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Topologically Mixing Suspension Flows

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ABSTRACT

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We find a set of conditions on a roof function to ensure topological mixing for suspension flows over a topological mixing base. In the measure theoretic case, such conditions have already been established for certain flows. Specifically, certain suspensions are topologically mixing if and only if the roof function is not cohomologous to a constant. We show that an analogous statement holds to establish topological mixing with the presence of dense periodic points. Much of the work required is to find properties specific to the equivalence class of functions cohomologous to a constant. In addition to these conditions, we show that the set of roof functions that induce a topologically mixing suspension is open and dense in the space of continuous roof functions.

Keywords: suspension flows, topological mixing

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Chapter 1. Introduction

The study of dynamical systems focuses on the behavior of a system as it evolves with time. Some systems have very basic structure and behavior, like the swinging of a pendulum or the rotation of a circle. Other systems have considerably more complicated behavior, like models for the weather or economy. Understanding and predicting systems that have complicated and chaotic properties motivates the study of dynamical systems. Two types of dynamical systems, are discrete and continuous systems.

Definition 1.1. Let X be a nonempty set. The map $f\colon X \to X$ is a *discrete-time dynamical system* or a *discrete dynamical system*.

In the study of discrete dynamical systems, we are interested in the trajectory of a point or a set under recursive iteration of our function. That is, we consider x, $f(x)$, $f \circ f(x)$, and so on. To simplify the notation we write

$$f^n(x) = f \circ f \circ \cdots \circ f(x) \text{ (composing } x \text{ in } f \ n \text{ times) and where } f^0(x) = x$$

It is by understanding the behaviors of the points in a space X under repeated compositions of f that we are able to form an understanding of the structure of the system (X, f). With these maps, the system evolves discretely.

The second type of dynamical system captures the notion of a system that changes over a continuous parameter.

Definition 1.2. A *continuous-time dynamical system*, *continuous dynamical system*, or *flow* is a mapping $\varphi\colon X \times \mathbb{R} \to X$ such that

(i) $\varphi(x, 0) = x$; and

(ii) $\varphi(\varphi(x, t), s) = \varphi(x, s + t)$.

We often write $\varphi(x,t) = \varphi^t(x)$ for convenience; however, we will use both notations as needed.

The space X is called the *phase space*. Here, by crossing our phase space with $\mathbb{R}$ we obtain a notion of time where negative values indicate the past state of the system, zero represents the present, and positive values correspond with the future state of the system. We take a point in the phase space and allow it to flow under φ with time $t \in \mathbb{R}$. Frequently, we are interested in the behavior of a point or set as time goes to infinity and whether predictions can be made about the evolution of a system. Without infinite precision, a dynamical system can appear to behave unexpectedly or randomly making predictions inaccurate or impossible to make. The study of dynamical systems attempts to quantify and understand this chaos and how to make conclusions when it is present in a system.

The systems we will consider are flows, which are typically generated by an ordinary differential equations or by a vector field on a manifold. Flows and discrete time dynamical systems can be thought of as the trajectories or motion dictated by a function in some space. In addition, these systems are deterministic. That is for any point, there is exactly one future for it. There is nothing random about the paths or trajectories of points in these kinds of systems; although "noise" can be added to a system to give it more random behavior, we do not consider any stochastic systems. This may suggest that deterministic systems are simple and can be easily understood; however, two nearby points could have vastly different behaviors and trajectories. Additionally, we would need an infinite level of precision to know the future for a given a point. This gives rise to the notion of deterministic chaos, and is a fundamental principle in the study of these systems.

1.1 History

The study of dynamical systems began when the French mathematician, Henri Poincaré in the 19th century as he studied and attempted to predict the movement of celestial bodies as governed by the laws of gravity. He found that for two bodies, the question is fairly simple.

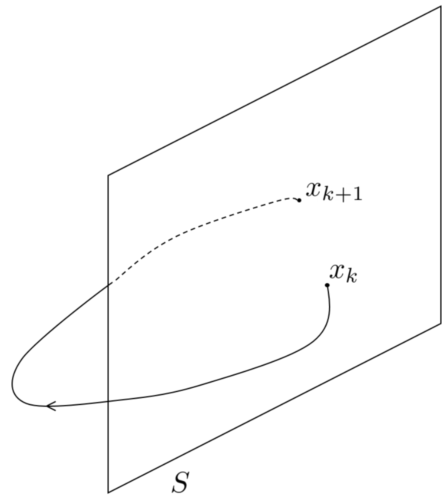

Figure 1.1: S, the Poincaré Section

However, for three or more bodies, finding a model that accurately predicted the behavior of the system in the long run was considerably more difficult. Prior to this work by Poincaré, mathematicians solved differential equations by finding an explicit solution or series solution. However, no closed form solution could be produced for the general n-body problem. Rather than find a solution Poincaré elected to study the orbits or trajectory of a point. This shift in focus made it possible to study a system, and its evolution over time without knowing the exact solution given a set of initial conditions of the system.

To assist his work on flows, Poincaré's introduced the foundations for the construction of a suspension flow, in *Mémoire sur les courbes définies par les équations différentielles* [19, p. I49]. Given a flow, we can find a transversal surface of the phase space of codimension one called a *Poincaré section*. This surface can be thought of as a manifold that does not run parallel along any orbit in the flow. From this cross section, we can obtain a *return map*, that represents where a point returns back to the transversal. This also allows us to define a discrete system on the surface by tracking how the path of the point intersects the Poincaré section. This map is often called the *Poincaré map* or *return map*. While the dynamical system on the Poincaré section is not a flow, since the Poincaré section has lower dimension, understanding the dynamics on this surface can be relatively easier than studying the flow.

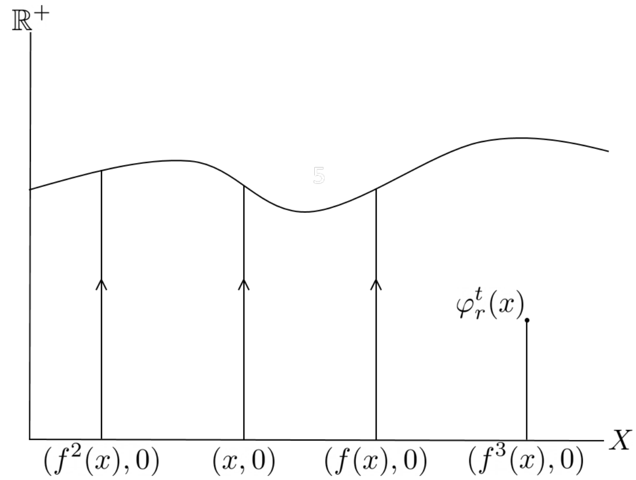

Figure 1.2: Suspension flow

From the Poincaré section, return map, and the Poincaré map of a flow, we can construct a suspension flow. Suspension flows or special flows are comprised of two components: a discrete system on a space X, and a function from X into the positive reals. The space X is referred to as the *base*, and the function mapping into the reals is called the *roof function*. We construct our phase space by considering the space under the roof and by identifying points on the roof with appropriate points in the base (see Definition 2.7). Points in the phase space flow at unit speed directly up to the roof as in Figure 1.2. When constructing a suspension from a flow, the Poincaré section is the base space, the return time is the roof function, and the Poincaré map or return map is the discrete system on the base. In this manner, we can construct suspension flows from a flow; a more rigorous treatment of this construction is found in Section 2.1.

This construction is often helpful because any flow on a measure space without fixed points is isomorphic to a suspension flow [3, p. 295]. Suspension flows are a useful tool when trying to understand the behavior of a system because they can simplify many problems that arise when studying a flow. Understanding the properties of a flow can be reduced to understanding the dynamics on the base and knowing the structure of the roof function of the corresponding suspension flow.

1.2 Mixing

The topological property of a dynamical system we will study is topological mixing.

Definition 1.3. A discrete time dynamical system $f: X \to X$ is said to be *topologically mixing* if for any two opens sets $U, V \subset X$, there exists an N, such that for all $n \geq N$ we have that $f^n(U) \cap V \neq \varnothing$. Similarly, a continuous time dynamical system $\varphi^t: X \to X$ is said to be *topologically mixing* if for any open sets $U, V \subset X$ there exists a T such that for all $t \geq T$, we have that $\varphi^t(U) \cap V \neq \varnothing$.

For compact metric spaces, there is an alternative definition that we may use [8, p. 56]. Here we only provide the definition for a flow, since the definition for discrete maps is similar.

Definition 1.4. A flow φ^t is *topologically mixing* if for all $\epsilon > 0$, there exists a $T \in \mathbb{R}$ such that for any $x, y \in X$ and $t \geq T$ we have that $\varphi^t(B(x, \epsilon)) \cap B(y, \epsilon) \neq \varnothing$.

The notion of mixing arose in physics and the study of mixing liquids or gas. The mathematical interpretation of mixing is meant to reflect the physical mixing process. For example, consider a glass of water with food coloring added to the glass. If we stir the glass enough, the food coloring will diffuse throughout the glass and become inseparable or indistinguishable from the water.

This is a topological approach to describing a mixing property, but there are analogous notions for different systems. If we are considering a system defined on a measure or probability space, we may think of mixing in a statistical manner.

Definition 1.5. Let $(X, \mathcal{A}, \mu)$ be a probability space. A measure preserving transformation or flow T is *(strong) mixing* if for any $A, B \in \mathcal{A}$ we have that

$$\lim_{t \to \infty} \mu(T^{-t}(A) \cap B) = \mu(A)\mu(B)$$

Equivalently, T is mixing if

$$\lim_{t\to\infty}\int_X f(T^t(x))\cdot g(x)d\mu = \int_X f(x)d\mu\cdot\int_X g(x)d\mu$$

For any bounded measurable functions f and g.

We may interpret this definition for mixing in our food coloring in water example. Suppose 90% of the solution is water and the remaining 10% is food coloring. Once we stir the solution and take a sample, we would expect that any sample would be approximately 90% water and 10% food coloring.

Some systems do not exhibit any mixing properties. Reusing the example of mixing fluids, suppose we have a glass of water with oil in it. If we stir this solution, the water and oil do not combine or mix in any manner because of the oils hydrophobic properties. The water and oil will always be separated. Again, considering the statistics, if 90% is water and the remaining 10% is oil, after stirring we would not expect that every sample of the solution would contain the same ratio of water and oil.

Mixing systems tend to have more complicated behaviors and structure than systems that are not mixing. Mixing is a strong property that implies a variety of other properties about the system. While there are a handful of mixing systems with predictable and basic behavior, most mixing systems have chaotic and unpredictable behavior. Because certain phase spaces have more structure than others, the system of interest leads to different notions and definitions of mixing can be used. Furthermore, there are a variety of definitions to capture the different levels of mixing a system exhibits. Since we are focusing on topological mixing we restrict our attention to compact metric spaces. To distinguish between the different types of mixing, we will always refer to topological mixing using the full term, and we will refer to strong mixing in Definition 1.5 simply as mixing.

1.3 Ergodic Theory

Poincaré's work on the n-body problem led him to utilizing probability theory and measure theory to determine statistical properties of systems. This work gave rise to ergodic theory and the study of measure preserving transformations. The statistical properties of a system can provide many insights into the long-term behavior of a system regarding the trajectories of points and the average values the points take on. Strong mixing is an example of how the statistics of a systems can provide information about the behavior of a system. Some of the most important and elementary results of ergodic theory are due to Poincaré, George Birkhoff, and John von Neumann. Here we provide the definition of an ergodic system.

Definition 1.6. Let $T\colon X \to X$ be a measure-preserving transformation (or flow) on a probability space $(X, \mathcal{A}, \mu)$. T is *ergodic* if for all $A \in \mathcal{A}$ with $T^{-1}(A) = A$, either $\mu(A) = 0$ or $\mu(A) = 1$.

Since the only sets that are fixed by T have zero measure or full measure, all other sets are spread throughout the space. Heuristically, this means that points in an ergodic system attain almost every possible state, and the orbits of points are spread throughout the system.

One fundamental result due to Poincaré deals with how often a point will return back to a set it is contained in.

Proposition 1.7. *Let T be a measure-preserving transformation on a probability space $(X, \mathcal{A}, \mu)$. If $A \in \mathcal{A}$, then for all almost every $x \in A$, there exists some $n \in \mathbb{N}$ such that $T^n(x) \in A$.*

This result implies that almost every point in a measurable set will return back to the set an infinite number of times. In particular, it also implies that non-periodic points will also return back to the set they started in.

Since ergodic systems exhibit the spreading of sets there is a connection between mixing and ergodicity. In particular, strong mixing implies ergodicity. Thus an ergodic system spreads sets throughout a space in a weaker manner than what a mixing system may exhibit.

1.4 Measure Theoretic Results

Because of the success and power of ergodic theory the study of dynamical systems on measure spaces became very popular. A great deal of research has been dedicated to studying suspension flows in the measure theoretic case. In this setting, suspension flows are typically referred to as "special flows." Because many of the useful results are known in the case of an ergodic transformation, study on how the roof function can affect the ergodic properties of a system has been a point of focus. Additionally, because mixing implies ergodicity, identifying roof functions that induce mixing special flows can help us show ergodicity in a system.

Major results often state that given a certain base, a special flow is mixing if the roof function has a certain property. Many of these results deal with special flows over some sort of rotation of the circle. Kochergin showed that special flows over a rotation of the circle under a roof function of bounded variation is not mixing [14]. This work would be furthered by Katok to special flows over a base that permutes fixed intervals of the circle [12]. Such a base is called an interval exchange transformation, and a special flow with such a base under a roof function of bounded variation is not mixing.

One major result states that an Anosov flow is mixing or is a special flow with a constant roof function modulo a time change [18]. This result provides a dichotomy for Anosov flows. Another result presents a similar dichotomy for roof functions over a certain class of parabolic flows. They state further that the set of roof functions that induce mixing flows is dense in the space of continuous roof functions [20]. These dichotomies can often provide simple methods or strategies to determine whether a system is mixing.

Although there are many results in this setting, there is a scarcity of results for suspensions for topological mixing. We will show that a similar dichotomy in the topological setting exists. While there are some differences in the topological case we will employ a similar strategy to obtain these results. Specifically, we will adapt the notion of cohomologous from the measure theoretic setting to classify roof functions.

Definition 1.8. [13, p. 717] Two roof functions r and s for a special flow over the same measure preserving transformation T are *cohomologous* if there is a measurable function g such that

$$r - s = g \circ T - g$$

Definition 1.9. Two functions $r\colon X \to \mathbb{R}$ and $s\colon X \to \mathbb{R}$ are said to be *cohomologous* if there exists a continuous function $g\colon X \to \mathbb{R}$ such that $r(x) - s(x) = g(f(x)) - g(x)$ for all $x \in X$. The function g is called a *transfer function.*

For the measure theoretic case, g is only required to be measurable. However, in the topological setting we require that g be continuous. This transfer function is important to how we compare roof functions, and finding a continuous transfer function for two roof functions can be extremely difficult.

Many of our results will hold for arbitrary functions on the base; however, we will use the existence of dense periodic points in the base to prove our main result, Theorem 4.9: a suspension flow is topologically mixing if and only if the roof function is not cohomologous to a constant. In addition, we will show that in this setting, the set of topologically mixing suspension flows is an open and dense set in Theorem 5.3. For clarity, the inclusion of dense periodic points will always be made in the statement of a result. While there is no universally accepted definition of chaos, the common definition attributed to Devaney includes a condition that a system must have dense periodic points [5, p. 50]. Hence, our main results apply to chaotic topologically mixing systems.

Chapter 2. Preliminaries

Here we state a handful of additional definitions that will provide the foundation for the more substantial results that will come later. We will also state several examples of systems that exhibit topological mixing or do not. Because symbolic dynamics contain spaces with a wide-variety of behaviors, they are a natural system to study for functions for the base of a suspension. We will provide the basics of symbolic dynamics and examples of systems with interesting topological mixing properties. We also include other basic examples to illustrate other kinds of systems and their topologically mixing behavior.

2.1 Definitions

Before we present any examples or results we will define several terms that will repeatedly appear throughout. These definitions will mainly focus on the structure of a system. In addition, we will give a rigorous definition of a suspension flow and the construction of a suspension from a flow.

Definition 2.1. Let $f\colon X \to X$ and let $x \in X$. The *positive semiorbit of x under f* is the set $\bigcup_{n\geq 0} f^n(x)$ and is denoted as $\mathcal{O}_f^+(x)$ or as $\mathcal{O}^+(x)$ if there is no confusion on the map.

If f is invertible, the *negative semiorbit of x under f* is the set $\bigcup_{n\leq 0} f^n(x)$ and is denoted as $\mathcal{O}_f^-(x)$ or as $\mathcal{O}^-(x)$. We write

$$f^{-n}(x) = f^{-1} \circ f^{-1} \circ \cdots \circ f^{-1}(x) \text{ (composing } n \text{ times).}$$

The *orbit of x under f* is the union of the positive and negative semiorbits and is denoted as $\mathcal{O}_f(x)$ or $\mathcal{O}(x)$.

Definition 2.2. Let $\varphi\colon X \times \mathbb{R} \to X$ be a flow and let $x \in X$. The *positive semiorbit of x under φ* is the set $\bigcup_{t\geq 0} \varphi^t(x)$ and is denoted as $\mathcal{O}_\varphi^+(x)$ or as $\mathcal{O}^+(x)$.

The *negative semiorbit of x under φ* is the set $\bigcup_{t\leq 0}\varphi^t(x)$ and is denoted as $\mathcal{O}^-_\varphi(x)$ or as $\mathcal{O}^-(x)$.

The *orbit of x under φ^t* is the union of the positive and negative semiorbits and is denoted as $\mathcal{O}_\varphi(x)$ or $\mathcal{O}(x)$.

The orbit of a point is the collection of all states that a point can attain in the past and future. The orbit of a point is a useful tool in analyzing the dynamics of a system as it can provide insights into the underlying structure of the system.

Definition 2.3. For a discrete time dynamical system, a point x is *periodic* if there exists an $N \geq 0$ such that $f^N(x) = x$. The smallest such N is the *period* of x.

Similarly, for a flow, a point x is *periodic* if there is a $t \in \mathbb{R}$ such that $t \geq 0$ and $\varphi^t(x) = x$. The smallest such t is the *period* of x.

For discrete time dynamical systems, periodic points have a finite orbit, and their existence will be important for us to obtain our main results. These orbits are useful because the information and dynamics on a periodic orbit are easier to use and understand than that of an infinite orbit. Poincaré state that periodic points "are the only breach by which we can penetrate a fortress hitherto considered inaccessible" [9, p. 284].

Infinite orbits can have varying and often unknown structure, but some infinite orbits can be very helpful.

Definition 2.4. A dynamical system defined on a topological space X is *transitive* if there exists a point $x \in X$ such that the positive semiorbit of x is dense in X. Equivalently, a continuous map of a compact metric space is *transitive* if for any non-empty open sets U and V, there exists a $n \geq 0$ such that $f^n(U) \cap V \neq \varnothing$ [8, p. 52].

The existence of a dense orbit means that our space cannot be expressed as a disjoint collection of open spaces that are invariant under the map [5, p. 49]. Transitivity is the topological analog to ergodicity as it provides a condition where the system cannot be decomposed. The existence of a dense orbit will prove to be a very useful tool in proving many of our results.

Topological mixing is a stronger property than transitivity, and topological mixing implies transitivity [2, p. 33]. Hence, any result we prove assuming transitivity will hold in the case for topological mixing. Although we are primarily concerned with topological mixing, many of our results will only rely on transitivity.

Definition 2.5. Two discrete dynamical systems $f\colon X \to X$ and $g\colon Y \to Y$ are *topologically conjugate* if there exists a homeomorphism $h\colon Y \to X$ such that $h \circ g = f \circ h$.

Similarly, a flow is transitive if there exists a point $x \in X$ such that the forward orbit is dense, or equivalently, if φ^t is a continuous flow on a compact metric space where for any two nonempty open sets U and V, there exists a $t > 0$ such that $\varphi^t(U) \cap V \neq \varnothing$. Two flows $\varphi\colon X \times \mathbb{R} \to X$ and $\psi\colon Y \times \mathbb{R} \to Y$ are *topologically conjugate* if there exists a homeomorphism $h\colon Y \to X$ such that $h(\psi(y,t)) = \varphi(h(y),t)$ for all $y \in Y$ and $t \in \mathbb{R}$.

Conjugate maps and flows have similar structures and properties including orbits structures and other topological properties. If we know that a flow is conjugate to another flow that is easier to study, then we restrict our attention to the flow that is easier to study because it will express the same properties.

The definition for conjugacy is more natural for discrete dynamical systems. For flows, there is a weaker and more natural definition for comparing two flows.

Definition 2.6. Two flows $\varphi\colon X \times \mathbb{R} \to X$ and $\psi\colon Y \times \mathbb{R} \to Y$ are *topologically equivalent* if there exists a homeomorphism $h\colon Y \to X$ that maps the orbits of ψ onto the orbits of φ and preserves the direction of time. That is, for all $y \in Y$ there exists a $\delta > 0$ such that if $0 < |s| < t \leq \delta$, and if s satisfies

$$h(\psi(y,t)) = \varphi(h(y),s)$$

then $s > 0$.

While this definition is more natural for flows, because it is weaker, in some instances, it no longer guarantees properties like topological mixing will be shared between two equivalent flows. Hence, for our purposes, we will mainly use topological conjugacy as a tool to compare flows.

With this groundwork, we are now ready to provide the formal definition of a suspension flow. In addition, we will demonstrate the process of obtaining a suspension flow from a flow.

Definition 2.7. [2, p. 21] Given a map $f\colon X \to X$, and a function $r\colon X \to \mathbb{R}^+$ bounded away from 0, consider the quotient space

$$M_r = \{(x,t) \in X \times \mathbb{R}^+\colon 0 \leq t \leq r(x)\}/\sim$$

where $\sim$ is the equivalence relation $(x, r(x)) \sim (f(x), 0)$. The *suspension* of f with *roof function* $r(x)$ is the flow $\varphi_t\colon M_r \to M_r$ by $\varphi_t(x,s) = (f^n(x), s')$, where n and s' satisfy

$$\sum_{j=0}^{n-1} r(f^j(x)) + s' = t + s, \qquad 0 \leq s' \leq r(f^n(x)).$$

We will refer to the set of points $\{(x,t)\colon 0 \leq t < r(x)\}$ as the *fiber* of x.

Definition 2.8. A *cross-section* of a flow $\varphi^t\colon X \to X$ is a subset $A \subset X$ such that the set $T_x = \{t \in \mathbb{R}\colon \varphi^t(x) \in A\}$ is a non-empty discrete subset of $\mathbb{R}$ for all $x \in X$.

Definition 2.9. If A is a cross section and $a \in A$, let $r(a) = \min T_a$. The function $r(a)$ is the called the *return time* to A and the *first return map* $f\colon A \to A$ by $f(a) = \varphi^{\tau(a)}(a)$. We define $f(a)$ to be the first point in the orbit of a under φ^t that intersects A. The first return map f is frequently called the *Poincaré map.* Note that f is a discrete dynamical system on A.

As discussed previously, given a flow $\varphi^t\colon X \to X$ there is a natural method of constructing a suspension flow from φ^t. We first choose a suitable cross section A. From this we can determine the first return map $f\colon A \to A$. The return time $r(a)$ is a function on A mapping to $\mathbb{R}^+$ and becomes the roof function for the suspension. We can determine many properties of φ^t by studying the properties of the suspension of f. If the system has a global cross section, rather than a local Poincaré section, the suspension flow is topologically mixing if and only if the original flow is topologically mixing [1].

The roof function plays a vital role in the behavior of the flow. In particular, the roof function influences whether a suspension flow is topologically mixing. In order to classify and study suspension flows we will inspect the roof functions.

2.2 Symbolic Dynamics

We now review a class of systems that provide useful examples of dynamical systems. Let $\mathcal{A}$ be some set. We refer to this set as the *alphabet* and the elements as *symbols*. The *full $\mathcal{A}$-shift* is the set $\mathcal{A}^{\mathbb{Z}} = \{(x_i)_{i\in\mathbb{Z}} \colon x_i \in \mathcal{A}\}$ and is the set of all two sided sequences whose terms are elements of $\mathcal{A}$. The set $\mathcal{A}$ can be infinite or finite; however, we are primarily concerned with finite alphabets. In the finite setting we let $\mathcal{A} = \{0, 1, \ldots N-1\}$, and we denote $\mathcal{A}^{\mathbb{Z}}$ by Σ_N. Elements of these spaces are of the form

$$x = \ldots x_{-2}x_{-1}x_0x_1x_2\ldots$$

When dealing with a specific point we use a period to identify where the zeroth term appears in the sequence. For example,

$$\ldots 1342.0394 \ldots$$

Here $x_{-2} = 4$, $x_{-1} = 2$, $x_0 = 0$, $x_1 = 3$, and so on. We call finite sequences of symbols *words*, and for $i \leq j$ we use the notation $x_{[i,j]}$ to denote the specific word $x_ix_{i+1}\ldots x_j$ that appears in the point x.

The set Σ_N can be endowed with the product topology and we define the *cylinder sets* to be

$$C^{n_1,\ldots,n_k}_{j_1,\ldots,j_k} = \{x = (x_l) \colon x_{n_i} = j_i \text{ and } i = 1, \ldots, k\}$$

where $n_1 < n_2 < \cdots < n_k$ are indices in $\mathbb{Z}$, and $j_i \in \mathcal{A}$. The cylinders form a basis for the topology. We may also endow a metric on this space that agrees with the topology.

$$d(x,y) = \begin{cases} 2^{-n} & \text{if } x \neq y \text{ and } n \text{ is maximal so that } x_{[-n,n]} = y_{[-n,n]} \\ 0 & \text{if } x = y. \end{cases}$$

Hence the distance between two points depends on how many symbols their central words agree upon.

Rather than consider two-sided sequences, we may study one-sided sequences instead. The construction and properties are defined in a similar manner. For a finite alphabet, we denote the space by Σ_N^+. It can also be endowed with the product topology, and the metric is defined as

$$d(x,y) = \begin{cases} 2^{-n} & \text{if } x \neq y \text{ and } n \text{ is maximal so that } x_{[0,n]} = y_{[0,n]} \\ 0 & \text{if } x = y. \end{cases}$$

We desire to study these spaces as dynamical systems. We naturally consider the shift map $\sigma \colon \Sigma_N \to \Sigma_N$ defined by

$$\sigma(\ldots x_{-2}x_{-1}.x_0x_1x_2\ldots) = \ldots x_{-1}x_0.x_1x_2x_3\ldots$$

Where σ simply shifts every term to the left. The behavior is similar for Σ_N^+, where the leading term of the sequence is deleted. Here $\sigma \colon \Sigma_N^+ \to \Sigma_N^+$, and we have that

$$\sigma(x_0x_1x_2\ldots) = x_1x_2x_3\ldots$$

In both settings, σ is a continuous function with the $d(\cdot,\cdot)$ metric. In addition, σ is an opening mapping on both Σ_N and Σ_N^+ because it maps cylinders to cylinders. Moreover, σ is a homeomorphism on Σ_N, but it is not on Σ_N^+ because it is not injective.

Some subsets of Σ_N can exhibit interesting dynamics, and we often study these subsets of Σ_N rather than the full-shift itself. A *subshift* is a subset of Σ_N or Σ_N^+ that is closed with respect to the topology and invariant under σ. That is, if $A \subseteq \Sigma_N$, then $\sigma(A) = A$. Subshifts can be constructed in a variety of different ways. They are often constructed by establishing a set of *forbidden words*, words that are not allowed appear in any element of the subshift. Alternatively, one can establish a set of *permissible words* and generate a subshift in an analogous fashion by including all possible sequences containing these words. We denote the set of forbidden words by $\mathcal{F}$, the set of permissible words as $\mathcal{B}(X)$, and $\mathcal{B}_n(X)$ is the set of permissible words of length n. Carefully constructing subshifts can yield systems with rich and astonishing dynamics.

Example 2.10. When we consider all of Σ_N, the function σ is topologically mixing on this space. We can see this by using the metric or by examining the behavior of σ on cylinders. However, for shift spaces, there is an equivalent definition for topological mixing [16, p. 129, 191].

Definition 2.11. A subshift X is *mixing* if, for every pair $u, v \in \mathcal{B}(X)$, there is an N such that for each $n \geq N$ there is a word $w \in \mathcal{B}_n(X)$ such that $uwv \in \mathcal{B}(X)$.

Since every word is in Σ_N we can easily see that σ is topologically mixing. We apply this result to Σ_N^+ and deduce that σ is topologically mixing on Σ_N^+ as well.

Example 2.12. One of the most commonly studied shifts is Σ_2^+, sequences of zeros and ones. Note that Σ_2^+ is homeomorphic to the Cantor set obtained by removing the middle third. This allows us to better understand the topological structure on Σ_2^+. For example, this tells us that Σ_2^+ is compact, totally disconnected, and contains no isolated points. Subshifts can inherit many of these properties, but we can see that all subshifts are compact because they are closed subsets of a compact space.

While the alphabet only contains two symbols, subshifts of Σ_2^+ can possess interesting dynamics. They can have ergodic or mixing properties. Subshifts can be constructed to only have periodic points of even period [6]. The majority of the following examples will be subshifts of Σ_2^+, and we will see what other kinds of behavior these subshifts can have. Since these subshifts can be constructed to exhibit a variety of different behaviors with only two symbols, Σ_2^+ and Σ_2 are often the preferred spaces to consider when working in symbolic dynamics.

Example 2.13. A *subshift of finite type* is a shift space that can be described by a finite set of forbidden words [16, p. 28]. That is $\mathcal{F}$ is finite. Subshifts of finite type can also be described with a directed graph and directed walks on the graph. Suppose X is the collection of all possible infinite walks moving from one vertex to another on a finite directed graph. This collection can also be thought of as set of all possible infinite sequences of edges. This set is closed and invariant, so it can be thought of as a shift, and is commonly called an *edge shift*. Every subshift of finite type is isomorphic to an edge shift and every edge shift is isomorphic to a subshift of finite type [2, p. 57]. Hence, a subshift of finite type may be characterized in multiple ways.

Because subshifts of finite type correspond to some sort of graph, we can analyze the adjacency matrix for the graph to determine the properties of the subshift. In particular, it has been shown that a subshift of finite type is topologically mixing if and only if the adjacency matrix A has the property that there exists a positive integer m_0 such that for all $i, j \in \{1, 2, \ldots, n\}$ such that $A_{ij}^m > 0$ for all $m \in \mathbb{Z}$ with $m > m_0$. Additionally, a subshift of finite type is transitive if and only if the corresponding adjacency matrix A is irreducible [11]. Strengthening the hypothesis, we have that the shift map on a subshift of finite type is topologically mixing if the adjacency matrix for the isomorphic edge shift is irreducible and the trace is nonzero [10]. Thus we see that for subshifts of finite type, given the adjacency matrix, there are multiple methods to determine whether a subshift of finite type is topologically mixing.

Definition 2.14. A dynamical system $f\colon X \to X$ is *minimal* if X does not contain any non-empty, proper, closed, f-invariant subset.

There are other equivalent definitions including the characterization that every orbit must be dense in X. A simple example of a minimal dynamical system is the orbit of a periodic point. Since there is only one orbit, there is only one invariant set. Minimal dynamical systems are important systems because every dynamical system is minimal or it contains a minimal system.

Example 2.15. For shift spaces, a minimal shift contains no proper subshift. Minimal shifts can be constructed to possess much more complex structure than a periodic orbit.

Definition 2.16. A point x is an *almost periodic point*, if for any neighborhood U of x, there exists an $N \in \mathbb{N}$ such that $\{f^{n+i}(x)\colon i = 0, 1, \ldots, N\} \cap U \neq \varnothing$ for all $n \in \mathbb{Z}^+$.

An almost periodic point returns near its initial position with bounded gaps in time. A shift space is minimal if and only if it is the closure of an almost periodic orbit [16, p. 457].

An example of a minimal shift with greater complexity than a periodic orbit is the Morse shift. The Morse shift is constructed from the Morse sequence.

$$x = 0110100110010110100101100110100 1\ldots$$

We use this sequence to construct a two-sided sequence and from there construct a shift space. Because this sequence is almost periodic the closure of its orbit yields a minimal shift.

The Morse sequence can be constructed in a variety of different manners. One method is by letting $B_0 = 0$ and recursively defining the words $B_{n+1} = B_n\overline{B_n}$, where $\overline{B_n}$ denotes the word where the 0's of B_n are replaced with 1's, and the 1's of B_n are replaced with 0's.

Another way to construct this sequence is by repeating a substitution where a 0 is replaced with 01 and 1 is replaced with 10. This kind of construction can be done in the general setting with more symbols and yields what are called substitution shifts. In the measure theoretic setting, it has been shown that substitution shifts are not measure theoretically strongly mixing [4]. Hence, some substitution shifts may be topologically mixing, but they are not strongly mixing.

Minimal shifts like the Morse shift contain no periodic orbits, and every orbit is dense in the space. So if such a system is the base of a suspension, we can determine many facts about the roof function by examining it along any orbit.

Example 2.17. In [17], a topologically mixing minimal shift is constructed. This shift has a somewhat stronger mixing property, in that for any open sets U and V, there exists an N such that $\sigma^n(U) \cap V \neq \varnothing$ for $|n| \geq N$. While we do not address topological entropy, this shift also exhibits zero topological entropy. There are many other examples of topologically mixing minimal shifts, and they often possess other interesting dynamical properties.

Example 2.18. Our main results hold when the periodic points are dense in the base. Because subshifts can be topologically mixing and not have dense periodic points they are a natural setting to investigate suspensions with less structure. We desire to know if a counterexample exists by constructing a roof function over a minimal topologically mixing base such that the suspension is not topologically mixing, but the roof function is not cohomologous to a constant. Another possible setting to consider is a base with a finite number of fixed points. There is an example of a shift space with only a fixed point, but is only weakly topologically mixing in [15]. Here we present the construction of a topologically mixing subshift with one fixed point and a period two orbit. This construction was inspired by the structure of the weakly topologically mixing shift and the construction in [7], but echoes many of the ideas in [17].

We will construct a subshift of Σ_2, but we will begin with an element a in Σ_N^+ that is almost periodic. We also require that a not be periodic and that $a_n \in \{3, 4, \ldots N\}$ for all $n \in \mathbb{N}$. In addition, because a is almost periodic $\mathcal{O}^+(a)$ does not limit on any periodic points. We will now think of a more as a sequence now rather than a point in Σ_N^+.

Let $u_1 = 10$, $u_2 = 1010$, $u_3 = 101010$, and so on. For $i \geq 1$, define $A_i = \{1^\beta, u_1, u_2, \ldots, u_i\}$ where $\beta > N$. Define $B_i = \{x \in \Sigma_2^+ \colon x = v_1 0^{a_1} v_2 0^{a_2} v_3 0^{a_3} \ldots \text{ where } v_j \in A_i\}$.

$$\text{Let } J_i = \bigcup_{k=0}^{\infty} \sigma^k(B_i).$$

Clearly $J_i \subseteq J_{i+1}$ for all $i \in \mathbb{N}$. Let $S = \overline{\bigcup_{i=1}^\infty J_i}$. S is clearly closed. Let $x \in \bigcup_{i=1}^\infty J_i$ with $x = x_1x_2x_3\ldots$ then the point $x' = x_0x_1x_2x_3 \cdots \in \bigcup_{i=1}^\infty J_i$ where $x_0 = 0$ if $x_1x_2 = 10$, or $x_0 = 1$ if $x_1x_2 = 11$ or 01. The only complication occurs when $x_1x_2 = 00$, since we have some restrictions on the lengths of words of zeros. This pair of zeros must be in some word of zero 0^{a_i} for some i. By construction each of these words are preceded by a word of repeating ones of 01 terms. In the instance that x_0 cannot be zero, then it must be a one. If x_0 cannot be one, then it must be a zero. Hence, for all $x \in \bigcup_{i=1}^\infty J_i$, there exists $x' \in \bigcup_{i=1}^\infty J_i$ such that $\sigma(x') = x$. By continuity of σ, we obtain that for all $x \in S$, there exists $x' \in S$ such that $\sigma(x') = x$. Hence $\sigma(S) = S$, so S is invariant and S is a subshift of Σ_2^+. Note further that by construction $\bigcup_{i=1}^\infty J_i$ is dense in S, and we will use this fact repeatedly.

Because of our choice of the sequence a the words of zeros do not appear in any periodic fashion with the exception of the u_i and the repeating 1s. This ensures that the only periodic orbits we can limit on are 1^∞ and $(01)^\infty$. So S only contains two periodic orbits.

Proposition 2.19. *σ is topologically mixing on S.*

Proof. Fix $N \in \mathbb{N}$. Let $x = v_1 0^{a_1} v_2 0^{a_2} \ldots$ and $y = w_1 0^{a_1} w_2 0^{a_2} \ldots$. By construction w_1 is either a word of repeating 01 terms or a word of repeating 1 terms. Without loss of generality, suppose that w_1 is a repeating word of 01 terms. Let u be the word of length $N' \geq N$ such that $x = ux_{N+1}x_{N+2}\ldots$ and such that the point $x' = u(01)^\infty$ is contained in $B(x, 2^{-N})$. Thus $\sigma^{N'}(x') = (01)^\infty$.

Since σ is an open map, there exists M sufficiently large, such that $B((01)^\infty, 2^{-M}) \subseteq \sigma^{N'}(B(x, 2^{-N}))$. For all $n \geq M$ there exist $z_n \in \bigcup_{i=1}^\infty J_i$ such that $z_n = (01)^n y$. Furthermore, $\sigma^{-N'}(z_n) \cap B(x, 2^{-N}) \neq \varnothing$. Hence, for all $n \geq M$, there exists $x_n \in B(x, 2^{-N})$ such that $\sigma^{N'}(x_n) = z_n$ and $\sigma^{N'+n}(x_n) = y_n$, so we may concluded that for all $t > N' + M$, $\sigma^t(B(x, 2^{-N})) \cap B(y, 2^{-N}) \neq \varnothing$.

This holds for arbitrary x and y. It clearly extends to any pair of points from $\mathcal{O}^+(x)$ and $\mathcal{O}^+(y)$. Since this holds for all point in $\bigcup_{i=1}^{\infty} J_i$ and because $\bigcup_{i=1}^{\infty} J_i$ is dense, it follow that σ is topologically mixing on all of S. □

From here we will use an inverse limit to construct X, a subshift of the full shift Σ_2. If X contained any other periodic points, S would also contain them as well. Because S only contains the fixed point and the period two orbit, X only contains these two orbits. Additionally, since σ is topologically mixing, it follows that for any permissible words u, v there exists an N such that for $n \geq N$ there is a word $w \in \mathcal{B}_n(S)$ such that $uwv \in \mathcal{B}(S)$. Every finite permissible word in X is also a finite permissible word of S, so we get that σ is topologically mixing on X. This shows that there exists a topologically mixing homeomorphism with exactly two periodic orbits.

2.3 Further Examples

Example 2.20. Let $f\colon X \to X$ be any discrete dynamical system. Let $r\colon X \to \mathbb{R}^+$ be constant. That is for all $x \in X$, $r(x) = k$ for some $k > 0$. We will show that the suspension over a constant roof function is not topologically mixing.

Proposition 2.21. *The suspension φ^t over f with roof function r is not topologically mixing.*

Proof. Let U and V be open sets of X. Let $U' = U \times (0, k/4)$ and $V' = V \times (0, k/4)$ open sets in the domain of φ^t. Since the roof function is constant, for any $a \in (0, k/4)$, the subset $U \times \{a\}$ of U' is identified to $f(U) \times 0$ simultaneously when $t = k - a$. Hence the vertical height of U' under the flow will always be of length $k/4$. Now, suppose there exists some T such that $\varphi^T(U') \cap V' \neq \varnothing$. Then $\varphi^{T+k/4}(U) \cap V = \varnothing$. φ^t is not topologically mixing. □

Suspensions with constant roof functions are not topologically mixing regardless of the system in the base. Even if f had been topologically mixing, the flow would not. However, because topological mixing implies transitivity, the suspension of φ^t would still be transitive. Thus we see that the suspension will reflect the same orbit structure as the base, but does not necessarily mimic other properties like topological mixing.

Example 2.22. The expanding map $E_m \colon S^1 \to S^1$ defined by $E_m(x) = mx \bmod 1$ for some $m > 1$. Although E_m is not a homeomorphism it is a topologically mixing discrete dynamical system. Let U be any open interval (a, b). The length of U is $b - a$. The length of $E_m(U)$ is $m(b - a)$, and in general, the length of $E_m^n(U)$ is $m^n(b - a)$. Since $m > 1$, there must exist an N such that for $n \geq N$, length of $E_m^n(U) > 1$. This implies that $S^1 \subseteq E_m^n(U)$. Thus for any pair of open sets U, V, there exists an N such that for $n \geq N$, $E_m^n(U) \cap V \neq \varnothing$.

We now prove some essential results that we will need throughout. The majority of these results will be about characteristics of cohomologous roof functions. If two roof functions are cohomologous we would like to know what they have in common. In particular, we would like to know what the relationship between the suspension flows of such roof functions may be. Furthermore, we would like to classify and provide a method to determine when two given roof functions are cohomologous.

To begin, we show that if two roof functions are cohomologous then the suspension flows are conjugate. However, we will first need the following lemma.

Lemma 3.1. *Let $M \in \mathbb{R}$, $M > 0$, $f \colon X \to X$ be a homeomorphism of a compact metric space, and $r \colon X \to \mathbb{R}$ be a roof function. For all $x \in X$ there exists a unique smallest N such that $\sum_{i=0}^{N} r(f^i(x)) > M$.*

Proof. M is finite and $r(f^i(x)) > 0$ for all x and all i and bounded from below. Hence we can find j$\in \mathbb{N}$ such that $\sum_{i=0}^{j} r(f^i(x)) > M$. Let A be the set of all such integers that satisfy this property. This is a set of positive integers, so it contains a smallest element N that is unique. □

Proposition 3.2. *Let $f \colon X \to X$ be a homeomorphism of a compact metric space. Let $r_1 \colon X \to \mathbb{R}$ be a roof function cohomologous to the function $r_2 \colon X \to \mathbb{R}$ by a transfer function $g \colon X \to \mathbb{R}$ such that $r_1(x) - r_2(x) = g(f(x)) - g(x)$. Let $\varphi_i \colon M_i \to M_i$ be the flow over r_i for $i \in \{1, 2\}$, then φ_{r_1} and φ_{r_2} are conjugate.*

Proof. Since r_1 and r_2 are cohomologous we have that $g(f(x)) - r_1(x) = g(x) - r_2(x)$. Define $h \colon M_1 \to M_2$ by

$$h(x,t) = \left(f^N(x), t + g(x) - \sum_{i=0}^{N-1} r_2(f^i(x)) \right)$$

where N is the smallest integer satisfying $t + g(x) < \sum_{i=0}^{N} r_2(f^i(x))$, where $t + g(x) > 0$. If $t + g(x) < 0$, we can find the smallest n such that $-\sum_{i=0}^{n} r_2(f^{-i}(x)) < t + g(x))$ and $h(x,t) = \big(f^{-n}(x), -t - g(x) - \sum_{i=0}^{n-1} r_2(f^i(x))\big)$.

We consider the case $h \circ \varphi_{r_1}\big(\overline{x}, s\big)$ where $\overline{x} = (x,t)$ and $t + g(x) > 0$. Here

$$h \circ \varphi(\overline{x}, s) = h \circ \varphi_{r_1}\big((x,t), s\big) = h\Bigg(f^a(x), s + t - \sum_{i=0}^{a-1} r_1(f^i(x))\Bigg)$$

$$= \Bigg(f^{a+k}(x), s + t - \sum_{i=0}^{a-1} r_1(f^i(x)) + g(f^a(x)) - \sum_{i=0}^{k-1} r_2(f^{a+i}(x))\Bigg)$$

$$= \Bigg(f^{a+k}(x), s + t - \sum_{i=0}^{a-2} r_1(f^i(x)) - r_1(f^{a-1}(x)) + g(f^a(x)) - \sum_{i=0}^{k-1} r_2(f^{a+i}(x))\Bigg)$$

$$= \Bigg(f^{a+k}(x), s + t - \sum_{i=0}^{a-2} r_1(f^i(x)) + g(f^{a-1}(x)) - r_2(f^{a-1}(x)) - \sum_{i=0}^{k-1} r_2(f^{a+i}(x))\Bigg)$$

$$= \Bigg(f^{a+k}(x), s + t + g(x) - \sum_{i=0}^{a-1} r_2(f^i(x)) - \sum_{i=0}^{k-1} r_2(f^{a+i}(x))\Bigg)$$

$$= \Bigg(f^{a+k}(x), s + t + g(x) - \sum_{i=0}^{a-1} r_2(f^i(x)) - \sum_{i=a}^{a+k-1} r_2(f^i(x))\Bigg)$$

$$= \Bigg(f^{a+k}(x), s + t + g(x) - \sum_{i=0}^{a+k-1} r_2(f^i(x))\Bigg)$$

where a, k are chosen so that $a + k$ is the smallest integer satisfying $\sum_{i=0}^{a+k} r_2(f^i(x)) > s + t + g(x)$.

We now consider $\varphi_{r_2}(h(\overline{x}), s)$. In this case,

$$\varphi_{r_2}(h(x,t), s) = \varphi_{r_2}\Bigg(\Bigg(f^N(x), t + g(x) - \sum_{i=0}^{N-1} r_2(f^i(x))\Bigg), s\Bigg)$$

$$= \Bigg(f^{N+M}(x), s + t + g(x) - \sum_{i=0}^{N-1} r_2(f^i(x)) - \sum_{i=0}^{M-1} r_2(f^{N+i}(x))\Bigg)$$

$$= \left(f^{N+M}(x), s + t + g(x) - \sum_{i=0}^{N-1} r_2(f^i(x)) - \sum_{i=N}^{N+M-1} r_2(f^i(x)) \right)$$

$$= \left(f^{N+M}(x), s + t + g(x) - \sum_{i=0}^{N+M-1} r_2(f^i(x)) \right)$$

where N and M are chosen so that $N + M$ is the smallest integer satisfying $\sum_{i=0}^{N+M} r_2(f^i(x)) > s + t + g(x)$.

By uniqueness, this implies that $a + k = N + M$, which means that $h \circ \varphi_{r_1} = \varphi_{r_2} \circ h$. A similar computation holds for when $t + g(x) < 0$. The flows φ_{r_1} and φ_{r_2} are conjugate. $\square$

Conjugate flows share the same topological properties, so knowing that two roof functions are cohomologous tells us that the suspensions have the same topological properties. If we know the behavior of a suspension for a certain roof function, then we can determine the behavior of another suspension if the roof functions are cohomologous. This can aid in the study of a system, because understanding the dynamics simplifies to knowing what kind of roof function the system has. The following is a standard result; we provide a proof for completeness.

Proposition 3.3. *Let $\varphi^t \colon X \to X$ and $\psi^t \colon Y \to Y$ be topologically conjugate flows. If φ^t is topologically mixing, then ψ^t is topologically mixing.*

Proof. Since φ and ψ are topologically conjugate, there exists a homeomorphism $h \colon Y \to X$ such that $h(\psi^t(y)) = \varphi^t(h(y))$. Let U and V be open sets in Y. $h(U)$ and $h(V)$ are open sets in X. There exists T such that for $t > T$ we have that $\varphi^t(h(U)) \cap h(V) \neq \varnothing$.

$\varphi^t(h(U)) = h(\psi^t(U))$ by definition. Plugging into h^{-1} we have the following.

$$\varnothing \neq h^{-1}(\varphi^t(h(U)) \cap h(V)) = h^{-1}(h(\psi^t(U)) \cap h(V)) = \psi^t(U) \cap V$$

ψ^t is topologically mixing. $\square$

Remark. By Example 2.20, Proposition 3.2, and this proposition, it follows that any suspension flow with roof function cohomologous to a constant is not topologically mixing. This is one direction of our main result. The converse is not as simple to establish. We will continue to focus on the roof function to create the foundation to prove the converse.

To aid us in the study of roof functions we may partition them into equivalence classes by showing that the cohomologous condition is an equivalence relation. This will greatly simplify the study of roof functions to equivalence classes.

Proposition 3.4. *The cohomologous condition is an equivalence relation.*

Proof. Let $f\colon X \to X$ be a base function and $r\colon X \to \mathbb{R}$, $s\colon X \to \mathbb{R}$, and $t\colon X \to \mathbb{R}$ be continuous roof functions.

Reflexivity: Let $g\colon X \to \mathbb{R}$ be any constant function.

$$r(x) - r(x) = 0 = g(f(x)) - g(x)$$

$r(x)$ is cohomologous to $r(x)$

Symmetry: Let $r(x)$ be cohomologous to $s(x)$. There exists a continuous function $g(x)$ such that $r(x) - s(x) = g(f(x)) - g(x)$. Define $G(x) = -g(x)$. $G(x)$ is continuous.

$$G(f(x)) - G(x) = -g(f(x)) + g(x) = -r(x) + s(x) = s(x) - r(x$$

$s(x)$ is cohomologous to $r(x)$.

Transitivity: Assume that $r(x)$ is cohomologous to $s(x)$ and $s(x)$ is cohomologous to $t(x)$. We show that $r(x)$ is cohomologous to $t(x)$. There exist continuous functions $g\colon X \to \mathbb{R}$ and $h\colon X \to \mathbb{R}$ such that $r(x) - s(x) = g(f(x)) - g(x)$ and $s(x) - t(x) = h(f(x)) - h(x)$. Define $G(x) = g(x) + h(x)$. $G(x)$ is continuous.

$$G(f(x)) - G(x) = g(f(x)) + h(f(x)) - g(x) - h(x) = g(f(x)) - g(x) + h(f(x)) - h(x)$$

$$= r(x) - s(x) + s(x) - t(x) = r(x) - t(x)$$

$r(x)$ is cohomologous to $t(x)$. □

Since we have shown that cohomologous roof functions induce conjugate flows, we may choose any desirable roof function from an equivalence class to determine the topological dynamics of the suspension flows with roofs in the same class. We know that roof functions cohomologous to a constant are not topologically mixing, but we do not know if a suspension that is not topologically mixing has a roof function that is cohomologous to a constant.

Now that we have shown that roof functions can be organized into equivalence classes, the natural next step is to determine whether two roof functions are elements of the same class. In this case, it depends on whether a transfer function exists. An important initial result shows the uniqueness of such a transfer function.

Proposition 3.5. *Let $f\colon X \to X$ be a transitive homeomorphism on a compact metric space. If $r(x)$ and $s(x)$ are continuous cohomologous roof functions, then the transfer function $g\colon X \to \mathbb{R}$ satisfying $r(x) - s(x) = g(f(x)) - g(x)$ is unique up to a constant.*

Proof. There exists a point $x_0 \in X$ that has a dense orbit in X under f by transitivity. For simplicity, let $g(x_0) = 0$. Suppose there exists another continuous function h such that $r(x) - s(x) = h(f(x)) - h(x)$. Denote $h(x_0) = k$ for $k \in \mathbb{R}$. So we have

$$h(f(x_0)) - h(x_0) = r(x_0) - s(x_0) = g(f(x_0)) - g(x_0) = g(f(x_0)),$$

and this implies

$$h(f(x_0)) - k = h(f(x_0)) - h(x_0) = g(f(x_0)).$$

So $h(f(x_0))$ and $g(f(x_0))$ differ by k and $-k = g(f(x_0)) - h(f(x_0))$.

Now consider

$$r(f(x_0)) - s(f(x_0)) = h(f^2(x_0)) - h(f(x_0)) = r(f(x_0)) - s(f(x_0)) = g(f^2(x_0)) - g(f(x_0))$$

$$h(f^2(x_0)) - h(f(x_0)) = g(f^2(x_0)) - g(f(x_0))$$

$$g(f^2(x_0)) = h(f^2(x_0)) - h(f(x_0)) + g(f(x_0)) = h(f^2(x_0)) - k.$$

Once again, $g(f^2(x_0))$ and $h(f^2(x_0))$ differ by k. We proceed by induction, let $j \geq 2$ and we assume that $-k = g(f^j(x_0)) - h(f^j(x_0))$. We show this to be true for $j+1$.

Following a similar procedure we have

$$h(f^{j+1}(x_0)) - h(f^j(x_0)) = r(f^{j+1}(x_0)) - s(f^{j+1}(x_0)) = g(f^{j+1}(x_0)) - g(f^j(x_0))$$

$$h(f^{j+1}(x_0)) - h(f^j(x_0)) = g(f^{j+1}(x_0)) - g(f^j(x_0))$$

$$h(f^{j+1}(x_0)) - k = h(f^{j+1}(x_0)) - h(f^j(x_0)) + g(f^j(x_0)) = g(f^{j+1}(x_0)).$$

So $h(f^{j+1}(x_0)) - k = g(f^{j+1}(x_0))$, and by mathematical induction for all $n \in \mathbb{N}$ we have that $g(x)$ and $h(x)$ differ by the same constant k on $\mathcal{O}^+(x_0)$. So $g(x)$ and $h(x) - k$ agree on a set that is dense in X. By continuity of $g(x)$ and $h(x) - k$ we have that $g(x) = h(x) - k$ for all $x \in X$. Therefore, g is unique up to a constant. $\square$

Note that given two cohomologous roof functions, $r(x)$ and $s(x)$ and a transfer function $g(x)$ satisfying $r(x) - s(x) = g(f(x)) - g(x)$, the function $G(x) = -g(x)$ is also a satisfactory transfer function. However, it would satisfy $s(x) - r(x) = -g(f(x)) + g(x) = G(f(x)) - G(x)$.

Here we provide a simple test to show that two roof functions are not cohomologous to one another.

Proposition 3.6. *Let $f\colon X \to X$ be a continuous base function. Let $r\colon X \to \mathbb{R}$ and $s\colon X \to \mathbb{R}$ be roof functions. Suppose x_0 is a periodic point with period p. If*

$$\sum_{j=0}^{p-1} r(f^j(x_0)) - s(f^j(x_0)) \neq 0$$

then $r(x)$ and $s(x)$ are not cohomologous.

Proof. Suppose that $r(x)$ and $s(x)$ are cohomologous. Then there exists a function $g\colon X \to \mathbb{R}$ such that $r(x) - s(x) = g(f(x)) - g(x)$. Note that

$$r(f(x)) - s(f(x)) + r(x) - s(x) = g(f^2(x)) - g(f(x)) + g(f(x)) - g(x) = g^2(f(x)) - g(x).$$

In general, we have the identity

$$\sum_{j=0}^{n-1} r(f^j(x_0)) - s(f^j(x_0)) = g(f^n(x)) - g(x).$$

Now consider x_0. By hypothesis we have

$$0 \neq \sum_{j=0}^{p-1} r(f^j(x_0)) - s(f^j(x_0)) = g(f^p(x_0)) - g(x_0) = g(x_0) - g(x_0) = 0,$$

which is a contradiction. The functions $r(x)$ and $s(x)$ are therefore not cohomologous. $\square$

The identity we used in this proof will be used repeatedly to prove many results. It can be very difficult to determine whether two roof functions are cohomologous; however, this proposition provides a simple way to show that two roof functions are not cohomologous. It is worth noting that this also holds for averages. That is, if $r(x)$ and $s(x)$ are cohomologous, then for a periodic point x_0 with period p,

$$\frac{1}{p}\sum_{j=0}^{p-1} r(f^j(x_0)) = \frac{1}{p}\sum_{j=0}^{p-1} s(f^j(x_0)).$$

Proposition 3.7. *[8, p. 239] Let $f\colon X \to X$ be a continuous function on a compact metric space X. Suppose there exists a point x_0 such that $\overline{\mathcal{O}^+(x_0)} = X$ and let $r\colon X \to \mathbb{R}$ and $s\colon X \to \mathbb{R}$ be continuous roof functions. If there is a function $g\colon \mathcal{O}^+(x_0) \to \mathbb{R}$ that is uniformly continuous and satisfies $r(x) - s(x) = g(f(x)) - g(x)$, then $r(x)$ and $s(x)$ are cohomologous.*

There are a variety of ways to determine whether two arbitrary functions are cohomologous or not. Being able to a construct a transfer function, in contrast, can be very difficult. However, the following result shows that certain conditions guarantee the existence or nonexistence of a transfer function. This result will be repeatedly used, and is a vital result for proving the converse of our main theorems. The drawback of this result is that it is not a practical method for constructing a roof function in application because it deals with an infinite sum.

Proposition 3.8. *Let $f\colon X \to X$ be a continuous transitive map on a compact metric space. Let $r\colon X \to \mathbb{R}$ and $s\colon X \to \mathbb{R}$. If $\overline{\mathcal{O}^+(x_0)} = X$ and the sum*

$$\sum_{j=0}^{\infty} r(f^j(x_0)) - s(f^j(x_0))$$

is bounded, then $r(x)$ and $s(x)$ are cohomologous.

Proof. There exists a point x_0 such that $\overline{\mathcal{O}^+(x_0)} = X$ by transitivity. Furthermore, there exists a subsequence such that the limit

$$\lim_{n\to\infty} \sum_{j=0}^{a_n-1} r(f^j(x_0)) - s(f^j(x_0))$$

exists because $\displaystyle\sum_{j=0}^{\infty} r(f^j(x_0)) - s(f^j(x_0))$ is bounded.

Define $g_n\colon \mathcal{O}^+(x_0) \to \mathbb{R}$ by $g_n(x) = \displaystyle\sum_{j=0}^{a_n-1} r(f^j(x)) - s(f^j(x))$. We know, by construction that $\lim_{n\to\infty} g_n(x_0)$ exists. That is, for all $\epsilon > 0$ there exists an N such that for all $n \geq N$ we have that $|g_n(x_0) - g(x_0)| < \epsilon$. We show that $g_n(x)$ converges for all points in the positive orbit of x_0. In addition, we show that the necessary N is the same for all points.

Consider

$$g_n(f(x_0)) = \sum_{j=0}^{a_n-1} r(f^j(f(x_0))) - s(f^j(f(x_0)))$$

$$= s(x_0) - r(x_0) + \sum_{j=0}^{a_n-1} r(f^j(x_0)) - s(f^j(x_0)) = s(x_0) - r(x_0) + g_n(x_0)$$

Hence the convergence of $g_n(f(x_0))$ depends only on the convergence of $g_n(x_0)$. Thus $g_n(f(x_0))$ converges and for $n \geq N$ we have that $|g_n(f(x_0)) - g(f(x_0))| < \epsilon$. The necessary value for N is the same.

We proceed by induction and assume that $g_n(f^k(x_0))$ converges and that for $n \geq N$ we have that $|g_n(f^k(x_0)) - g(f^k(x_0))| < \epsilon$. We show that this holds for the $k+1$ case.

$$g_n(f^k(x_0)) = \sum_{j=0}^{a_n-1} r(f^j(f^{k+1}(x_0))) - s(f^j(f^{k+1}(x_0)))$$

$$= s(f^{k+1}(x_0)) - r(f^{k+1}(x_0)) + \sum_{j=0}^{a_n-1} r(f^j(f^k(x_0))) - s(f^j(f^k(x_0)))$$

$$= s(f^{k+1}(x_0)) - r(f^{k+1}(x_0)) + g_n(f^k(x_0))$$

The convergence depends only on that of $g_n(f^k(x_0))$. By the inductive hypothesis it follows that $g_n(f^{k+1}(x_0))$ converges and the necessary N is the same as that of $g_n(x_0)$. So by induction we have that for all $\epsilon > 0$ there exists an N such that for all $n \geq N$ and all $x \in \mathcal{O}^+(x_0)$ we have that $|g_n(x) - g(x)| < \epsilon$, so g_n converges uniformly by definition.

We show that $s(x) - r(x) = g(f(x)) - g(x)$ on $\mathcal{O}^+(x_0)$.

$$g(f(x)) - g(x) = \lim_{n\to\infty} g_n(f(x)) - \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} g_n(f(x)) - g_n(x)$$

$$= \lim_{n\to\infty} \sum_{j=0}^{a_n-1} r(f^j(f(x))) - s(f^j(f(x))) - \sum_{j=0}^{a_n-1} r(f^j(x)) - s(f^j(x))$$

$$= \lim_{n\to\infty} s(x) - r(x) + \sum_{j=0}^{a_n-1} r(f^j(x)) - s(f^j(x)) - \sum_{j=0}^{a_n-1} r(f^j(x)) - s(f^j(x))$$

$$= \lim_{n\to\infty} s(x) - r(x) = s(x) - r(x)$$

It readily follows that $g(f^k(x)) - g(x) = \sum_{j=0}^{k-1} s(f^j(x)) - r(f^j(x))$.

Finally, we show that $g(x)$ is uniformly continuous on $\mathcal{O}^+(x_0)$, which is dense in X. By uniform convergence there exists an N' such that $|g_{N'}(x) - g(x)| < \epsilon/3$ for all $x \in \mathcal{O}^+(x_0)$. In addition, $g_{N'}$ is a finite sum of uniformly continuous functions. Thus it too is uniformly continuous. Thus there exists some $\delta > 0$ such that for all x, y where $d(x, y) < \delta$ implies that $|g_{N'}(x) - g_{N'}(y)| < \epsilon/3$ and d is the metric on X. Now let x, y satisfy $d(x, y) < \delta$

$$|g(x) - g(y)| = |g(x) - g_{N'}(x) + g_{N'}(x) - g_{N'}(y) + g_{N'}(y) - g(y)|$$

$$\leq |g(x) - g_{N'}(x)| + |g_{N'}(x) - g_{N'}(y)| + |g_{N'}(y) - g(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Hence, $g(x)$ is uniformly continuous. So, there exists a continuous extension of g from $\mathcal{O}^+(x_0)$ to a continuous function $\hat{g}(x)$ on all of X, where $\hat{g}$ satisfies $\hat{g}(f(x)) - \hat{g}(x) = s(x) - r(x)$ by Proposition 3.7. Therefore, $r(x)$ and $s(x)$ are cohomologous by definition. $\square$

Chapter 4. Main Result I

Our first main result shows that a suspension flow is topologically mixing if and only if the roof function is cohomologous to a constant provided that the base is topologically mixing and periodic points are dense. To establish this dichotomy, Proposition 3.8 will prove to be a key result throughout this section because we will need to construct or prove the existence of transfer functions based on the behavior of the flow. To find candidates for transfer functions, we will need certain structure in the base. We will need the base to be transitive and have dense periodic points. Since topological mixing in the base is required to obtain topologically mixing in the flow, this is a property that would be necessary in the general setting. However, the added assumption of dense periodic points makes our hypothesis stronger than that of the most general setting.

4.1 Shearing

Here we introduce the notion of shearing. Let $r\colon X \to \mathbb{R}$ be some roof function. Let $\varphi\colon M_r \to M_r$ be a flow under r. Let U be an open set in the domain of φ. We choose two distinct points $\hat{x}, \hat{y} \in U$ such that $\hat{x} = (x, a)$ and $\hat{y} = (y, a)$ where $x, y \in X$ and $a \in \mathbb{R}$.

We desire to approximate the change in vertical position caused by φ_r. Initially, they have the same vertical height a, but as $\hat{x}, \hat{y}$ flow under φ we can calculate the change in vertical position by the difference $r(x) - r(y)$ once both points have flowed out of the initial fibers and into the $f(x)$ and $f(y)$ fibers. We can continue this process to determine the change in vertical positioning for the n-th iteration by the sum $\sum_{j=0}^{n-1} r(f^j(x)) - r(f^j(y))$. As we allow n to approach infinity and we can determine the total shearing caused by the roof. This is what we will refer to as the shearing experienced by $\hat{x}$ and $\hat{y}$ under φ_r. While the infinite sum may not converge, there are instances where the shearing will be bounded. We will use the shearing of by the roof function to determine the topological mixing properties of the flow.

The notion of shearing we will use and some of the other summations we will rely on are similar to the Birkhoff sums in ergodic theory. Given a measure-preserving transformation $t\colon X \to X$ for X a finite measure space, and f a real valued function, the Birkhoff average is the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(x)).$$

This sum has great importance in the measure theoretic setting. One of the fundamental theorems in ergodic theory is Birkhoff's Ergodic Theorem [3]. It states that if $\mu(X)$ is positive and finite, and if f is a measurable function and T is ergodic, then for almost every x

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(x)) = \frac{1}{\mu(X)} \int f \, d\mu.$$

If f is the characteristic function for a subset U of X, then we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbf{1}_U(T^j(x)) = \frac{\mu(U)}{\mu(X)}.$$

The right hand side of this equation represents the spatial average and the left hand side represents the time average. Thus, for an ergodic transformation, the space and time averages are equal. This means that for almost any point in the space X, will spend $\mu(U)/\mu(X)$ of its time in the set U.

Although our results are not concerned with spatial averages, we will use a similar summation to determine the average length of a fiber. From this information, we can determine whether a roof function is cohomologous to a constant, what kind of shearing we can expect, and make conclusions about the topological mixing properties of the system.

Our first result addresses the case when the shearing is bounded. In general, if the shearing induced by a roof function is bounded, then the suspension flow will not be topologically mixing. To obtain this result we will use topological equivalence to assist us in the proof.

Proposition 4.1. *Let $r\colon X \to \mathbb{R}$ be any continuous roof function on a compact metric space, and let $R\colon X \to \mathbb{R}$ be given by $R(x) = r(x) + k$ for some $k > 0$. If $\varphi^t\colon M_r \to M_r$ is the suspension flow under r, and $\psi^t\colon M_R \to M_R$ is the suspension flow under R, then φ^t and ψ^t are topologically equivalent flows.*

Proof. Note that elements of M_r and M_R are tuples $(x,t) \in X \times \mathbb{R}$. Since X is compact, r attains a minimum $a = \min_{x\in X} r(x)$.

Let $h\colon M_R \to M_r$ be defined as follows

$$h(x,t) = \begin{cases} (x, \frac{a}{a+k}t) & \text{if } t \in [0, a+k) \\ (x, t-k) & \text{if } t \in [a+k, R(x)). \end{cases}$$

If π_i is the projection function onto the i^{th} component, we get that $\pi_1 \circ h$ is continuous and $\pi_2 \circ h$ is continuous. Hence h is continuous.

We now show that h is a bijection. Suppose that $h(x,t) = h(y,s)$. This immediately implies that $x = y$. There are now three cases we must consider.

Case 1: $\frac{a}{a+k}t = \frac{a}{a+k}s \implies t = s$.

Case 2: $t - k = s - k \implies t = s$.

Case 3: $t - k = \frac{a}{a+k}s$. If this is the case, then $t \in [a+k, R(x))$ and $s \in [0, a+k)$. Hence $t - k \geq a$ and $\frac{a}{a+k}s < a$. This is a contradiction. This case and the case for $s - k = \frac{a}{a+k}t$ cannot occur. Hence, h is injective.

We now show that h is surjective. Let $(x,t) \in M_r$. There are two cases. $t \in [0,a)$, or $t \in [a, r(x))$. If $t \in [0,a)$, the $h(x, \frac{a+k}{a}t) = (x,t)$. If $t \in [a, r(x))$, then $h(x, t+k) = (x,t)$. Thus h is surjective. In addition, these computations give us the inverse function for h where

$$h^{-1}(x,t) = \begin{cases} (x, \frac{a+k}{a}t) & \text{if } t \in [0, a) \\ (x, t+k) & \text{if } t \in [a, r(x)). \end{cases}$$

h^{-1} is continuous in each component, so it is continuous. Hence, h is a homeomorphism.

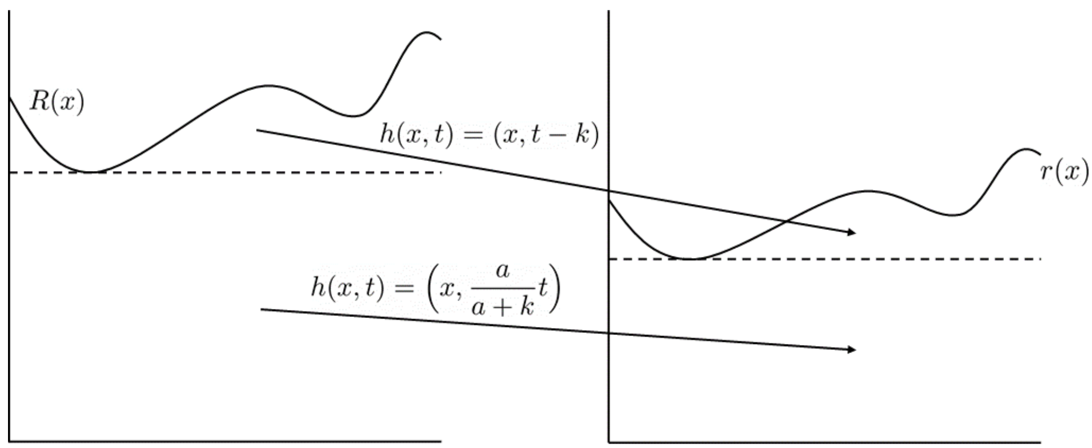

Figure 4.1: The mapping of the homeomorphism h

Finally, we must show that the flow satisfies the final condition of the definition. That is the flow of time is preserved by h. There are two cases.

Case 1: Suppose $(x, t_0) \in M_R$ with $t_0 \in [0, a+k)$. Let $\delta = a + k - t_0$. If $0 < |s| \leq t < \delta$ with s such that

$$h(\psi^t(x, t_0)) = \varphi^s(h(x, t_0)).$$

Evaluating the right-hand and left-hand side of the equality we have

$$h(\psi^t(x, t_0)) = h(x, t_0 + t) = \left(x, \frac{a}{a+k}(t + t_0)\right)$$

and

$$\varphi^s(h(x, t_0)) = \varphi^s\left(x, \frac{a}{a+k}t_0\right) = \left(x, \frac{a}{a+k}t_0 + s\right).$$

Thus,

$$\left(x, \frac{a}{a+k}(t + t_0)\right) = \left(x, \frac{a}{a+k}t_0 + s\right).$$

This implies that $s = \frac{a}{a+k}t > 0$, as desired.

Case 2: Suppose $(x, t_0) \in M_R$ with $t_0 \in [a+k, R(x))$. Let $\delta = R(x) - t_0$. If $0 < |s| \leq t < \delta$ with s such that

$$h(\psi^t(x, t_0)) = \varphi^s(h(x, t_0)).$$

Evaluating the right-hand and left-hand side of the equality we have

$$h(\psi^t(x, t_0)) = h(x, t_0 + t) = (x, t_0 + t - k)$$

and

$$\varphi^s(h(x, t_0)) = \varphi^s(x, t_0 - k) = (x, t_0 - k + s).$$

Thus,

$$(x, t_0 + t - k) = (x, t_0 - k + s).$$

This implies $s = t > 0$ as desired. Therefore, ψ and φ are topologically equivalent by definition. □

While we assumed the existence of an s satisfying the condition

$$h(\psi^t(x, t_0)) = \varphi^s(h(x, t_0))$$

we see that for the appropriate δ, for all $0 < t < \delta$, there exists $s > 0$ that satisfies the equality. We extend this with the following result.

Proposition 4.2. *If ψ^t and φ^t are as in Proposition 4.1, and $t > 0$, then there exists an s satisfying $0 < s \leq t$ such that*

$$h(\psi^t(x, t_0)) = \varphi^s(h(x, t_0)).$$

Proof. By the work from Proposition 4.1, the proof here reduces to two simple cases.

Case 1: Suppose $(x, t_0) \in M_R$ with $t_0 \in [0, a + k)$. Let $t = a + k - t_0$. We show that there exists an $s > 0$ such that

$$h(\psi^t(x, t_0)) = \varphi^s(h(x, t_0)).$$

Let $s = a - \frac{a}{a+k}t_0$, and note that when we evaluate the right-hand and left-hand side of the equality we have

$$h(\psi^t(x, t_0)) = h(x, t_0 + t) = h(x, a + k) = (x, a)$$

and

$$\varphi^s(h(x, t_0)) = \varphi^s\Big(x, \frac{a}{a+k}t_0\Big) = \Big(x, \frac{a}{a+k}t_0 + s\Big) = (x, a).$$

Thus, $h(\psi^t(x, t_0)) = \varphi^s(h(x, t_0))$. Since $t_0 < a + k$, we have that $s = a - \frac{a}{a+k}t_0 > a - a = 0$.

Case 2: Suppose $(x, t_0) \in M_R$ with $t_0 \in [a + k, R(x))$. Let $t = R(x) - t_0$. We show that there exists an $s > 0$ such that

$$h(\psi^t(x, t_0)) = \varphi^s(h(x, t_0)).$$

Let $s = k - t_0 + r(x) = R(x) - t_0 = t > 0$, and note that when we evaluate the right-hand and left-hand side of the equality we have

$$h(\psi^t(x, t_0)) = h(x, t_0 + t) = h(x, R(x)) = h(f(x), 0) = (f(x), 0)$$

and

$$\varphi^s(h(x, t_0)) = \varphi^s(x, t_0 - k) = (x, t_0 - k + s) = (x, r(x)) = (f(x), 0).$$

Therefore, for any $t > 0$, there exists an s satisfying $0 < s \leq t$ such that $h(\psi^t(x, t_0)) = \varphi^s(h(x, t_0))$. $\square$

Having established this existence result, we are now ready to show that topological mixing is preserved for suspensions whose roof functions differ by a constant. From there we will prove that bounded shearing results in suspensions that are not topologically mixing.

Proposition 4.3. *If ψ^t and φ^t are as in Proposition 4.1, and if φ^t is topologically mixing, then ψ^t is topologically mixing.*

Proof. First we show that for all $S > 0$, there exists a $T > 0$ such that for all $t > T$ there exists $s > S$ such that

$$h(\psi^t(x, t_0)) = \varphi^s(h(x, t_0)).$$

Fix $S > 0$ and $(x, t_0) \in M_R$. Consider $h^{-1}(\varphi^S(x, t_0))$. There exists some T such that $\psi^T(x, t_0) = h^{-1}(\varphi^S(h(x, t_0)))$. Hence for $t > T$, we have that the s satisfying $h(\psi^t(x, t_0)) = \varphi^s(h(x, t_0))$ must be larger than S.

Let U and V be any open sets in M_R. Let $U' = h(U)$ and $V' = h(V)$. U' and V' are open in M_r because h is a homeomorphism. There exists S such that for all $s > S$ we have that $\varphi^s(U') \cap V' \neq \varnothing$ since φ^t is topologically mixing. Let

$$T = \sup_{x \in U}\{t \in \mathbb{R} \colon \psi^t(x, t_0) = h^{-1}(\varphi^S(h(x, t_0)))\}.$$

T exists and is finite because the number of times the projection of U' onto X iterates under f in time S can be uniformly bounded. Hence for $t > T$, we have that $\psi^t(U) \cap V \neq \varnothing$ since $h(\psi^t(U) \cap V) = \varphi^s(U') \cap V' \neq \varnothing$. ψ is topologically mixing. □

Proposition 4.4. *Let $f \colon X \to X$ be a homeomorphism on a compact metric space and let $r \colon X \to \mathbb{R}$ be a continuous roof function. If there exist an open set $U \subset X$ and a real number $M > 0$ such that for all $x, y \in U$ we have that*

$$\Big|\sum_{j=0}^{\infty} r(f^j(x)) - r(f^j(y))\Big| \leq M$$

then the suspension φ^t with roof function r is not topologically mixing.

Proof. Let $R \colon X \to \mathbb{R}$ be defined as $R(x) = r(x) + 4M$ and let ψ^t be the suspension flow with roof $R(x)$. We see that for all $x, y \in U$

$$\Big|\sum_{j=0}^{\infty} R(f^j(x)) - R(f^j(y))\Big| = \Big|\sum_{j=0}^{\infty} r(f^j(x)) - r(f^j(y))\Big| \leq M.$$

Consider the open set $U' = U \times (0, M)$, and let V be some open set of X and consider the open set $V' = V \times (0, M)$. If there exists some n such that $f^n(U) \cap V \neq \varnothing$. Then there exists some T such that $\psi^T(U') \cap V' \neq \varnothing$. However, $\psi^{T+2M}(U') \cap V' = \varnothing$. Since the shearing never exceeds the minimum value of R, it cannot be topologically mixing. By Proposition 4.3, φ^t is not topologically mixing. $\square$

The existence of an open set that experiences bounded shearing is one way to identify a system that is not topologically mixing. However, this result does not tell us anything about the structure of the roof function or what equivalence class it belongs to. We would like to take this a step further and relate bounded shearing to roof functions cohomologous to a constant.

Proposition 4.5. *If X is a compact space and $f\colon X \to X$ is a homeomorphism, and $r\colon X \to \mathbb{R}$ is a roof function cohomologous to a constant, then the shearing induced by r is bounded.*

Proof. Since r is cohomologous to k, there exists a continuous transfer function $g\colon X \to \mathbb{R}$ such that $r(x) - k = g(f(x)) - g(x)$. Since X is compact, $|g| \leq M$ for some $M > 0$. Let $x, y \in X$ and consider

$$\Big|\sum_{j=0}^{n-1} r(f^j(x)) - r(f^j(y))\Big| \leq \Big|\sum_{j=0}^{n-1} r(f^j(x)) - k\Big| + \Big|\sum_{j=0}^{n-1} k - r(f^j(y))\Big|$$

$$= |g(f^n(x)) - g(x)| + |-g(f^n(y)) + g(y)| \leq 4M.$$

This holds for all n, so the shearing induced by r is bounded. $\square$

We desire to show that roof functions that yield bounded shearing are cohomologous to a constant. In order to find a candidate for the constant, we will need to look at the average fiber length of a periodic point. In order to prove the converse, we will need the added assumption of dense periodic points. We require this to leverage the average value of a roof function on a periodic orbit and Proposition 3.8.

Proposition 4.6. *Let X be a compact metric space, $f\colon X \to X$ a topologically mixing homeomorphism with dense periodic points, and $r\colon X \to \mathbb{R}$ be a continuous roof function. If there exist an open set $U \subset X$ and a real number $M > 0$ such that for all $x, y \in U$ we have*

$$\Big|\sum_{j=0}^{\infty} r(f^j(x)) - r(f^j(y))\Big| \leq M,$$

then r is cohomologous to a constant.

Proof. First note that $\lim_{n\to\infty} \frac{1}{n}\big|\sum_{j=0}^{n-1} r(f^j(x))\big|$ exists for all $x \in U$ where x is a periodic point. Now for all $x, y \in U$

$$\lim_{n\to\infty} \frac{1}{n}\Big|\sum_{j=0}^{n-1} r(f^j(x)) - r(f^j(y))\Big| \leq \lim_{n\to\infty} \frac{1}{n}M = 0.$$

Thus for all $x, y \in U$ we have that $\lim_{n\to\infty} \frac{1}{n}\big|\sum_{j=0}^{n-1} r(f^j(x))\big| = \lim_{n\to\infty} \frac{1}{n}\big|\sum_{j=0}^{n-1} r(f^j(y))\big| = k$ for some $k \in \mathbb{R}$.

Since f is topologically mixing, there exists $x_0 \in U$ such that $\overline{\mathcal{O}^+(x_0)} = X$. We consider $\lim_{n\to\infty} \big|\sum_{j=0}^{n-1} r(f^j(x_0)) - k\big|$. If the limit is bounded, then we may conclude that $r(x)$ is cohomologous to a constant.

Let y be a periodic point in U. We have that

$$\Big|\sum_{j=0}^{\infty} r(f^j(x_0)) - r(f^j(y))\Big| \leq M$$

by hypothesis. Since y is a periodic point, we have that

$$\Big|\sum_{j=0}^{\infty} r(f^j(y)) - k\Big| \leq M.$$

Thus we have that

$$-M \leq \sum_{j=0}^{n-1} r(f^j(x_0)) - r(f^j(y)) \leq M$$

$$-M \leq \sum_{j=0}^{n-1} r(f^j(x_0)) - k + k - r(f^j(y)) \leq M$$

$$-M + \sum_{j=0}^{n-1} r(f^j(y)) - k \leq \sum_{j=0}^{n-1} r(f^j(x_0)) - k \leq M + \sum_{j=0}^{n-1} r(f^j(y)) - k.$$

Allowing $n \to \infty$, it follows that $\sum_{j=0}^{\infty} r(f^j(x_0)) - k$ is bounded. Therefore, $r(x)$ is cohomologous to a constant by Proposition 3.8. $\square$

We have shown that a roof function is cohomologous to a constant if and only if the shearing is bounded. Moreover, we know that a roof function that is not cohomologous to a constant must cause unbounded shearing. We now prove an important property about the shearing induced by a roof function that is not cohomologous to a constant. Again, we will need dense periodic points to obtain this result.

Proposition 4.7. *Let X be a compact metric space and $f\colon X \to X$ a topologically mixing homeomorphism. Assume that periodic points are dense in X under f. Let $r\colon X \to \mathbb{R}$ be a continuous roof function that is not cohomologous to a constant. For any open set U, there exist $x, y \in U$ such that*

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} r(f^j(x)) - r(f^j(y))$$

diverges to infinity.

Proof. Let U be an open set in X. Let $x, y \in U$, where x has a dense orbit, and y is a periodic point. We must show that for all $M > 0$, there exists an N, such that for all $n \geq N$ the sum

$$\Big|\sum_{j=0}^{n-1} r(f^j(x)) - r(f^j(y))\Big| > M.$$

Suppose that this is not the case. Then there would exist an $M > 0$ such that for any N, there would exist an $n \geq N$ such that

$$\Big|\sum_{j=0}^{n-1} r(f^j(x)) - r(f^j(y))\Big| \leq M.$$

Hence there would exist an increasing subsequence a_n of natural numbers such that for all n,

$$\Big|\sum_{j=0}^{a_n-1} r(f^j(x)) - r(f^j(y))\Big| \leq M.$$

Since y is a periodic point, $r(f^j(y))$ can only take on a finite number of distinct values for any j. Hence, there exists a subsequence b_n of a_n, such that

$$\sum_{j=0}^{b_n-1} r(f^j(y)) = b_n k$$

for some $k > 0$. This implies that,

$$\Big|\sum_{j=0}^{b_n-1} r(f^j(x)) - k\Big| \leq M.$$

It follows by Proposition 3.8 that $r(x)$ is cohomologous to a constant. A contradiction. The shearing must diverge to infinity. □

Here we concern ourselves with the total shearing we obtain from a roof function. We can alternatively consider the average shearing by evaluating

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} r(f^j(x)) - r(f^j(y)).$$

It can readily be seen that if the average is nonzero, then the total shearing is unbounded and diverges to positive of negative infinity. However, if this average shearing is zero, we cannot easily conclude anything about the total shearing caused by the roof function for the system.

4.2 Topological Mixing

The results for the shearing caused by roof functions over dense periodic points shows that there are two types of shearing. The shearing is either bounded, or it diverges to infinity.

Since roof functions with bounded shearing are cohomologous to a constant, all other roof functions must induce unbounded shearing. These facts about the roof functions will provide us with the necessary foundation to complete the dichotomy for the topological mixing behavior of suspension flows. As the previous results required dense periodic points, the following results will also include this structure in the base.

We have already proven the first part of the dichotomy, a suspension flow with roof cohomologous to a constant is not topologically mixing. This is true without the presence of dense periodic points. However, to establish the converse, we will show that a suspension with a roof function that is not cohomologous to a constant is topologically mixing. To accomplish this, we will need dense periodic points because the proof depends on Proposition 4.7.

Proposition 4.8. *Let $f\colon X \to X$ a topologically mixing homeomorphism of a compact metric space with dense periodic points. If $r\colon X \to \mathbb{R}$ is a continuous roof function that is not cohomologous to a constant, then the suspension flow under r is topologically mixing.*

Proof. X is a compact metric space, so every open cover of X has a finite subcover. Let $\epsilon > 0$ and define $O_x = \{x_0 \in X\colon d(x, x_0) < \epsilon\}$. The union $\bigcup_{x\in X} O_x$ is an open cover of X for any choice ϵ, and it has a finite subcover. Let $\bigcup_{n=0}^{M} O_{x_n}$ be a finite covering of X. Also, as $r(x)$ is continuous, it is bounded. Let $R = \sup_{x\in X} r(x)$.

Let U be an open set in the domain of φ_r and let U' be the projection of U onto X. There exist points $\hat{x}, \hat{y} \in U$ such that $\hat{x} = (x, a)$ and $\hat{y} = (y, a)$ where $x, y \in X$ and $a \in \mathbb{R}$, which satisfy the conclusion of Proposition 4.7. Let l be the set of points in U whose height is a and are not $\hat{x}$ and $\hat{y}$. Note that l limits onto $\hat{x}$ and $\hat{y}$. By Proposition 4.7 we know there exists a N_1 such that for all $n > N_1$ we have that

$$\sum_{j=0}^{n} r(f^j(x)) - r(f^j(y)) > RM.$$

Recall that M is the number of open set required to cover X. This sum represents that amount of shearing caused by φ_r. By hypothesis, for any V' in the open cover of X, there is an N_2 such that $n > N_2$, $f^n(U') \cap V' \neq \varnothing$ since f is topologically mixing. Choose $N = \max\{N_1, N_2\}$. Choose T sufficiently large so that both x and y have iterated N times under f.

For all $t > T$, the projection of l into X intersects every open set in our subcover of X. Also, by the shearing of $\hat{x}$ and $\hat{y}$ we get that the vertical difference of l exceeds RM, so the image of U under φ_r will intersect any open set of radius ϵ in the domain of φ_r. Therefore φ_r is topologically mixing by Definition 1.4. $\square$

Combining all of these results, we are ready to state the dichotomy and prove our first main result.

Theorem 4.9. *Let $f\colon X \to X$ is a topologically mixing homeomorphism with dense periodic points. A suspension flow over X is topologically mixing if and only if the roof function is not cohomologous to a constant.*

Proof. We show that if a suspension flow is topologically mixing then the roof function is not cohomologous to a constant by proving the contrapositive. Let $\varphi_r(x)$ be a suspension flow with roof function $r\colon X \to \mathbb{R}$. Suppose that the roof function is cohomologous to a constant k. Let φ_k we the suspension flow with roof function $s\colon X \to \mathbb{R}$ where $s(x) = k$ for all $x \in X$.

By Proposition 3.2 we have that φ_r and φ_s are conjugate. Thus they have the same topological mixing properties by Proposition 3.3. φ_s is not topologically mixing; therefore, φ_r is not topologically mixing.

The converse is proved by Proposition 4.8. $\square$

Chapter 5. Main Result II

Our second result shows that the set of topologically mixing suspension flows is an open and dense set provided the base has dense periodic points. We will accomplish this by showing that the set of roof functions that are not cohomologous to a constant is an open and dense set. We will first show that this set is open and then show that it is dense.

5.1 Open

The open property gives us an important insight to the collection of topologically mixing suspensions. If we have a topologically mixing suspension flow with dense periodic points, then for any $r \geq 0$ any C^r perturbation that is C^0 close will still yield a topologically mixing flow. Furthermore, the set of flows that are not topologically mixing flows is closed, hence slight changes in the roof function could cause the suspension to be topologically mixing.

Proposition 5.1. *Let $f\colon X \to X$ be a topologically mixing continuous base function on a compact metric space X with dense periodic points. Let $r_n\colon X \to \mathbb{R}$ be a sequence of roof functions that converge uniformly to a limit function $r(x)$. If all $r_n(x)$ are cohomologous to a constant k_n, then $r(x)$ is cohomologous to a constant.*

Proof. By hypothesis, $r_n(x)$ is cohomologous to k_n. In addition, there exists a point x_0 such that $\overline{\mathcal{O}^+(x_0)} = X$. Further, there exists a sequence of continuous function $g_n\colon X \to \mathbb{R}$ such that $r_n(x) - k_n = g_n(f(x)) - g_n(x)$ and $g_n(x_0) = 0$.

First we show that k_n is bounded. Suppose for contradiction that k_n is unbounded. Without loss of generality suppose that $k_n \to \infty$. Let y be a periodic point with period p. Thus

$$\sum_{j=0}^{p-1} r_n(f^j(y)) = pk_n.$$

In addition, $r(x)$ is continuous, so it is bounded. Thus

$$\sum_{j=0}^{p-1} r(f^j(y))$$

is also bounded. However,

$$\sum_{j=0}^{p-1} r(f^j(y)) = \lim_{n\to\infty} \sum_{j=0}^{p-1} r_n(f^j(y)) = \lim_{n\to\infty} pk_n = \infty$$

a contradiction. Hence, k_n is bounded.

Because k_n is bounded, there exists a convergent subsequence $k_{n_i} \to k$. We define $s_i\colon X \to \mathbb{R}$ by $s_i(x) = g_{n_i}(f(x)) - g_{n_i}(x) + k$. We show that $s_i(x)$ converges uniformly to $r(x)$. For all $\epsilon > 0$, there exists I_1 such that for $i \geq I_1$ we have that $|k_{n_i} - k| < \epsilon/2$. In addition, there exists I_2 such that for $i \geq I_2$ we have that for all x, $|r_{n_i}(x) - r(x)| < \epsilon/2$. Let $i \geq \max\{I_1, I_2\}$. It follows that

$$|s_i(x) - r(x)| = |s_i(x) - r_{n_i}(x) + r_{n_i}(x) - r(x)| \leq |s_i(x) - r_{n_i}(x)| + |r_{n_i}(x) - r(x)|$$

$$= |g_{n_i}(f(x)) - g_{n_i}(x) + k - g_{n_i}(f(x)) + g_{n_i}(x) - k_{n_i}| + |r_{n_i}(x) - r(x)|$$

$$= |k - k_{n_i}| + |r_{n_i}(x) - r(x)| < \epsilon.$$

We now consider the limit of $g_{n_i}(x)$. In general, $g(f^a(x_0)) = \lim_{i\to\infty} \sum_{j=0}^{a-1} s_i(f^j(x_0)) - k = \sum_{j=0}^{a-1} r(f^j(x_0)) - k$. Suppose $f^{a_n}(x_0) \to y \in \mathcal{O}^+(x_0)$ and

$$\lim_{n\to\infty} \sum_{j=0}^{a_n-1} r(f^j(x_0)) - k = \infty.$$

Then $\lim_{i\to\infty} g_{n_i}(y) = \infty$, which implies that $\lim_{i\to\infty} g_{n_i}(f^c(y)) = \infty$ for all $c \in \mathbb{Z}$. This would imply that $\lim_{i\to\infty} g_{n_i}(x_0) = \infty$, a contradiction. Therefore, the sum

$$\sum_{j=0}^{a_n-1} r(f^j(x_0)) - k$$

is bounded. Therefore, $r(x)$ is cohomologous to a constant by Proposition 3.8. □

Under the assumption of the existence of dense periodic points, this shows that the set of roof functions that are not cohomologous to a constant is open.

5.2 Dense

Now we show that the set of continuous roof functions that are not cohomologous to a constant are dense. Thus the likelihood of selecting a roof function not cohomologous to a constant is relatively high. Furthermore, this implies that any continuous roof function can be approached by roof functions that are not cohomologous to a constant. Note that this result will not require dense periodic points, nor does it require topological mixing in the base either.

Proposition 5.2. *Let $f: X \to X$ be continuous and X be a compact metric space. The set of roof functions cohomologous to a non-constant function is dense in the set of continuous roof functions.*

Proof. Let $\epsilon > 0$ and let $r: X \to \mathbb{R}$ be an arbitrary roof function cohomologous to a constant k. Then $r(x) = g(f(x)) - g(x) + k$ for some continuous function g. Furthermore, define T to be the set of all continuous functions cohomologous to a non-constant function.

There exists some continuous functions $s(x)$ that is not cohomologous to a constant. Define the sequence of $s_n(x) = \frac{1}{n}s(x) + k$. We show that $s_n(x)$ converges uniformly to k.

$s(x)$ is bounded so there exists some $M > 0$ such that $|s(x)| < M$. Let $\epsilon > 0$ and let $N > \frac{M}{\epsilon}$. For $n \geq N$ we have that $|s_n(x) - k| = |\frac{1}{n}s(x) + k - k| < |\frac{M}{n}| < \frac{M}{N} < \epsilon$. So $s_n(x)$

converges uniformly to k.

We can now construct a sequence of roof functions $t_n(x) = g(f(x)) - g(x) + s_n(x)$. Each t_n is not cohomologous to a constant since each $s_n(x)$ is not cohomologous to a constant. Note that for $n \geq N$

$$|t_n(x) - r(x)| = |g(f(x)) - g(x) + s_n(x) - (g(f(x)) - g(x) + k)| = |s_n(x) - k| < \epsilon.$$

So the sequence $t_n(x)$ converges to $r(x)$ uniformly. Since $r(x)$ is arbitrary, for any roof function cohomologous to a constant, there exists a sequence of roof functions not cohomologous to a constant function that converges uniformly to $r(x)$. So the set of roof functions cohomologous to a constant is contained in the closure of T. Since all continuous functions are cohomologous to either a constant or a nonconstant function, $\overline{T}$ is equal to the set of all continuous roof functions. Therefore, T is dense in the set of continuous roof functions. $\square$

We now compile these results to state our second main result regarding the space of topologically mixing suspension flows.

Theorem 5.3. *Let $f \colon X \to X$ be a topologically mixing homeomorphism of a compact metric space with dense periodic points. The space of topologically mixing suspension flows is open and dense.*

Proof. We know that a the space of roof functions cohomologous to a constant is closed by Proposition 5.1. Hence its complement, the space of functions not cohomologous to a constant must be open. This implies that the space of topologically mixing flows must also be open.

The space of roof functions not cohomologous to a constant are dense in the space of continuous roof functions by Proposition 5.2. All flows associated with this space of functions induce topologically mixing flows. Hence the space of topologically mixing flows must be dense. $\square$

Chapter 6. Conclusion

We have shown that with the inclusion of dense periodic points, a suspension flow is topologically mixing if and only if the roof function is not cohomologous to a constant. In this case, the set of topologically mixing suspension flows is open and dense. Many of our major results establishing the topological mixing properties of a suspension depend on Proposition 3.8, which gives us a criteria to determine if a transfer function exists between any two roof functions. While this result does not require the existence of dense periodic points, applying this result to construct a transfer function or even to find a candidate for a transfer function without the underlying structure of dense periodic points has proven to be a difficult task. In the general setting it could be possible that a larger class of roof functions induce non-mixing suspension flows.

We employed the strategy of cohomologous roof functions because of its use in the measure theoretic case [18]. While this definition is suited for a probabilistic setting, this characterization may not be appropriate in the topological case in full generality. Using a characterization on the roof function other than equivalence classes of cohomologous functions may be necessary to obtain a desired dichotomy in the most general case. A candidate is the notion of bounded shearing we have introduced. Other equivalent conditions could be useful in proving a general result.

In addition, it is not known whether the roof functions that induce topologically mixing suspension flows is an open and dense set in the set of continuous roof functions in the absence of dense periodic points. This leads to the following opens questions:

(i) If a topologically mixing base does not have dense periodic points, can the suspension have a roof function that is not cohomologous to a constant and be non-mixing?

(ii) If such a suspension exists, what kind of periodic structure can the base have? For instance, could there exist periodic point that are not dense, or must periodic points be absent from the system?

(iii) Does the existence or absence of periodic points in the base affect the manner in which we classify topologically mixing suspensions.

(iv) Given a topologically mixing base, is there an open and dense set of roof functions such that the suspension flow is topologically mixing?

(v) Is there a criterion to determine if a roof function is cohomologous to a constant given any topologically mixing base?

(vi) Can zero average shearing, $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} r(f^j(x)) - r(f^J(y)) = 0$, lead to any conclusions about the roof function or topological mixing properties of a system?

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