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Student Use of Mathematical Content Knowledge During Proof Production

Chelsey Lynn Van de Merwe
Brigham Young University

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Chelsey Lynn Van de Merwe

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Steven Williams, Chair
Daniel Siebert
Steven Jones

Department of Mathematics Education

Brigham Young University

ABSTRACT

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Chelsey Lynn Van de Merwe
Department of Mathematics Education, BYU
Master of Arts

Proof is an important component of advanced mathematical activity. Nevertheless, undergraduates struggle to write valid proofs. Research identifies many of the struggles students experience with the logical nature and structure of proofs. Little research examines the role mathematical content knowledge plays in proof production. This study begins to fill this gap in the research by analyzing what role mathematical content knowledge plays in the success of a proof and how undergraduates use mathematical content knowledge during proofs. Four undergraduates participated in a series of task-based interviews wherein they completed several proofs. The interviews were analyzed to determine how the students used mathematical content knowledge and how mathematical content knowledge affected a proof's validity. The results show that using mathematical content knowledge during a proof is nontrivial for students. Several of the proofs attempted by the students were unsuccessful due to issues with mathematical content knowledge. The data also show that students use mathematical content knowledge in a variety of ways. Some student use of mathematical content is productive and efficient, while other student practices are less efficient in formal proofs.

Keywords: proof, mathematical content knowledge, undergraduate education

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CHAPTER 1: INTRODUCTION

Proof writing is an important component of advanced mathematical activity and learning (Herbst, 2002; Stylianides, et al., 2017). In fact, proof is one of the features of mathematics that distinguishes it from other disciplines. The myriad of purposes for mathematical proof include verification, explanation, discovery, systematization, communication, and intellectual challenge (de Villiers, 1999). Hersh (1993) agrees with de Villiers that there are many purposes of mathematical proof, including convincing a student that a statement is true and providing the student with mathematical insight about an assertion. The emphasis on proof is apparent even in the early grades.

In the United States, the National Council of Teachers of Mathematics (NCTM, 2000) includes reasoning and proof as one of five process standards students should engage in when learning mathematics. The NCTM proposes that reasoning and proof be incorporated into mathematics classrooms beginning in kindergarten because by reasoning about the mathematics, "students should see and expect that mathematics makes sense" (p. 4). According to NCTM, informal proof construction should begin as early as kindergarten. Formal proof construction is usually introduced in a middle or high school geometry class (Herbst, 2002; Moore, 1994). The proofs introduced at the secondary level are typically limited to direct, two-column proofs. Thus, proof at the secondary level provides students with brief exposure to proof.

Students become more engaged in the formal practice of proving in undergraduate courses. Proof comprehension and production is heavily emphasized in advanced mathematics courses. Students who pursue a bachelor's degree in mathematics or mathematics education are usually required to be competent in a variety of proof writing techniques. Professors place great value on proof in undergraduate courses, often making proof the primary method of assessing student performance (Weber, 2001). In preparation for these proof-intensive courses, an increasing number of primary and secondary mathematics curricula are designed to provide students with proof writing experience.

Despite the importance of proof in advanced mathematics, it is well-established that university undergraduates struggle to write mathematical proofs (Duval, 2007; Epp, 2003; Moore, 1994; Stylianides, et al., 2017). The literature on proof suggests that in addition to mathematical content knowledge, students need to develop an understanding of the basic nature of proof (Bae, et al., 2018; Duval, 2007; Weber, 2001). Understanding the nature of proof includes understanding the rules of logic, knowing how to use prior knowledge advantageously, and familiarity with various proof structures and techniques. Issues with rules of logic are evident in students' struggle to understand the components of a logical argument. For example, students demonstrate a misunderstanding of the nature of proof if they struggle to understand the relationship between the premise and the conclusion of a theorem. In order to produce a deductive argument, students must assume the premise and then use mathematical facts and rules of deduction as stepping stones to arrive at the conclusion. Misunderstanding the nature of proof is also seen when students struggle to make logical, purposeful connections between the mathematical facts used in the argument. Although students may have a robust understanding of the mathematics being used, they may still struggle to put that knowledge together to form a proof. Thus, they miss some of the "logical stepping stones" needed to reach the conclusion. Another evidence that students misunderstand the nature of proof is lack of strategic knowledge (Weber, 2001), meaning students do not know how to use their mathematical knowledge advantageously to produce a proof. Students lacking strategic knowledge often do not know how to use definitions or theorems in a logical argument. Strategic knowledge also refers to competence in proof writing skills such as selecting and correctly using a proof structure.

While most literature focuses on the struggles students experience with the logical nature of proof, it is impossible to study mathematical proof without attending to mathematical content. The logical nature of proof refers to the logical structures and components present in a mathematical proof. It is worthwhile to consider how characteristics of mathematical content knowledge may affect student proof production (Moore, 1994; Weber, 2001). For example, students may lack knowledge of important theorems or definitions, making it difficult to reason about the mathematics. A student asked to write a proof about the divisibility of integers needs to be familiar with the definition of division with integers. Without this mathematical knowledge, the proof may be quite difficult to construct.

In response to the struggles students experience writing proofs, many mathematics departments have created courses to introduce students to proofs (Moore, 1994). Most universities with mathematics departments have an introduction-to-proof (ITP) course (David & Zazkis, 2017). ITP courses are designed to help students make the transition from computation-heavy courses to proof-intensive courses (Moore, 1994). The course is usually required for mathematics and mathematics education majors. While there may be several intended learning goals for ITP courses (such as that students learn to communicate their mathematical understanding in the formal language of proof), the main focus of ITP courses is to teach students to comprehend and write valid mathematical proofs. ITP courses are designed to primarily address student struggles understanding the nature of proof as a logical entity. While mathematical content, such as set theory or number theory, is typically used as a context for writing proofs, the mathematical content varies between courses (David & Zazkis, 2017). The mathematical content seems to be considered only as a necessary medium through which to learn to write proofs. Thus, the courses generally do not directly address deficiencies in mathematical content knowledge. Research on proof similarly focuses on students' struggle with the nature of proof, largely ignoring the mathematics (Dawkins & Karunakaran, 2016). This is evidenced by the fact that, in the context of proof production, there is substantially less literature on mathematical content knowledge than on the logical nature of proof.

I hypothesize that this lack of attention to mathematical content knowledge in ITP classes is hindering students from writing valid proofs. Focusing too much on the logic and proof structure (as perhaps is the case in many ITP courses) may cause students to intellectually separate the practice of proving from other important mathematical practices such as, making conjectures, exploring mathematical relationships, and discovering relationships (Herbst, 2002). This separation may hinder students from using their mathematical content knowledge to construct valid proofs. Additionally, if deficiencies in mathematical content knowledge are not addressed, students may be incapable of constructing valid arguments because they will be trying to use their incomplete mathematical knowledge to construct a proof. This study will provide insights into how students' mathematical knowledge is (or is not) being used by students in one ITP course. These insights are important because issues with mathematical content knowledge may be a major reason students struggle to write proofs. I hope this study will begin to fill a gap in the research about the role of mathematical content in proof production.

This study draws on part of diSessa's (2018) Knowledge-in-Pieces (KiP) framework as a theoretical context. According to KiP, knowledge is best understood as being held in individual pieces. These pieces are linked together to form concepts. An important aspect of the KiP framework that I will use is that the *context* of a problem affects which pieces of knowledge are used (or *activated*) in order to solve it. For example, consider a student working within the context of a mathematics homework assignment. In this particular context, he is likely to activate pieces of knowledge that he recently studied in class. The context may lead him to ignore pieces of knowledge he did not learn in class. In this study I consider proof writing as a new mathematical context and explore how pieces of knowledge are (or are not) used in the context of proof writing.

KiP is a particularly useful framework for this study because it is flexible. There is little research about how mathematical knowledge influences proof production. Thus, there is no existing framework within the proof literature that fits this study. The general nature of KiP also lends itself nicely to this study because KiP can be used to examine how pieces of mathematical knowledge are used by students in different contexts. KiP also allows me to distinguish between the two issues students experience with mathematical content knowledge: (a) deficient mathematical content knowledge and (b) deficient logical connections between mathematical principles. Using KiP, I consider the pieces of knowledge (mathematical knowledge) as individual entities that are connected by logic. My analysis sheds light on how students struggle with the mathematical pieces of knowledge and the logical connections between the pieces.

The data for this qualitative study were gathered from students in an ITP course at Brigham Young University. The data were gathered during units of the course about proofs with integers. A content analysis of the course textbook was performed to determine what mathematical knowledge about integers could be used to write the proofs in the textbook. I chose four research subjects and administered a pretest to the students to determine their prior mathematical knowledge with respect to the knowledge about integers that is needed for the course. I observed the class during portions of the units about integers to provide context for the interviews. Interviews were conducted after three class periods to determine how the participants' mathematical knowledge affected their ability to write valid proofs. The task-based interviews allowed me to analyze how students activated their mathematical content knowledge within the context of proof production. As part of the final interview students were asked to reflect on how they used mathematical knowledge in different contexts. Further details of the data collection and analysis will be given in Chapter 3.

CHAPTER 2: BACKGROUND

In this chapter I review existing literature about proof comprehension and production by undergraduate mathematics students. I discuss the ambiguity associated with the meaning of proof and describe the view of proof this study adopts. Then I discuss what mathematics education researchers already know about student struggles with proof production. The framework for this study (KiP) is then described in detail. The chapter concludes with an example of how KiP will be used to answer my research questions.

What Counts as a Mathematical Proof?

In a comprehensive review of the literature on teaching and learning proof, Stylianides, et al. (2017) state there is ambiguity in the research about what constitutes a mathematical proof. Some researchers argue that proof should be an argument that convinces both the writer and reader of a statement's truthfulness (Sowder & Harel, 1998). The ambiguity arises because mathematicians use different criteria to decide what convinces them a statement is true. For example, a student may provide extensive examples illustrating the truthfulness of a statement and thereby convince themself that the statement is true. However, most mathematicians would remain unconvinced that a general statement is true based on a set of limited examples.

Weber (2014) proposes that proof is a cluster concept, meaning there is no precise definition of a proof. Rather, there is a cluster of characteristics an argument must satisfy to be considered a proof. Weber suggests six characteristics to consider when determining if an argument is a mathematical proof. Specifically, a proof should: (a) be convincing, (b) be deductive, (c) be transparent (the reader could fill in any potential gaps in the argument), (d) be clear (e) follow communal norms, and (f) be accepted by the mathematical community. These characteristics are not new to the field; Weber simply compiles these characteristics into a new model. Proofs are considered typical if they contain all the cluster characteristics; atypical proofs satisfy only some of the cluster characteristics. Further study by Weber (2016) with the idea of proof as a cluster concept showed that mathematicians agree about the classification of typical proofs but are divided about atypical proofs. Because of this widespread ambiguity about proof, researchers are encouraged to be explicit about the definition of proof that is adopted in a given study. The view of proof assumed in my research and the cluster characteristics satisfied by this view are discussed below.

Reid & Knipping (2010) provide a model for thinking about proof using three dimensions: a) whether proofs are considered to be written artifacts, mental actions and reasoning, or products of discourse; b) to what extent a proof must be convincing, deductive, or formal; and c) the philosophy of mathematics assumed. The first dimension refers to what format the proof takes on. Proof can be observed in proof-text (written work), reasoning (verbal or written reasoning), or discourse (arguments made in mathematical discourse). The second dimension refers to how broad the researcher's view of proof is. A narrow view of proof implies that a proof must be deductive, convincing, and at least semi-formal; a broad scope adopts only one or two of these characteristics. The third dimension is what philosophy of mathematics the researcher assumes. Knipping and Reid suggest four possible philosophical perspectives: a priorist (belief that axioms refer to real objects in the world), infallibillist (axioms are accepted as true and deductive arguments conserve truth), quasi-empiricist (mathematics, including proof, is fallible and subject to revision), and social constructivist (belief that the truthfulness of a proof is determined by a mathematical community). I will now discuss my research perspective on proof with respect to each of these dimensions.

Regarding dimension one, I analyzed proofs of two different forms: verbal arguments that are representative of student reasoning and written arguments. Verbal arguments rooted in student reasoning were presented during interviews. A verbal proof tended to be less formal than, and sometimes preliminary to, a proof-text. Written arguments (or proof-texts) were presented by the students while working on homework prompts. A verbal proof was often constructed concurrently with the written argument. That is, students provided reasoning about the steps in their proof either before or after writing the proof step on paper.

With respect to dimension two, I considered proof from a broad perspective by requiring a proof to be both convincing and deductive (thus satisfying the first two of Weber's, 2014, cluster characteristics). However, I did not require all proofs to be formal. For example, proofs given verbally were often informal (i.e. non-technical words or structures were used). However, I believe even these informal proofs must be convincing and deductive. Sowder and Harel (1998) suggest that the defining characteristic of a proof is its ability to convince the writer and reader that the conclusion is true. I agree with these researchers and will thus require a proof to be convincing.

In terms of dimension three, I adopt a social constructivist view of mathematics when determining what counts as a proof. From this perspective, the mathematical community accepts that mathematics, including the very proofs that mathematicians construct and rely on, are social constructions. This perspective accepts the basically social nature of proof, including what counts as a proof. At the same time, we can expect some uniformity in what the mathematical community accepts as proof, especially in the context of an ITP class. Thus there is general agreement in the mathematical community about whether or not a verbal or written argument constitutes a proof, particularly for simple proofs of the kind encountered in an ITP Class. For example, in an ITP course, the professor represents that mathematical community, and teaches his students what constitutes a proof from that perspective. The students' proofs are then judged against the criteria established in the course. Adopting this perspective satisfies characteristics e (adopting communal norms) and f (accepted by the mathematical community) in the list of Weber's (2014) cluster characteristics.

Using an interview protocol helped to satisfy characteristics c (be transparent) and d (be clear) in Weber's (2014) cluster characteristics. I asked questions during the interviews to push student reasoning until it was transparent to me what the student was thinking. At times the direction of student reasoning was so direct that no further questioning was needed to enable me fill in gaps in student reasoning. Questioning also helped clarify student thinking.

The purpose of this research is to study how mathematical knowledge affects students' construction of proof. As discussed, I view a proof as a social construction, one that is meaningful only to a particular community of practice. However, the purpose of my study is not to study how proofs are socially constructed, nor to describe how proof as a social or discursive entity is situated in a particular community. Instead, I see the nature and meaning of proof as largely taken-as-shared in the larger mathematical community. Thus the purpose of the ITP class in which I gathered data was to help students become proficient at producing a proof according to the rules and perspectives of the mathematical community at large. Having gone through a lengthy apprenticeship to become a proficient proof-writer from this perspective, I am competent to judge the validity of the students' proofs from the perspective of the mathematical community.

Why Students Struggle to Write Proofs

The next several paragraphs discuss specific issues students have writing proof. First, a discussion about student proof schemes is presented (Sowder & Harel, 1998). This is followed by a discussion of student comprehension of proof (Mejia-Ramos, et al., 2012; Stylianides, et al., 2017). Then I discuss issues students experience with respect to the nature of mathematical proof (see, for example, Chamberlain & Vidakovic, 2016; Dawkins, 2017; Duval, 2007; Weber, 2001). Finally, I argue that student competence with mathematical content knowledge is a critical component to proof production. I will discuss how these issues influence a student's ability to write proofs. However, this study will focus mainly on how students' mathematical content knowledge is used in proof production. It is important to note that a student's ability (or inability) to write proofs is likely influenced by all of these potential issues. Thus, struggles that do not pertain to mathematical content are discussed here because they may affect or be affected by mathematical knowledge.

Proof Schemes

According to Sowder and Harel (1998), a *proof scheme* is whatever a student believes makes an argument convincing and persuasive. A variety of proof schemes ranging from providing examples to formally building on mathematical axioms are prevalent in student work (Sowder & Harel, 1998). Some of these different schemes may be indicative of student experience with proof (i.e. a student is likely to start learning to prove by considering several different examples before moving to general statements).

Another attempt to classify students' methods of proving is found in Balacheff's 1991 *levels of proof activity* as described by Knuth & Elliot (1998). There are four levels of proof activity: (a) making assertions based on a limited number of specific cases, (b) making assertions based on experimenting with a non-general case, (c) making assertions based on the behavior of a general example, (d) making assertions based on logic that is distinct from examples. While some of the levels of proof activity described seem less desirable, the authors do not make explicit which level students should attain to be considered competent proof writers. There is some overlap between proof schemes (Sowder & Harel, 1998) and levels of proof activity. One of the biggest differences between these two constructs is that levels of proof imply a progression of proof activity, whereas proof schemes only describe ways that students conceptualize proof. Sowder and Harel do not attempt to describe how a student may progress through a hierarchy of proof activity. There may be ambiguity about what constitutes an ideal proof in certain contexts. However, there is a fairly clear goal of instruction in an undergraduate ITP course. Students at this level should be writing proofs that could be classified as using analytic proof schemes (Sowder & Harel, 1998) and as level 4 proof activity (Balacheff, 1991).

Undergraduate Comprehension of Proof

Undergraduates have difficulty understanding proofs (Mejia-Ramos, et al., 2012; Stylianides, et al., 2017). Understanding mathematical proofs would seem to be a necessary prerequisite to writing valid, original proofs. Thus, if undergraduates cannot comprehend a proof, they are less capable of producing valid arguments themselves. Research has been conducted in an attempt to identify what it means to comprehend a proof. Meija-Ramos, et al. (2012) studied the beliefs of mathematicians and proposed four components to proof comprehension: identifying the main idea of the proof, understanding the discrete parts of the proof, applying the proof in another context, and reasoning about the proof using examples. Problems with proof comprehension and production have been noted in both mathematics and mathematics education students (Chamberlain & Vidakovic, 2016; Duval, 2007; Herbst, 2002). These problems have also been observed in mathematics courses (Epp, 2003; Moore, 1994).

Logical Nature of Proof

Undergraduates may struggle to write proofs because they do not understand the logical component of proof (Dawkins, 2017). The logical component of proof refers to the different pieces of a logical argument such as premise, conclusion, quantifiers, and mathematical statements. If these logical components are properly employed, any reader familiar with the mathematical content will be able to follow the proposed argument. The deductive reasoning used in mathematical proof may be foreign to students and therefore difficult to understand (Duval, 2007). In order for students to use logic and deduction correctly, teachers must specifically address the integration of logic with mathematics (Dawkins & Cook, 2016). For example, students may struggle to make appropriate connections of mathematical ideas to produce a valid deductive argument (Duval, 2007). Consider the student who is writing a geometric proof of the Pythagorean Theorem. She may construct a right triangle and squares off each of the sides of the triangle to represent the quantities a^2, b^2, and c^2. However, if the student never explicitly states that the areas of the squares represent the specified quantities, she has failed to make the necessary connections to produce a deductive argument. Bourreau, et al. (1998) proposed that another issue students experience with respect to the logical component of proof is misunderstanding the relationships between the premise and conclusion of a theorem (as cited in Duval, 2007). Students may misunderstand that in a direct proof the premise is assumed and then deductive connections are made to reach the conclusion. For example, a student proving the Pythagorean Theorem may assert both the premise of the theorem (the triangle is right) and the conclusion ($a^2 + b^2 = c^2$) at the beginning of the theorem. One interpretation of this mistake is that the student does not understand that, although the conclusion is true, it needs to be deduced from the premise and other accepted mathematical facts.

Students can also struggle to write proofs because they are unaware of how to use their mathematical content knowledge advantageously. Weber (2001) calls the application of existing knowledge *strategic knowledge*. Strategic knowledge refers to logical knowledge that governs when and how mathematical content knowledge is used during proof production. For example, when undergraduates begin writing proofs, they often struggle to identify which theorems are useful in a given proof. Weber found that it was insufficient for an undergraduate to simply know the theorems; an additional type of knowledge was needed to allow the student to recall the theorem at appropriate times. The student needs both mathematical knowledge of the theorem and logical knowledge of how to use the theorem effectively. Students may also struggle to use mathematical definitions advantageously. Even when a student can correctly define a mathematical object, she may still struggle to use the definition in an argument (Edwards & Ward, 2004). That is to say, knowing definitions and using them to reason are distinct practices. Misuse of definitions is sometimes caused because definitions are abbreviated using mathematical notation that is foreign to students and difficult to "unpack," making the definition unclear to the student (Moore, 1994). Often, the student unknowingly uses a definition without explicit appeal to a definition or theorem. Definitions are frequently used haphazardly when the student has no other justification for the validity of a statement. This lack of justification is problematic (Chamberlain & Vidakovic, 2016).

While undergraduates often have the content and syntactic knowledge necessary to produce proofs, they still struggle to produce valid proofs because they have deficient understanding of the formal structure of proof (Weber, 2001). There is much more to writing a proof than knowing the theorems and mathematics involved. Undergraduates often lack knowledge of the domain's proof techniques and structures. The different structures of mathematical proof (i.e. proof by induction, proof by contradiction, etc.) require different proof techniques that students must develop (Brown, 2013; Lee, 2015; Stylianides, et al., 2017). Undergraduates inexperienced in the domain have a difficult time choosing an appropriate proof structure (Weber, 2001). Once a proof structure has been selected, students may struggle to execute the proof properly. For example, if a proof by cases structure is selected, the student must attend to each individual case appropriately. He must also argue that the cases presented represent all possible cases. Chamberlain & Vidakovic (2016) found in one case that once an undergraduate learned a proof structure, he began to use the structure as a template, procedurally constructing proofs without considering the argument. Thus, students in courses that predominantly use one type of proof structure (i.e. the epsilon-delta proof structures in real analysis) are more comfortable writing proofs (Bae, et al., 2018). Specifically, the students felt more confident starting proofs in real analysis because the proofs they constructed in class and on their homework all began the same way.

The type of proof structure taught in a course affects the learning that takes place. For example, Herbst (2002) provides an extensive history of the use of two-column proofs in geometry. He concludes that while the two-column proof structure provides stability for geometry, the proof structure allows students to dissociate the mathematics from the proof. The extensive focus on the form of the proof leads to a decreased awareness of the substance of the proof (i.e. the mathematics). One of Herbst's students said "We did proofs in school but we never proved anything" (p. 307). A two-column proof structure may not be the best way to encourage intellectual associations between mathematics and argument. An overemphasis on using a specific proof structure may shift the focus from mathematics to proving procedures.

This is problematic because one of the purposes of mathematical proof is to provide students with a deeper understanding of the mathematics being discussed (Mejia-Ramos, et al., 2012).

Mathematical Content Knowledge

Undergraduates may struggle to write proofs because they lack the mathematical content knowledge necessary to produce valid proofs. This deficiency makes it difficult for students to use properties of a mathematical object that are critically important to the development of the proof. For example, a student writing a proof involving integers should be familiar with what an integer is and the basic properties of integers. Without this knowledge, the student is unlikely to know how to move forward in the proof. Similarly, the lack of robust knowledge of seminal theorems means that students do not have quick access to the key ideas that are needed in the proof (Weber, 2001). For example, the students in my study wrote proofs about integer division. It was critical that students had a robust understanding of the division algorithm in order to successfully complete the proofs. Students who didn't understand the division algorithm theorem struggled to write valid proofs.

One specific area of mathematical content that students struggle with is understanding mathematical definitions, a basic building block of mathematics (Edwards & Ward, 2004). In his study about student's transition to formal proof, Moore (1994) found that students did not understand definitions or how to use them in a proof. Other research has even concluded that students do not conceptualize definitions in the same way mathematicians do (Edwards & Ward, 2004). One study found that 30% of proof chunks (e.g. meaningful units of reasoning) depended on the use of mathematical definitions (Savic, 2011). Some students do not understand precise definitions and thus fail to use definitions correctly in a proof.

A parallel argument about the importance of mathematical content can be made using the van Hiele levels in geometry. The van Hiele levels provide a way to characterize student thinking in geometry based on the way students engage in mathematical activity (Burger & Shaughnessy, 1986). There are five van Hiele levels: visualization, analysis, abstraction, deduction, and rigor. It is believed that each level is discrete. In order to reach a new level of thinking, students must first master the previous levels. The first three levels of the van Hiele model attend to geometric (mathematical) content. For example, a student in the second level informally analyzes the properties of geometric shapes. The fourth and fifth levels of the van Hiele model attend to proof techniques and logic. For example, a student in the fourth level writes formal proofs that rely on axioms and logical systems. This widely-accepted model suggests that before proof production can be mastered, mathematical content knowledge must be thoroughly developed.

Theoretical Framework: Knowledge in Pieces

This study draws on a subset of diSessa's (2018) Knowledge-in-Pieces (KiP) framework. KiP proposes that knowledge is best described as comprising small, individual pieces. A piece a knowledge is *activated* when it is used to reason about a specific situation (Hammer, 2000). In any given situation, there will be some pieces of knowledge activated and some left dormant. The *inferential net* (defined in more detail below) refers to all knowledge that has the potential to be activated (diSessa, et al., 2016). In the context of this study, a student's inferential net consists of all the pieces of knowledge the student has that could be used during production of a specific proof. There are several factors that influence how/when a piece of knowledge is activated. One factor that is particularly relevant to this study is *contextuality* (diSessa, 2018).

Contextuality refers to how the context of a problem influences which pieces of knowledge are activated (diSessa, 2018). For example, a student may have ample knowledge about operating with integers, but when asked to construct a formal argument about the behavior of integers, she may not activate pieces of knowledge about the commutative or associative properties. The context also limits which pieces of knowledge are useful to the given situation. For example, a student may know that multiplying two numbers together yields a product that is bigger than the multiplier and multiplicand, which is true when dealing with positive integers. However, if the context is changed so either the multiplicand or the multiplier is a negative number, then the product will be less than at least one of the numbers being multiplied. The piece of knowledge (multiplication always produces a bigger number) was useful in one context, but not in another.

Pieces of knowledge are grouped together to form *concepts* (diSessa, 2018). A *coordination class* is a type of concept that organizes pieces of knowledge into a complex system that provides the learner with strategies to interpret the world (diSessa, 2004, 2018; diSessa & Sherin, 1998). The main purpose of a coordination class is to gather information and implications from a given situation (diSessa & Sherin, 1998). An example of a non-coordination class concept is a category concept; the purpose of a category concept is to determine if a specific entity fits into the category. For example, ‘integers’ could be considered a category concept because the main purpose of ‘integer’ is to determine if a number does or does not belong to the set of integers. However, ‘operations with integers’ could be considered a coordination class concept because integer operations are used to gather information, such as the product of two integers. This research study will specifically consider coordination class concepts because the way mathematics is used in proof writing correlates with coordination class concepts. For example, when reasoning about even and odd numbers, students needed to be able to identify an even/odd number and recognize how to use the integer’s parity to make mathematical inferences. Thus, the student's knowledge of integer parity allowed him to gather information (identify even/odd numbers) and implications (this number is even because it can be written as 2k for some integer k) from a situation.

From a KiP perspective, *learning* does not always refer only to the acquisition of new pieces of knowledge, but also to the reorganization of existing pieces of knowledge into new or more advanced coordination classes (diSessa, 2004, 2018; diSessa & Wagner, 2005). This reorganization allows the learner to interpret the world around him more accurately. This method of interpretation is called an *extraction* (also referred to as a *readout strategy* in other work, diSessa, et al., 2016). Thus, learning means increasing the accuracy of the interpretation yielded during extraction. Extractions become more accurate as the learner obtains knowledge resources from different contexts. Over time, the learner comes to use the pieces of knowledge in ways more similar to experts. The learner's progress is measured by how similar his extractions are to the extractions of experts. In this sense, KiP is consistent with a constructivist learning theory in that the learner is not expected to learn an objective truth, but rather to align his thinking with experts' understanding. This study will not use the idea of extraction in the analysis. Extraction is a helpful way to measure learning; this study does not aim to measure learning but rather to capture student understanding at a specific point in time.

The *inferential net* (also referred to as the *causal net* in other work) refers to all the pieces of knowledge held in the coordination class (diSessa, et al., 2016). The inferential net also includes knowledge of the rules of inference that connect pieces of knowledge. It allows the learner to draw conclusions during extraction. Other research using KiP attempts to provide extensive maps to describe activity within a student's inferential net. This research will focus on the pieces of knowledge within the inferential rather than the links between pieces. During extraction, the learner chooses to activate pieces of knowledge in his coordination class to produce a *concept projection*. The concept projection refers to all parts of the coordination class that are being used to interpret a given situation. That is, the concept projection is the way the learner applies his knowledge (diSessa & Wagner, 2005).

The development of a coordination class can be “tested” by observing the learner’s concept projection, that is, the knowledge that is brought to bear in a given situation (diSessa & Wagner, 2005). It should be noted that since coordination classes are constantly evolving, the learner may apply the concept expertly in one context and fail to do so in another. This misapplication does not imply the learner does not “have” the coordination class, but rather that it is still in its early developmental stages. Since coordination classes are so complex, it does not make sense to say an individual “has” or “does not have” a coordination class. Rather, it is helpful to think of a coordination class as a system that is constantly evolving and developing. A coordination class is modified as the learner reorganizes the pieces of knowledge in the system or adds new pieces of knowledge to the system. There is no way to determine if a coordination class is complete. In fact, diSessa and Wagner suggest even experts do not need to have all possible concept projections. Expert inferential nets are thoroughly developed to create appropriate concept projections as needed. As a coordination class becomes more expert, the learner’s concept projections more accurately match the real world.

This study will use only a subset of the components of the KiP framework. The ideas of context, activation of knowledge, inferential nets, and coordination classes are central to my work. I consider proof production as a new context in which students are experiencing mathematics. I will examine how students activate their mathematical knowledge within the context of proof production. Other studies that adopt a KiP perspective tend to provide in-depth explanations of how knowledge is activated by examining why knowledge is or is not activated in a given situation. My study differs from that norm; I will seek to describe what knowledge is activated but will not provide extensive descriptions as of why students choose to activate a piece of knowledge. Thus, my study will rely heavily on student's inferential nets, omitting discussion of students' extraction strategies and concept projections.

The inferential nets described in this study reside within coordination classes. As described previously, inferential nets contain pieces of knowledge as well as logical links between the pieces. For the purposes of this study, I categorize the pieces of knowledge to be part of a student's understanding of mathematical content knowledge. I categorize the links between pieces of knowledge to be part of a students' logical knowledge. My focus in this study will be on the pieces of knowledge within the inferential nets. I acknowledge the logical links within a coordination, but will not focus on them in this research. When I discuss mathematical knowledge it always refers to pieces of knowledge. Logical knowledge refers to logical links within mathematical coordination classes or pieces of knowledge within a logical coordination class. It should be noted that this categorization of mathematical content knowledge was purposefully chosen to ensure that anything categorized as "mathematical" was undoubtedly mathematical and not logical. As a consequence of this categorization mathematical content knowledge may be underrepresented.

Example

I will now provide a hypothetical example of how KiP can be used to analyze proof production. Consider the proposition: If x is an odd integer, than x^2 is also odd. Figure 1 gives an example of one way to prove this statement.

Figure 1

Proof that the Square of an Odd Integer is also Odd

Proposition: If x is an odd integer, then x^2 is also odd.
Proof:
(1) Let x be an odd integer.
(2) Then $x = 2k + 1$ where k is an integer.
(3) Then $x^2 = (2k + 1)^2$
(4) Then $(2k + 1)^2 = 4k^2 + 4k + 1$
(5) $4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$
(6) Note that $2k^2 + 2k$ is an integer.
(7) Therefore, x^2 is odd.

We will consider the work of a student who fails to complete line 5 of the suggested proof and is thus incapable of completing the proof. Below I provide brief descriptions of what the student's extraction, inferential net, and concept projection might be in this situation.

- Extraction: The student recognizes that she must use knowledge that resides in the following coordination classes: parity of integers and arithmetic with integers.
- Inferential Net: The student knows the algebraic definition of an odd integer which allows her to assert that $x = 2k + 1$ where k is an integer. Her inferential net likely contains knowledge of squaring, multiplying binomials, factoring out constants, and other basic single-variable algebra. However, suppose she does not activate the piece of knowledge about factoring the constant 2 out of the first two terms of expression (line 5). Without doing so, she cannot appeal to the algebraic definition of odd numbers to claim that x^2 is odd.
- Concept Projection: The student has activated the following pieces of knowledge: algebraic definition of an odd number, substitution of two equivalent expressions, and squaring a binomial.

At this point it makes sense to consider what pieces of knowledge (if activated) would allow the students to complete the proof (i.e. the knowledge of "chunking" a trinomial into a binomial and a monomial, and of factoring a constant out of a binomial). Although the hypothetical student whose work is discussed above undoubtedly knows she can factor a constant out of the expression, she does not activate it during proof production. She is thus unlikely to accurately complete the proof. This is a simple example of how the language of KiP will allow me to consider the pieces of knowledge that students are using as they write proofs.

The following two research questions are addressed in this study.

1. How does mathematical content knowledge contribute to the success of a proof?
2. How is mathematical content knowledge used during proof production?

CHAPTER 3: METHODS

Context of the Study

As discussed in Chapter 1, I gathered data from an introduction to proof (ITP) course. The ITP class (Math 290: Fundamentals of Mathematics) wherein I collected data was taught at Brigham Young University during the Fall 2019 semester. The class met three times a week for 50 minute class sessions. There were 30 students enrolled in the class. The students in the class I observed were primarily mathematics or mathematics education majors. The students had completed (or were currently enrolled in) Calculus 1. The textbook used in this specific course was *A Transition to Advanced Mathematics* (Doud & Nielsen, 2018), authored by two professors in the Math Department at BYU. The professor of the class I observed has taught the ITP course multiple times and is one of the authors of the textbook that was used. The book was specifically written for this course at BYU.

The data for this study were gathered from students using three different data collection methods: a pre-test assessing prior knowledge, student interviews, and classroom observations. The data were gathered during a unit that focused on elementary number theory, specifically with respect to integers. This proof topic was selected because most of the students had substantial knowledge about integers when they entered the course, and the proofs were likely to be fairly straightforward. The students were very comfortable working with integers in non-proof contexts. Thus, it provided a good opportunity to see how knowledge about integers is activated in the context of proof production. I did not expect a large difference between students' knowledge of integers at the beginning of the class. My analysis focused on how knowledge about integers was activated when students begin to write proofs about this familiar content. The data were gathered during the students' experience with Chapter 3 and Chapter 5 of the textbook.

Chapter 3 (seven sections) uses integers as a context to introduce basic proof techniques; Chapter 5 (three sections) discusses proofs with integers exclusively. For the most part, one section was covered during each class period. Homework problems accompanied each section.

Content Selection

I first performed a content analysis of chapters three and five of the course textbook to inform subject selection. The purpose of the analysis was twofold. First, the content analysis was necessary to give me (and the reader) context about the mathematical content being used in proof production. I carefully studied the chapters in the textbook (including the accompanying homework problems) and made a list of the mathematical ideas that are used. Second, the content analysis helped me create a list of the pieces of mathematical knowledge students could potentially use when writing the proofs from the chapters. The data showed that mathematical knowledge from both the course itself and from previous mathematical courses was used. Based on a preliminary analysis of the textbook, I created a rough description of the mathematical content that would likely be used by students during the proofs. Figure 2 lists the pieces of mathematical content knowledge I anticipated students to use during the course of the interviews. These descriptions were adjusted after data collection. Updated descriptions are found in Chapter 4.

Figure 2

Pieces of Mathematical Content Knowledge Needed for the Proofs

Pieces of Mathematical Content Knowledge Necessary for Successful Completion of the Proofs
• Definition of an integer • Parity of the integers (even numbers vs. odd numbers) • Operations with integers • Definition of a factor • Absolute value • Divisibility • Long division • Greatest common factor/divisor • Linear combinations (at least an informal prior knowledge of this concept) • Prime numbers • Composite numbers • Modular arithmetic • Factorization & prime factorization

Note. This figure is a list of the pieces of knowledge I hypothesized students would use to successfully complete the proofs. This list was compiled prior to data collection. Adjusted descriptions of necessary mathematical content knowledge are provided in Chapter 4.

Four students were selected as research subjects based on their willingness to participate in the study. As mentioned above, I expected most students would have similar amounts of knowledge about integers. Throughout the interviews, the student subjects displayed varying levels of proof writing skills. Thus the data allow me to make conclusions about a wide range of students. I call the four research subjects TJ, Gracie, Matthew, and Ally.

Data Collection

I administered a pretest to the students at the beginning of data collection. The pretest can be found in Appendix A. I designed questions that assessed student understanding of the topics I identified as critical after completing the content analysis. The pretest was administered before instruction. Topics that the textbook introduced as new mathematical concepts were not assessed in the pretest. The purpose of the pretest was to determine the current state of their knowledge about integers before any new mathematical instruction was delivered in the ITP course. I was also interested in knowing if there were some pieces of knowledge that students activated in the non-proof contexts, but not in proof contexts. The pretests did not provide any surprising data because the mathematical content knowledge assessed in the pretest was neither new nor complex. That is to say, the students did not struggle with the mathematical content knowledge assessed in the pretest. The students struggled with the new mathematical material taught in the ITP course. More detail about student struggles with specific mathematical content is discussed in Chapter 4.

In this study I will consider the activation of knowledge in two different contexts: proof contexts and non-proof contexts. Proof contexts refer to formal proof presented by students. The product of work in a proof context is a proof, as described previously. Non-proof contexts refer to other mathematical activity. For example, non-proof contexts include computational mathematics, informal justifications, and searching for a correct answer.

I observed the class periodically during the units about integers. My primary purpose during observations was to provide context for future interviews. By observing instruction, I learned about the norms established by the class and gained insight into how theorems, definitions, and notation were used by the professor during instruction. I focused on observing rather than participating in the class. I participated only as needed. During the observations I took detailed field notes describing the nature of the mathematics discussed in class. I also noted how the research subjects interacted with the professor, other students, and course material.

Classroom observations allowed me to observe what view of proof was assumed by the professor and students. This view is described in Chapter 4. Since these observations were only to provide context, they were not recorded or coded.

Interviews were conducted with the research subjects after three of the class periods about integers. The interview questions can be in the Appendix B. I chose to conduct three rounds of interviews with the same students because it allowed students to attempt writing proofs about several different topics within elementary number theory. This variety required students to use different CCs. Part of my analysis describes how mathematical CCs were activated and used by the students. The purpose of these interviews was to determine how mathematical content knowledge affected the success of student proofs and how the students used mathematical content knowledge about integers while writing proofs. The interviews were task-based. I asked the students to bring their homework to the interview and to wait to work on the problems until the interview. Students were instructed to do specific homework problems and verbalize their reasoning throughout the writing process. I selected homework problems that covered a broad range of topics. I selected both problems that I considered easy and problems that I considered challenging. This allowed me to see how students used mathematical content knowledge in more and less challenging contexts. I asked clarifying questions about their reasoning as appropriate. At the end of the final interview, I asked students to reflect on how they activated mathematical knowledge in different contexts. Throughout the interviews, I paid particular attention to how the students activated their mathematical knowledge in the context of proof production. The interviews were audio recorded, transcribed, and coded.

The data included both written proofs and verbal proofs. Copies of written student homework were collected and analyzed. The transcriptions of the interview are representative of verbal reasoning proofs. The written work helped me better understand the student's reasoning when the verbal reasoning was unclear.

Threats to Validity

One major threat to validity in this study was my experience in the ITP offered at the same university where the study was conducted. I completed the course in Fall 2014, five years prior to this research. My experience in the class was not positive. I did not feel that instruction prompted the activation of my prior mathematical knowledge. Although I received a good grade in the class, I didn't feel I had a strong understanding of the logical or mathematical content discussed in the class. Because of this experience, I have a tendency to expect that other students have similar feelings about the way mathematical content is used in the course. In order to minimize this researcher bias, I gathered data from a class taught by a different professor than the professor who taught my ITP class. I collected data with the expectation that the structure and content of the course had likely changed since I took the class (for example, a new textbook was being used). I conducted member checks throughout my study to ensure that I was not interpreting interview responses to fit into any preconceived thoughts I have about the course.

My presence as a researcher was also a validity threat. The teacher may have altered his teaching style because he was being observed for a research study. Another threat to validity regarding the professor is that he knew, at least to some extent, what my study is about. Thus, he may have made extra effort to activate students' mathematical content knowledge. To minimize these threats, I provided the professor with adequate, but minimal information about the specifics of the study so he remained as impartial as possible. Since I am not much older than most of the students enrolled in the class, I did not expect that my presence would significantly change their behavior.

The way I conducted interviews was also a potential validity threat. If I had disclosed my personal opinions about the class to research subjects, then they may have been more (or less) likely to answer questions to satisfy my viewpoints. The student research subjects were freshman undergraduates. Since I am a graduate student, they may not have felt comfortable with me and thus provided short answers. The rich data I was looking for required detailed responses from my interviewees. To combat these interview-induced validity threats I strived to establish good rapport with my interviewees. I helped them feel comfortable during the interview by being friendly and explaining the purpose of the study. By the end of the interviews, I felt that I had established a professional, comfortable relationship with the interviewees. I used a non-intrusive audio recording device to encourage an open conversation with my interviewees rather than a formal meeting. Finally, I asked the interviewees to be honest in their responses because the best kind of data is honest data.

Nature of the Proofs Analyzed

The statements students were asked to prove during the interviews were not meant to be long or complex. The logical structure was clear and the mathematical content needed was meant to be accessible to the students. In each case, there was a fairly clear way to successfully complete the proof. Based on the course requirements and content, all students were considered capable of writing a valid proof for each statement. Thus, in instances when a proof was invalid, it was not difficult to determine what was lacking to produce a valid proof. I was open to the possibility that students may assume an atypical approach to the proof. In instances where the student took an atypical approach, I attempted to understand the student reasoning from this new perspective.

The proofs analyzed in this study were all situated within the context of elementary number theory. Students needed to use only two sets of numbers to successfully complete the proofs: integers and natural numbers. Algebraic language and notation were used frequently to express mathematical definitions. Parity, divisors, and prime number were important components of many of the proofs in the study. Three different kinds of equivalence relationships were explored during classroom instruction: equations, divisibility statements, and congruence statements.

Data Analysis

The data from the interviews were coded using two different units of analysis. The first unit of analysis was the individual proof as a whole. During each interview, the students completed 2-4 proofs. Each research subject wrote 8-9 proofs over the course of the interviews. A total of 34 proofs were analyzed. These proofs were analyzed and coded individually (see below). The second unit of analysis was a sentence. While transcribing the interviews, I separated the dialogue into sentences. A key purpose of the separation into sentences was to accurately represent the process the student went through in constructing a proof. Thus, I attempted to separate the dialogue into sentences that reflected what I perceived to be the intended purpose of the interview participants. Individual sentences were then coded (see below).

Analysis at the Proof Level

Each proof was coded as *valid* or *invalid*. A proof was considered valid if it included all mathematical and logical components necessary to adequately prove the statement as well as appropriate connections between the mathematical and logical components. Validity was determined by the researcher, based on the proof definition used by the research subjects and the professor teaching the ITP course. This definition is explained in the next section. Any proof of the statement that was not logically and mathematically sound from the perspective of this definition was coded as invalid.

Each invalid proof was further analyzed to determine whether the proof was complete or incomplete. Completeness was determined from the perspective of the student: Did they consider what they had produced to be a complete proof, with all the necessary parts to make it valid? If an invalid proof was considered incomplete, it was coded as incomplete and went through a second round of coding (see the next paragraph). If an invalid proof was considered complete, I determined whether the proof's lack of validity was due to mathematical or logical issues. In every case but one, I identified the first problematic move in the proof. A move refers to a statement that indicated the next step in the proof or a statement ending a proof. A problematic move refers to a move that was incorrect, either mathematically or logically. In one case I considered the second problematic move rather than the first. In this case the student's second problematic move gave insight into how he was using mathematical content knowledge, and so provides richer answers to my research questions. Problematic moves found in the data were further classified as mathematical (e.g. neglected parts of the domain of the set), or logical (e.g. ending the proof prematurely, or starting down an unproductive path). In the case of a mathematical issue, the first problematic move referred to the student making an incorrect mathematical statement or failing to make a correct mathematical statement. For example, Ally was asked to prove that the GCD of two expressions was 2. She attempted to use the Euclidean algorithm to prove the statement, but her incomplete knowledge of the algorithm kept her from providing adequate justification as to why the GCD was 2. She knew that the Euclidean algorithm involved looking at remainders, but she neglected to prove that 2 was the last non-zero remainder. In this example, Ally's proof was coded as invalid for mathematical reasons. By contrast, in the case of a logical issue, the problematic move often referred to statements that didn't logically follow based on the previous statements or statements that indicated an incorrect interpretation of a logical statement. For example, TJ was asked to prove that no integer is both even and odd. He began his proof by introducing two distinct, arbitrary integers, one even and one odd. Because he never asserted that the two integers were equal, I identified this as the first problematic step in his proof. A correct approach to this statement using a proof by contradiction would be to introduce one arbitrary integer and claim that it is both even and odd. The rest of TJ's proof built on his incorrect interpretation of the hypothesis. In this example, TJ's proof was coded as invalid for logical reasons.

If an invalid proof was considered incomplete from the perspective of the student, it was coded as incomplete. I further analyzed each incomplete invalid proof to determine if the incompleteness was due to mathematical or logical issues. To do this, I identified what the next move would be if the student were to continue. The next step was determined by considering the steps the student had already taken in the proof. I observed the approach the student was taking to the proof and inferred what the next move would be if the student had continued along the same path he/she started. I then determined whether the next move in the proof was mathematical or logical. The student's inability to produce the next move in the proof was classified as either "mathematical issue" or "logical issue" depending on the classification of the next step. Any proof coded as invalid because of incompleteness was given a second code of "mathematical issues" or "logical issues." This "double" coding was implemented in order to analyze the different ways insufficient mathematical knowledge affected a proof.

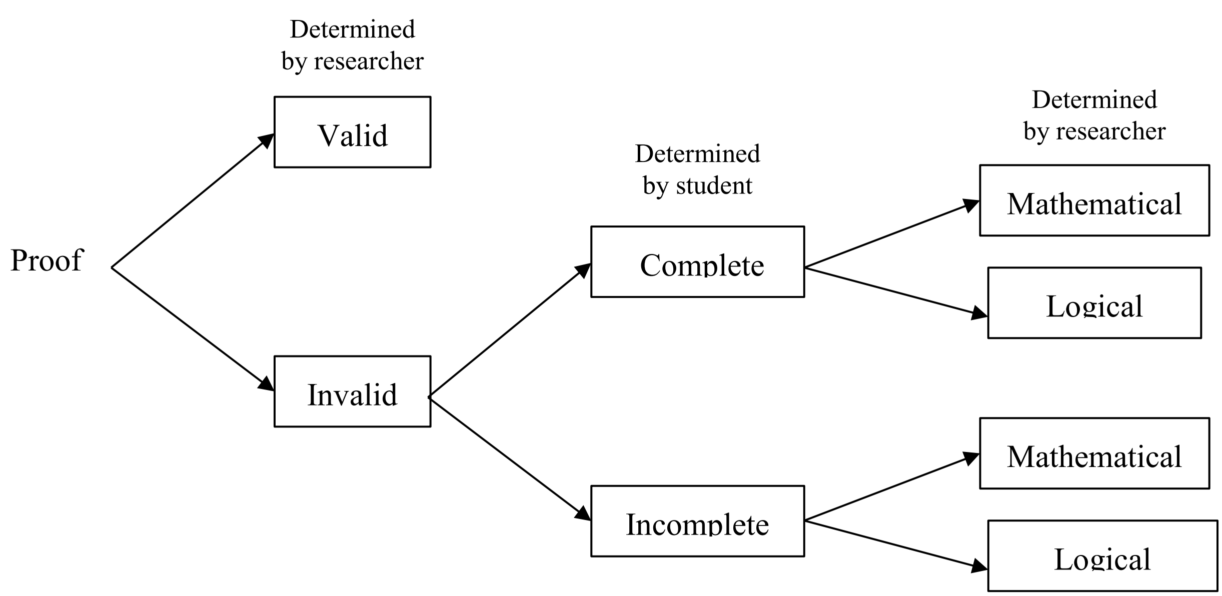
The flowchart in Figure 3 shows a summary of the coding process each proof underwent. It should be noted that a substantial number of proofs in the study received only one proof code:

valid. Student activity in these proofs was captured by the sentence level coding. Further coding of the invalid proofs provided additional insight into how mathematical content knowledge affected the success of a proof. Each proof gathered during the course of data collection was given exactly one of the following sets of codes:

- Valid
- Invalid, complete, mathematical issues
- Invalid, complete, logical issues
- Invalid, incomplete, mathematical issues
- Invalid, incomplete, logical issues

Figure 3

The Process for Coding Each of the 34 Proofs



Mathematical and logical issues are related to the development of a coordination class (CC). A mathematical issue indicates that the CC is either missing important pieces of knowledge or the pieces of knowledge held in the CC are incorrect/incomplete. A logical issue indicates that pieces of knowledge are missing from a logical CC or that the logical structure within a mathematical CC is incomplete. I classify the issue of failing to make connections between mathematical ideas as an issue related to the logical nature of proof. The purpose of this study is to focus on the use of mathematical content knowledge. Consequently, I provide detailed descriptions of the mathematical CCs that were used in the study. I consider all logical pieces of knowledge to be part of a broad CC of logical proof structures. For example, in many instances, students were required to construct proofs by induction. This required them to select and use an appropriate inductive hypothesis. The activity of selecting and using an inductive hypothesis is contained with the student's logical CC. I consider logic that is needed to connect the pieces of mathematical knowledge to be contained within the mathematical CCs. For example, a student may multiply two binomials together. The student may then factor a 2 out of the resulting polynomial to claim that the polynomial is even. The mathematical content knowledge used in this example is arithmetic. The student made logical decisions about when to multiply the two binomials together as well as how to rewrite the product polynomial in a way that demonstrates evenness. The logical decisions and connections found between mathematical moves are part of the mathematical CC.

In most instances it was easy to determine whether a mathematical or logical code applied. In the few cases where it was difficult to code the reason for the failure of the proof I defaulted to coding the proof as a logical issue. One purpose of this study is to argue that mathematical content knowledge is a critical component of successful proofs. By defaulting to a "logical issue" code in the case of ambiguity, I am confident that the invalid proofs coded as mathematical issues are decidedly caused by mathematical issues.

Analysis at the Sentence Level

The sentences of each transcript were coded to determine how the student used mathematical content knowledge. Only dialogue spoken by the research subject was coded. Sentences were only coded if mathematical knowledge was being used in the sentence. For example, restating the hypothesis ("givens") as the first line of the proof was not coded. The codes were formulated after data collection. These codes are described in detail in the results section.

Construction of Typical Proofs

I constructed a proof for each of the prompts given to the students. I attempted to construct proofs similar to the ones I expected students to construct. These typical, researcher constructed proofs were then coded using the sentence codes that were derived from the interviews. I then analyzed relationships between what mathematical content knowledge was used and how it was used by students and the researcher.

CHAPTER 4: RESULTS

In this chapter I provide a detailed account of how the data were analyzed and provide answers to my research questions. First I will discuss the normative meaning of proof used in the study in order to provide readers with an understanding of what students in this study understood to be a proof. Second, I will answer the first research question by discussing how mathematical content knowledge affects the success of a proof by discussing the mathematical and logical issues students experienced when writing proofs. Next, I will describe what mathematical content knowledge was used by students during the study. I will then answer the second research question by describing how the mathematical content knowledge was used during proof production. I will explain in detail the sentence codes that were developed to describe student use of mathematical content knowledge. I conclude this chapter by providing two examples of student proofs that illustrate how mathematical content knowledge was used by students as well as how that knowledge affected the success of the proof.

Proof Norms for the ITP Course

I observed the classroom five times and made note of specific proof norms that were discussed explicitly or implicitly. One such norm was that a proof should be written in full and complete sentences. The professor used complete sentences while writing proofs on the board during class. Common mathematical notation was used to shorten the length of the sentences. The first sentence of the proof was typically used to name variables (for example, “let x be an integer”). The second sentence usually stated what type of proof structure would be used (for example, “I will work contrapositively”). After these two introductory sentences, the students proceeded to the body of the proof. The students were learning what a proof is through social apprenticeship. The professor gradually taught them what a proof was and what constituted a valid proof in his class.

At the time of the research study, the students had a fairly accurate understanding of what counts as a proof in the mathematical community. This is evidenced by the fact that, with only one exception, if one of my student subjects felt he/she had produced an incomplete proof, I also considered the proof invalid. I determined a student's perception of completeness based on the way they talked about the proof once they had finished working on it. At the end of each interview I asked the student how they felt about the proofs they had written; their reflection helped me determine whether they felt the proof was complete. The one exception occurred in a proof where the research subject had produced a valid proof, but she felt the proof was incomplete because she had made big "jumps" in her reasoning. In every other instance of an incomplete proof, it was clear to me that additional steps were needed to produce a valid proof. The converse is not true. There were proofs I considered invalid that the students considered complete. All this suggests that although the students were being apprenticed into the field of mathematical proof, they were still able to recognize when a proof was problematic in some cases.

How Mathematical Content Knowledge Contributes to the Success of a Proof

In this section I will address how mathematical content knowledge affects the success of a proof. I will analyze the results of my proof-level coding, which sheds light on the impact mathematical content knowledge has on proof production. Figure 4 is a summary of the distribution of proof-level codes. Subsequent discussions will analyze the implications of the proof-level codes.

Figure 4

Distribution of Proof-Level Codes

		Proof Count	Percent of Total Proofs
Valid		17	50%
Invalid			
	Invalid, complete, mathematical issues	2	6%
	Invalid, complete, logical issues	5	15%
	Invalid, incomplete, mathematical issues	3	9%
	Invalid, incomplete, logical issues	7	21%
		34	**100%**

Note. The total number of proof analyzed was n=34.

Figure 4 shows how many of the 34 proofs gathered and analyzed during this study were valid. It is interesting to note that exactly half the proofs were valid while the other half were invalid. All students were given the same interview prompts. Every student produced at least one valid proof.

A significant subset of the total proofs were invalid because the proofs were incomplete (30%; see Figure 4). There are many reasons why a student may not have completed a proof including time restraints during the interview, feelings of discomfort during the interview, or insufficient mathematical or logical knowledge. Analysis of the transcripts of the interview suggest that time restraints did not have a significant effect on incomplete proofs; at no point did a student have to stop writing a proof because the allotted interview time had elapsed. I believe the students felt generally comfortable during the interviews. One research subject, Ally, demonstrated feelings of discomfort about verbalizing her reasoning. Thus, it may be possible to attribute some of her incomplete proofs to feelings of discomfort. However, based on the fact that in most instances the interview subjects demonstrated feelings of comfort (i.e. joking with me and discussing non-interview related material), I believe that emotions were not a critical factor in incomplete proofs. I believe every incomplete proof was caused by insufficient logical or mathematical knowledge.

More proofs were incomplete due to logical issues than due to mathematical issues, as shown in Figure 4. This is unsurprising because the research subjects are enrolled in an ITP course wherein one of the main learning outcomes is to become familiar with new logical structures. Moreover, the mathematical content of elementary number theory was likely chosen by the professor to minimize mathematical problems the student would have writing proofs. Thus, it is not shocking that when a student is unable to move forward in a proof it is often because he is unable to make an appropriate logical move. These failed logical moves refer to instances when the logical structure within a mathematical CC is deficient or the student's logical CC did not contain necessary pieces of knowledge. Instances where logical issues refer to deficient logical connections within a mathematical CC suggest that the inferential nets within the students' CCs are still developing as the students struggle to organize pieces of knowledge into logical structures. An example of deficient logical connections occurred in an interview with Gracie. Gracie was trying to prove that no integer is both even and odd. During the course of her interview she talked about the algebraic definition of even and odd numbers, divisibility statements, and the division algorithm. It was evident that her understanding of the division algorithm was not robust because she stated that she wasn't exactly clear what the division algorithm concludes. This lack of understanding of a key theorem certainly contributed to her inability to complete the proof. However, she could have completed the proof without the algorithm. Although she mentioned key pieces of mathematical content knowledge needed for the proof, she was unable to connect these pieces of knowledge into a logical argument. Her proof was incomplete because her mathematical CCs lacked logical connections. In instances such as the one just discussed, the student's mathematical CC contained all the necessary pieces of mathematical knowledge but lacked a robust understanding of the rules of logic that connect the pieces.

The three proofs that were incomplete due to mathematical reasons were proofs about the Euclidean Algorithm and modular arithmetic. It is important to note that these mathematical ideas were first introduced formally to the students in the ITP course where these data were gathered. One potential reason for the students' mathematical issues is that the students' inferential nets are not robust enough to use the knowledge in the context of proof production. That is, the students have not had enough exposure to the ideas, which renders the student incapable of valid extraction and application of the knowledge. For example, TJ was trying to prove that, with one exception, every prime number is congruent to 1 or -1 modulo 3. He made a compelling argument that only one prime number is congruent to 0(mod3). However, he couldn't produce a satisfactory reason why all other prime numbers must then be congruent to 1 or -1 modulo 3. He did not verbalize that every integer belongs in one of the three congruence classes. He likely knew that all other prime numbers must be congruent to 1 or -1 modulo 3, but he wasn't able to leverage that knowledge appropriately in a proof context. One way to develop a more robust inferential net is to experience the concept in several different contexts; the students have only experienced the Euclidean Algorithm and modular arithmetic in the context of proof production. Perhaps it is necessary that the students experience the concept in other non-proof concepts before attempting to write a proof requiring the activation of pieces of knowledge related to the concept.

Figure 4 shows what percentage of the proofs were invalid due to mathematical issues and logical issues. It is clear that issues with the logical nature of proof play a critical role in the production of a valid proof. However, the data suggest that the use of mathematical content knowledge during proof production is nontrivial. It is significant that 15% of the total proofs analyzed were invalid because of mathematical issues. Three of the four research subjects produced at least one proof that failed for a mathematical issue. This suggests that the mathematical issues cannot be attributed to only one student's lack of mathematical knowledge, but rather a more general struggle to apply mathematical content knowledge in the context of proof production. I returned to the five proofs that failed for mathematical reasons to determine what mathematical knowledge was needed to complete the proof. The five proofs failed for insufficient understanding of the following pieces of mathematical knowledge: the Euclidean Algorithm, the definition of a divisibility statement (i.e. $a|b$), and modular arithmetic. The reason for the misapplication of these specific pieces of knowledge can be attributed to underdeveloped inferential nets, as explained previously and demonstrated with the example of TJ's failure to prove that with one exception every prime number is congruent to 1 or -1 modulo 3. Students have not had extended exposure to these ideas, which may disable the students from using the ideas in proof contexts.

Mathematical Content Knowledge Used During Proof Production

In this section I will describe what mathematical content knowledge I hypothesized to be used by students during the interviews. The purpose of this section is to explain *what* mathematical content knowledge is used; the next section will explicitly answer the research question by explaining *how* this mathematical content is used during proof production. In this section I will first provide a brief description of what coordination classes (CCs) of knowledge were likely used. Next I will compare student activation of CCs and pieces of knowledge within the CCs to the activation that would be necessary to complete a valid proof.

Remember that CCs are extensive systems of knowledge. Here I will focus only on the pieces of the CC that were relevant to the proofs in this study. Figure 5 lists the relevant CCs and their associated components. I chose to combine arithmetic and algebra into one CC because students could often accomplish the same purpose by activating knowledge of either arithmetic or algebra. The arithmetic and algebra content knowledge activated in this study were similar enough that I chose to categorize them as part of the same CC. In the next section I separate arithmetic and algebra when I describe how students use mathematical content knowledge during proof production. I chose to separate them later because students used algebra and arithmetic differently while writing proofs. While the mathematical content knowledge involved in arithmetic and algebra is similar enough to combine them into one CC, the ways students leverage arithmetic and algebra were different enough that each merited its own unique code during coding for student use of mathematical content knowledge.

Figure 5

List of the CCs Hypothesized to be Used in the Study

Coordination Class (CC)	Relevant Pieces of Knowledge
Integer Division	● Division algorithm ● Divisibility statements ($a\|b$) and translating divisibility statements into equations ● Experiential knowledge (e.g. the remainder will always be less than the divisor) ● Divisors and greatest common divisors ● GCD Switching Theorem
Parity of Integers	● Algebraic definition of even and odd numbers ● Experiential knowledge (every number is either even or odd)
Arithmetic & Algebra	● Addition, subtraction, multiplication, division ● Distributive, commutative, and associative properties ● Performing arithmetic with variables ● Perform substitutions of equivalent statements
Modular Arithmetic	● Switching representations from a congruence statement to an equation ● Every integer can be placed in a congruence class.
Prime Numbers	● Definition of a prime number

Note. Each CC is accompanied by a list of the pieces of knowledge from the CC that were important in the construction of the proofs.

I identified which CCs of knowledge were needed to successfully complete each proof by determining what CCs I used to construct each typical proof. With one exception, every proof required the activation of pieces of knowledge from two CCs. One proof required three CCs. I then compared the CCs used in the typical proof to CCs the students appeared to use when attempting to prove the same statement. It was not difficult to determine what CCs students used during proof production. Whenever a piece of knowledge was activated, I could easily determine what CC held that piece of knowledge using the CC descriptions in Figure 5. For example, I could easily identify that a student was working in the integer division CC whenever they referred to a greatest common divisor, a divisibility statement, or the division algorithm. The large size of each CC allowed students to use the same CC while approaching the proof in unique ways. For example, in order to successfully complete one proof, a student could appeal to a divisibility statement or the division algorithm; both of these pieces of knowledge reside in the same CC, so the two proofs were recognized as similar because they relied on the same CC: integer division. It should be noted that although it was easy to recognize the content knowledge used in these proofs, the CCs described here are hypothetical. I did not perform careful interviews with students about their mathematical content knowledge in order to provide an accurate map of their content knowledge.

Sometimes students activated knowledge from CCs that were not necessary to the successful completion of the proof. This was the case in eight of the 34 proofs. The activation of non-necessary CCs did not always have a negative impact on the proof's validity. The activation of a non-necessary CC may simply refer to an instance when the student provided an additional justification for the validity of a step by appealing to another class of knowledge. The activation of a non-necessary CC may also indicate that the student was unable to strategically use the knowledge from the CC used in the typical proof and was looking for help from other CCs. This ability to move flexibly from one CC to another is a valuable skill in proof production.

Occasionally students failed to activate a CC that was critical for the successful completion of the proof. The failure to activate a necessary CC was rare; it only occurred in three of the 34 analyzed proofs. All three of these proofs were written in response to the same interview question. This suggests that the students generally had sufficient understanding of each of the CCs listed above to activate the CC at an appropriate time. Thus, the extraction components of the students' CCs were sufficient to enable students to identify useful CCs.

However, even when an appropriate CC was activated, there were times when the CC itself was not robust enough to allow successful completion of the proof. This issue indicates that the inferential nets within the students' CCs were insufficient. For example, the CC may lack important pieces of knowledge or the logical connections between the pieces. For example, consider the example of TJ presented in the previous section. His CC of modular arithmetic was not developed enough to assert that every integer resides in one of the following congruence classes: 0 (mod3), 1 (mod3), or -1 (mod3).

In summary, most students activated all necessary mathematical CCs that would allow them to successfully complete a specific proof. However, it was often problematic that students did not activate the appropriate pieces of mathematical content knowledge within a CC. When appropriate mathematical content knowledge was not activated, it usually meant students needed to develop the inferential net of an existing mathematical CC.

How do Students use Mathematical Content Knowledge during Proof Production?

In this section I describe each of the sentence-level codes I developed to describe how students use mathematical content knowledge during proof production. The internal codes were derived from the data. Thus, the codes are specific to this study. The next several paragraphs describe the ten sentence level codes. Figure 6 provides a summary of these ten codes. After a description of the nature of the codes, I discuss the frequency with which the codes were used in student proofs and in researcher-constructed proofs.

Algebraic Substitution

Sentences coded as algebraic substitution indicate instances when students use a mathematical definition to rewrite a term (or group of terms) using a new variable. In order to be considered algebraic substitution, the variable being used must change. For example, a student may claim that n is an even number and substitute $n=2k$. This is an algebraic substitution because the definition of even number is used to change the variable representing the even number. The algebraic substitution code also refers to instances when the student uses a mathematical definition to put boundaries on variables. For example, a student may state that $0<n<3$. This is also coded as algebraic substitution because a mathematical definition is being used to restrict a variable's values. Note that if a student uses a theorem to put boundaries on a variable, the sentence is *not* coded as algebraic substitution, but rather as "theorem/algorithm" (see below). The algebraic substitution code includes assigning a specific value to a variable. Algebraic substitutions require students to depend on the pieces of knowledge in an inferential net to identify mathematically correct substitutions and the logical connections in an inferential net to make beneficial substitutions.

Mathematical Inference

The mathematical inference code indicates student use of a definition to make a classification statement. For example, a student may infer that an expression is odd if it is of the form $2k+1$ for some integer k. A student may also infer that the greatest common divisor of 1 and 2 is 1. Student conclusions about specific terms or expressions were often coded as mathematical inferences. Mathematical inferences rely on identifying pertinent pieces of mathematical knowledge in the inferential net.

Representation Switch

A sentence was coded as a representation switch when the student translated information from one equivalence relationship to another. That is to say, students expressed the same idea using a new representation. For example, a student may rewrite $a|b$ as $b = a * n$ for some integer n, thus switching the representation from a divisibility statement to an equation. Similar to algebraic substitutions, representation switches rely on both the mathematical and logical components of an inferential net.

Reword

The reword code identifies instances where students use definitions to reword a statement. A common use of the reword code is to restate a written statement symbolically. For example, a student may translate the statement "a divides b" into symbols ($a|b$) and/or verbally ("that means b is a multiple of a"). Rewording is distinct from a representation switch because it relies on the translation of symbols to words, words to symbols, or words to words; a representation switch translates one set of symbols into a different set of symbols. Rewording relies on robust understanding of the pieces of mathematical content knowledge in an inferential net.

Theorem/Algorithm

When students state or use a mathematical theorem or algorithm, the action is coded as "theorem/algorithm." Using a theorem/algorithm may look very similar to algebraic substitutions, mathematical inferences, representational switches, or rewording; the distinction between other codes and the theorem/algorithm code is that other codes rely on mathematical *definitions* while items coded with the theorem/algorithm code rely on mathematical *theorems* or *algorithms*. The theorem/algorithm code includes using a theorem or algorithm to restrict the values of a variable. Using a theorem or algorithm requires the student perform an appropriate extraction of the situation to identify that the theorem or algorithm is applicable. The student must then activate knowledge of the theorem or algorithm.

Arithmetic Operation

This code refers to instances when arithmetic is used to manipulate an expression or equation. The goal of the arithmetic operation is usually to make an expression resemble a definition. For example, the student may use arithmetic operations to show that $(2k + 1)^2 = 2(2k^2 + 2k) + 1$ and then argue that the expression is odd. Choosing to perform an arithmetic operation is governed by the logical connections in an inferential net; carrying out the appropriate arithmetic relies on proper activation of pieces of mathematical content knowledge.

Example

Students often construct examples to convince themselves of the truthfulness of the statement being proven. For example, when proving that if s^3 is odd, then s is odd, one student first considered the example of $s^3 = 27$ and $s = 3$. This example convinced him that the statement was true and he continued to search for a rigorous way to prove the statement in general. Examples are also constructed to convince the student that a particular step in a proof is valid. Throughout the study, students sometimes put too much stock in the examples they constructed by considering the example to be a rigorous proof. For example, when attempting to prove that no integer is even and odd, a student provided one example as his proof of this statement. The ability to produce an appropriate example suggests that the student's inferential net is sufficiently developed to spontaneously produce examples of a mathematical phenomenon. These examples are likely enhanced by previous experience with the CC and the pieces of knowledge contained therein.

Partial Use

Occasionally when students use a theorem, algorithm, or definition they ignore an important part of the statement that would be helpful in the given proof. In some instances of partial use, the student would later realize the part of the statement they had previously ignored and use it effectively. Sometimes a partial use code caused the student's proof to be invalid. In other cases students were able to find other ways to prove the statement. A common example of partial use in the data was the incomplete use of the division algorithm. The division algorithm states that any integer n can be written can be written as $n = qd + r$, where q, d, and r are unique and $0 \leq r < d$. Several students neglected to use the division algorithm to restrict the value of r. Some students provided alternative reasons why $r < d$. These justifications were correct, although less efficient than appealing to the division algorithm. Partial use codes generally indicate the pieces of mathematical knowledge within an inferential net are not robust. However, the logical connections are intact because the student recognized that the theorem, algorithm, or definition was beneficial to furthering the logical argument.

Structure

Students use mathematical knowledge to select or set up a proof structure. For example, in a proof about prime numbers, a student attempted a proof by cases. There were two cases: one that attended to prime numbers and one that attended to composite numbers. He used his mathematical knowledge that every integer is either prime or composite to construct the two cases. It is important to note that students often used logical knowledge to make decisions about the proof structure. For example, the phrase "no integer divides" seemed to influence the students to do a proof by contradiction. I could not discern any specific mathematical content knowledge that was being used to make this structural decision. The structure code was only applied when mathematical knowledge was used in the decision making. Using mathematics to select a proof structure relies on the logical connections between pieces of mathematical content knowledge.

Extra Mathematics

Students may introduce new concepts or variables in a proof that could easily be completed without the new concept or variable. This is considered extra mathematics. For example, when asked to prove that no integer is even and odd, one student constructed the set of all integers. He then attempted to partition the set into the set of odd integers and the set of even integers. Even though the introduction of set theory is not wrong, set theory is not necessary to the valid completion of this proof. This action was coded as extra mathematics. The extra mathematics code accounted for any instance when new mathematical ideas were used in place of an idea that was already part of the proof. For example, one student ignored part of the division algorithm during her proof. Because of this, she had to provide additional justification. This justification was coded as extra mathematics because the division algorithm, when used completely, provided sufficient justification for the proof. The extra mathematics code was often found in conjunction with a partial use code. The introduction of extra mathematics is indicative of one of the following: (1) the student has sufficient understanding of mathematical ideas, but lacks the necessary logical connections to harness the mathematics effectively or (2) the student lacks pieces of mathematical content knowledge.

Figure 6

Sentence Codes Derived from the Data

Code	Description	Example
Algebraic Substitution	Rewrite a term using a new variable	$n = 2k$
Mathematical Inference	Make a mathematical inference	$2k+1$ is odd if k is an integer
Switch Representation	Switch equivalence relationships	$p \equiv 1(mod3)$ then $p - 1 = 3k$ for some integer k
Reword	Reword a statement	$a\|b$ means that b is a multiple of a
Theorem/Algorithm	Use or state a theorem or algorithm	If $a = bx + c$, then GCD(a,b)=GCD(b,c)
Arithmetic Operation	Use arithmetic to rewrite an expression	$(2k + 1)^2 = 2(2k^2 + 2k)+1$
Example	Use an example to convince	I believe that if s^3 is odd then s is odd because if $s^3 = 27$ then $s = 3$
Partial use	Using only part of a theorem or definition	Using the division algorithm to claim $n = 2q + r$ without placing any stipulations of the value of r
Structure	Make a decision about proof structure using mathematical knowledge	A proof by cases where case 1 attends to prime numbers and case 2 attends to composites
Extra Mathematics	Performing extra mathematical steps not necessary to the proof	Constructing a formal set of even numbers to talk about even numbers.

Note. The codes describe ways mathematical content knowledge was used during proof production.

Analysis of Sentence Codes

Figure 7 provides information about how frequently the codes were used. I analyzed the codes from each of the 34 proofs in the data and determined which codes each proof contained. I then calculated what percent of the total proofs contained each of the codes. This allowed me to see which codes were most frequently used by the research subjects. I chose to calculate frequency as how many proofs contained the code rather than how many sentences contained the code because I felt it would be a more accurate frequency. Some students spent several sentences describing the same proof move that another student spent one sentence describing. Thus, a code describing the proof move may appear several times in the first student's proof and only once in the second student's proof. I did not want the differing lengths of explanations of the same proof move to skew the analysis of how frequently a code appeared in the data. I also coded the proofs I constructed and recorded how the frequency with which the codes appeared in my proofs.

Figure 7

Frequency of Each Code Found in Student Proofs and in Typical Proofs

Code	**Percentage of total student proofs containing the code**	**Percentage of typical proofs containing the code**
Algebraic Substitution	82%	67%
Mathematical Inference	94%	100%
Switch Representation	44%	22%
Reword	24%	0%
Theorem/Algorithm	29%	33%
Arithmetic Operation	77%	44%
Example	15%	0%
Partial use	6%	0%
Structure	32%	0%
Extra Mathematics	35%	0%

The three most commonly used codes were mathematical inference, algebraic substitution, and arithmetic operation. In all but two student proofs, students used a mathematical definition to make a mathematical inference. Both of the proofs without a mathematical inference code were invalid. Every typical, researcher-constructed proof contained a mathematical inference code. This was unsurprising given the general nature of the mathematical inference code. It is necessary to make at least one mathematical inference in a valid proof. The high frequency with which the algebraic substitution and arithmetic operation codes were used highlight the importance of using algebra and arithmetic in proofs about elementary number theory. Algebra and arithmetic were used frequently by students, even when they were not necessary. I anticipate that the frequent use of these mathematical concepts is specific to number theory. If the study were to be replicated in another proof course, such as graph theory, I do not anticipate algebra and arithmetic to be used as frequently.

With the exception of 2 codes, the frequency with which codes appeared in student proofs was more than the frequency with which I used codes. This suggests that students used mathematical content in more ways than I did when writing proofs. There are several reasons for this. One reason is that the student proofs included the verbal reasoning and the final product proof; my proofs were only final products. I may have used mathematical content knowledge in similar ways as the students during the reasoning and proof preparation phase, but my proofs only included the final proof I constructed. By nature of the data collection, my proofs contained much less reasoning and justification than the student proofs.

Many of the discrepancies between the frequencies of students' and my use of codes were substantial (a difference of at least 15%). Seven codes, including algebraic substitution, representation switch, and arithmetic operations, had substantial differences between student use and researcher use. Students seemed more likely to activate mathematical content knowledge they were familiar with, such as arithmetic and algebra. These familiar uses of mathematical content knowledge were not always necessary to the proof. I hypothesize that students have had success using these familiar mathematical ideas in other mathematics problems, in both proof and non-proof contexts. This may encourage students to activate pieces of knowledge that have been useful in the past.

Five codes appeared at least once in student proofs, but not at all in the researcher-constructed proofs. These codes were: structure, reword, example, partial use, and extra mathematics. It may seem odd that none of my proofs contained a structure code because from my perspective, it was obvious that each of my proofs was situated within a proof structure. In many instances I used logical cues to choose a structure (e.g. a statement such as "4 does not divide a^2 prompted me to use a proof by contrapositive). The structure code only refers to instances of using *mathematics* to select a proof structure. Thus, my use of logic to choose a structure did not provoke a structure code. Additionally, my proofs were final products and did not contain any background reasoning. Selecting a proof structure was a background activity, so is not captured in my proofs. However, student background activity was captured in the interviews through students' verbal reasoning. I consider the structure code to be an efficient use of mathematical content knowledge during verbal proofs. The other four codes used only by students played multiple roles in student proofs. The reword and example codes helped students unpack the prompt and the proof. These two codes allowed students to make progress in the proof, although often that progress was not the most efficient way to move through the proof. The extra mathematics code occasionally helped students move forward in the proof in a similarly inefficient way. The partial use and extra mathematics codes tended to be indicators that the student author had insufficient mathematical or logical knowledge. Proofs that included either of these codes tended to be inefficient proofs.

I want to make clear that using strategies that I describe as inefficient in the previous paragraph are productive ways to reason through a proof. I cautiously label the codes as inefficient simply to communicate that the proofs could have been completed more efficiently. Mathematicians value efficiency in a final, published proof. However, using mathematical content knowledge in these less efficient ways is an effective way for students to use mathematical content knowledge, particularly when students are learning how to write proof. I attempted to write the typical proofs to be efficient, which explains why they do not contain instances of less efficient use of mathematical content knowledge. Additionally, student interviews represented both the final product proof and the background reasoning that gave way to the final product. Some of the inefficient uses of mathematical content knowledge were used only during the background reasoning phase of the proof and not during the final proof.

Examples

In this section I provide an in depth example of how student use of mathematical content knowledge contributed to the success of three different proofs. I will present three proofs: a typical proof I constructed, a successful proof constructed by TJ, and an unsuccessful proof constructed by Matthew. Each proof will be followed by a description of how mathematical content knowledge was used during proof production. This discussion includes specific reference to the codes just described. I will also discuss what CCs were activated in each proof. The statement that was being proved is “Every integer is even or odd.”

Typical Proof

The proof found in Figure 8 is considered a typical proof of the statement. I constructed this proof and the professor of the class from which the data were gathered confirmed that the proof is appropriate and acceptable for the course. This first proof will differ from the other two discussed in this section because I include only the final product. The next two examples of student-generated proofs are much lengthier because they include the dialogue and reasoning that allowed the student to produce his proof.

Figure 8

Typical Proof that Every Integer is Even or Odd

		Coordination Class Activated	Code
1	Let n be an integer.		
2	Apply the division algorithm with $d=2$.	-Integer division	-Theorem/Algorithm
3	So n can be written as $2q+r$, where $r = 0$ or 1	-Integer division	-Theorem/Algorithm
4	Then $n = 2q$ or $n = 2q+1$ for some integer q.	-Integer division	-Theorem/Algorithm
5	Numbers of the form $2n$ are even, and numbers of the form $2n+1$ are odd	-Parity of integers	-Mathematical Inference
6	Therefore, n is even or odd.	-Parity of integers	-Mathematical Inference

My use of mathematical content knowledge in this proof is categorized by two different codes: theorem/algorithm and mathematical inference. In lines 2, 3, and 4 I use the division algorithm. Activation of the division algorithm, which is part of the integer division coordination class, is caused by a hint in the prompt that suggests using the division algorithm with d=2. The purpose of introducing this algorithm is to assert that if n is an integer, then it can be written in exactly one of the following two ways: $n=2q$ or $n=2q+1$. The familiar forms of $2q$ and $2q+1$ where q is an integer activates knowledge of the algebraic definition for even and odd integers (i.e. if $n=2k$ then n is even, etc.). Lines 5 and 6 of my proof are coded as a mathematical inference. In these lines I combine lines 4 and 5 to conclude that every integer is even or odd.

One research subject, Ally, wrote a proof that was very similar to the proof presented above. She had already completed the homework assignment containing the prompt, which allowed her to complete the proof very quickly during the interview. Her interview provides little insight into her thinking because the transcript is almost synonymous with the final product proof she turned in. She activated knowledge from her integer division and parity of integers CCs. The codes used to describe her use of mathematical content knowledge were theorem, arithmetic operation, and mathematical inference. She used theorems and mathematical inferences similarly to the way I did, as described in the previous paragraph. She refers to arithmetic operations once during the interview when she provides an additional justification for why the remainder will be 0 or 1 when any integer is divided by 2.

Successful Proof: TJ

In this section I analyze a successful proof written by TJ. TJ constructed a total of 9 proofs during his interviews, 7 of which were valid (78%). Figures 9-11 shows the interview transcript from TJ's interview.

Figure 9

Excerpt 1 from TJ's Successful Proof that Every Integer is Even or Odd

		Dialogue	**Coordination Class Activated**	**Code(s)**
8	T	The numerator [equals] the denominator times the quotient plus the remainder.	Integer division	Theorem/Algorithm
9		And so we're saying that-use the division algorithm with $d=2$.	Integer division	
10		So $n=2q+$ (pause) r is either 1 or 0.	Integer division	Theorem/Algorithm
11	I	How do you know that?		
12	T	Because, umm, (pause)		
13		Yeah, I have to think about that.		
14		Umm, so (pause) I mean it's-yeah it's kinda one of those things you never really think about.	Integer division	Partial Use

Note. The line numbers match the original manuscript. The excerpt does not include lines from the interview that are irrelevant to the analysis. (T=TJ, C=Chelsey).

The first thing TJ does in his proof is turn to his integer division coordination class and activate knowledge about the division algorithm. This activation of knowledge is likely due to the prompt recommending the use of the division algorithm with $d=2$. This allows TJ to rewrite n as $n=2q+r$ where r is 0 or 1 (Figure 9, line 10). When asked why the remainder (r) is restricted to only 0 and 1, TJ does not appeal to the division algorithm to support his claim that these are the only possible values for the remainder. Using the complete division algorithm would certainly have been the most efficient way to justify TJ's assertion. TJ read the algorithm from his notes or the book, but for some reason he did not use the algorithm in its completeness. The transcript of TJ's interview suggests that he may have simply overlooked the part of the algorithm that restricts the remainder. However, the fact that he spent a large portion of his proof searching for a justification for the restrictions on the remainder indicates that his CC of the integer division is not yet expert. It seems that his inferential net was developed enough to use the division algorithm, but only partially. A potential reason for his partial use of the division algorithm is that he had not used the division algorithm in the context of proving. Perhaps unbeknownst to him, he had used the algorithm to calculate a quotient and a remainder, but he had never had to make any general claims about the quotient or the remainder. Experiencing the division algorithm in more contexts, especially in the context of proof production, will make his CC of the integer division more robust because he will have a better understanding of a critical component of the CC: the division algorithm.

Figure 10

Excerpt 2 from TJ's Successful Proof that Every Integer is Even or Odd

		Dialogue	Coordination Class Activated	Code(s)
21		Umm, or I could say, rather, $n=2q+r$ and (pause)	Integer division	Theorem/Algorithm
22		We're gonna say r is greater than or equal to 0 just, like, for the sake of the scope that we're working in.		Algebraic Substitution
23		Is that good enough?		
24	I	Yeah		
25	T	Umm, so it could be $n=2q+r$ -not r- 0 or $2q+1$, but then when you get to $2q+2$ that can also be written as, umm.	Integer division	-Partial Use -Extra Mathematics
26		If our d was 2 we could say it's, like, $d+1$ times q plus 0.	Arithmetic & Algebra	Arithmetic Operation
31		Umm, so that way you can't have-and then when you do, like, $d+1$, $q+1$ and then repeat it again, you have, like, $d+2$.	-Arithmetic & Algebra -Integer division	-Extra Mathematics -Arithmetic Operation

Note. The line numbers match the original manuscript. The excerpt does not include lines from the interview that are irrelevant to the analysis. (T=TJ, C=Chelsey).

TJ's partial use of the division algorithm led him to activate several other pieces of knowledge in an effort to provide a justification for why the only possible remainders were 0 and 1 (Figure 10). His justifications are coded as extra mathematics because if he had applied the division algorithm in its completeness, he need not have provided any additional justification.

However, his justification is valid and demonstrates his ability to move between pieces of knowledge with his integer division CC. The pattern of an extra mathematics code following a partial use code was not unique to TJ's proof. It surfaced in other student proofs. After deciding that r must be greater than 0 (Figure 10, line 22), he begins to consider each potential remainder, one at a time (i.e. $n = 2q$, $n = 2q+1$, $n = 2q+2$). His generation of multiple different remainder possibilities, each of which is a non-negative integer, is evidence of the activation of pieces of knowledge from his integer division CC. He knows that the remainder of a division problem is usually restricted to nonnegative integers. Next, he activates knowledge from his arithmetic CC by rewriting each potential expression for n to be in the form $n = 2k$ or $n = 2k+1$ where k is an integer (Figure 10, line 25, 26, 31). He knows that if he can prove that every integer can be written in one of these two forms, he can appeal to the algebraic definition of even and odd numbers to claim that the integer is even or odd. Thus, his extensive work in the integer division and arithmetic CCs is spurred by envisioning the end of the proof.

Figure 11

Excerpt 3 from TJ's Successful Proof that Every Integer is Even or Odd

		Dialogue	Coordination Class Activated	Code(s)
35		Umm, what did we say in the notes? (pause while he checks notes)		
36		Okay we said 0 is less-or is greater-or 0 is less than or equal to *r*, which is less than *d*.	Integer division	Theorem/Algorithm
37		And that's just kinda the definition of the whole n plus *qd+r* (I think he means *n=qd+r*)	Integer division	Theorem/Algorithm
38	I	Okay.		
39	T	So, (pause) that way you can have- when *d=2*, you can have *2q+0*.	Integer division	
44	T	And then our second case is *n=2q+1*	-Proof structures -Integer division	Structure
45		But then our third case would be *2q+2* but we already established that *r* has to be less than *d*, so those are our only 2 cases, so (pause) um, our *q*'s can, like, become our *l*'s and *k*'s depending on, like, which one we're talking about.	-Proof structures -Integer division	-Algebraic Substitution -Theorem/Algorithm -Structure -Extra Mathematics
46		So our *n*, which is like our numerator, is *2k* or *n* is *2k+1*	Integer division	Mathematical Inference
51		So, we know with the just division algorithm that *r* has to be greater than or equal to 0, but less than *d*.	Integer division	Theorem/Algorithm
52		And so when *d=2*, the only options that fit are r=*0* and *r=1*.	Integer division	Theorem/Algorithm
53		And plugging those into the division algorithm creates the scenarios *n=2k* or *n=2k+1*.	-Integer division -Parity of integers	-Theorem/Algorithm -Mathematical Inference

Note. The line numbers match the original manuscript. The excerpt does not include lines from the interview that are irrelevant to the analysis. (T=TJ, C=Chelsey).

Eventually TJ rereads his notes and realizes that the division algorithm does restrict the remainder in the way he needs it to (Figure 11, line 36). He uses the division algorithm to conclude that $n = 2q$ or $n = 2q+1$. He structures this assertion within a proof by cases. After assigning case 1 and case 2 to the instances where $r = 0$ and $r = 1$, he says that the next case would be $n = 2q+2$ (Figure 11, line 45). He explains that this case is not necessary because the remainder must be less than the divisor. It is interesting that he proposes a third case at all because the division algorithm clearly states that only the first two cases are necessary. He uses the division algorithm to disprove the existence of a third case, but the fact that he even feels the need to present a third case suggests that his understanding of the division algorithm is lacking. It seems that he is constructing the cases based on other pieces of knowledge (i.e. that the possible remainders in a division problem are all the natural numbers) and then choosing which cases are valid by activating knowledge of the division algorithm.

It is interesting to note that even before TJ reread the division algorithm and used it to explain why the remainder was either 0 or 1, he provided an adequate justification for the remainder restrictions. This justification was coded as extra mathematics and provided interesting insight into TJ's thinking. The problem solving exhibited in his argument includes both logical and mathematical moves. The justification was made possible because of the mathematical content knowledge held in TJ's integer division CC and arithmetic CC. I find it particularly interesting that TJ felt comfortable introducing prior mathematical knowledge in the context of proof production. This suggests that pieces of his CC of integer division and of arithmetic are sufficiently developed for him to feel comfortable activating appropriate mathematics in the context of proof production.

Another research subject, Gracie, wrote a proof that was very similar to TJ's. Like TJ, she did not appeal to the division algorithm to assert that if the divisor is 2, the remainder must be 0 or 1 (i.e. any integer can be written as $2q$ or $2q+1$). Gracie's partial use of the division algorithm caused her to activate knowledge in her integer division CC. She referred to the fact that when you divide by 2, the remainder is always 0 or 1. This knowledge seemed to be based on her past experience with integer division. She also activated knowledge from her integer classification CC when she said the number 2 is composed of two 1s. She used this decomposition of the number 2 to make claims about what the remainder looks like if the divisor is 2. Gracie, like TJ, used her integer division CC to provide adequate justification for why an integer n can be written $n = 2q$ or $n = 2q+1$.

Unsuccessful Proof: Matthew

In this section I analyze an unsuccessful proof written by Matthew. Matthew constructed a total of 8 proofs during his interviews, 1 of which was valid (13%). Figures 12-14 show the interview transcripts from Matthew's interview.

Figure 12

Excerpt 1 from Matthew's Unsuccessful Proof that Every Integer is Even or Odd

			Coordination Class Activated	Code(s)
2	M	I kinda think this is gonna be one of those 2 case things.	Proof structures	
3		Umm, although at the same time it's kinda weird because every one is odd or even. (pause)		
4		Uhh, okay, I'm just gonna try it with 2 cases for the even		
5	I	K		
6	M	So I'm thinking with this one, what I want to do is based off of the assumption that $a=2k$, I could show that, uh, whatever k is, as long as it divides (pause) half of a (pause)	Parity of integers	
7		Nope, my mind's thinking of it, but it's not coming. (pause) So (pause)		
9	I	So what are you thinking about right now?		
10	M	I'm kinda thinking that maybe if I just reverse the equation-the $a=2k$ then I might be able to show that whatever integer k is, it will equal a.	Parity of Integers	Arithmetic Operation
16	M	(pause) Hmm, based off the section I would need to do-need to use divisors [p2] but I'm not sure how that works in this situation.	Integer division	

Note. The line numbers match the original manuscript. The excerpt does not include lines from the interview that are irrelevant to the analysis. (M=Matthew, C=Chelsey).

Matthew begins his proof by deciding he wants to pursue a proof by cases. This decision uses knowledge from his proof structures CC. I believe this knowledge was activated because one of the proof norms in his class is that the proof structure is made explicit at the beginning of the proof. He knows he needs to choose a proof structure before beginning the body of the proof.

I believe he chooses proof by cases because the prompt states that every integer must fall into two categories: even and odd. Proof by cases is often used when the domain is separated into different categories.

Matthew next turns to the CC at the center of this proof: parity of the integers. He appeals to the algebraic definition of even numbers and suggests some arithmetic operations that may be useful. However, it seems he does not find these ideas satisfying because he immediately turns to the textbook to see if it provides any helpful examples. His brief review of the textbook section prompts him to use divisors (integer division CC). He decides to attempt using the idea of divisors in the context of set theory. In Matthew's proof experience, set theory is used as a proof structure. His spontaneous implementation of set theory suggests that his proof structures CC is developing because he feels comfortable introducing this structure into the proof.

Figure 13

Excerpt 2 from Matthew's Unsuccessful Proof that Every Integer is Even or Odd

			Coordination Class Activated	Code(s)
18	M	Or, I might be able to use a big long set. [p4]	Set theory	Extra Mathematics*
19	I	What do you mean "a big long set"?		
20	M	Well if I had a set of all the numbers, I could turn that into 2 different sets-ones that's divisor is 2 and the other that is a divisor of 2 mod1.	-Integer division -Parity of integers	Extra Mathematics

Note. The line numbers match the original manuscript. The excerpt does not include lines from the interview that are irrelevant to the analysis. (M=Matthew, C=Chelsey).

Matthew's introduction of set theory is coded as extra mathematics because there were already two ideas to approach this proof without set theory on the table: his idea to use divisors and the prompt's hint to use the division algorithm. Thus, his spontaneous implementation of set theory, although not incorrect, is considered extra mathematics. The remainder of his proof is situated within set theory. Matthew considers the set of all numbers (integers) and suggests splitting the set into two distinct sets "one that's divisor is 2 and the other that is a divisor of 2 mod1 (Figure 13, line 20)." He gives an informal, verbal description of both sets as "one where 2 goes into all these numbers" and "one where 2 goes into all of them mod1." Here I assume that he is talking about the elements of the set; namely, one set contains elements that have 2 as a divisor, not that 2 is a divisor of the set itself. It is interesting to note Matthew's incorrect introduction of modular arithmetic. Other work he produced during the interviews suggest that Matthew's CC of modular arithmetic is still in its beginning stages. Occasionally he can interpret modular arithmetic expressions correctly, but frequently he uses modular arithmetic incorrectly. In this proof he seems to be using "mod1" as a synonym for "remainder 1." I consider his description of both sets to be activation of knowledge from his integer division CC with incorrect notation.

Figure 14

Excerpt 3 from Matthew's Unsuccessful Proof that Every Integer is Even or Odd

			Coordination Class Activated	Code(s)
62	M	So *a* divides *b*.	Integer division	-Extra Mathematics
63		If I remember right, it's *a* (pause). I think it's *ac* times some *c* equals *b*, but I don't remember for sure.	Integer division	-Extra Mathematics -Switch Representations
64		So if I'm gonna go with that, then it's going to be (pause) 2 times some integer equals *k* for this set-or then *k* would be this set.	Integer division	Mathematical Inference
65		And then if it's 2 times some integer plus 1 would be *k* in this set.	Integer division	Mathematical Inference
66	I	So you said *k* would *be* this set for this one and then *k* would be *in* this set for this one. Is *k* gonna be the set or, like-		
67	M	Uh no, *k*'s gonna be in the set.		
73		I'm gonna introduce a new variable.		Extra Mathematics
74		I'm gonna call it *j*.		
75		It's gonna be some integer.		Algebraic Substitution
76		So I'm gonna say $2j=k$ being an element of the set from negative infinity, -4, -2, 0, 2, 4, and so on.	Parity of integers	Algebraic Substitution
77		So basically all the even numbers.		Mathematical Inference
78		And then for this one, it's going to be 2□ + 1is going to equal *k*, which is an element of another set, which basically all the even ones-er not the even-all the odd numbers.	Parity of integers	-Algebraic substitution -Mathematical inference
87	I	Do you feel like that proves it?		
88	M	Not sure if it proves it, but I think it's pretty close.		

89	I	Okay. What do you feel like it's missing? Could you, like, point out what it's missing?		
90	M	Uhh being set up formally (laughter)	Proof structures	

Note. The line numbers match the original manuscript. The excerpt does not include lines from the interview that are irrelevant to the analysis. (M=Matthew, C=Chelsey).

Matthew spends a substantial amount of time searching for a satisfying way to define the even set and the odd set. His next idea is to attempt to use arithmetic operations to demonstrate the way 2 divides or doesn't divide the elements of the set. This causes him to introduce divisibility statements (i.e. $a|b$; Figure 14, line 62), which reside in his integer division CC. The most productive steps in Matthew's proof so far have come from his activation of knowledge from the integer division CC, which encourages him to continue using this CC. The introduction of divisibility statements is coded as extra mathematics because Matthew had already introduced other ways to discuss divisibility with divisibility statements.

Matthew continues to be dissatisfied with his set definitions, which results in the introduction of a new variable (Figure 14, line 73). This was coded as extra mathematics. Introduction of an unnecessary variable occurred in several of the proofs in this study. I believe the purpose of this new variable was to allow him to jump from using the integer division CC to using the parity of integers CC because the new variable allowed him to create expressions that resembled the algebraic definitions of even and odd numbers.

After using his new variable to define the even and odd sets, he feels satisfied that the only thing lacking in his proof is that the proof be set up formally (Figure 14, line 88). While there is a lack of formality in his proof, there are also other issues. The main issue is logical. While the idea of sorting the integers into even and odd sets is productive, he makes no argument that *every* integer will be included in at least one of the sets. I consider this to be a logical issue because he does not attend to the possibility that an integer does not belong in either group, which is precisely what he is trying to prove.

CHAPTER 5: DISCUSSION

In this section I summarize the answers to my two research questions: (1) How does mathematical content knowledge contribute to the success of a proof? (2) How is mathematical content knowledge used during proof production? Then I explain the limitations of the study, describe the implications for instruction the results suggest, and provide ideas for further research.

How does Mathematical Content Knowledge Contribute to the Success of a Proof?

Many pieces of mathematical content knowledge must be activated in order to successfully complete a proof. These pieces of knowledge are situated within multiple CCs. In this study, five CCs contained all the mathematical content knowledge needed to successfully complete the proofs. Sometimes students neglected to activate pieces of a necessary CC, which caused the proof to be invalid. Sometimes students activated pieces of knowledge from extra, non-necessary CCs. These "extra" CCs were usually used to help students provide more detailed justifications for the steps of their proof. In most cases, students activated all necessary CCs. Invalid proofs could often be attributed to failing to activate a necessary piece of knowledge in a CC that was already activated. As expected, it was critical that students activated the appropriate CCs and pieces of knowledge within the CCs in order to accurately prove each statement.

Mathematical content knowledge is a critical component to a successful proof. The data from this study suggest that using mathematical content knowledge appropriately in a proof is non-trivial. Five of the 17 invalid proofs (29.4%) were invalid due to issues with mathematical content knowledge. These invalid proofs represent proofs that were incomplete as well as proofs that were considered complete by the student but were formally invalid. Consistent with previous research, a large portion of the proofs were invalid due to issues with the logical nature of a proof. However, the fact that almost a third of the invalid proofs were invalid due to mathematical issues suggests that proof educators must attend not only to students' logical knowledge, but also to students' mathematical knowledge. The proofs in this study used basic ideas such as parity of integers and integer division. Some requisite pieces of knowledge were acquired during the course in which the research subjects were enrolled. These new concepts often contributed to the failure of a proof. For example, many students experienced issues using knowledge about modular arithmetic in their proofs.

How is Mathematical Content Knowledge Used during Proof Production?

Students use mathematical content knowledge in a multitude of ways during a proof. A set of internal codes were developed from this study to represent how the research subjects used mathematical content knowledge. The frequency with which students' used these codes was compared to the frequency with which I used these codes when writing typical proofs of the same statements. Discrepancies between the two frequencies showed that students tended to use mathematical content knowledge in unnecessary, yet familiar, ways, such as performing extra arithmetic operations. A subset of the codes were never used in my proofs; these codes were described as inefficient ways to use mathematical content knowledge in a formal proof. These inefficient codes were valuable to students as they reasoned through a proof. For example, the use of an example to convince oneself of the veracity of a statement was considered inefficient in a formal proof, but very useful when constructing a verbal argument. The ways students used mathematical content knowledge in this study are hypothesized to be specific to proofs in the domain of elementary number theory.

Two in-depth examples of student proofs are provided in the "Results" section. The purpose of these examples is to provide evidence of student use of specific CCs and sentence codes. These examples provided an opportunity to synthesize the results of the study and answer both research questions simultaneously.

Limitations of the Study

There are several limitations of this study that should be noted. First, the mathematical context of number theory significantly influences the answer to the second research question. The way mathematical content is used in number theory proof is potentially very different from the way mathematical content is used in other domains, such as graph theory. This idea is discussed in detail in previous chapters. Second, I purposefully selected a simple, familiar mathematical context for data collection. This choice influenced the way students used mathematical content knowledge. In some instances, a familiar mathematical context may have enabled students to use their mathematical content knowledge more readily and efficiently. In other instances, the familiar mathematics may have been difficult to activate in the unfamiliar context of proofs. Third, I conducted all my interviews during a very short period of time. The data would likely have been different if the students participated in the interviews throughout the course. A prolonged interview schedule may have allowed me to analyze how the role mathematical content knowledge in proof production changed throughout the semester.

Implications for Instruction

Although this study et al. on student thinking and reasoning with mathematics, it also provides important implications for mathematics instructors. Efficiency is a quality that is valued by most mathematics professors. However, results of this study show that students use mathematical content knowledge in a variety of inefficient ways while reasoning about a proof. These inefficient uses of mathematical content knowledge should not be seen as negative, but rather as helpful pedagogical strategies. For example, students in this study often constructed examples to help them reason about a proof. These examples enabled further progress with the proof. Professors may consider providing examples to students in an effort to help students understand the proof more easily. Professors should also avoid devaluing student work that contains inefficient use of mathematical content knowledge, especially when students are still being apprenticed into the practice of proof construction.

Relatively simple mathematical contexts such as elementary number theory are often used in introduction to proof courses. Professors and textbooks often assume that since the mathematical context is simple, the students will not experience any issues with respect to the corresponding mathematical content knowledge and that therefore the ITP course can be et al. solely on the logical nature of proof. Mathematics professors should not assume that mathematical content knowledge will not be problematic for students. In addition to attending to student understanding of the logical nature of proof, professors should be aware that students may also experience struggles with mathematical content knowledge. Measures can be taken to better support student development of mathematical content knowledge including facilitating class discussions about key definitions and theorems, presenting examples of the mathematical principles in non-proof contexts, and making space during class for students to ask questions about mathematical content knowledge.

The mathematics used in this study (elementary number theory) was fairly familiar to the students; that is, they had dealt with most of the required concepts in other contexts such as prior mathematics classes. However, the proofs that were invalid tended to require the use of mathematical content knowledge that was new to the students (e.g. modular arithmetic). It was difficult for students to use new mathematical content knowledge in proofs. Perhaps students must experience mathematical principles in non-proof contexts before competently using the principles in proof contexts. Professors can support students by providing opportunities to engage with new mathematical principles in non-proof contexts before requiring students to use new mathematical content knowledge during proof production.

Ideas for Future Research

This study focused on how students use mathematical content knowledge in the context of proof production. One trend in the data is that students struggled to activate knowledge about less familiar mathematical ideas, such as modular arithmetic, during proof production. It is interesting to wonder whether students would activate pieces of knowledge about unfamiliar mathematical ideas in non-proof contexts. This would help explain the relationship between using mathematical content knowledge in proof contexts and non-proof contexts. Another potential research study could examine if mathematical principles need to be mastered in non-proof contexts before they can be used proficiently in proof contexts. Many of the issues students experienced in this study were related to mathematical ideas that were taught during the ITP course. Perhaps if students were familiar with the concepts outside of proof writing they would be more capable of using the knowledge within the context of proofs.

I hypothesize that the ways students used mathematical content knowledge in this study are specific to proofs of number theory. Future research could examine how students use mathematical content knowledge in proofs in other domains, such as graph theory or real analysis. Perhaps there is some overlap between the ways mathematical definitions, theorems, and procedures are used among different domains. A research study examining student proofs in several different domains would provide a more robust answer to the question of how students use mathematical content knowledge during proof production.

One important result from this study is that the use of mathematical content knowledge in proofs is nontrivial. This result leads to the question of how teachers in an ITP course should address mathematical content knowledge. The primary purpose of an ITP course is to apprentice students into the art of writing mathematical proofs. The focus of these courses is the logical structure and nature of proof. Mathematics is not the focus of instruction. However, this study shows that even when very basic mathematical contexts like elementary number theory are used students still struggle to activate mathematical content knowledge. Future research can identify ways that teachers can attend to the development of mathematical content knowledge in a course that is focused on the development of logical knowledge.

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APPENDIX A

Pretest Administered to Research Subjects Before Interviews

Please answer the following questions. Please don't look up any of the answers. The purpose of this questionnaire is <u>not</u> to assess your mathematical ability, but rather to see how you think.

1. What is a natural number?
2. What is an integer?
3. What is the definition of an even integer? What is the definition of an odd integer?
4. What does it mean if *a* divides *b*?
5. What is a common divisor of two integers?
6. What is the greatest common divisor of two integers? What is the greatest common factor?
7. What is the least common divisor of two integers?
8. What's the greatest common divisor of 15 and 27? 7 and 36?
9. What does it mean to find the prime factorization of a number?
10. What is the prime factorization of 54?
11. Can two different numbers have the same factorization?
12. What is a prime number? What is a composite number?
13. Is 1 a prime number? Explain.
14. How many factors can an integer have?
15. When you're doing long division, how small can the remainder be? How great can the remainder be?

APPENDIX B

Interview Questions

Interview #1

1. Prove the following: Let s be an integer. Prove that s is odd if and only if s^3 is odd.
2. Let a, b, c, d be integers. Prove that if $a|c$ and $b|d$, then $ab|cd$.
3. Let a be an integer. Prove that if 4 does not divide a^2, then a is odd.

Interview #2

1. Let a be an integer. Recall that a is even if there is some k in the integers such that $a=2k$, and a is odd if there is some l in the integers such that $a=2l+1$. Prove the following statements, which we took for granted previously. (Hint: Use the division algorithm with $d=2$).
   a. Every integer is even or odd.
   b. No integer is both even and odd.
2. Recall that the Fibonacci numbers are defined by the relations $F_1 = 1, F_2 = 1$ and for $n > 2$, the recursion $F_n = F_{n-1} + F_{n-2}$. Prove by induction that for every n in the natural numbers we have $GCD(F_{n+1},\ F_n) = 1$.
3. Let n be in the integers. Prove that $GCD(6n + 2,\ 12n + 6) = 2$.

Interview #3

1. Prove that, with only one exception, every prime number is congruent to either 1 or -1 modulo 3.
2. Prove that for any n in the natural numbers and any $a_1, \dots, a_n$ in the integers, if each $a_i \equiv 1 (mod 3)$, the product $a_1 a_2 \cdots a_n \equiv 1 (mod 3)$. (Use induction).