Isogeometric Shell Analysis: Multi-patch Coupling and Overcoming Locking

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ABSTRACT

Isogeometric Shell Analysis: Multi-patch Coupling and Overcoming Locking

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The fundamental advantages of applying Isogeometric Analysis (IGA) to shell analysis have been extensively demonstrated across a wide range of problems and formulations. However, a phenomenon called numerical locking is still a major challenge in IGA shell analysis, which can lead to dramatically deteriorated analysis accuracy. Additionally, for complex thin-walled structures, a simple and robust coupling technique is desired to sew together models composed of multiple patches. This dissertation focuses on addressing these challenges of IGA shell analysis.

First, an isogeometric dual mortar method is developed for multi-patch coupling. This method is based on Bézier extraction and projection and can be employed during the creation and editing of geometry through properly modified extraction operators. It is applicable to any spline space which has a representation in Bézier form. The error in the method can be adaptively controlled, in some cases recovering optimal higher-order rates of convergence, by leveraging the exact refineability of the proposed dual spline basis without introducing any additional degrees-of-freedom into the linear system. This method can be used not only for shell elements but also for heat transfer and solid elements, etc.

Next, a mixed formulation for IGA shell analysis is proposed that addresses both shear and membrane locking and improves the quality of computed stresses. The starting point of the formulation is the modified Hellinger-Reissner variational principle with independent displacement, membrane, and shear strains as the unknown fields. To overcome locking, the strain variables are interpolated with lower-order spline bases while the variations of the strain variables are interpolated with the proposed dual spline bases. As a result, the strain variables can be condensed out of the system with only a slight increase in the bandwidth of the resulting linear system and the condensed approach preserves the accuracy of the non-condensed mixed approach but with fewer degrees-of-freedom.

Finally, as an alternative, new quadrature rules are developed to release membrane and shear locking. These quadrature rules asymptotically only require one point for Reissner-Mindlin (RM) shell elements and two points for Kirchhoff-Love (KL) shell elements in B-spline and NURBS-based isogeometric shell analysis, independent of the polynomial order $p$ of the elements. The quadrature points are Greville abscissae and the quadrature weights are calculated by solving a linear moment fitting problem in each parametric direction. These quadrature rules are free of spurious zero-energy modes and any spurious finite-energy modes in membrane stiffness can be easily stabilized by using a higher-order Greville rule.

Keywords: Isogeometric analysis, Mortar methods, Dual basis, Locking, Reduced quadrature
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CHAPTER 1. INTRODUCTION

1.1 Background, motivation and challenges

Through decades a large number of numerical methods have been developed for engineering analysis, of which the finite element analysis (FEA) is the most widely used method due to its effectiveness and general applicability. Standard FEA employs Lagrange polynomials as the basis for analysis while the Computer Aided Design (CAD) uses spline basis for geometry description. Therefore, a transformation between CAD and FEA models is inevitable. This process introduces not only model errors but also burdensome work for creating analysis-suitable geometry, which is estimated to take up 80% of the total analysis time for complex engineering designs [2]. To bridge the gap between design and analysis, Hughes et al. [3] proposed isogeometric analysis (IGA) in 2005, which directly takes the spline basis of CAD geometry as the basis for analysis. This unifying paradigm has the potential to eliminate the costly geometry clean-up and mesh generation steps which encumber traditional simulation pipelines. In addition, it also improves the simulation accuracy through geometrically exact analysis model and higher-order continuous spline basis. Due to its superiority in various aspects over FEA, IGA has attracted a large amount of attention from both CAD and CAE communities and has been successfully used in numerous fields, such as structural mechanics [4–7], optimization [8, 9], electromagnetics [10, 11] and fluid-structure interaction [12, 13], etc.

CAD technologies like Non-Uniform Rational B-splines (NURBS) [14], subdivision surfaces [15], and T-splines [16, 17] usually make use of surface representations to model geometrical objects, while engineering analysis of general three-dimensional (3D) structures often requires a volumetric description of the geometry. Extra efforts are necessary to address this inconsistency between design and analysis. Two major attempts to deal with this inconsistency are reconstructing CAD models with volumetric splines [18, 19] and employing special numerical methods for analysis, for example, the boundary element method [20–22]. As an exception, shell analysis,
an important branch of computational engineering used widely to analyze thin-walled structures, does not require a volumetric description of the underlying geometry. Therefore, the integration of design and shell analysis is more straightforward.

Thin shell structures are sensitive to geometric imperfections [23] which are inevitable in classical FEA as the finite element mesh is only an approximation of the CAD geometry. In contrast, IGA directly employs the CAD geometry as the analysis model, so quantities that are crucial in shell analysis, such as curvature, tangents, and normals are immediately accessible and exact. Additionally, the CAD technologies mentioned above make it possible to define smooth higher-order basis functions over both structured and unstructured meshes, which has opened the door to new shell formulations based on both RM and KL shell kinematics [4, 6, 24–27].

The fundamental advantages of IGA shell analysis have been demonstrated through a large amount of work [4–6, 28]. However, numerical locking is still a challenge in using higher-order smooth splines to discretize shell theories. The primary sources of locking are shear and membrane locking. In shear locking, pure inextensional bending modes are polluted by parasitic shear strains and, in membrane locking, pure inextensional bending modes are polluted by parasitic membrane strains. While not a major contributor to locking for low-order (linear) shell elements, membrane locking is particularly vexing for higher-order curved shell elements. In this sense, shell elements based on smooth splines are more prone to locking than their low-order counterparts. All locking effects can be ameliorated by increasing the degree of the basis functions, but for quadratic and cubic basis functions, the workhorse degrees in isogeometric shell analysis, locking can still destroy the accuracy of computed shell solutions, especially in thin shell regimes. In particular, computed stresses can be completely spurious. Figure 1.1 illustrates the pathology of locking through a simple example of a straight cantilever beam under end shear force. The model is discretized with six quadratic IGA Timoshenko beam elements. Figure 1.1b shows that the maximum deflection is underestimated as the slenderness increases, which indicates that the straight beam is severely locked. Additionally, the shear force, as shown in Figure 1.1c, oscillates dramatically even with slenderness $\frac{L}{t} = 100$. To take full advantage of higher-order IGA, effective techniques to remove the locking problems in IGA shell analysis are essential.
In addition to the notorious locking phenomena, another problem in isogeometric shell analysis is the multi-patch coupling of nonconforming higher-order smooth meshes. For a complex engineering structure, the underlying geometry usually consists of multiple patches. These patches are normally non-conforming along their interfaces, i.e., either the control points of different patches are not conforming along the interfaces or the underlying parameterizations are not the same. As an example, Figure 1.2 shows a multi-patch non-conforming T-spline model of the hatch structure of a container ship. The non-conforming geometry is generally not a problem in CAD, but usually an issue for IGA because some physical fields themselves need to be continuous. To introduce additional flexibility into the isogeometric approach, weak coupling techniques are often used to sew together models composed of multiple patches. These approaches can accommodate patches with differing parameterizations and trimming [29–39]. However, if not done properly, these coupling techniques can negatively impact the accuracy and robustness of the analy-
sis [29, 30, 39]. Therefore, effective and robust multi-patch coupling techniques are also necessary to saw together the physical fields between different patches.

![A container ship.](image1.jpg) ![T-spline model of the hatch structure.](image2.jpg)

Figure 1.2: A container ship and the CAD model of its hatch structure.

1.2 State of the art

Various techniques have been proposed to address locking in finite element shells. Prominent among these approaches are mixed formulations based on generalized variational principles [40–42] with displacement, strain and/or stress unknowns, the reduced and selective integration techniques [43–45], the assumed strain method [46–51], and the enhanced assumed strain method [52, 53]. To some extent, the last three methods are equivalent to a mixed formulation under specific conditions [54–56]. The reduced and selective integration techniques are dominant in commercial low-order (linear) finite elements due to their simplicity, efficiency, and robustness. However, the reduced integration method may introduce spurious zero energy modes, which is a serious defect requiring some form of stabilization [57]. The selective reduced integration technique is less efficient than the reduced integration approach in terms of the computational efficiency and is usually restricted to cases where the material properties of the shell do not vary through the thickness. The assumed strain method interpolates selected strain components with specially constructed lower-order basis functions, while the enhanced assumed strain method attaches additional terms to the selected strains and introduces extra strain variables that can then be condensed out on the element level.
Since IGA was first proposed in 2005, numerous IGA shell formulations have been developed based on both Reissner-Mindlin and Kirchhoff-Love shell kinematics [4, 6, 24–27]. Most traditional finite element techniques listed above for mitigating locking have been extended to IGA shells [1, 26, 58–61]. Although these methods can remove shear and/or membrane locking for $C^0$ continuous shell elements, unfortunately, they are not as effective if higher-order continuous basis is used. By leveraging $C^1$-smoothness, several new shear deformable shell formulations have been proposed which are based on a hierarchical parameterization of the rotation in the deformed configuration [28, 62–64]. These approaches eliminate shear locking \textit{a priori}. Recently, Bieber et al. [65] proposed the mixed displacement method, which uses the gradient of auxiliary displacement variables to represent mixed strain components. In this way, both shear and membrane locking can be alleviated while the method retains many of the features of a pure displacement-based formulation (i.e., no lower-order spaces have to be explicitly constructed). Unfortunately, the application of the gradient operator to the displacement auxiliary variables introduces additional zero energy modes into the system which must be removed with appropriate boundary conditions. In the fully unstructured spline setting, it is still uncertain how to devise these boundary conditions.

There is a vast literature on the subject of weak coupling of multiple patches. The main categories are the penalty method, the Lagrange multiplier method, the Nitsche method and the dual mortar method. The penalty method [29, 30, 66] weakly imposes a coupling constraint by introducing a penalty term into the variational formulation. It is simple to implement and it does not introduce any additional degrees-of-freedom. The drawback is that, to get an accurate result, a problem and mesh dependent penalty parameter must be selected. This parameter, if not properly adjusted during mesh refinement, results in ill-conditioned linear systems [29].

The Lagrange multiplier method employs a field of Lagrange multipliers to weakly enforce a coupling constraint. For structural mechanics problems, the field of Lagrange multipliers can be interpreted as the traction forces across an interface. In the context of mesh coupling, this method is also called the mortar method [67]. The additional Lagrange multiplier field leads to a saddle point variational formulation, which requires that the Lagrange multiplier space satisfy inf-sup stability and ideally have enough approximability to recover optimal convergence rates [67, 68]. In the context of IGA, the mortar method was first used to couple multiple non-uniform rational B-splines (NURBS) patches by Dornisch et al. [69], and then applied in nonlinear elasticity by
Hesch and Betsch [33]. Brivadis et al. [39] explored several choices for the Lagrange multiplier space theoretically and numerically.

The Nitsche method [70], originally introduced for the weak treatment of Dirichlet boundary conditions, is a method that has a variational structure between the Lagrange multiplier and penalty methods. In this approach, the Lagrange multiplier in the variational formulation is replaced by the normal flux, and an extra penalty-like stabilization term is added to restore the coercivity of the bilinear form. This method has been applied to the coupling of non-conforming meshes in many areas, including IGA [29, 31, 32, 36]. Like the penalty method, the stabilization term contains a parameter which must be estimated [29].

The approach proposed in [71, 72] embeds the coupling constraints into the finite element space directly, thus leading to a positive definite nonconforming variational problem. Based on [71, 72], Wohlmuth [73, 74] proposed a local dual Lagrange multiplier space and called the resulting formulation a dual mortar method. In contrast to a standard Lagrange multiplier method, in a dual mortar method the Lagrange multipliers can be eliminated easily leading to greater computational efficiency. In addition, the compact support of the local dual basis along with the biorthogonality preserves the sparsity of the stiffness matrix. Unfortunately, it is not easy to construct a local dual basis that possesses a high-order polynomial reproduction property [75, 76].

Dornisch et al. [34] developed a dual mortar method based on a global B-spline dual basis, and derived a relation matrix which enabled a condensation of the Lagrange multiplier degrees-of-freedom. A similar relation matrix is derived by Coox et al. [35] by inserting virtual knots on either side of an interface. This method is mathematically identical to the global dual method in [34] but is more efficient. However, it is limited to the case where the neighboring patches have the same degree and parameterization along the interface, which is a very restrictive requirement. Seitz et al. [37] proposed a local dual mortar method based on a NURBS basis for both patch coupling and contact mechanics. In this case, the local dual basis does not satisfy the polynomial reproduction property, so only reduced convergence rates are obtained. Other types of local dual basis functions, such as, the explicit de Boor-Fix dual basis [77–79] and the approximate dual basis [80] are explored in [38]. The de Boor-Fix dual basis functions have the same support as the B-spline basis functions. However, the polynomial reproduction property does not hold, leading to significantly deteriorated convergence rates. The approximate dual basis fulfills the
polynomial reproduction property but not biorthogonality. Therefore, the fully populated inverse matrix of the original mortar matrix must be approximated by a diagonal matrix to maintain the locality. This implies that the coupling constraints are not imposed exactly. Even though several numerical examples show that the approximate dual mortar method achieves convergence rates which are comparable to the global dual mortar method, a mathematical analysis of the effects of the approximation is still missing.

### 1.3 Objectives and contributions

The main objective of this dissertation is to develop effective and robust techniques to further improve the efficiency of isogeometric shell analysis. This goal is achieved by introducing extra flexibilities to analyze complex shell structures composed of multi-patch nonconforming meshes and by alleviating shear and membrane locking to improve the analysis accuracy of IGA shell formulations. The primary contributions are as follows:

- A local Bézier dual mortar method based on Bézier extraction and projection is proposed. This method does not add any additional degrees-of-freedom and preserves the sparsity of the resulting stiffness matrix. The coupling accuracy can be adaptively controlled by refining the proposed dual spline basis. This coupling technique is simple and robust, and can be applied to not only IGA shells but also other structural elements.

- A mixed geometrically nonlinear isogeometric shell formulation is developed which is capable of alleviating both shear and membrane locking and improving the quality of shear and membrane stresses. An efficient technique utilizing the dual spline basis adapted from the proposed multi-patch coupling technique is used to condense out the strain variables in the mixed shell formulation. This technique preserves the sparsity of the resulting stiffness matrix and preserves the accuracy of displacement and stress solutions produced by the non-condensed mixed formulation. This formulation is applied to both KL and RM shell formulations.

- Simple and efficient quadrature schemes, called Greville quadrature rules, are developed. These quadratures alleviate both shear and membrane locking and requires asymptotically
only one point for RM shell elements and two points for KL shell elements for all degrees. They are free of zero-energy modes regardless of mesh size, degrees, and smoothness. Any spurious finite-energy modes in the membrane stiffness can be easily stabilized by applying higher-order Greville quadratures, which leads to selective and reduced quadrature schemes for KL and RM shells, respectively. The method is applicable to non-uniform knot vectors of arbitrary degree and smoothness. The higher the continuity the more effective the method is at releasing locking and reducing the number of quadrature points.

1.4 Organization of this dissertation

The outline of this dissertation is as follows. Chapter 2 provides an introduction to the fundamental concepts of splines, including Bézier curves, B-splines and NURBS, and briefly reviews isogeometric analysis method. These preliminaries serve as the foundation of subsequent work. Chapter 3 introduces basic isogeometric shell theories based on both RM and KL assumptions. Emphasis is placed on the definitions of the RM shell formulation as this formulation allows $C^0$ continuous basis functions and facilitates imposing rotational boundary conditions through the rotational degree-of-freedom. Chapter 4 describes multi-patch domain decomposition problem, and proposes the isogeometric Bézier dual mortar method for coupling nonconforming meshes that only have $C^0$ continuity requirement. In Chapter 5, efficient locking-free shells based on the mixed Bubnov-Galerkin and Petrov-Galerkin formulations are developed, respectively. In Chapter 6, new quadrature schemes for IGA shell analysis are proposed for alleviating both shear and membrane locking. Finally, we draw conclusions and outline directions for future research in Chapter 7.
CHAPTER 2.  FUNDAMENTALS OF ISOGEOOMETRIC ANALYSIS

In this chapter, we provide in Section 2.1 a brief overview of Bézier, B-spline and NURBS curves, and surfaces focusing on the concepts and properties which are critical to the isogeometric analysis framework introduced in Section 2.3 and to further developments on isogeometric multi-patch coupling and shell analysis. Section 2.2 introduces the Bézier extraction concept, which is the cornerstone of the Bézier dual basis proposed later for multi-patch coupling and locking-free shells. For more detailed descriptions of the geometric concepts and isogeometric analysis see [81] and [82], respectively.

2.1 Spline preliminaries

2.1.1 Bézier curves

![Bézier curves](image)

(a) A univariate Bernstein basis.  (b) A Bézier curve and its control polygon.

Figure 2.1: A univariate Bernstein basis and a Bézier curve, $p = 2$.

The $I$th univariate Bernstein polynomial of degree $p$ on $[\xi_1, \xi_2]$ can be defined as

$$B^p_I(\xi) = \binom{p}{I-1} \left( \frac{\xi_2 - \xi}{\xi_2 - \xi_1} \right)^{p-I+1} \left( \frac{\xi - \xi_1}{\xi_2 - \xi_1} \right)^{I-1}, \quad (2.1)$$
where \( \binom{p}{I-1} = \frac{p!}{(I-1)!(p-I+1)!} \) is the binomial coefficient.

The set of Bernstein polynomials \( B(\xi) = \{ B^p_I(\xi) \}_{I=1}^{p+1} \) forms a basis for the space of polynomials of degree \( p \). Figure 2.1(a) shows a univariate quadratic Bernstein basis on \([0, 1]\). The Bernstein polynomials \( B(\tilde{\xi}) \) defined on \([\tilde{\xi}_1, \tilde{\xi}_2]\), can be related to the Bernstein basis \( B(\xi) \) defined on \([\xi_1, \xi_2]\) through the relation

\[
B(\tilde{\xi}) = (M)^{-T}B(\xi)
\]

(2.2)

with \( M \) the transformation matrix. A formula for the inverse of the transformation matrix \( M \) can be found in [83] and is written as

\[
(M)^{-1}_{IJ} = \sum_{l=\max(1,I+J-p-1)}^{\min(I,J)} B^I_{l-1}(\xi_2)B^{p-I+1}_{J-l+1}(\xi_1), \quad 1 \leq I, J \leq p+1.
\]

(2.3)

A degree \( p \) Bézier curve in \( \mathbb{R}^d \) can be written as

\[
x(\xi) = \sum_{I=1}^{p+1} P_I B^p_I(\xi), \quad \xi \in [\xi_1, \xi_2]
\]

(2.4)

where \( P_i \) is called a control point, as shown in Figure 2.1(b) for a quadratic Bézier curve.

### 2.1.2 B-spline curves

A univariate B-spline basis is defined by a knot vector \( \Xi = \{ \xi_1, \xi_2, \ldots, \xi_{n+p+1} \} \), which consists of a non-decreasing sequence of real numbers, \( \xi_I \leq \xi_{I+1}, I = 1, \ldots, n + p + 1 \), where \( p \) is the degree of the B-spline basis functions and \( n \) is the number of basis functions. The \( I \)th B-spline basis function of degree \( p \), denoted by \( N^p_I(\xi) \), can be recursively defined by

\[
N^0_I(\xi) = \begin{cases} 
1, & \text{if } \xi_I \leq \xi < \xi_{I+1} \\
0, & \text{otherwise}
\end{cases}
\]

\[
N^p_I(\xi) = \frac{\xi - \xi_I}{\xi_{I+p} - \xi_I} N^{p-1}_I(\xi) + \frac{\xi_{I+p+1} - \xi}{\xi_{I+p+1} - \xi_{I+1}} N^{p-1}_{I+1}(\xi).
\]
B-spline basis possesses the following properties:

- **Partition of unity:**

  \[ \sum_{I=1}^{n} N_I^p(\xi) = 1, \quad \xi \in [\xi_I, \xi_{n+p+1}] \quad (2.5) \]

- **Pointwise nonnegativity:**

  \[ N_I^p(\xi) \geq 0, \quad I = 1, 2, \ldots, n \quad (2.6) \]

- **Linear independence:**

  \[ \sum_{I=1}^{n} c_I N_I^p(\xi) = 0 \leftrightarrow c_I = 0, \quad I = 1, 2, \ldots, n \quad (2.7) \]

- **Compact support:**

  
  \[
  \begin{cases}
  N_I^p(\xi) > 0, & \text{if } \xi \in (\xi_I, \xi_{I+p+1}) \\
  N_I^p(\xi) = 0, & \text{otherwise}
  \end{cases}
  \quad (2.8)
  \]

These properties are important in Computer Aided Geometric Design (CAGD) and also desirable in the context of finite element analysis. For example, the linear independence and the compact support properties ensures a well conditioned and sparse stiffness matrix. An example of a univariate quadratic B-spline basis for a knot knot vector \( \Xi = \{0, 0, 0, 1, 2, 3, 4, 5, 5, 5\} \) is illustrated in Figure 2.2(a). Note that if a knot \( \xi_I \) has multiplicity \( k \) (i.e., \( \xi_I = \xi_{I+1} = \cdots = \xi_{I+k-1} \)), the basis functions are \( C^{p-k} \)-continuous at \( \xi_I \). When \( k = p \), the basis is \( C^0 \) and interpolatory at that location. The derivatives of a B-spline basis function can be given in terms of the derivatives of \( N_{I-1}^p \) and \( N_{I+1}^{p-1} \) as

\[
\frac{dN_I^p(\xi)}{d\xi} = \frac{p}{\xi_{I+p} - \xi_I} N_{I-1}^{p-1}(\xi) - \frac{p}{\xi_{I+p+1} - \xi_{I+1}} N_{I+1}^{p-1}(\xi).
\quad (2.9)
\]

A B-spline curve of degree \( p \) can be written as
(a) A univariate B-spline basis.

(b) A B-spline curve and its control polygon.

Figure 2.2: A quadratic univariate B-spline basis (a) and a B-spline curve (b) generated from a knot vector \( \Xi = \{0, 0, 0, 1, 2, 3, 4, 4, 5, 5, 5\} \).

\[
x(\xi) = \sum_{l=1}^{n} P_l N^p_l(\xi), \quad \xi \in [\xi_1, \xi_{n+p+1}] \tag{2.10}
\]

where \( P_l \) is called a control point. Figure 2.2(b) shows a quadratic B-spline curve as a linear combination of the B-spline basis functions in Figure 2.2(a).
For a degree $p$ B-spline with knot vector $\Xi = \{ \xi_1, \xi_2, \ldots, \xi_{n+p+1}\}$, the $I$th Greville abscissae [84] is given by

$$x_I = \frac{1}{p}(\xi_{I+1} + \xi_{I+2} + \cdots + \xi_{I+p}), \quad I = 1 \cdots n. \tag{2.11}$$

Figure 2.3 illustrates the Greville abscissae for a univariate quadratic B-spline associated with a knot vector $\Xi = \{0,0,0,1,2,3,4,4,4\}$. Note that the number of Greville points is equal to the number of B-spline basis functions.

Figure 2.3: Greville abscissae corresponding to the knot vector $\Xi = \{0,0,0,1,2,3,4,4,4\}$.

### 2.1.3 NURBS curves

Figure 2.4: Quadratic B-spline curves with knot vector $\Xi = \{0,0,0,1,2,3,4,4,4,5,5,5\}$ and different control weights $w_3 = 0.1, 0.5, 1$ and 1.5.
A $p$th-degree NURBS curve can be written as

$$x(\xi) = \sum_{I=1}^{n} P_I w_I R_I^p(\xi), \quad \xi \in [\xi_1, \xi_{n+p+1}]$$  \hspace{1cm} (2.12)

where the NURBS basis function $R_{I,p}$ is defined by

$$R_I^p(\xi) = \frac{N_I^p(\xi)}{W(\xi)}$$  \hspace{1cm} (2.13)

where $N_I^p(\xi)$ is the $I$th $p$-degree B-spline basis function,

$$W(\xi) = \sum_{I=1}^{n} w_I N_I^p(\xi)$$  \hspace{1cm} (2.14)

is the weighting function, and $w_I$ is the weight corresponding to $N_I^p(\xi)$. Since a NURBS curve is a rational polynomial it can be used to exactly represent conic sections. Note that as the partition of unity of the B-spline basis, if all weights $w_I$ are set to be 1, a NURBS curve degenerates to a B-spline curve. Therefore, NURBS is a generalization of the B-spline in fact. Figure 2.4 shows four different quadratic NURBS curves associated with a knot vector $\Xi = \{0,0,0,1,2,3,4,4,5,5,5\}$, and different control weights $w_3 = 0.3, 0.5, 1,$ and $1.5$, respectively.

### 2.1.4 NURBS surfaces

Higher dimensional analogs to spline curves described above can be created using tensor products of multiple univariate spline curves. Let $N_I^p(\xi)$ and $M_J^q(\eta)$ be degree $p$ and $q$ univariate B-spline basis functions defined on the knot vectors $\Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\}$ and $\eta = \{\eta_1, \eta_2, \ldots, \eta_{m+q+1}\}$. The $K$th bivariate NURBS basis function is defined

$$R_K^{p,q}(\xi, \eta) = \frac{M_J^q(\eta)N_I^p(\xi)w_K}{\sum_{I=1}^{n} \sum_{J=1}^{m} M_J^q(\eta)N_I^p(\xi)w_K},$$  \hspace{1cm} (2.15)

where

$$K(I,J) = m(I-1)+J \quad \text{with} \quad I \in 1,2,\cdots,n \quad \text{and} \quad J \in 1,2,\cdots,m$$  \hspace{1cm} (2.16)
Figure 2.5: 2D NURBS basis functions with all weights set to be one except for $w_1 = w_{16} = w_{18} = 2$. Knot vectors are $\mathcal{E} \times \mathcal{H} = \{0, 0, 0, 1, 2, 3, 4, 4\} \times \{0, 0, 0, 1, 2, 3, 4, 4\}$. 

(a) Parametric domain.
(b) 2D NURBS basis functions.
is an index mapping from the two univariate B-spline basis functions (or control points) index to the global bivariate NURBS basis functions (or control points). Similarly, a NURBS surface is defined as the linear combination of bivariate NURBS basis functions

\[
x(\xi, \eta) = \sum_{I=1}^{n \times m} R^p_0(\xi, \eta) P_I.
\]

(2.17)

Figure 2.5 illustrates the two-dimensional NURBS basis functions associated with knot vectors \( \Xi \times \mathcal{H} = \{0,0,0,1,2,3,4,4\} \times \{0,0,0,1,2,3,4,4\} \), of which all the weights are set to be one except for \( w_1 = w_{16} = w_{18} = 2 \).

### 2.2 Bézier extraction

The Bézier extraction process [85, 86] generates a linear operator, called the extraction operator, that maps a Bernstein basis onto a B-spline basis. In the context of one-dimensional B-splines, the extraction operator encodes the result of repeated knot insertion [87] such that the multiplicity of all interior knots of a knot vector is \( p \). A brief derivation is given as follows. For detailed algorithms, please refer to [85].

Assume that there is a set of knots \( \{\bar{\xi}_1, \bar{\xi}_2, ..., \bar{\xi}_m\} \) needed to be inserted into the original knot vector to complete the Bézier decomposition. The knot insertion process [87] needs to be repeated \( m \) times, and the final set of control points, \( P^b \), which defines the Bézier elements of the decomposition, is obtained by

\[
P^b = C^T P
\]

where \( C \) is the Bézier extraction operator generated from the knot insertion processes, \( P \) is a vector of the original B-spline control points. Since the knot insertion is essentially a change of vector space basis, the curve itself is not changed either geometrically or parametrically. Thus, we have a B-spline curve

\[
x(\xi) = P^T N(\xi) = (P^b)^T B(\xi) = (C^T P)^T B(\xi) = P^T C B(\xi)
\]
From the equation above, we get

\[ N(\xi) = CB(\xi). \]

The computation of Bézier extraction operator \( C \) is independent with either the control points or the basis functions and can be applied to the NURBS directly. Let \( \{ P_i / w_i \}_{i=1}^n \) be a set of NURBS control points, where \( P_i \) is the \( i \)th B-spline control points and \( w_i \) the associated weight. A NURBS curve is defined as

\[ x(\xi) = \sum_{i=1}^{n} w_i P_i N_{i,p}(\xi) / \sum_{i=0}^{n} w_i N_{i,p}(\xi) = \frac{P^T N(\xi)}{W^T N(\xi)} = \frac{P^T CB(\xi)}{W^T CB(\xi)} \]  

To formulate the control points of the Bézier elements, we define the vector \( w^b = W^T C = \{ w^b_1, w^b_2, \ldots, w^b_m \} \), and the diagonal matrix

\[ W^b = \begin{bmatrix} w^b_1 & 0 & 0 & \cdots & 0 \\ 0 & w^b_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & w^b_m \end{bmatrix}. \]

Equation (2.18) can be rewritten as

\[ x(\xi) = \frac{W^b (W^b)^{-1} P^T CB(\xi)}{(w^b)^T B(\xi)} = \frac{W^b (P^b)^T B(\xi)}{(w^b)^T B(\xi)} \]  

where \( P^b = C^T P (W^b)^{-1} \) are the new Bézier control points. Equation (2.19) can be interpreted as that the NURBS curve \( x(\xi) \) is decomposed into rational Bézier elements, whose control points are \( P^b = \{ P^b_i \} \), and the corresponding weights are \( w^b = \{ w^b_i \} \).

Thus far, we have shown that both B-splines and NURBS curves can be represented in terms of a set of Bézier elements. However, it is noted that the global extraction is never constructed in practice. Because in standard FEA frameworks, most quantities are calculated on the element
level. Element extraction operators $C^e$ can be obtained locally such as

$$N^e = C^eB.$$  \hspace{1cm} (2.20)

For higher-dimensional splines, such as surface and solid, the Bézier element extraction operator can be written as the tensor product of univariate element extraction operators in different parametric directions.

### 2.3 Isogeometric Analysis

In this section, we briefly introduce the IGA framework through a simple boundary-value problem. Suppose the differential equation for $u$ is defined on a one-dimensional domain $\Gamma$ as

$$\begin{cases}
\Delta u = f, & \text{on } \Gamma \\
u|_{\partial \Gamma} = 0
\end{cases}$$  \hspace{1cm} (2.21)

where $f : \Gamma \mapsto \mathbb{R}$ is a scalar function. After standard FEA procedures, (2.21) can be transformed into its equivalent variational form as:

Given $f : \Gamma \mapsto \mathbb{R}$. Find $u \in \mathcal{S}$ such that, for all $\delta u \in \mathcal{V}$,

$$\int_{\Gamma} \delta u \nabla u d\Gamma = \int_{\Gamma} \delta u f d\Gamma,$$  \hspace{1cm} (2.22)

where $\delta u$ is the testing function, $\mathcal{V}$ is the testing function space and $\mathcal{S}$ is the trial solution space. In this case, these two function spaces are identical, i.e.,

$$\mathcal{S} = \mathcal{V} = \{ \delta u \in H^1(\Gamma), \delta u|_{\partial \Gamma} = 0 \}.$$  \hspace{1cm} (2.23)

The key point of isogeometric analysis is that both the trial solution $u$ and the testing function $\delta u$ are discretized with the same smooth spline basis which is also used for representing the geometry,
i.e.,

\[ x = \sum_{A} x_{A}N_{A}, \quad (2.24) \]

\[ u \approx \sum_{A} u_{A}N_{A}, \quad (2.25) \]

\[ \delta u \approx \sum_{A} \delta u_{A}N_{A}. \quad (2.26) \]

The difference between the classical isoparametric FEA and IGA is that, for FEA, both the geometry and unknown fields are approximated by piecewise Lagrangian polynomials. Figure 2.6a and b illustrate a univariate quadratic B-spline basis and a univariate quadratic piecewise Lagrange polynomial basis that are used for IGA and FEA, respectively.
CHAPTER 3. ISOGEOOMETRIC SHELL ANALYSIS

In this chapter, fundamentals of isogeometric shell analysis based on both RM and KL shell theories are introduced. Emphasis is placed on RM shell as this shell formulation is more robust in different aspects. One major advantage is that RM shell allows $C^0$ continuous basis functions, which are inevitable in representing sharp features in complex CAD models. In contrast, KL shell requires at least $C^1$ continuous basis functions, otherwise extra $C^1$ continuity constraints have to be imposed. Additionally, the rotational degree-of-freedom can be leveraged to easily accommodate rotational boundary conditions, geometry with kinks, and non-manifold connections, thus facilitating modeling complex structural assemblies.

For RM shell, the definition of the current director is crucial for accuracy and simplicity of the resulting shell formulations. Two commonly used approaches, i.e., the continuous and discrete rotational approaches [26, 88–92], are introduced in Section 3.3, where an efficient scheme for updating the current director and its derivatives is proposed to reduce the storage cost and improve the computational efficiency.

3.1 Shell kinematics

The reference and current configurations of a shell-like body, as illustrated in Figure 3.1, are parameterized as

$$
X(\xi^1, \xi^2, \xi^3) = \bar{X}(\xi^1, \xi^2) + \xi^3 D(\xi^1, \xi^2),
$$

$$
\mathbf{x}(\xi^1, \xi^2, \xi^3) = \bar{\mathbf{x}}(\xi^1, \xi^2) + \xi^3 \mathbf{d}(\xi^1, \xi^2)
$$

where $-\frac{h}{2} \leq \xi^3 \leq \frac{h}{2}$. $\bar{X}$ and $\bar{\mathbf{x}}$ denote the midsurfaces, and $D$ and $\mathbf{d}$ denote the directors in the reference and current configurations, respectively. For conciseness, we will omit the parametric
coordinate $\xi^i$ in the following description, and adopt the established convention for Latin and Greek indices (i.e., $i = 1, 2, 3$ and $\alpha = 1, 2$).

The base vectors of the midsurface can be written as

$$
A_\alpha = \bar{X}_\alpha, \quad A_3 = D = \frac{A_1 \times A_2}{|A_1 \times A_2|},
$$

(3.3)

$$
a_\alpha = \bar{x}_\alpha, \quad a_3 = d
$$

(3.4)

where $(\cdot)_\alpha$ denotes $\partial(\cdot)/\partial \xi^\alpha$. The covariant base vectors at any point in the shell continuum are defined as

$$
G_\alpha = X_\alpha = \bar{X}_\alpha + \xi^3 D_\alpha = A_\alpha + \xi^3 D_\alpha \quad G_3 = X_3 = D,
$$

(3.5)

$$
g_\alpha = x_\alpha = \bar{x}_\alpha + \xi^3 d_\alpha = a_\alpha + \xi^3 d_\alpha \quad g_3 = x_3 = d.
$$

(3.6)

The Green-Lagrange strain tensor $E$ is defined as

$$
E = \frac{1}{2}(F^T F - I)
$$

(3.7)
where $\mathbf{F} = \frac{d\mathbf{x}}{d\mathbf{X}}$ is called the material deformation gradient and $\mathbf{I}$ is the identity tensor. In components, the Green-Lagrange strain can be written as

$$E_{ij} = \frac{1}{2}(g_{ij} - G_{ij}) \quad (3.8)$$

where

$$g_{ij} = g_i \cdot g_j, \quad G_{ij} = G_i \cdot G_j. \quad (3.9)$$

Substituting (3.5) and (3.6) into (3.8) we can write the components of the Green-Lagrange strain as

$$E_{ab} = \frac{1}{2} \left( a^a d^b + \xi^3 a^a \cdot d_\beta + \xi^3 d_\alpha \cdot a^a \right) - \left( A^a \cdot A_\beta + \xi^3 A^a \cdot D_\beta + \xi^3 D_\alpha \cdot A_\beta \right), \quad (3.10)$$

$$E_{a3} = \frac{1}{2} \left( a^a \cdot d + \xi^3 d_\alpha \cdot d - A^a \cdot D - \xi^3 D_\alpha \cdot D \right), \quad (3.11)$$

$$E_{33} = \frac{1}{2} \left( d \cdot d - D \cdot D \right), \quad (3.12)$$

where higher-order terms of $\xi^3$ are neglected.

The inextensibility assumption of the director, i.e. $||\mathbf{d}|| = 1$, leads to

$$\mathbf{d} \cdot \mathbf{d} = D \cdot D = 1, \quad (3.13)$$

$$\mathbf{d}_\alpha \cdot \mathbf{d} = D_\alpha \cdot D = 0 \quad (3.14)$$

and according to the definition of $\mathbf{D}$ we also have that $A^a \cdot D = 0$ and $A^a \cdot D_\beta = -D \cdot A^a_\beta = D_\alpha \cdot A_\beta$.

### 3.1.1 Reissner-Mindlin shell

The Reissner-Mindlin shell theory assumes that fibers straight and normal to the midsurface prior to the deformation remain straight during the deformation. In other words, the current director remains straight during deformation, but it is not necessary perpendicular to the midsur-
face. Therefore, the shear strain $\gamma_\alpha$ may be non-zero and the resulting shell formulation is shear deformable.

As a consequence of the inextensibility assumption and the definition of the reference directors, (3.10) to (3.12) become

$$E_{\alpha\beta} = \frac{1}{2} \left[ (a_\alpha \cdot a_\beta - A_\alpha \cdot A_\beta) + \xi^3 (a_\alpha \cdot d_\beta + d_\alpha \cdot a_\beta - 2D_\beta \cdot A_\alpha) \right],$$  \hspace{0.5cm} (3.15)

$$E_{\alpha 3} = \frac{1}{2} a_\alpha \cdot d,$$  \hspace{0.5cm} (3.16)

$$E_{33} = 0.$$  \hspace{0.5cm} (3.17)

Rewriting the non-zero strains with Voigt notation results in

$$\mathbf{E} = \begin{bmatrix} \epsilon + \xi^3 \kappa \\ \gamma \end{bmatrix}$$  \hspace{0.5cm} (3.18)

where $\epsilon$, $\kappa$ and $\gamma$ are the membrane, bending, and shear strains, respectively, which are defined as

$$\epsilon = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix}, \quad \kappa = \begin{bmatrix} \kappa_{11} \\ \kappa_{22} \\ 2\kappa_{12} \end{bmatrix}, \quad \text{and} \quad \gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$  \hspace{0.5cm} (3.19)

where

$$\epsilon_{\alpha\beta} = \frac{1}{2} (a_\alpha \cdot a_\beta - A_\alpha \cdot A_\beta),$$  \hspace{0.5cm} (3.20)

$$\kappa_{\alpha\beta} = \frac{1}{2} (a_\alpha \cdot d_\beta + d_\alpha \cdot a_\beta - D_\beta \cdot A_\alpha),$$  \hspace{0.5cm} (3.21)

$$\gamma_\alpha = a_\alpha \cdot d.$$  \hspace{0.5cm} (3.22)
3.1.2 Kirchhoff-Love shell

The Kirchhoff-Love shell theory assumes the current director is perpendicular to the mid-surface and defined as

\[
d = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|},
\]

(3.23)

In this case, the shear strain \( \gamma_\alpha \) vanishes and equations (3.10) to (3.12) can be simplified to

\[
E_{\alpha\beta} = \frac{1}{2} \left[ \left( \mathbf{a}_\alpha \cdot \mathbf{a}_\beta - \mathbf{A}_\alpha \cdot \mathbf{A}_\beta \right) + 2 \xi^3 (\mathbf{a}_\alpha,\beta \cdot \mathbf{d} + \mathbf{A}_\alpha,\beta \cdot \mathbf{D}) \right]
\]

(3.24)

\[
= \varepsilon_{\alpha\beta} + \xi^3 \kappa_{\alpha\beta}
\]

(3.25)

\[
E_{\alpha3} = E_{33} = 0
\]

(3.26)

where

\[
\kappa_{\alpha\beta} = -\mathbf{a}_{\alpha,\beta} \cdot \mathbf{d} + \mathbf{A}_{\alpha,\beta} \cdot \mathbf{D}.
\]

(3.27)

Note that compared with the bending strain for RM shell in (3.21), the derivatives of the director \( \mathbf{d} \) and \( \mathbf{D} \) are transferred to the tangent vector \( \mathbf{a}_\alpha \) and \( \mathbf{A}_\alpha \) for KL shell as shown in (3.27).

3.2 Variational formulation

For a Saint-Venant Kirchhoff material, the constitutive relation satisfies

\[
\mathbf{S} = \mathbf{C} \mathbf{E} \quad \text{on } \Omega
\]

(3.28)

where \( \mathbf{C} \) is the elasticity matrix defined in the curvilinear coordinate system, which can be calculated as

\[
\mathbf{C} = \mathbf{T}^T \begin{bmatrix} C_p & 0 \\ 0 & C_s \end{bmatrix} \mathbf{T},
\]

(3.29)
where \( T \) is the strain transformation matrix from local Cartesian components to covariant components \([93]\) and

\[
C_p = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1 - \nu}{2} \end{bmatrix}, \quad C_s = \kappa_s \frac{E}{2(1 + \nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

(3.30)

with Young’s modulus \( E \), Poisson’s ratio \( \nu \), and shear correction factor \( \kappa_s \).

The total potential energy can be written as

\[
\Pi(u) = \frac{1}{2} \int_{\Omega} E^T S d\Omega - \int_{\Omega} u^T F_b d\Omega - \int_{\Gamma_t} u^T F_t d\Gamma_t,
\]

(3.31)

where \( S \) is the second Piola-Kirchhoff stress, \( F_b \) and \( F_t \) are the body force and traction which act on the continuum body \( \Omega \) in the reference configuration and the traction boundary \( \Gamma_t \), respectively.

Equilibrium is the stationary point of the total potential functional \( \Pi(u) \) and can be written as

\[
\delta \Pi(u) = 0.
\]

(3.32)

For simplicity, we assume that the loads \( F_b \) and \( F_t \) are independent of the body deformation. In that case, (3.32) can be written as

\[
\delta \Pi(u, \delta u, \Delta u) = \int_{\Omega} \delta E^T C E d\Omega - \int_{\Omega} \delta u^T F_b d\Omega - \int_{\Gamma_t} \delta u^T F_t d\Gamma_t = 0.
\]

(3.33)

The linearization of the above variational form of equilibrium yields

\[
L[\delta \Pi(u, \delta u)] := \delta \Pi(u, \delta u) + \Delta \delta \Pi(\Delta u, \delta u) = 0
\]

(3.34)

where

\[
\Delta \delta \Pi(\Delta u, \delta u) = \int_{\Omega} \frac{\delta E^T C \Delta E}{\text{material stiffness}} + \frac{\Delta \delta E^T C E}{\text{geometric stiffness}} d\Omega.
\]

(3.35)
For a KL shell, the shear strains are zero, and analytical preintegration along the thickness direction can be used directly since the so-called shifter tensor [94] is the identity. Therefore, (3.33) and (3.35) can be simplified to

\[
\delta \Pi(\mathbf{u}, \delta \mathbf{u}) = h \int_A \delta \mathbf{e}^T \tilde{\mathbf{C}} \mathbf{e} d\Omega + \frac{h^3}{12} \int_A \delta \kappa^T \tilde{\mathbf{C}} \kappa d\Omega - \int_\Omega \delta \mathbf{u} \cdot \mathbf{F}_b d\Omega - \int_{\Gamma_t} \delta \mathbf{u} \cdot \mathbf{F}_t d\Gamma_t = 0,
\]

(3.36)

\[
\Delta \delta \Pi(\Delta \mathbf{u}, \delta \mathbf{u}) = \int_\Omega \underbrace{h \delta \mathbf{e}^T \tilde{\mathbf{C}} \Delta \mathbf{e}}_{\text{material stiffness}} + \frac{h^3}{12} \delta \kappa^T \tilde{\mathbf{C}} \Delta \kappa + h \Delta \delta \mathbf{e}^T \tilde{\mathbf{C}} \mathbf{e} + \frac{h^3}{12} \Delta \delta \kappa^T \tilde{\mathbf{C}} \kappa d\Omega,
\]

(3.37)

where

\[
\tilde{\mathbf{C}} = \hat{T}^T C_p \hat{T}
\]

(3.38)

and \(\hat{T}\) is the in-plane strain transformation matrix from local Cartesian components to covariant components [93].

### 3.3 Director definition for RM shell

In this section, we describe both a continuous and discrete approaches to define shell directors in the current configuration. In [26, 89, 92], the continuous approach calculates the current director by rotating the initial director at each quadrature point, while the discrete approach calculates the current director by interpolating current nodal directors which are also obtained by rotating the initial nodal directors. In both cases, a rotation tensor and its derivative need to be updated and stored at each quadrature point or node, respectively.

The present work follows the same update framework, however, the current directors at quadrature points in the continuous approach or at nodes in the discrete approach are obtained by rotating the previous directors at the corresponding locations. In this way, we only need to calculate and store the current director and its derivative information at each quadrature point or node rather than a rotation tensor and its derivatives, which reduces computational cost and storage. It can be easily shown that the proposed method is equivalent to that in [26, 89, 91, 92]. The derivations of
the variations of the directors and their derivatives follow standard approaches [26, 89, 92]. For additional details see Appendices B.1 and B.2.

3.3.1 The continuous approach

In this case, the current director is defined as

$$d^i(\xi^\alpha) = \Delta R d^{i-1}(\xi^\alpha)$$ (3.39)

where the superscripts $i$ and $i - 1$ indicate the Newton-Raphson iterate, $\Delta R$ is the incremental rotation tensor, and

$$\Delta R = I + c_1 \Delta \Omega + c_2 \Delta \Omega^2$$

$$c_1 = \frac{\sin \Delta \omega}{\Delta \omega}, \quad c_2 = \frac{1 - \cos \Delta \omega}{\Delta \omega^2}$$

$$\Delta \omega = |\Delta \omega|$$ (3.40)

where $\Delta \omega = \Delta \omega(\xi^\alpha) \in \mathbb{R}^d$ is the axial vector of the global incremental rotation at each quadrature point and

$$\Delta \Omega = \text{skew} \Delta \omega = \begin{bmatrix} 0 & -\Delta \omega_3 & \Delta \omega_2 \\ \Delta \omega_3 & 0 & -\Delta \omega_1 \\ -\Delta \omega_2 & \Delta \omega_1 & 0 \end{bmatrix}.$$ (3.41)

The derivatives of the director can be written as

$$d^i_{,\alpha} = \Delta R_{,\alpha} d^{i-1} + \Delta R d^{i-1}.$$ (3.42)

where

$$\Delta R_{,\alpha} = c_{1,\alpha} \Delta \Omega + c_1 \Delta \Omega_{,\alpha} + c_{2,\alpha} \Delta \Omega^2 + c_2 (\Delta \Omega_{,\alpha} \Delta \Omega + \Delta \Omega \Delta \Omega_{,\alpha})$$ (3.43)

$$\Delta \Omega_{,\alpha} = \text{skew} \Delta \omega_{,\alpha}$$

$$c_{1,\alpha} = \Delta \omega_{,\alpha} \frac{\Delta \omega \cos \Delta \omega - \sin \Delta \omega}{\Delta \omega^2}$$

$$c_{2,\alpha} = \Delta \omega_{,\alpha} \frac{\Delta \omega \sin \Delta \omega - 2 + 2 \cos \Delta \omega}{\Delta \omega^3}$$ (3.44)

$$\Delta \omega_{,\alpha} = \frac{\Delta \omega_{,\alpha} \cdot \Delta \omega}{\Delta \omega}.$$ (3.45)
Note that when $\Delta \omega$ is very small we choose

$$
c_1 = 1, \quad c_2 = \frac{1}{2} \quad \text{and} \quad c_{1,\alpha} = c_{2,\alpha} = \Delta \omega, \alpha = 0
$$

(3.46)

to ensure numerical stability.

It can be shown that the above approach to calculate $d^i$ and $d^i,\alpha$ is equivalent to the approach in [26, 89, 92], where

$$
d^i(\xi^\alpha) = R^iD(\xi^\alpha) \quad \text{and} \quad d^i,\alpha = R^i,\alpha D + R^iD,\alpha.
$$

(3.47)

Here the total rotation tensor $R^i$ is updated in a multiplicative manner as

$$
R^i = \Delta RR^i^{-1}
$$

(3.48)

to avoid the singularity of large rotations. Its derivatives can be written as

$$
R^i,\alpha = \Delta R,\alpha R^i^{-1} + \Delta RR^i,\alpha^{-1}.
$$

(3.49)

In this way, $R^i$ and $R^i,\alpha$ are required to be calculated and stored at every iteration. Also, $D$ and $D,\alpha$ need to be calculated at each iteration. However, the approach proposed here does not need to calculate these quantities and only requires $d^i$ and $d^i,\alpha$ to be stored. Therefore, except for the initial nodal director information, the number of variables to be stored at each quadrature point or node are reduced from 27 to 9.

### 3.3.2 The discrete approach

In this case, the current director is defined through the interpolation of the current nodal directors $d_I$ as

$$
d_h = \sum_I N_I d_I \quad \text{and} \quad d_{h,\alpha} = \sum_I N_I,\alpha d_I
$$

(3.50)
where

\[ \mathbf{d}_I = \Delta \mathbf{R}(\Delta \omega_I) \mathbf{d}_I^{i-1} \]  \hspace{1cm} (3.51)

\[ \Delta \omega_I = \mathbf{T}_{3 I} \Delta \beta_I. \]  \hspace{1cm} (3.52)

Note that \( \Delta \mathbf{R}(\Delta \omega_I) \) is calculated by inserting the nodal increment of the axial vector \( \Delta \omega_I \) into (3.40) and \( \mathbf{T}_{3I} \) is defined in (B.7). Again, we only store nodal director information at nodes.
CHAPTER 4. ISOGEOOMETRIC MULTI-PATCH COUPLING

In this chapter, we develop a new isogeometric multi-patch coupling method. This method is based on Bézier extraction and projection and is applicable to any spline space which can be represented in Bézier form (i.e., NURBS, T-splines, LR-splines, etc.). For this reason, we call the method the isogeometric Bézier dual mortar method. The approach weakly enforces the $C^0$ continuity of the solution at patch interfaces and the error can be adaptively controlled by leveraging the refineability of the underlying slave dual spline basis without introducing any additional degrees-of-freedom. As a consequence, optimal higher-order convergence rates can be achieved without the need for an expensive shared master/slave segmentation step. We also develop weakly continuous geometry as a particular application of isogeometric Bézier dual mortaring. A weakly continuous model is a CAD model in which weak continuity constraints are embedded directly into the geometry description. In this way, multi-patch models can be processed in a solver directly, without having to employ a mortaring approach during the solve.

In Section 4.1, we describe the multi-patch domain decomposition and the model problem that we use to define our method. Isogeometric Bézier dual mortaring is described in Section 4.2. We define weakly continuous geometry and its relationship to Bézier dual mortaring in Section 4.3. We demonstrate the utility of the approach in Section 4.4 through several challenging benchmark problems, including steady-state heat equations, and geometrically linear and nonlinear elasticity problems with two-dimensional solid and RM shell elements.

Note that the main work of this chapter has been presented in our work [95] except that here we also demonstrate the applicability of the proposed coupling technique to RM shell elements. For KL shells, extra effort is required to apply the $C^1$ continuity constraint along patch interfaces, which is not addressed in this dissertation. But we note that the proposed Bézier dual mortar method can be used for coupling KL shells and other coupling problems with higher-order continuity requirements. Interested readers are referred to [96] for additional details about this.
4.1 Problem description

4.1.1 Domain decomposition

Let $\Omega$ be a bounded domain decomposed into $K$ non-overlapping subdomains $\Omega^k$, i.e.,

$$\Omega = \bigcup_{k=1}^{K} \Omega^k, \text{ and } \Omega^i \cap \Omega^j = \emptyset, i \neq j.$$ 

We define the interfaces as the interior of the intersections of the boundaries, i.e., $\Gamma^k = \partial \Omega^i \cap \partial \Omega^j$. On each $\Omega^k$ the solution space $\mathcal{S}^k$ is defined as

$$\mathcal{S}^k = \{u^k \in H^1(\Omega^k), u^k|_{\partial \Omega^i \cap \partial \Omega^k} = u_0\}$$

where $H^1(\Omega^k)$ are the standard Sobolev spaces and $u_0$ are the Dirichlet boundary conditions. The corresponding weighting function spaces, $\mathcal{V}^k$, are similarly defined with homogeneous boundary conditions on $\partial \Omega^i \cap \partial \Omega^k$. The displacement solution space on $\Omega$ is the broken Sobolev space $\mathcal{S}$ defined as $\mathcal{S} = \prod_{k=1}^{K} \mathcal{S}^k$, along with continuity conditions defined along the interfaces. To simplify the exposition of the proposed mortaring technique, we employ a two-patch geometry with one interface, i.e., $K = 2$. The interface is denoted as $\Gamma = \partial \Omega^m \cap \partial \Omega^s$, where the superscripts $m$ and $s$ are used to denote the master and slave patches, respectively.

4.1.2 A linear elastic model problem

To ground our approach in a practical example, we consider the following linear elasticity problem:

$$\text{div } \sigma + b = 0 \quad \text{in } \Omega = \Omega^m \cup \Omega^s$$

(4.1a)

$$u = u_0 \quad \text{on } \Gamma_u = \Gamma_u^m \cup \Gamma_u^s$$

(4.1b)

$$\sigma \cdot n = t_0 \quad \text{on } \Gamma_\sigma = \Gamma_\sigma^m \cup \Gamma_\sigma^s$$

(4.1c)

$$u^m = u^s \quad \text{on } \Gamma$$

(4.1d)

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where \( \Gamma_u \cap \Gamma_\sigma = \emptyset, \Gamma_u \cap \Gamma = \emptyset, \Gamma_\sigma \cap \Gamma = \emptyset \), \( \sigma \) is the stress tensor, \( b \) is the body force, \( u_0 \) and \( t_0 \) are the prescribed Dirichlet and Neumann boundary conditions applied on \( \Gamma_u \) and \( \Gamma_\sigma \), respectively, and \( n \) is the unit outward normal vector on \( \partial \Omega \), see Figure 4.1. The kinematic coupling condition \( u^m = u^s \) is introduced along the interface \( \Gamma \) where \( u^m \) and \( u^s \) are the master and slave interface displacements, respectively.

Figure 4.1: A schematic of the linear elastic model problem.

The total potential energy \( \Pi \) of the system \( \Omega \) is

\[
\Pi(u) = \Pi^m(u^m) + \Pi^s(u^s) + \int_\Gamma \Phi \cdot (u^m - u^s) \, ds \tag{4.2}
\]

where \( \Pi^m \) and \( \Pi^s \) are the potential energy on \( \Omega^m \) and \( \Omega^s \), respectively, and \( \Phi \) is a Lagrange multiplier weakly enforcing the continuity constraint along the interface. Invoking the stationarity of \( \Pi \) with respect to \( u^m \), \( u^s \) and \( \Phi \), we obtain the weak formulation of (4.1) that reads as: find \( u^m \in \mathcal{V}^m \), \( u^s \in \mathcal{V}^s \) and \( \Phi \in \mathcal{V}^\ell \) such that for all variations \( \delta u^m \in \mathcal{V}^m \), \( \delta u^s \in \mathcal{V}^s \) and \( \delta \Phi \in \mathcal{V}^\ell \)
\[ \delta \Pi^*(u, \delta u^m) = \delta \Pi^m(u^m, \delta u^m) + \int_{\Gamma} \Phi \cdot \delta u^m \, d\Gamma = 0 \quad (4.3a) \]
\[ \delta \Pi^*(u, \delta u^s) = \delta \Pi^s(u^s, \delta u^s) - \int_{\Gamma} \Phi \cdot \delta u^s \, d\Gamma = 0 \quad (4.3b) \]
\[ \delta \Pi^*(u, \delta \Phi) = \int_{\Gamma} \delta \Phi \cdot (u^m - u^s) \, ds = 0 \quad (4.3c) \]

where \( \mathcal{S}^m \), \( \mathcal{S}^s \) and \( \mathcal{S}^\ell \) are the displacement solution approximation spaces on \( \Omega^m \) and \( \Omega^s \) and the Lagrange multiplier space, respectively, and \( \mathcal{V}^m \), \( \mathcal{V}^s \) and \( \mathcal{V}^\ell \) are the corresponding weighting function spaces. Note that in (4.2) we define the interface energy on the parametric domain of the slave interface, denoted by \( \hat{\Gamma} \), which results in the interface continuity condition (4.3c). As will be shown subsequently, this will allow us to define a dual basis which is independent of geometry, an important simplification which improves the efficiency of the approach.

### 4.2 Isogeometric Bézier dual mortaring

#### 4.2.1 Dual basis fundamentals

Suppose \( \mathcal{B}_p \) is a \((p + 1)\)-dimensional linear space generated by a set of linearly independent functions \( \{b_i\}_{i=1}^{p+1} \) of maximal degree \( p \). Given an inner product \( \langle \cdot, \cdot \rangle : \mathcal{B}_p \times \mathcal{B}_p \to \mathbb{R} \), the functions from the set

\[ \lambda_p := \{\lambda_i\}_{i=1}^{p+1} \quad (4.4) \]

satisfying the following conditions

\[
\begin{cases}
\text{span} \lambda_p = \mathcal{B}_p, \\
\langle b_i, \lambda_j \rangle = \delta_{ij}, \quad 1 \leq i, j \leq p + 1
\end{cases}
\quad (4.5)
\]

form the so-called dual basis corresponding to the basis \( \{b_i\}_{i=1}^{p+1} \) with respect to the inner product \( \langle \cdot, \cdot \rangle \). The first condition in (4.5) is called the reproduction property of order \( p \) and the second property is called the biorthogonality property. With local dual mortar method, \((p - 1)\)-order
reproduction property of the Lagrange multiplier space is required to guarantee the optimality of
the finite element space of order $p$ [76].

4.2.2 Polynomial dual basis functions

We will choose the Lagrange multiplier spaces to be those spanned by a dual spline basis
defined over $\hat{\Gamma}$ which emanate from the Bézier extraction and projection framework. A weighted
dual basis for each element domain $\hat{\Gamma}^e$ is defined as

$$\hat{N}^e = \text{diag}(\omega^e) (R^e)^T (G^e_{B,B})^{-1} B^{e,s}$$  \hspace{1cm} (4.6)

$$= D^e B^{e,s}$$  \hspace{1cm} (4.7)

where $B^{e,s}$ is the set of Bernstein polynomials defined on the $e$th slave interface element and

$$G^e_{B,B} = \left[ \int_{\hat{\Gamma}^e} B_i^{e,s}(\xi) B_j^{e,s}(\xi) \, ds \right]$$  \hspace{1cm} (4.8)

is the Gramian matrix for the Bernstein basis [97, 98], $R^e$ is the element reconstruction oper-
ator [97] and $D^e$ is called a dual element extraction operator. Note that $R^e$ is restricted to the
element boundary $\hat{\Gamma}^e$. We use the standard Bézier projection weighting, i.e.,

$$\omega^e = \frac{\int_{\hat{\Gamma}^e} N_i^{e,s} \, ds}{\int_{\hat{\Gamma}^I} N_i^I \, ds}$$  \hspace{1cm} (4.9)

where $\hat{\Gamma}^I$ is the domain of support for the interface basis function $N_i^I$ and $I(i,e)$ is a standard
mapping from element nodal indexing to a global index $I$. While other weightings could be used
this weighting has been shown to give particularly accurate results [97]. We can easily show that
the proposed dual basis satisfies the biorthogonality condition (4.5) by noting that

$$\int_{\hat{\Gamma}^e} \hat{N}^e (N^{e,s})^T \, ds = \text{diag}(\omega^e)$$  \hspace{1cm} (4.10)
and

\[ A_e \left[ \int_{\Gamma^e} \bar{N}^e (N^{e,s})^T ds \right] = I. \]  \hspace{1cm} (4.11)

where \( A \) is the usual finite element assembly operator. In other words,

\[ \int_{\Gamma} \bar{N}_I N^e_j ds = \delta_{IJ} \]  \hspace{1cm} (4.12)

as desired.

Note that even though this dual basis does not possess a higher-order polynomial reproduction property \[99\], optimal higher-order rates can be easily recovered through a simple refinement step. A theoretical explanation for this behavior is beyond the scope of this paper and will be pursued in a subsequent publication.

### 4.2.3 Rational dual basis functions

If rational basis functions are used, we define the dual basis as

\[ \tilde{R}_I = \frac{W}{w_I} \bar{N}_I \]  \hspace{1cm} (4.13)

where \( W \) is the rational weight given in (2.14). Now

\[ \int_{\Gamma} \tilde{R}_I R^s_j ds = \int_{\Gamma} \bar{N}_I N^s_j ds = \delta_{IJ}. \]  \hspace{1cm} (4.14)

### 4.2.4 Discretization

Over the slave interface we introduce the discretizations

\[ u^m = \sum_I N^m_I (\phi(\bar{\xi}^s)) \, d_I^m \]  \hspace{1cm} (4.15)

\[ u^s = \sum_I N^s_I (\bar{\xi}^s) \, d_I^s \]  \hspace{1cm} (4.16)

\[ \delta \Phi = \sum_I \bar{N}_I (\bar{\xi}^s) \, \delta \Phi_I \]  \hspace{1cm} (4.17)
where $\xi^s \in \Gamma^s$ is a parametric position on the slave interface and $\varphi : \Gamma^s \to \Gamma^m$ is a compositional mapping defined to be

$$\varphi = (x^m)^{-1} \circ x^s$$

(4.18)

where $x^s : \tilde{\Omega}^s \to \Omega^s$ and $x^m : \tilde{\Omega}^m \to \Omega^m$ are the slave and master geometric mappings, respectively, as shown in Figure 4.2. Note that we say the master and slave parameterizations are matched if the mapping $\varphi$ is linear, otherwise, we say the master and slave parameterizations are mismatched. In the mismatched case $\varphi$ can be computed using the Newton-Raphson algorithm.

![Figure 4.2: Slave and master geometric mappings, x^s and x^m.](image)

Note that discretizing (4.3c) and leveraging the biorthogonality property of the dual basis gives

$$d^s = G_{\tilde{N},Nm} d^m.$$  

(4.19)

where

$$G_{\tilde{N},Nm} = \int_{\Gamma} \tilde{N}_i(\xi^s) N^m_j (\varphi(\xi^s)) Ids$$

(4.20)

is a diagonal matrix.
The matrix form of (4.3) can be written as

\[
\begin{bmatrix}
K^m & 0 & (K^{\ell m})^T \\
0 & K^s & -(K^{\ell s})^T \\
K^{\ell m} & -K^{\ell s} & 0
\end{bmatrix}
\begin{bmatrix}
d^m \\
d^s \\
d^\ell
\end{bmatrix}
= \begin{bmatrix}
f^m \\
f^s \\
0
\end{bmatrix}
\] (4.21)

where \( K^m \) and \( K^s \) are standard patch-level stiffness matrices, \( f^m \) and \( f^s \) are the corresponding force vectors, and \( K^{\ell m} \) and \( K^{\ell s} \) are stiffness matrices with all entries equal to zero except for those related to the \( I \)th Lagrange multiplier basis \( \bar{N}_I \) and the \( J \)th master and slave interface basis \( N^m_J \) and \( N^s_J \), respectively. In other words, the nonzero entries of \( K^{\ell m} \) and \( K^{\ell s} \) can be written as

\[
K^{\ell m}_{\bar{N}_I, N^m_J} = \int_{\Gamma} \bar{N}_I N^m_J \, ds = (4.22)
\]

and

\[
K^{\ell s}_{\bar{N}_I, N^s_J} = \int_{\Gamma} \bar{N}_I N^s_J \, ds = \delta_{IJ}. \quad (4.23)
\]

The displacement vectors \( d^m \) and \( d^s \) can be split such that

\[
d^m = \begin{bmatrix} d^m_d \\ d^m_c \end{bmatrix} \quad \text{and} \quad d^s = \begin{bmatrix} d^s_d \\ d^s_c \end{bmatrix} \quad (4.24)
\]

and the corresponding stiffness matrices \( K^m, K^s, K^{\ell m} \) and \( K^{\ell s} \) are

\[
K^m = \begin{bmatrix} K^m_{dd} & K^m_{dc} \\ K^m_{cd} & K^m_{cc} \end{bmatrix} \quad K^s = \begin{bmatrix} K^s_{dd} & K^s_{dc} \\ K^s_{cd} & K^s_{cc} \end{bmatrix} \quad K^{\ell m} = \begin{bmatrix} 0 \\ \hat{K}^{\ell m}_{cd} \end{bmatrix}^T \quad \text{and} \quad K^{\ell s} = \begin{bmatrix} 0 \\ \hat{K}^{\ell s}_{cd} \end{bmatrix}^T \quad (4.25)
\]

where the subscript \( d \) indicates the distinct degrees-of-freedom internal to each patch, and \( c \) indicates the degrees-of-freedom along the interface. The entries of \( \hat{K}^{\ell m} \) and \( \hat{K}^{\ell s} \) are defined in (4.22) and (4.23), respectively. Substituting (4.24) and (4.25), and the relation (4.19) between \( d^s_c \) and \( d^m_c \) into (4.21) allows us to statically condense the Lagrange multiplier coefficients \( d^\ell \) and the slave
patch degrees-of-freedom $d_i$ leading to the simplified system

$$
\begin{bmatrix}
K_{dd}^m & K_{dc}^m & 0 \\
K_{cd}^m & K_{cc}^m + (\mathbf{G}_{N,Nm})^T \mathbf{K}_{cc}^s \mathbf{G}_{N,Nm} & (\mathbf{G}_{N,Nm})^T \mathbf{K}_{cd}^s \\
0 & K_{dc}^s \mathbf{G}_{N,Nm} & K_{dd}^s
\end{bmatrix}
\begin{bmatrix}
d_d^m \\
d_c^m \\
d_d^s
\end{bmatrix}
= 
\begin{bmatrix}
f_d^m \\
f_c^m + (\mathbf{G}_{N,Nm})^T f_c^s
\end{bmatrix}.
\quad (4.26)
$$

Figure 4.3: Refinement of a slave interface and corresponding control points (CPs).
4.2.5 Refinement of the dual basis

If the master and slave parameterizations are matched, the underlying basis have the same degrees, and the knots along the master interface are contained in the slave interface the interface constraint (4.3c) is imposed exactly. In this case, $G_{RN}$ is a standard spline refinement operator. In any case, the approximation can be improved without adding additional degrees-of-freedom to the global system by refining the slave interface and dual basis. We highlight that if the slave interface is refined, quadrature error accumulates if the new lines of reduced continuity in the slave interface are not accounted for in the element domains $\hat{\Omega}^{s,e}$ which touch the slave interface. For example, in Figure 4.3, two quadratic, linearly parameterized B-spline patches meet at a common interface. A refinement is performed in which all knots in the master interface which are not already present in the slave interface are added, i.e., the knot $\frac{1}{2}$ is inserted into the slave interface. To properly account for the new line of reduced continuity in the slave interface element $e_2$, it is subdivided into two elements $e_{21}$ and $e_{22}$ and quadrature is performed on both subelements. The nodes whose basis functions are supported by element $e_{21}$ are depicted in Figure 4.3. Note that the method is stable under refinement since dual basis refinement adds the same number of Lagrange multiplier degrees-of-freedom as slave interface degrees-of-freedom. As a result, the matrix $K^{fx}$ in (4.21) has full row rank.

For the case where many mismatched patches are required to define a geometric domain more advanced local refinement schemes could be utilized such as those commonly used for analysis-suitable T-splines [100]. In fact, the process described in Figure 4.3 for inserting master knots into the slave interface is a particular type of local T-spline knot insertion. For most of the cases considered in this paper, two global uniform refinements of the slave interface was enough to recover optimal higher-order convergence rates.

4.3 Weakly continuous geometry

Since the weak continuity constraint is defined on the parametric domain of the slave interface Bézier dual mortaring can also be viewed as an isogeometric design methodology for building multi-patch geometry where the weak continuity constraint is built into the space spanned by the geometric basis. In this case, weak geometric compatibility is preserved for any choice of control
points and the dual mortaring no longer needs to be incorporated into the finite element assembly algorithm. To build the weak continuity constraint into the element extraction operators we start by noticing that

\[ N^m = (G_{N,Nm})^T N^e \]  

which can be localized to each element on the interface

\[ N^{m,e} = (G_{N,Nm}^e)^T \tilde{R}^e B^{e,s} \]  

\[ = \tilde{R}^e B^{e,s} \tag{4.29} \]

where \( \tilde{R}^e \) is called a weakly continuous element extraction operator. Since

\[ \tilde{R}^e = (G_{N,Nm}^e)^T R^e \]  

\[ \tilde{R}^e = (G_{N,Nm}^e)^T R^e \]  

40
it is clear that each row of $\tilde{\mathbf{R}}$ (which corresponds to a master basis function) is formed by taking a linear combination of rows in $\mathbf{R}$ (which correspond to slave basis functions) where the weighting in the linear combination comes from the columns of $\mathbf{G}_{N,N_m}$. If the slave interface is refined, (4.27)

\begin{align*}
\mathbf{N}^m &= (\mathbf{G}_{N,N_m})^T \mathbf{N}
\end{align*}

Figure 4.5: Weakly continuous basis functions along an interface corresponding to the mesh shown in Figure 4.3.
where $\mathbf{N}'$ is the refined slave interface basis vector. Similarly, (4.28) can be written as

$$ \mathbf{N}^{m,e} = \left( \mathbf{G}_{N^e,N^m}^e \right)^T \mathbf{R}^{e,r} \mathbf{B}^{e,r} = \left( \mathbf{G}_{N^e,N^m}^e \right)^T \mathbf{R}^{e,r} \mathbf{M}^{-T} \mathbf{B}^{e,s} $$

(4.32)

where $\mathbf{R}^{e,r}$ is the standard element extraction operator defined on the refined slave interface, and $\mathbf{M}$ is the Bernstein basis transformation matrix defined in (2.3). Therefore, the weakly continuous element extraction operator can be written as

$$ \tilde{\mathbf{R}}^e = \left( \mathbf{G}_{N^e,N^m}^e \right)^T \mathbf{R}^{e,r} \mathbf{M}^{-T}. $$

(4.33)

Figure 4.4 shows the action of (4.33) for the interface element $e_{21}$ in Figure 4.3 and the resulting weakly continuous two-dimensional basis functions along the interface are shown in Figure 4.5. The full expressions for the weakly continuous element extraction operators $\tilde{\mathbf{R}}^e$ are given in the A.

### 4.4 Numerical examples

We evaluate the performance of the Bézier dual mortar method on several challenging benchmark problems. In all cases, dual basis refinement is employed where, during the first step of refinement, the master knots are projected into the slave interface. Subsequent refinement utilizes uniform element subdivision of the slave interface. To avoid the well-known mortar integral error in calculating $\mathbf{G}_{N^e,N^m}^e$ in (4.20) when no refinement of the dual basis is employed, the master knots are still projected into the slave interface and the integration is performed on the combined knot intervals as described in [38]. Note that the first step of refinement might generate subelements with high aspect ratios if the projected master knots are too close to the original slave interface knots. These knots can be easily filtered out. In this work, if the parametric interval of a slave subelement is less than $\frac{1}{20}$th of the original parametric interval the projected master knot is ignored. Note that during the first step of refinement, uniform subdivision could also be employed. However, a small error would be incurred due to the integration error in calculating $\mathbf{G}_{N^e,N^m}^e$. For complex parameterizations in three-dimensions (including volumetric parameterizations) this error may be an acceptable trade-off to avoid an expensive combined master/slave segmentation step. We compare
our method to a global dual mortar method [34, 38], where the global dual basis is computed using \( L^2 \) projection.

### 4.4.1 Steady-state heat equations

**A manufactured solution on a square domain**

We first solve the Laplace equation, \(-\Delta u = 0\), on the square domain, \( \Omega = (0, 1) \times (0, 1) \). The domain is modeled with two maximally smooth quadratic B-spline patches where the left patch is the master and the right patch is the slave as shown in Figure 4.6. Two different boundary conditions, shown in Figure 4.6a and b, are considered to demonstrate that, in contrast to the global dual mortar method, the Bézier dual mortar method does not suffer from the so-called crosspoint problem [38, 39]. This superior performance is due to the locality of the dual basis functions. Both boundary conditions satisfy the manufactured solution, \( u(x, y) = \sin(\pi y) \sinh(\pi x) \). The ratio of master to slave element size is initially chosen to be 2 : 3. The master and slave interface boundaries are matched but the underlying meshes are nonconforming.

![Figure 4.6: Two quadratic maximally smooth nonconforming B-spline patches.](image)
The sparsity patterns for the stiffness matrices for the proposed method, a global dual mortar method, and a standard conforming method are shown in Figure 4.7a, 4.7b, and 4.7c, respectively, after four applications of uniform global refinement. It is clear that the proposed method generates a sparse stiffness matrix with only a slight increase in bandwidth when compared to a conforming method.

![Figure 4.7](image-url)

Figure 4.7: Stiffness matrix sparsity patterns for (a) the proposed method, (b) a global dual mortar method, and (c) a standard conforming method. The nonconforming examples (i.e., (a) and (b)) are generated from nonconforming meshes after uniformly refining the initial mesh shown in Figure 4.6 four times. The resulting mesh has 13862 nodes. The conforming example in (c) is generated from a mesh with 13860 nodes.

We first present results for the boundary condition shown in Figure 4.6a. The convergence rates of the displacement error in the $L^2$-norm for several different degrees are shown in Figure 4.8a. The proposed approach is compared to a global dual mortar method. As can be seen, the global dual mortar method only gets optimal rates for $p = 1$, and for $p = 2, 3, 4$ the convergence rates are 2, which is suboptimal. This reduction in the rates can be attributed to the crosspoint problem [38, 39]. In other words, the Lagrange multiplier space is bigger than the primal space due to the crosspoint, which, in this case, corresponds to the points where the interface and Dirichlet boundary conditions intersect. As a result, inf-sup stability is lost. Without refinement of the dual basis, the proposed method achieves optimal rates for $p = 1$, and slightly deteriorated rates for $p = 2$. For $p = 3, 4$, the convergence rates are reduced but still converge faster than the global dual mortar method. To demonstrate the insensitivity of the method to master and slave selection,
we change the mesh ratio to \( m : s = 3 : 2 \). The convergence rates are shown in Figure 4.8b. As can be seen, the convergence rates are close to the previous case, \( m : s = 2 : 3 \).

We now refine the proposed dual space to improve the accuracy. The convergence rates are shown in Figure 4.9. As expected, with one refinement of the interface dual basis the proposed method obtains optimal convergence rates for all degrees \( p = 1, 2, 3, \) and 4 for both mesh ratios \( m : s = 2 : 3 \) and \( m : s = 3 : 2 \).

![Convergence rates for a square domain with two non-conforming patches, full Dirichlet boundary conditions (Figure 4.6a) and matched parameterizations.](image)

Figure 4.8: Convergence rates for a square domain with two non-conforming patches, full Dirichlet boundary conditions (Figure 4.6a) and matched parameterizations.

The second boundary condition case, shown in Figure 4.6b, allows for a comparison of the proposed method with the global dual method without crosspoint pollution. As shown in Figure 4.10, the optimality of the global dual mortar method can be observed for this case, while the proposed method behaves in a similar fashion as in the first full Dirichlet boundary condition case. Again, with one refinement, the Bézier dual mortar method obtains optimal rates as shown in Figure 4.11. This demonstrates that the proposed method is relatively insensitive to crosspoint pollution. This superior behavior can be attributed to the locality of the dual basis. The reduced rates in the proposed method without refinement is due to the lack of higher-order polynomial reproduction in the dual basis.
Figure 4.9: Convergence rates for a square domain with two non-conforming patches, full Dirichlet boundary conditions (Figure 4.6a) and matched parameterizations where the dual space is refined \( n \) times, \( n = 0, 1 \).

Figure 4.10: Convergence rates for a square domain with two non-conforming patches, Dirichlet-Neumann boundary conditions (Figure 4.6b), and matched parameterizations.

We next investigate the impact of mismatched parameterizations on the results. Mismatched parameterizations can be created by perturbing the position of the control points along the master and/or slave interfaces. To avoid crosspoint pollution we only consider the Dirichlet-Neumann boundary condition case. The convergence results for mesh ratio \( m : s = 2 : 3 \) without
Figure 4.11: Convergence rates for a square domain with two non-conforming patches, Dirichlet-
Neumann boundary conditions (see Figure 4.6b), matched parameterizations where the dual space
is refined \( n \) times, \( n = 0, 1 \).

Refinement are shown in Figure 4.12. Again, the global dual mortar method obtains the optimal
rates for \( p = 2, 3 \) and 4. Without refinement of the dual basis, the Bézier dual mortar method
behaves in a manner which is similar to the matched parameterization case. To improve solution
behavior we refine the dual basis. The resulting convergence rates are shown in Figure 4.13. Since
the geometric mapping is no longer linear, the continuity constraint (4.3c) cannot be imposed ex-
actly by refining the dual basis once. Therefore, optimal rates cannot be achieved. However, we
can improve the accuracy by simply refining the dual space additional times. As shown in Fig-
ure 4.13, for \( p = 2, 3 \), uniformly refining the dual space once recovers optimal rates and refining
twice recovers optimal rate for \( p = 4 \). Recall that regardless of how many times the dual space is
refined the number of global degrees-of-freedom remains fixed.

We next study the case where both the parameterizations and degrees are mismatched. In
this case, the degree of the master patch is one order higher than the slave patch. As shown in
Figure 4.14, without refinement of the dual basis, optimal rates are only obtained for the case \( p^m =
3, p^s = 2 \) and mesh ratio \( m : s = 2 : 3 \). Note that the optimal rates are bounded by the lower degree
among the different patches. As expected, refining the dual basis improves its approximation
ability. Therefore, optimal rates are obtained for \( p^m = 4 \), and \( p^s = 3 \) with both mesh ratios, \( m : s =
2 : 3 \) and \( m : s = 3 : 2 \).
Figure 4.12: Convergence rates for a square domain with two non-conforming patches, Dirichlet-
Neumann boundary conditions (Figure 4.6b) and mismatched parameterizations, master/slave
mesh ratio, $m : s = 2 : 3$.

Figure 4.13: Convergence rates for a square domain with two non-conforming patches, Dirichlet-
Neumann boundary conditions (Figure 4.6b) and mismatched parameterizations where the dual
space is refined $n$ times, $n = 0, 1, 2$, master/slave mesh ratio, $m : s = 2 : 3$.

**A manufactured solution on an annular domain**

We now solve the Poisson equation, $-\Delta u = f$, on the annular domain, $\Omega = \{(r, \phi) \mid 0.4 \leq r \leq 4, \pi/2 \leq \phi \leq \pi \}$. This example tests the effectiveness of rational dual basis functions. The
domain is composed of two NURBS patches as shown in Figure 4.15. The internal force and
the boundary conditions correspond to the manufactured solution, $u(x,y) = \sin(\pi x) \sin(\pi y)$. As
Figure 4.14: Convergence rates for a square domain with two non-conforming patches, Dirichlet-Neumann boundary conditions (see Figure 4.6b), mismatched parameterizations and degrees $p^m = p^s + 1$, where the dual space is refined $n$ times, $n = 0, 1, 2$.

Figure 4.15: An annular domain composed of two quadratic nonconforming NURBS patches.

shown in Figure 4.15, there are no crosspoints in this problem. Note that we only consider matched parameterizations in this example.

The convergence rates in the $L^2$-norm of the displacement are shown in Figure 4.16 for $p = 2, 3, 4$, without refining the dual basis. As can be seen, the global dual mortar method achieves
Figure 4.16: Convergence rates for an annular domain with two non-conforming NURBS patches, Dirichlet-Neumann boundary conditions (see Figure 4.15) and matched parameterizations.

Figure 4.17: Convergence rates for an annular domain with two non-conforming NURBS patches, Dirichlet-Neumann boundary conditions (see Figure 4.15) and refined matched parameterizations.

the optimal rates for all degrees and mesh ratios $m : s = 2 : 3$ and $m : s = 3 : 2$. The proposed method achieves the optimal rates for $p = 2, 3$ and $m : s = 2 : 3$, and slightly deteriorated convergence rate for $p = 4$. For mesh ratio $m : s = 3 : 2$, the proposed method experiences reduced convergence rates for $p = 3, 4$. However, one refinement recovers optimal rates for $p = 4$ with mesh ratio $m : s = 2 : 3$, and $p = 3$ and $4$ with mesh ratio $m : s = 3 : 2$, as shown in Figure 4.17.
4.4.2 2D solid problems

Infinite elastic plate with a circular hole

We next simulate the classical infinite elastic plate with a circular hole benchmark problem. In this case, we apply a constant traction in the $x$-direction at infinity. Due to symmetry, only one quarter of the plate is modeled as shown in Figure 4.19, where $T_x$ is the traction, $R$ is the radius of the hole, $L$ is the length of each side of the plate, $E$ is Young’s modulus, and $v$ is Poisson’s ratio.
An analytical solution to this problem can be found in [82] and is reproduced here for completeness.

\[
\sigma_{rr}(r, \theta) = \frac{T_s}{2} \left(1 - \frac{R^2}{r^2}\right) + \frac{T_s}{2} \left(1 - 4\frac{R^2}{r^2} + 3\frac{R^4}{r^4}\right) \cos 2\theta, \quad (4.34)
\]

\[
\sigma_{\theta\theta}(r, \theta) = \frac{T_s}{2} \left(1 + \frac{R^2}{r^2}\right) - \frac{T_s}{2} \left(1 + 3\frac{R^4}{r^4}\right) \cos 2\theta, \quad (4.35)
\]

\[
\sigma_{r\theta}(r, \theta) = -\frac{T_s}{2} \left(1 + 2\frac{R^2}{r^2} - 3\frac{R^4}{r^4}\right) \sin 2\theta. \quad (4.36)
\]
As shown in Figures 4.20a and b, we first decompose the geometry into two patches with matched and mismatched parameterizations, respectively. The convergence rates of the stress component $\sigma_{xx}$ in the $L^2$-norm are optimal for the global dual mortar method for all cases in Figure 4.21 and 4.23 due to the absence of crosspoints. For the matched parameterization case without refinement, the Bézier dual mortar method only achieves optimal rate for $p = 2$, $m : s = 2 : 3$. With one refinement of the dual basis, the proposed method recovers optimal rates for all cases as shown in Figure 4.22. For mismatched parameterizations without refinement, the Bézier dual mortar method exhibits similar reduced convergence rates as for the matched parameterization case as shown in Figure 4.23. However, as shown in Figure 4.24a, after one refinement the proposed method recovers the optimal rates for all degrees, $m : s = 3 : 2$, and refining the dual basis twice results in optimal rates for $p = 2, 3$, $m : s = 3 : 2$, as shown in Figure 4.24b.

![Figure 4.21: Stress convergence rates for a quarter plate with a circular hole decomposed into two nonconforming NURBS patches with matched parameterizations (Figure 4.20a).](image)

![Figure 4.21: Stress convergence rates for a quarter plate with a circular hole decomposed into three nonconforming NURBS patches with matched and mismatched parameterizations as shown in Figures 4.25a and b, respectively.](image)
Figure 4.22: Stress convergence rates for a quarter plate with a circular hole decomposed into two nonconforming NURBS patches with refined matched parameterizations (see Figure 4.20a).

(a) Master/slave mesh ratio, $m : s = 2 : 3$

(b) Master/slave mesh ratio, $m : s = 3 : 2$

Figure 4.23: Stress convergence rates for a quarter plate with a circular hole decomposed into two nonconforming NURBS patches with mismatched parameterizations (Figure 4.20b).

(a) Master/slave mesh ratio, $m : s = 2 : 3$

(b) Master/slave mesh ratio, $m : s = 3 : 2$

interface/Dirichlet intersections and the interface/interface intersection in the middle. For matched parameterizations without refinement, the Bézier dual mortar method achieves optimal rate for $p = 2$, and slightly deteriorated rate for $p = 3$ as shown in Figure 4.26a. Refining once recovers the optimal rates for $p = 3, 4$ (see Figure 4.26b). For mismatched parameterizations, optimal rates are achieved for $p = 2$ by refining the dual space once and for $p = 3$ by refining the dual space
Figure 4.24: Stress convergence rates for a quarter plate with a circular hole decomposed into two nonconforming NURBS patches with refined mismatched parameterizations (see Figure 4.20b).

twice (see Figure 4.27b). Figure 4.28 shows plots of the stress component $\sigma_{xx}$ for both matched and mismatched parameterizations for $p = 2$. It can be seen that even for the coarse initial meshes shown in Figures 4.25a and b, for both matched and mismatched parameterizations, the stress concentration in the circular cutout is very close to the analytical solution $\sigma_{xx} = 30$, as shown in Figure 4.28a and Figure 4.28c.

Figure 4.25: NURBS meshes for a quarter plate with a hole.
Figure 4.26: Stress convergence rates for a quarter plate with a circular hole decomposed into three nonconforming NURBS patches with matched parameterizations (Figure 4.25a).

Figure 4.27: Stress convergence rates for a quarter plate with a circular hole decomposed into three nonconforming NURBS patches with mismatched parameterizations (Figure 4.25b).
Weakly continuous geometry: Large deformations in two-dimensions

We now employ Bézier dual mortaring to build weakly continuous geometry as described in Section 4.3. Since the weak continuity constraint is embedded into the geometric description, a standard finite element code can be employed to process the weakly continuous basis in exactly the same manner as a standard conforming basis. To demonstrate the effectiveness of the approach, we compare the displacements computed on a weakly continuous mesh to those computed on a similar continuous mesh for a large deformation, plane strain problem.

The initial geometry and the location of the interface are shown in Figure 4.29. For the weakly continuous mesh, the discretization does not match at the interface between the two patches and continuity is enforced weakly by building appropriately modified extraction operators. For the
continuous mesh, the interface is treated as a $C^0$ interface. In both cases, the basis functions are quadratic maximally smooth B-splines. A Bézier element representation of the coarsest weakly continuous mesh is also shown in Figure 4.29. As can be seen, there is one additional element on the right side of the interface in the vertical direction. As the mesh is refined, the size of the elements on the left side of the interface is cut in half in each direction and the right side is refined so that there is always one additional element in the vertical direction. The continuous meshes are refined such that the element size is always the same as the element size on the left side of the weakly continuous meshes for a given refinement level.

We compare the computed results for the three load cases with associated boundary conditions that are shown in Figure 4.30. The deformation is governed by the strain energy density functional that is given by

$$
\psi = \lambda \left( \frac{1}{4} (J^2 - 1) - \frac{1}{2} \ln J \right) + \frac{1}{2} \mu (\text{tr}[\mathbf{b}] - 3 - 2 \ln J)
$$

(4.37)

where $\lambda$ and $\mu$ are the typical Lamé parameters with

$$
\lambda = \frac{E \nu}{(1 + \nu)(1 - 2\nu)}
$$

(4.38)

$$
\mu = \frac{E}{2(1 + \nu)}
$$

(4.39)
for Young’s Modulus, $E$, and Poisson’s ratio, $\nu$. We use $E = 30 \times 10^9$ and $\nu = 0.48$ for the results presented here. In addition,

$$ J = |F| \quad \text{and} \quad b = FF^T \quad (4.40) $$

where $F$ is the deformation gradient, and $b$ is the left Cauchy-Green tensor.

The pressure boundary condition, $p$, is applied as a dead load in the reference configuration and is increased in twenty equal load increments to a maximum value of $100 \times 10^9$. At each load increment, the nonlinear problem is solved using a Newton-Raphson scheme with convergence satisfied when the residual is reduced by a factor of $10^8$.

![Figure 4.30: Load cases](image)

The results of the computations are shown in Figures 4.31 through 4.33. Each figure shows the unscaled deformation at the final load increment. On the left, the color scale indicates the magnitude of the displacement. On the right, each patch is shown as a distinct color so that the deformation of the interface between the two patches can clearly be seen. In all cases, the deformation of the interface is severe, but there is nothing in the displacement plot that indicates the presence of the weak interface.

To quantify the accuracy of the weak geometry approach we compare the displacements, $u^{h,w}$, computed on the weakly continuous mesh to the displacements, $u^{h,c}$, computed on the con-
Figure 4.31: Vertical displacement and deformed configuration - Case 1.

Figure 4.32: Vertical displacement and deformed configuration - Case 2.

Figure 4.33: Horizontal displacement and deformed configuration - Case 3.
tinuous mesh. We define the relative error, $e_r$, to be the $L^2$-norm of the difference between the two considered displacements, i.e

$$e_r = \|u^{h,w} - u^{h,c}\|_{L^2}.$$  \hfill (4.41)

Using the triangle inequality,

$$\|u^{h,w} - u\|_{L^2} \leq \|u^{h,w} - u^{h,c}\|_{L^2} + \|u^{h,c} - u\|_{L^2};$$  \hfill (4.42)

we see that the absolute error of the solution computed on the weakly continuous mesh case is bounded by the sum of the relative error and the absolute error of the solution computed on the continuous mesh case. Now, assuming that the solution computed on the continuous mesh case converges optimally, by (4.42), if the relative error converges optimally, we know that the absolute error of the solution computed on the weakly continuous mesh case also must converge optimally. The convergence rates of the relative error of the last load step are plotted in Figure 4.34 for the three load cases. This figure clearly shows that the convergence rate of the relative error are cubic, which is the optimal rate for quadratic basis functions.

Figure 4.34: Convergence rates of the $L^2$-relative error of the last load step for the three large deformation load cases.
4.4.3 3D shell problems

Weakly continuous geometry: Scordelis-Lo roof

In this section, we apply the weakly continuous geometry concept to multi-patch Reissner-Mindlin shell analysis of the Scordelis-Lo roof problem. As shown in Figure 4.35, an 80° arc of a cylinder with radius $R = 25$, length $L = 50$, and thickness $t = 0.25$ is supported on each end by a rigid diaphragm. It is loaded with its own weight $q = 90$. The material has Young’s Modulus $E = 4.32 \times 10^8$, and Poisson’s ratio $\nu = 0$. The maximum displacement occurs on the free edge at $\frac{L}{2}$. There have been multiple theoretical solutions reported in the literature. The usual FEA solution converges to 0.3024 [101].

![Figure 4.35: Schematic for the Scordelis-Lo roof problem.](image)

The whole model is decomposed into two NURBS patches with initial conforming and nonconforming meshes as shown in Figure 4.36a and b, respectively, which are uniformly refined afterwards for convergence study. Through the Bézier dual mortar method, we build a weakly continuous geometry on the nonconforming mesh and analysis is directly conducted on it. Figure 4.37 shows the convergence of the maximum displacement for these two different meshes with degrees ranging from $p = 2$ to 4. As can be seen, the nonconforming meshes converge comparably with the conforming meshes, which demonstrates the effectiveness of the weakly continuous geometry method. In fact, for each refinement the nonconforming mesh achieves slightly better results than
Figure 4.36: Initial meshes for the Scordelis-Lo roof problem decomposed into two NURBS patches. Blue curves indicate the interfaces.

Figure 4.37: Convergence of the maximum displacement with $C^0$-continuous (conforming mesh) and weakly continuous (nonconforming mesh) geometries modeled with RM shell with continuous rotation of director, $p = 2, 3$ and 4.

the conforming mesh. This can be contributed to the fact that, for the nonconforming case, the mesh density of the slave patch is bigger than the conforming case.
CHAPTER 5. LOCKING-FREE SHELLS BASED ON MIXED FORMULATION

In this chapter, a mixed geometrically nonlinear isogeometric shell formulation is developed for the analysis of thin-walled structures that leverages Bézier dual basis functions to address both shear and membrane locking and to improve the quality of computed stresses. This formulation is based on the modified Hellinger-Reissner variational principle with independent displacement, membrane, and shear strains as the unknown fields. To overcome locking, the strain variables are interpolated with lower-order spline bases. However, to efficiently condense out the strain variables, the variations of the strain variables are interpolated with the proposed dual spline bases. The condensed approach preserves the accuracy of the non-condensed mixed approach but with fewer degrees-of-freedom. From a practical point of view, since the proposed dual spline basis is completely specified through Bézier extraction, any spline space that admits Bézier extraction can utilize the proposed approach directly.

The outline of this chapter is as follows. Section 5.1 introduces the mixed variational formulations which are the starting point for our formulations. We describe the interpolation scheme for the assumed shear and membrane strains in Section 5.2 and propose an efficient technique, based on the proposed dual spline basis functions, to condense out the assumed strain variables in Section 5.3. Several challenging benchmark problems are solved with KL and/or RM shell elements in Section 5.4 to verify the effectiveness of the proposed formulation at releasing shear and membrane locking. In addition, it also allows us to compare the accuracy of the continuous and discrete rotational approaches for RM shells without the pollution of locking.
5.1 Variational formulation

5.1.1 The Hu-Washizu variational theorem

The Hu-Washizu functional can be written as

$$\Pi_{HW}(\mathbf{u}, \tilde{\mathbf{E}}, \tilde{\mathbf{S}}) = \frac{1}{2} \int_{\Omega} \mathbf{\tilde{E}}^T \tilde{\mathbf{S}} \, d\Omega - \int_{\Omega} \tilde{\mathbf{S}}^T (\tilde{\mathbf{E}} - \mathbf{E}) \, d\Omega - \int_{\Gamma_u} (\mathbf{\tilde{S}n})^T (\mathbf{u} - \mathbf{u}_0) \, d\Gamma + \Pi_{ext}(\mathbf{u}) \tag{5.1}$$

where

$$\Pi_{ext}(\mathbf{u}) = - \int_{\Omega} \mathbf{u}^T \mathbf{F}_b \, d\Omega - \int_{\Gamma_f} \mathbf{u}^T \mathbf{F}_f \, d\Gamma_f \tag{5.2}$$

and \(\mathbf{u}, \tilde{\mathbf{E}},\) and \(\tilde{\mathbf{S}}\) are the unknown displacement, strain, and stress variables, \(\mathbf{E}\) is the Green-Lagrange strain defined in (3.18). Again, for simplicity, we assume that the loads \(\mathbf{F}_b\) and \(\mathbf{F}_f\) are independent of the body deformation.

The Hu-Washizu variational theorem can be simply written as

$$\Pi_{HW}(\mathbf{u}, \tilde{\mathbf{E}}, \tilde{\mathbf{S}}) = \text{Stationary.} \tag{5.3}$$

5.1.2 The Hellinger-Reissner variational theorem

For Saint-Venant Kirchhoff materials, if we let the displacement boundary condition and the stress-strain relation be satisfied strongly, i.e.,

\[
\mathbf{u} = \mathbf{u}_0 \quad \text{on } \Gamma_u \tag{5.4}
\]
\[
\tilde{\mathbf{S}} = \mathbf{C} \tilde{\mathbf{E}} \quad \text{on } \Omega \tag{5.5}
\]

where \(\mathbf{C}\) is the elasticity matrix and \(\mathbf{u}_0\) is the prescribed displacement on boundary \(\Gamma_u\), the Hu-Washizu functional can be written as

$$\Pi_{HR}(\mathbf{u}, \mathbf{E}) = \int_{\Omega} \left( \mathbf{E}^T \mathbf{C} \mathbf{E} - \frac{1}{2} \mathbf{E}^T \mathbf{CE} - \mathbf{u}^T \mathbf{F}_b \right) \, d\Omega - \int_{\Gamma_f} \mathbf{u}^T \mathbf{F}_f \, d\Gamma_f. \tag{5.6}$$
This is often called a modified Hellinger-Reissner functional as the strains, instead of stresses, are taken as the extra unknowns. At this point, (5.6) no longer depends on unknown stresses $\bar{\mathbf{S}}$ since they have been eliminated through the strong satisfaction of the constitutive law in (5.5).

The Hellinger-Reissner variational theorem can be written as

$$\Pi_{HR}(\mathbf{u}, \bar{\mathbf{E}}) = \text{Stationary.} \quad (5.7)$$

Invoking the stationarity of $\Pi_{HR}(\mathbf{u}, \bar{\mathbf{E}})$ results in

$$\delta \Pi_{HR}(\mathbf{u}, \delta \mathbf{u}, \bar{\mathbf{E}}, \delta \bar{\mathbf{E}}) = \int_{\Omega} \delta \mathbf{E}^T \mathbf{C} \bar{\mathbf{E}} - \delta \bar{\mathbf{E}}^T \mathbf{C} (\bar{\mathbf{E}} - \mathbf{E}) d\Omega$$

$$- \int_{\Omega} \delta \mathbf{u}^T \mathbf{F}_b d\Omega - \int_{\Gamma_t} \delta \mathbf{u}^T \mathbf{F}_t d\Gamma_t = 0. \quad (5.8)$$

The linearization of the above stationary conditions yields

$$L[\delta \Pi_{HR}(\mathbf{u}, \delta \mathbf{u}, \bar{\mathbf{E}}, \delta \bar{\mathbf{E}})] := \delta \Pi_{HR}(\mathbf{u}, \delta \mathbf{u}, \bar{\mathbf{E}}, \delta \bar{\mathbf{E}}) + \Delta \delta \Pi_{HR}(\Delta \mathbf{u}, \delta \mathbf{u}, \Delta \bar{\mathbf{E}}, \delta \bar{\mathbf{E}}) = 0 \quad (5.9)$$

where

$$\Delta \delta \Pi_{HR}(\Delta \mathbf{u}, \delta \mathbf{u}, \Delta \bar{\mathbf{E}}, \delta \bar{\mathbf{E}}) = \int_{\Omega} - \delta \mathbf{E}^T \mathbf{C} \Delta \bar{\mathbf{E}} + \delta \bar{\mathbf{E}}^T \mathbf{C} \Delta \mathbf{E} + \delta \mathbf{E}^T \mathbf{C} \Delta \mathbf{E} \quad \text{material stiffness}$$

$$+ \int_{\Gamma_t} \Delta \delta \mathbf{E}^T \mathbf{C} (\bar{\mathbf{E}} - \mathbf{E}) \mathbf{d} \Gamma_t. \quad (5.10)$$

5.2 Discretization of the mixed variational form

In this section, we introduce a proper way to define and discretize the strain variables. For generality, we assume both shear and membrane locking exist and the RM shell theory is used for subsequent development. If the KL shell theory is used, the shear strains are zero as mentioned in Section 3.1.2. In this case, all the following procedures can be followed in a straightforward way except that treatments on shear strains are not required.
5.2.1 Strain variable discretization

To eliminate membrane and shear locking, the assumed strains are defined as

\[
\bar{\mathbf{E}} = \begin{bmatrix} \bar{\varepsilon} + \bar{\kappa}^3 \mathbf{k} \\ \bar{\gamma} \end{bmatrix}
\]

(5.11)

where the bending strains, \(\kappa\), are not modified. If membrane locking is negligible, the membrane strain components, \(\bar{\varepsilon}\), can also remain unchanged, i.e., \(\bar{\varepsilon} = \varepsilon\). The assumed membrane and shear strains are interpolated with carefully chosen lower-order bases as

\[
\bar{\varepsilon} = \begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_3 \end{bmatrix} = \begin{bmatrix} \sum_{l=1}^{n_a} R_{l}^{p-1,q} \bar{\varepsilon}_{l1} \\ \sum_{j=1}^{n_b} R_{j}^{p,q-1} \bar{\varepsilon}_{j2} \\ \sum_{k=1}^{n_c} R_{K}^{p-1,q-1} \bar{\varepsilon}_{K3} \end{bmatrix}, \quad \bar{\gamma} = \begin{bmatrix} \bar{\gamma}_1 \\ \bar{\gamma}_2 \end{bmatrix} = \begin{bmatrix} \sum_{l=1}^{n_a} R_{l}^{p-1,q} \bar{\gamma}_{l1} \\ \sum_{j=1}^{n_b} R_{j}^{p,q-1} \bar{\gamma}_{j2} \end{bmatrix},
\]

(5.12)

where \(n_a\), \(n_b\) and \(n_c\) are the basis function numbers of the selectively reduced lower-order bases \(\{R_{l}^{p-1,q}\}_{l=1}^{n_a}\), \(\{R_{j}^{p,q-1}\}_{j=1}^{n_b}\), and \(\{R_{K}^{p-1,q-1}\}_{K=1}^{n_c}\) with \(p\) and \(q\) the degrees of the primal basis in \(\xi\) and \(\eta\) directions, respectively. The first variation of the assumed strains can be written as

\[
\delta \bar{\varepsilon} = \begin{bmatrix} \sum_{l=1}^{n_a} R_{l}^{p-1,q} \delta \bar{\varepsilon}_{l1} \\ \sum_{j=1}^{n_b} R_{j}^{p,q-1} \delta \bar{\varepsilon}_{j2} \\ \sum_{k=1}^{n_c} R_{K}^{p-1,q-1} \delta \bar{\varepsilon}_{K3} \end{bmatrix}, \quad \delta \bar{\gamma} = \begin{bmatrix} \sum_{l=1}^{n_a} R_{l}^{p-1,q} \delta \bar{\gamma}_{l1} \\ \sum_{j=1}^{n_b} R_{j}^{p,q-1} \delta \bar{\gamma}_{j2} \end{bmatrix},
\]

(5.13)

and the second variations vanish.

This choice of interpolation of the strain variables is inspired by the mixed displacement method [65], where some derivatives of the original primal basis are used to interpolate different strain components. It can be easily verified that the spans of the lower-order bases used here are the same as the derivative spaces utilized in [65] with appropriate constraints. These lower-order bases have also been used in [28] and [59] to release membrane locking for geometrically linear problems. The effectiveness of these interpolation schemes for releasing both shear and membrane
locking is demonstrated numerically in Section 5.4 for geometrically linear and nonlinear problems. A theoretical exploration of the inf-sup stability of the resulting discretizations [102, 103] in the regime of higher-order continuous isogeometric shell elements will not be explored in this paper.

5.2.2 Matrix formulation

Substituting (3.18) and (5.11) into the linearized stationary condition (5.9) gives

\[
\int_{\Omega} - \begin{bmatrix} \delta \bar{\varepsilon} + \xi^3 \delta \kappa \\ \delta \bar{\gamma} \end{bmatrix}^T \mathbf{C} \begin{bmatrix} \Delta \bar{\varepsilon} + \xi^3 \Delta \kappa \\ \Delta \bar{\gamma} \end{bmatrix} + \begin{bmatrix} \delta \bar{\varepsilon} + \xi^3 \delta \kappa \\ \delta \bar{\gamma} \end{bmatrix}^T \mathbf{C} \begin{bmatrix} \Delta \varepsilon + \xi^3 \Delta \kappa \\ \Delta \gamma \end{bmatrix} + \begin{bmatrix} \delta \varepsilon + \xi^3 \delta \kappa \\ \delta \gamma \end{bmatrix}^T \mathbf{C} \left( \begin{bmatrix} \bar{\varepsilon} + \xi^3 \kappa \\ \bar{\gamma} \end{bmatrix} - \begin{bmatrix} \varepsilon + \xi^3 \kappa \\ \gamma \end{bmatrix} \right)
\]

\[
\begin{bmatrix} \Delta \bar{\varepsilon} + \xi^3 \Delta \kappa \\ \Delta \bar{\gamma} \end{bmatrix} + \begin{bmatrix} \Delta \varepsilon + \xi^3 \Delta \kappa \\ \Delta \gamma \end{bmatrix}^T \mathbf{C} \begin{bmatrix} \bar{\varepsilon} + \xi^3 \kappa \\ \bar{\gamma} \end{bmatrix} - \begin{bmatrix} \varepsilon + \xi^3 \kappa \\ \gamma \end{bmatrix}
\]

\[d\Omega = - \int_{\Omega} \begin{bmatrix} \delta \varepsilon + \xi^3 \delta \kappa \\ \delta \gamma \end{bmatrix}^T \mathbf{C} \begin{bmatrix} \bar{\varepsilon} + \xi^3 \kappa \\ \bar{\gamma} \end{bmatrix} + \begin{bmatrix} \delta \bar{\varepsilon} + \xi^3 \delta \kappa \\ \delta \bar{\gamma} \end{bmatrix}^T \mathbf{C} \left( \begin{bmatrix} \bar{\varepsilon} + \xi^3 \kappa \\ \bar{\gamma} \end{bmatrix} - \begin{bmatrix} \varepsilon + \xi^3 \kappa \\ \gamma \end{bmatrix} \right) d\Omega
\]

\[+ \int_{\Omega} \delta \mathbf{u}^T \mathbf{F}_b d\Omega + \int_{\Gamma} \delta \mathbf{u}^T \mathbf{F}_d \mathbf{d} \gamma d\Gamma_e. \quad (5.14)
\]

We define the primal displacement/rotation and assumed strain nodal vectors by

\[
\hat{\mathbf{U}} = \begin{bmatrix} \mathbf{u}_1 \\ \beta_1 \\ \vdots \\ \mathbf{u}_I \\ \beta_I \\ \vdots \\ \mathbf{u}_{n_p} \\ \beta_{n_p} \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{E}} = \begin{bmatrix} \hat{\mathbf{E}}_1 \\ \hat{\mathbf{E}}_2 \\ \hat{\mathbf{E}}_3 \end{bmatrix}, \quad (5.15)
\]

respectively, where \( \mathbf{u}_I \) and \( \beta_I \) are the displacement and rotation unknowns corresponding to the \( I \)th primal node, \( n_p \) is the number of primal nodes, \( \hat{\mathbf{E}}_1 \), \( \hat{\mathbf{E}}_2 \) and \( \hat{\mathbf{E}}_3 \) are the assumed strain nodal vectors.
associated with the three different lower-order bases in (5.12), which are defined as

\[
\begin{align*}
\hat{\mathbf{E}}_1 &= \begin{bmatrix} \hat{e}_{11} \\ \hat{\gamma}_{11} \\ \vdots \\ \hat{e}_{I1} \\ \hat{\gamma}_{1} \\ \vdots \\ \hat{e}_{n_1} \\ \hat{\gamma}_{n_1} \end{bmatrix}, & \hat{\mathbf{E}}_2 &= \begin{bmatrix} \hat{e}_{12} \\ \hat{\gamma}_{12} \\ \vdots \\ \hat{e}_{I2} \\ \hat{\gamma}_{2} \\ \vdots \\ \hat{e}_{n_2} \\ \hat{\gamma}_{n_2} \end{bmatrix}, & \text{and } \hat{\mathbf{E}}_3 &= \begin{bmatrix} \hat{e}_{13} \\ \hat{\gamma}_{13} \\ \vdots \\ \hat{e}_{I3} \\ \hat{\gamma}_{3} \\ \vdots \\ \hat{e}_{n_3} \end{bmatrix}
\end{align*}
\tag{5.16}
\]

where \( \hat{\mathbf{e}}_{iI} \), \( \hat{\gamma}_{Ii} \) are membrane and shear strain unknowns of the \( I \)th assumed strain node of the \( i \)th lower-order basis.

Inserting the interpolations of the variations of displacement, assumed strains, and Green-Lagrange strain components (C.6), (C.7), and (C.8) into the linearized stationary condition (5.14) leads to the linear system solved at each Newton-Raphson iteration

\[
\begin{bmatrix}
K_{11}^{\text{mat}} + K_{12}^{\text{geom}} & K_{12}^{\text{mat}} \\
K_{21}^{\text{mat}} & K_{22}^{\text{mat}}
\end{bmatrix}
\begin{bmatrix}
\Delta \hat{U} \\
\Delta \hat{E}
\end{bmatrix}
= \begin{bmatrix}
\hat{F}_{\text{ext}} - \hat{F}_{\text{int}}^{b} \\
-\hat{F}_{\text{int}}^{ms}
\end{bmatrix},
\tag{5.17}
\]

where the material stiffness matrices \( K_{\alpha \beta}^{\text{mat}} \) are defined as

\[
\begin{align*}
K_{11}^{\text{mat}} &= \int_{\Omega} (\mathbf{B}^b)^T \mathbf{C} \mathbf{B}^{ms} + (\mathbf{B}^b + \mathbf{B}^{ms})^T \mathbf{C} \mathbf{B}^b \, d\Omega, \\
K_{12}^{\text{mat}} &= \int_{\Omega} (\mathbf{B}^{ms})^T \mathbf{C} \mathbf{B}^{ms} \, d\Omega, \\
K_{21}^{\text{mat}} &= \int_{\Omega} (\mathbf{B}^{ms})^T \mathbf{C} \mathbf{B}^{ms} \, d\Omega, \\
K_{22}^{\text{mat}} &= \int_{\Omega} -(\mathbf{B}^{ms})^T \mathbf{C} \mathbf{B}^{ms} \, d\Omega.
\end{align*}
\tag{5.18-5.21}
\]

Here the superscripts \( b \) and \( ms \) indicate that the strain-displacement matrices \( \mathbf{B}^b \) and \( \mathbf{B}^{ms} \) are derived from the variations of bending and membrane and shear strains, respectively. These strain-
displacement matrices can be written as

\[
B^b_l = \begin{bmatrix}
R_{l,1}d_1^{hT} & \bar{x}_1^{hT}T_{l,1} \\
R_{l,2}d_2^{hT} & \bar{x}_2^{hT}T_{l,2} \\
R_{l,2}d_1^{hT} + R_{l,1}d_2^{hT} & \bar{x}_1^{hT}T_{l,1} + \bar{x}_2^{hT}T_{l,2} \\
0 & 0
\end{bmatrix}
\]

and

\[
B^{ms}_l = \begin{bmatrix}
R_{l,1}\bar{x}_1^{hT} & 0 \\
R_{l,1}\bar{x}_2^{hT} & 0 \\
R_{l,1}d_1^{hT} + R_{l,2}\bar{x}_1^{hT} & \bar{x}_1^{hT}T_l \\
R_{l,1}d_2^{hT} & \bar{x}_2^{hT}T_l
\end{bmatrix}
\]

(5.22)

The strain-displacement matrix \( \bar{B}^{ms} \) is derived from the variations of assumed \textit{membrane} and \textit{shear} strains (5.13). As there are three types of assumed strain nodes, it is written at the element level as

\[
\bar{B}^{ms}_e = \begin{bmatrix}
R_1^{p-1,q} & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
R_l^{p-1,q} & 0 & 0 & 0 & 0 \\
0 & R_1^{p,q-1} & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & R_m^{p,q-1} & 0 & 0 & 0 \\
0 & 0 & R_1^{p-1,q-1} & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & R_n^{p-1,q-1} & 0 & 0 \\
0 & 0 & 0 & R_1^{p-1,q} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & R_l^{p-1,q} & 0 \\
0 & 0 & 0 & 0 & R_1^{p,q-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & R_m^{p,q-1}
\end{bmatrix}
\]

(5.23)

where \( l, m, \) and \( n \) denote the numbers of the three different strain node types defined over the element.
The $IK^{th}$ nodal submatrix of the geometric stiffness matrix $K_{geom}$ can be written as

$$K_{IK}^{geom} = \int_{\Omega} \begin{bmatrix} K_{IK_{11}}^{geom} & K_{IK_{12}}^{geom} \\ K_{IK_{21}}^{geom} & K_{IK_{22}}^{geom} \end{bmatrix} d\Omega$$

where

$$K_{IK_{11}}^{geom} = (R_{I,1}R_{K,1}\dddot{S}_{11} + R_{I,2}R_{K,2}\dddot{S}_{22} + R_{I,1}R_{K,2}\dddot{S}_{12} + R_{I,2}R_{K,1}\dddot{S}_{21})I,$$

$$K_{IK_{12}}^{geom} = \xi^3 S_{11} R_{K,1} T_{K,1} + \xi^3 S_{22} R_{K,2} T_{K,2} + \xi^3 S_{12} (R_{I,1} T_{K,2} + R_{I,2} T_{K,1}) + (\dddot{S}_{13} R_{I,1} + \dddot{S}_{23} R_{I,2}) T_K,$$

$$K_{IK_{21}}^{geom} = \xi^3 S_{11} R_{K,1} T_{K,1}^T + \xi^3 S_{22} R_{K,2} T_{K,2}^T + \xi^3 S_{12} (R_{K,1} T_{I,2}^T + R_{K,2} T_{I,1}^T) + (\dddot{S}_{13} R_{K,1} + \dddot{S}_{23} R_{K,2}) T_I^T,$$

$$K_{IK_{22}}^{geom} = \xi^3 S_{11} \hat{m}_I^{\beta\beta} (\dddot{X}_1) + \xi^3 S_{22} \hat{m}_K^{\beta\beta} (\dddot{X}_2) + \xi^3 S_{12} (\hat{m}_I^{\beta\beta} (\dddot{X}_1^h) + \hat{m}_K^{\beta\beta} (\dddot{X}_2^h)) + \dddot{S}_{13} \hat{q}_I^{\beta\beta} (\dddot{X}_1) + \dddot{S}_{23} \hat{q}_K^{\beta\beta} (\dddot{X}_2).$$

Here $\dddot{S}_{ij}$ are the components of second Piola-Kirchhoff stress $S$, and $\dddot{S}_{ij}$ are the stress components calculated from the assumed strains, i.e.,

$$\dddot{S} = CE.$$

The internal force vectors $\hat{F}_{int}^b$ and $\hat{F}_{int}^{ms}$ are defined as

$$\hat{F}_{int}^b = \int_{\Omega} (B_{ms}^{\dddot{S}})^T \dddot{S} + \xi^3 (B^{\dddot{b}})^T S d\Omega$$

$$\hat{F}_{int}^{ms} = \int_{\Omega} (\dddot{B}_{ms}^{\dddot{S}})^T (S - \dddot{S}) d\Omega.$$

The external force vector $\hat{F}_{ext}$ is calculated using standard approaches for RM shells [104]. We use numerical integration through the thickness of the shell. Analytical preintegration can be used if the shifter tensor is assumed to be the identity [26].

### 5.3 Condensation of assumed strain variables

The mixed shell formulation described previously effectively alleviates locking as will be shown numerically in Section 5.4. However, it can introduce up to five additional degrees-of-
freedom per assumed strain node. In traditional $C^0$ RM shell elements, the assumed strain fields are discontinuous along element interfaces and can be easily condensed out at the element level. However, for higher-order smooth shell elements this is no longer possible and alternative approaches must be employed. We devise an approach based on the dual spline basis functions proposed in Section 4.2 that preserves the sparsity of the resulting stiffness matrix and accuracy of the original mixed formulation.

### 5.3.1 Strain variable discretization and matrix formulation

We now define the variations of the assumed strains $\delta \bar{\epsilon}$ and $\delta \bar{\gamma}$ as

$$
\delta \bar{\epsilon} = \frac{1}{h} C^{-T} \begin{bmatrix}
\sum_{I=1}^{n_a} \bar{R}_I^{p-1,q} \delta \bar{\epsilon}_I \\
\sum_{J=1}^{n_b} \bar{R}_J^{p,q-1} \delta \bar{\gamma}_J \\
\sum_{K=1}^{n_c} \bar{R}_K^{p-1,q-1} \delta \bar{\epsilon}_K 
\end{bmatrix}, \quad \text{and} \quad
\delta \bar{\gamma} = \frac{1}{h} C^{-T} \begin{bmatrix}
\sum_{I=1}^{n_a} \bar{R}_I^{p-1,q} \delta \bar{\gamma}_I \\
\sum_{J=1}^{n_b} \bar{R}_J^{p,q-1} \delta \bar{\gamma}_J \\
\sum_{K=1}^{n_c} \bar{R}_K^{p-1,q-1} \delta \bar{\epsilon}_K 
\end{bmatrix},
$$

where $\bar{R}_I^{p-1,q}$, $\bar{R}_J^{p,q-1}$ and $\bar{R}_K^{p-1,q-1}$ are the dual spline bases corresponding to $R_I^{p-1,q}$, $R_J^{p,q-1}$ and $R_K^{p-1,q-1}$ and $h$ is the shell thickness. Note that as the integral in the variational form is defined on the physical domain, the dual basis used here are also defined on the physical domain so that we have the biorthogonality condition as

$$
\int_{\Omega} \bar{R}_I R_J d\Omega = \delta_{IJ}.
$$

The dual spline basis defined on the physical domain can be easily constructed by following the procedures given in Section 4.2 except that the Gramian matrix in Section 4.2 should be redefined on the physical domain as

$$
G_{B,B}^f = \left[ \int_{\Omega^e} B_i^f(\xi) B_j^f(\xi) \, ds \right].
$$

Since the assumed strain variables (5.12) and its variations (5.27) are discretized using different function spaces this is a so-called Petrov-Galerkin formulation. In this case, the structure
of the resulting linear systems remains the same as (5.17) where now $K_{21}^{\text{mat}}$ and $K_{22}^{\text{mat}}$ are defined as

$$K_{21}^{\text{mat}} = \int_{\Omega} (\tilde{B}^{ms})^T B^{ms} \, d\Omega$$

$$K_{22}^{\text{mat}} = \int_{\Omega} - (\tilde{B}^{ms})^T B^{ms} \, d\Omega$$

where $B^{ms}$ and $\tilde{B}^{ms}$ are the same as those defined in Section 5.2.2, and $\tilde{B}^{ms}$ is defined at element level as

$$\tilde{B}_{e}^{ms} = \frac{1}{h} \begin{bmatrix} \tilde{R}_1^{p-1,q} & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{R}_l^{p-1,q} & 0 & 0 & 0 & 0 \\ 0 & \tilde{R}_1^{p,q-1} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \tilde{R}_m^{p,q-1} & 0 & 0 & 0 \\ 0 & 0 & \tilde{R}_n^{p-1,q-1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \tilde{R}_1^{p,q-1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \tilde{R}_m^{p,q-1} \end{bmatrix}^T.$$  \hspace{1cm} (5.32)

When calculating $K_{22}^{\text{mat}}$ we assume that the basis vectors $G_i$ are constant through the thickness. This assumption and the biorthogonality condition of the dual basis (5.28) means that

$$K_{22}^{\text{mat}} = \int_{\Omega} - (\tilde{B}^{ms})^T B^{ms} \, d\Omega = -I.$$  \hspace{1cm} (5.33)
Figure 5.1: Stiffness matrix sparsity patterns for the Scordelis-Lo roof problem in Section 5.4.1, 64 elements per side. (a) a standard displacement-based RM formulation, (b) the mixed Bubnov-Galerkin formulation with strain variables condensed out, (c) the proposed Petrov-Galerkin formulation with strain variables condensed out.

Therefore, the assumed strain variables can be easily condensed out leading to the linear system

\[
\hat{\mathbf{K}}_{\text{mat}} + \hat{\mathbf{K}}_{\text{mat}}^{12} \hat{\mathbf{K}}_{\text{mat}}^{21} + \hat{\mathbf{K}}_{\text{geom}} \Delta \hat{\mathbf{U}} = \hat{\mathbf{F}}_{\text{ext}} - \hat{\mathbf{F}}_{\text{int}}^{b} - \hat{\mathbf{K}}_{12} \hat{\mathbf{F}}_{\text{int}}^{ms}. \tag{5.34}
\]

Notice that no inverse matrix is required and the final stiffness matrix

\[
\hat{\mathbf{K}} = \hat{\mathbf{K}}_{11}^{\text{mat}} + \hat{\mathbf{K}}_{12}^{\text{mat}} \hat{\mathbf{K}}_{21}^{\text{mat}} + \hat{\mathbf{K}}_{\text{geom}} \tag{5.35}
\]
remains sparse with a slightly wider bandwidth. Figure 5.1 shows the sparsity patterns for the stiffness matrices computed for the Scordelis-Lo roof problem in Section 5.4.1. The geometric stiffness matrix $K_{geom}$, the external force vector $\hat{F}_{ext}$, and the internal force vector $\hat{F}_{int}^b$ remain unchanged and are computed as described in Section 5.2.2. However, the internal force vector $\hat{F}_{int}^{ms}$ is now defined as

$$\hat{F}_{int}^{ms} = \int_\Omega (\hat{B}_{ms})^T C^{-1} (S - \bar{S}) \, d\Omega. \quad (5.36)$$

Within each Newton-Raphson iteration the assumed strain unknowns $\Delta \hat{E}^{ms}$ can be recovered as

$$\Delta \hat{E}^{ms} = K_{mat}^{21} \Delta \hat{U} + \hat{F}_{int}^{ms}. \quad (5.37)$$

5.4 Numerical examples

We now evaluate the performance of the proposed mixed shell formulations on several benchmark problems. Several different shell elements are compared:

- KL: *Kirchhoff-Love* shell [4].
- KLM: *Kirchhoff-Love* shell based on the mixed Bubnov-Galerkin formulation.
- KLMP: *Kirchhoff-Love* shell based on the mixed Petrov-Galerkin formulation.
- RMC: *Reissner-Mindlin* shell with *continuous* rotations of the director [26, 89, 92].
- RMD: *Reissner-Mindlin* shell with *discrete* rotations of the director [26, 91, 92].
- RMCMD: *Reissner-Mindlin* shell with *continuous* rotations of the director, based on the *mixed* displacement method [65].
- RMCM: *Reissner-Mindlin* shell with *continuous* rotations of the director, based on the *mixed* Bubnov-Galerkin formulation.
- RMDM: *Reissner-Mindlin* shell with *discrete* rotations of the director, based on the *mixed* Bubnov-Galerkin formulation.
• RMCMP: *Reissner-Mindlin* shell with *continuous* rotations of the director, based on the *mixed Petrov-Galerkin* formulation.

### 5.4.1 Geometrically linear problems

**Cylindrical shell subject to transverse loading in the radial direction**

![Figure 5.2: A cylindrical shell subject to transverse loading in the radial direction.](image)

Figure 5.2 shows the schematic for a cylindrical shell subject to transverse loading in the radial direction. This model has a radius $R = 10$, a width $b = 1$, and a varying thickness. Young’s modulus and Poisson’s ratio are $E = 1000$ and $\nu = 0$, respectively. The cylindrical shell is clamped at one end and subject to a traction, $q_x = 0.1r^3$, at the other end. An analytical solution based on Bernoulli beam theory gives an approximate value of 0.942 for the radial displacement at the free end. This problem is modeled with quadratic and cubic NURBS of maximal smoothness. The initial mesh has $2 \times 1$ elements, and uniform refinement is applied in the circumferential direction. Figure 5.3 shows the convergence of the maximum tip displacement $u_x$ with varying slendernesses $\frac{R}{t} = 100, 1000, \text{ and } 10000$. As can be seen, the proposed RMCM shell nearly obtains the reference solution with the initial mesh for $p = 2, 3$ and all slendernesses. The proposed RMCMP shell obtains the same accuracy as the RMCM shell but with strain variables condensed out.
Figure 5.3: A cylindrical shell modeled with elements based on the RM shell theory, convergence of maximum tip displacement $u_x$ with increasing slenderness $\frac{R}{t} = 100, 1000, 10000$ and degree $p = 2, 3$.

In contrast, the standard isogeometric shells RMC and RMD lock severely for the coarse meshes. The locking increases as the slenderness increases. For $p = 2$, RMC and RMD shells are comparable. For $p = 3$, and $\frac{R}{t} = 100$ the RMD shell achieves better results than the RMC shell as shown in Figure 5.3a. This can be attributed to the fact that the error from the discrete rotation of the director softens the system response [26]. However, this improvement disappears as the slenderness increases as shown in Figure 5.3b and c. The RMDM shell converges faster than the standard RMC and RMD shells for $p = 2$ and all slendernesses, but it is less accurate.
than the RMCM shell. This is because, once locking has been alleviated, the inaccuracy inherent in the discrete rotation of the director manifests itself. This inaccuracy worsens with increasing degree [26]. The proposed mixed shell formulations also accurately capture stresses. This is in stark contrast to standard isogeometric shells. Figure 5.4 shows the computed membrane force $n_{\phi\phi}$ for quadratic $C^1$ cylindrical shell consisting of 10 × 1 RMC, RMCM and RMCMP elements, respectively. Exact membrane force: max = 0; bin = −0.1$r^3$.

Figure 5.4: A cylindrical shell modeled with elements based on the RM shell theory, membrane force $n_{\phi\phi}$ for a quadratic $C^1$ cylindrical shell consisting of 10 × 1 RMC, RMCM and RMCMP elements, respectively. Exact membrane force: max = 0; bin = −0.1$r^3$.

stark contrast to standard isogeometric shells. Figure 5.4 shows the computed membrane force $n_{\phi\phi}$ for quadratic RMC, RMCM and RMCMP shell elements. The analytical solution of the membrane force is 0 at the free end and $-0.1r^3$ at the fixed end. As can be seen, with 10 elements the membrane force for the RMC shell oscillates significantly, especially for increasing slenderness,
and the maximum and minimum forces are several orders of magnitude off from the analytical solution. In contrast, the proposed RMCM and RMCMP shells achieve almost the same smooth and accurate results. Figure 6.10 shows the convergence of the maximum tip displacement $u_x$ for

![Graphs showing convergence of maximum tip displacement $u_x$ for different slenderness ratios $R/t$.](image)

(a) Slenderness $R/t = 100$.  
(b) Slenderness $R/t = 1000$.  
(c) Slenderness $R/t = 10000$.

Figure 5.5: A cylindrical shell modeled with elements based on the KL shell theory, convergence of maximum tip displacement $u_x$ with increasing slenderness $R/t = 100, 1000, 10000$ and degree $p = 2, 3$.

KL, KLM and KLMP shells. As can be seen, KLM and KLMP shells behave similarly as the RMCM and RMCMP shell as shown in Figure 5.3 and converge much faster than KL shell, which indicates that they effectively release membrane locking. Figure 5.6 also shows superior computed membrane force quality of KLM and KLMP shells over standard KL shell.
Figure 5.6: A cylindrical shell modeled with elements based on the KL shell theory, membrane force $n_{\phi\phi}$ for a quadratic $C^1$ cylindrical shell consisting of $10 \times 1$ KL, KLM, and KLMP elements, respectively. Exact membrane force: max = 0; bin = $-0.1r^3$.

Scordelis-Lo roof

The Scordelis-Lo roof problem is part of the shell obstacle course [101] and tests a shell element’s ability to handle both membrane and bending modes. The problem setup can be seen in Section 4.4.3. In addition to thickness $t = 0.25$, in this section, thickness $t = 0.025$ is also used as it exhibits more serious membrane locking. In this case, the maximum displacement on the free edge at $\frac{L}{2}$ is 32 given in [59].
Figure 5.7: Scordelis-Lo roof modeled with elements based on the RM shell theory, convergence of the maximum displacement, \( p = 2, 3 \), maximally smooth elements.

(a) \( t = 0.25 \)  
(b) \( t = 0.025 \)

Figure 5.8: Scordelis-Lo roof modeled with elements based on the RM shell theory, membrane force \( n_{\phi \phi} \), \( t = 0.25 \), maximally smooth elements.

(c) RMCM, pts=17 \times 17, \( p = 2 \).  
(d) RMCMP, pts=17 \times 17, \( p = 2 \).
Figure 5.9 shows the convergence behavior of the maximum displacement for both thicknesses $t = 0.25$ and $0.025$ in terms of degrees-of-freedom, where only one quarter of the geometry is modeled due to symmetry. The proposed RMCM shell converges faster than the standard RMC shells for all cases in terms of the number of degrees-of-freedom as shown in Figure 5.7a and b. Again, RMCMP shell achieves the same results as RMCM shell but with less degrees-of-freedom as strain variables are condensed out.

The proposed shell elements are not only superior in terms of displacement but also in their ability to represent stress fields. A numerical reference for the membrane force $n_{\phi \phi}$ computed using maximally smooth $p = 5$ isogeometric KL shell elements [4] with $50 \times 50$ control points for the whole model is shown in Figure 5.8a for $t = 0.25$. Figure 5.8b to d show the membrane force $n_{\phi \phi}$ for $t = 0.25$, $p = 2$. As can be seen, with $17 \times 17$ maximally smooth control points both RMCM and RMCMP obtain accurate and smooth stress resultants while the stresses computed using standard RMC shells oscillate significantly.

Figures 5.9 shows the convergence behavior of the maximum displacement for KL, KLM and KLMP shells, where, in this case, the whole geometry is modeled. Again, KLM and KLMP obtain comparable results and they release membrane locking effectively compared to the standard KL shell. Figure 5.10 shows the membrane force $n_{\phi \phi}$ for these types of elements with $13 \times$
Figure 5.10: Scordelis-Lo roof modeled with elements based on the KL shell theory, membrane force $n_{\phi\phi}$, $t = 0.25$, maximally smooth quadratic elements with control points number pts = 13 x 13.

13 maximally smooth quadratic control points and thickness $t = 0.25$. As can be seen, KLM and KLMP achieve smooth membrane force, while the standard KL shell gets highly oscillating membrane force.

5.4.2 Geometrically nonlinear problems

Straight cantilever shell subjected to end moment

In this section, we study the roll-up of the straight cantilever beam (modeled as a shell) shown in Figure 5.11. This problem is widely used to test the ability of a shell element to handle large rotations. The beam has a length of $L = 12$, width $b = 1$, and thickness $t = 0.1$. It is fixed
\[ L = 12 \]
\[ b = 1 \]
\[ t = 0.1 \]
\[ E = 1.2 \times 10^6 \]
\[ \nu = 0 \]

Figure 5.11: A cantilever shell subjected to an end moment.

on one end and subjected to a uniform line moment \( m = M/b \) on the free end. The material has Young’s modulus, \( E = 1.2 \times 10^6 \), and Poisson’s ratio, \( \nu = 0.3 \). The analytical solutions of the displacements from the classical flexural theory at the free end are given by 
\[
\begin{align*}
\Delta u_x (M) &= \left[ \sin \left( \frac{M}{M_0} \right) \frac{M_0}{M} - 1 \right] L \\
\Delta u_z (M) &= \left[ 1 - \cos \left( \frac{M}{M_0} \right) \right] \frac{M_0}{M}
\end{align*}
\]
with \( M_0 = \frac{EI}{L^2} = \frac{25}{3} \) [105]. The cantilever beam should roll up into an exact circle for \( M = 2\pi M_0 \) with a free end rotation of \( \theta = 2\pi \). In this test, the moment

Figure 5.12: Cantilever beam subjected to an end moment, convergence of the endpoint rotation, \( p = 2, 3 \), maximally smooth elements, 10 load steps.

\( M = 2\pi M_0 \) is applied in 10 load steps and the convergence of the free end rotation for maximally smooth B-splines of degree \( p = 2 \) and 3 are shown in Figure 5.12. The initial mesh consists of \( 4 \times 1 \) elements. As can be seen, the proposed RMCM shell element obtains \( 2\pi \) with very coarse meshes for both \( p = 2 \) and 3. The RMCMP shell obtains the same results as the RMCM shell but with
Table 5.1: Straight cantilever beam subjected to an end moment: Newton-Raphson iteration behavior of the last load step for RMC, RMCM and RMCMP shells, $p = 2$.

A residual norm of $1 \times 10^8$ is used as the tolerance for convergence.

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<tr>
<th>Last load step iterations</th>
<th>Norm of the global residual vector</th>
</tr>
</thead>
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<td>1.7221171e+04</td>
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<tr>
<td>3</td>
<td>2.6892551e+03</td>
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<tr>
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<td>0.0023499</td>
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<tr>
<td>11</td>
<td>1.2197837e-09</td>
</tr>
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</table>

Load step # 10 7 7
Total iteration # 110 51 50
Element # $64 \times 1$ $5 \times 1$ $5 \times 1$

less degrees-of-freedom. However, the RMDM shell element does not converge for the coarse mesh and even behaves worse than the standard RMC and RMD shells. This can be attributed to the inaccuracies inherent in the discrete rotations for coarse meshes and large rotations. For 10 load steps, the RMC shell requires 64 quadratic elements to converge to a circle, as shown in Figure 5.13a, while the RMCM and RMCMP shells only need 5 quadratic elements and 7 load steps to achieve a circle, as shown in Figure 5.13b and c, respectively. The total Newton-Raphson iteration counts and the iteration behavior of the last load step for the different shell formulations are shown in Table 5.1. The RMCM and RMCMP shells achieve a circle with only 51 and 50 total Newton-Raphson iterations, respectively, compared to 110 iterations for the RMC shell. Figure 5.14 shows the deflection at each load step for the RMCMP shell, which tracks the analytical solution very well. This example demonstrates the superior behavior of the proposed RMCM and RMCMP shells in handling large rotations.
Figure 5.13: Deformed configurations at each load step for different shell elements, $p = 2$.

**Hemispherical shell with hole**

The hemispherical shell problem tests a shell element’s ability to represent combined membrane and bending modes [106]. The geometry is a hemisphere with radius, $R = 10$, thickness, $t = 0.04$, and an 18° hole as shown in Figure 5.15. The hemisphere is first loaded with four point loads, $P = 2$, on the equator with alternating sign. Only one quarter of the hemisphere is modeled due to symmetry. The radial displacement at point $B$ is monitored and compared against a reference solution of 0.094 [106]. The convergence behavior of the different shell formulations are shown in Figure 5.16. As can be seen, the proposed RMCM and RMCMP converge much faster than the RMC shell for all cases. For $p = 2$, the RMCMP shell obtains better results than the RMCM shell.
Figure 5.14: Cantilever beam subjected to an end moment, load-deflection, 5 quadratic maximally smooth elements, 7 load steps.

Figure 5.15: Schematic for the hemispherical shell problem.

and they are comparable for $p = 3$. Additionally, we note that the proposed RMCM shell achieves almost the same results as the mixed displacement method.

Figure 5.17 shows the contour plots of the midsurface membrane stress $\sigma_{\theta\theta}$ (see Figure 5.15 for the definition of the $\theta$ direction) for RMC, RMCMD, RMCM, RMCMP with $16 \times 16$ maximally smooth cubic elements. The membrane stresses are smooth but since equal loads are applied on the equator with alternating signs and the geometry is symmetric, the membrane stress $\sigma_{\theta\theta}$ at points $A$ and $B$ (shown in Figure 5.15) should have the same absolute value but different signs, and the midpoint on the equator between $A$ and $B$ should be 0. Therefore, as can been seen
in Figures 5.17c and d, the proposed RMCM, RMCMP shells give reasonable stress states similar to the RMCMD shell while the stress state for the RMC shell is completely wrong.

**Figure 5.16**: Hemispherical shell, convergence of the maximum displacement, $p = 2, 3$, maximally smooth elements.

We then applied the load, $P = 200$, which results in large deformations and rotations. In this case, the reference solution of the radial displacement $u_y$ at point $B$ is $-5.86799$ [26]. Figure 5.18 shows the convergence of the radial displacement at point $B$ for different formulations. The load is applied with 10 equal load steps for $p = 2$ and $3$, respectively. In this case, the proposed RMCMP shell converges quickly to the reference solution for both $p = 2$ and $3$, which is faster than the RMCM shell. Again, the proposed RMCM shell achieved identical results as the RMCMD shell. To achieve the relative displacement error $|u_B - u_{ref}|/|u_{ref}| < 5\%$ at point $B$, the proposed RMCMP shell requires only $7 \times 7$ quadratic elements compared to the standard RMC shell which requires $28 \times 28$ elements. The RMCM and RMCMD shell both need $9 \times 9$ elements. The deformed configurations of the whole hemisphere are created by mirroring the quarter deformed configurations through the symmetric planes, as shown in Figure 5.19. The Newton-Raphson iteration information for the shell formulations is listed in Table 5.2. As can be seen, even though RMCM and RMCMP shells require far fewer elements than the RMC shell, they only require 60 total iterations which is less than the 89 iterations required by the RMC shell. RMCMD shell requires more total iterations than the RMCM shell in this case. This is due to the difference between the selectively reduced lower order bases in RMCM and the lower order spaces constructed by taking derivatives.
of the primal basis in RMCMD. Even though these two bases span the same function space, the individual basis functions differ, which will affect the results slightly.

Figure 5.17: Hemispherical shell, membrane stress $\sigma_{\theta \theta}$, $p = 3$, $16 \times 16$ maximally smooth elements.

5.4.3 Multi-patch coupling problems

In this section, we verify that the proposed RMCM and RMCMP shells can be applied to weakly continuous geometries.
Figure 5.18: Hemispherical shell with hole. Convergence of the displacement at point $B$, $p = 2, 3$, maximally smooth elements and 10 load steps.

Figure 5.19: Hemispherical shell with hole. Deformed configurations for RMC, RMCM, RMCMD and RMCMP shells with maximally smooth, quadratic elements and 10 load steps, $|u_B - u_{ref}|/|u_{ref}| < 5\%$ at point $B$. 
Table 5.2: Hemispherical shell with hole: Newton-Raphson iteration behavior for RMC, RMCMD, RMCM and RMCMP shells with 10 load steps, \( p = 2 \). A residual norm of \( 1 \times 10^7 \) is used as the tolerance for convergence.

<table>
<thead>
<tr>
<th>Last load step iterations</th>
<th>Norm of the global residual vector</th>
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<td>9 \times 9</td>
<td>9 \times 9</td>
<td>7 \times 7</td>
</tr>
</tbody>
</table>

**Scordelis-Lo roof**

![Graph of maximum displacement](image)

(a) \( t = 0.25 \).

![Graph of maximum displacement](image)

(b) \( t = 0.025 \).

Figure 5.20: Scordelis-Lo roof modeled with a weakly continuous geometry built on two patches of nonconforming meshes as shown in Figure 4.36b, convergence of the maximum displacement for RMC, RMCM and RMCMP shells, \( p = 2, 3 \).

The problem setup is described in Section 4.4.3 and the weakly continuous geometry is discretized with two patches of nonconforming meshes as shown in Figure 4.36b. Figure 5.20
illustrates the convergence behavior of the maximum displacement for RMC, RMCM and RMCMP shells with thicknesses $t = 0.25$ and $0.025$, and degrees $p = 2$ and $3$. As can be seen, the RMCM and RMCMP shells converge faster than the standard RMC shell. Again, the RMCM and RMCMP shells obtain comparable results but the latter uses less degrees-of-freedom.
CHAPTER 6. GREVILLE QUADRATURE: A ONE-POINT QUADRATURE SCHEME FOR ISOGEOOMETRIC SHELL ANALYSIS

In this chapter, we develop effective and robust quadrature rules, called Greville quadrature rules, to release membrane and shear locking. The process of determining the quadrature points and weights is simple and efficient. The one-dimensional quadrature points are chosen a priori to be the Greville abscissae associated with a corresponding knot vector and the weights are determined by solving a linear moment fitting system. The two-dimensional quadrature points and weights are generated by a simple tensor product of the corresponding one-dimensional quantities.

This chapter is organized as follows: Section 6.1 describes the Greville quadrature scheme. Section 6.2 explores the stability and accuracy of the Greville quadrature. Numerical examples are given in Section 6.3.

6.1 Definition of Greville quadrature

Numerical integration of a univariate function, \( f(x) \), can be written as

\[
\int_{\hat{\Gamma}} f d\hat{\Gamma} \approx \sum_{i=1}^{n} f(x_i)w_i,
\]  

(6.1)

where \( f \) is the integrand, \( \hat{\Gamma} \) is the integral domain, \( \{x_i\}_{i=1}^{n} \) are the \( n \) quadrature points, and \( \{w_i\}_{i=1}^{n} \) are the weights. Given a univariate \( p \)-degree \( (p \geq 2) \) B-spline basis \( \{N_i\}_{i=1}^{n} \) with an open knot vector \( \Xi = \{\xi_1, \xi_2, \ldots, \xi_m, \ldots, \xi_{n+p+1}\} \) and \((p + 1) < m \leq n\), we propose a way to determine the quadrature points and weights as follows:

1. If the basis functions have maximal continuity, i.e., \( C^{p-1} \) continuity, at all interior knots, the Greville abscissae \( \{x_i\}_{i=1}^{n} \), calculated by (2.11), are chosen to be the quadrature points and the weights \( \{w_i\}_{i=1}^{n} \) are determined so that the quadrature rule can integrate all linear combinations of the univariate B-spline basis \( \{N_i\}_{i=1}^{n} \). This can be accomplished by solving
the following moment fitting system of equations

\[
\begin{bmatrix}
\int_{\Gamma} N_1(\xi) d\xi \\
\int_{\Gamma} N_2(\xi) d\xi \\
\vdots \\
\int_{\Gamma} N_n(\xi) d\xi \\
\end{bmatrix}
\begin{bmatrix}
N_1(x_1) & N_1(x_2) & \cdots & N_1(x_n) \\
N_2(x_1) & \cdots & \cdots & N_2(x_n) \\
\vdots & \vdots & \vdots & \vdots \\
N_n(x_1) & \cdots & \cdots & N_n(x_n) \\
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
\vdots \\
w_n \\
\end{bmatrix},
\] (6.2)

where the left-hand side are the moments \([107, 108]\), and are computed using full Gauss quadrature.

2. If the continuity of the basis functions are reduced at knot \(\xi_m\), for example, the knot vector \(\Xi = \{\xi_1, \xi_2, \ldots, \xi_m, \ldots, \xi_{n+p+1}\}\) is segmented into two knot vectors as follows:

\[
\Xi_1 = \{\xi_1, \xi_2, \ldots, \xi_m\} \quad \text{and} \quad \Xi_2 = \{\xi_m, \ldots, \xi_{n+p+1}\}. \quad (6.3)
\]

The Greville abscissae associated with \(\Xi_1\) and \(\Xi_2\) are taken as the quadrature points and the corresponding weights are determined by solving (6.2) with the nonzero basis functions over \([\xi_1, \xi_m]\) and \([\xi_m, \xi_{n+p+1}]\), respectively.

![Figure 6.1: Greville quadrature points and weights for a quadratic B-spline basis with knot vector \(\Xi = \{0, 0, 0, 1, 2, 3, 4, 4, 4\}\). Red dots denote the locations of the quadrature points and \((\cdot, \cdot)\) indicates \((x_i, w_i), i = 1, 2, \ldots, 6\).](image)

Note that when the smoothness of the basis functions is reduced, integration accuracy with the quadrature determined by the first rule may decrease. However, the segmentation technique described by the second rule helps to maintain the integration accuracy which is verified numerically in Section 6.3.1. Figure 6.1 shows the Greville quadrature points and weights for a univariate quadratic B-spline basis associated with the knot vector \(\Xi = \{0, 0, 0, 1, 2, 3, 4, 4, 4\}\) and Figure 6.2
illustrates the segmentation approach for a univariate quadratic B-spline basis with initial knot vector \( \Xi = \{0, 0, 0, 1, 2, 2, 3, 4, 4, 4\} \). Note that the basis functions are \( C^0 \)-continuous at \( \xi = 2 \).

Figure 6.2: Segmented Greville quadrature points and weights for a quadratic B-spline basis with knot vector \( \Xi = \{0, 0, 0, 1, 2, 2, 3, 4, 4, 4\} \). Red dots denote Greville quadrature points and \((\xi, \cdot)\) indicates \((x_i, w_i), i = 1, 2, \cdots, 8\).

Note that the Greville quadrature points and weights are calculated with respect to the global parametric domain of the patch. To utilize the method in existing FEA routines we can easily map these quadrature points into a parent element coordinate system through an affine mapping. The mapped quadrature weights can be obtained by dividing computed weights \( w_i \) by the determinant of the Jacobian matrix that maps the parametric coordinate system into the parent co-
For a bivariate B-spline or NURBS basis, the quadrature points and weights are efficiently obtained through a simple tensor product of the corresponding univariate quantities. Figure 6.4 illustrates the distribution of Greville quadrature points for a bivariate B-spline basis which are generated through a tensor product of the univariate quantities shown in Figure 6.1. Detailed Greville quadrature point and weight information in terms of the parent coordinate system is shown in Table 6.1.

### 6.2 Stability and accuracy of Greville quadrature

Numerical stability requires that the discrete Galerkin forms be rank sufficient under quadrature. Reduced quadrature rules used in traditional $C^0$-continuous FEA usually lead to unstable systems, i.e., the numerically integrated stiffness matrix is rank-deficient and the system produces spurious zero-energy modes. The proposed Greville quadrature can be regarded as a reduced
Table 6.1: Parent element quadrature points and weights corresponding to Figure 6.4.

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quadrature rule for shell analysis with higher-order smooth spline discretizations. In this section, we investigate its stability and accuracy by studying the generalized eigenvalue problem in the context of a linear elasticity problem discretized with KL shell elements.

6.2.1 The generalized eigenvalue problem

The discrete generalized eigenvalue problem takes the form

$$(\mathbf{K} - \lambda \mathbf{M}) \Psi = 0,$$  \hspace{1cm} (6.4)
where \( \lambda \) is called the eigenvalue corresponding to the eigenvector \( \Psi \), \( K \) is the material stiffness matrix of the KL shell [109, 110], and \( M \) is the mass matrix where

\[
M_{ij} = h \int_A \rho N_i N_j \, dA
\]  

(6.5)

with \( h \) the shell thickness and \( \rho \) the mass density. For the displacement-based KL shell formulation considered above, both \( K \) and \( M \) are \( n_{eq} \times n_{eq} \) square matrices, where \( n_{eq} \) denotes the number of degrees-of-freedom. In this case, there exists \( n_{eq} \) eigenvalues and corresponding eigenvectors that satisfy (6.4).

We examine the generalized eigenvalue problem (6.4) on a square elastic plate with material modulus \( E = 1000 \), Poisson’s ratio \( \nu = 0 \), mass density \( \rho = 1000 \), thickness \( t = 0.1 \), and edge length \( L = 10 \). The bending and membrane modes are explored separately by applying different Dirichlet boundary conditions

\[
\begin{align*}
u_x = \nu_y &= 0 \quad \text{and} \quad \nu_z = 0,
\end{align*}
\]  

(6.6)

which result in three out-of-plane and three in-plane rigid body modes, respectively, as shown in Figures 6.5a and b. In these cases, the stiffness matrix \( K \) is positive semi-definite and the mass matrix is positive definite. As a consequence, all eigenvalues should be nonnegative real numbers, and the number of zero eigenvalues for each case should equal to the number of rigid body modes. Note that as the mass matrix does not introduce locking it is usually calculated with full Gauss quadrature. However, we observed that similar results are obtained when the mass matrix is calculated using Greville quadrature.

Numerical results with the proposed Greville quadrature verify that the resulting bending and membrane stiffness matrices have rank deficiency 3 for all mesh sizes and degrees from \( p = 2 \) to 5, which is equal to the number of rigid body modes as expected. Additionally, Figure 6.6 shows the convergence of the smallest eigenvalues normalized by a reference solution from full Gauss quadrature with \( 50 \times 50 \) maximally smooth quintic elements. These results illustrate that the proposed Greville quadrature does not introduce spurious zero-energy modes. Tables 6.2, 6.3, 6.4 and 6.5 list the lowest 50 nonzero eigenvalues corresponding to the bending modes for \( p = 2 \) to 5 with \( 50 \times 50 \) elements. As can be seen, the Greville quadrature scheme produces
(a) Out-of-plane rigid body modes for $u_x = u_y = 0$.

(b) In-plane rigid body modes for $u_z = 0$.

Figure 6.5: Rigid body modes for different boundary conditions. Dashed lines indicate initial configurations.

Figure 6.6: Convergence of the smallest eigenvalue normalized by a reference solution from full Gauss quadrature with $50 \times 50$ maximally smooth quintic elements.

Eigenvalues which are all close to those obtained from full Gauss quadrature, and the accuracy improves as the polynomial degree increases. Tables 6.6, 6.7, 6.8 and 6.9 list the lowest 50 nonzero eigenvalues corresponding to the membrane modes. In contrast, even though the proposed Greville quadrature accurately captures most of the eigenvalues for all degrees from $p = 2$ to 5, there are
a few unmatched ones. However, as the polynomial degree increases, the integration accuracy improves and the lower order unmatched eigenvalues disappear. These unmatched eigenvalues correspond to spurious modes with finite energy [111]. Figure 6.7 shows the first two spurious finite-energy modes for $p = 2$. To illustrate these two spurious modes more clearly, a light source is added to the mode plots and the $z$ coordinates are elevated by the function

$$z = 5\sqrt{2} - \sqrt{(x-5)^2 + (y-5)^2}.$$  

(6.7)

It is interesting to note that for $p = 2$ the proposed Greville quadrature rule has the same number of quadrature points as the so-called optimal reduced quadrature rule proposed in [1], and the locations of interior quadrature points are the same. Therefore, it can be inferred that the optimal reduced quadrature rule in [1] also produces spurious modes, several of which are shown in Figure 6.8.

### 6.2.2 Stabilization of spurious finite-energy modes

The spurious finite-energy modes are local oscillating modes produced by inaccurate quadrature. They cannot be completely suppressed through boundary conditions or by increasing the integration accuracy of boundary elements. These modes can be removed by either local stabilization techniques or by improving the integration accuracy. In this work, we propose a simple way to improve the integration accuracy by using a higher-order Greville quadrature along each parametric direction. This approach removes the spurious finite-energy modes. Given a univariate $p$-degree B-spline basis with knot vector $\Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\}$, instead of using the Greville abscissae and quadrature weights associated with this basis, the stabilized Greville abscissae and weights are determined from the degree $p + 1$ B-spline basis with knot vector $\Xi = \{\xi_1, \xi_1, \xi_2, \ldots, \xi_{n+p+1}, \xi_{n+p+1}\}$ through the same procedure described in Section 6.1. Note that the new knot vector is defined by increasing the end knot multiplicity by one. The number of Greville abscissae is increased by one compared with the original degree $p$ basis. We call this new quadrature rule the stabilized Greville quadrature.
Table 6.2: List of lowest non-zero eigenvalues corresponding to bending modes from full Gauss quadrature and Greville quadrature, \( p = 2 \) and \( 50 \times 50 \) maximally smooth elements (\( k \)—nonzero mode number).

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Table 6.5: List of lowest non-zero eigenvalues corresponding to bending modes from full Gauss quadrature and Greville quadrature, \( p = 5 \) and \( 50 \times 50 \) maximally smooth elements (\( k \)—nonzero mode number).

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Table 6.6: List of lowest non-zero eigenvalues corresponding to membrane modes from full Gauss quadrature and Greville quadrature, $p = 2$ and $50 \times 50$ maximally smooth elements ($k$—nonzero mode number, $s$—spurious mode).

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Table 6.7: List of lowest non-zero eigenvalues corresponding to membrane modes from full Gauss quadrature and Greville quadrature, $p = 3$ and $50 \times 50$ maximally smooth elements ($k$—nonzero mode number, $s$—spurious mode).

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Table 6.8: List of lowest non-zero eigenvalues corresponding to membrane modes from full Gauss
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elements ($k$—nonzero mode number, $s$—spurious mode).

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Table 6.9: List of lowest non-zero eigenvalues corresponding to membrane modes from full Gauss quadrature and Greville quadrature, $p = 5$ and $50 \times 50$ maximally smooth elements ($k$—nonzero mode number, $s$—spurious mode).

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Figure 6.7: Spurious finite-energy mode shapes produced by Greville quadrature, $p = 2$. A light source is added and the $z$ coordinates are elevated by $z = 5\sqrt{2} - \sqrt{(x-5)^2 + (y-5)^2}$. The black curves indicate the physical knot lines.

Figure 6.8: Spurious finite-energy mode shapes produced by the optimal reduced quadrature proposed in [1], $p = 2$. A light source is added and the $z$ coordinates are elevated by $z = 5\sqrt{2} - \sqrt{(x-5)^2 + (y-5)^2}$. The black curves indicate the physical knot lines.

The stabilized Greville quadrature rule removes the spurious finite energy modes in the membrane stiffness for all degrees from $p = 2$ to 5. Figure 6.9 illustrates the stabilized Greville quadrature points and weights for a quadratic basis with knot vector $\Xi = \{0,0,1,2,3,4,4,4\}$. 

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Tables 6.10 and 6.11 list the lowest 50 nonzero eigenvalues corresponding to the membrane modes for $p = 2$ and 3, respectively. Notice that all eigenvalues now closely match those computed with full Gauss quadrature.

One side effect of the stabilized Greville quadrature rule is that it introduces an instability into the bending modes of a KL shell discretized by a quadratic maximally smooth B-spline basis, because the second derivatives of the quadratic B-spline basis are indeterminate at interior knots, which happen to also be the stabilized Greville quadrature point locations as shown in Figure 6.9. To circumvent this issue for the KL shell, we only apply the stabilized Greville rule to the membrane stiffness while using the standard Greville rule for the bending stiffness. This can be considered to be a type of selective reduced integration scheme. For the RM shell, only first derivatives are required. If the basis functions are $C^0$ at some knots, the scheme described in Section 6.1 will produce two identical quadrature points at these knots and distribute them into the neighboring elements. In this way, the discontinuity in the first derivative of the basis functions can be captured accurately.

Figure 6.9: Stabilized quadrature points and weights for a quadratic B-spline basis with knot vector $\Xi = \{0, 0, 0, 1, 2, 3, 4, 4, 4\}$. The dots denote the Greville abscissae calculated from the knot vector $\Xi = \{0, 0, 0, 0, 1, 2, 3, 4, 4, 4, 4\}$ and $(\cdot, \cdot)$ indicates $(\xi_i, w_i), i = 1, 2, \cdots, 7$.

6.3 Numerical examples

We now evaluate the performance of the proposed Greville quadrature rules on several benchmark problems. Several different quadrature rules are compared:

- GAUSS: full Gauss quadrature;
- GREVI: unstabilized Greville quadrature used for both RM and KL shells;
- SGREVI: stabilized Greville quadrature used for RM shells;
Table 6.10: List of lowest non-zero eigenvalues corresponding to membrane modes from full Gauss quadrature and stabilized Greville quadrature, $p = 2$ and $50 \times 50$ maximally smooth elements ($k$—nonzero mode number).

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Table 6.11: List of lowest non-zero eigenvalues corresponding to membrane modes from full Gauss quadrature and stabilized Greville quadrature, \( p = 3 \) and 50 \times 50 maximally smooth elements (\( k \)—nonzero mode number).

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<td>6.31533e-01</td>
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<td>1.57914e+00</td>
<td>1.57916e+00</td>
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</tr>
</tbody>
</table>
• SR: selective reduced Greville quadrature used for KL shells, which uses SGREVI for the membrane stiffness and GREVI the for bending stiffness.

Two shell elements are used:

• KL: Kirchhoff-Love shell [109];

• RMC: Reissner-Mindlin shell with a continuous rotation of the director [26].

6.3.1 Geometrically linear problems

Cylindrical shell subject to transverse loading in the radial direction

The problem setup of a cylindrical shell subject to transverse loading in the radial direction is given in Figure 5.2 of Section 5.4.1. Figure 6.10 shows the convergence of the maximum tip displacement $u_x$ for the KL shell with varying quadrature rules, GAUSS, GREVI and SR, varying degrees $p = 2$ to $5$, and increasing slenderness $\frac{R}{t} = 100, 1000, \text{ and } 10000$. The initial mesh is composed of $2 \times 1$ maximally smooth elements and uniform refinement is used afterwards. As can be seen from Figures 6.10a, c and e, with the quadrature rule GREVI the tip displacement converges faster than GAUSS for all degrees and slendernesses except for the case $\frac{R}{t} = 100$ and $p = 5$. In this case, both GREVI and GAUSS almost achieve the reference solution with the initial mesh. This is due to the fact that the shell is not very thin and higher order bases release locking effectively. It can be also seen that for a fixed degree, the GREVI rule does not lock as the slenderness increases. This is not the case for the GAUSS rule which locks severely as the slenderness increases. This demonstrates that the GREVI rule eliminates membrane locking effectively. The SR rule does not completely remove membrane locking, but is still effective, especially after the first refinement as shown in Figures 6.10b, d, and f. Again, for slenderness $\frac{R}{t} = 100$ and $p = 3$ to $5$, locking is not severe and SR obtains results comparable to GAUSS. Figure 6.11 shows the convergence results for the RMC shell with GAUSS, GREVI and SGREVI rules. As can be seen, the behavior of the GREVI and SGREVI rules for the RMC shell is almost identical to the KL shell.
Figure 6.10: A cylindrical KL shell, convergence of maximum tip displacement $u_x$ with different quadratures, increasing slenderness $\frac{R}{t} = 100, 1000, 10000$, various degrees $p = 2$ to $5$, and maximally smooth elements. The initial mesh consists of $2 \times 1$ elements and uniform refinement is then used in the longitudinal direction.
Figure 6.11: A cylindrical RMC shell, convergence of maximum tip displacement \( u_x \) with different quadratures, increasing slenderness \( \frac{R}{t} = 100, 1000, 10000 \), various degrees \( p = 2 \) to 5, and maximally smooth elements. The initial mesh consists of \( 2 \times 1 \) elements and uniform refinement is then used in the longitudinal direction.
We now analyze a clamped square plate subject to a uniform distributed load as shown in Figure 6.12a. The square plate has length $L = 10$, Young’s modulus $E = 1000$, and Poisson’s ratio $\nu = 0.3$. The thickness $t$ is varied to give a slenderness ratio $\frac{L}{t}$. The distributed load is set to $q = t^3$. The maximum displacement at the center of the plate is monitored. The exact solution is $u_{\text{max}} = -0.138173$. The geometry is initially modeled with $4 \times 4$ maximally smooth regular elements as shown in Figure 6.12b and refined uniformly afterwards. For a flat plate, the membrane and bending strains are decoupled. Therefore, no membrane locking is present in this
Figure 6.13: A clamped square plate under a distributed load modeled with the RMC shell, maximum deflection $u_z$ with quadratures GAUSS, GREVI and SGREVI, increasing slendernesses $\frac{L}{t} = 100, 1000, 10000$ and various degrees $p = 2$ to 5. The initial mesh consists of $4 \times 4$ maximally smooth regular elements as shown in 6.12b and uniform refinement is used afterwards.

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Figure 6.14: A clamped square plate under a distributed load modeled with the RMC shell, maximum deflection $u_z$ with quadratures GAUSS and SGREVI, slendernesses $\frac{R}{t} = 1000$ and $10000$, and various degrees $p = 2$ to 5. The initial mesh consists of $4 \times 4$ maximally smooth skewed elements as shown in Figure 6.12c and uniform refinement is used afterwards.

Figure 6.13a shows that the GREVI rule achieves slightly better results than the GAUSS rule for $p = 2, 3$ and slenderness $\frac{R}{t} = 100$. They both almost reach the reference solution with the initial mesh for $p = 4$ and 5. This is because, in these cases, the plate is not very thin, therefore shear locking is less pronounced, and a higher order basis releases shear locking effectively. However, as can be seen in Figures 6.13c and e, the GREVI rule is superior to the GAUSS rule for all degrees as the slenderness increases. The GAUSS rule locks severely for $p = 2$ and 3. The stabilized Greville rule, SGREVI, is similar to the GREVI rule except for converging from below for $p = 2$ and all slendernesses as shown in Figures 6.13b, d, and f. We then explore the accuracy of the SGREVI rule for the skewed initial mesh shown in Figure 6.12c. Again, for slenderness $\frac{R}{t} = 100$, SGREVI and GAUSS are similar and are therefore omitted. However, the superiority of SGREVI is clear for $\frac{R}{t} = 1000$ and 10000 and degrees $p = 2$ to 5 as shown in Figures 6.14a and b.

We next investigate the effect of reduced continuity by inserting a $C^0$ or $C^1$ line in the middle of each parametric direction of the initial regular mesh. As can be seen in Figures 6.15a, c and e, when $C^0$ lines are inserted, SGREVI converges faster than GAUSS for $p = 2, 3$ and all
Figure 6.15: A clamped square plate under a distributed load modeled with the RMC shell, maximum deflection $u_x$ with GAUSS and SGREVI quadratures, slendernesses $\frac{L}{T} = 100$, 1000 and 10000. A reduced continuity line ($C^0$ on the left and $C^1$ on the right) is inserted in the middle of each parametric direction of the initial regular mesh as shown in Figure 6.12b.
(a) Slenderness $\frac{L}{t} = 100$, with $C^0$ lines.
(b) Slenderness $\frac{L}{t} = 100$, with $C^1$ lines.
(c) Slenderness $\frac{L}{t} = 1000$, with $C^0$ lines.
(d) Slenderness $\frac{L}{t} = 1000$, with $C^1$ lines.
(e) Slenderness $\frac{L}{t} = 10000$, with $C^0$ lines.
(f) Slenderness $\frac{L}{t} = 10000$, with $C^1$ lines.

Figure 6.16: A clamped square plate under a distributed load modeled with the RMC shell, comparisons of SGREVI quadrature with and without knot vector segmentation at the locations of reduced continuity, slendernesses $\frac{L}{t} = 100, 1000$ and 10000. A reduced continuity line ($C^0$ on the left and $C^1$ on the right) is inserted in the middle of each parametric direction as shown in Figure 6.12b.
slendernesses. When \(C^1\) lines are inserted, both GAUSS and SGREVI behave similarly to the \(C^0\) cases as shown in Figure 6.15, on the right.

In this problem, we also show the importance of segmenting knot vectors when the smoothness of basis functions is reduced as described in Section 6.1. As shown in Figures 6.16a, c and e, for SGREVI, even though not segmenting knot vectors seems to release locking more effectively than segmenting knot vectors for \(p = 2\) with \(C^0\)-continuous lines, the former is less stable than the latter for \(p = 3\) to 5 and all slendernesses. This behavior is also observed when \(C^1\) lines are inserted as shown in Figures 6.16b, d, and f. We also observed similar behavior for other benchmarks, but omitted them for the sake of conciseness. Figure 6.17 gives the computational time comparisons between the GAUSS and GREVI rules in computing the stiffness matrix for the clamped square plate problem. Computational efficiency is significantly improved by the GREVI rule. Observed speedups are approximately 60\% for \(p = 2\) and more than 90\% for \(p = 5\), as the mesh is refined. Note that the computational time spent on calculating the quadrature points and weights are also included in the time comparisons. The SGREVI rule behaves similarly.

**Scordelis-Lo roof**

The problem setup for the Scordelis-Lo roof can be seen in Section 4.4.3. The initial mesh of the whole model consists of \(4 \times 4\) maximally smooth uniform elements as shown in Fig-
Figure 6.18: Initial uniform mesh of the Scordelis-Lo roof problem.

Figure 6.19: Scordelis-Lo roof with the KL shell, convergence of the maximum displacement $u_z$ with different quadratures, various degrees $p = 2$ to 5, and maximally smooth elements. A whole model is modeled with an initial $4 \times 4$ uniform mesh.
Figure 6.20: Scordelis-Lo roof with the RMC shell, convergence of the maximum displacement $u_z$ with different quadratures, various degrees $p = 2$ to 5, and maximally smooth elements. A whole model is modeled with an initial $4 \times 4$ uniform mesh.

Figure 6.18. The maximum displacement on the free edge at $\frac{L}{2}$ is monitored and results for the KL shell are shown in Figure 6.19. For $p = 2$ and $t = 0.25$, as shown in Figure 6.19a, the GREVI rule shows significant improvement over the GAUSS rule. As the degree increases, locking is alleviated largely and GREVI obtains nearly coincident results with GAUSS for $p = 3$ and 5, but slightly worse results for $p = 4$ with the initial mesh. As the shell thins, i.e., $t = 0.025$, the membrane locking becomes more severe. In this case, the GREVI rule achieves superior results for all degrees as shown in Figure 6.19c. Similarly, the SR quadrature rule also performs better than the GAUSS rule for all cases except for $p = 4, 5$ and thickness $t = 0.25$ where results hardly differ from each

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other as shown in Figures 6.19b and d. It is interesting to note that for $p = 2$, the SR quadrature behaves softer than GREVI as results with SR converge from above.

Since this problem is membrane dominated, shear locking is not significant. Figures 6.20a and c demonstrate that with GREVI quadrature the RMC shell converges in the same way as the KL shell shown in Figures 6.19a and c, except that a jump occurs after the first refinement for $p = 2$ and $t = 0.025$, which is stabilized by the SGREVI quadrature as shown in Figure 6.20d. Results in Figure 6.20d also show that SGREVI slightly locks the RMC elements with the initial mesh when compared with results obtained with GREVI in Figure 6.20c.

![Graph 1](attachment:Graph1.png)  
(a) GAUSS vs GREVI.  
(b) GAUSS vs SGREVI.

Figure 6.21: Scordelis-Lo roof with the RMC shell, convergence of the maximum displacement $u_z$ with thickness $t = 0.025$, various degrees $p = 2$ to 5 and a $C^0$ line inserted in the middle of each parametric direction. The whole geometry is modeled with an initial $4 \times 4$ uniform mesh.

We then insert a $C^0$ line down the middle of each parametric direction of the initial geometry. Figure 6.21 shows the convergence results of the RMC shell with different quadratures and thickness $t = 0.025$. As can be seen in Figure 6.21a, with GREVI, results converge faster than those with GAUSS for all degrees from $p = 2$ to 5. Figure 6.21b also shows that with SGREVI results converge faster than those for GAUSS for $p = 2, 3$ and 4, and are almost identical for the initial mesh, which is due to the fact that, in this case, the number of segmented Greville abscissae is not significantly reduced when compared to the number of GAUSS points. Therefore, locking is not released as effectively. However, as the mesh is refined, this behavior disappears as ex-
pected. When a \( C^1 \) line is inserted, similar phenomena for \( p = 3, 4 \) and 5 are observed as shown in Figure 6.22.

![Graphs showing convergence of maximum displacement with thickness and degrees of polynomial interpolation](image)

(a) GAUSS vs GREVI.  
(b) GAUSS vs SGREVI.

Figure 6.22: Scordelis-Lo roof with the RMC shell, convergence of the maximum displacement \( u_z \) with thickness \( t = 0.025 \), various degrees \( p = 3 \) to 5 and a \( C^1 \) line inserted in the middle of each parametric direction. The whole geometry is modeled with an initial \( 4 \times 4 \) uniform mesh.

![Initial non-uniform mesh](image)

Figure 6.23: An initial non-uniform mesh.

Next, we explore the accuracy of SGREVI for non-uniform meshes. The initial mesh of the whole model is now composed of \( 4 \times 4 \) maximally smooth non-uniform elements as shown in Figure 6.23. Figures 6.24 shows that SGREVI converges faster than GAUSS for all thicknesses \( t = 0.25, 0.025 \) and all degrees \( p = 2 \) to 5 except that both of them achieve almost the reference solution for the case \( p = 5 \) and \( t = 0.25 \). These observations are nearly identical to the results
obtained with uniform meshes as shown in Figures 6.20b and d, which indicates that non-uniform meshes do not affect the accuracy of the SGREVI rule.

Figure 6.24: Scordelis-Lo roof with the RMC shell, convergence of the maximum displacement $u_z$ with SGREVI and GAUSS, various degrees $p = 2$ to 5 and maximally smooth elements. The whole geometry is modeled with an initial $4 \times 4$ non-uniform mesh.

6.3.2 Geometrically nonlinear problems

Straight cantilever shell subjected to an end moment

The problem setup for a straight cantilever shell subjected to an end moment is the same as described in Section 5.4.2. In this section, the initial mesh consists of $2 \times 1$ maximally smooth B-spline elements with degrees ranging from 2 to 5, and only the RMC shell is used as it facilitates the application of the end moment exactly.

In this problem, three different quadratures are applied. Figure 6.25 shows the convergence of the end rotation with the RMC shell. As can be seen, both GREVI and SGREVI perform better than GAUSS in terms of the convergence rate of the end rotation. The SGREVI rule shows some locking with the initial mesh for $p = 2$ and 3 as shown in Figure 6.25b, but it is quickly eliminated through mesh refinement.

Even though the GREVI rule is more effective at releasing locking than the SGREVI rule for coarse meshes, Figures 6.26b, e, and h show that for quadratic maximally smooth elements the
Figure 6.25: A cantilever beam subjected to an end moment with the RMC shell, convergence of the endpoint rotation with various degrees $p = 2$ to $5$, maximally smooth elements and 10 load steps. The initial mesh consists of $2 \times 1$ elements.

Figure 6.26: The deformed configurations at each load step for different quadratures, RMC shell, $p = 2$. 
The deformed configuration starts to oscillate as the rotation increases. This is due to the local instability in the membrane modes as described in Section 6.2.1. In contrast, the SGREVI rule is stable and free of displacement oscillations as shown in Figures 6.26c, f, and i. Once the polynomial degree is increased, the membrane modes can be represented more accurately with the GREVI rule and the oscillations in the deformed configuration disappear as shown in Figures 6.27b, e and h. In this case, the GREVI and SGREVI rules achieve a closed circle with only 4 and 8 elements, respectively. However, the GAUSS rule requires more elements to close the circle.

Figure 6.28 shows the load-deflection curves of the free end. As can be seen, with 16 maximally smooth quadratic elements the GREVI and SGREVI rules trace the exact solutions accurately at each load step even though the GREVI rule exhibits local instabilities. In contrast, the GAUSS rule fails to accurately predict the end deformation as the rotation increases.
Figure 6.28: A cantilever beam subjected to an end moment with the RMC shell, load-deflection curve for different quadratures, 16 maximally smooth quadratic elements, and 10 load steps.

Table 6.12: A cantilever shell subjected to an end moment, Newton-Raphson iteration information for the last step for the RMC shell for GAUSS, GREVI and SGREVI rules with 16 × 1 maximally smooth quadratic elements and 10 load steps. A residual norm of $1 \times 10^8$ is used as the tolerance for convergence.

<table>
<thead>
<tr>
<th>Last load step iteration</th>
<th>Norm of the global residual vector</th>
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<tr>
<td></td>
<td>GAUSS</td>
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<tr>
<td>1</td>
<td>3.0229989</td>
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<tr>
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</tr>
<tr>
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</tr>
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<tr>
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</tr>
<tr>
<td>9</td>
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<td>10</td>
<td>1.0417200e-10</td>
</tr>
<tr>
<td>11</td>
<td>2.6471600e-10</td>
</tr>
</tbody>
</table>

Table 6.12 gives the total iteration count and Newton-Raphson iteration information for the last load step with 16 maximally smooth quadratic B-spline elements. The GREVI and SGREVI rules take approximately 110 iterations in total to achieve a full circle and they achieve the optimal quadratic convergence rate of the Newton-Raphson method. Note that even though the GAUSS
rule takes only 98 iterations, a full circle is not obtained as shown in Figure 6.26g. More elements and iterations are required to predict the correct deformation.

**Hemispherical shell with a hole**

The problem setup for a hemispherical shell with a hole is given in Section 5.4.2. One quarter of the hemisphere is modeled with an initial $4 \times 4$ mesh. Figure 6.29 shows the convergence of the radial displacement $u_y$ at point $B$, as shown in Figure 5.15, for different quadratures with one-step point loads, $P = 2$, on the equator with alternating sign. As can be seen, for the RMC shell, GREVI and SGREVI again obtain converged results faster than GAUSS for coarse meshes with various degrees $p = 2$ to 4, and the SGREVI rule releases locking but not as effectively as the GREVI rule for these cases. For $p = 5$, locking is largely released by the higher-order basis, therefore all three quadratures achieve identical results.

![Graph](image)

(a) GAUSS vs GREVI.

(b) GAUSS vs SGREVI.

Figure 6.29: A hemispherical shell and the RMC shell, convergence of the maximum displacement $u_y$ at point $B$ with quadratures GAUSS, GREVI and SGREVI, various degrees $p = 2$ to 5, and maximally smooth elements. One quarter of the hemisphere is modeled with an initial $4 \times 4$ mesh.

Figure 6.30 shows the convergence of the radial displacement at point $B$ for different quadrature rules with 10 equal load steps. Behavior consistent with previous benchmarks are observed for this problem, i.e., superior convergence rates are obtained for $p = 2$ to 4 and comparable
results are achieved for $p = 5$ due to the higher-order basis. To achieve a relative displacement error $|u_B - u_{ref}|/|u_{ref}| < 5\%$ at point $B$, the proposed GREVI rule requires only $10 \times 10$ quadratic maximally smooth elements compared to $28 \times 28$ elements required by the GAUSS rule. The SGREVI rule requires $21 \times 21$ elements. The deformed configurations of the whole hemisphere are created by mirroring the quarter deformed configurations through symmetric planes, as shown in Figure 6.31. The Newton-Raphson iteration information for different quadratures is listed in Table 6.13. As can be seen, even though the proposed integration rules require fewer elements and far fewer quadrature points than the GAUSS rule, they still require approximately the same number of total iterations as the GAUSS rule. Again, the optimal quadratic convergence rates are observed for both the GREVI and SGREVI rules.

6.3.3 Multi-patch coupling problems

In this section, we demonstrates that the Greville quadrature can be applied to weakly continuous geometry discretized with nonconforming meshes.
Figure 6.31: A hemispherical shell and the RMC shell. Deformed configurations with different quadrature rules, maximally smooth quadratic B-spline elements and 10 load steps. $|u_y - u_{ref}|/|u_{ref}| < 5\%$ at point $B$.

Table 6.13: A hemispherical shell and the RMC shell: Newton-Raphson iteration behavior for GAUSS, GREVI and SGREVI rules with maximally smooth quadratic B-spline elements and 10 load steps. A residual norm of $1 \times 10^7$ is used as the tolerance for convergence.
**Scordelis-Lo roof**

Figure 6.32 illustrates the convergence behavior of the maximum displacement for GAUSS and SGREVI with RMC shells, where the whole model is represented by a weakly continuous geometry built on two patches of nonconforming meshes as shown in Figure 4.36b. As can be seen, SGREVI converges faster than GAUSS for all cases except for thickness $t = 0.25$, and $p = 4, 5$. These observations are consistent with those obtained previously for single-patch geometries.

Figure 6.32: Scordelis-Lo roof modeled with a weakly continuous geometry built on two patches of nonconforming meshes as shown in Figure 4.36b, convergence of the maximum displacement for GAUSS and SGREVI, $p = 2, 3, 4$ and 5.
CHAPTER 7. CONCLUSIONS AND FUTURE WORK

7.1 Conclusions

In this dissertation, we have first introduced a new approach for the coupling of non-conforming higher-order smooth spline patches which we call the isogeometric Bézier dual mortar method. The construction of the underlying dual spline basis is based on Bézier extraction and projection and is applicable to any spline description which has a Bézier representation (i.e., NURBS, T-splines, LR-splines, etc.). The dual basis is refineable and the associated mortaring strategy preserves the sparsity of the stiffness matrix. The accuracy of the coupling can be adaptively controlled by employing a dual basis refinement scheme which can be used to recover optimal convergence rates without adding any additional degrees-of-freedom to the global system. As a particular application of Bézier dual mortaring, we introduced weakly continuous geometry, where the weak continuity constraint is built into properly modified extraction operators. This allows for the use of weakly coupled multi-patch geometry in design and as a basis for standard finite element frameworks which do not employ any mortaring algorithms. We applied the isogeometric Bézier dual mortar method to steady-state heat equations, standard linear and nonlinear elastic test cases with solid and shell elements, and B-spline and NURBS geometries. All tests show that the isogeometric Bézier dual mortar method is robust and accurate, works for arbitrary master/slave pairings, and arbitrary parameterizations.

We have then introduced a mixed isogeometric shell formulation, based on the modified Hellinger-Reissner variational principle, to overcome both shear and membrane locking and stress oscillations in higher-order continuous elements. In addition to the displacements and rotations, the membrane and shear strains are chosen as unknown fields, which are interpolated with carefully constructed lower-order spline bases. Leveraging the orthogonality property of the proposed dual spline basis, the assumed strain variables in the mixed shell formulation can be condensed out efficiently. This technique preserves the sparsity of the global stiffness matrix. Both linear and
nonlinear numerical examples show that the condensation approach actually improves the accuracy of the mixed approach on a per degree of freedom basis. Additionally, we have also investigated the accuracy of the mixed shell formulations for both the continuous and discrete rotation concepts. Numerical examples demonstrate that the continuous and discrete rotation concepts achieve similar accuracy for standard RM shells with lower-order bases and small deformations. However, once the shear and membrane locking is removed by the proposed formulation, the discrete rotation concept is less accurate, and even unstable for large deformation problems. This can be attributed to the inaccuracies inherent in projecting the continuous surface normal to the control points, which do not interpolate the surface. We have also proposed a more computationally efficient scheme to update the current director.

Finally, we have introduced Greville quadrature rules for isogeometric KL and RM shell analysis. The quadrature points are chosen to be the standard Greville abscissae for B-splines and NURBS. The quadrature weights are determined by solving a simple moment fitting system of equations. A study of a generalized eigenvalue problem shows that the unstabilized Greville quadrature does not introduce zero-energy modes commonly seen in reduced quadrature, therefore resulting in a solvable system of equations. It produces accurate bending modes, but also produces spurious finite-energy modes in the membrane stiffness, which can negatively affect the analysis accuracy if excited. Fortunately, these spurious modes can be easily removed by using a higher-order Greville quadrature, called the stabilized Greville quadrature. For the KL shell and a quadratic $C^1$ basis, the second derivatives are discontinuous at knot locations so a selective Greville rule is proposed, which uses the unstabilized Greville quadrature for the bending stiffness but a stabilized Greville quadrature for the membrane stiffness. Numerical examples show that the proposed Greville quadrature rules alleviate membrane and shear locking effectively for KL and RMC shells. They are not only superior in terms of analysis accuracy but are also efficiency as they asymptotically require only one point for RM shell elements and two points for KL shell elements.

7.2 Future work

There exist many promising future research directions based on the work in this dissertation. They are briefly outlined as follows:
• Extending the Bézier dual mortar method to the coupling of trimmed surfaces. Trimmed surface technology is widely used in CAD systems. Unfortunately, it appears that the existing coupling techniques for trimmed surfaces in IGA are not simple and robust. The Bézier dual mortar method has the potential to simplify this process.

• Applying the Bézier dual mortar method to the coupling of different types of structural elements. Structural analysis for complex structures usually involves different element types including solid, beam, and shell elements, etc. In this work, we only explored the coupling of the same types of elements, but the proposed method can be used for coupling different types of elements.

• Extending the mixed locking-free shell formulation to material nonlinear problems. The proposed locking-free shell formulation is in the linear material regime. It would be very valuable to extend it to material nonlinear due to its high efficiency at releasing locking and improving the stress quality.

• Studying the stability of the mixed locking-free shell formulation. The proposed mixed shell formulation turns the original minimization problem into a saddle point problem. Therefore, rigorous mathematical analysis of the stability is desired, which would be helpful for further improvements of the formulation.

• Constructing local quadrature schemes based on the Greville quadrature rules. For unstructured meshes, the Greville abscissae are not well defined so the proposed Greville quadratures cannot be used directly. In this case, a efficient local quadrature is preferable. The layout of Greville quadrature points and the corresponding quadrature weights on structured meshes shown in this work can be a starting point for devising local quadrature schemes.
REFERENCES


APPENDIX A. A DERIVATION OF THE WEAKLY CONTINUOUS EXTRACTION
OPERATOR FOR ELEMENT $e_{21}$ FROM FIGURE 4.3.

For the example shown in Figure 4.3, the basis relation matrix $G_{N_r,N_m}$ defined by (4.27) and the localized counterpart $G^e_{N_r,N_m}$ for the interface of element $e_{21}$ are

$$G_{N_r,N_m} = \begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 1 \end{bmatrix}^T \quad \text{and} \quad G^e_{N_r,N_m} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}. \quad (A.1)$$

The standard Bézier extraction operator $R^e_{\xi_1}$ and the basis transformation matrix $M$ are

$$R^e_{\xi_1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}, \quad (A.2)$$

and the weakly continuous one-dimensional interface element extraction operator $\tilde{R}^e_{\xi_1}$ for element $e_{21}$ is

$$\tilde{R}^e_{\xi_1} = (G^e_{N_r,N_m})^T R^e_{\xi_1} M^{-1} = \begin{bmatrix} \frac{1}{9} & -\frac{1}{9} & \frac{1}{9} \\ \frac{2}{3} & \frac{2}{3} & 0 \\ \frac{2}{9} & \frac{4}{9} & \frac{8}{9} \end{bmatrix}. \quad (A.3)$$
The two standard one-dimensional Bézier extraction operators for the original slave patch element $e_2$ are

$$\begin{bmatrix}
\frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 1 & \frac{1}{2} \\
0 & 0 & \frac{1}{2}
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
\frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.$$ (A.4)

As shown in Figure 4.3, the interior basis functions of element $e_{21}$ are identical to those of element $e_2$, and only the interface basis functions are replaced by the refined interface basis. We decompose $R_{\xi_2}^e$ into two submatrices $R_1$ and $R_2$ such that

$$R_{\xi_2}^e = \begin{bmatrix} R_1 \\ \cdots \\ R_2 \end{bmatrix} = \begin{bmatrix}
\frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},$$ (A.5)

where $R_1$ is related to the interior basis functions and $R_2$ is related to the interface basis functions.

The weakly continuous patch element extraction operator $\tilde{R}^e$ for element $e_{21}$ can be computed as follows:

$$\begin{bmatrix}
\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{9} & -\frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\
0 & 0 & 0 & 0 & 0 & \frac{2}{9} & \frac{2}{9} & 0 & \frac{8}{9} \\
0 & 0 & 0 & 0 & 0 & \frac{2}{9} & \frac{4}{9} & \frac{8}{9}
\end{bmatrix}.$$ (A.6)
Note that the only difference between the weakly continuous element extraction operator and the standard element extraction operator is that the last three rows are modified. These rows correspond to interface basis functions.
B.1 Discretization based on the continuous rotation concept

The first variation of \( d \) can be written as

\[
\delta d = W^T \delta \omega, \quad W = \text{skew } d
\]  

(B.1)

and the derivatives can be written as

\[
\delta d,\alpha = W,\alpha \delta \omega + W^T \delta \omega,\alpha, \quad W,\alpha = \text{skew } d,\alpha.
\]  

(B.2)

The second variation of \( d \) can be written as

\[
h \cdot \Delta d = \delta \omega^T M(h) \Delta \omega, \quad M(h) = \frac{1}{2} (d \otimes h + h \otimes d) - (d \cdot h) I
\]  

(B.3)

and the derivatives can be written as

\[
h \cdot \Delta d,\alpha = \delta \omega^T M(h) \Delta \omega + \delta \omega^T M,\alpha(h) \Delta \omega + \delta \omega^T M(h) \Delta \omega,\alpha,
\]  

(B.4)

\[
M(h),\alpha = \frac{1}{2} (d,\alpha \otimes h + h \otimes d,\alpha) - (d,\alpha \cdot h) I.
\]  

(B.5)

See [89] for additional details.

The increment of the axial vector and its derivatives are discretized as

\[
\Delta \omega^h = \sum_{l} N_l \Delta \omega_l = \sum_{l} N_l T_{3l} \Delta \beta_l \quad \text{and} \quad \Delta \omega^h,\alpha = \sum_{l} N_l,\alpha \Delta \omega_l = \sum_{l} N_l,\alpha T_{3l} \Delta \beta_l
\]  

(B.6)
where

\[ T_{3I} = \begin{cases} [a_{1I} \ a_{2I}] & \text{for nodes in the smooth regions} \\ I_{3 \times 3} & \text{for nodes along kinks} \end{cases} \tag{B.7} \]

\[ a_{\alpha I} = R(\Delta \omega_I) a_{\alpha I}^{-1}, \quad a_{\alpha I}^{-1} \text{ is the current nodal basis} \tag{B.8} \]

\[ \Delta \omega_I = T_{3I} \Delta \beta_I. \tag{B.9} \]

Note that here we choose to store the current nodal bases so that we do not need to store the rotation matrix \( R^{i-1} \).

The first variation of the director and its derivatives are discretized as

\[ \delta d^h = W^{hT} \delta \omega^h = \sum_{l=1}^{n_p} T_l \delta \beta_l \tag{B.10} \]

\[ \delta d^h_{\alpha} = \sum_{l=1}^{n_p} T_{l,\alpha} \delta \beta_l \tag{B.11} \]

where

\[ T_l = W^{hT} N_l T_{3I} \tag{B.12} \]

\[ T_{l,\alpha} = [W^{hT} N_l + W^{hT} N_{l,\alpha}] T_{3I}. \tag{B.13} \]

The second variation of the director and its derivatives are discretized as

\[ h \cdot \Delta \delta d^h = \sum_{l=1}^{n_p} \sum_{K=1}^{n_p} \delta \beta_l^T T_{3I}^T N_l M^h(h) N_K T_{3K} \delta \beta_K = \sum_{l=1}^{n_p} \sum_{K=1}^{n_p} \delta \beta_l^T \hat{q}_{lK}^\beta (h) \delta \beta_K \tag{B.14} \]

\[ h \cdot \Delta \delta d^h_{\alpha} = \sum_{l=1}^{n_p} \sum_{K=1}^{n_p} \delta \beta_l^T T_{3I}^T [N_{l,\alpha} M^h(h) N_K + N_l M_{\alpha}^h(h) N_K + N_l M^h(h) N_{K,\alpha}] T_{3K} \delta \beta_K \]

\[ = \sum_{l=1}^{n_p} \sum_{K=1}^{n_p} \delta \beta_l^T \hat{m}_{lK,\alpha}^\beta (h) \delta \beta_K \tag{B.15} \]
where

\[ \hat{q}_{IK}^{\beta \beta}(h) = T_{3l}^T N_j M^h(h) N_K T_{3K} \]  \hspace{1cm} (B.16)

\[ \hat{m}_{IK,\alpha}^{\beta}(h) = T_{3l}^T [N_{I,\alpha} M^h(h) N_K + N_j M_{I,\alpha}^h(h) N_K + N_j M^h(h) N_{K,\alpha}] T_{3K} \]  \hspace{1cm} (B.17)

and \( M^h(h) \) and \( M_{I,\alpha}^h(h) \) are calculated by inserting interpolated values of \( d^h \) and \( d_{I,\alpha}^h \) into (B.3) and (B.4), respectively.

**B.2 Discretization based on the discrete rotation concept**

The first variation of the director can be discretized as

\[ \delta d^h = \sum_I N_I \delta d_I \quad \text{and} \quad \delta d_{I,\alpha}^h = \sum_I N_{I,\alpha} \delta d_I \]  \hspace{1cm} (B.18)

where

\[ \delta d_I = W^T_I \delta \omega_I. \]  \hspace{1cm} (B.19)

Following (B.10) to (B.11), we have that

\[ \delta d^h = \sum_{I=1}^{n_p} T_I \delta \beta_I \]  \hspace{1cm} (B.20)

\[ \delta d_{I,\alpha}^h = \sum_{I=1}^{n_p} T_{I,\alpha} \delta \beta_I \]  \hspace{1cm} (B.21)

where

\[ T_I = N_I W_I^T T_{3l} \]  \hspace{1cm} (B.22)

\[ T_{I,\alpha} = N_{I,\alpha} W_I^T T_{3l}. \]  \hspace{1cm} (B.23)

The second variation of the director can be discretized as

\[ \Delta \delta d^h = \sum_I N_I \Delta \delta d_I \quad \text{and} \quad \Delta \delta d_{I,\alpha}^h = \sum_I N_{I,\alpha} \Delta \delta d_I \]  \hspace{1cm} (B.24)
where $\Delta \delta d_I$ is calculated at each node using (B.3) as

$$h \cdot \Delta \delta d_I = \delta w_I \cdot M_I(h) \Delta w_I = \delta \omega_I^T M_I(h) \Delta \omega_I.$$  

(B.25)
APPENDIX C. DISCRETIZATION OF THE STRAIN FOR RM SHELL

The first variation of the strain components in (3.20-3.22) can be written as

\[
\delta \varepsilon_{\alpha \beta} = \frac{1}{2} (\delta \ddot{x}_\alpha \cdot \dot{x}_\beta + \delta \dot{x}_\beta \cdot \ddot{x}_\alpha),
\]
\[
\delta \kappa_{\alpha \beta} = \frac{1}{2} (\delta \ddot{x}_\alpha \cdot \ddot{d}_\beta + \delta \dot{d}_\beta \cdot \dot{x}_\alpha + \delta \ddot{d}_\alpha \cdot \dot{x}_\beta + \delta \dot{x}_\beta \cdot \ddot{d}_\alpha),
\tag{C.1}
\]
\[
\delta \gamma_\alpha = \delta \ddot{x}_\alpha \cdot d + \delta d \cdot \ddot{x}_\alpha
\]

and the second variation as

\[
\Delta \delta \varepsilon_{\alpha \beta} = \frac{1}{2} (\delta \ddot{x}_\alpha \cdot \Delta \dot{x}_\beta + \delta \dot{x}_\beta \cdot \Delta \ddot{x}_\alpha),
\tag{C.2}
\]
\[
\Delta \delta \kappa_{\alpha \beta} = \frac{1}{2} (\delta \ddot{x}_\alpha \cdot \Delta \ddot{d}_\beta + \delta \dot{d}_\beta \cdot \Delta \dot{x}_\alpha + \delta \ddot{d}_\alpha \cdot \Delta \dot{x}_\beta + \delta \dot{x}_\beta \cdot \Delta \ddot{d}_\alpha)
\tag{C.3}
\]
\[
+ \ddot{x}_\alpha \cdot \Delta \delta d_\beta + \ddot{x}_\beta \cdot \Delta \delta d_\alpha,
\tag{C.4}
\]
\[
\Delta \delta \gamma_\alpha = \delta \ddot{x}_\alpha \cdot \Delta d + \delta d \cdot \Delta \ddot{x}_\alpha + \ddot{x}_\alpha \cdot \Delta \delta d.
\tag{C.5}
\]

Substituting \(\delta d^h\) and \(\delta d^h_\alpha\), computed with B.1 or B.2, into (C.1) gives the discretized first variation of the strains as

\[
\delta e^h = \sum_{l=1}^{n_p} \begin{bmatrix}
N_{l,1} \ddot{x}_1^h T \quad N_{l,2} \ddot{x}_2^h T \\
N_{l,1} \ddot{x}_2^h T + N_{l,2} \ddot{x}_1^h T
\end{bmatrix} \delta u_l
\]
\[
\delta \kappa^h = \sum_{l=1}^{n_p} \begin{bmatrix}
N_{l,1} d_1^h T \quad \ddot{x}_1^h T_{l,1} \\
N_{l,2} d_2^h T \quad \ddot{x}_2^h T_{l,1} + \ddot{x}_2^h T_{l,2}
\end{bmatrix} \begin{bmatrix}
\delta u_l \\
\delta \beta_l
\end{bmatrix}
\tag{C.6}
\]
\[
\delta \gamma^h = \sum_{l=1}^{n_p} \begin{bmatrix}
N_{l,1} d_1^h T \quad \ddot{x}_1^h T_l \\
N_{l,2} d_2^h T \quad \ddot{x}_2^h T_l
\end{bmatrix} \begin{bmatrix}
\delta u_l \\
\delta \beta_l
\end{bmatrix}.
\]
The second variation of the membrane strains (C.2) are independent of \( \mathbf{d} \) so their discretization can be written as

\[
\Delta \delta \varepsilon_{\alpha \beta}^h = \frac{1}{2} (\delta \bar{x}_\alpha^h \cdot \Delta \bar{x}_\beta^h + \delta \check{x}_\beta^h \cdot \Delta \check{x}_\alpha^h) = \frac{1}{2} \sum_{l=1}^{n_s} \sum_{K=1}^{n_p} \frac{1}{2} \delta \mathbf{u}_l^T (N_{l,\alpha} N_{K,\beta} + N_{l,\beta} N_{K,\alpha}) \mathbf{I} \Delta \mathbf{u}_K. \tag{C.7}
\]

The bending strains (C.3) contain both first and second variation information. To simplify the derivation, their discretization can be written in two parts as

\[
\Delta \delta \kappa_{\alpha \beta}^h = \frac{1}{2} (\delta \bar{x}_\alpha^h \cdot \Delta \bar{d}_\beta^h + \delta \check{x}_\beta^h \cdot \Delta \check{d}_\alpha^h + \delta \bar{d}_\alpha^h \cdot \Delta \bar{x}_\beta^h + \delta \check{d}_\beta^h \cdot \Delta \check{x}_\alpha^h + \bar{x}_\alpha^h \cdot \Delta \delta \bar{d}_\beta^h + \check{x}_\beta^h \cdot \Delta \delta \check{d}_\alpha^h)
\]

\[
= \underbrace{\Delta \delta \hat{\kappa}_{\alpha \beta}^h}_{\text{first variation}} + \underbrace{\Delta \delta \check{\kappa}_{\alpha \beta}^h}_{\text{second variation}} \tag{C.8}
\]

where

\[
\Delta \delta \hat{\kappa}_{\alpha \beta}^h = \frac{1}{2} (\delta \bar{x}_\alpha^h \cdot \Delta \bar{d}_\beta^h + \delta \check{x}_\beta^h \cdot \Delta \check{d}_\alpha^h + \delta \bar{d}_\alpha^h \cdot \Delta \bar{x}_\beta^h + \delta \check{d}_\beta^h \cdot \Delta \check{x}_\alpha^h)
\]

\[
\text{and}
\]

\[
\Delta \delta \check{\kappa}_{\alpha \beta}^h = \frac{1}{2} (\check{x}_\alpha^h \cdot \Delta \delta \bar{d}_\beta^h + \bar{x}_\beta^h \cdot \Delta \delta \check{d}_\alpha^h). \tag{C.9}
\]

Substituting \( \delta \bar{d}_\alpha^h \) and \( \delta \check{d}_\beta^h \) derived in B.1 or B.2 into (C.9) leads to the discretized variation of the strains

\[
\Delta \delta \hat{\kappa}_{\alpha \beta}^h = \frac{1}{2} \sum_{l=1}^{n_s} \sum_{K=1}^{n_p} \{ \delta \mathbf{u}_l^T [N_{l,\alpha} T_{K,\beta} + N_{l,\beta} T_{K,\alpha}] \Delta \beta_K + \delta \beta_l^T [N_{K,\beta} T_{l,\alpha}^T + N_{K,\alpha} T_{l,\beta}^T] \Delta \mathbf{u}_K \}.
\]

For the continuous rotation concept substituting (B.15) into (C.10) results in

\[
\Delta \delta \hat{\kappa}_{\alpha \beta} = \frac{1}{2} \sum_{l=1}^{n_s} \sum_{K=1}^{n_p} \delta \beta_l^T (\hat{m}_{lK,\beta}^\beta + \hat{m}_{lK,\alpha}^\beta) \Delta \beta_K.
\]
For the discrete rotation concept substituting the second equation of (B.24) into (C.10) results in

\[
\Delta \delta \hat{x}_{\alpha \beta}^h = \frac{1}{2} (\delta x_{\alpha}^h \cdot \Delta \delta d_{\beta}^h + \delta x_{\beta}^h \cdot \Delta \delta d_{\alpha}^h)
\]

\[
= \frac{1}{2} \sum_{l=1}^{n_p} (\hat{x}_{\alpha N_l, \beta}^h + \hat{x}_{\beta N_l, \alpha}^h) \cdot \Delta \delta d_l
\]

\[
= \frac{1}{2} \sum_{l=1}^{n_p} \delta \omega_l^T M_l (\hat{x}_{\alpha N_l, \beta}^h + \hat{x}_{\beta N_l, \alpha}^h) \Delta \omega_l
\]

\[
= \frac{1}{2} \sum_{l=1}^{n_p} \delta \beta_l^T T_{3l}^T M_l (\hat{x}_{\alpha N_l, \beta}^h + \hat{x}_{\beta N_l, \alpha}^h) T_{3l} \Delta \beta_l
\]

\[
= \frac{1}{2} \sum_{l=1}^{n_p} \sum_{K=1}^{n_p} \delta \beta_l^T T_{3l}^T M_l (\hat{x}_{\alpha N_l, \beta}^h + \hat{x}_{\beta N_l, \alpha}^h) T_{3l} \Delta \beta_K
\]

\[
= \frac{1}{2} \sum_{l=1}^{n_p} \sum_{K=1}^{n_p} \delta \beta_l^T (\hat{m}_{lK, \alpha}^{\beta \beta} (\hat{x}_{\alpha}^h) + \hat{m}_{lK, \alpha}^{\beta \beta} (\hat{x}_{\beta}^h)) \Delta \beta_K
\]

where

\[
\hat{m}_{lK, \alpha}^{\beta \beta} (\hat{x}_{\alpha}^h) = \delta_{lK} N_{l, \alpha} T_{3l}^T M_l (\hat{x}_{\alpha}^h) T_{3l}.
\]

Similarly, we can get the shear part as

\[
\Delta \delta \gamma_{\alpha}^h = \delta \gamma_{\alpha}^h \cdot \Delta d + \delta d \cdot \Delta \gamma_{\alpha}^h + \delta \gamma_{\alpha}^h \cdot \Delta \delta d
\]

\[
= \frac{\Delta \delta \gamma_{\alpha}}{\text{first variation}} + \frac{\Delta \delta \gamma_{\alpha}}{\text{second variation}}
\]

where

\[
\Delta \delta \gamma_{\alpha} = \sum_{l=1}^{n_p} \sum_{J=1}^{n_p} \delta \gamma_{lJ}^T N_{l, \alpha} T_{K} \Delta \beta_K + \delta \beta_l^T N_{K, \alpha} T_{3l}^T \Delta u_k
\]

and

\[
\Delta \delta \tilde{\gamma}_{\alpha} = \sum_{l=1}^{n_p} \sum_{J=1}^{n_p} \delta \beta_l^T \tilde{q}_{lJ}^{\beta \beta} (\hat{x}_{\alpha}^h) \Delta \beta_K
\]
where $\hat{q}_{IK}^{\beta \beta}(\bar{x}, \alpha)$ is defined in (B.16) for the continuous rotation of the director and

$$\hat{q}_{IK}^{\beta \beta}(\bar{x}, \alpha) = \delta_{IK} N_I T_{3I}^T M_I(\bar{x}, \alpha) T_{3I} \quad (C.13)$$

for the discrete rotation of the director.