Teachers' Mathematical Meanings: Decisions for Teaching Geometric Reflections and Orientation of Figures

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Teachers’ Mathematical Meanings: Decisions for Teaching
Geometric Reflections and Orientation of Figures

Porter Peterson Nielsen

A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of
Master of Arts

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ABSTRACT

Teachers’ Mathematical Meanings: Decisions for Teaching Geometric Reflections and Orientation of Figures

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Teachers’ instructional decisions are important for students’ mathematics learning as they determine the learning opportunities for all students. This study examines teachers’ decisions about the activities and tasks they choose for students’ mathematics learning, the ordering and connecting of mathematics topics, and the mathematics within curricula not to cover. These decisions are referred to as curricular decisions. I also identify teachers’ mathematical schemes, referred to as mathematical meanings, in relation to geometric reflections and orientation of figures and examine teachers’ reasoning with their mathematical meanings as they make these curricular decisions. Additionally, based on the results of this study I identify several productive and unproductive mathematical meanings in relation to geometric reflections and orientation of figures. Describing productive mathematical meanings as providing coherence to student mathematical understanding and preparing students for future mathematics learning (Thompson, 2016). These findings can be used to better understand why teachers make the curricular decisions they do as well as help teachers identify whether or not their mathematical meanings are productive in an effort to foster productive mathematical meanings for students.

Keywords: curricular reasoning, mathematical meanings, geometric reflections, orientation of figures, grade 8 teachers
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CHAPTER 1: RATIONALE

While much research has been done on teachers’ instructional decisions, there remains much we do not know. In the body of research dedicated to understanding teachers’ decision, there is a growing set of research dedicated to teachers’ knowledge (e.g., Empson & Junk, 2004; Hill et al., 2008; Johnson & Larsen, 2012) with an attempt to find a statistical link between teacher’s content knowledge and students’ mathematics learning (Thompson, 2016). Such research has been vital to the mathematics education field, allowing us to better understand the role teachers’ knowledge plays in mathematics education (Ball et al., 2008). This research has helped us understand what teachers’ instructional decisions are and their influences; however, there is still very little known about why teachers make the instructional decisions they do.

While planning and enacting mathematics lessons, mathematics teachers make a variety of instructional decisions involving curricula. Some of these are decisions about the activities and tasks that are beneficial for students’ mathematics learning, decisions about the ordering and connecting of mathematics topics, and decisions about what mathematics within curricula not to cover. Dingman et al. (in review) referred to such decisions as curricular decisions.

Teachers’ curricular decisions are important to mathematics learning as they influence the learning opportunities for students and we know students do not learn that which they are not given the opportunity to learn (Hiebert, 1999). Due to the fundamental importance of teachers’ decisions, much research has been done to better understand these decisions. For example, we know beliefs, attitudes, provided curriculum, teachers’ mathematical knowledge, and student mathematical thinking that emerges during a lesson all influence curricular decisions (e.g., Handal & Herrington, 2003; Remillard, 2004; Remillard & Heck, 2014).
A move away from focusing on teachers’ mathematical knowledge, which I define as a teacher’s declarative knowledge of mathematics described as factual information retrieved from memory such as mathematical algorithms or vocabulary, towards a focus on teachers’ mathematical meanings may help us better understand teachers’ reasoning for their curricular decisions. Mason and Spence (2000) determined that knowing is more useful than knowledge for thinking about teaching and learning as knowledge (i.e., declarative knowledge) leads to the separation of the knower from what is known. They suggested that by focusing on teachers’ acts of knowing, we are better able to describe sources of teachers’ curricular decisions and actions as teachers need to make a plethora of decisions using their declarative knowledge; deciding what mathematics is important for students to learn or not to learn. In alignment with Mason and Spence’s suggestion to focus on acts of knowing, Thompson (2013) proposed a move towards focusing on teachers’ mathematical meanings. Thompson described mathematical meanings through Piagetian schemes - mathematical meanings are schemes, organizations of an individual’s interpretations of their personal experiences that serve as guides in how to act. Liang (2020) suggests that by changing focus from teachers’ content knowledge (i.e., declarative knowledge) to their mathematical meanings allows the field to stop theorizing about the declarative knowledge teachers should possess and start examining how teachers come to know something.

To illustrate how a move to mathematical meanings is beneficial to understanding teachers’ curricular decisions I refer to an example from my data. I found when I asked two teachers to define the geometric orientation of a figure they provided identical answers. However, their mathematical meanings of orientation of a figure were very different from each other. One teachers’ mathematical meanings led the teacher to think of a figure's orientation as a
diagnostic tool, using orientation to verify whether or not a geometric reflection was performed correctly. The other teacher’s mathematical meanings led her to think of orientation of a figure as a fundamental characteristic connecting all rigid transformations to geometric reflections. Both teachers would assess to have similar declarative knowledge about the orientation of a figure and both could perform similarly in answering questions regarding the concept. However, both made drastically different curricular decisions that could not be understood through focusing only on their declarative knowledge. However, through focusing on the teachers’ mathematical meanings we can better understand the teachers’ thinking and the reasons for their unique decisions.

Much research has been done regarding teachers’ declarative knowledge and the influence upon curricular decisions; however, we still know little about why teachers make the curricular decisions they do. Byerly and Thompson (2017) argued that teachers’ mathematical meanings are one of the most important factors in constraining or affording teachers’ curricular decisions. Additionally, teachers’ mathematical meanings are a fundamental means of providing explanations for teachers’ curricular decisions (Teuscher et al., 2016). To better understand the reasons for teachers’ curricular decisions I examined teachers’ mathematical meanings as they reason about curricular decisions in their planning and enacting their lessons on geometric reflections and orientation of figures.
CHAPTER 2: BACKGROUND

Literature Review

In this chapter I discuss teachers’ use of curriculum, teachers’ declarative knowledge, and the theoretical framework used for my study. In the section about teachers’ use of curriculum, I define curriculum, curricular decisions, curricular reasoning and discuss the Instructional Pyramid model (Dingman et al., in review). I then discuss teachers’ declarative knowledge; specifically, in regard to what we know and what we do not know about the influence of teachers’ declarative knowledge on their instructional decisions. For the theoretical framework, I use teachers’ curricular reasoning with their mathematical meanings to better understand why teachers make the curricular decisions they do. I discuss mathematical meanings and the research related to mathematical meanings. Providing additional justification for studying teachers’ reasoning with their mathematical meanings.

Teachers’ Use of Curriculum

Curriculum is a fundamental part of teaching and learning. Curricula is often thought of as textbooks, but can be expanded to include any materials used by teachers for mathematics instruction. Curriculum is the content of learning, the “what” students learn as opposed to the “how” it is taught (Stein et al., 2007). Gehrke, Knapp, and Sirotnik (1992), in order to better understand what teachers “really” teach conducted a review on themes found within curriculum and curriculum use spanning the fields of social studies, language arts, mathematics, and science. Though each content area contained sundry differences in their content and use, Gehrke et al. (1992) identified two distinctly different and important types of curriculum; planned curriculum and enacted curriculum. Planned curriculum includes teachers’ goals and planned activities for a
given lesson. Enacted curriculum refers to the curriculum that is taught within the classroom to students.

**Teachers’ Curricular Decisions**

While planning and enacting curriculum, teachers make multiple curricular decisions; decisions regarding what content will be taught. While planning curriculum teachers must decide what mathematics to teach or not to teach, what problems from textbooks or other resources to use or not use, what supplementary resources to use or not use, how to sequence the mathematics, what level of mathematical rigor to require of students, what mathematical syntax to use or not use, how to describe or define mathematical terms and concepts, and other similar mathematical considerations. After planning curriculum, teachers then enact the curriculum, making more curricular decisions as the teacher’s plans interact with students, often in unanticipated ways (Gehrke et al., 1992).

As teachers make curricular decisions as they plan and enact curriculum, no one teacher will enact the same lesson as another teacher, even when using the same curriculum materials. Brown (2009) demonstrates this by drawing parallels between teaching and performing music. In his research, Brown looked at several artists’ rendition of the song, *Take the A Train*, given the same written sheet music. The different artists’ renditions contained noticeable similarities yet each had distinctly different sounds. Teaching requires much pre-planning, post reflecting, and similar to playing music teaching is also an in-the-moment performance requiring adjustments, spontaneous considerations, decisions and expertise. Curriculum materials, similar to sheet music, act as guides for performing an activity but are not the activity itself (Brown, 2009).

Teachers development of their planned and enacted curriculum is an iterative and adaptive process. Brown (2009) asserts that teaching is design work in which the teacher, no
matter the curriculum, must make a plethora of purposeful curricular decisions. Stein et al. (2007) sought to clarify the multiple meanings of curriculum and how different types of curriculum interact with one another suggested that in the course of planning, teachers first start with a written curriculum and through the teacher's curricular decision-making process plan curriculum to use in the classroom. While enacting the planned curriculum the teacher makes a variety of in-the-moment curricular decisions informed by a multiplicity of reasons that results in an enacted curriculum that varies from the planned curriculum. The enacted curriculum provides students with learning opportunities and subsequently the teacher may reflect and make curricular decisions for future lessons, at which point the cycle of curricular decision-making and planning begins anew.

Researchers have sought to identify connections between teachers’ curricular decisions and their enacted curriculum; however, few have investigated explicit reasons for teachers’ curricular decisions. Remillard (2018) suggested that variation of enacted curriculum is related to the teacher’s ability to interpret the curriculum and mathematics found therein, to make curricular decisions, and to use the curriculum as a tool to aid in the development of instruction. Remillard recognized that as different teachers enacted curriculum they adopt different styles and pacing. These teachers emphasized different mathematics or ignored entire concepts. Each enactment not only was different but there was also variety in the quality of enactment.

Researchers have also studied how teachers use curriculum, specifically examining similarities and differences in teachers’ curricular decisions, across different types of curriculum and the impact it has on students. Choppin (2011) examined three teachers’ use of curriculum materials, what the teachers understood about the curriculum materials, and how they connected student thinking with the curriculum materials. Choppin found that though similarities existed in
the teachers’ goals and understanding of the curriculum, the teachers differed in how they adapted tasks in their lessons and in how they made connections with student thinking and the curriculum materials. Yet, we know little about why these teachers made different curricular decisions about adapting tasks and making connections with student thinking and the curriculum materials.

Curricular Reasoning Framework and the Instructional Pyramid Model.

As teachers make curricular decisions as they plan and enact their lessons they rely on their curricular reasoning. Roth McDuffie and Mather (2009) defined curricular reasoning as “a specific form of pedagogical reasoning that teachers employ while working with curriculum materials to plan, implement, and reflect on instruction” (p. 304).

Roth McDuffie and Mather (2009) explain that as teachers work with curriculum materials they are guided by goals and activities as they consider a plethora of educationally pertinent questions: whether or not the curriculum will match their students learning needs, will the curriculum build mathematical understandings now and into the future, how will the curriculum influence learning over time, and other similar questions. The authors developed the definition of curricular reasoning as they conducted research in which a coterie of teachers planned lessons, were interviewed, and then reflected upon their enactment of the lessons.

Research about teachers’ engagement in curricular reasoning while making curricular decisions has been further refined by Dingman et al. (in review). Building on others’ research (Cohen et al., 2003; Roth McDuffie & Mather, 2009), Dingman et al. identified five curricular reasoning aspects teachers reason with when making curricular decisions about mathematics, students, and curriculum. These five curricular reasoning aspects are (a) viewing mathematics from the learner’s perspective, (b) mapping learning trajectories, (c) considering mathematical
meanings, (d) analyzing curriculum materials, and (e) revising curriculum materials. Three of the five aspects came directly from Roth McDuffie and Mather’s (2009) research, the other two Dingman et al. (in review) added and were identified through an opening-coding process. For my study I focused on teachers’ reasoning with the considering mathematical meaning aspect and will discuss this curricular reasoning aspect more in-depth later in this section.

Dingman et al. (in review) developed the Instructional Pyramid model to visually display the curricular reasoning framework. Figure 1 is the Instructional Pyramid model in which the top vertex represents the teacher as they make decisions about the bottom vertices representing mathematics, students, and curriculum. The edges in the model represent the interactive relationship between any two elements (vertices) and represent different curricular reasoning aspects teachers reason with as they make curricular decisions. Figure 2 is the Instructional Pyramid model with the edges labeled with the different curricular reasoning aspects. The Instructional Pyramid model provides an organized structure to allow meaningful communication and contemplation about teachers’ reasons and curricular decisions.

Figure 1

*The Instructional Pyramid a Model for Curricular Reasoning*
For my study I focused on teachers’ reasoning with the considering mathematical meaning aspect which is one of the most used aspects that teachers reason with that is also on two edges. Dingman et. al. described teachers’ mathematical meanings as teachers’ understanding of the mathematics or the anticipated meanings students will develop as a consequence of their learning. One edge is situated between the teacher and mathematics and the other edge between mathematics and students. As a teacher reasons with their mathematical meanings they may consider whether or not (correctly or incorrectly) certain mathematics is important and whether or not they should teach the mathematical concepts. As a teacher reasons in this manner, they are reasoning on one edge of the Instructional Pyramid model between teachers and mathematics. As teachers consider what mathematics they want students to learn, what mathematics is in reach of student mathematical abilities, and what mathematics may support students’ future mathematics learning, they reason on another edge between mathematics and students. Data from the National Science Foundation grant (NSF #1561569) indicates that teachers primarily reason on the front three edges of the Instructional Pyramid which are the considering mathematical meaning (TM), considering mathematical meaning (SM), and viewing mathematics from the learner’s perspective (A/A) (see Dingman et al. (in review) for
descriptions of these and other curricular reasoning aspects); thus, my study was centered on one of the most used aspects that is also on two edges.

**Teachers’ Declarative Knowledge**

In this section I discuss portions of what we know about teachers’ declarative knowledge in relation to mathematics education and what we do not know. I will explain how considering teachers’ mathematical meanings may help us understand teachers’ curricular reasoning and decisions differently than considering teachers’ declarative knowledge.

For the past two decades teachers’ declarative knowledge has been a great concern for many researchers. It is not sufficient for teachers to have a lot of declarative knowledge, they must have certain types of knowledge. Ball et al. (2008) suggested the mathematical knowledge needed for teaching allows teachers to anticipate student misunderstandings, comprehend the mathematics behind student errors so that context appropriate clarification can be given, and make conceptual connections.

Within the field of mathematics education there has been an important focus on attempting to find a statistical link between teacher’s declarative knowledge and students’ mathematics learning, implying that if we can identify what declarative knowledge is important for teaching we can improve mathematics instructional quality and students’ mathematics learning (Thompson, 2016). However, such a link has been difficult to uncover.

Some studies have found that teachers’ declarative knowledge has a significant relation with a teacher’s quality of instruction; however, other studies have found that declarative knowledge is not the only factor in determining the quality of instruction. For example, Copur-Gencturk (2015) studied 21 Kindergarten to grade 8 teachers enrolled in a master’s program and found a statistically significant relation between teachers’ mathematical knowledge (i.e.,
declarative knowledge) and the quality of their instruction. Whereas, Hill et al. (2008) while studying the associations between teachers’ mathematical knowledge (i.e., declarative knowledge) and the quality of mathematical instruction found several anomalies occurred in which teachers with low levels of assessed mathematical knowledge (i.e., declarative knowledge) occasionally provided high levels of instruction. Implies that declarative knowledge was not the only factor influencing teachers’ curricular decisions. Hill et al. speculated that curriculum materials, participation in professional development and other similar considerations may be influencing teachers’ instruction.

Supporting Hill et. al.’s (2008) speculation that there are other factors influencing teachers’ instruction, Stein and Kaufman (2010) researched how a group of teachers used two different curricula over the course of two years in an attempt to begin to answer the question: “What curricular materials work best under what kinds of conditions?” (Stein & Kaufman, 2010, p. 663). The authors considered teachers’ perceptions of mathematics and capacity, which included declarative knowledge, and found that teachers were better at implementing one curriculum over another; yet, Stein and Kaufman found unexpectedly that mathematical knowledge (i.e., declarative knowledge) had little impact upon the teachers’ use of the curriculum, suggesting that teachers use of the curriculum may be “… more important than the education, experience, and knowledge that…” the teacher poses (Stein & Kaufman, 2010, p. 688).

Remillard and Kim (2017) proposed a framework for identifying mathematical knowledge (i.e., declarative knowledge) teachers use while working with curriculum. In their study, they analyzed specific tasks that teachers provided while teaching, and examined teachers’ curriculum materials and curricular decisions. The researchers studied what the teachers choose
to use or not use of the provided materials through examining audio recorded interviews in which the participatory teachers were prompted to reflect upon their planning. The authors found that teachers, while making curricular decisions with the provided materials, indeed used their mathematical knowledge (i.e., declarative knowledge) and they also found that teachers were using their mathematical meanings as they made curricular decisions.

Despite the voluminous amount of research conducted on preservice and in-service teachers’ declarative knowledge, it has been difficult to directly link declarative knowledge with teachers’ curricular decisions and reasoning (e.g., Stein & Kaufman, 2010; Shechtman et al., 2010; Thompson & Thompson, 1996). We do not understand well how teachers use their declarative knowledge to reason about the mathematics they plan and enact as they make curricular decisions. As recognized by Hill et al. (2008), the difficulty in directly linking mathematical knowledge (i.e., declarative knowledge) with curricular decisions may be because teachers are using other resources (e.g., state and school policies, curriculum, professional development) as a mediation between their mathematical knowledge (i.e., declarative knowledge) and their teaching. This may be the case, if a teacher obtains a new textbook they may use the text to make decisions. However, Remillard and Kim (2017) found it might also be that teachers’ mathematical meanings are mediating their decisions. Similar to conductors using the same sheet music to lead music yet producing distinctly different renditions of the provided song, teachers with similarly assessed declarative knowledge and given the same textbooks will plan and enact distinctly different lessons (Brown, 2009). Having sufficient declarative knowledge does not inoculate a teacher from occasionally making poor instructional decisions nor does having impoverished levels of declarative knowledge condemn teachers to languish in
perpetual poor practices, teachers reason with a multitude of different aspects; including their mathematical meanings.

**Theoretical Framework**

I used teachers’ curricular reasoning with their mathematical meanings as the framework to better understand why teachers make the curricular decisions they do. I first describe constructivism upon which mathematical meanings are built; defining important terms and concepts. I then describe Thompson’s (2016) mathematical meanings for teaching, upon which two edges (mathematical meanings) of the curricular reasoning framework are built. I then explain how I use constructivism and mathematical meanings to help me understand and identify teachers’ curricular reasoning with their mathematical meanings as they made curricular decisions about geometric reflections and orientation of figures.

**Constructivism and Schemes**

We are constantly drawing conclusions as we interpret our experiences. As we encounter new experiences that perturb our prior interpretations, we may modify our interpretations to accommodate the new experiences. If our interpretations fit our observations and reliably enable us to solve problems, the interpretations are referred to as viable. Thus, from a constructivist perspective, learning is the construction of viable interpretations of experiences from which one acts (von Glasersfeld, 1983).

As an individual repeatedly reasons, interprets personal experiences, and reconstructs their interpretations an individual constructs schemes that organize their experiences and interpretations. Thompson et al. (2014) described schemes, as an organization of actions, operations, and images with many entry points that trigger action and anticipation of outcomes. Thompson clarified his meaning of actions and images by using Piaget’s description, as “all
movement, all thought, or all emotions that responds to a need” (Piaget & Elkind, 1968, p. 6).

Thompson characterized image as

By “image” I mean much more than a mental picture. Rather, I mean “image” as the kind of knowledge that enables one to walk into a room full of old friends and expect to know how events will unfold. An image is constituted by coordinated fragments of experience from kinesthesia, proprioception, smell, touch, taste, vision, or hearing. It seems essential also to include the possibility that images entail fragments of past affective experiences, such as fearing, enjoying, or puzzling, and fragments of past cognitive experiences, such as judging, deciding, inferring, or imagining. Images are less well delineated than are schemes of actions or operations (Cobb & von Glasersfeld, 1983). They are more akin to figural knowledge (Johnson, 1987; Thompson, 1985) and metaphor (Goldenberg, 1988).

A person’s images can be drawn from many sources, and hence they tend to be highly idiosyncratic. (Thompson, 1994, pp. 229-230)

Therefore, an individual’s schemes are consistent and personal to the individual (Thompson, 2013; Thompson et al., 2014), that guide individuals in how to act, think, move, and respond to a context and any need found therein.

**Assimilation and Accommodation of Schemes**

Two mental processes that enable individuals to adapt and revise their schemes are *assimilation* and *accommodation* (von Glasersfeld, 1995). Assimilation is the integration of external stimuli (e.g. experiences, contexts, students’ thinking) into an existing scheme. Piaget is known to have said while talking about his meaning for assimilation, “A rabbit that eats a cabbage doesn't become cabbage; it's the cabbage that becomes rabbit—that's assimilation. It's the same thing at the psychological level. Whatever a stimulus is, it is integrated with internal
structures” (Bringuier, 1980, p. 42). Assimilation is absorption in the sense that it is not the scheme that adapts to the external elements but rather it is the external elements that are adapted to fit the individual’s schemes (Piaget, 1970). Therefore, assimilation is structured by the individual’s existing schemes, acting as a lens to determine what an individual attends to (Liang, 2020). Accommodation is the mental process of scheme revision. As an individual encounters external stimuli that perturbs their current schemes, the individual may restructure their schemes to reconcile the perturbation through accommodation (Liang, 2020).

Assimilation and accommodation are codependent. Piaget (1970) explained that cognitive adaptation is a balancing act of both assimilation and accommodation and one cannot exist without the other. The two mental processes do not need to have a specific percentile ratio of use but if one mental process is absent in an activity, an individual’s cognition is thrown into disequilibrium. Resulting in undesirable schemes that are either not viable or not adaptable. For example, if an individual primarily uses assimilation, their schemes and cognition develop in an egocentric direction. Meaning, reality becomes subordinate to the individual’s schemes, resulting in a distortion of how the individual interprets external stimuli to fit their schemes rather than reality (Piaget, 1951). Conversely, if accommodation is used without assimilation, an individual will fall into a state of imitation. Mimicking the movements and actions of others without being able to adapt to new external stimuli (Piaget, 1970). Assimilation and accommodation are both necessary for an individual to adapt and revise their schemes.

**Teachers’ Mathematical Meanings**

Thompson (2016) suggested that as a field our understanding of why teachers act and make the decisions they do will grow if we consider how teachers reason with their mathematical meanings. Thompson interpreted Piaget and von Glasersfeld’s notion of scheme as synonymous
with meaning. Defining teachers’ mathematical meanings as reflected schemes which are ways of acting that guide teachers in interacting with students as they strive to develop the meanings the teacher intends (Silverman & Thompson, 2008). Byerly and Thompson (2017) describe mathematical meanings as schemes and ways of coordinating schemes such that teachers can act adaptively in accordance to their goals and instructional situations. Therefore, in accordance with Piaget and von Glasersfeld’s notion of schemes, teachers’ mathematical meanings are personal to the individual, can adapt or change through assimilation and accommodation, and are inseparable from meaning.

Thompson (2016) provides an example of teachers’ mathematical meanings in the consideration of two teachers. Teacher 1 has a mathematical meaning that an equation is anything with an equal sign. As steps are taken to solve an equation each subsequent expression is an equation as long as there exists an equal sign. Teacher 2, on the other hand, has a mathematical meaning of equations not only as an equality statement, but also containing a question prompting an answer. Though both teachers may answer questions similarly involving equations and performing the mathematical computations required to solve equations; both teachers most likely would provide entirely different learning opportunities to students based on their mathematical meanings.

Dingman et al. (in review) describe mathematical meanings similar to Thompson (2016) and Byerly and Thompson (2017), “teacher’s internal interpretation of the mathematics under study” (Dingman et al., in review, p. 17). This includes how teachers understand and think about specific mathematical concepts and are dynamic, constantly changing, and unique to every teacher. Data from the National Science Foundation grant (#1561569) indicates that it is one of the most utilized aspects that middle school teachers reason with as they make decisions.
Additionally, it lies on two edges of the Instructional Pyramid model; a characteristic only shared with the curricular reasoning aspect viewing mathematics from the learner’s perspective.

I chose to focus on the curricular reasoning aspect of mathematical meaning for my study because although declarative knowledge influences teachers’ instructional decisions, it is through teachers’ mathematical meanings that teachers reason. Determining what mathematics concepts are important for students to learn, ultimately determining what learning opportunities students experience. Thompson (2016) also suggests that as a field our understanding of teachers’ mathematical knowledge (i.e., declarative knowledge) will be deepened if we start considering teachers’ mathematical meanings. Teachers’ mathematical meanings are important to study if we hope to better understand teachers’ curricular reasoning and understand how teachers think about the mathematics they teach.

**Teachers’ Mathematical Meanings as a Theoretical Framework**

Using teachers’ mathematical meanings as a theoretical framework I gained access to how the teachers make sense of the mathematics themselves, how the teachers’ want students to make sense of the mathematics, what mathematics they want students to learn, and what mathematics they think is valuable and not valuable for students to learn. Subsequently, I was able to better understand why teachers reason with their mathematical meanings as they made decisions about geometric reflections and orientation of figures and I also examined how teachers adapted their mathematical meanings through assimilation or revised their mathematical meanings through accommodation as they reasoned and made curricular decisions.
Research Questions

For my research, I wanted to better understand teachers’ reasoning with their mathematical meanings as they make curricular decisions. Therefore, my research question was:

What mathematical meanings do middle grades teachers’ reason with when making curricular decisions about geometric reflections and orientation of figures?
CHAPTER 3: METHODS

This chapter explains the methods I used to answer my research question. My research question was: What mathematical meanings do middle grades teachers’ reason with when making curricular decisions about geometric reflections and orientation of figures? First, I describe the contextual mathematics used for my study as well as the reason for providing participant teachers with specific curricula. Second, I describe the data collection for my study. Third, I describe my participants, two middle grade teachers and why these individuals were chosen. Fourth, I explain the data analysis and how it helped answer my research question.

Mathematics Content

I chose to study teachers’ decisions and reasoning about geometric reflections and orientation of figures primarily as a methodological choice, serving as a lens by which to better understand teachers’ curricular reasoning. However, this content is also mathematically important, laying the mathematical foundations for congruence and similarity and this content has a significant number of important opportunities for teachers to make curricular decisions. For example, the data from the National Science Foundation grant (NSF #1561569), despite the mathematical importance of geometric reflections, indicated that middle-grades students struggled learning geometric reflections and teachers struggled teaching geometric reflections. Geometric reflections are a nuanced mathematical concept that requires a variety of curricular decisions that makes for rich content in which to study teachers’ curricular reasoning.

The Common Core State Standards for Mathematics (CCSSM; National Governors Association Center for the Best Practices and Council of Chief State School Officers, 2010) affirm the importance transformations play in grades 6-8. The CCSSM modified the grade-level at which students learn rigid transformations (i.e., translations, rotations, and reflections),
resulting in geometry standards constituting one of the most observable changes to the middle grades mathematics curriculum (Teuscher et al., 2015; Wang & Smith, 2011). Prior to this change it was typical for students to learn rigid transformations in high school, yet not connected to congruence (Wang & Smith, 2011); however, CCSSM defines congruence using rigid transformations (CCSSI, 2010; Teuscher et al., 2015). Despite this new trajectory of connecting congruence to rigid transformations, many curricula have placed congruence prior to rigid transformations (Wang & Smith, 2011). Many teachers and students are using these curricula and therefore may not make these connections between rigid transformations and congruence.

Crites, Dougherty, Slovin, and Karp (2018) recently outline how teachers can aid students in developing essential understandings of geometry to make connections between rigid transformations and congruence. Crites et al. (2018) recommend that (a) students should learn and use the properties and characteristics (including orientation of figures) of geometric transformations to identify whether figures are reflections, translations, or rotations; (b) students should reflect figures over many different lines of reflection (e.g., horizontal, vertical, and oblique) as well as reflect figures that intersect the line of reflection; and (c) students should perform geometric transformations with and without the coordinate grid, but not develop coordinate rules until high school. The authors not only outline salubrious teaching methods for teaching transformations, they also consider common deleterious practices.

In 2010 many states adopted CCSSM, which meant schools needed new curricula that were aligned with these new standards. Teuscher, and Kasmer (2016) analyzed six middle school textbooks that were reported to align with CCSSM and found the majority of the textbooks lacked in their alignment of three core concepts related to geometric transformations: properties of transformations, congruence in terms of transformations, and orientation of figures. Five of
the six textbooks referenced orientation of a figure; however, the description of orientation of a figure diverge into two different conceptualizations of orientation of a figure. Despite supposedly being aligned to the same set of standards every analyzed textbook varied greatly from one another and though such an analysis may not be broadly generalizable to all textbooks, such an analysis does suggest some of the difficulties and challenges teachers face as they teach transformations. These ideas provide additional motivation for why I examined teachers’ mathematical meanings as they made curricular decisions about these topics. We know that teachers use their curriculum materials, with 71% of teachers across the U.S. reported to use textbooks as guides in their classrooms (Banilower et al., 2013), yet if the textbooks teachers are using are not aligned with CCSSM and are lacking mathematical content, it is important to understand teachers’ reasoning in spite of such difficulties.

Although much research regarding geometry and geometric transformations exists, little research has been conducted involving mathematical meanings and geometric transformations. Most research investigating mathematical meanings has focused on algebra, calculus, or statistics. DeJarnette et al. (2016), though not specifically researching mathematical meanings, classified different student conceptualization of reflections. In their research five different student conceptualizations of reflections were identified: (a) rotating conception, (b) flipping conception, (c) visual conception, (d) paper folding conception, and (e) perpendicular bisector conception. Many of these conceptions proved to be problematic for the students, such as the rotating conception in which students thought reflections were rotations, and many conceptions proved to be beneficial for the students such as the paper folding conception and the perpendicular bisector conception that allowed students to consistently and accurately perform reflections. Though this research did not focus on mathematical meanings, it does hint at the
potential in researching mathematical meanings and geometric transformations. Just as the students held so many diverse and impactful conceptualizations of geometric reflections, the teachers in my research also held varied conceptualization and mathematical meanings of geometric reflections.

**Provided Curriculum**

Participant teachers were provided either *University of Chicago Mathematics Project (UCSMP) Geometry* (Benson et al., 2009) curriculum or *Utah Middle Grades Mathematics* curriculum (Rossi et al., 2013) and asked to use the curriculum in planning their unit on geometric transformations. Both curricula were selected for a variety of reasons. Both curricula were unfamiliar to the respective teachers to whom the curricula were provided and both curricula aligned with the CCSSM grade 8 geometry transformation standards. Additionally, both curricula went beyond the scope of the grade 8 CCSSM, creating opportunities for teachers to decide what to teach in the geometric transformation unit. Thus, I was able to study teachers’ curricular decisions and their reasoning with their mathematical meanings for these decisions.

**Data Collection**

I collected data from two grade 8 mathematics teachers. Each teacher completed at least three lesson-cycles that include a pre-interview, lesson observation, and a post-interview for lessons in their geometric transformation unit. The content of the three lesson-cycles were (a) geometric reflections, (b) rotations, and (c) sequencing of transformations. If teachers spanned multiple days on one or more of these lessons, then additional lesson-cycle data were collected. These three lessons were selected because of the richness of the mathematics contained therein and the variety of decisions teachers make about the content in these lessons.
One lesson-cycle was dedicated to geometric reflections; however, all lesson-cycles were useful in the scope of my thesis as each lesson referenced geometric reflections and provided additional data above that which the lesson-cycle on reflections provided. For example, lesson-cycles about geometric rotations included discussions about orientation, which lead to teachers’ reasoning about reflections because reflections change a figure’s orientation. Additionally, data from the National Science Foundation grant (NSF #1561569) indicated students often confuse rotations and reflections. By including the lesson-cycle on rotations I gained access to teachers’ reasoning about why their students may or may not make such mathematical misattributions and the teachers’ mathematical meanings for why reflections and rotations are mathematically similar and different from one another. Lesson-cycles about sequences of transformations were also useful for my study as they included teachers’ reasoning about reflections in conjunction with other geometric transformations. It was also during this lesson that students made connections that rotations were obtained through a sequence of two reflections over intersecting lines and translations were obtained through the sequence of two reflections over parallel lines. Such lesson-cycles further provided data useful in understanding teachers’ mathematical meanings regarding their conceptualization of reflections in relation to other geometric transformations.

The pre- and post-interviews were video recorded to capture the teacher’s decisions and reasoning as they planned and enacted lessons. Interviews followed a specified protocol designed to elicit teachers’ reasoning for their curricular decisions (see Appendix A for pre-interview protocol and Appendix B for post-interview protocol). The pre-interview protocol specifically focused on the teacher’s decisions and reasoning that contributed to the formulation of the planned curriculum. The post-interview protocol focused on the teacher’s decisions and
reasoning that contributed to the enacted curriculum, decisions that differed between their
planned curriculum and enacted curriculum, and possible changes teachers made to future
lessons based on how the lesson went. In addition to the interview protocol, follow-up questions
were used to clarify the teacher's responses and better understand teacher’s modifications
between planned and enacted curriculum.

The lesson observation within each lesson-cycle was also video recorded. The teacher
wore a microphone in order to ensure that all conversations with students were clearly heard. The
video captured the teacher and his/her curricular decisions during the lesson. Student work and
small group conversations were captured if they were presented to the whole class or the teacher
was in the small group. In addition to the video recorded lesson-cycles, several other data were
collected: teachers’ concept map, teachers’ anticipated goals for each lesson, and student tasks. I
outline the importance of these data for my study and how I used these data to answer my
research questions.

The teachers’ concept maps outline their ways of thinking about the mathematics and the
connections within the content. The interviewer used the teacher concept maps during the
interviews to ask questions regarding teachers’ reasoning and mathematical meaning. As I
analyzed the interviews and lessons I used the teachers’ concept maps to help contextualize the
teachers’ responses, decisions, and reasoning.

Teachers’ anticipated goals were a list of the mathematics they want their students to
understand. Goals may change throughout the course of teaching and were captured in the
interview process as the teachers reflected on and revised his/her goals. Teachers’ anticipated
goals were used as reference to contextualize interviews.
The student tasks were given to students during the lesson. These provided reference and context to the teacher’s lesson, anticipated student thinking, and the mathematics the teacher prioritizes in his/her lessons. Similar to the teachers’ anticipated goals, student tasks were used to aid in identifying teachers’ mathematical meanings by displaying the mathematics the teacher valued and the mathematics the teacher wanted students to understand.

Participants

For my thesis, I selected two teachers from the National Science Foundation grant (NSF #1561569) pseudo named Collin and Judy. In this section, I provide their background as mathematics teachers and explain the reasons I selected these particular teachers.

One reason for selecting these two teachers for my study was because of the diversity between the two teachers; both in curricular reasoning and mathematical meanings as well as disposition and teaching experience. Collin took an alternative route in becoming a mathematics teacher, first receiving a Bachelor's degree in history education and then a Masters’ degree in mathematics education from an online university. He had taught junior high mathematics for 5 years. He participated in four lesson-cycles, spending two days on reflections. Collin set performing transformations on the coordinate grid as his prominent goal for students, requiring reflections to be performed primarily over horizontal or vertical lines of reflections. Collin taught at a junior high with high socio-economics, being one of the wealthier areas in the school district. Collin spent the majority of class time giving instruction and working sample problems with students. After working an example problem or two, Collin had students work similar problems with each other and ask him questions. After instruction, Collin typically gave students time to start on their homework, work with peers or by themselves, and ask questions.
Collin participated in the NSF study for one year and was provided with the *UCSMP* (Benson et al., 2009) curriculum and supplemental student tasks. He used portions of one of the supplemental student tasks (see Appendix C). Initially, Collin used similar definitions for reflections and orientations as the *UCSMP* (Benson et al., 2009) curriculum, but as the lesson-cycles progressed, he referred less to orientation of a figure and the definition of reflection. He also used the *UCSMP* (Benson et al., 2009) curriculum less and referred more to his *Glencoe Math* (Carter et al., 2013) curriculum he had used in the past.

Judy received her Bachelor’s degree in mathematics education from a four-year undergraduate university in the Mountain West, and taught mathematics at the same school for over 10 years. She participated in five lesson-cycles instead of the standard three because her mathematical focus was distinctly different than other teachers in the study and the researchers wanted to gather as much data as possible in order to better understand her reasoning that led to these differences. Judy set orientation of geometric figures as a prominent goal; whereas, many of the other teachers in the larger NSF study introduced orientation briefly and rarely referred to orientation of figures except in a perfunctory manner. Judy wanted students to identify geometric transformations, justify their reasoning, and perform a variety of transformations on and off the coordinate grid. She also required students to perform reflections over a variety of lines, including oblique, horizontal, and vertical lines. Judy teaches at a junior high with moderate socio-economics, neither high nor low when compared to surrounding areas. Judy spent the majority of class time having students work on and discuss mathematical tasks. Students worked on the tasks in small group settings and discussed their work and thinking together as an entire class; coming to the board to show their work, explaining their thinking, and asking questions of their peers.
Unlike Collin, Judy participated in the NSF (#1561569) study for two years and for my study I used data from her second year of participation in the larger study. During her first year, as part of the NSF study, Judy was given the UCSMP (Benson et al., 2009) curriculum along with supplemental student tasks. Judy used the mathematical ideas found in the UCSMP curriculum as she developed her own student tasks. During her second year of participation in the grant, Judy was given the Utah Middle Grades Mathematics curriculum (Rossi et al., 2013). Instead of using the given curriculum, she used much of the same mathematical tasks as she did in the first year of NSF study, but refined the student tasks from the previous year. Her definitions of geometric reflections and orientation of a figure were similar to UCSMP’s definitions.

I selected these teachers for my study because both had rich amounts of data, having participated in four or more lesson-cycles. Data were collected on these two teachers because of the interesting curricular decisions they made throughout the teaching process and they decided to spend more time on mathematical concepts related to geometric reflections, rotations, and sequence of transformations. Both teachers were thorough in responding during the interview process. They often extended their response to add more depth and clarity to their response then the original questions elicited. They were also forthright in their responses, answering they “did not know” when asked a question that they in fact did not know the answer too. These rich data allowed deeper analysis to occur than if teachers with cursory responses were selected, and the teachers’ propensity for forthrightness increases the validity of the data by adding further confidence in the accuracy of the teachers’ responses.

Another reason for selecting these two teachers is because of my familiarity with both teachers and my involvement in the data collection process. I was one of the primary data
collectors for both teachers. This familiarity with the teachers and their data is advantageous as I performed data analysis because I was better situated to understand the context of the data and reliably analyze the data than if I had selected teachers whom I was not familiar with.

**Data Analysis**

Data analysis began with the coding of the two teachers’ interviews and their lessons. The interviews and lessons were coded differently and used different units of analysis. I explain how the different data were coded and provide justification for the coding protocol.

**Interviews**

The pre- and post-interviews were coded for teachers’ decisions and reasoning with their mathematical meanings about geometric reflections or orientation of figures. For the coding process, I first divided the interviews into questions posed by the interviewer and responses given by the participants. Next the teachers’ responses were further divided into whether or not they included a mathematical decision or reflection (DR). Next the teacher responses that included a mathematical decision were further coded for whether or not they contained decisions regarding the definition of geometric reflections or decisions regarding orientation of figures which I called big decisions (BD). These big decisions were then coded for whether or not they contained teachers’ reasoning with their mathematical meanings (MM) determined by whether or not the teacher expressed their understanding or interpretation of the mathematics under discussion or expressed what mathematics students needed to know. For example, the following transcription was coded as DR, BD, and MM.

[Reflected figures] are congruent as far as the shape is that they have the same angles [measures] and the same sides [lengths], so they are the same the same shape and the same size so they are congruent in that aspect but … they [reflected figures] are not the
same in terms of orientation, where they are actually flipping their orientation in a reflection, whereas rotation and translation keep that orientation.

The preceding instance was coded as DR because Collin is explaining his decision of why reflected images are congruent, it was coded as BD because it contained Collin reasoning about orientation of a figure, and it was coded as MM because Collin expressed how he understood a figure's orientation in relationship to the transformation. Instances, such as the previous example, that contain DR, BD, and MM served as my unit of analysis and these data were used to describe each teacher’s mathematical meanings for their definition of geometric reflections and orientation of figures. Figure 3 illustrates the iterative process for coding.

**Figure 3**

*Coding Process Steps for Teachers’ Decisions and Reasoning*

<table>
<thead>
<tr>
<th>Step 1: Divide video into questions posed and responses given</th>
</tr>
</thead>
<tbody>
<tr>
<td>Questions and Responses</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Step 2: Divide the question and responses into mathematical decisions or reflections (DR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Questions and Responses</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Step 3: Identify DR instances about geometric reflections or orientation (BD)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Questions and Responses</td>
</tr>
<tr>
<td>BD: Geometric Reflections</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Step 4: Identify BD instances that include teachers reasoning with their mathematical meanings (MM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Questions and Responses</td>
</tr>
<tr>
<td>BD: Geometric Reflections</td>
</tr>
<tr>
<td>MM: Mathematical Meanings</td>
</tr>
</tbody>
</table>
Lessons

For the lesson observations I first identified sections of the video recordings that involved class discussions and instruction about geometric reflections, orientation of figures, or both. These sections were then further divided into smaller instances according to specific characteristics of the definition of geometric reflections or orientation of figures. For example, a class discussion about the definition of geometric reflections may span twenty minutes. I first identified the entire discussion as a single instance. Within this instance there may exist several sub-discussions and instructions about different characteristics of reflection (e.g., equidistance, perpendicularity, congruence). I then divided the larger instance into these smaller sub-discussions about the sundry characteristics. I inferred that the teachers’ mathematical meanings was present because the teacher was discussing the mathematics with their student(s). I verified this to be the case as each instance contained different characteristics for reflections or orientations and it included teachers’ reasoning with their mathematical meanings. Thus, for the lessons, my unit of analysis was the instances containing class sub-discussion about specific characteristics regarding the definition of geometric reflection or orientation of figures that inherently also contained the teachers’ mathematical meanings about geometric reflections or orientation of figures. These data were used, similar to the interviews’ data, to describe each teachers’ mathematical meanings for their definition of geometric reflections and orientation of figures. Figure 4 illustrates identified instances that represent my unit of analysis.

Figure 4

Lesson Coded for Class Discussion about Geometric Reflection or Orientation

<table>
<thead>
<tr>
<th>Classwide Discussion: Reflection</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classwide Discussion: Orientation</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>
In order to increase the reliability of the coding for the pre-and post-interviews, coding was performed by two individuals. After both individuals performed the aforementioned coding, they came together and reconciled to create a consensus coding. The lessons were coded by a single individual (myself); however, I frequently counseled with my advisor during the coding process to maintain reliability in my coding process.

The purpose of my data analysis was to identify the teachers’ mathematical meanings as they made curricular decisions regarding the definition of geometric reflections and about orientation of figures. After coding the interviews and lessons, I established each teacher’s mathematical meanings in relation to the description of geometric reflection and orientation of figures by systematically analyzing these data found in the interviews and lessons. Using this analysis I created models of teachers’ mathematical meanings.

The models of teachers’ mathematical meanings for geometric reflections and orientation of figures were constructed in the following manner. First, I created a detailed explanation of the teachers’ mathematical meanings conveyed in every video recorded instance representing the two units of analysis, both for interviews and lessons. Next, these explanations were analyzed for common themes for the teachers’ unique mathematical meanings. The teachers’ unique mathematical meanings were then categorized according to their themes and organized according to those that occurred the most and those that occurred the least. I inferred that the themes that occurred the most were critical to the teachers’ mathematical meanings and the themes that occurred the least, though potentially part of the teachers’ mathematical meanings did not provide enough evidence to consider them critical. At the end of this process I had a model of the teachers’ mathematical meanings composed of a list of their mathematical meanings organized into categories that were ordered by importance. These models made the teachers’ mathematical
meaning clear as I determined their primary ways of thinking about geometric reflections and orientation. Additionally, the data analysis from the lessons served as validation for the data analysis from the interviews as the teachers’ mathematical meanings were frequently consistent in both the lessons and the interviews.

After the teachers’ models were constructed, I connected the teachers’ decisions with their mathematical meanings found in the models. Such connections were further analyzed to identify how the teachers used their mathematical meanings to reason about geometric reflections and orientation of figures.
CHAPTER 4: RESULTS

In this chapter I answer my research question: What mathematical meanings do middle grades teachers’ reason with when making curricular decisions about geometric reflections and orientation of figures? I first describe the teachers’ mathematical meanings for geometric reflections and orientations of figures. I then provide models for their mathematical meanings. I then discuss the curricular decisions Collin and Judy made, and how they reasoned with their mathematical meanings, providing further evidence of their mathematical meanings. Additionally, I present the results by alternating between Collin and Judy to highlight differences and similarities in their mathematical meanings, curricular decisions, and curricular reasoning.

Development of Models of Teachers’ Mathematical Meanings

Through data analysis, Collin and Judy’s mathematical meanings of geometric reflections and orientation of figures were determined. I inferred their mathematical meanings from their interviews and lesson observations as they discussed how they understood geometric reflections or orientation of figures, explained why they thought such mathematics were important (or not important), and expressed how they wanted students to understand the mathematics under discussion. These inferred mathematical meanings, taken together, were used to create models of Collin and Judy’s mathematical meanings. I also inferred that the more frequently Collin and Judy expressed any particular mathematical meaning the more central it was to their reasoning. I illustrate how I created the teachers’ model of mathematical meaning by using Collin as an example.

In an interview, Collin defined geometric reflections as a transformation in which “...every point is exactly the same distance across a reflection line at a perpendicular angle [from the line of reflection].” From this instance and many like it, I identified through inference, that
Collin’s mathematical meanings held that geometric reflections were comprised of two separate concepts: (a) corresponding points are *equidistant* from the line of reflection and (b) the line segments connecting the corresponding points are *perpendicular* to the line of reflection.

Although Collin defined geometric reflections through the concepts of equidistance and perpendicularity, he did not give equal importance to both concepts. Collin referred to geometric reflections through the concept of perpendicularity 31% of the instances (n=51) in which he demonstrated his mathematical meanings for geometric reflections. However, many of these instances in which Collin referred to the concept of perpendicularity involved him explaining why he did not need to attend to perpendicularity as it was implicit in the coordinate grid. Whereas, Collin referred to the concept of equidistance 55% of the instances in which he demonstrated his mathematical meanings for geometric reflections. For example, Collin stated:

It’s more [the idea of] equidistance… Since the grid is set up [the way it is], each [of] the axes are perpendicular [therefore] we don’t have to worry so much about that definition as much as just looking at the equidistance. Which is why the grid is … nice. The definition fits with what we are already seeing.

Through instances such as this, I determined that Collin’s mathematical meanings, though understanding geometric reflections are defined with the concepts of equidistance and perpendicularity, viewed the concept of perpendicularity as implicit because the definition (including perpendicularity) “fits with what we are already seeing” on the coordinate grid.

In juxtaposition to Collin’s mathematical meanings holding perpendicularity as implicit, Collin’s mathematical meanings held the concept of equidistance as the essence of what geometric reflections are. Collin explained in an interview:
The idea that we’re going to look at each point and see how far away [it is] from our reflection line and we’re going to go on the other side [of the line of reflection] … that same distance. That’s why we came up with that definition. Because that’s the essence of what a reflection is and how we find it.

In this excerpt Collin explained the reason for determining a definition for geometric reflections was so that the students could use the concept of equidistance to perform geometric reflections because “that’s the essence of what a reflection is and how we find it.” Collin did not reference perpendicularity or discuss how it might be useful in conjunction with the concept of equidistance in performing geometric reflections. I inferred from this instance and many similar instances (55% of the instances) that Collin’s mathematical meanings considered the concept of equidistance in relation to geometric reflections as being of primary importance. Using Collin’s decisions and reasoning, I determined through inferences portions of his mathematical meanings for geometric reflections. Using these mathematical meanings, I began to create a model of his mathematical meanings. This model included that although Collin defined geometric reflections through the concept of equidistance and perpendicularity, he viewed equidistance as the primary characteristic of geometric reflections and perpendicularity as an implicit characteristic that his students did not need and so he did not emphasis it.

I also determined through inferences that Collin’s mathematical meanings included the understanding that geometric reflections should primarily be reflected over the x- or y-axis. Collin expressed understanding that geometric reflections could be reflected over any line of reflection, showing one example of what a reflection over an oblique line might look like because a student asked about reflecting over other lines. However, this single instance appeared to be aberrant as Collin referred to reflecting over the x- or y-axis in 39% of the instances in
which he expressed his mathematical meanings for geometric reflections. He also never required students to perform geometric reflection over oblique lines. While initially defining geometric reflections with his students, Collin stated geometric reflections are defined as “points being equidistance from the axis and perpendicular to the axis” (emphasis added). I inferred from this and many similar instances that Collin’s mathematical meanings conflated line of reflection with axis because he primarily thought of geometric reflections as being performed over the x- and y-axis. Collin further clarified that “in this class we are going to focus on the axis when [performing geometric reflections].” Implying that he understood that reflecting over oblique lines is possible but not necessarily important for his students to understand. Thus, as part of the model of Collin’s mathematical meanings I inferred that Collin’s mathematical meanings by in large equated lines of reflection with the x- or y-axis.

As a final example of how I created Collin’s model of mathematical meanings, I refer to Collin’s use of orientation within his class. Collin defined orientation similar to UCSMP (Benson et al., 2009), using the order of vertices to determine if two figures had the same orientation. While verifying whether or not he had correctly performed a geometric reflection with his class, he stated, “How about some of the other criteria? Let’s talk about orientation. What happens to our orientation? [Ok], it changes... So this reflection meets that criteria as well.” From this and several similar instances, I inferred that Collin’s mathematical meanings held that orientation changes in reflections and can be used as a diagnostic tool to determine whether or not a geometric reflection was performed correctly. Collin referenced orientation changing in geometric reflections 36% of instances (n=36) in which he referenced the concept of orientation. Therefore, this reasoning was added to Collin’s model of his mathematical meanings.
This section is not intended to be an exhaustive explanation of how Collin’s model of mathematical meanings was constructed or an exhaustive explanation of his mathematical meanings. Instead, this is intended to give the reader a sense of the process in which I created the teachers’ models of their mathematical meanings.

**Model of Collin’s Mathematical Meanings**

From the data I inferred that Collin’s mathematical meanings held that geometric reflections were composed of two separate concepts: (1) corresponding points are *equidistant* from the line of reflection and (2) the line segments connecting the corresponding points are *perpendicular* to the line of reflection. Though Collin was aware of both concepts, perpendicularity was implicit as it was visually apparent to Collin both on and off the grid. Resulting in Collin primarily thinking about geometric reflections through the concept of equidistance. For Collin’s mathematical meanings, geometric reflections were actions to be performed (i.e., reflecting figures over horizontal and vertical lines). The definition of geometric reflections, particularly the notion of equidistance, was a tool to inform how one might perform geometric reflections. Although he recognized one can reflect over any line, Collin emphasized reflecting over an axis, going as far as defining geometric reflections as a transformation over the axis instead of the line of reflection. Finally, Collin’s mathematical meanings for geometric reflections held that geometric reflections preserved congruence but not orientation.

From the data, I inferred that Collin described orientation similar to *UCSMP* (Benson et al., 2009), using the order of vertices to determine if two figures had the same orientation. His mathematical meanings held that orientation was visually obvious and therefore related to congruence also a visual characteristic. Additionally, Collin’s mathematical meaning held that
orientation changed in reflected figures and could be used to determine if something was reflected correctly.

**Model of Judy’s Mathematical Meanings**

From the data I inferred that Judy’s mathematical meanings held that geometric reflections were a combination of characteristics. These characteristics included (a) reflected figures are congruent, (b) corresponding reflected points are equidistant from the line of reflection, (c) line segments between corresponding points are parallel to each other and perpendicular to the line of reflection, and (d) the orientation of reflected figures change. For Judy, these characteristics were important and not implicit. Judy’s mathematical meanings also held that the line of reflection is a perpendicular bisector. Additionally, for Judy’s mathematical meanings, geometric reflections are mathematical concepts to be understood through which one can perform a variety of actions including identifying geometric reflections,

From the data, I inferred that Judy described orientation similar to *UCSMP* (Benson et al., 2009), using the order of vertices or sides to determine if two figures had the same orientation. To Judy’s mathematical meanings, orientation of a figure and the idea that orientation changes in geometric reflections was a fundamental characteristic of geometric reflections that connected geometric reflections with other geometric transformations. Her mathematical meanings also held that orientation is a means to identify whether or not something is a geometric reflection.

To summarize Collin and Judy’s mathematical meanings Table 1 organizes the teachers’ models of their mathematical meanings. While there are some similarities across the two teachers there are also differences in their mathematical meanings. In the following sections I provide
further evidence of Collin and Judy’s mathematical meanings as I present my findings regarding their decisions and reasoning with their mathematical meanings.

Table 1

Summary of Teachers’ Mathematical Meanings

<table>
<thead>
<tr>
<th>Model of Collin’s Mathematical Meanings</th>
<th>Model of Judy’s Mathematical Meanings</th>
</tr>
</thead>
<tbody>
<tr>
<td>▪ Geometric Reflections</td>
<td>▪ Geometric Reflections</td>
</tr>
<tr>
<td>o Equidistant</td>
<td>o Congruent</td>
</tr>
<tr>
<td>o Perpendicular (implicit and visually apparent)</td>
<td>o Equidistant</td>
</tr>
<tr>
<td>▪ Line of Reflection</td>
<td>▪ Line of reflection</td>
</tr>
<tr>
<td>o Axis</td>
<td>o Perpendicular bisector</td>
</tr>
<tr>
<td>o Horizontal or vertical</td>
<td></td>
</tr>
<tr>
<td>▪ Reflections are actions to be performed</td>
<td>▪ Reflections are concepts to be understood</td>
</tr>
<tr>
<td>o Equidistance is a tool to perform reflection</td>
<td>o Identify geometric reflections</td>
</tr>
<tr>
<td>▪ Orientation</td>
<td>▪ Orientation</td>
</tr>
<tr>
<td>o Described orientation through order of vertices</td>
<td>o Described orientation through order of vertices</td>
</tr>
<tr>
<td>o Orientation is visually obvious</td>
<td>o Orientation is a fundamental characteristic of geometric reflections</td>
</tr>
<tr>
<td>o Reflections preserve congruence but not orientation</td>
<td>▪ Orientation changes in reflected figures</td>
</tr>
<tr>
<td>o Orientation used to confirm reflections are performed correctly</td>
<td></td>
</tr>
</tbody>
</table>

Reasoning with Mathematical Meanings

I identified two visible ways in which Collin and Judy reasoned with their mathematical meanings. The two ways are through their definition of geometric reflections and through their decisions they made during classroom instruction. These two ways emerged during data analysis and serve as platforms to juxtapose the differences between Collin and Judy’s mathematical meanings. I also identified, through inferences, gaps in their mathematical meanings as they reasoned about their decisions. The examples I share below are selected because they represent
the types of decisions and reasoning that the two teachers used multiple times throughout the data collection.

**Collin’s Reasoning with his Definition of Geometric Reflections**

During the first interview, Collin defined geometric reflections as a transformation in which “...every point is exactly the same distance across a reflection line at a perpendicular angle [from the line of reflection].” For Collin’s definition of geometric reflections there were two separate concepts: (a) corresponding points are *equidistant* from the line of reflection and (b) the line segments connecting the corresponding points are *perpendicular* to the line of reflection. I will refer to these concepts as equidistance and perpendicularity, respectively. Collin’s definition of geometric reflections had significant meaning to him as illustrated by him referring back to the definition in all eight interviews and four lessons.

To illustrate Collin reasoning with his mathematical meaning as he made curricular decisions about the definition of geometric reflections I refer to an episode in class in which Collin developed the definition with his students using the first few questions from a task from the supplemental materials that were given to him (see Appendix C). In this task, students used patty paper to perform reflections and make observations. After students worked on the task, Collin listed the students’ observed characteristics of geometric reflections on the board, writing (a) congruent, (b) points are opposite, (c) symmetrical, (d) equidistance, and (e) inverted. I inferred his meaning of these terms to be (a) corresponding reflected lines of figures are *congruent*, (b) reflected *points* x- or y-coordinates are *opposite* depending upon which axis a figure is reflected over [referring to coordinate rules], (c) reflected figures are *symmetrical* to one another, (d) corresponding points are *equidistant* from the line of reflection, and (e) reflected figures orientation changes.
After listing the students’ observed characteristics, Collin stated, referring to the list of characteristics, “… these are excellent characteristics of what a reflection is… let’s solidify this and come up with an actual definition.” Two students provided the following definitions. Student A stated, a reflection is “an alternative perspective of the original shape but the points are inverted from the original.” This student referenced characteristics b and e. Student B stated, a reflection is “like a mirror that looks the same but…are inversed,” Student B also referenced characteristic e from the list.

Although these definitions were not mathematically correct or complete, Collin dismissed these definitions, which I inferred was because the students’ definitions did not align with his mathematical meaning. Collin circled the fourth characteristic from the list; the only one that aligned with his mathematical meaning, and stated that “there’s a little bit more than … reflected corresponding points] are the same distance.” Collin then drew on the board an image similar to Figure 5 as he further explained “there is one key characteristic that we are missing … [drawing in the dotted line] …this line [dotted line] will always do something.” To which a student responded “that line is… always 90 degrees.”

**Figure 5**

*Recreated Image of Collin’s Board Work*
Having heard a student response that aligned with his mathematical meanings, Collin completed the definition of geometric reflections; writing, “Points being the same distance from the axis and perpendicular to the axis.” Thus, we see that while developing the definition of geometric reflections in his lesson, Collin decided to elevate the characteristics that aligned with his mathematical meaning.

**Judy’s Reasoning with her Definition of Geometric Reflections**

Judy did not use a formal-written definition to define geometric reflections, rather, she described geometric reflections through listing different characteristics. Judy was asked how she planned to define geometric reflections and she responded that she was going to use characteristics to describe geometric reflections. These characteristics included (a) reflected figures are congruent, (b) corresponding reflected points are equidistant from the line of reflection, (c) line segments between corresponding points are parallel to each other and perpendicular to the line of reflection, and (d) the orientation of reflected figures change.

To illustrate how Judy reasoned with her mathematical meanings as she made curricular decisions about the definition of geometric reflections I refer to three lessons in which Judy explored the definition with her students. The reason for providing three examples from Judy is because she developed geometric reflections over several lessons. Judy’s first lesson was an introduction to transformations in which students sorted cards into four different categories and then justified their choices. Students autonomously sorted the cards (see Appendix D) into the four transformations: translations, reflections, rotations, and dilations. After which Judy asked the students to proffer reasons for sorting the cards in the fashion they did. The list of characteristics students produced for geometric reflections included: (a) mirrored, (b) reflected, and (c) the same size, which Judy interpreted to mean congruent. When students said mirrored
and reflected, Judy wrote the words down with minimal verbal acknowledgement. However, when students said “same size”, Judy responded by saying, “same size, I like it” before writing it down. Thus, emphasizing to the students that “same size” was a noteworthy characteristic. Judy made the decision to verbally emphasize the concept of congruent as it aligned with her mathematical meanings and listed it as a characteristic of reflection at the start of every lesson.

During the first lesson, Judy and her students also discussed the concept of orientation. While discussing rotations, a student stated that rotated figures “face the same direction.” Judy made the decision to pursue a conversation about this statement as she connected it with orientation, which aligned with her mathematical meaning of the definition of geometric reflections as a characteristic.

Judy: Larry [pseudonym] just said something that I want to point out. Larry, say that again.

Larry: They [the rotated figures] face the same direction.

Judy: They face the same direction. [The class expresses agreement, disagreement, and confusion] Do they face the same direction [indicating the rotated figures]?... I want to make sure we [talk about this]. Okay, Gerry [pseudonym] you say yes they do face the same direction?

Gerry: Yes, because even though [the rotated figure] is tilted the [figure] is the same orientation as the original.

Judy: [Upon hearing the word orientation] Woahhh! Love it. You just got a really big word there for me. Orientation.
Judy discussed briefly what orientation meant with the students. She then discussed whether or not the orientation of a figure changes for geometric translations and geometric reflections.

Judy made the decision to follow-up with a student who said “they face the same direction” because she saw it as a connection to orientation. Judy then made the decision to emphasize the word orientation by exclaiming, “Woahhh! Love it.” Judy made the decision to connect the concept of orientation to geometric reflections because her mathematical meaning defined geometric reflections (in part) through the characteristic that the orientation of a reflected figure changes.

After the first lesson, Judy determined students struggled to explain the meaning of orientation. They were conflating orientation with the direction an object was facing or rotated. After observing the students’ difficulties, Judy decided to restructure her unit plan by adding an additional day dedicated entirely to understanding the concept of orientation. The additional lesson constituted Judy’s second lesson and the decision to restructure her unit to contain this lesson fit Judy’s mathematical meanings as she defined geometric reflection through characteristics and reasoned that students needed more time to understand orientation than originally planned.

During the third lesson, Judy made a variety of decisions that allowed her students to explore all the different characteristics that were aligned with her mathematical meanings for the definition of geometric reflections. First, Judy decided to introduce what a line of reflection was, reasoning, “[students] always start off with saying [the line of reflection] has to be in the middle.” True to her prediction, the first student to proffer an explanation stated, “the line of reflection is going to be the middle … it's like when you open a book, the spine of the book
would be the reflection line.” This decision aligned with Judy’s mathematical meaning that corresponding reflected points are equidistant from the line of reflection. Second, Judy decided to present figure 6 and ask students to justify why the two figures were geometric reflections. Students provided explanations that the figures were both congruent and the figures’ orientation were different from one another. The students’ observations aligned with Judy’s mathematical meanings about the definition of geometric reflections and she did not pursue a conversation about these points further as the students expressed understanding. However, Judy, not hearing a reference to line segments connecting corresponding points being parallel, which was also part of her mathematical meanings for the definition of geometric reflections, decided to continue asking questions until students made a connection between geometric reflections and parallel lines. Third, figure 7 is a question that Judy decided to ask students to draw a line of reflection between two points in order to explore the concept of perpendicularity. Judy reasoned, that “if the [concept of perpendicularity] hasn’t come out yet… then it will have to, if [the students] only have one [set of points, figure 7].” This decision also aligned with Judy’s mathematical meanings as she thought about the line of reflection as perpendicular to line segments formed between reflected corresponding points. Thus, during the third lesson, Judy made a variety of curricular decisions that aligned with her mathematical meanings regarding the definition of geometric reflections as her decisions required students to discuss all four characteristics that constituted geometric reflection for Judy.
Collin’s Reasoning with His Mathematical Meanings During Mathematical Instruction

Although Collin defined geometric reflections as including both concepts of equidistance and perpendicularity, he emphasized the concept of equidistance while unknowingly de-emphasizing the concept of perpendicularity. The following example demonstrates Collin’s reasoning with his mathematical meaning within his decision to perform geometric reflections on the coordinate grid. When asked in an interview if reflecting over the axes on the coordinate grid would require his entire definition of geometric reflections or just the portion involving equidistance, Collin responded:

It’s more [the idea of] equidistance… Since the grid is set up [the way it is], each [of] the axes are perpendicular [therefore] we don’t have to worry so much about that definition
as much as just looking at the equidistance. Which is why the grid is … nice. The
definition fits with what we are already seeing.
Collin reasoned that the coordinate grid was “nice” because it allowed him and his students not
“to worry so much about that definition as much as just [look] at the equidistance.” Collin also
reasoned that perpendicularity was inherently observable on the coordinate grid and that he and
his students could see the perpendicularity, stating “the definition fits with what we are already
seeing.” Therefore, Collin decided that students would only perform geometric reflections on the
coordinate grid and never required his students to reflect over oblique lines. Such a decision
aligned with Collin’s mathematical meanings for geometric reflections because he thought of
geometric reflections as containing both notions of perpendicularity and equidistance. However,
because the grid intrinsically addressed the concept of perpendicularity, in Collin’s mind, he only
needed to address the concept of equidistance.

Collin repeated similar reasoning and meaning to his students. While teaching how to
reflect a figure over a line of reflection that was the x-axis, Collin stated:

The awesome thing about the coordinate grid is that your x- and y-axes are perpendicular
to each other, so all this grid pattern, all those lines are intersecting each other at 90
degrees. So we are going to use these grid lines to make sure we are at 90 degrees and we
are perpendicular.

Collin reasoned that the “grid lines” could be used to make sure that a reflection was at 90
degrees but never explained what he meant by this besides reflecting points over the x- or y-
axis. Thus, Collin reasoned that the coordinate grid made the concept of perpendicularity explicit
in the minds of his students, which aligned with his mathematical meanings about geometric
reflections, and unknowingly de-emphasized this concept with his students. Relying primarily on
the concept of equidistance as he instructed students how to reflect over an axis.

Although Collin mathematical meanings held that perpendicularity was readily
observable from the coordinate grid and thus a concept that did not need explicit attention,
student actions indicated that perpendicularity may actually have been obscured by the
cooridinate grid. After showing students how to perform a reflection over the x-axis; a student,
struggling to understand how perpendicularity related to performing geometric reflections, asked
how one might reflect over a diagonal line as opposed to an axis. Collin had previously stated
that they were going to only perform reflections over horizontal or vertical lines (primarily over
the x- and y-axes); however, he made the decision to take up the student’s question because he
could use it to further show the importance of equidistance, a notion that was aligned with his
mathematical meanings.

Collin responded to the student by using the preimage (a triangle) from a previous
explanation and quickly drew an oblique reflection line. While performing the geometric
reflection, he proceeded point by point around the triangle drawing a corresponding point that
appeared to be reflected across the line of reflection. Emphasizing the distance from the points to
the line of reflection by tracing back and forth between the two with his pencil. He then stated:

That same concept still applies [referring to his previous explanation of how to reflect a
figure]. The idea that we’re going to look at each point and see how far away [it is] from
our reflection line and we’re going to go on the other side [of the line of reflection] …
that same distance. That’s why we came up with that definition. Because that’s the
essence of what a reflection is and how we find it.
Although the quickly reflected points appeared to be perpendicular, Collin never mentioned perpendicularity or its role in the process of performing geometric reflections. Rather, he reasoned with his mathematical meanings that the “essence of what a reflection is and how we find it” is completely related to the concept of equidistance. Further illustrating his decision to emphasize equidistance while unknowingly de-emphasizing perpendicularity.

Collin reasoned with his mathematical meaning to use the concept of equidistance as a tool used to perform geometric reflections. As previously illustrated, Collin thought that the concept of equidistance was not only the “essence of what a reflection is” but also “how we find it.” While instructing students how to perform geometric reflections, Collin gave students a triangle that was on the coordinate grid that was displayed on the document camera. Figure 8 is a recreation of the image used. Collin explained that the class was going to reflect the triangle over the x-axis, stating:

...if we [use] the definition [of geometric reflections] we are going to be exactly the same length away from our reflection line… we can just count down how many units [to the line of reflection] and go beyond [the line of reflection] the same amount.

Figure 8

*Reflection Over the x-axis*
Making a decision to show students how to procedurally perform geometric reflections, Collin reasoned with his mathematical meanings that the definition could be used as a tool to perform reflections. Counting down from each point to the line of reflection and then counting an equal number of squares past the line of reflection. Once again, not mentioning the idea of perpendicularity during the process. Thus, in alignment with his mathematical meaning, Collin emphasized the concept of equidistance while placing the concept of perpendicularity in the periphery; using equidistance as a tool to perform geometric reflections.

After teaching students how to use equidistance to perform geometric reflections, Collin abstracted the procedure into a set of coordinate rules that could be applied to vertices of a figure to produce a geometric reflection over the x- or y-axis. Collin decided to teach coordinate rules in relation to equidistance and never wrote a formal coordinate rule for geometric reflections (i.e., \((x, y) \rightarrow (x, -y)\) or \((x, y) \rightarrow (-x, y)\)). Instead, Collin had students compare points of geometric reflections over the x-axis and the y-axis and identify patterns that could be abstracted into rules. Practicing on a list of coordinate pairs reflected over either the x- or y-axis.

Collin was asked in an interview how the generalized rules connected to geometric reflections, for which he responded, “it’s fairly loose, when you're doing reflections over vertically or horizontally you can still kind of make those connections, at least vaguely into them.” Following-up on what Collin meant by “vaguely” to which he further explained, “... if you have vertical lines not on the y-axis, you can still have the basic idea of distance, like the distances still have to be the same. It meets that criteria of reflection.” Once again, Collin’s decisions to emphasize how the coordinates change were formed, in part, by Collin’s mathematical meaning about geometric reflections - emphasizing the concept of equidistance while performing geometric reflections.
Judy’s Reasoning with Her Mathematical Meanings During Mathematical Instruction

Judy used her mathematical meaning as a tool to understand geometric reflections. Judy intended to teach geometric reflections both on and off the coordinate grid; however, she expressed uncertainty of how she wanted to proceed as she was trying to determine whether she wanted to start on the coordinate grid or whether she wanted to start off the coordinate grid, stating:

I can see being able to go to [the coordinate grid], that way we can talk about the things that we have talked about so far, like parallel lines, it will just be there… we can just do that on a graph. I just hope then, taking away the graph for the reflection… that’s where I am having a hesitation... I can see the benefits of [exploring reflections on a coordinate grid] but I also want to be able to do that task (see Figure 7) where I made them find the line of reflection not on a graph to get them to see perpendicular lines... they can get [the characteristic of perpendicularity] from a graph too, but for some reason in my mind… it [exploring the characteristic of perpendicularity] makes more sense [off a coordinate grid]. Maybe that’s where I am biased to say, it makes more sense to just say, let’s actually do it [explore the characteristic of perpendicularity] first, let’s not worry about the graph first, let’s do it [explore the characteristic of perpendicularity] and then we will put it on a graph.

Judy was at a crossroad, she could either start on the coordinate grid in order to talk about characteristics such as parallel lines or she could start off the coordinate grid so that students could “see perpendicular lines.” Judy was having a dilemma because both decisions were supported by her mathematical meanings. To Judy, parallel lines comprised, in part, her definition of geometric reflections and she thought parallel lines were readily visible via the
coordinate grid. On the other hand, to Judy, the definition of geometric reflections also contained the concept of perpendicularity and though Judy acknowledged perpendicularity could be taught on the coordinate grid she explicitly stated “It makes more sense” to teach the characteristic of perpendicularity off the coordinate grid.

Ultimately, Judy decided to start by teaching geometric reflections off the coordinate grid. Judy reasoned with her mathematical meaning that it was more important for students to understand geometric reflection than to perform geometric reflections.

I am trying to get [students] to see it [characteristics]… rather than do it [perform geometric reflections] … To me, as I read the core and talk to [the 9th grade teachers], my goal is to get [students] to see it [characteristics] and to prove why [reflections are made up of those characteristics] … If I can get the properties out there so that [the students] can use them next year, to say ‘oh yeah, those lines have to be perpendicular to the line of reflection or whatever we get out of this.

Judy wanted students to see the different characteristics of geometric reflections and use those characteristics to prove whether or not a geometric transformation was a geometric reflection. As previously mentioned, to Judy, teaching off a coordinate grid better allowed her students to see these characteristics, particularly the concept of perpendicularity, as it “made more sense” to Judy.

The concept of equidistance, perpendicularity, and the orientation of a figure in relation to geometric reflections were of primary importance to Judy and her definition of geometric reflections and subsequently received more instructional time. Whereas, the concept of line segments between corresponding points being parallel received less instructional time. Therefore, I discuss Judy’s decisions regarding these characteristics (equidistance,
perpendicularity, and orientation) and not the concepts of line segments between corresponding points being parallel.

The characteristics of equidistance and perpendicularity were both important to Judy’s mathematical meanings of geometric reflections. During several interviews, Judy described the line of reflection as a perpendicular bisector. A description that requires both concepts of equidistance and perpendicularity. Judy also wanted students to understand the concepts of perpendicularity and equidistance in relation to geometric reflections. When explaining what she wanted students to understand about the line of reflection, Judy stated, “The details [of understanding the line of reflection] will start [with] finding the line of reflection and being able to say it is perpendicular and all that good stuff.” By “find the line of reflection” Judy meant, given two reflected figures, students were to identify lines of reflection. By “all that good stuff” Judy meant the concept of the line of reflection being a perpendicular bisector; as she explained, “I would have [the students] look at the perpendicular bisector.” Therefore, the concept of the line of reflection being a perpendicular bisector was an integral part of Judy’s mathematical meanings for geometric reflections.

Reasoning with her mathematical meanings, Judy made the decision to emphasize the concept of equidistance in several lessons. As previously discussed, Judy purposefully made the decision to begin her lesson about geometric reflections by asking students what they thought the line of reflection was, anticipating, “[students] always start off with saying [the line of reflection] has to be in the middle.” Judy also decided to emphasize equidistance as she required her students to identify lines of reflection and formalize characteristics of geometric reflections. Such decisions aligned with her mathematical meanings as she thought of lines of reflection as perpendicular bisectors, which imply the concept of equidistance.
To illustrate one such decision in which Judy emphasized equidistance, I refer to an episode in a lesson where Judy selected a student to come to the board and explain how he drew in a line of reflection.

Judy: How did you draw your line [of reflection]?

Student: [Drew a very rough line of reflection with no explanation.]

Judy: But how did you know where to start and where to end?

Student: [Indicated a line segment between two corresponding points and stated he]

“started in the middle of that.”

Judy: There was something big he said there, did anybody catch what he said? He said the middle, is this [the line of reflection] important that we draw this in the middle?

[Class discussed why they thought it was important to draw the line of reflection in the middle of line segments formed by corresponding points.]

Judy was quick to attend to the concept of equidistance as she heard the presenting student say “in the middle”. She emphasized equidistance by equating the concept to “something big” or important. She also decided to give students time to discuss why it was “important that we draw this [line of reflection] in the middle.” Equidistance was important to Judy and she wanted her students to understand its relation to geometric reflections.

Judy also reasoned with her mathematical meanings as she made the decision to emphasize the concept of perpendicularity as it was important to her mathematical meanings. During the previously discussed episode in which the student came to the board and drew in a line of reflection, Judy decided to push the students in an attempt to elicit a connection to perpendicularity. She indicated a non-perpendicular line (with a yardstick) that bisected one line
segment that connected corresponding points but not all line segments between corresponding points. Figure 9 represents both the student’s line of reflection and Judy’s representation, through using a yardstick, of a non-perpendicular line.

Judy: So if I found the middle of this guy right here [pointing to the point equidistant between A and A’] and I say it kind of goes like that, [moves the yardstick to no longer be perpendicular] is that a good line of reflection to draw?

Class: It has to be the same distance. It has to be in the middle

Judy: It has to be in the middle of those. Do you understand why it has to be in the middle of those?

Student: It has to be in the middle because if it is angled or is off from the middle, they [the reflected figures] won’t be symmetrical and if it is a reflection it will be symmetrical every time.

Judy attempted to get students to see perpendicularity as she asked, “do you understand why it has to be in the middle of those?” but students were firm in attending to equidistance explained that if the line of reflection is “off from the middle” then the reflected figures would no longer be symmetrical, which was problematic to the student.
Judy anticipated that students would have difficulties recognizing the characteristic of perpendicularity; explaining, “the [characteristic of] perpendicular doesn’t always come out immediately.” Therefore, as the idea of perpendicularity was not coming up with the students, Judy dropped the idea for the time being; having previously planned a question that would further elicit the notion of perpendicularity as students were asked to draw a line of reflection between only two points (see figure 7). Therefore, Judy reasoning with her mathematical meanings determined that the concept of perpendicularity was important for students to understand and decided to provide multiple opportunities for students to understand perpendicularity.

For Judy, orientation of a figure was an important characteristic of geometric reflections. This was the second year Judy taught orientation of figures and she recognized the concept’s
importance in understanding geometric reflections. Judy decided to explore orientation much earlier this year than last year. Deciding to explore orientation from the first lesson to the last lesson. When asked if she thought bringing up orientation was a good thing, Judy stated, “I think it has [been a good thing]. All the way through it has [been a good thing].” Judy reiterated in another interview why she thought teaching orientation was beneficial, explaining, “to me, it’s just the fact that orientation came out earlier is going to help [students’] whole explanations of things and properties.... Although [orientation] was there last year, I made sure [orientation] was out there earlier [this year].” Judy’s mathematical meanings held that orientation of figures was an essential characteristic for geometric reflections, resulting in Judy deliberately deciding to make “sure orientation was out there earlier.”

For herself, Judy described orientation similar to UCSMP (Benson et al., 2009), using the order of vertices or sides to determine if two figures had the same orientation. Judy referred to such a description frequently during interviews and during lessons. However, prior to the first lesson, Judy was unsure how students would describe orientation because, in accordance to her pedagogical penchant of not explicitly providing students with a description, Judy decided to let students come up with their own descriptions. Judy stated, “It will be interesting to see what [the students] come up with because to me orientation means something, but to them it doesn’t.”

After the first lesson, students had not come up with a description for orientation that aligned with Judy’s mathematical meaning of orientation of a figure. Despite students recognizing that orientation was changing in reflections, by the end of the first lesson, students did not have a clear explanation of what they meant by orientation and referred to orientation through the direction a figure was “facing”. Judy explained that her other classes “were set on what their definition of orientation was” but the observed class was “still up in the air.”
In an attempt to solidify her students’ description of orientation, Judy decided to add an extra day to her unit devoted to exploring the meaning of orientation in alignment to her mathematical meanings that considered orientation of a figure as an important characteristic of geometric reflections. Having taught orientation to her satisfaction in other classes, Judy had several plans of how she might elicit students to make connections in regards to orientation that aligned with her meanings. She explained:

What I found is, in some of my classes, and again this didn’t happen in [this class] but [the students] were okay with the fact that a reflection was flipping [figures] over and that’s kind of what we based our orientation discussion about. Okay, if it flips what happened there, how is it different than a translation or a rotation? Where you could just pick it up, move it and we are good rather than flipping it over. And so, getting [the students] to eventually see the going around the shape is the best place to go is what I have seen in my other classes.

Judy determined that her conversation about reflections being “flips” in previous classes was the enabling factor that aided students to realize that orientations could be described through the order of the points “going around the shape.”

In alignment with her thoughts, during the lesson about orientation, students proffered the ideas that rotated figures could be mapped to one another by spinning one figure similar to a pinwheel. Judy, identified an opening and decided to steer the conversation towards reflections by showing figure 10, an image of two geometrically reflected squirrels, and asked, “Does that same thing [mapping a figure through rotation] work on this?” To which a student responded, “Whatever direction [the squirrel] spun, it wouldn’t be the same [as the other squirrel] ... it’s
flipped” Judy decided to use the student’s explanation as a springboard to describe orientation, stating:

If we start here [Judy pointed to the head of the leftmost squirrel in figure 10] and go clockwise we are going head, tail, feet, head, tail, feet, head … [as Judy said this she pointed to the corresponding part of the indicated squirrel]. Okay if we go clockwise over here [Judy indicated the rightmost squirrel] it goes head, feet, tail… There is no way without flipping him because we are flipping this guy over… so his orientation is different. So on a reflection, his orientation, his head, tail, feet, head, tail, feet, is different.

Judy decided to describe orientation similar to her mathematical meaning; using the order of vertices, or in the case of the squirrels, the order of the head, feet, and tail to determine the orientation of a figure.

**Figure 10**

*Example of Reflected Figures Used to Describe Orientation*

Judy decided to ask students to identify geometric reflections and justify their choices.

Given a variety of geometric transformations, students accurately identified which
transformations were geometric reflections and which transformations were not, citing orientation (and the fact that orientation changes in a geometric reflection) as their reasoning. We asked Judy if she was okay with this reasoning and she stated,

I still think for right now it’s okay [to only reference orientation] but as we get into…

[Judy paused in thought] I think even then [students] can use orientation to [identify geometric reflections]. I think that’s okay… I think it is helpful to have that frame of reference [indicating orientation as a tool to understand what makes a geometric reflection].”

This reasoning was aligned with Judy’s mathematical meanings in that she thought orientation was important and “helpful” to understand geometric reflections. Judy used orientation as a tool to understand geometric reflections.

**Gaps in Teachers’ Mathematical Meanings**

In my analysis of the two teachers' mathematical meanings, I identified gaps in their mathematical meanings that I inferred influenced their decisions about the mathematical content that they chose to teach their students. These gaps in the teachers' mathematical meanings may explain why teachers made the decisions that they did in planning and enacting their lessons. From an observer perspective these decisions may not have made sense, yet understanding the gaps in teachers' mathematical meanings provides an explanation for why the teachers may have made the decisions they did. Often these gaps in the teachers' mathematical meanings resulted in the mathematics being disconnected for the teacher and students. These gaps often times caused the teachers to not interpret student responses in ways that would help students make sense of the mathematics content.
Collin’s Gaps in His Mathematical Meanings Related to Geometric Reflections

By primarily focusing on the concept of equidistance while performing geometric reflections and de-emphasizing perpendicularity; Collin had an inflexible mathematical meaning that prevented him from making sense of his students' thinking. Such inflexibility is illustrated by a question Collin posed to his students on a homework assignment. The question read, “[True or False] Point A is exactly 2 inches away from a reflection line at its closest point. Point A’ will also be exactly 2 inches away from the reflection line.” The question is ambiguous, leaving the students unsure as to what is being asked. Are the points intended to be assumed as reflections of one another? Are the points equidistant from the line of reflection and the reader is supposed to determine if they are reflections of one another?

When asked in an interview how he expected students to respond to this question, Collin stated, “[are the points] going to be the same distance [from the reflection line]? That is what I am trying to drive at. [Is point A] the same distance away from our reflection line as our other point? [moving his hands to indicate an image similar to figure 1]” continuing on to explain he wanted students to respond that the question was true. In Collin’s response, it was clear that he wanted students to associate equidistance and the line of reflection with one another. However, to Collin there was no ambiguity because this aligned with his mathematical meanings, if two points are equidistant from a line of reflection then the points are reflections and if two points are reflected then the two points are equidistant from the line of reflection. I inferred that Collin thought that equidistant from the line of reflection was sufficient to assume reflection. However, figure 11 shows a counter example to such thinking. Points are equidistant or even “2 inches away from a reflection line at its closest point” and yet these points are not a reflection.
Collin’s mathematical meaning about geometric reflections had conflated points being equidistant from a line of reflection as being synonymous with reflections. Such a gap in his mathematical meanings may have contributed to the previously illustrated decisions in which equidistance was emphasized and perpendicularity was unknowingly de-emphasized. Or how equidistance was used as a tool to perform geometric reflections where perpendicularity was not. Additionally, the true-or-false question itself came about because of Collin’s mathematical meaning. The question was created by Collin. He made the decision on how it was worded and that it was important to ask students, reasoning there was only one way to envision equidistant from a line of reflection.

Collin viewed the characteristics of geometric reflections as criteria to verify if a geometric reflection was performed correctly. After performing a geometric reflection for his students, Collin used the characteristics to check his work; stating:

Let’s re-evaluate some of this and look to see if it fits all of our criteria as well. [Ok], we can already see, and we have already talked about our definition that these [reflected figures] will be the same distance away... from our reflection line. So D and D' are the same distance away from our reflection line along with [the other points] ... So it meets that criteria. How about some of the other criteria? Let’s talk about orientation. What
happens to our orientation? [Ok], it changes... So this reflection meets that criteria as well. Let’s look at our last criteria, are these congruent? [Ok], they look congruent, but do we know for sure? There are some things we are not sure of, we don’t have [angle measures] here. We do have one line we can look for sure [indicating a side length that is horizontal on the coordinate grid and its length can be counted] ... So we can see that these side lengths are the same, they are congruent. [Ok], and that happens when it is a true reflection, that all the side lengths are congruent... So we can see that it also meets that criteria.

Collin used the word criteria interchangeably with characteristic, directly referencing the students’ list of characteristics for geometric reflections. It is interesting to note that the first criteria Collin referenced was equidistance and yet did not mention perpendicularity as a criterion. Further providing evidence that Collin reasoned with his mathematical meanings that equidistance was important and perpendicularity was implicit in reflections.

Additionally, it is interesting to note that Collin made a decision to only check one side length when verify congruence, reasoning it was sufficient because in a true reflection the other sides must be congruent as well and since the figures were equidistant from the line of reflection, orientation had changed, and one side was already congruent his performed transformation was indeed a true reflection. However, when asked in an interview if students could perform a reflection and state the two figures were congruent, Collin responded:

I would want a little bit more… if [students] are just saying, yeah [the reflections are] congruent, that’s just telling me they are just getting the basic understanding, it’s kind of that lower end idea of it, but someone actually going more in depth, [stating the figures] are congruent because the angles are the same, [there] is a line of symmetry, and gives
that full explanation [referring to the criteria], that’s someone I can clearly see is at that higher end... understanding that full concept.

Thus, it seemed that Collin did not want students to use the transformation of a reflection to infer characteristics about figures, such as inferring two reflected figures were congruent because they were reflections. Rather, Collin wanted students to use characteristics to verify geometric reflections were performed correctly.

**Collin’s Gaps in His Mathematical Meanings Related to Orientation of a Figure**

This was the first year Collin taught the orientation of a figure. He recognized that orientation was in the Utah Core Standards (Mathematics Core, n.d.), and that it was a unique characteristic of geometric reflections; therefore, he decided to teach it. Reasoning, “I decided to add it this year because it is a unique property [of geometric reflections] and I did see it after reading through the [Utah Core Standards] again.” When asked if he was only teaching it because it was in the Utah Core Standards he stated, “a little bit, yeah.” But further reasoned with his mathematical meanings that there were some practical uses to understand the orientation of a figure. Reasoning, “[students] will have to identify if [a figure is a reflection] and [the transformed figures] might look like they are reflections but their orientations might be the same or you might have very similar shapes that look the same and might be translation verses a reflection and you can describe why or why not.”

Collin did not have a clear image of the importance of orientation. He referenced the Utah Core Standards (Mathematics Core, n.d.) as the primary reason for teaching the concept and although he understood orientation to be a unique characteristic of geometric reflection, he was unsure how orientation connected to geometric reflections beyond being a unique characteristic. Because Collin had an incohesive mathematical meaning of how the orientation of
a figure was related to geometric transformations, and subsequently his decisions about orientation were incohesive. Collin decided to teach orientation in relation with congruence, two concepts that are not normally related to one another. Collin also decided to teach orientation as a diagnostic tool to verify if geometric reflections had been properly performed as his mathematical meanings placed importance on the performance of geometric reflections.

Collin defined congruence as, “all the [angle measures] are the same and all the side lengths are the same.” He also understood that geometric reflections preserved congruence, citing it as one of the two properties of geometric reflections he would teach. The second property being orientation. However, previous to teaching geometric reflections and subsequently orientation, Collin had only informally taught congruence. Explaining that if asked what congruence meant, his students “would probably say they [geometric figures] are the same. They are exactly the same.” To Collin’s mathematical meanings, congruence was primarily identified through two figures appearing visually the same.

For Collin, orientation was related to congruence because like congruence, orientation was a visually recognizable characteristic. When asked how he planned to talk about orientation, Collin explained:

[First] we’ll have [an] activity where we will just basically understand reflection and see a reflection. Hopefully, we’re going to lead that discussion into, is this exactly the same… relate it back to translations where we are seeing the exact same image… Hopefully, someone will catch that, no it's not quite. I am expecting [students] aren’t going to be able to explain exactly [what orientation is], but [I do expect students to] be able to identify, yeah something is a little off. And then hopefully build into that definition of why [orientation] is not the same [in geometric reflections].”
Collin’s mathematical meanings held translated figures were “the exact same”, meaning translated figures are congruent and the same orientation. However, within geometric reflections, orientation is changing. Collin assumed that before describing orientation, students would naturally identify “something is a little off” with geometric reflections in relation to geometric translations because orientation was visually recognizable.

Collin decided to make a distinction between congruence as observed in geometric translations and congruence in geometric reflections, calling the latter “not congruent by… orientation.” A phrase that holds no conventional mathematical meaning. Collin was asked in a post interview what he meant by this phrase in which he explained:

They [reflected figures] are congruent as far as the shape … they have the same angles, and same sides so that they are the same shape and the same size. So they are congruent in that aspect… but they are not the same in terms of orientation, where they are actually flipping their orientation in a reflection. Whereas, rotation and translation keep that orientation. That is what I was probably trying to drive at.

Collin stated that reflected figures are congruent because they are “the same shape and the same size”; however, he stated they were only congruent in “that aspect” but “not the same in terms of orientation”. Collin decided to teach orientation in relation to congruence because he had intertwined the meaning of orientation with congruence. According to Collin’s mathematical meanings, both concepts were visual aspects of geometric figures and therefore interrelated. It is interesting to note, that even Collin was a little unsure what he meant by “congruent but not by orientation”, saying “that is what I was probably trying to drive at.” Providing further evidence that Collin did not have a clear meaning of the importance of orientation in relation to geometric reflections.
Despite not having a clear mathematical meaning of orientation’s importance in relation to geometric transformations, Collin had a well-developed description of orientation. In a pre-interview Collin predicted that “possibly we will have a student be able to identify the points, that they are not in the same order, the same pattern. Which is how you can identify whether or not something has that orientation.” Collin described orientation similar to UCSMP (Benson et al., 2009), using the order of vertices to determine if two figures had the same orientation.

As previously explained, Collin’s mathematical meanings held orientation as a property of geometric reflections and decided to use orientation the same way he used other properties: (a) as an aspect of geometric transformations, and (b) as a criterion to check that a geometric reflection was performed correctly. At the start of every class, Collin decided to spend a scant amount of time listing out previously taught geometric transformations and their corresponding properties. After the first lesson, whether orientation was maintained or changed was always present on every geometric transformation. Geometric reflections, Collin would write that orientation is changed but on geometric translations, rotations, and dilations Collin wrote that orientation was not changed. Collin also decided to teach orientation as a verifiable procedure because, in part, his mathematical meanings viewed properties as criteria used to check whether or not a geometric reflection was performed correctly. During one particular lesson, after performing a geometric reflection, Collin verified that the new figure was indeed a geometric reflection by checking three criteria. (a) equidistance, (b) orientation, and (c) congruence. While checking orientation Collin stated, “Let’s talk about orientation. What happens to our orientation? [Ok], it changes... So this reflection meets that criteria as well.” Collin reasoned with his mathematical meanings to decide to use orientation as a criterion to check whether or not a geometric reflection was performed correctly.
Collin’s mathematical meanings about orientation also made it difficult for Collin to understand students’ perspective in relation to orientation. During a lesson Collin presented an image of two figures which was a rotation. Collin, referenced the two rotated figures as he stated to the class, “our orientation [between the two figures] is not changing. So could we have a reflection in here?” The students responded both “yes” and “no”, to which Collin explained “if I have a reflection, understand I would have to have two reflections...the best way to probably describe [what transformation occurred] is to understand here our image is rotating. It has to be changed somehow, so there has to be a rotation.” Collin did not attend to why students said “yes” the figures with different orientations were reflections. When asked what he thought the students were thinking when they said “yes”, Collin responded, “I don’t think [students] know how to identify multiple characteristics of how it is changing. this was one thing I really wasn’t expecting.” Collin’s mathematical meaning viewed orientation as a diagnostic tool to verify if a geometric reflection was performed correctly but did not view orientation as a tool to identify what transformation was performed. Collin’s decisions reflected such mathematical meanings.

Judy’s Gaps in Her Mathematical Meanings Related to Geometric Reflections

Judy’s mathematical meanings held the characteristics of geometric reflections as important and she was quick to connect student language to the different characteristics. When students said, “same size” Judy connected the student language to congruence, when students mentioned the direction a figure was “facing” Judy connected the student language to orientation, and when students mentioned the line of reflection being in the “middle” Judy connected the student language to equidistance. Although Judy decided to teach the characteristic of perpendicularly, she had a much more difficult time connecting the student language and thinking with the characteristic. For example, Judy did not relate perpendicularly
with students’ descriptions of “straight” lines. Additionally, Judy explained that one of her students had determined whether or not figures were geometric reflections by folding their paper to determine if figures lined up through the paper. The concept of perpendicularity is implicit in such a strategy, requiring the fold to be the line of reflection as otherwise the figures would not be transformed an equidistance and perpendicular to the fold. However, Judy did not recognize why more students did not attempt a similar strategy, despite not having explicated perpendicularity yet at that time in her unit. She stated, “I don’t know [why more students did not fold their papers]. I really don’t know.” Perpendicularity was important to Judy’s mathematical meanings, but she did not connect it with student language or certain student strategies.

Discussion

In my discussion, I discuss productive and unproductive mathematical meanings and discuss how teachers assimilate or accommodate their schemes about geometric reflections and orientation of figures. Through my analysis of the data I identified four themes in which Collin and Judy’s mathematical meanings were productive or unproductive that I will discuss. These four themes are (a) the connection between equidistance and perpendicularity, (b) geometric reflections over oblique lines or over axes, (c) the relation between geometric reflections and congruence, and (d) orientation as a characteristic of geometric reflections.

Productive vs. Unproductive Mathematical Meanings

Productive mathematical meanings are described as providing coherence to student mathematical understanding and preparing students for future mathematics learning (Thompson, 2016). The more coherent a teachers’ mathematical meaning, the greater the opportunities for students to learn mathematics coherently; thus, such mathematical meanings are referred to as
productive. Similarly, the less coherent the teachers’ meanings, the fewer opportunities for students to learn mathematics coherently; thus, such mathematical meanings are referred to as unproductive (Byerly & Thompson, 2017). For example, we cannot say that thinking about geometric reflections as flips is not true, but we can say such meaning is unproductive as it is useful in limited circumstances.

Productive teachers’ mathematical meanings are identifiable. An indicator of productive teachers’ mathematical meanings is if the mathematical meanings can be used to understand a wide range of mathematical contexts and problems (Byerly & Thompson, 2017). For example, understanding that a geometric reflection is a rigid transformation that transforms a figure over a line of reflection such that the line of reflection is a perpendicular bisector to any line segment connecting corresponding points on the pre-image and image is mathematically more useful in a wide range of contexts and problems than knowing that a geometric reflection is a flip. The first meaning positions one to perform reflections, identify lines of reflections, and construct geometric reflections; whereas, the meaning of flip positions one to imprecisely estimate the position of a flipped figure, not attending to the relation between the transformation or “flip” and the line of reflection.

I determined whether or not Collin or Judy’s mathematical meanings were productive by comparing whether or not their mathematical meanings aligned with Crites et al. (2018) recommendations for teaching geometric reflections or whether or not their meanings aligned with CCSSM’s definition of congruence. Crite et al. recommended: (a) students should learn and use properties and characteristics (including orientation of figures) of geometric transformations to identify whether figures are reflections, translations, or rotations; (b) students should reflect figures over many different lines of reflection (e.g., horizontal, vertical, and oblique) as well as
reflect figures that intersect the line of reflection; and (c) students should perform geometric transformations with and without the coordinate grid, but not develop coordinate rules until high school. CCSSM defines congruence using rigid transformations (CCSSI, 2010; Teuscher et al., 2015).

**The Connection Between Equidistance and Perpendicularity**

Thinking about geometric reflections through the concept of equidistance and perpendicularity is not the same as thinking of the line of reflection as a perpendicular bisector. Though similar, both meanings have subtle differences. Thinking about geometric reflections through the concept of equidistance and perpendicularity situates the reflected figures as the object of consideration in relation to the line of reflection. For example, the line of reflection is not considered as acting as a bisector, rather the distance from the figure to the line of reflection is the notion under consideration. Additionally, it is the line segments formed between corresponding points of the reflected figures that are considered perpendicular to the line of reflection as illustrated by Collin’s statement, “perpendicular to the line of reflection.” Collin never said that the line of reflection was perpendicular to the line segments. The understanding that the line of reflection is a perpendicular bisector situates the line of reflection as the object of consideration in relation to the reflected figures. Meaning, the line of reflection is the actor, bisecting line segments formed by connecting corresponding reflected points. The line of reflection is considered perpendicular in relation to the formed line segments that connect corresponding points on the pre-image and image.

It stands to reason that teachers who think about the line of reflection as a perpendicular bisector may think of reflected figures through the concepts of equidistance and perpendicularity and vice versa. In fact, Judy’s language suggested this very thing as she alternated between
discussing geometric reflections through the concept of a perpendicular bisector and the concepts of equidistance and perpendicularity. However, the frame of reference in which one’s mathematical meanings actively thinks about geometric reflections appears to matter. Thinking about the line of reflection as a perpendicular bisector appears to inseparably connect the concepts of equidistance and perpendicularity; whereas, thinking about geometric reflections through the concepts of equidistance and perpendicularity appears to separate the concept of equidistance and perpendicularity. Such a separation may not be inherently unproductive; however, as in Collin’s case, such a separation may lead to an emphasis of one concept and a de-emphasize of the other.

Mathematical meanings that emphasize equidistance and de-emphasize perpendicularity appear to be unproductive and mathematical meanings that connect both equidistance and perpendicularity (i.e., meanings that consider the line of reflection as a perpendicular bisector) appear to be productive. Although Collin defined geometric reflections with both equidistant and perpendicularity, I inferred that his mathematical meanings held the notion of perpendicularity as implicit and something that was visually obvious. Collin reasoned with his mathematical meanings to only perform geometric reflections on a grid, contrary to Crites et al. (2018) recommendation to perform geometric reflections both on and off the coordinate grid. Thus, Collin’s mathematical meaning to view perpendicularity as implicit and emphasize the characteristic of equidistance was unproductive, not resulting in learning opportunities for students to perform reflections off the grid. Judy’s mathematical meanings held that the line of reflection was a perpendicular bisector and geometric reflections were defined through the characteristics of equidistance and perpendicularity. This resulted in Judy deciding to ask students to use characteristics to identify geometric reflections as well as perform geometric
reflections on and off the coordinate grid. Judy reasoned that she wanted to start off the
cordinate grid because, to her, perpendicularity was not obvious and the coordinate grid
obfuscated perpendicularity by making it implicit. Therefore, Judy made curricular decisions,
which aligned with Crites et al. (2018) recommendations, to both identify figures that are
gometric reflections and perform geometric reflection on and off the coordinate grid. Thus,
Judy’s mathematical meaning that emphasized the importance of both equidistance and
perpendicularity was productive.

**Geometric Reflections as Concepts to be Understood or Actions to be Performed**

Mathematical meanings that hold geometric reflections as a concept to be understood are
productive. For Collin’s mathematical meanings, geometric reflections were actions to be
performed. Although he recognized one can reflect over any line, Collin emphasized reflecting
over an axis, going as far as defining geometric reflections as a transformation over the axis
instead of the line of reflection. Such a curricular decision did not align with the Crites et al.
(2018) recommendation to reflect over a variety of lines, including oblique lines. For Judy’s
mathematical meanings, geometric reflections were a mathematical concept to be understood.
Judy decided to give students a variety of problems in the attempt to allow students to make a
variety of mathematical connections to help identify a geometric reflection. Thus, Judy required
students to perform geometric reflections over a variety of lines of reflection, not primarily over
the axes. This decision aligned with the Crites et al. recommendation. Thus, it appears that
mathematical meanings that hold geometric reflections primarily as actions to be performed are
unproductive as they do not provide students with opportunities to reflect over a variety of lines
and mathematical meanings that hold geometric reflections as concepts to be understood are
productive because they do provide students with opportunities to reflect over a variety of lines.

**The Connection Between Geometric Reflections and Congruence**

CCSSM defines congruence using rigid transformations (CCSSI, 2010; Teuscher et al., 2015). Neither Collin nor Judy’s mathematical meanings were aligned with such a definition. Rather, Collin and Judy defined congruence through corresponding side lengths and corresponding angle measures being equal. Thus, Collin and Judy’s mathematical meanings about the relationship among congruence, geometric reflections, and transformations in general appeared unproductive for them to make decisions aligned with CCSSM to define congruence through rigid transformations. Additionally, Collin’s mathematical meaning regarding orientation and congruence appeared to be unproductive in that it did not align with either CCSSM or traditional mathematical definitions of congruence.

**Orientation as a Characteristic of Geometric Reflections**

Mathematical meanings that view orientation as an important characteristic of geometric reflections are productive. Collin and Judy both described orientation similarly and both made curricular decisions to teach orientation of figures; however, both did so for different reasons, which resulted in different learning opportunities for students. Collin taught orientation primarily because it was in the Utah Core Standards (Mathematics Core, n.d.). Judy taught orientation because, to her mathematical meanings, it was a fundamental characteristic of geometric reflections. Collin spent a very limited amount of instructional time dedicated to orientation of a figure; whereas, Judy spent several days on the topic. Collin’s students used orientation in a limited capacity to verify if they performed a geometric reflection correctly. Whereas, Judy’s students used orientation to identify geometric reflections. Judy’s students’ use of orientation
existed in part because Judy gave students many opportunities to identify geometric reflections, but it also appears to be because Judy emphasized the importance of orientation as a characteristic of geometric reflections. Collin’s mathematical meanings resulted in decisions that were not aligned with the Crites et al. (2018) recommendation to use properties and characteristics (including orientation of figures) of geometric transformations to identify whether figures are reflections; whereas, Judy’s decisions regarding orientation were aligned to the Crites et al. recommendation. Thus, Collin’s mathematical meanings for orientation, which did not consider orientation as an important characteristic of geometric reflections, were unproductive as it did not provide students opportunities to use orientation to identify geometric reflections. Whereas, Judy’s mathematical meanings for orientation, that held orientation as an important characteristic of geometric reflections, were productive in providing students opportunities to use orientation to identify geometric reflections as suggested by Crites et al. Therefore, it is not enough for teachers to know what orientation is, they must have mathematical meanings that hold orientation as important in relation to geometric reflections.

Assimilation vs. Accommodation

Collin has five years of teaching experience and in his planning he seemed to determine what mathematics he wanted students to perform, seldom anticipating student thinking. The tasks he designed or provided to students were also frequently designed to practice mathematics skills and were not intended to elicit specific mathematics thinking. Students’ incorrect solutions to problems were frequently not attended to, or assessed to be deficiencies in students' understanding and not because students may have alternate ways of thinking about mathematics. Collin appeared to primarily assimilate and not accommodate.
Judy has over ten years of teaching experience and in her planning she seemed to reflect deeply on how students think about mathematics and do mathematics. She designed her tasks so when student thinking came up she did not need to accommodate her mathematical meanings to student thinking because she had already considered multiple viewpoints and various ways of thinking. Judy had a few instances in which she demonstrated a willingness to accommodate her mathematical meanings, but she appeared to primarily assimilate and not accommodate.

Aligned with Piaget’s (1970) sentiments, the ratio between assimilation and accommodation does not need to be a specific ratio. Both Collin and Judy appeared to assimilate more than they accommodated; however, how they planned and enacted curriculum was very different. Collin appeared not to accommodate because he planned primarily based on his thinking; whereas, Judy appeared not to accommodate because she had proactively prepared for student thinking. It may have also been the case that Judy, with more than ten years of teaching experience, had already frequently accommodated her mathematical meanings resulting in cohesive and coherent mathematical meanings that resulted in different learning opportunities than those provided by Collin.
CHAPTER 5: CONCLUSION

While planning and enacting mathematics instruction, mathematics teachers engage in curricular reasoning as they make a myriad of curricular decisions that subsequently influence the mathematics students have the opportunity to learn. However, very little is known about why teachers make the curricular decisions they do. By examining teachers’ curricular reasoning, specifically through their mathematical meanings we can understand how teachers make sense of the mathematics themselves, how the teachers’ want students to make sense of the mathematics, what mathematics teachers want students to learn, and what mathematics teachers think is valuable for students to learn. Additionally, we are better able to understand what mathematical meanings are productive (or unproductive) for student mathematics learning. Through explicitly asking two teachers about their curricular decisions, I created models of two teachers’ mathematical meanings related to geometric reflections and orientation of figures to understand why they made the curricular decisions they did. Mathematical meanings that emphasize both equidistance and perpendicularity in relation to geometric reflections and consider the line of reflection as a perpendicular bisector are productive. Additionally, mathematical meanings that hold the concept of orientation as fundamental to understanding geometric reflections is also productive.

Contributions

Thompson (2016) suggests that as a field our understanding of teachers’ mathematical knowledge (i.e., declarative knowledge) may increase as we study teachers’ mathematical meanings. Though I did not directly study nor assess Collin and Judy’s declarative knowledge, my study contributes to the field’s understanding of teachers’ declarative knowledge as I examined Collin and Judy’s mathematical meanings and their curricular reasoning. Collin and
Judy both appeared to have similar declarative knowledge about geometric reflections. Both used the concepts of equidistance and perpendicularity to describe geometric reflections and both had similar descriptions for orientation of figures. However, despite Collin and Judy’s apparent similarities in their declarative knowledge, they made different decisions as they taught geometric reflections. Collin and Judy’s declarative knowledge was not the only determining factor of their instructional practices. Their different decisions and reasoning resulted in different mathematics learning opportunities for students. Assessing teachers' declarative knowledge may not give us the details of why teachers are making decisions that go against their declarative knowledge. Thus, my study contributes to our understanding of teachers’ declarative knowledge by demonstrating teachers’ declarative knowledge, though vitally important in mathematics teaching, is mediated by teachers’ mathematical meanings and that it may be beneficial to study teachers’ mathematical meanings.

My study also contributes to the field by expanding our understanding of how teachers think about geometric reflections. Much of the recent research on geometric reflections has largely focused on students’ conceptualization of geometric reflections or recommendations for teaching geometric reflections (e.g., Crites et al., 2018; DeJarnette, González, Deal, & Lausell, 2016). However, little to no research has been conducted on teachers’ mathematical meanings of geometric reflection or how teachers conceptualize geometric reflections. As a contribution of this study, I provided specific models of two teachers’ mathematical meanings about geometric reflections and orientation of figures, identifying productive and unproductive mathematical meanings, and identifying what decisions teachers may make as they reason with their mathematical meanings about geometric reflections and orientation of figures.
I determined four main productive mathematical meanings teachers should have in relation to geometric reflections. First, that teachers’ mathematical meanings that connect both concepts of equidistance and perpendicularity are productive. Resulting in curricular decisions that emphasize both concepts and provide students with opportunities to learn geometric reflections in a similar manner and not solely through the concept of equidistance. Second, that mathematical meanings that hold geometric reflections as a concept to be understood are productive; potentially resulting in decisions to perform geometric reflections over several different lines of reflection, not just horizontal or vertical lines. Third, that teachers' mathematical meanings that define congruence through geometric reflections are productive as teachers may reason with such meanings to make decisions that connect across multiple mathematics concepts. For example, teachers who define congruence through geometric transformations may use geometric transformations as part of mathematical proofs. Fourth, that teachers’ mathematical meanings that value orientation as a fundamental characteristic of geometric reflections are productive as teachers may reason with such meanings to make decisions that connect geometric transformations to one another. Such decisions may include defining geometric translations as two geometric reflections over parallel lines that preserve orientation or geometric rotations as two geometric reflections of two intersecting lines that also preserve orientation. Thus, my study provides a contribution to the field by identifying productive (or unproductive) mathematical meanings about geometric reflections and orientation of figures. However, despite identifying four productive mathematical meanings we want teachers to have, within the large NSF (#1561569) study we found that teachers’ mathematical meanings may more frequently be aligned with Collin’s unproductive meanings. Indicating a
need for teachers to identify unproductive mathematical meanings within their own meanings and develop productive mathematical meanings.

**Implications for Practice**

An implication for teaching is that through supporting teachers in the development of productive mathematical meanings through professional development, teachers can make decisions that provide desirable mathematics learning opportunities for all students. Much research regarding mathematics teachers’ professional development has centered on improving teachers’ declarative knowledge (e.g., Hill & Ball, 2004; Rogers et al., 2007). However, as discussed in the contributions of my study, teachers’ declarative knowledge, though important, is not the only determining factor in instructional quality as teachers reason with their mathematical meanings. Musgrave et al. (2015) believe teachers cannot support students in developing deeper mathematical understandings beyond the mathematical meanings teachers possess. This is aligned with my findings as teachers make decisions they reason with their mathematical meanings and teachers with unproductive mathematical meanings make decisions that provide incohesive mathematics learning opportunities for students. Thus, it is imperative mathematics teachers develop productive mathematical meanings. Teacher educators may use these findings to help mathematics teachers become aware of their own mathematical meanings and develop productive meanings in relation to geometric reflections and orientation of figures. Such productive mathematical meanings include connecting both equidistance and perpendicularity in relation to reflections and holding orientation as a fundamental characteristic in understanding geometric reflections. Thus, by developing productive teachers’ mathematical meanings, teacher educators may positively affect teachers' reasoning and decisions, resulting in desirable mathematics learning opportunities for all students.
Implications for Research

Middle grades mathematics textbooks developed to support productive mathematical meanings may influence teachers’ curricular reasoning and curricular decisions, which subsequently affects the mathematics learning opportunities for students. Teuscher and Kasmer (2016) conducted an analysis of six middle grades textbooks that purported to align with CCSSM and found that many of the textbooks were lacking in several areas in relation to CCSSM. These areas included properties of geometric transformations, congruence, and orientation of a figure. Such findings aligned with my findings that productive or unproductive mathematical meanings about geometric reflections also center on how teachers think about geometric reflections through properties (e.g. equidistance and perpendicularity), how teachers thought about congruence, and how teachers thought about orientation of figures. However, one would need to study how teachers make sense of textbook materials to understand how textbooks are contributing to these productive and unproductive mathematical meanings.

Dingman et al. (in review) posit a hypothesis that teachers who reason with multiple curricular reasoning aspects make different decisions and subsequently provide different mathematics learning opportunities than teachers who reason with fewer curricular reasoning aspects. Though not studying the other curricular reasoning aspects, it appeared that Judy reasoned with more than just her mathematical meanings. Along with her mathematical meanings, Judy often considered students’ mathematical thinking and her designed tasks elicited desired student thinking. Thus, it appears that in accordance with Dingman et al.’s hypothesis, Judy, who appeared to reason with more than just her mathematical meanings, made different decisions than Collin who appeared to reason primarily through his mathematical meanings. This suggests that we need to study teachers’ reasoning with multiple curricular reasoning aspects as
they make curricular decisions and how reasoning with multiple curricular reasoning aspects may impact teachers’ curricular decisions and students’ opportunities to learn.

**Limitations and Direction**

Dingman et. al. (in review) explained that teachers reason with a variety of different aspects along with their mathematical meanings. A limitation of this study is that I examined one aspect in which teachers reason, their mathematical meanings and though I was able to determine how Collin and Judy reasoned with their mathematical meanings in relation to geometric reflections and orientation of figures, I did not examine how they reasoned with multiple aspects of the curricular reasoning framework. As previously explained, it appeared Colin infrequently reasoned with more than one or two curricular reasoning aspects at a time; whereas, Judy appeared to frequently reason with multiple curricular reasoning aspects as she made curricular decisions. However, I am only able to make conjectures about how Collin and Judy reasoned with multiple curricular reasoning aspects and how these different aspects influenced how they reasoned with their mathematical meanings as they made curricular decisions. Further research is needed into how teachers reason with multiple curricular reasoning aspects.

Another limitation to my study is that I examined how teachers reason with their mathematical meanings as a single notion. Dingman et. al. (in review) divide reasoning with mathematical meanings into two types of reasoning that lie on two edges of the Instructional Pyramid model. One edge, representing how teachers reason with their mathematical meanings about the mathematics under study, is situated between the vertices representing decisions about the teacher and the mathematics. Another edge, representing how teachers reason with their mathematical meanings about what mathematics is important for students to learn, is situated between the vertices representing decisions about mathematics and students. Though I examined
both types of reasoning, I combined teachers’ reasoning with their mathematical meanings into a single notion instead of considering the two different edges as different types of reasoning. Therefore, I did not examine how teachers reason with their mathematical meanings in different ways, and how these different ways of reasoning with their mathematical meanings influence teachers' curricular decisions and the mathematics learning opportunities provided to students. Further research into how teachers reason with their mathematical meanings on the two edges of the Instructional Pyramid model between the vertices teachers and mathematics and between the vertices mathematics and students is needed and may provide further understanding of how teachers reason with their mathematical meanings.

The content area in which I studied teachers’ curricular reasoning with their mathematical meanings was constrained to geometric reflections and orientation of figures. Although such a constraint of content area is not a limitation, there exists a need for research about how teachers reason with their mathematical meanings in other content areas other than geometric reflection and orientation of figures. Some content areas that may provide insight into teachers’ mathematical meanings are angle measures, exponential functions, and fractions as research has shown that these are areas in which teachers struggle teaching and students struggle learning (e.g., Confrey & Smith, 1995; Moore, 2010; Moore, 2013; Izsák, 2008).

Conclusion

The reason for performing this study was to understand why teachers make the curricular decisions they do. I accomplished this, in part, by examining teachers' explicit reasoning for their decisions and analyzing how they reasoned with their mathematical meanings as they made curricular decisions. This resulted in an understanding of how teachers think about geometric
reflections and orientation of figures and why they make the decisions they do in relation to geometric reflections and orientation of figures.
REFERENCES


APPENDIX A

- Teachers’ decisions as they planned the lesson mathematics
- Mathematical Goals for Student
- Placement of lesson sequence (captured in the Pre-Interview storyline)
- Analyzing from the learner’s perspective, Mapping learning trajectories, Reflecting and revising plans

CONTENT
- Tell me about your decision not to include __________ in the UCSMP materials (or supplemental materials) in your lesson plan?
- What is your understanding of the common core state standards related to transformations?
- How does your lesson align or not align with the common core state standards related to transformations?

Lesson 4.1
- Definition of Reflection
  - Why do you think the definition of reflection is important for students to understand?
  - How does knowing the definition of reflection assist students in creating reflected images?

Lesson 4.2
- Properties of Reflection
  - What are properties in general?
  - Why are you teaching (or not teaching) all the properties of reflection in your lesson?
- Orientation of a figure
  - What is the orientation of a figure? Is this a different definition than what you have previously used? How so?
  - Do you consider orientation of a figure to be a property of reflections? Why or why not?
  - Why do you think that the orientation of a figure is important for students to learn?

Lesson 4.5
- Rotations
  - How was this lesson different or the same from teaching rotations before?
  - What did you learn from planning this lesson that was new to you?
  - What important aspects of rotations were prominent for you as you planned this lesson?
  - What was mathematically challenging for you as you studied this lesson?

Lesson 4.7
- Isometries and Glide Reflections
  - How do you see the properties of reflection connected to congruence?
  - What did you learn about isometries?
- What was your understanding of glide reflections prior to planning this lesson?
- Given that glide reflections are not in the Grade 8 Common Core Standards, why did you choose to teach this lesson?

**RESOURCES**
- In what ways did the resources that you used inform or assist you in the design of your lesson?

**DECISIONS during planning**

| Goals          | - How did you decide on the mathematical goals you selected for your students?  
|                |   - What is it that you want students to know and understand about mathematics as a result of this lesson?  
|                |   - How are your goal(s) related to the common core state standards?  
| R&R            | - Is this the first time using these materials to plan your lesson?  
|                | - Is this the first time teaching the lesson?  
|                |   - If so, do you have any areas of concern about using these materials where you think there might be some holes? What are those areas? Do you have any modifications that you prepared to address these holes?  
|                |   - If not, what modifications did you make from the last time you used these materials?  
|                | - What parts of the lesson do you expect to go well? Why?  
|                | - What parts of the lesson do you expect to be rough? Why?  
| A-LP           | - Why did you decide to use the task you choose in your lesson?  
|                |   - Where did you get the student task that you are using?  
|                |   - Are you intending to have students do the entire task? If not, which questions will you skip and why will you skip them?  
|                |   - Did you complete/do the student task as part of your planning?  
| A-LP           | - What aspects of the UCSMP materials do you feel comfortable teaching? Why?  
|                | - What aspects of the UCSMP materials do you feel uncomfortable teaching? Why?  
| A-LP           | - Which of all the ways that you anticipated students solving the task do you think your students will use? Why?  
|                |   - Of the misconceptions that you listed in the LPP, which ones will you address in your lesson and how will you address them?  
|                |   - Why will you not address specific misconceptions?  
| Mapping        | - Why did you sequence the lesson in the way that you chose?  
|                | - follow-up to response on Lesson Plan Protocol (LPP) questions about past lessons and upcoming lesson
APPENDIX B

Lesson 4.1
- Definition of Reflection
  - What aspects of the definition of reflection do you feel like your students understand? Why?
  - How did understanding the definition of reflection assist your students in reflecting images?

Lesson 4.2
- Properties of Reflection
- Orientation of a figure
  - How did understanding orientation of a figure assist your students in reflecting images?

Lesson 4.5
- Rotations
  - How did teaching rotations as reflections over two intersecting lines help or hinder your students’ understanding of this transformation?
  - What aspects of rotations did your students understand better than in past years?
  - What aspects of rotations did your students not understand better from past years?
  - What was mathematically challenging for your students as you taught the lesson?

Lesson 4.7
- Isometries and Glide Reflections
  - How did this lesson connect the prior lessons together?
  - What was mathematically challenging for your students as you taught the lesson?

DECISIONS during implementation of lesson

<table>
<thead>
<tr>
<th>R&amp;R</th>
<th>What do you feel went well in your lesson? Why do you think it went well?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>What do you feel didn’t go well in your lesson? Why do you think it didn’t go well?</td>
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<table>
<thead>
<tr>
<th>R&amp;R</th>
<th>What surprised you about the implementation of the lesson?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>What surprised you about your students’ learning of the content?</td>
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| R&R | Which of the mathematical goals did you feel your students achieved? How do you know? |

<table>
<thead>
<tr>
<th>R&amp;R</th>
<th>Do you feel your task promoted student learning in the way you had hoped? How so?</th>
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<tr>
<td></td>
<td>[If necessary] Why did you choose to modify or change the task as you did?</td>
</tr>
<tr>
<td>R&amp;R</td>
<td>• [If necessary] Why did you choose to change the sequence of the lesson?</td>
</tr>
<tr>
<td>R&amp;R</td>
<td>• [If necessary] Why did you choose to address or not address specific misconceptions?</td>
</tr>
<tr>
<td>R&amp;R</td>
<td>• If you were to teach this lesson again, what changes would you make to the lesson? Why?</td>
</tr>
<tr>
<td>R&amp;R</td>
<td>• What do you plan to teach tomorrow based on how this lesson went today?</td>
</tr>
</tbody>
</table>
| A-LP       | • Compare the UCSMP materials with ones you have used to teach this content before.  
            |   • In what ways did they better address the mathematics content?  
            |   • In what ways did they not address the mathematics content? |
| Mapping    | • How do you see this lesson fitting in the broader grade 8 curricula?  
            |   • Specifically, when is Congruence and Similarity taught? |

- Tell me about any decisions you made while teaching the lesson. What propelled you to make these decisions?
  - Follow up with decisions observer noticed. [Observer protocol]
    - Tell me about your decision not to include _________ from your lesson plan?
    - Tell me about your decision to change ______ from your lesson plan?
    - Tell me about your decision to add _________ to your lesson plan?
APPENDIX C

Properties of Reflecting Figures

1. Draw a non-regular polygon $WXYZ$ on a piece of patty paper – either on the left or the right side of the patty paper.
2. Fold the patty paper to make a line of reflection that does not pass through the polygon $WXYZ$.
3. Trace over the top of polygon $WXYZ$.
4. Unfold the patty paper and draw the reflected polygon labeling it $W'X'Y'Z'$.
5. Compare your two polygons
   a. How do the points in the preimage compare to the points in the image?
   b. How do the lengths of the sides in the preimage compare to the lengths in the sides of the image?
   c. How do the angles in the preimage compare to the angles in the image?
6. Are the two figures a reflection? Why or Why not?