(\gamma z), \quad z \in \mathbb{H}.$$

We call a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ a *modular form of weight* k if

$$f[\gamma]_k = f, \quad \forall \gamma \in SL_2(\mathbb{Z}),$$

and f is holomorphic at ∞. The set of modular forms of weight k is denoted by $\mathcal{M}_k(SL_2(\mathbb{Z}))$. A well-known fact of $\mathcal{M}_k(SL_2(\mathbb{Z}))$ is that it is a finite dimensional vector space over $\mathbb{C}$, see, e.g., [5, p. 88].

Let f be a modular form of weight k. Then $f(z+1) = f(z)$ since $z + 1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z$. Hence f admits a *Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n(f) q^n, \quad q = e^{2\pi i z}.$$

If $a_0(f) = 0$, we call f a *cusp form of weight* k. The set of cusp forms of weight k is denoted by $\mathcal{S}_k(SL_2(\mathbb{Z}))$.

Modular forms can also be defined for *congruence subgroups*, i.e., a subgroup of $SL_2(\mathbb{Z})$ that contains the *principal subgroup*

$$\Gamma(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{M} \right\}$$

for some $M \geq 1$. Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. A holomorphic function

$f : \mathbb{H} \to \mathbb{C}$ is said to be a *modular form of weight k with respect to* Γ if

$$f[\gamma]_k = f, \quad \forall \gamma \in \Gamma,$$

and $f[\alpha]_k$ is holomorphic at ∞ for all $\alpha \in SL_2(\mathbb{Z})$. If in addition, $a_0 = 0$ in the Fourier expansion of $f[\alpha]_k$ for all $\alpha \in SL_2(\mathbb{Z})$, then f is a *cusp form of weight k with respect to* Γ. The set of modular (resp., cusp) forms of weight k with respect to Γ is denoted as $\mathcal{M}_k(\Gamma)$(resp., $\mathcal{S}_k(\Gamma)$). The space $\mathcal{M}_k(\Gamma)$ is finite dimensional over $\mathbb{C}$ as well (see[5, p. 88]).

Let Γ be a congruence subgroup. The space $\mathcal{S}_k(\Gamma)$ admits an inner product, called the *Petersson inner product*, which is defined as follows.

Let

$$d\mu(z) = \frac{dxdy}{y^2}, \quad z = x + iy \in \mathbb{H}$$

be the *hyperbolic measure* on $\mathbb{H}$. It is invariant under all the linear transforms induced by elements in $GL_2^+(\mathbb{R})$. For any $f, g \in \mathcal{S}_k(\Gamma)$, their Petersson inner product on $\mathcal{S}_k(\Gamma)$) is

$$\langle f, g \rangle = \frac{1}{V_\Gamma} \int_{\mathcal{D}_\Gamma} f(z)\overline{g(z)}(\mathrm{Im}(z))^k d\mu(z),$$

where $\mathcal{D}_\Gamma$ is a fundamental domain of $\mathbb{H}$ under the action of Γ and

$$V_\Gamma = \int_{\mathcal{D}_\Gamma} d\mu(z)$$

is the volume of $\mathcal{D}_\Gamma$ under the measure $d\mu$.

We now fix, once and for all, a positive integer M and a positive even integer k. A tool of great importance in the study of modular forms is the *Hecke operators*(see [8]). Let $N \geq 1$ be an integer and

$$\Gamma_0(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : M \mid c \right\}$$

be a congruence subgroup. For $f \in \mathcal{M}_k(\Gamma_0(M))$, define

$$f|T_p(z) = \sum_{j=0}^{p-1} p^{-1} f\left(\frac{z+j}{p}\right) + p^{2k-1} f(pz),$$

see [1]. Let p and q be primes such that $p \nmid M$ and $q \nmid M$. Then we have

(i) $T_p\mathcal{M}_k(\Gamma_0(M)) \subseteq \mathcal{M}_k(\Gamma_0(M))$, $T_p\mathcal{S}_k(\Gamma_0(M)) \subseteq \mathcal{S}_k(\Gamma_0(M))$, and

(ii) $T_pT_q = T_qT_p$.

A modular form $f \in \mathcal{M}_k(\Gamma_0(M))$ is called an *eigenform* (or *Hecke eigenform*) if it is a common eigenvector for all the Hecke operators T_p, $p \nmid M$. An eigenform is said to be normalized if $a_1(f) = 1$.

The Hecke operators T_p, $p \nmid M$, and the Petersson inner product are related by the following theorem.

Theorem (Petersson). *For any prime $p \nmid M$, the Hecke operator T_p is self-adjoint with respect to the Petersson inner product.*

Since the Hecke operators T_p commute with each other and $\mathcal{S}_k(\Gamma_0(M))$ is a finite-dimensional inner product space, the space has an orthogonal basis of common eigenvectors for all T_p. We state this result as following.

Theorem. *The space $\mathcal{S}_k(\Gamma_0(M))$, endowed with the Petersson inner product, has an orthogonal basis of eigenforms.*

For each positive integer d, let

$$F|B_d = d^{-k} F\left[\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}\right]_k.$$

If $f \in \mathcal{S}_k(\Gamma_0(M))$, one can verify directly that $f|B_d \in \mathcal{S}_k(\Gamma_0(dM))$. Let $\mathcal{S}_k(\Gamma_0(M))^-$ be the subspace of $\mathcal{S}_k(\Gamma_0(M))$ generated by $g|B_d$ where $g \in \mathcal{S}_k(\Gamma_0(M'))$, $dM' \mid M$. Since T_p

commutes with B_d if $(p,d)=1$, we have

$$T_p : \mathcal{S}_k(\Gamma_0(M))^- \to \mathcal{S}_k(\Gamma_0(M))^- \quad \text{if} \quad p \nmid M.$$

Let $\mathcal{S}_k(\Gamma_0(M))^+$ be the orthogonal complement of $\mathcal{S}_k(\Gamma_0(M))^-$ in $\mathcal{S}_k(\Gamma_0(M))$. Then T_p : $\mathcal{S}_k(\Gamma_0(M))^+ \to \mathcal{S}_k(\Gamma_0(M))^+$ for all $(p,M)=1$. The space $\mathcal{S}_k(\Gamma_0(M))^+$ thus has an orthogonal basis of eigenforms. We call an eigenform in $\mathcal{S}_k(\Gamma_0(M))^+$ a *newform*.

Let $f(z)=\sum_{n=1}^{\infty} a_n q^n \in \mathcal{S}_k(\Gamma_0(M))$ be a cusp form of weight k. Its associated L-function is

$$L(s,f)=\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

which converges absolutely when $\mathrm{Re}(s) > k/2+1$. If f is further a normalized newform, by [11, Theorem 3], $L(s,f)$ admits the Euler product

$$L(s,f)=\prod_{q|M}(1-a_q q^{-s})^{-1}\prod_{p\nmid N}(1-a_p p^{k-1-2s})^{-1}. \tag{1.3}$$

To obtain analytic properties of $L(s,f)$, we need the following operator:

$$f|H_M = f\left[\begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}\right]_k \quad \text{for} \quad f \in \mathcal{S}_k(\Gamma_0(M)).$$

The operator H_M satisfies:

(i) $H_M^2 = id$, and

(ii) $H_M T_p = T_p H_M$ for all $(p,M)=1$.

If $f|H_M = \mu f$ for some constant μ, by (i), we have $\mu = \pm 1$. According to [11, Remarks (ii), p. 304], the normalized new form in $\mathcal{S}_k(M)$ are exactly those which are eigenvectors of H_M and whose associated L-function have Euler product (1.3). The following theorem characterizes analytic properties of $L(s,f)$.

Theorem. *Let* $f(z) = \sum_{n=1}^{\infty} a_n q^n \in \mathcal{S}_k(\Gamma_0(M))$ *and*

$$\Lambda_f(s) = M^{s/2}(2\pi)^{-s}\Gamma(s)L(s,f).$$

If $f = C \cdot i^k \cdot f|H_M$ *with* $C = \pm 1$, *then* Λ_f *has analytic continuation in the whole* s*-plane and satisfies*

$$\Lambda_f(s) = C\Lambda_f(k-s).$$

A converse to the above theorem, that is, a Dirichlet series with analytic continuations and functional equations must come from a modular form, also holds. See, e.g., [27].

Let $\phi(z) = \sum_{n=1}^{\infty} a_n q^n \in \mathcal{S}_k(\Gamma_0(M))$ be a normalized newform and

$$\Lambda_\phi(s) = M^{s/2}(2\pi)^{-s}\Gamma(s)L(s,\phi).$$

By the above discussion, the function $\Lambda_\phi(s)$ extends to an entire function and satisfies the functional equation

$$\Lambda_\phi(s) = C\Lambda_\phi(k-s), \tag{1.4}$$

where $C = 1$ or -1, a constant depending on ϕ. Moreover, the order of the entire function $\Lambda_\phi(s)$ is 1. This could be derived from the following observation:

$$\begin{aligned}
\Lambda_\phi(s) =& M^{s/2}\int_0^\infty \phi(it)t^s\frac{dt}{t} = \int_0^\infty \phi(it/\sqrt{M})t^s\frac{dt}{t} \\
&= \int_1^\infty \phi(it/\sqrt{M})t^s\frac{dt}{t} + \int_0^1 \phi(it/\sqrt{M})t^s\frac{dt}{t} \\
&= \int_1^\infty \phi(it/\sqrt{M})t^s\frac{dt}{t} + C\int_0^1 \phi|H_M(it/\sqrt{M})t^s\frac{dt}{t} \\
&= \int_1^\infty \left(\phi(it/\sqrt{M})t^s + C\phi(it/\sqrt{M})t^{k-s}\right)\frac{dt}{t}
\end{aligned}$$

for $\mathrm{Re}(s) > k/2+1$. Applying the estimate $\phi(it) = O(e^{-2\pi t})$ to the last integral, we deduce that

$$\Lambda_\phi(s) = O(e^{c|s|\log|s|}) \tag{1.5}$$

for some $c > 0$.

By the above discussion, we conclude that $L(s,\phi)$ admits an analytic continuation to the whole s-plane with simple zeros at $s = -n$, $n \in \mathbb{N}$. These zeros are called the *trivial zeros* of $L(s,\phi)$. The other zeros are called the *nontrivial zeros* of $L(s,\phi)$. By (1.3) and (1.4), they lie in the vertical strip $\{0 \le \mathrm{Re}(s) \le k\}$ and $L(k,\phi) \ne 0$. A conjecture similar to the Riemann Hypothesis states that all the nontrivial zeros of $L(s,\phi)$ lie on the vertical line $\mathrm{Re}(s) = k/2$.

Define $Z_\phi f(x) = \sum_{n=1}^{\infty} a_n f(nx)$ for $f \in \mathcal{H}_0$. Then $Z_\phi \mathcal{H}_0 \subseteq \mathcal{H}_-$ (Theorem 2.9). The quotient space $\mathcal{H}^\phi := \mathcal{H}_-/Z_\phi\mathcal{H}_0$ is the Hilbert-Pólya space with respect to $L(s,\phi)$. Also, the differential operator D satisfies $DZ_\phi\mathcal{H}_0 \subseteq Z_\phi\mathcal{H}_0$, thus it induces an operator D_-^ϕ on $\mathcal{H}^\phi$.

Let $g \in \mathcal{H}_0$. Then

$$\widehat{Z_\phi g}(s) = \int_0^\infty \left(\sum_{n=1}^{\infty} a_n g(nx)\right) x^{s-1}dx = \sum_{n=1}^{\infty} a(n) \int_0^\infty g(y)\left(\frac{y}{n}\right)^{s-1} d\left(\frac{y}{n}\right) = L(s,\phi)\widehat{g}(s)$$

when $\mathrm{Re}(s) > k/2 + 1$.

The set of eigenvalues of D_-^ϕ on $\mathcal{H}^\phi$ is exactly the set of nontrivial zeros of $L(s,\phi)$ as follows.

Theorem 1.4. *A complex number ρ is an eigenvalue of D_-^ϕ on $\mathcal{H}^\phi$ if and only if ρ is a nontrivial zero of $L(s,\phi)$. Moreover, the geometric multiplicity of every eigenvalue ρ is one and its algebraic multiplicity is equal to the vanishing order of $L(s,\phi)$ at ρ.*

1.3 Outline

In Chapter 2, we lay some basic properties for zeta operators. These properties will be used many times in later chapters.

In Chapter 3, we show various Poisson summation formulas for functions in $\mathcal{H}_0, \mathcal{H}_\cap$ and $\mathcal{H}_\cap^\chi$. In particular, we notice surprisingly that there is a connection between newforms and Bessel functions.

In Chapter 4, we prove Theorem 1.2.

In Chapter 5, we prove Theorem 1.3.

In Chapter 6, we prove Theorem 1.4.

2.1 A Description of $\mathcal{H}_0$ and $\mathcal{H}_-$

Lemma 2.1. *Let $f \in \mathcal{H}_0$ and*

$$\widehat{f}(s) = \int_0^\infty f(x)x^{s-1}dx$$

be the Mellin transform of f. Then $\widehat{f}$ admits a meromorphic extension to the whole complex plane and its only singularities are simple poles at a subset of non-positive integers. Moreover, for any $n \in \mathbb{N}$, the function $\widehat{f}(s)$ is holomorphic at $s = -n$ if and only if $f^{(n)}(0) = 0$.

Proof. Let $\mathrm{Re}(s) > 0$. For any $k \in \mathbb{N}$, integrating by parts several times yields

$$\begin{aligned}\widehat{f}(s) &= \int_0^\infty f(x)x^{s-1}dx = \frac{x^s}{s}f(x)\Big|_0^\infty - \frac{1}{s}\int_0^\infty f'(x)x^s dx \\ &= -\frac{1}{s}\int_0^\infty f'(x)x^s dx = \cdots = \frac{(-1)^k}{s(s+1)\cdots(s+k-1)}\int_0^\infty f^{(k)}(x)x^{s+k-1}dx.\end{aligned}$$

Since the integral $\int_0^\infty f^{(k)}(x)x^{s+k-1}dx$ exists when $\mathrm{Re}(s) > -k$, we can extend $\widehat{f}(s)$ meromorphically to the whole complex plane with possible simple poles at $s = 0, -1, -2, \cdots$.

The remaining assertion follows by noticing that, for any $n \in \mathbb{N}$, the function $\widehat{f}(s)$ is holomorphic at $s = -n$ if and only if

$$0 = \int_0^\infty f^{(n+1)}(x)x^{-n+n}dx = \int_0^\infty f^{(n+1)}(x)dx = -f^{(n)}(0).$$

This completes the proof of the Theorem. □

For $-\infty < a < b < \infty$, let $\Omega_{a,b}$ denote the vertical strip $\{s \in \mathbb{C} | a \leq \mathrm{Re}(s) \leq b, |\mathrm{Im}(s)| \geq 1\}$.

Let f be a function on $\mathbb{C}$. We say that f is *essentially bounded* if $s^m f(s)$ is bounded in $\Omega_{a,b}$ for any $-\infty < a < b < \infty$ and $m \in \mathbb{N}$.

Theorem 2.2. *Assume $f \in \mathcal{H}_0$. Then the function $\widehat{f}(s)$ is meromorphic, essentially bounded and its singularities are simple poles at a subset of non-positive integers.*

On the other hand, let $h(s)$ be an essentially bounded meromorphic function on $\mathbb{C}$ and assume its singularities are simple poles at a subset of non-positive integers. Then h is the Mellin transform of some function in $\mathcal{H}_0$.

Proof. By Lemma 2.1, $\widehat{f}(s)$ is a meromorphic function on $\mathbb{C}$ whose singularities are simple poles at a subset of non-positive integers.

Pick $n \in \mathbb{N}$ such that $-n-1 < a < b < \infty$. We can find a constant $C > 0$ depending only on a and b such that

$$|s^m| \leq C \cdot |(s+n+1)(s+n+2)\cdots(s+n+m)|$$

for $s \in \Omega_{a,b}$. Hence

$$\begin{aligned} |s^m F_n(s)| \leq & C \cdot \left|(s+n+1)\cdots(s+n+m)\int_0^\infty f^{(n+1)}(x)x^{s+n}dx\right| \\ = & C \cdot \left|\int_0^\infty f^{(n+m-1)}(x)x^{s+n+m}dx\right| \end{aligned}$$

is bounded in $\Omega_{a,b}$ as $f^{(n+m+1)}(x)$ decays rapidly when $x \to \infty$ and $\mathrm{Re}(s+n+m) \geq a+n+m > -1$.

On the other hand, let $h(s)$ be an essentially bounded meromorphic function on $\mathbb{C}$ and its possible poles are simple poles at $s = 0, -1, -2, \cdots$. Define

$$g(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h(s)x^{-s}ds$$

for $x > 0, c \in \mathbb{R}$. Since h is essentially bounded, $g(x)$ is independent of the choice of c. Moreover, for any $n \in \mathbb{N}$,

$$g^{(n)}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -s(-s-1)\cdots(-s-n-1)h(s)x^{-s-n}ds,$$

and the values $g^{(n)}(x)$ are also independent of the choice of c. Hence

$$\begin{aligned}|x^m g^{(n)}(x)| =& \left|\frac{1}{2\pi i}\int_{m+1-i\infty}^{m+1+i\infty} -s(-s-1)\cdots(-s-n-1)h(s)x^{-s-n+m}ds\right| \\ \leq& \frac{1}{2\pi i}\int_{m+1-i\infty}^{m+1+i\infty} \frac{1}{x}\cdot|-s(-s-1)\cdots(-s-n-1)h(s)|\,ds \to 0\end{aligned}$$

as $x \to \infty$ for any $m, n \in \mathbb{N}$.

It remains to show that $\lim_{x\to 0+} g^{(n)}(x)$ exists for all $n \in \mathbb{N}$. Notice that the function

$$h_n(s) := -s(-s-1)\cdots(-s-n-1)h(s)$$

has at most a simple pole at $s = -n$ in the region $\mathrm{Re}(s) > -n-1$ and $s^m h_n(s)$ is essentially bounded. Consider the integral of $h_n(s)$ along the boundary of the rectangle with vertices $-n-\frac{1}{2}-iy$, $-n-\frac{1}{2}+iy$, $\frac{1}{2}-iy$ and $\frac{1}{2}+iy$. By the residue formula and letting $y \to \infty$, we have

$$g^{(n)}(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} h_n(s)x^{-s-n}ds = \frac{1}{2\pi i}\int_{-n-\frac{1}{2}-i\infty}^{-n-\frac{1}{2}+i\infty} h_n(s)x^{-s-n}ds + \mathrm{res}_{s=-n}h_n(s)x^{-s-n}.$$

Notice that

$$\left|\frac{1}{2\pi i}\int_{-n-\frac{1}{2}-i\infty}^{-n-\frac{1}{2}+i\infty} h_n(s)x^{-s-n}ds\right| \leq \frac{1}{2\pi i}\int_{-n-\frac{1}{2}-i\infty}^{-n-\frac{1}{2}+i\infty} x^{1/2}|h_n(s)|ds \to 0$$

as $x \to 0+$. We derive

$$\lim_{x\to 0+} g^{(n)}(x) = \mathrm{res}_{s=-n}h_n(s)x^{-s-n} = \lim_{s\to -n}(s+n)h_n(s),$$

which exists since the singularities of $h_n(s)$ are simple poles.

This completes the proof of the theorem. □

Corollary 2.3. *If $g \in \mathcal{H}_0$ and $\widehat{g}(s_0) = 0$, then there is an $h \in \mathcal{H}_0$ such that $\widehat{g}(s) = (s-s_0)\widehat{h}(s)$.*

Proof. Let

$$F(s) = \frac{\widehat{g}(s)}{s - s_0}.$$

Since $\widehat{g}(s_0) = 0$, the singularities of F are thus the same as that of $\widehat{g}$. By Theorem 2.2, the function $\widehat{g}$ is essentially bounded. Hence F is essentially bounded as well. By Theorem 2.2 again, the function F is the Mellin transform of some function $h \in \mathcal{H}_0$ as desired. □

Theorem 2.4. *An entire function G is the Mellin transform of a function in $\mathcal{H}_-$ if and only if G is essentially bounded.*

Proof. Suppose $G = \widehat{g}$ for some $g \in \mathcal{H}_-$. Then

$$\begin{aligned} |\widehat{g}(s)| = \left| \int_0^\infty g(x)x^{s-1}dx \right| &\leq \int_0^1 |g(x)x^{s-1}|dx + \int_1^\infty |g(x)x^{s-1}|dx \\ &\leq \int_0^1 |g(x)x^a|dx + \int_1^\infty |g(x)x^{b+1-2}|dx \leq C_1 + C_2, \end{aligned}$$

for $s \in \Omega_{a,b}$, where

$$C_1 = \sup_{0<x<1} |g(x)x^a| \quad \text{and} \quad C_2 = \sup_{x \geq 1} |g(x)x^{b+1}|.$$

Hence $\widehat{g}$ is bounded in $\Omega_{a,b}$.

Next we show that the function $s^m \widehat{g}(s)$ is bounded in S for all $m \in \mathbb{N}_{>0}$. Notice that $\frac{s}{s+j}$ is bounded in $\Omega_{a,b}$ for $j = 0, 1, \cdots, m-1$. Thus we can find a constant $C > 0$ so that

$$|s^m| \leq C|s(s+1)\cdots(s+m-1)|.$$

Integrating by parts yields

$$\begin{aligned} |s^m \widehat{g}(s)| \leq & C\,|s(s+1)\cdots(s+m-1)\widehat{g}(s)| \\ =& C \left| s(s+1)\cdots(s+m-1) \int_0^\infty g(x) x^{s-1} dx \right| \\ =& C \left| \int_0^\infty g^{(m)}(x) x^{s+m-1} dx \right|. \end{aligned}$$

The last integral is the Mellin transform of the function $x^m g^{(m)}(x)$. Since $g \in \mathcal{H}_-$, this integral is bounded in any $\Omega_{a,b}$. Hence $s^m \widehat{g}(s) = s^m G(s)$ is bounded in any $\Omega_{a,b}$. Therefore G is essentially bounded.

Conversely, suppose G is essentially bounded. Let

$$h(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G(s) x^{-s} ds, \quad x > 0,\ c \in \mathbb{R}$$

be the Mellin inverse transform (see [25, Theorem 28, p. 46]) of G. The above integral exists for all $c \in \mathbb{R}$ and is independent of the choice of c. It is enough to check that $h \in \mathcal{H}_-$.

Since G decays rapidly on every vertical line,

$$h^{(n)}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G(s) x^{-s-n} (-s)(-s-1)\cdots(-s-n+1) dx$$

for any$n \in \mathbb{N}$. Also, for any $|k|, n \in \mathbb{N}$, choose $c = k - n$, we have

$$\begin{aligned} \left| x^k h^{(n)}(x) \right| =& \left| \frac{1}{2\pi i} \int_{k-n-i\infty}^{k-n+i\infty} G(s)(-s)(-s-1)\cdots(-s-n+1) x^{k-n-s} ds \right| \\ \leq & \left| \frac{1}{2\pi i} \int_{k-n-i\infty}^{k-n+i\infty} G(s)(-s)(-s-1)\cdots(-s-n+1) ds \right| < \infty. \end{aligned}$$

Hence $h \in \mathcal{H}_-$.

This completes the proof of the theorem. □

By Theorem 2.4, we can deduce the following corollary, which will be used many times later.

Corollary 2.5. *Let $f \in \mathcal{H}_-$. If $\widehat{f}(s_0) = 0$ for some $s_0 \in \mathbb{C}$, then there is a $g \in \mathcal{H}_-$ such that $\widehat{f}(s) = (s - s_0)\widehat{g}(s)$.*

2.2 Approximations of ζ and L-functions

Write $s = \sigma + it$, $\sigma, t \in \mathbb{R}$.

Theorem 2.6 (Phragmen-Lindelöf principle, [10, p. 150]). *Let f be analytic function on an open neighborhood of a strip $a \le \sigma \le b$, for some real numbers $a < b$, such that $|f(s)| \ll \exp(|s|^A)$ for some $A \ge 0$ and $a \le \sigma \le b$.*

(1) Assume that $|f(s)| \le M$ for all s on the boundary of the strip, i.e. for $\sigma = a$ or $\sigma = b$. Then we have $|f(s)| \le M$ for all s in the strip.

(2) Assume that

$$\begin{aligned} |f(a+it)| \le& M_a(1+|t|)^{\alpha}, \\ |f(b+it)| \le& M_b(1+|t|)^{\beta} \end{aligned}$$

for $t \in \mathbb{R}$. Then

$$|f(\sigma+it)| \le M_a^{l(\sigma)} M_b^{1-l(\sigma)} (1+|t|)^{\alpha l(\sigma) + \beta(1-l(\sigma))}$$

for all s in the strip, where l is the linear function such that $l(a) = 1$, $l(b) = 0$.

Theorem 2.7. *The Stirling asympototic formula*

$$\Gamma(s) = \left(\frac{2\pi}{s}\right)^{\frac{1}{2}} \left(\frac{s}{e}\right)^{s} \left(1 + O\left(\frac{1}{|s|}\right)\right) \tag{2.1}$$

is valid in the angle $|\arg s| \le \pi - \varepsilon$ with the implied constant depending on ε.

Recall the notation $\Omega_{a,b} = \{s = \sigma + it | a \le \sigma \le b, |t| \ge 1\}$ for real numbers $a < b$. By (2.1),

$$\Gamma(\sigma+it) = \sqrt{2\pi}(it)^{\sigma-1/2} e^{-\frac{\pi}{2}|t|} \left(\frac{|t|}{e}\right)^{it} \left\{1 + O\left(\frac{1}{|t|}\right)\right\} \tag{2.2}$$

holds uniformly in $\Omega_{a,b}$.

A function f is said to be *polynomially bounded* in $\Omega_{a,b}$ if there is a polynomial P such that

$$|f(s)| \leq |P(s)|$$

for $s \in \Omega_{a,b}$.

Theorem 2.8. *The Riemann ζ-function, the Dirchelet L-functions $L(s,\chi)$ and the L-function $L(s,\phi)$ associated with a normalized newform ϕ in $\mathcal{S}_k(\Gamma_0(M))$ are polynomially bounded in $\Omega_{a,b}$ for any $a < b$.*

Proof. We first show that $\zeta(s)$ is polynomially bounded in $\Omega_{a,b}$ for any $a < b$. The polynomial bound of $L(s,\chi)$ can be proved by a similar argument.

Since $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ converges absolutely when $\mathrm{Re}(s) > 1$, $\zeta(s)$ is polynomially bounded in the right half plane $\{s|\mathrm{Re}(s) \geq 2\}$. By the functional equation of $\zeta(s)$,

$$\zeta(s) = \pi^{s-1}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\zeta(1-s).$$

Applying (2.2), it follows that $\zeta(s)$ is polynomially bounded in $\Omega_{a,b}$ for $a < b \leq -1$.

For the remaining strip $\{s| -1 \leq \mathrm{Re}(s) \leq 2\}$, consider the function $F(s) = \zeta(s)/(s+10)$. We see that $F(s)$ is bounded in the boundary $\{s|\mathrm{Re}(s) = 2\}$. By the functional equation of $\zeta(s)$ and formula (2.2), the function F is also bounded in the other boundary $\{s|\mathrm{Re}(s) = -1\}$. Notice that $s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ and $1/\Gamma(s/2)$ are both entire functions of order 1. Hence we have $F(s) \ll \exp(|s|^A)$ for some $A \geq 0$ when $-1 \leq \mathrm{Re}(s) \leq 2$. Therefore the polynomial bound of $\zeta(s)$ in this strip follows from Theorem 2.6.

Next we show that $L(s,\phi)$ is polynomially bounded in $\Omega_{a,b}$ for any $a < b$.

Since $L(s,\phi) = \sum_{n=1} a_n n^{-s}$ converges absolutely when $\mathbb{R}(s) > k/2+1$, it is polynomially bounded in the half plane $\{s|\mathrm{Re}(s) \geq k/2+2\}$. By the functional equation of $L(s,\phi)$, we have

$$L(s,\phi) = M^{k/2-s}(2\pi)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}L(k-s,\phi).$$

According to (2.2), $L(s,\phi)$ is polynomially bounded in $\Omega_{a,b}$ with $a < b \le k/2-2$.

For the remaining strip $\{s|k/2-2 \le \mathrm{Re}(s) \le k/2+2\}$, let $G(s) = L(s,\phi)/(s+k+2)^4$. Then $G(s)$ is bounded in the boundary $\{s|\mathrm{Re}(s) = k/2+2\}$. By the functional equation of $L(s,\phi)$ and formula (2.2), $G(s)$ is also bounded in the boundary $\{s|\mathrm{Re}(s)k/2-2\}$. Since Λ_ϕ and $1/\Gamma$ are both entire functions of order 1, we have $G(s) \ll \exp(|s|^B)$ for some $B \ge 0$ when $k/2-2 \le \mathrm{Re}(s) \le k/2+2$. Hence $L(s,\phi)$ is polynomially bounded in $\Omega_{k/2-2,k/2+2}$ by Theorem 2.6.

This completes the proof of the theorem. □

2.3 Some relations on Hilbert-Pólya spaces

Theorem 2.9. *We have*

$$Z\mathcal{H}_\cap \subseteq \mathcal{H}_-, \quad Z_\chi \mathcal{H}_\cap^\chi \subseteq \mathcal{H}_- \quad \text{and} \quad Z_\phi \mathcal{H}_0 \subseteq \mathcal{H}_-.$$

Proof. Let $g_1 \in \mathcal{H}_\cap$. By Lemma 2.1, $\widehat{g_1}(s)$ is meromorphic on $\mathbb{C}$ and its only singularities are simple poles at a subset of negative even integers. Since $\zeta(-2n) = 0$ for $n = 1, 2, \cdots$, the function

$$\widehat{Zg_1}(s) = \widehat{g_1}(s)\zeta(s)$$

is entire. By Theorems 2.2 and 2.8, the function $\widehat{g_1}(s)\zeta(s)$ is essentially bounded. Hence $Zg_1 \in \mathcal{H}_-$ by Theorem 2.4 .

Let $g_2 \in \mathcal{H}_\cap^\chi$. Then

$$\widehat{Z_\chi g_2}(s) = \widehat{g_2}(s)L(s,\chi).$$

is an entire function by comparing the zeros and poles of $\widehat{g_2}(s)$ and $L(s,\chi)$. By Theorems 2.2 and 2.8, $\widehat{g_2}(s)L(s,\chi)$ is essentially bounded. Hence $Z_\chi g_2 \in \mathcal{H}_-$.

Let $g_3 \in \mathcal{H}_0$. Then

$$\widehat{Z_\phi g_3}(s) = \widehat{g_3}(s)L(s,\phi).$$

By Lemma 2.1, the only singularities of $\widehat{g_3}(s)$ are simple poles at a subset of $0, -1, -2, \cdots$. Since $L(s, \phi)$ has simple poles at $0, -1, -2, \cdots$, the function $\widehat{Z_\phi g_3}(s)$ is entire. Also, the function $\widehat{Z_\phi g_3}(s)$ is essentially bounded by Theorems 2.2 and 2.8. Applying Theorem 2.4 one more time, we obtain $Z_\phi g_3 \in \mathcal{H}_-$.

This completes the proof of the theorem. □

For $k = 1, 2, \cdots$, define the operators $J_k f(x) = x^{-k} f(x^{-1})$ on the space of functions on $(0, \infty)$. In particular, we denote J_1 by J. By the definition of $\mathcal{H}_-$, we have $J_k \mathcal{H}_- \subseteq \mathcal{H}_-$. Notice that J_k^2 is the identity. It follows that $J_k \mathcal{H}_- = \mathcal{H}_-$. Moreover,

$$\widehat{J_k f}(s) = \int_0^\infty x^{-k} f(x^{-1}) x^{s-1} dx = \int_0^\infty f(x) x^{k-s-1} dx = \widehat{f}(k-s) \tag{2.3}$$

for $f \in \mathcal{H}_-$.

Theorem 2.10. *We have*

$$JZ\mathcal{H}_\cap = Z\mathcal{H}_\cap, \quad JZ_\chi \mathcal{H}_\cap^\chi = Z_{\overline{\chi}} \mathcal{H}_\cap^{\overline{\chi}} \quad \textit{and} \quad J_k Z_\phi \mathcal{H}_0 = Z_\phi \mathcal{H}_0.$$

Proof. Let $f \in \mathcal{H}_\cap$. By Theorem 2.9, $Zf \in \mathcal{H}_-$. Applying (1.1) and (2.10), we have

$$\widehat{JZf}(s) = \widehat{Zf}(1-s) = \widehat{f}(1-s)\widehat{\zeta}(1-s) = \widehat{f}(1-s)\pi^{\frac{1}{2}-s} \frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})} \zeta(s).$$

Let

$$F(s) = \widehat{f}(1-s)\pi^{\frac{1}{2}-s} \frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}.$$

Since $f \in \mathcal{H}_\cap$, the function $\widehat{f}(1-s)$ is meromorphic on $\mathbb{C}$ and its singularities are simple poles at a subset of $\{2n+3 | n \in \mathbb{N}\}$, and $\widehat{f}(1) = 0$. Note that $\Gamma(s)$ is meromorphic and non-vanishing on $\mathbb{C}$ with simple poles at $0, -1, -2, \cdots$. It follows that $F(s)$ is meromorphic on $\mathbb{C}$ and its singularities are simple poles at a subset of $-2, -4, \cdots$, and $F(1) = 0$. Also, by Theorem 2.2 and formula (2.2), the function F is essentially bounded. It follows from

Theorem 2.2 that F is the Mellin transform of some function g in $\mathcal{H}_\cap$, that is, $\widehat{JZf} = \widehat{g}$ for some $g \in \mathcal{H}_\cap$. Hence $\widehat{JZf} = \widehat{Zg}$ and $JZf \in Z\mathcal{H}_\cap$.

For the remaining part, using functional equations (1.2) and (1.4), Theorem 2.2, and formula (2.2), we can show $JZ_\chi\mathcal{H}_\cap^\chi = Z_{\overline{\chi}}\mathcal{H}_\cap^{\overline{\chi}}$ and $J_k Z_\phi \mathcal{H}_0 = Z_\phi \mathcal{H}_0$ in a similar fashion. $\square$

Chapter 3. Poisson Summation Formulas on the Function Spaces

The functions in the function spaces $\mathcal{H}_0, \mathcal{H}_\cap$ and $\mathcal{H}_\cap^\chi$ admit various Poisson summation formulas. In this chapter, we prove these formulas from the functional equations of the Riemman zeta function and the L-functions. We start with functions in $\mathcal{H}_0$.

Theorem 3.1. *Let $f \in \mathcal{H}_0$. Let $\phi(z) = \sum_{n=1}^{\infty} a_n q^n \in \mathcal{S}_k(\Gamma_0(M))$ be a normalized newform of weight k (an even positive integer). Let*

$$L(s, \phi) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad \textit{and} \quad \Lambda_\phi(s) = M^{s/2}(2\pi)^{-s}\Gamma(s)L(s,\phi)$$

with

$$\Lambda_\phi(k-s) = C\Lambda_\phi(s)$$

for $C = \pm 1$. Then

$$\sum_{n=1}^{\infty} a_n f(n) = C \cdot \sum_{n=1}^{\infty} a_n H(f)(n) \tag{3.1}$$

where

$$H(f)(y) = -y^{-\frac{k}{2}} \int_0^\infty f(x) J_k\left(4\pi\sqrt{\frac{xy}{M}}\right) x^{\frac{k}{2}-1} dx.$$

Here

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

is the Bessel function *of the first kind.*

Proof. Since $f \in \mathcal{H}_0$, we have

$$f(x) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=k+1} \widehat{f}(s) x^{-s} ds$$

for $x > 0$. Hence

$$\begin{aligned}\sum_{n=1}^{\infty} a_n f(n) =& \sum_{n=1}^{\infty} a_n \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=k+1} \widehat{f}(s) n^{-s} ds \\ =& \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=k+1} \widehat{f}(s) \sum_{n=1}^{\infty} a_n n^{-s} ds \\ =& \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=k+1} \widehat{f}(s) L(s,\phi) ds,\end{aligned}$$

where the change of the summation and the integral follows from the absolute convergence of $\sum_{n=1}^{\infty} a_n n^{-s}$ when $\mathrm{Re}(s) > k/2+1$. By the functional equation (1.4),

$$\begin{aligned}\frac{1}{2\pi i} \int_{\mathrm{Re}(s)=k+1} \widehat{f}(s) L(s,\phi) ds =& \frac{C}{2\pi i} \int_{\mathrm{Re}(s)=k+1} \widehat{f}(s) M^{\frac{k}{2}-s} (2\pi)^{2s-k} L(k-s,\phi) \frac{\Gamma(k-s)}{\Gamma(s)} ds \\ =& \frac{C}{2\pi i} \int_{\mathrm{Re}(s)=k/2-2} \widehat{f}(s) M^{\frac{k}{2}-s} (2\pi)^{2s-k} L(k-s,\phi) \frac{\Gamma(k-s)}{\Gamma(s)} ds.\end{aligned}$$

Here we can change the path of the integral because for any polynomial $P(s)$, the function

$$P(s)\widehat{f}(s) M^{\frac{k}{2}-s} (2\pi)^{2s-k} L(k-s,\phi) \frac{\Gamma(k-s)}{\Gamma(s)}$$

is bounded in any vertical strip (by Theorems 2.2 and 2.8, and Stirling's formula (2.2)). Since $\sum_{n=1}^{\infty} a_n n^{-s}$ converges absolutely when $\mathrm{Re}(s) > k/2+1$,

$$\begin{aligned}&\frac{1}{2\pi i} \int_{\mathrm{Re}(s)=k/2-2} \widehat{f}(s) M^{\frac{k}{2}-s} (2\pi)^{2s-k} L(k-s,\phi) \frac{\Gamma(k-s)}{\Gamma(s)} ds \\ &= \sum_{n=1}^{\infty} \frac{a_n}{2\pi i} \int_{\mathrm{Re}(s)=k/2-2} \widehat{f}(s) M^{\frac{k}{2}-s} (2\pi)^{2s-k} \frac{\Gamma(k-s)}{\Gamma(s)} n^{s-k} ds.\end{aligned}$$

The above sum converges absolutely since

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=k/2-2}\widehat{f}(s)M^{\frac{k}{2}-s}(2\pi)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}n^{s-k}ds\\
&\quad\ll n^{-\frac{k}{2}-2}\int_{-\infty}^{\infty}\left|\widehat{f}\left(\frac{k}{2}-2+it\right)M^{2-it}(2\pi)^{-4+2it}\frac{\Gamma(\frac{k}{2}+2-it)}{\Gamma(\frac{k}{2}-2+it)}\right|dt\\
&\quad=O(n^{-\frac{k}{2}-2}),
\end{aligned}$$

where the implied constant depends on f, k and M.

To prove (3.1), it suffices to show that

$$\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=k/2-2}\widehat{f}(s)M^{\frac{k}{2}-s}(2\pi)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}n^{s-k}ds=H(f)(n) \tag{3.2}$$

for any $f\in\mathcal{H}_0$.

Let $f_\alpha(x)=f(x)e^{-\alpha x^2}$ for $\alpha>0$. We first verify (3.2) holds for f_α. Consider the integral

$$\int_{\mathrm{Re}(s)=L+1/2}\widehat{f_\alpha}(s)M^{\frac{k}{2}-s}(2\pi)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}n^{s-k}ds$$

for an odd integer $L>k$. When $\mathrm{Re}(s)=L$, we have

$$\widehat{f_\alpha}(s)\ll\int_0^\infty e^{-\alpha x^2}x^{L-\frac{1}{2}}dx\ll\Gamma\left(\frac{2L+1}{4}\right)\ll\left(\frac{L-1}{2}\right)!$$

where the implied constant depends on f and α. Notice that

$$\Gamma(s)=s\Gamma(s-1)=\cdots=s(s-1)\cdots(s-(2L+1-k)+1)\Gamma(s-(2L+1-k)).$$

Since $\Gamma(s-(2L+1-k))=\overline{\Gamma(k-s)}$ for $s=L+\frac{1}{2}+it$, we thus obtain

$$\left|\frac{\Gamma(k-s)}{\Gamma(s)}\right|=\left|\frac{1}{s(s-1)\cdots(2s-k+1)}\right|\ll\left|\frac{1}{s(s-1)(L-2)!}\right|$$

By the estimate $L! \sim \sqrt{2\pi L}\left(\frac{L}{e}\right)^L$, we conclude that

$$\lim_{L\to\infty}\int_{\mathrm{Re}(s)=L+1/2}\widehat{f_\alpha}(s)M^{\frac{k}{2}-s}(2\pi)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}n^{s-k}ds = 0.$$

Applying the residue theorem yields

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=k/2-2}\widehat{f_\alpha}(s)M^{\frac{k}{2}-s}(2\pi)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}n^{s-k}ds\\
&=\sum_{m=k}^{\infty}\widehat{f_\alpha}(m)M^{\frac{k}{2}-m}(2\pi)^{2m-k}\frac{\mathrm{res}_{s\to m}\Gamma(k-s)}{\Gamma(m)}n^{m-k}\\
&=\sum_{m=k}^{\infty}\int_0^\infty f_\alpha(x)x^{m-1}M^{\frac{k}{2}-m}(2\pi)^{2m-k}\frac{(-1)^{m+1}}{m!(m-k)!}n^{m-k}dx\\
&=\int_0^\infty f_\alpha(x)\sum_{m=k}^{\infty}x^{m-1}M^{\frac{k}{2}-m}(2\pi)^{2m-k}\frac{(-1)^{m+1}}{m!(m-k)!}n^{m-k}dx\\
&=\int_0^\infty f_\alpha(x)x^{k-1}M^{-\frac{k}{2}}(2\pi)^k\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{m!(m+k)!}\left(\frac{4\pi^2xn}{M}\right)^m dx.
\end{aligned}$$

Thus we obtain

$$\sum_{n=1}^{\infty}a_nf_\alpha(n) = C\cdot\sum_{n=1}^{\infty}a_nH(f_\alpha)(n),$$

where

$$\begin{aligned}
H(f_\alpha)(y) &= \int_0^\infty f_\alpha(x)x^{k-1}M^{-\frac{k}{2}}(2\pi)^k\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{m!(m+k)!}\left(\frac{4\pi^2xy}{M}\right)^m dx\\
&= -\,y^{-\frac{k}{2}}\int_0^\infty f_\alpha(x)J_k\left(4\pi\sqrt{\frac{xy}{M}}\right)x^{\frac{k}{2}-1}dx.
\end{aligned}$$

We now turn back to f. Recall the classical fact (see [26, p. 19]) that

$$J_k(x) = \frac{1}{2\pi}\int_0^{2\pi}\cos(k\theta - x\sin\theta)d\theta$$

when n is an integer. Then

$$f_\alpha(x)J_k\left(4\pi\sqrt{\frac{xy}{M}}\right)x^{\frac{k}{2}-1}$$

is bounded by $|f(x)|x^{\frac{k}{2}-1}$, which is absolutely integrable on $(0,\infty)$. By Lebesgue's dominated convergence theorem,

$$\lim_{\alpha\to 0+} H(f_\alpha)(n) = H(f)(n), \quad \forall n \geq 1.$$

To prove (3.2) for f, it remains to show that

$$\begin{aligned}
&\lim_{\alpha\to 0+}\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=k/2-2}\widehat{f}_\alpha(s)M^{\frac{k}{2}-s}(2\pi)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}n^{s-k}ds \\
=&\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=k/2-2}\widehat{f}(s)M^{\frac{k}{2}-s}(2\pi)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}n^{s-k}ds.
\end{aligned}\tag{3.3}$$

To avoid possible singularities, we pass the above integrals to the vertical line $\{s|\mathrm{Re}(s) = k/2\}$. Then (3.3) becomes

$$\begin{aligned}
&\lim_{\alpha\to 0+}\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=k/2}\widehat{f}_\alpha(s)M^{\frac{k}{2}-s}(2\pi)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}n^{s-k}ds \\
=&\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=k/2}\widehat{f}(s)M^{\frac{k}{2}-s}(2\pi)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}n^{s-k}ds.
\end{aligned}\tag{3.4}$$

Let $\mathrm{Re}(s) = k/2$. For any $g \in \mathcal{H}_0$,

$$\begin{aligned}
\widehat{g}(s) =& \int_0^\infty g(x)x^{s-1}dx = \frac{1}{s}x^s g(x)\Big|_0^\infty - \frac{1}{s}\int_0^\infty g'(x)x^s dx \\
=& -\frac{1}{s(s+1)}g'(x)x^{s+1}\Big|_0^\infty + \frac{1}{s(s+1)}\int_0^\infty g''(x)x^{s+1}dx \\
=&\frac{1}{s(s+1)}\int_0^\infty g''(x)x^{s+1}dx.
\end{aligned}$$

Also, we can find some constant $C > 0$, depending only on f, such that $|f_\alpha''(x)| \leq C(|f(x)| + |f'(x)| + |f''(x)|)$ for any $x \in (0,\infty)$ and $\alpha \in (0,1)$. Denote

$$A := \int_0^\infty C(|f(x)| + |f'(x)| + |f''(x)|)x^{\frac{k}{2}+1}dx$$

Then

$$\widehat{f}_\alpha(s)M^{\frac{k}{2}-s}(2\pi)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}n^{s-k}$$

is bounded by the function

$$\left|\frac{A}{s(s+1)}n^{-\frac{k}{2}}\right|,$$

where A is independent of s. Since $\lim_{\alpha\to 0+}\widehat{f}_\alpha(s)=\widehat{f}(s)$, applying Lebesgue's dominated convergence theorem to

$$\lim_{\alpha\to 0+}\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=k/2}\widehat{f}_\alpha(s)M^{\frac{k}{2}-s}(2\pi)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}n^{s-k}ds$$

yields (3.4).

This completes the proof of the theorem. □

The Poisson summation formulas for functions in $\mathcal{H}_\cap$ and $\mathcal{H}_\cap^\chi$ can be derived similarly by the functional equations of $\zeta(s)$ and $L(s,\chi)$. They coincide with the classical Poisson summation formulas, see [22, Theorem 3.1, p. 154] and [3, (1.10), p. 8]. We state them as follows.

Theorem 3.2. *If $f\in\mathcal{H}_\cap$, then*

$$\sum_{n=1}^{\infty}f(n)=\sum_{n=1}^{\infty}H_1(f)(n), \tag{3.5}$$

where

$$H_1(f)(y)=2\int_0^\infty f(x)\cos(2\pi xy)dx.$$

Theorem 3.3. *Let χ be a primitive Dirichlet character mod $N>1$. Let $f\in\mathcal{H}_\cap^\chi$. Then*

$$\sum_{n=1}^{\infty}\chi(n)f(n)=\sum_{n=1}^{\infty}\overline{\chi(n)}H_2(f)(n),$$

where

$$H_2(f)(n)=\frac{2\chi(-1)\tau(\chi)}{N}\int_0^\infty f(x)\cos(2\pi xy/N)dx.$$

Here

$$\tau(\chi) = \sum_{n \pmod N} \chi(n) e^{2\pi i n/N}$$

is the Gauss Sum*.*

Chapter 4. Spectral interpretation of the Riemann zeta function

4.1 Distribution of Eigenvalues of D_- on $\mathcal{H}$

Lemma 4.1. *Let λ be an eigenvalue of D_- on $\mathcal{H}$. Then both $\overline{\lambda}$ and $1-\lambda$ are eigenvalues of D_- on $\mathcal{H}$.*

Proof. For any given λ, an eigenvalue of D_- on $\mathcal{H}$, there is a function $f \in \mathcal{H}_-$ not in $Z\mathcal{H}_\cap$ and a $g \in \mathcal{H}_\cap$ such that

$$-xf'(x) = \lambda f(x) + Zg(x). \tag{4.1}$$

Taking the conjugate, we obtain

$$-x\overline{f'(x)} = \overline{\lambda}\overline{f(x)} + \overline{Zg(x)}.$$

Note that $\overline{f} \in \mathcal{H}_- \setminus Z\mathcal{H}_\cap$ and $\overline{g} \in \mathcal{H}_\cap$. It follows that $\overline{\lambda}$ is an eigenvalue of D_- on $\mathcal{H}$.

Next we verify that $1-\lambda$ is also an eigenvalue of D_- on $\mathcal{H}$. We achieve it by showing that Jf is an eigenfunction with eigenvalue $1-\lambda$. Since $J\mathcal{H}_- = \mathcal{H}_-$ and $JZ\mathcal{H}_\cap = Z\mathcal{H}_\cap$ (by Theorem 2.10), we have $Jf \in \mathcal{H}_- \setminus Z\mathcal{H}_\cap$. Applying the Mellin transform to (4.1) yields

$$s\widehat{f}(s) = \lambda\widehat{f}(s) + \zeta(s)\widehat{g}(s).$$

Substituting s by $1-s$ and applying (2.3) we obtain

$$\begin{aligned}(1-s)\widehat{f}(1-s) =& \lambda\widehat{f}(1-s) + \zeta(1-s)\widehat{g}(1-s)\\ =& \lambda\widehat{f}(1-s) + \widehat{Zg}(1-s) = \lambda\widehat{f}(1-s) + \widehat{JZg}(s).\end{aligned}$$

By Theorem 2.10, we can find $g_1 \in \mathcal{H}_\cap$ such that $JZg = Zg_1$. Since $\widehat{Jf}(s) = \widehat{f}(1-s)$, the above formula becomes

$$s\widehat{Jf}(s) = (1-\lambda)\widehat{Jf}(s) - \widehat{Zg_1}(s).$$

Taking the Mellin inverse transform we derive

$$-x(Jf)'(x) = (1-\lambda)Jf(x) + Z(-g_1)(x).$$

Hence $1-\lambda$ is an eigenvalue of D_- on $\mathcal{H}$.

This completes the proof of the lemma. □

Lemma 4.2. *Let $g \in \mathcal{H}_\cap$. If $\widehat{g}(s_0) = 0$ when $s_0 \neq 1$, or $ord_{s=1}\widehat{g}(s) \geq 2$ when $s_0 = 1$, then we can find an $h \in \mathcal{H}_\cap$ such that $\widehat{g}(s) = (s-s_0)\widehat{h}(s)$.*

Proof. By Corollary 2.3, there is an $h \in \mathcal{H}_0$ such that

$$\widehat{g}(s) = (s-s_0)\widehat{h}(s).$$

It is suffices to check that $h \in \mathcal{H}_\cap$. Since $\widehat{g}(s_0) = 0$, the two functions $\widehat{h}$ and $\widehat{g}$ share the same singularities. Also, we have $\widehat{h}(1) = 0$ as $\widehat{g}(1) = 0$. It then follows from the definition of $\mathcal{H}_\cap$ that $h \in \mathcal{H}_\cap$.

This completes the proof of the lemma. □

Corollary 4.3. *Let λ be an eigenvalue of D_- on $\mathcal{H}_-$. Choose $f \in \mathcal{H}_- \setminus Z\mathcal{H}_\cap$ and $g \in \mathcal{H}_\cap$ such that*

$$-xf'(x) = \lambda f(x) + Zg(x)$$

for $x > 0$. Then $\widehat{g}(\lambda) \neq 0$. Consequently, $\zeta(\lambda) = 0$ and $\widehat{f}(s)/\zeta(s)$ has a simple pole at $s = \lambda$.

Proof. Assume that $\widehat{g}(\lambda) = 0$. Applying the Mellin tranform to $-xf'(x) = \lambda f(x) + Zg(x)$ we obtain

$$(s-\lambda)\widehat{f}(s) = \widehat{g}(s)\zeta(s).$$

Then

$$\frac{\widehat{f}(s)}{\zeta(s)} = \frac{\widehat{g}(s)}{s-\lambda}.$$

Since $\zeta(s)$ has a simple pole at $s = 1$, if $\lambda = 1$, then $\mathrm{ord}_{s=1}\widehat{g}(s) \geq 2$. By Lemma 4.2, we can find $h \in \mathcal{H}_\cap$ so that $\widehat{g}(s) = (s-\lambda)\widehat{h}(s)$. Hence

$$\frac{\widehat{f}(s)}{\zeta(s)} = \frac{\widehat{g}(s)}{s-\lambda} = \widehat{h}(s).$$

This implies that $f = Zh$, in contradiction to the assumption that $f \notin Z\mathcal{H}_\cap$. Therefore $\widehat{g}(\lambda) \neq 0$.

This completes the proof of the corollary. □

4.2 Proof of Theorem 1.2

Proof of Theorem 1.2. First we show that a complex number ρ is an eigenvalue of D_- on $\mathcal{H}$ if and only if ρ is a nontrivial zero of the Riemann zeta function.

Let ρ be a nontrivial zero of $\zeta(s)$. We see that ρ is an eigenvalue of D_- by Theorem 1.1.

Conversely, let ρ be an eigenvalue of D_-. By Corollary 4.3, we see that $\zeta(\rho) = 0$. In order to show that ρ is a nontrivial zero of $\zeta(s)$, we assume ρ is a trivial zero, that is, $\rho = -2n$ for some $n \in \mathbb{N}_{>0}$. By Lemma 4.1, $1 - \rho = 2n + 1$ is also an eigenvalue of D_- acting on $\mathcal{H}$. Thus $\zeta(2n + 1) = 0$ by Corollary 4.3, which is absurd. Hence ρ is a nontrivial zero of $\zeta(s)$.

It remains to show that the algebraic multiplicity of the eigenvalue ρ is equal to the vanishing order of $\zeta(s)$ at ρ. Suppose that the function $\zeta(s)$ has vanishing order $m_0 > 0$ at ρ. We show that the algebraic multiplicity of D_- at ρ is also m_0.

Let $F_1 \in \mathcal{H}_- \setminus Z\mathcal{H}_\cap$ and $g_1 \in \mathcal{H}_\cap$ such that $-xF_1'(x) = \rho F_1(x) + Zg_1$. By Corollary 4.3, the function $\widehat{F}_1(s)/\zeta(s)$ has a simple pole at ρ. So $\widehat{F_1}(s)$ has vanishing order $m_0 - 1$ at ρ.

By Theorem 2.4, the function $\widehat{F_1}(s)$ is essentially bounded. Define

$$H_i(s) = \frac{\widehat{F_1}(s)}{(s-\rho)^i}, \quad 0 \le i < m_0.$$

Then $H_0, H_1, \cdots, H_{m_0-1}$ are essentially bounded entire functions. By Theorem 2.4, the function H_i is the Mellin transform of a function F_{i+1} in $\mathcal{H}_-$ for $0 < i \le m_0$.

Let

$$V_n = \left\{ f \in \mathcal{H}_- \middle| (D-\rho)^n f \in Z\mathcal{H}_\cap \right\}$$

for any $n \ge 1$. Then the dimension over $\mathbb{C}$ of the subspace $\bigcup_{n=1}^{\infty} V_n / Z\mathcal{H}_\cap$ in $\mathcal{H}$ is equal to the algebraic multiplicity of the eigenvalue ρ of D_- acting on $\mathcal{H}$. Define $W_n = V_n / Z\mathcal{H}_\cap$. To prove our theorem, it is sufficient to show

$$\dim W_n = \begin{cases} n, & 1 \le n \le m_0 \\ m_0, & n > m_0. \end{cases} \tag{4.2}$$

Now we prove formula (4.2). First we show that $V_n = V_{n-1}$ for $n > m_0$, which implies that $W_n = W_{n-1}$ for $n > m_0$. If there is an element $f \in V_n \setminus V_{n-1}$, we will show that $n \le m_0$.

As $f \in V_n \setminus V_{n-1}$, we have $(D-\rho)^n f \in Z\mathcal{H}_\cap$ and $(D-\rho)^{n-1} f \notin Z\mathcal{H}_\cap$. Denote by f_1 the function $(D-\rho)^{n-1} f$. Then there is some function $g \in \mathcal{H}_\cap$ such that

$$-x f_1'(x) = \rho f_1(x) + Z g(x)$$

for $x > 0$. By Corollary 4.3, the function $\widehat{f_1}(s)/\zeta(s)$ has a simple pole at ρ. Since $\zeta(s)$ has vanishing order m_0 at ρ, the vanishing order of the function $\widehat{f_1}(s) = (s-\rho)^{n-1}\widehat{f}(s)$ at ρ is $m_0 - 1$. Because $\widehat{f}$ is analytic at ρ, we must have $n \le m_0$. Hence $W_n = W_{n-1}$ for $n > m_0$.

Next we show that $\dim W_n = n$ for $1 \le n \le m_0$. The case $n = 1$ follows from Theorem 1.1. For the case $1 < n_0 \le m_0$, it suffices to show that $\dim(V_n/V_{n-1}) = 1$ for $1 < n \le m_0$. Recall that $F_n \in \mathcal{H}_-$ is the Mellin inverse transform of H_{n-1}. Applying the Mellin transform to $(D-\rho)^n F_n$ and $(D-\rho)^{n-1}F_n$, we obtain

$$\widehat{(D-\rho)^n F_n}(s) = (s-\rho)^n \widehat{F_n}(s) = (s-\rho)^n H_{n-1}(s) = (s-\rho)\widehat{F_1}(s)$$

and

$$\widehat{(D-\rho)^{n-1} F_n}(s) = (s-\rho)^{n-1} \widehat{F_n}(s) = (s-\rho)^{n-1} H_{n-1}(s) = \widehat{F_1}(s).$$

Since $F_1 \notin Z\mathcal{H}_\cap$ but $(D-\rho)F_1 \in Z\mathcal{H}_\cap$, we have $F_n \in V_n \setminus V_{n-1}$. Hence $\dim(V_n/V_{n-1}) \geq 1$.

Finally, we show that $\dim(V_n/V_{n-1}) \leq 1$. Let G_1 and G_2 be two functions in $V_n \setminus V_{n-1}$. Then $(D-\rho)^{n-1}G_1, (D-\rho)^{n-1}G_2 \in V_1 \setminus Z\mathcal{H}_\cap$. By Theorem 1.1, there is a complex number μ such that

$$(D-\rho)^{n-1}G_1 - \mu(D-\rho)^{n-1}G_2 = (D-\rho)^{n-1}(G_1 - \mu G_2) \in Z\mathcal{H}_\cap.$$

So $G_1 - \mu G_2 \in V_{n-1}$. Hence $\dim(V_n/V_{n-1}) \leq 1$. Therefore $\dim(W_n/W_{n-1}) = \dim(V_n/V_{n-1}) = 1$ for $1 \leq n \leq m_0$ and (4.2) holds.

This completes the proof of Theorem 1.2. □

Chapter 5. Spectral interpretation of Dirichlet L-functions

Throughout this chapter, χ denotes a primitive Dirichlet character modulo $N > 1$ and $\epsilon = \frac{1-\chi(-1)}{2}$.

5.1 Nontrivial zeros of $L(s,\chi)$ are eigenvalues of D_-^χ

Let ρ be a nontrivial zero of $L(s,\chi)$. We show in this section that ρ is an eigenvalue of D_-^χ acting on $\mathcal{H}^\chi$.

If $\chi(-1) = 1$, let $\eta_\chi(x) = e^{-\pi x^2}$. Then $\eta_\chi \in \mathcal{H}_\cap^\chi$ and

$$\begin{aligned}\widehat{\eta}_\chi(s) = &\int_0^\infty e^{-\pi x^2} x^{s-1} dx = \frac{1}{2}\int_0^\infty e^{-\pi x} x^{\frac{s}{2}-1} dx\\ =&\frac{1}{2}\int_0^\infty e^{-x}\left(\frac{x}{\pi}\right)^{\frac{s}{2}} \frac{dx}{x} = \frac{1}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right).\end{aligned}$$

If $\chi(-1) = -1$, let $\eta_\chi(x) = xe^{-\pi x^2}$. Then we have $\eta_\chi \in \mathcal{H}_\cap^\chi$ and

$$\begin{aligned}\widehat{\eta}_\chi(s) = &\int_0^\infty e^{-\pi x^2} x^{s} dx = \frac{1}{2}\int_0^\infty e^{-\pi x} x^{\frac{s-1}{2}} dx\\ =&\frac{1}{2}\int_0^\infty e^{-x}\left(\frac{x}{\pi}\right)^{\frac{s+1}{2}} \frac{dx}{x} = \frac{1}{2}\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right).\end{aligned}$$

Put

$$F_{\rho,\chi}(x) = \int_1^\infty Z_\chi \eta_\chi(tx) t^{\rho-1} dt.$$

The following theorem asserts that ρ is an eigenvalure of D_-^χ on $\mathcal{H}^\chi$ with eigenfunction $F_{\rho,\chi}$.

Theorem 5.1. *The function $F_{\rho,\chi}$ lies in $\mathcal{H}_- \setminus Z_\chi \mathcal{H}_\cap^\chi$ and*

$$-xF'_{\rho,\chi}(x) = \rho F_{\rho,\chi}(x) + Z_\chi \eta_\chi(x).$$

Proof. Notice that

$$F_{\rho,\chi}(x) = \int_1^\infty Z_\chi \eta_\chi(tx) t^{\rho-1} dt = x^{-\rho} \int_x^\infty Z_\chi \eta_\chi(t) t^{\rho-1} dt.$$

Hence

$$\begin{aligned} -xF'_{\rho,\chi}(x) =& -x \left(-\rho x^{-\rho-1} \int_x^\infty Z_\chi \eta_\chi(t) t^{\rho-1} dt - x^{-\rho} Z_\chi \eta_\chi(x) x^{\rho-1} \right) \\ =& \rho \int_x^\infty Z_\chi \eta_\chi(t) t^{\rho-1} dt + Z_\chi \eta_\chi(x) = \rho F_{\rho,\chi}(x) + Z_\chi \eta_\chi(x). \end{aligned}$$

It remains to show that $F_{\rho,\chi} \in \mathcal{H}_- \setminus Z_\chi \mathcal{H}_\cap^\chi$.

Applying the Mellin transform to $-xF'_{\rho,\chi}(x) = \rho F_{\rho,\chi}(x) + Z_\chi \eta_\chi(x)$ we obtain

$$(s-\rho)\widehat{F_{\rho,\chi}}(s) = \widehat{Z_\chi \eta_\chi}(s) = \widehat{\eta}_\chi(s) L(s,\chi).$$

Since ρ is a nontrivial zero, we have $0 \le \mathrm{Re}(\rho) \le 1$ and $\rho \neq 0$. By Lemma 2.1, the function $\widehat{\eta}_\chi(s)$ is holomorphic at $s = \rho$. Hence

$$\widehat{Z_\chi \eta_\chi}(\rho) = \widehat{\eta}_\chi(\rho) L(\rho,\chi) = 0.$$

Then it follows from Corollary 2.5 that $F_{\rho,\chi} \in \mathcal{H}_-$.

Next we show $F_{\rho,\chi} \notin Z_\chi \mathcal{H}_\cap^\chi$. Assume the contrary then

$$\frac{\widehat{F_{\rho,\chi}}(s)}{L(s,\chi)} = \frac{\widehat{\eta}_\chi(s)}{s-\rho} = \frac{1}{2}\pi^{-\frac{1+\epsilon}{2}} \frac{\Gamma(\frac{s+\epsilon}{2})}{s-\rho}$$

is the Mellin transform of a function in $\mathcal{H}_\cap^\chi$. Hence

$$\frac{\Gamma(\frac{s+\epsilon}{2})}{s-\rho}$$

is holomorphic at $s = \rho$. However, $\Gamma(s)$ vanishes nowhere. This is a contradiction. Therefore

$F_{\rho,\chi} \notin Z_\chi \mathcal{H}_\cap^\chi$.

This completes the proof of the theorem. $\square$

5.2 Distribution of Eigenvalues of D_-^χ on $\mathcal{H}^\chi$

Lemma 5.2. *Let λ be an eigenvalue of D_-^χ on $\mathcal{H}^\chi$. Then $\overline{\lambda}$ and $1-\lambda$ are eigenvalues of $D_-^{\overline{\chi}}$ on $\mathcal{H}^{\overline{\chi}}$.*

Proof. For any given λ, an eigenvalue of D_-^χ on $\mathcal{H}^\chi$, there is a function $f \in \mathcal{H}_-$ not in $Z_\chi \mathcal{H}_\cap^\chi$ and a $g \in H_\cap^\chi$ such that

$$-xf'(x) = \lambda f(x) + Z_\chi g(x). \tag{5.1}$$

Taking the conjugate, we obtain

$$-x\overline{f'(x)} = \overline{\lambda}\overline{f(x)} + \overline{Z_\chi g(x)},$$

i.e.,

$$D\overline{f} = \overline{\lambda f} + Z_{\overline{\chi}}\overline{g}.$$

Hence $\overline{\lambda}$ is an eigenvalue of D_-^χ acting on $\mathcal{H}^\chi$.

Next we show that $1-\lambda$ is an eigenvalue of $D_-^{\overline{\chi}}$ acting on $\mathcal{H}^{\overline{\chi}}$. Since $J\mathcal{H}_- = \mathcal{H}_-$ and $JZ_\chi \mathcal{H}_\cap^\chi = Z_{\overline{\chi}}\mathcal{H}_\cap^{\overline{\chi}}$, we have $Jf\imath H_- \setminus Z_{\overline{\chi}}\mathcal{H}_\cap^{\overline{\chi}}$. Applying the Mellin transform to (5.1) we obtain

$$s\widehat{f}(s) = \lambda\widehat{f}(s) + L(s,\chi)\widehat{g}(s).$$

Replacing s by $1-s$ and applying (2.3) we have

$$\begin{aligned}(1-s)\widehat{f}(1-s) =& \lambda\widehat{f}(1-s) + L(1-s,\chi)\widehat{g}(1-s)\\ =& \lambda\widehat{f}(1-s) + \widehat{Z_\chi g}(1-s) = \lambda\widehat{f}(1-s) + \widehat{JZ_\chi g}(s).\end{aligned}$$

By Theorem 2.10, there is an $h \in \mathcal{H}_\cap^{\overline{\chi}}$ such that $JZ_\chi g = Z_{\overline{\chi}} h$. So

$$s\widehat{f}(1-s) = (1-\lambda)\widehat{f}(1-s) - L(s,\overline{\chi})\widehat{h}(s).$$

Since $\widehat{Jf}(s) = \widehat{f}(1-s)$, the above formula becomes

$$s\widehat{Jf}(s) = (1-\lambda)\widehat{Jf}(s) + L(s,\overline{\chi})\widehat{(-h)}(s).$$

Applying the Mellin inverse transform to the above formula yields

$$-x(Jf)'(x) = (1-\lambda)Jf(x) + Z_{\overline{\chi}}(-h)(x).$$

Hence $1-\lambda$ is an eigenvalue of $D_-^{\overline{\chi}}$ on $\mathcal{H}^{\overline{\chi}}$.

This completes the proof of the lemma. □

Lemma 5.3. *Let $g \in \mathcal{H}_\cap^\chi$. If $\widehat{g}(s_0) = 0$ for some $s_0 \in \mathbb{C}$, then we can find an $h \in \mathcal{H}_\cap^\chi$ such that $\widehat{g}(s) = (s-s_0)\widehat{h}(s)$.*

Proof. By Corollary 2.3, there is an $h \in \mathcal{H}_0$ such that

$$\widehat{g}(s) = (s-s_0)\widehat{h}(s).$$

Since $\widehat{g}(s_0) = 0$, the singularities of $\widehat{h}$ are the same as that of $\widehat{g}$. Hence $h \in \mathcal{H}_\cap^\chi$By the definition of $\mathcal{H}_\cap^\chi$.

This completes the proof of the lemma. □

Corollary 5.4. *Let λ be an eigenvalue of D_-^χ on $\mathcal{H}_-^\chi$. Choose $f \in \mathcal{H}_- \setminus Z_\chi \mathcal{H}_\cap^\chi$ and $g \in \mathcal{H}_\cap^\chi$ such that*

$$-xf'(x) = \lambda f(x) + Z_\chi g(x)$$

for $x > 0$. Then $\widehat{g}(\lambda) \neq 0$. Consequently, $L(\lambda,\chi) = 0$ and $\widehat{f}(s)/L(\lambda,\chi)$ has a simple pole at $s = \lambda$.

Proof. Assume that $\widehat{g}(\lambda) = 0$. Applying the Mellin tranform to $-xf'(x) = \lambda f(x) + Z_\chi g(x)$ we obtain

$$(s-\lambda)\widehat{f}(s) = \widehat{g}(s)L(s,\chi).$$

So

$$\frac{\widehat{f}(s)}{L(s,\chi)} = \frac{\widehat{g}(s)}{s-\lambda}.$$

By Lemma 4.2, we can find $h \in \mathcal{H}_\cap^\chi$ such that $\widehat{g}(s) = (s-\lambda)\widehat{h}(s)$. Hence

$$\frac{\widehat{f}(s)}{L(s,\chi)} = \frac{\widehat{g}(s)}{s-\lambda} = \widehat{h}(s).$$

This implies that $f = Z_\chi h$, which contradicts the assumption that $f \notin Z_\chi \mathcal{H}_\cap^\chi$. Therefore $\widehat{g}(\lambda) \neq 0$.

This completes the proof of the corollary. □

5.3 Proof of Theorem 1.3

Proof of Theorem 1.3. We use the same type of argument here as the proof of Theorem 1.2.

First we show that a complex number ρ is an eigenvalue of D_-^χ on $\mathcal{H}^\chi$ if and only if ρ is a nontrivial zero of $L(s,\chi)$.

Let ρ be a nontrivial zero of $L(s,\chi)$. We see that ρ is an eigenvalue of D_-^χ by Theorem 5.1.

Conversely, let ρ be an eigenvalue of D_-^χ. By Corollary 5.4, we see that $L(\rho,\chi) = 0$. In order to show that ρ is a nontrivial zero of $L(s,\chi)$, assume that ρ is a trivial zero, that is, $\rho = -2n - \epsilon$ for some $n \in \mathbb{N}_{>0}$. By Lemma 5.2, $1 - \rho = 2n + \epsilon + 1$ is also an eigenvalue of $D_-^{\overline{\chi}}$ acting on $\mathcal{H}^\chi$. Thus $L(2n+\epsilon+1, \overline{\chi}) = 0$ by Corollary 5.4. This is absurd. Hence ρ is a nontrivial zero of $L(s,\chi)$.

Next we show that the geometric multiplicity of every eigenvalue ρ is one and the algebraic multiplicity of the eigenvalue ρ is equal to the vanishing order of $L(s,\chi)$ at ρ.

Suppose that $L(s,\chi)$ has vanishing order $m_0 > 0$ at ρ. Let

$$V_n = \left\{ f \in \mathcal{H}_- \middle| (D-\rho)^n f \in Z_\chi \mathcal{H}_\cap^\chi \right\}$$

for $n \geq 1$. Let $W_n = V_n / Z_\chi \mathcal{H}_\cap^\chi$. Then $\dim_{\mathbb{C}} W_1$ is the geometric multiplicity of ρ and the dimension of the subspace $\bigcup_{n=1}^\infty W_n$ is equal to the algebraic multiplicity of the eigenvalue ρ of D_-^χ acting on $\mathcal{H}^\chi$. By an argument similar to the proof of Theorem 1.2, we can derive

$$\dim W_n = \begin{cases} n, & 1 \leq n \leq m_0 \\ m_0, & n > m_0. \end{cases} \tag{5.2}$$

This completes the proof of Theorem 1.3. □

Chapter 6. Spectral interpretation of L-functions associated with newforms

Let $\phi(\tau) = \sum_{n=1}^{\infty} a_n q^n \in \mathcal{S}_k(\Gamma_0(M))$ be a normalized newform, where $M > 0$ is an integer and k is an even positive integer.

6.1 Nontrivial zeros of $L(s, \phi)$ are eigenvalues of D_-^{ϕ}

Let ρ be a nontrivial zero of $L(s, \phi)$. We prove in this section that ρ is an eigenvalue of D_-^{ϕ} acting on $\mathcal{H}^{\phi}$.

Let $\eta_\phi(x) = e^{-x}$. Then $\widehat{\eta}(s) = \Gamma(s)$. Put

$$F_{\rho,\phi}(x) = \int_1^{\infty} Z_\phi \eta_\phi(-tx) t^{\rho-1} dt.$$

The following proposition shows that ρ is an eigenvalure of D_-^{ϕ} on $\mathcal{H}^{\phi}$ with eigenfunction $F_{\rho,\phi}$.

Theorem 6.1. *The function $F_{\rho,\phi}$ lies in $\mathcal{H}_- \setminus Z_\phi \mathcal{H}_0$ and*

$$-xF'_{\rho,\phi}(x) = \rho F_{\rho,\phi}(x) + Z_\phi \eta_\phi(x).$$

Proof. Notice that

$$F_{\rho,\phi}(x) = \int_1^{\infty} Z_\phi \eta_\phi(tx) t^{\rho-1} dt = x^{-\rho} \int_x^{\infty} Z_\phi \eta_\phi(t) t^{\rho-1} dt.$$

Hence

$$\begin{aligned} -xF'_{\rho,\phi}(x) =& - x \left(-\rho x^{-\rho-1} \int_x^{\infty} Z_\phi \eta_\phi(t) t^{\rho-1} dt - x^{-\rho} Z_\phi \eta_\phi(x) x^{\rho-1} \right) \\ =& \rho \int_x^{\infty} Z_\phi \eta_\phi(t) t^{\rho-1} dt + Z_\phi \eta_\phi(x) = \rho F_{\rho,\phi}(x) + Z_\phi \eta_\phi(x). \end{aligned} \tag{6.1}$$

It remains to show that $F_{\rho,\phi} \in \mathcal{H}_- \setminus Z_\phi \mathcal{H}_0$.

Applying the Mellin transform to (6.1) yields

$$(s-\rho)\widehat{F_{\rho,\phi}}(s) = \widehat{Z_\phi \eta_\phi}(s) = \widehat{\eta}_\phi(s) L(s,\phi).$$

Since ρ is a nontrivial zero, we have $0 \leq \mathrm{Re}(\rho) \leq k$ and $\rho \neq 0$. By Lemma 2.1, $\widehat{\eta}_\phi(s)$ is holomorphic at $s = \rho$. Hence

$$\widehat{Z_\phi \eta_\phi}(\rho) = \widehat{\eta}_\phi(\rho) L(\rho,\phi) = 0.$$

Then by Corollary 2.5 we have $F_{\rho,\phi} \in \mathcal{H}_-$.

Suppose that $F_{\rho,\phi} \in Z_\phi \mathcal{H}_0$. Then

$$\frac{\widehat{F_{\rho,\phi}}}{L(\rho,\phi)} = \frac{\widehat{\eta}_\phi(s)}{s-\rho} = \frac{\Gamma(s)}{s-\rho}$$

is the Mellin transform of a function in $\mathcal{H}_0$. Hence

$$\frac{\Gamma(s)}{s-\rho}$$

is holomorphic at $s = \rho$, in contradiction to the fact that $\Gamma(s)$ vanishes nowhere. Therefore $F_{\rho,\phi} \notin Z_\phi \mathcal{H}_0$.

This completes the proof of the theorem. □

6.2 Distribution of Eigenvalues of D_-^ϕ on $\mathcal{H}^\phi$

Lemma 6.2. *Let λ be an eigenvalue of D_-^ϕ on $\mathcal{H}^\phi$. Then $k-\lambda$ is an eigenvalue of D_-^ϕ on $\mathcal{H}^\phi$.*

Proof. Let λ be an eigenvalue of D_-^ϕ on $\mathcal{H}^\phi$. Then we can find a function $f \in \mathcal{H}_-$ not in

$Z_\phi\mathcal{H}_0$ and a $g \in H_0$ such that

$$-xf'(x) = \lambda f(x) + Z_\phi g(x). \tag{6.2}$$

We will show that $k-\lambda$ is an eigenvalue of D_-^ϕ acting on $\mathcal{H}^\phi$ with eigenfunction $J_k f$.

Since $J_k\mathcal{H}_- = \mathcal{H}_-$ and $J_k Z_\phi \mathcal{H}_0 = Z_\phi\mathcal{H}_0$, $J_k f \in \mathcal{H}_- \setminus Z_\phi\mathcal{H}_0$. Applying the Mellin transform to (6.2) we obtain

$$s\widehat{f}(s) = \lambda\widehat{f}(s) + L(s,\phi)\widehat{g}(s).$$

Replacing s by $k-s$ and applying (2.3) we have

$$\begin{aligned}(k-s)\widehat{f}(k-s) =& \lambda\widehat{f}(k-s) + L(k-s,\phi)\widehat{g}(k-s)\\ =& \lambda\widehat{f}(k-s) + \widehat{Z_\phi g}(k-s) = \lambda\widehat{f}(k-s) + \widehat{J_k Z_\phi g}(s).\end{aligned}$$

By Theorem 2.10, there is an $h \in \mathcal{H}_0$ such that $J_k Z_\phi g = Z_\phi h$. So

$$s\widehat{f}(k-s) = (k-\lambda)\widehat{f}(k-s) - L(s,\phi)\widehat{h}(s).$$

Since $\widehat{J_k f}(s) = \widehat{f}(k-s)$, the above identity can be written as

$$s\widehat{J_k f}(s) = (1-\lambda)\widehat{J_k f}(s) + L(s,\overline{\chi})\widehat{(-h)}(s).$$

Applying the Mellin inverse transform we obtain

$$-x(J_k f)'(x) = (1-\lambda)J_k f(x) + Z_\phi(-h)(x).$$

Hence $k-\lambda$ is an eigenvalue of D_-^ϕ on $\mathcal{H}^\phi$.

This completes the proof of the lemma. □

Lemma 6.3. *Let λ be an eigenvalue of D_-^{ϕ} on $\mathcal{H}_-^{\phi}$. Choose $f \in \mathcal{H}_- \setminus Z_\phi \mathcal{H}_0$ and $g \in \mathcal{H}_0$ such that*

$$-xf'(x) = \lambda f(x) + Z_\phi g(x)$$

for $x > 0$. Then $\widehat{g}(\lambda) \neq 0$. Consequently, $L(\lambda, \phi) = 0$ and $\widehat{f}(s)/L(\lambda, \phi)$ has a simple pole at $s = \lambda$.

Proof. Assume that $\widehat{g}(\lambda) = 0$. Applying the Mellin tranform to $-xf'(x) = \lambda f(x) + Z_\phi g(x)$ we obtain

$$(s - \lambda)\widehat{f}(s) = \widehat{g}(s)L(s, \phi).$$

So

$$\frac{\widehat{f}(s)}{L(s, \phi)} = \frac{\widehat{g}(s)}{s - \lambda}.$$

By Lemma 2.3, we can find $h \in \mathcal{H}_0$ such that $\widehat{g}(s) = (s - \lambda)\widehat{h}(s)$. Hence

$$\frac{\widehat{f}(s)}{L(s, \phi)} = \frac{\widehat{g}(s)}{s - \lambda} = \widehat{h}(s).$$

This implies that $f = Z_\phi h$, which contradicts the assumption that $f \notin Z_\chi \mathcal{H}_0$. Therefore $\widehat{g}(\lambda) \neq 0$.

This completes the proof of the lemma. □

6.3 Proof of Theorem 1.4

Proof of Theorem 1.4. We use the same type of argument as the proof of Theorem 1.2.

First we show that a complex number ρ is an eigenvalue of D_-^{ϕ} on $\mathcal{H}^{\phi}$ if and only if ρ is a nontrivial zero of $L(s, \phi)$.

Let ρ be a nontrivial zero of $L(s, \phi)$. We see that ρ is an eigenvalue of D_-^{χ} by Theorem 6.1.

Conversely, let ρ be an eigenvalue of D^{ϕ}_{-}. By Lemma 6.3, we see that $L(\rho, \phi) = 0$. In order to show that ρ is a nontrivial zero of $L(s, \phi)$, assume that ρ is a trivial zero, that is, $\rho = -n$ for some $n \in \mathbb{N}$. By Lemma 6.2, $k - \rho = k + n$ is also an eigenvalue of D^{ϕ}_{-} acting on $\mathcal{H}_{-}$. Thus $L(k + n, \phi) = 0$ by Lemma 6.3, which contradicts the fact that $L(s, \phi)$ does not vanish when $\mathrm{Re}(s) > k$. Hence ρ is a nontrivial zero of $L(s, \phi)$.

Next we show that the geometric multiplicity of every eigenvalue ρ is one and the algebraic multiplicity of the eigenvalue ρ is equal to the vanishing order of $L(s, \phi)$ at ρ. Suppose that $L(s, \phi)$ has vanishing order $m_0 > 0$ at ρ.

Let $W_n = V_n / Z_{\phi}\mathcal{H}_0$. Then $\dim_{\mathbb{C}} W_1$ is the geometric multiplicity of the eigenvalue ρ and the dimension of the subspace $\bigcup_{n=1}^{\infty} W_n$ in $\mathcal{H}_{-}$ is equal to the algebraic multiplicity of the eigenvalue ρ of D^{ϕ}_{-} acting on $\mathcal{H}^{\phi}$. By an argument similar to the proof of Theorem 1.2 we can derive that

$$\dim W_n = \begin{cases} n, & 1 \leq n \leq m_0 \\ m_0, & n > m_0. \end{cases} \tag{6.3}$$

This completes the proof of Theorem 1.4. □

Bibliography

[1] A. Atkin and J. Lehner: Hecke operators on $\Gamma_0(m)$. Math. Ann. **185**, 134-160 (1970)

[2] E. Bombieri: New progress on the zeta function: From old conjectures to a major breakthrough. Proc. Natl. Acad. Sci. U.S.A. **116**. 11085-11086, 2019

[3] D. Bump: Automorphic forms and representations, Cambridge University Press, 1998

[4] A. Connes: Trace formula in noncommutative geometry and the zeros of the Riemman zeta function, Selecta Math. **5**, 29–106 (1999)

[5] F. Diamond and J. Shurman: A first course in modular forms, Springer, 2005

[6] L. Ge: Lectures on "The Riemann zeta function and Connes's noncommutative geometry", 29th Canadian Symposium On Operator Algebras 2001, https://www.msri.org/workshops/96/schedules/518

[7] M. Griffin, K. Ono, L. Rolen and D. Zaiger, Jensen polynomials for the Riemann zeta function and other sequences. Proc. Natl. Acad. Sci. U.S.A. **116**. 11103-11110, 2019

[8] E. Hecke: Mathematische Werke. Göttingen: Vandenhoeck und Ruprecht 1959

[9] H. Heilbronn: Zeta-functions and L-functions, Algebraic Number Theory, Edited by J. W. S. Cassels and A. Fröhlich

[10] H. Iwaniec, E. Kowalski: Analytic Number Theory, American Mathematical Society, 2013

[11] W. W. Li: Newforms and functional equations. Math. Ann. **212**, 285–315 (1975)

[12] X.-J. Li: On spectral theory of the Riemann zeta function. Sci. China Math. 62, 2317–2330 (2019)

[13] R. Meyer: A spectral interpretation for the zeros of the Riemann zeta function, Math. Institut, Georg-August-Univ, Göttingen: Seminars Winter Term 2004/2005, 117–137, Univ. Göttingen, Göttingen, 2005

[14] R. Meyer: On a representation of the idele class group related to primes and zeros of L-Functions, Duke Math. J. **127**, 519–595 (2005)

[15] H. L. Montgomery and R. C. Vaughan: Multiplicative number theory I. Classical Theory, Cambrigr University Press, 2007

[16] H. L. Montgomery: The pair correlation of zeros of the zeta function, Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972), pp. 181–193. Amer. Math. Soc., Providence, R.I., 1973

[17] C. M. Newman: Fourier transforms with only real zeroes, Proc. Amer. Math. Soc. **61**, 246-251(1976)

[18] B. Riemann: Über die Anzahl der Primzahlen unter einer gegebenen Grösse. Monatsberichte der Berliner Akademie, November 1859, 671-680.

[19] B. Rodgers and T. Tao: The De Bruijn-Newman constant is non-negative, Forum Math. Pi **8**, 2020

[20] W. Rudin, Principals of mathematical analysis, Third Edition, 1976

[21] A. Selberg, Harmonic analysis and discontinuous groups on weakly symmetric Riemannian spaces with applications to Dirichlet series, J. Indian Math. Soc. **20**, 47–87 (1956)

[22] E. M. Stein and R. Shakarchi: Fourier analysis. An introduction. Princeton Lectures in Analysis, 1. Princeton University Press, Princeton, NJ, 2003

[23] E. M. Stein and R. Shakarchi, Real analysis. Measure theory, integration, and Hilbert spaces. Princeton Lectures in Analysis, 3. Princeton University Press, Princeton, NJ, 2005

[24] E. C. Titchmarsh: The theory of the Riemann zeta-function, Second Edition, Oxford, 1986

[25] E. C. Titchmarsh: Introduction to the theory of Fourier integrals, Second Edition, Oxford, 1986

[26] G. N. Watson: A Treatise on the Theory of Bessel Functions, Second Edition, Cambridge, 1944

[27] A. Weil: Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen. Math. Ann. **168**, 149-156 (1967)