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Almost Homeomorphisms and Inscrutability

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ABSTRACT

Almost Homeomorphisms and Inscrutability

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"Homeomorphic" is the standard equivalence relation in topology. To a topologist, spaces which are homeomorphic to each other aren't merely similar to each other, they are the same space. We study a class of functions which are homeomorphic at "most" of the points of their domains and codomains, but which may fail to satisfy some of the properties required to be a homeomorphism at a "small" portion of the points of these spaces. Such functions we call "almost homeomorphisms." One of the nice properties of almost homeomorphisms is the preservation of almost open sets, i.e. sets which are "close" to being open, except for a "small" set of points where the set is "defective." We also find a surprising result that all non-empty, perfect, Polish spaces are almost homeomorphic to each other.

A standard technique in algebraic topology is to pass between a continuous map between topological spaces and the corresponding homomorphism of fundamental groups using the π_1 functor. It is a non-trivial question to ask when a specific homomorphism is induced by a continuous map; that is, what is the image of the π_1 functor on homomorphisms?

We will call homomorphisms in the image of the π_1 functor "tangible homomorphisms" and call homomorphisms that are not induced by continuous functions "intangible homomorphisms." For example, Conner and Spencer [1] used ultrafilters to prove there is a map from HEG to $\mathbb{Z}_2$ not induced by any continuous function $f\colon \mathrm{HE}\to Y$, where Y is some topological space with $\pi_1(Y) = \mathbb{Z}_2$. However, in standard situations, such as when the domain is a simplicial complex, only tangible homomorphisms appear.

Our job is to describe conditions when intangible homomorphisms exist and how easily these maps can be constructed. We use methods from Shelah [3] and Pawlikowski [5] to prove that Conner and Spencer could not have constructed these homomorphisms with a weak version of the Axiom of Choice. This leads us to define and examine a class of pathological objects that cannot be constructed without a strong version of the Axiom of Choice, which we call the class of inscrutable objects. Objects that do not need a strong version of the Axiom of Choice are scrutable. We show that the scrutable homomorphisms from the fundamental group of a Peano continuum are exactly the homomorphisms induced by a continuous function.

Keywords: scrutability, inscrutability, tangible, intangible, almost homeomorphism, Polish, almost open, meager, nowhere dense, dense, Cantor, Gödel, Shelah

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Contents

Chapter 1. Preliminaries

1.1 Almost Open Sets

The intuition to have on almost open sets is that an almost open set A is slightly defective in fulfilling the requirements to be an open set. There exists a "small amount" of elements that A needs to either gain or lose to become an open set; the precise meaning of "small amount" is formalized in the following definitions.

Definition 1.1 (Nowhere dense set)**.** A set N is *nowhere dense in* X if every nonempty open set of X contains a nonempty open set that does not intersect N. This is equivalent to the closure of N having empty interior.

Definition 1.2 (Meager set)**.** A set $M \subset X$ is *meager* in X if it is the countable union of nowhere dense sets in X. The complement of a meager set is called a *comeager* set.

Definition 1.3 (Almost open set)**.** A set $A \subset X$ is *almost open* if it is the symmetric difference of an open set of X with a meager set of X.

1.1.1 Nowhere Dense Sets.

Definition 1.4 (Dense set)**.** A set D is *dense in* X if $\overline{D}_X = X$.

Lemma 1.5. *Here are a few properties of nowhere dense sets.*

(a) If $N \subset X \subset Y$ and N is nowhere dense in X then N is nowhere dense in Y.

(b) Let $X = \Pi_i X_i$, where each X_i is first countable. Suppose that A_j is nowhere dense in X_j. Then $A = X_1 \times X_2 \times \cdots \times X_{j-1} \times A_j \times X_{j+1} \times \cdots$ is a nowhere dense set. The singleton sets of a Hausdorff space with no isolated points are nowhere dense. A nowhere dense set of X cannot contain an isolated point of X.

Proof. (a) Let $U \subset Y$ be open and nonempty. If $(U \cap X)$ is empty, then U contains itself as a nonempty subset that has empty intersection with N. If $(U \cap X)$ is nonempty, then $(U \cap X)$ is open in X, so it contains nonempty open V such that $(N \cap V) = \emptyset$. There exists open $V' \subset Y$ such that $(V' \cap X) = V$, so $(V' \cap U) \cap N = \emptyset$ and $(V' \cap U)$ is nonempty and open in Y. So N is nowhere dense in Y.

(b) Since $\overline{A_j}$ closed in X_j, its complement in X_j is open in X_j, which implies that $X_1 \times X_2 \times \cdots \times X_{j-1} \times \overline{A_j} \times X_{j+1} \times \cdots$ is open in X.

Pick any point in $\overline{A}$, call it $x = (x_1, x_2, \cdots)$. Pick a sequence $\{y_i\} \subset X_j$ converging to x_j such that $y_i \notin A_j$ for any i. Then the sequence $z_i = \{(x_1, x_2, \cdots, x_{j-1}, y_i, x_{j+1}, \cdots)\}$ is a sequence converging to x. Each z_i is not in A, because the j^{th} component of Z_i is not in A_j, so x is in the boundary of A. Since X was chosen arbitrarily, all points of A are boundary points of A. So A is a set whose closure has empty interior, so A is nowhere dense. □

Example 1.6. It is possible for $N \subset X \subset Y$ such that N is nowhere dense in Y, but dense in X. This is a counterexample to the converse of Lemma 1.5(a).

Proof. Suppose that $N = X = \{0\}$ and $Y = \mathbb{R}$. Then N is nowhere dense in Y, since Y is a Baire, Hausdorff space and singleton sets are nowhere dense in such spaces. However, $N = X$ and no space is nowhere dense in itself. □

1.1.2 Meager Sets.

Definition 1.7 (Net)**.** Suppose that $A_i \subset X_i$. The *net* of the $\{A_i\}$'s in $\Pi_i X_i$ is the set

$$\bigcup_i X_1 \times X_2 \times \cdots \times X_{i-1} \times A_i \times X_{i+1} \times \cdots$$

and is denoted net $(\{A_i\})$.

Lemma 1.8. *Here are a few properties of meager sets.*

(a) If $M \subset X \subset Y$ and M is meager in X, then M is meager in Y.

$$\{x \in X \mid A \cap (\{x\} \times Y) \textit{ is meager in } (\{x\} \times Y)\}$$

is comeager in X.

(b) Let $X = \Pi_i X_i$ be a countable product of first countable spaces and let $A_i \subset X_i$ be meager in X_i. Then net$(\{A_i\})$ is meager in X.

Proof. Let M be meager and $M = \cup_i N_i$, where each of the countably many N_i is nowhere dense.

(a) Since each N_i is nowhere dense in X, Lemma 1.5(a) implies that each N_i is nowhere dense in Y, so M is the union of countably many nowhere sets sets of Y, so M is meager in Y.

(b) Each A_i is a countable union of nowhere dense sets of X_i, call one of these N. Lemma 1.5(b) implies that

$$\bigcup_i X_1 \times X_2 \times \cdots \times X_{i-1} \times N \times X_{i+1} \times \cdots$$

is nowhere dense in X. This implies that

$$\bigcup_i X_1 \times X_2 \times \cdots \times X_{i-1} \times A_i \times X_{i+1} \times \cdots$$

is meager in X. The countable union of all of these sets over all of the A_i's is therefore meager. □

1.1.3 Borel Sets.

Definition 1.9 (G_δ set)**.** A G_δ *subset* of a topological space X is a countable intersection of open sets of X.

Definition 1.10 (F_σ set)**.** An F_σ *subset* of a topological space X is a countable union of closed sets of X.

Definition 1.11 ($\sigma-$algebra)**.** A $\sigma-$algebra is a set $\mathcal{A}$ of subsets of a set X satisfying the following properties:

(a) $X \in \mathcal{A}$

(b) For all $A \in \mathcal{A}$ the set A^c_X is an element of $\mathcal{A}$.

(c) For any countable subset of $\mathcal{A}$, call it $\{A_i\}$, the union $\bigcup_i A_i$ is an element of $\mathcal{A}$.

Definition 1.12 (Borel set)**.** The collection of *Borel sets* of a topological space X is the smallest sigma algebra containing the open sets of X. A *Borel set* of X is a member of this collection.

Definition 1.13 (Coinfinite)**.** A subset A of a set X is *coinfinite* if $X \setminus A$ is infinite.

Definition 1.14 (Cocountable)**.** A subset A of a set X is *cocountable* if $X \setminus A$ is countable.

1.1.4 Almost Open Sets.

Lemma 1.15. *Here are a few properties of almost open sets.*

(a) The almost open subsets of X form an abelian group under $\triangle$.

(b) The almost open sets of a topological space form a sigma algebra.

(c) Borel sets are almost open.

(d) If $A \subset X \subset Y$ with A almost open in X and X a G_δ subset of Y, then A is almost open in Y.

(e) *If $A \subset X \subset Y$ with A almost open in X and X comeager in Y, then A is almost open in Y.*

(f) *Let $h\colon X \to Y$ be a homeomorphism. A set $A \subset X$ is almost open if and only if $h(A)$ is almost open.*

Proof.

(a) Suppose that $A = U\triangle M$, $B = V\triangle N$, where A, B are almost open, U, V are open, and M, N are meager. The boundary of an open set is nowhere dense, so the calculations below show that $A\triangle B$ is almost open.

$$A\triangle B = (U\triangle M)\triangle(V\triangle N) = (U\triangle V)\triangle(M\triangle N)$$

$$= \big(((U \cup V) \setminus \overline{(U \cap V)}) \cup \mathrm{bd}\overline{(U \cap V)}\big)\triangle(M\triangle N)$$

$$= \big(((U \cup V) \setminus \overline{(U \cap V)})\big)\triangle(\mathrm{bd}\overline{(U \cap V)}\triangle M\triangle N)$$

So the set of almost open sets is closed under $\triangle$. Every multiplicatively closed subset of $(\mathcal{P}(X), \triangle)$ is a normal subgroup of $(\mathcal{P}(X), \triangle)$, so the almost open sets form a normal subgroup of $(\mathcal{P}(X), \triangle)$.

(b) The properties of a $\sigma-$algebra T of a set X are (1) $X \in T$, (2) T is closed under complements, and (3) T is closed under countable unions, so we prove each of these.

(1) Since every space is open in itself, every space is almost open in itself. So the almost open sets contain the space as an element.

(2) If A is an almost open set of the space X, then $A\triangle X$ is both the complement of A and an almost open set. So the almost open sets are closed under complements.

(3) Suppose that $\{A_i = U_i\triangle M_i\}$ is a countable collection of almost open sets, where U_i is open and M_i is meager for all i.

Define $U = \bigcup_i U_i$, $M_{in} = (\bigcup_i (M_i \setminus U_i)) \setminus U$, and $M_{out} = \bigcap_i (M_i \cap U_i)$. It is obvious that U is open and M_{in} and M_{out} are both meager, so $U \triangle (M_{in} \triangle M_{out})$ is an almost open set. We intend to prove that $\bigcup_i A_i = U \triangle (M_{in} \triangle M_{out})$.

We see that $M_{out} \subset U$, so $U \triangle (M_{in} \triangle M_{out}) = (U \cup M_{in}) \setminus M_{out}$.

Suppose that $x \in \bigcup_i A_i$. Either there is some U_i such that $x \in (U_i \setminus M_i)$ or there exists some M_i such that $x \in (M_i \setminus U_i)$. Consider the first case. Then $x \in U$ and $x \notin M_{out}$, so $x \in (U \cup M_{in}) \setminus M_{out}$. Consider the second case, then either $x \in M_{in}$ or $x \in U$, but $x \notin M_{out}$, so $x \in (U \cup M_{in}) \setminus M_{out}$. So $(\bigcup_i A_i) \subset (U \cup M_{in}) \setminus M_{out}$.

Suppose that $x \in (U \cup M_{in}) \setminus M_{out}$. Then $x \in U \setminus M_{out}$ or $x \in M_{in} \setminus M_{out}$. Consider the first case. Then there exists some U_i such that $x \in (U_i \setminus M_i)$, which is a subset of A_i, so $x \in \bigcup_i A_i$. Consider the second case. Then $x \notin U$ and $x \in M_i$ for some M_i. So $x \in M_i \setminus U_i \subset A_i \subset \bigcup_i A_i$. So $((U \cup M_{in}) \setminus M_{out}) \subset (\bigcup_i A_i)$. So $(\bigcup_i A_i) = (U \cup M_{in}) \setminus M_{out} = U \triangle (M_{in} \triangle M_{out})$ is an almost open set, so the almost open sets are closed under countable unions.

(c) The Borel sets are the smallest sigma algebra containing the open sets. Part (b) shows that the almost open sets are a sigma algebra, and since open sets are almost open, then form a sigma algebra containing the open sets.

(d) Let $A = U \triangle M$, where U is open in X and M is meager in X. By Lemma 1.8(a), we know that M is meager in Y. There exists V, an open subset of Y, such that $V \cap X = U$. There also exists a countable collection of open subsets of Y, call them $\{O_i\}$, such that $\bigcap_i O_i = X$. So $U = V \cap X = V \cap (\bigcup_i O_i)$. So U is a Borel subset of Y, so U is almost open in Y. So $A = U \triangle M$ is almost open in Y.

(e) Write A as $U \triangle M$, where U is open in X and M is meager in X. By Lemma 1.8(a), we know that M is meager in Y. There exists V, an open subset of Y, such that $V \cap X = U$. Let $N = V \setminus X$, so $U = V \triangle N$. N is a subset of $Y \setminus X$, so N is meager in

Y. So $A = U \triangle M = (V \triangle N) \triangle M = V \triangle (N \triangle M)$, where V is open in Y and $(N \triangle M)$ is meager in Y.

(f) Suppose that $A = U \triangle M$, where U is open and M is meager. Then $f(A) = f(U \triangle M) = f(U) \triangle f(M)$ by injectivity. Homeomorphisms preserve meager sets, so $f(M)$ is meager, so $f(U) \triangle f(M) = f(A)$ is almost open. $\square$

1.2 Baire Spaces and Polish Spaces

Definition 1.16 (Baire Space)**.** A topological space X is a *Baire Space* if for any countable collection $\{N_i\}$ of closed sets with empty interior, $\bigcup_i N_i$ has empty interior.

1.2.1 Baire Spaces.

Lemma 1.17. *Here are a few properties of Baire Spaces.*

(a) Suppose that $N \subset X \subset Y$, where X is comeager in Y and Y is a Baire space. Then N is nowhere dense in X if and only if N is nowhere dense in Y.

(b) Suppose that $M \subset X \subset Y$, where X is comeager in Y and Y is a Baire space. Then M is meager in X if and only if M is meager in Y.

(c) Let Y be a Baire space, X a comeager subset of Y and $A \subset X \subset Y$. If A is almost open in Y then A is almost open in X.

Proof.

(a) ($\Rightarrow$) Let N be nowhere dense in X. Lemma 1.5(a) implies that N is nowhere dense in Y.

($\Leftarrow$) Suppose that N is nowhere dense in Y. Assume towards contradition that N somewhere dense in X. Then $\text{int}_X(\overline{N}_X) \neq \emptyset$. So there exists $V \subset Y$ open in Y such that $(V \cap X) = \text{int}_X(\overline{N}_X)$. The closure of a nowhere dense set is nowhere dense, so $\overline{N}_Y$ is nowhere dense in Y. Subsets of nowhere dense sets are nowhere dense, so $(V \cap X) = \text{int}_X(\overline{N}_X) \subset \overline{N}_X \subset \overline{N}_Y$ implies that $(V \cap X)$ is nowhere dense in Y. This implies that $V = (V \cap X) \cup (V \setminus X)$ is both open in Y and meager in Y, which is a contradiction.

(b) ($\Rightarrow$) Let M be meager in X. Lemma 1.8(a) implies that M is meager in Y.

($\Leftarrow$) Since M is meager in Y, there exists a countable collection of nowhere dense subsets of Y, call them N_i. Part(a) implies that each N_i is nowhere dense in X, so M is meager in X.

(c) ($\Rightarrow$) Lemma 1.15(e) implies that if A is almost open in X then A is almost open in Y.

($\Leftarrow$) Suppose that $A = U \triangle M$ is almost open in Y, and that U is open in Y and M is meager in Y. Consider that $U = (U \cap X) \cup (U \setminus X) = (U \cap X) \triangle (U \setminus X)$, since $(U \cap X)$ and $(U \setminus X)$ are disjoint. The subset of a meager set is meager, so $(U \setminus X) \subset X_Y^c$ is meager in Y. The meager sets are closed under $\triangle$, so $\big((U \setminus X) \triangle M\big)$ is meager in Y. So $A = (U \cap X) \triangle \big((U \setminus X) \triangle M\big)$, where $(U \cap X)$ is open in X and $\big((U \setminus X) \triangle M\big)$ is meager in Y. $\square$

1.2.2 Polish Spaces.

Definition 1.18 (Polish space). A topological space X is a *Polish Space* if it is separable and completely metrizable.

Lemma 1.19. *Here are some properties of Polish spaces.*

(a) Polish spaces are Baire spaces by the Baire Category Theorem. [10, P.41]

(b) Closed sets of a Polish space may be written uniquely as the disjoint union of a perfect set and a countable set. This is the Cantor–Bendixson theorem. [10, P.32]

1.2.3 Baire Groups and Polish Groups.

The intuition is that topological groups are both groups and topological spaces where the group operation and inverses preserve nice properties of the topology. Further, Baire groups and Polish groups are topological spaces with nice topological properties, but they are also groups with operations that interact well with the Baire and Polish topological properties.

Definition 1.20. A *topological group* is a group, call it G, whose underlying set is imbued with a topology such that $F\colon G \times G \to G$, given by $F(x, y) = x \cdot y$, and $i\colon G \to G$, given by $i(x) = x^{-1}$, are continuous functions.

Definition 1.21 (Baire group/Polish group)**.** A *Baire group* is a topological group that is also a Baire space. Likewise, a *Polish group* is a topological group that is also a Polish space. We notice that since all Polish spaces are Baire spaces, all Polish groups are Baire groups.

Lemma 1.22. *Let G be a Baire group and K be a subgroup of G. Here are some properties of Baire Groups.*

(a) Let X be a Baire group and $A \subset X$. A contains an almost open, non-meager subset if and only if $\{ab^{-1} : a, b \in A\}$ contains an open set of the identity of X. [18, p.211]

(b) K is open if and only if it contains a non-meager, almost open set.

(c) If K is a countably indexed non-open subgroup of G then K is not almost open.

(d) There exists a subgroup of a Polish group that is not almost open.

Proof. (a) This a result by Banach, Kuratowski, and Pettis. [18, p.211]

(b) ($\Rightarrow$) Suppose K is open. Open sets are almost open and nonempty open subsets of a Baire space are non-meager, so K contains itself as a non-meager, almost open set.

($\Leftarrow$) Suppose K contains a non-meager, almost open subset. By Lemma 1.22(a), there exists an open set U such that $e \in U \subset \{ab^{-1} \colon a, b \in K\}$, where e is the identity of G. Multiplication by a fixed element preserves openness and subgroups are closed under both multiplication and inversion, so for every $k \in K$ we get $k = ke \in kU \subset k\{ab^{-1} \colon a, b \in K\} \subset K$. So K is open in G.

(c) If K were meager then B would be meager, which is a contradiction.

Part (b) implies that every non-meager subset of A is not almost open. Since A is a non-meager subset of itself, A is not almost open.

(d) We shall take the Cantor set to be the underlying topological space of $\mathbb{Z}_2^{\mathbb{N}}$. The Cantor set is a Polish space, and Lemma 1.19(a) says that Polish spaces are Baire spaces, so $\mathbb{Z}_2^{\mathbb{N}}$ is a Baire group.

Let S be the extension of $\{e_i \colon i \in \mathbb{N}\}$ to a basis of $\mathbb{Z}_2^{\mathbb{N}}$. We construct a homomorphism $\phi \colon \mathbb{Z}_2^{\mathbb{N}} \to \mathbb{Z}_2$ by mapping all of the e_i's of $\mathbb{Z}_2^{\mathbb{N}}$ to 1, making arbitrary assignments for the images of the remaining elements of S, and then assigning the images of the non-basis elements of $\mathbb{Z}_2^{\mathbb{N}}$ in a manner consistent with the basis assignments. Then $\ker(\phi)$ is not an open subgroup, since $(e_i) \to 0$, but $\phi(e_i) = 1$. The index of $\ker(\phi)$ is two, so by Part (c), $\ker(\phi)$ is not almost open in $\mathbb{Z}_2^{\mathbb{N}}$.

As a corollary, we notice that this is also an example of a subset of a Polish space that is not almost open. The homomorphism in this example is from Conner and Spencer [1, p. 225]. $\square$

Notice that we used the Axiom of Choice to construct the homomorphism in Lemma 1.22 (d) when we extended $\{e_i \colon i \in \mathbb{N}\}$ to a basis of $\mathbb{Z}_2^{\mathbb{N}}$. Later in the paper we will show that there is no way to construct this homeomorphism using weak versions of the Axiom of Choice.

1.2.4 The Cantor Set.

Definition 1.23 (The Cantor group $\mathcal{K}$)**.** We denote the topological group $\mathbb{Z}_2^{\mathbb{N}}$ as $\mathcal{K}$ in this paper. The underlying topological space for this topological group is the Cantor space.

Lemma 1.24. *The Cantor space $\mathcal{K}$ is homeomorphic to both $\mathcal{K}^n$ for any $n \in \mathbb{N}$ and $\mathcal{K}^{\mathbb{N}}$*

Proof. Since our Cantor space $\mathcal{K}$ is the countably infinite product $\mathbb{Z}_2^{\mathbb{N}}$, the spaces $\mathcal{K}^n$ and $\mathcal{K}^{\mathbb{N}}$ are merely $(\mathbb{Z}_2^{\mathbb{N}})^n$ and $(\mathbb{Z}_2^{\mathbb{N}})^{\mathbb{N}}$, which are both countably infinite products of copies of $\{0, 1\}$, which is the definition of $\mathcal{K}$. $\square$

Chapter 2. Almost Homeomorphisms

The intuition for almost homeomorphisms is that they are "slightly defective" in fulfilling the requirements to be homeomorphisms. The precise meaning of "slightly defective" is formalized in the following definition.

Note 2.1. We will be working with subspaces extensively and this necessitates notation to differentiate between the interior, exterior, closure, etc of a set in a space and in the subspace. We will use subscripts (e.g. $\mathrm{int}_X(A), \overline{A}_Y$) to indicate which space this operation is respecting.

Definition 2.2 (Almost homeomorphic)**.** A bijection $f\colon X \to Y$ between topological spaces is an *almost homeomorphism* if there exist comeager sets $C \subset X$ and $D \subset Y$ such that $f\mid_C\colon C \to D$ is a homeomorphism. When there exists an almost homeomorphism between two topological spaces, we say the spaces are *almost homeomorphic*.

Lemma 2.3. *Here are some properties of almost homeomorphisms.*

(a) Homeomorphic spaces are almost homeomorphic.

(b) Suppose that Y is a non-empty, perfect, Baire, Hausdorff space and that X is a co-countable subset of Y. Then X and Y are almost homeomorphic.

(c) Almost homeomorphisms between Baire spaces preserve almost open sets.

(d) Suppose that $F\colon X_1 \times X_2 \times \cdots \times X_n \to Y_1 \times Y_2 \times \cdots \times Y_n$ is given by

$$F(x_1, x_2, \cdots, x_n) = (f_1(x_1), f_2(x_2), \cdots, f_n(x_n))$$

and that each $f_i(x_i)\colon X_i \to Y_i$ is an almost homeomorphism. Then F is an almost homeomorphism.

(e) *Let X and Y be topological spaces such that X is the disjoint union of a meager set M with a countable collection of open sets A_i and Y is the disjoint union of a meager set N with a countable collection of open sets B_i. Suppose also that M and N have the same cardinality. If there exists an almost homeomorphism $f_i\colon A_i \to B_i$ for every $i \in \mathbb{N}$, then there exists an almost homeomorphism $F\colon X \to Y$.*

(f) *The property of being almost homeomorphic is an equivalence relation on Baire spaces.*

(g) *The following spaces are almost homeomorphic: the Cantor set, the closed interval $[0,1]$, $\mathbb{R}$, the finite product of $[0,1]$'s, $\mathbb{R}^n$, the Hilbert Cube and $\mathbb{R}^{\mathbb{N}}$.*

Proof.

(a) Suppose X and Y are homeomorphic. Every space is comeager in itself, so $X \subset X$ and $Y \subset Y$ are homeomorphic comeager subsets of their respective spaces, so X and Y are almost homeomorphic.

(b) Let $A = X^c_Y$ and B be an infinite, coinfinite, countable subset of X. We handle the cases where A is finite and A is infinite separately.

(1) If A is finite with cardinality n, the following function $f\colon Y \to X$ is an almost homeomorphism.

$$f(x) = \begin{cases} x & x \notin A \cup B \\ b_i & x = a_i \in A \\ b_{i+n} & x = b_i \in B \end{cases}$$

(Injective) Suppose that $f(x) = f(y)$ for some $x, y \in Y$. $f(A \cup B) \cap f((A \cup B)^c) = \emptyset$, so either $x, y \notin A \cup B$ and $x = y$ or $x, y \in A \cup B$, so $f(x) = f(y) = b_i$ so $x = y = a_i$ for $i \leq n$ or $x = y = b_{i-n}$ for $i > n$. So the function is injective.

(Surjective) Suppose that $y \in X$. Then $f(y) = y$ if $y \notin (A \cup B)$, or $f(a_i) = b_i$ if $y = b_i$ for $i \leq n$, or $f(b_{i-n}) = b_i$ for $y = b_i$ for $i > n$. So the function is surjective.

(Restriction) Since the union of meager sets is meager, $Y \setminus (A \cup B)$ is a comeager subset of Y. Lemma 1.17(b) says that a comeager subset of a Baire space that is contained in a comeager subspace is comeager in that subspace, so $X \setminus B$ is comeager in X.

The restriction of f to $f\,|_{Y \setminus (A \cup B)} \colon Y \setminus (A \cup B) \to (X \setminus B)$ is the identity, so it is a homeomorphism. So the function f is an almost homeomorphism.

(2) If A is infinite, the following function $f \colon Y \to X$ is an almost homeomorphism.

$$f(x) = \begin{cases} x & x \notin A \cup B \\ b_{2i-1} & x = a_i \in A \\ b_{2i} & x = b_i \in B \end{cases}$$

(Bijective) Suppose that $f(x) = f(y)$ for some $x, y \in Y$. $f(A \cup B) \cap f((A \cup B)^c) = \emptyset$, so either $x, y \notin A \cup B$ and $x = y$ or $x, y \in (A \cup B)$, so $f(x) = f(y) = b_i$ so $x = y = a_{(i+1)/2}$ for odd i or $x = y = b_{i/2}$ for even i. So the function is injective.

(Surjective) Suppose that $y \in X$. Then $f(y) = y$ if $y \notin (A \cup B)$ or $y = b_i$ for some b_i. If i is odd then $f(a_{(i+1)/2}) = b_i$ and if i is even then $f(b_{i/2}) = b_i$. So the function is surjective.

(Restriction) Since the union of meager sets is meager, $Y \setminus (A \cup B)$ is a comeager subset of Y. Lemma 1.17(b) says that a comeager subset of a Baire space that is contained in a comeager subspace is comeager in that subspace, so $X \setminus B$ is comeager in X.

So X and Y are almost homeomorphic.

(c) Suppose that $h \colon X \to Y$ is an almost homeomorphism between Baire spaces X and Y and $A \subset X$ is almost open. Then there exist subsets $C \subset X$ and $D \subset Y$, which are comeager in their respective spaces, such that $h\,|_C\, C \to D$ is a homeomorphism.

Notice that $A \setminus C$ is meager. Lemma 1.15(a) says that the almost open sets are closed under $\triangle$, so $(A \cap C) = (A \triangle (A \setminus C)) \subset C$ is almost open in X. Lemma 1.17(c) implies that $(A \cap C)$ is almost open in C, because C is comeager and X is a Baire space. Lemma 1.15(f) says that homeomorphisms preserve almost open sets, so $h(A \cap C)$ is almost open in D. Lemma 1.17(c) implies that $h(A \cap C)$ is almost open in Y because D is comeager in Y and Y is a Baire space.

Notice that $h(A \setminus C)$ is a subset of the meager set D^c_Y, so it is meager. So

$$h(A \cap C) \triangle h(A \setminus C) = h((A \cap C) \triangle (A \setminus C)) = h(A)$$

is almost open in Y.

Let $f\colon \big(X \setminus (M \cup g^{-1}(N))\big) \to \big(Y \setminus (h(M) \cup N)\big)$ be the restriction of h to $\big(X \setminus (M \cup g^{-1}(N))\big)$. So f is a homeomorphism that is also a restriction of h to a comeager set. So X and Y are almost homeomorphic.

(d) A function which is bijective in each coordinate is a bijection, so all that remains is to show that some restriction of F to a comeager subspace is a homeomorphism.

Let $A_i \subset X_i$ be a meager set such that $f_i \mid_{(X_i \setminus A_i)} (X_i \setminus A_i) \to (Y_i \setminus f_i(A_i))$ is a homeomorphism. Lemma 1.8(b) implies that $\mathrm{net}(\{A_i\})$ and $\mathrm{net}\left(\{f_i(A_i)\}\right) = F\left(\mathrm{net}(\{A_i\})\right)$ are meager in X and Y respectively.

Since $f_i \mid_{(X_i \setminus A_i)} (X_i \setminus A_i) \to (Y_i \setminus f_i(A_i))$ is a homeomorphism, the further restriction

$$f_i \mid_{(X_i \setminus \pi_i(\mathrm{net}(\{A_i\})))} (X_i \setminus \pi_i(\mathrm{net}(\{A_i\}))) \to (Y_i \setminus f_i(\pi_i(\mathrm{net}(\{A_i\}))))$$

is a homeomorphism. So $F \mid_{(X \setminus (\mathrm{net}(\{A_i\})))}$ is the product of homeomorphisms, so it is a homeomorphism. So F is an almost homeomorphism.

(e) Let g be a bijection from M to N. Define F as follows:

$$F(x) = \begin{cases} g(x) & x \in M \\ f_i(x) & x \in A_i \end{cases}$$

Pasting bijections together results in a bijection, so F is a bijection. What remains to be shown is the property that there exist comeager sets, one in the domain and the other in the codomain, which are homeomorphic to each other.

Each f_i corresponds to a pair of meager sets M_i and N_i, with M_i in the domain and N_i in the codomain, such that $f_i \mid_{(A_i \setminus M_i)}$ is a homeomorphism with its image. $M \cup \left(\bigcup_{i \in \mathbb{N}} M_i\right)$ and $N \cup \left(\bigcup_{i \in \mathbb{N}} N_i\right)$ are meager because countable unions of meager sets are meager. Pasting homeomorphisms together results in a homeomorphism, so $F \mid_{\left(X \setminus \left(M \cup \left(\bigcup_{i \in \mathbb{N}} M_i\right)\right)\right)}$ is a homeomorphism with its image, so F is an almost homeomorphism.

(f) (1) A space is homeomorphic to itself and is comeager in itself, therefore a space is almost homeomorphic to itself.

(2) Being homeomorphic is an equivalence relation, so if $A \subset X$ homeomorphic to $B \subset Y$ then B is homeomorphic to A. Both A and B are comeager in their respective spaces, so Y is almost homeomorphic to X.

(3) Suppose that X is almost homeomorphic to Y and Y is almost homeomorphic to Z. Then there exist $A \subset X$, $B \subset Y$, $C \subset Y$, and $D \subset Z$ such that A and B are homeomorphic and C and D are homeomorphic, and each set is comeager in its respective space.

The intersection of countably many comeager sets is comeager, so $(B \cap C)$ is comeager in Y. Lemma 1.17(b) says that a subset of a comeager subspace of a Baire space is comeager in the subspace if and only if it is comeager in the Baire space, so $(B \cap C)$ is comeager in both B and C.

Since B is homeomorphic to A, the image of $(B \cap C)$ in A is comeager in A. So the image of $(B \cap C)$ in A is comeager in X.

Similarly, since C is homeomorphic to D, the image of $(B \cap C)$ in D is comeager in D, so the image of $(B \cap C)$ in D is comeager in Z. So there exist comeager subsets of X and Z that are homeomorphic to each other, namely the images of $(B \cap C)$ in X and Z.

(g) We will prove that the Cantor set is almost homeomorphic to both $[0, 1]$ and the Hilbert cube, and prove that $[0, 1]$ is almost homeomorphic to $\mathbb{R}$; Part (f) will then imply that all of these spaces are almost homeomorphic to each other.

(1) The Cantor set, $[0, 1]$ and $\mathbb{R}$ are almost homeomorphic to each other.

Let D be the dyadic rationals of $[0, 1]$, $C = [0, 1] \setminus D$, A the accessible points of $\mathbb{Z}_2^{\mathbb{N}}$ and B the inaccessible points of $\mathbb{Z}_2^{\mathbb{N}}$.

Let $g\colon \mathbb{Z}_2^{\mathbb{N}} \to [0, 1]$ be given by

$$a = (a_1, a_2, \cdots) \mapsto \begin{cases} \sum_i \frac{a_i}{2^i} & a \in B \\ \frac{1}{2} + \sum_i \frac{a_i}{2^{i+1}} & a \text{ eventually } \overline{1} \\ \sum_i \frac{a_i}{2^{i+1}} & a \text{ eventually } \overline{0} \end{cases}$$

g is a bijection that maps A to D and B to C.

We claim that $g\restriction_B\colon B \to C$ is a homeomorphism. If we consider the elements of C to be real numbers in their binary representations, then it is clear that both B and C are countably infinitely long strings of zeros and ones that are neither eventually zero nor eventually one, and that $g\restriction_B\colon B \to C$ preserves the natural order of these strings. So $g\restriction_B$ is a homeomorphism. So g is an almost homeomorphism.

(2) Part (b) states that a non-empty, perfect, Baire Hausdorff space is almost homeomorphic to any cocountable subspace of itself, so $[0,1]$ is almost homeomorphic to $(0,1)$, and Part (a) says that homeomorphic spaces are almost homeomorphic to each other, so $(0,1)$ is almost homeomorphic to $\mathbb{R}$.

(3) The Cantor set, $[0,1]^n$, the Hilbert cube, $\mathbb{R}^n$, and $\mathbb{R}^{\mathbb{N}}$ are almost homeomorphic to each other.

All cases are done analogously to the Hilbert cube case, so only that case will be shown. Let $f\colon \mathcal{K} \to [0,1]$ be an almost homeomorphism. The function $F\colon \mathcal{K}^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ with each coordinate function being f is an almost homeomorphism by Part (d). Lemma 1.24 states that $\mathcal{K}$ is homeomorphic to $\mathcal{K}^{\mathbb{N}}$, so the Hilbert cube is almost homeomorphic to $\mathcal{K}$. $\square$

2.1 Manifolds

A manifold is a special topological space with several nice qualities. Manifolds are metrizable, are separable, and are locally homeomorphic to Euclidean space. They also have a uniform dimension and are Hausdorff. These are other nice properties allow us to learn a lot about manifolds. We shall prove some results about almost homeomorphisms using manifolds as a familiar starting point.

Definition 2.4 (Manifold)**.** A *manifold* is a second countable Hausdorff space that is locally homeomorphic to $\mathbb{R}^n$ for some fixed $n \in \mathbb{N}$. The number n is called the *dimension* of M.

Definition 2.5 (Euclidean half-space)**.** The *Euclidean half-space* of dimension n is a subspace of $\mathbb{R}^n$ defined to be $\{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.

Definition 2.6 (Manifold with boundary)**.** A *manifold with boundary* is a second countable Hausdorff space that is locally homeomorphic to $\mathbb{R}^n_+$ for some fixed $n \in \mathbb{N}$. The number n is called the *dimension* of M.

Note 2.7. We notice that all manifolds are manifolds with boundary, even though not all manifolds with boundary are manifolds. However, our theorems and lemmas regarding manifolds are true of manifolds with boundary as well, so we will use the convention in this paper of calling both manifolds and manifolds with boundary "manifolds."

Some papers omit the second countability requirement for a manifold, but this paper does not.

2.1.1 Manifold Decomposition.

We will show a technique for decomposing an arbitrary manifold into a disjoint collection of subsets that will allow us to construct an almost homeomorphism to the real line.

This technique is intuitively like taking an $n-$dimensional ice cream scoop to the manifold and scooping out a countable collection of picturesque, round, Euclidean scoops. We shall show that it is possible to scoop out a countably infinite set of Euclidean ice cream scoops and that what is left is negligible in the sense that what is left over is meager.

Lemma 2.8. *A manifold M of dimension $n > 1$ can be decomposed into a pairwise disjoint union of a set of the carinality of the continuum which is meager in M and countably infinitely many homeomorphic copies of $\mathbb{R}^n$.*

Proof. Let M be an n dimensional manifold. Since M is second countable there can be at most countably many disjoint open sets in M. This means that there can be at most countably many pairwise disjoint balls in M that are homeomorphic to $\mathbb{R}^n$. Let $\mathcal{A}$ be a maximal collection of pairwise disjoint open balls in M, each ball being homeomorphic to $\mathbb{R}^n$. We shall show that the complement of $\bigcup_{A\in\mathcal{A}} A$ is meager.

First, we notice that since $\bigcup_{A\in\mathcal{A}} A$ is open, its complement is closed. Suppose that $x \in \operatorname{int}\left(\left(\bigcup_{A\in\mathcal{A}} A\right)^c\right)$. Then there exists an open set U containing x that does not intersect $\bigcup_{A\in\mathcal{A}} A$, which further implies that there exists a ball homeomorphic to $\mathbb{R}^n$ containing x, which is itself contained in U. Since U does not intersect any element of $\mathcal{A}$, this is a ball homeomorphic to $\mathbb{R}^n$ which is disjoint from any element of $\mathcal{A}$. The existence of this ball contradicts the maximality of $\mathcal{A}$. So int $\left(\left(\bigcup_{A\in\mathcal{A}} A\right)^c\right)$ is empty, so $\left(\bigcup_{A\in\mathcal{A}} A\right)^c$ is meager.

We have decomposed M into the countable disjoint union of the elements of $\mathcal{A}$, each of which is a ball homeomorphic to $\mathbb{R}^n$, and the meager set $\left(\bigcup_{A\in\mathcal{A}} A\right)^c$.

If $\left(\bigcup_{A\in\mathcal{A}} A\right)^c$ is not of the cardinality of the continuum, then $\left(\bigcup_{A\in\mathcal{A}} A\right)^c$ has cardinality less than the continuum, because manifolds are of the cardinality of the continuum (so long as the manifold is not zero dimensional). So we need to add points to $\left(\bigcup_{A\in\mathcal{A}} A\right)^c$ while retaining its meagerness. We shall do this by dividing an element of $\mathcal{A}$ into two homeomorphic copies of $\mathbb{R}^n$ and a meager set, then redefining $\mathcal{A}$ to reflect the new collection of homeomorphic copies of $\mathbb{R}^n$. Take one ball from $\mathcal{A}$. This ball is homeomorphic to $\mathbb{R}^n$. We can divide this ball into three subsets, $(-\infty,0)\times\mathbb{R}^{n-1}$, $\{0\}\times\mathbb{R}^{n-1}$, and $(0,\infty)\times\mathbb{R}^{n-1}$. Both of $(-\infty,0)\times\mathbb{R}^{n-1}$ and $(0,\infty)\times\mathbb{R}^{n-1}$ are homeomorphic to $\mathbb{R}^n$, while $\{0\}\times\mathbb{R}^{n-1}$ is meager in the ball. Lemma 1.8(a) implies that $\{0\}\times\mathbb{R}^{n-1}$ is meager in M, because it is meager in a subspace of M. So

$$\left(\{0\}\times\mathbb{R}^{n-1}\right)\cup\left(\bigcup_{A\in\mathcal{A}} A\right)^c$$

is meager in M. Since the cardinality of $\{0\}\times\mathbb{R}^{n-1}$ is the cardinality of the continuum, the newly constructed meager set is also of the cardinality of the continuum. We redefine $\mathcal{A}$.

Suppose that there are only finitely many elements of $\mathcal{A}$. Then we can take one element of $\mathcal{A}$ and decompose it into the countably infinitely many sets of the form $(m_1, m_1+1)\times(m_2, m_2+1)\times\cdots\times(m_n, m_n+1)$ where $m_i\in\mathbb{Z}$ for all i. We shall call the collection of these sets $\mathcal{B}$ and note that each element of $\mathcal{B}$ is homeomorphic to $\mathbb{R}^n$.

The complement of the union of the elements of $\mathcal{B}$ is the set

$$\left(\bigcup_{B\in\mathcal{B}} B\right)^c_{\mathbb{R}^n} = \{x\in\mathbb{R}^n \mid\ x \text{ has at least one integer coordinate}\}$$

which is the net of n copies of $\mathbb{Z}$ in $\mathbb{R}^n$, so it is meager by Lemma 1.8(b).

The union of $\{x \in \mathbb{R}^n \mid \ x \text{ has at least one integer coordinate}\}$ and $\left(\bigcup_{A\in\mathcal{A}} A\right)^c$ is meager, while the union of $\mathcal{A}$ and $\mathcal{B}$ is a countably infinite collection of homeomorphic copies of $\mathbb{R}^n$. □

2.1.2 Manifolds are Almost Homeomorphic.

Theorem 2.9. *All manifolds of at least dimension 1 are almost homeomorphic to each other.*

Proof. We shall split into the one dimensional case and the more than one dimensional case.

(1) Suppose that X is a one dimensional manifold. The one dimensional manifolds are countable disjoint unions of $\mathbb{R}$, $[0,1]$, $[0,1)$, and S^1. It is possible to remove at most countably many points from a one dimensional manifold to leave an at most countable collection of homeomorphic copies of $\mathbb{R}$. This allows us to apply Lemma 2.3(b).

    If there are countably infinitely many copies of $\mathbb{R}$ after removing the aforementioned points, then each is homeomorphic to $(m, m+1), m \in \mathbb{N}$ and by the pasting properties of almost homeomorphisms in Lemma 2.3(e) we have an almost homeomorphism.

    If there are only finitely many copies of $\mathbb{R}$ after the point removal, but at least 2 copies, then a similar technique is used, except that one of the copies is homeomorphed to $(-\infty, 0)$ and another is homeomorphed to (n, ∞), for an appropriate $n \in \mathbb{N}$. The remaining copies, if any, are mapped to $(m, m+1)$ for $m \in \mathbb{N}$ but less than n.

    If there is only one copy of $\mathbb{R}$ then we are done.

(2) Lemma 2.8 implies that a manifold of dimension at least 2 can be decomposed into a disjoint union of a meager set of cardinality of the continuum and countably infinitely many homeomorphic copies of $\mathbb{R}^n$. Call the meager set M and let $\{A_i\}$ be the countably many homeomorphic copies of $\mathbb{R}^n$.

    Let g be a bijection from M to the usual "middle thirds" Cantor set embedded in the closed interval $[0,1]$. There are countably many open intervals in the complement of the Cantor subset of $[0, 1]$, and each of these open intervals are almost homeomorphic to $\mathbb{R}^n$ by Lemma 2.3(g). Let $f_i \colon A_i \to (a_i, b_i)$ be an almost homeomorphism between A_i and the i^{th} open interval in the complement of the Cantor subset of $[0, 1]$.

Lemma 2.3(e) implies that

$$F(x) = \begin{cases} g(x) & x \in M \\ f_i(x) & x \in A_i \end{cases}$$

is an almost homeomorphism.

So all manifolds of at least dimension 1 are almost homeomorphic to each other. $\square$

2.2 Perfect Polish Spaces

Definition 2.10 (Perfect Polish space)**.** A *perfect Polish space* is a Polish space with no isolated points.

2.2.1 Polish Space Decomposition.

Theorem 2.11. *If Y is an imperfect Polish space containing a non-almost open set, then there exists a perfect Polish subspace of Y which contains a non-almost open set.*

Proof. Let A be the non-almost open subset of Y. Since Polish spaces are Hausdorff, singleton sets of Y are closed and countable unions of closed sets are almost open by Lemma 1.15(b), which states that the almost open sets form a sigma algebra. So a non-almost open set would need to be of uncountable cardinality. This means that Y can be decomposed into a perfect set X and a countable set P by Part (b).

The set $(A \setminus X)$ is a countable subset of the Polish space Y, so it is an almost open subset of Y. We see that

$$A = (A \cap X) \cup (A \setminus X) = (A \cap X) \triangle (A \setminus X)$$

because $(A \cap X)$ and $(A \setminus X)$ are disjoint. Suppose that $(A \cap X)$ were almost open in Y. Then by Lemma 1.15(a), which states that the almost open sets form a group under $\triangle$, A is almost open. This is a contradiction. So $(A \cap X)$ is not almost open in Y.

X is the complement of a countable set, so X is a G_δ subset of Y. Lemma 1.15(d) states that a set which is almost open in a G_δ subspace must be almost open in the superspace, so the contrapositive to Lemma 1.15(d) implies that since $(A \cap X)$ is not almost open in Y, it cannot be almost open in X.

So X is a Polish subspace of Y containing a non-almost open set and containing no isolated points. □

Definition 2.12 (Borel measurable)**.** A function $f\colon X \to Y$ is a *Borel measurable* function if for every Borel set $B \subset Y$ the preimage $f^{-1}(B)$ is Borel in X.

Definition 2.13 (Baire measurable)**.** A function $f\colon X \to Y$ is a *Baire measurable* function if for every open set $U \subset Y$, the preimage $f^{-1}(U)$ is almost open in X.

Lemma 2.14. *Borel measurable functions are Baire measurable functions.*

Proof. Suppose that $f\colon X \to Y$ is Borel measurable. Let U be open in Y. Then $f^{-1}(U)$ is Borel in X. Borel sets are almost open by Lemma 1.15(c). So f is Baire measurable. □

Lemma 2.15. *If X is an non-empty, perfect Polish space, then there exists a dense G_δ subset of X which is homeomorphic to $\mathbb{N}^{\mathbb{N}}$.*

Proof. Let $f\colon D \to X$ be a continuous bijection from a closed subset of $\mathbb{N}^{\mathbb{N}}$. This is possible by Theorem 2.6.9 of [11, P. 77] which states that every Polish space is the one-to-one continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$. Suppose that $J \subset D$ is a Borel set. Then $f(J)$ is Borel by Theorem 4.5.4 in [11, P. 153], which says that if X, Y are Polish, $A \subset X$, and the function $f\colon A \to Y$ is one-to-one and continuous then $f(A)$ is Borel. The implication that J being Borel causes $f(J)$ to be Borel is equivalent to f^{-1} being Borel measurable.

Every Borel measurable function is Baire measurable by Lemma 2.14, so Proposition 3.5.8 [11, P. 110] implies that there exists $A \subset X$ which is comeager in X such that $f^{-1}|_A\colon A \to D$

is continuous. Since f was one-to-one and continuous, the restriction $f^{-1}|_A\colon A \to f^{-1}(A)$ is a homeomorphism. By a theorem of Lavrentiev [11, P. 55] there exist G_δ sets B and C such that $f^{-1}(A) \subset C \subset \mathbb{N}^\mathbb{N}$ and $A \subset B \subset X$ such that $f^{-1}|_A\colon A \to f^{-1}(A)$ can be extended to a homeomorphism $g\colon C \to B$.

We notice that neither B nor C have any isolated points, because B is a dense G_δ subset of X and X has no isolated points.

Let Q be a countable dense subset of C. We shall show that $C \setminus Q$ is a G_δ subset of $\mathbb{N}^\mathbb{N}$ with no isolated points such that any compact subset of $C \setminus Q$ has empty interior.

C is a G_δ subset of $\mathbb{N}^\mathbb{N}$ so its complement in $\mathbb{N}^\mathbb{N}$ is an F_σ. Points are closed in $\mathbb{N}^\mathbb{N}$, so $C^c_{\mathbb{N}^\mathbb{N}} \cup Q^c_C$ is a countable union of closed sets of $\mathbb{N}^\mathbb{N}$, so the complement of $C^c_{\mathbb{N}^\mathbb{N}} \cup Q^c_C$, namely $C \setminus Q$ is a G_δ subset of $\mathbb{N}^\mathbb{N}$.

Suppose that $K \subset C \setminus Q$ is compact and has nonempty interior in $C \setminus Q$. Then we can pick a nonempty open set $O \subset C$ such that $K \cap O = (C \setminus Q) \cap O$. By the density of Q, there exists a point $x \in O \cap Q$. Since C has no isolated points we can choose a sequence $\{x_n\} \subset C$ which converges to x and does not intersect Q. Eventually $\{x_n\}$ is in O, so it is eventually in $O \cap K = (C \setminus Q) \cap O$. This implies that the metric on K induced from C is not complete, which contradicts every metric on a compact metric space being complete.

So $C \setminus Q$ is a G_δ subset of $\mathbb{N}^\mathbb{N}$, so it is a Polish space that has no isolated points. A theorem of Alexandrov and Urysohn [10, Theorem 7.7] implies that $C \setminus Q$ is homeomorphic to $\mathbb{N}^\mathbb{N}$. We notice that $g(C \setminus Q)$ is a dense G_δ subset of X that is homeomorphic to $\mathbb{N}^\mathbb{N}$. $\square$

Lemma 2.16. *An non-empty, perfect Polish space contains a nowhere dense set of the cardinality of the continuum.*

Proof. Let X be an uncountable Polish space without isolated points. Proposition 2.6.1 in [11] states that every uncountable Polish Space without isolated points contains a subset which is homeomorphic to the Cantor set, call the subset A and the homeomorphism $f\colon \mathcal{K} \to A$.

The Cantor set, denoted $\mathcal{K}$, is homeomorphic to $\mathcal{K}^2$ by Lemma 1.24, call the homeomorphism $g\colon \mathcal{K}\times\mathcal{K}\to\mathcal{K}$. So $(f\circ g)\colon \mathcal{K}\times\mathcal{K}\to A$ is a homeomorphism.

Pick a point in $\mathcal{K}$, call it x. By Lemma 1.5(b) we know that $\{x\}\times\mathcal{K}$ is nowhere dense in $\mathcal{K}\times\mathcal{K}$. Homeomorphisms preserve nowhere density, so $(f\circ g)(\{x\}\times\mathcal{K})$ is nowhere dense in A. Restrictions of homeomorphisms are homeomorphisms, so $(f\circ g)\restriction_{\{x\}\times\mathcal{K}}\colon \{x\}\times\mathcal{K}\to (f\circ g)(\{x\}\times\mathcal{K})$ is a homeomorphism. But $\{x\}\times\mathcal{K}$ is homeomorphic to $\mathcal{K}$, so $(f\circ g)(\{x\}\times\mathcal{K})$ is a homeomorphic copy of $\mathcal{K}$ that is nowhere dense in A.

Lemma 1.5(a) states that being nowhere dense in a subspace implies being nowhere dense in a superspace, so $(f\circ g)(\{x\}\times\mathcal{K})$ is a nowhere dense subset of X which is homeomorphic to $\mathcal{K}$. □

2.2.2 Non-Empty Perfect Polish Spaces are Almost Homeomorphic. This result is strictly stronger than Theorem 2.9, but relies on a stronger knowledge of descriptive set theory than the proof of the manifold case does. The manifold case uses a topological argument that will feel more natural to a topologist and give intuition for this stronger result.

The theorem shows that all uncountable Polish spaces without isolated points are almost homeomorphic. Since meager sets cannot contain isolated points, it isn't possible to strengthen the theorem much more than this. If two uncountable Polish spaces share the same cardinality of isolated points in an immediate corollary, but Polish spaces with differing numbers of isolated points cannot be almost homeomorphic because no isolated point (which is non-meager as a singleton set) can be sent to a non-isolated point (which wound be nowhere dense) by an almost homeomorphism.

Theorem 2.17. *All non-empty, perfect Polish spaces are almost homeomorphic.*

Proof. Suppose X and Y are uncountable Polish spaces without isolated points. There exist nowhere dense sets $\mathcal{K}_X\subset X$ and $\mathcal{K}_Y\subset Y$ which are each homeomorphic to the Cantor set $\mathcal{K}$. Then $X\setminus\mathcal{K}_X$ and $Y\setminus\mathcal{K}_Y$ are Polish spaces with no isolated points, so by Lemma 2.15

there exist dense G_δ subsets $Z_X \subset X \setminus \mathcal{K}_X$ and $Z_Y \subset Y \setminus \mathcal{K}_Y$ which are homeomorphic to $\mathbb{N}^\mathbb{N}$, therefore they are homeomorphic to each other. Further, $X \setminus Z_X$ and $Y \setminus Z_Y$ are each of cardinality $2^{\aleph_0}$ and are meager in X and Y respectively, so X and Y are almost homeomorphic. $\square$

Chapter 3. Inscrutability

There is a long history in mathematics of categorizing the complexity of various objects or difficulty of various tasks. Bachmann [14] and Landau [15] developed the Bachmann–Landau notation "Big O" classification for the amount of time it would take for a computer to solve a problem. Turing [16] and Church defined rigorously what it means for a problem to be computable. The definition they devised gives a useful boundary between what we can reasonably consider to be solvable by some computer and what cannot be solved by computer.

We parallel these definitions to give a rigorous definition of what it means for a human to be able to visualize or picture a mathematical object. Intuitively, a *scrutable* object is able to be constructed in countably many decisions; a countably infinite number of successively improving approximations can be made which approach the actual object arbitrarily well.

On the other hand, an *inscrutable* object is an object that cannot be constructed using countably many choices, where these choices need not be made at the same time and which may not be algorithmically related. Bases for $\mathbb{R}$ as a vector space over $\mathbb{Q}$, a set of group theoretic cosets of $\mathbb{Q}$ in $\mathbb{R}$, and non-principle ultra-filters are examples of such objects.

3.1 Axiom Systems

The definition of inscrutability is based on the relationship between the Zermelo-Fraenkel axioms and the Axiom of Choice, so we will review these in preparation for the formal definition of inscrutability.

Definition 3.1 (Inconsistent)**.** An axiomatic system is said to be *inconsistent* if a contradiction can be derived from the axioms.

Definition 3.2 (Consistent)**.** An axiomatic system is said to be *consistent* if it is not inconsistent

Example 3.3. The axiom system $\{P, \neg P\}$ allows the statement "P and $\neg P$" to be derived, so this axiom system contains a contradiction.

Definition 3.4 (Equiconsistent)**.** Two axiomatic systems, $\mathcal{A}$ and $\mathcal{B}$, are *equiconsistent* if the consistency of $\mathcal{A}$ implies the consistency of $\mathcal{B}$ and the consistency of $\mathcal{B}$ implies the consistency of $\mathcal{A}$.

Definition 3.5 (Addition of an axiom to an axiomatic system)**.** If $\mathcal{A}$ is an axiomatic system and P is an axiom, it is possible to form a new (possibly inconsistent) axiomatic system that contains all of the axioms of $\mathcal{A}$ and the axiom P. This concatenated system is denoted $\mathcal{A} + P$

Definition 3.6 (Concatenation of axiomatic systems)**.** If $\mathcal{A}$ and $\mathcal{B}$ are axiomatic systems, it is possible to form a third (possibly inconsistent) axiomatic system that contains all of the axioms of $\mathcal{A}$ and $\mathcal{B}$. This concatenated system is denoted $\mathcal{A} + \mathcal{B}$

Note 3.7. We notice that for axiomatic systems $\mathcal{A}, \mathcal{B}$, and $\mathcal{A} + \mathcal{B}$, the consistency of $\mathcal{A} + \mathcal{B}$ implies the consistency of $\mathcal{A}$ and $\mathcal{B}$.

That is, you may always delete axioms from a consistent system while preserving consistency.

3.1.1 Zermelo-Fraenkel Axioms.

The Zermelo-Fraenkel axioms are the most commonly used axiom system in mathematics currently, although most mathematicians will add the Axiom of Choice to the original set that Zermelo and Fraenkel originally proposed.

Definition 3.8 (Zermelo-Fraenkel Axioms)**.** The *Zermelo-Frankel axioms* [21], denoted ZF, are the following eight axioms:

Extensionality $\forall A, \forall B[\forall x(x \in A \Leftrightarrow x \in B) \Rightarrow A = B]$

This axiom states that two sets are equal if they have exactly the same elements.

Foundation $\forall A(A \neq \emptyset \Rightarrow \exists x \in A(x \cap A = \emptyset))$

This axiom states that every nonempty set A contains an element x such that the intersection of A and x as sets is empty. This is not to be confused with the intersection of A and the set containing x, denoted $\{x\}$, which would have x as an element and would be nonempty.

Specification $\forall A, \forall p, \exists B, \forall x(x \in B \Leftrightarrow \exists x(x \in A \wedge \phi(x,p))$

This axiom states that for any property ϕ with parameter p and any set A there exists a subset of A consisting of the elements of A which satisfy ϕ.

Union $\forall A, \exists B, \forall x(x \in B \Leftrightarrow \exists y(y \in A \wedge x \in y))$

This axiom states that for any set A the union of the elements of A as sets can be formed.

Pairing $\forall x, \forall y, \exists A, \forall z(z \in A \Leftrightarrow (x = a \vee x = b))$

This axiom states that for any two objects a set containing exactly those two elements can be formed.

Replacement $\forall x, \forall y, \forall z[\phi(x,y,p) \wedge \phi(x,z,p) \Rightarrow y = z] \Rightarrow \forall X, \forall Y, \forall y[y \in Y \Leftrightarrow (\exists x \in X)\phi(x,y,p)]$

This axiom states that if F is a function and X is a set then there exists an image of X under F.

Power Set $\forall A \exists B \forall x(x \in B \Leftrightarrow x \subset A)$

This axiom allows for the creation of the power set of a set.

Infinity $\exists A[\emptyset \in A \wedge (\forall x \in A)](x \cup \{x\}) \in S$

This axiom states that there exists an infinite set.

3.1.2 Choice Axioms.

A choice axiom dictates the types of choices allowed in an axiomatic system. The most powerful choice axiom is the Axiom of Choice (AC). There are several weaker choice axioms, with the most commonly used of these being the Axiom of Dependent Choice (DC). The Axiom of Choice allows an arbitrary number of choices to be made, while the Axiom of Countable Choice only allows countably many choices to be made. The Axiom of Dependent Choice is strictly stronger than the Axiom of Countable Choice, and strictly weaker than the Axiom of Choice; it allows a countable number of choices to be made, like ACC, but allows each choice to be made individually with its output depending on the choices that were made previously.

Here are two equivalent definitions of multivalued function, and their corresponding equivalent statements of the Axiom of Dependent Choice.

Definition 3.9 (Multivalued Function)**.** Let S be a set. A *multivalued function* on S is a relation $F \subseteq S \times S$ such that for all $s \in S$ there exists $s' \in S$ such that $(s, s') \in F$.

Axiom 3.10 (Dependent Choice)**.** *The* Axiom of Dependent Choice *(DC) states that for any nonempty set S and multivalued function F on S there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ such that $(s_n, s_{n+1}) \in F$. Note that if F is simply a function we do not require DC to construct such a sequence, for we may select $s_0 \in S$ and define by recursion $F(s_n) = s_{n+1}$.*

Definition 3.11 (Multivalued Function)**.** Let S be a set. A *multivalued function* on S is a function $f\colon S \to \mathcal{P}(S)$.

Axiom 3.12 (Dependent Choice)**.** *The* Axiom of Dependent Choice *(DC) states that for any nonempty set S and multivalued function f on S there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ such that $s_{n+1} \in f(s_n)$. Note that if F is simply a function we do not require DC to construct such a sequence, for we may select $s_0 \in S$ and define by recursion $F(s_n) = s_{n+1}$.*

Definition 3.13 (Zermelo-Fraenkel Axioms with the Axiom of Dependent Choice)**.** The *Zermelo-Fraenkel Axioms with the Axiom of Dependent Choice* is ZF + DC.

Axiom 3.14 (The Axiom of Choice)**.** *The* Axiom of Choice *(AC) states that for any set X of nonempty sets there exists a choice function on X.*

Definition 3.15 (Zermelo-Fraenkel Axioms with the Axiom of Choice)**.** The *Zermelo-Fraenkel Axioms with the Axiom of Choice*, denoted ZFC are defined as follows ZFC = ZF + AC

3.2 Formal Definition of Inscrutability

Definition 3.16 (Inscrutable)**.** A class P is *inscrutable* if

$$\text{ZFC + "}P\text{ is empty" is inconsistent}$$
$$\text{and}$$
$$\text{ZF + DC + "}P\text{ is empty" is equiconsistent with ZFC.}$$

Definition 3.17 (Scrutable)**.** A class P is *scrutable* if ZF + DC + "P is empty" is inconsistent.

3.3 Inscrutable Subsets of Polish Spaces

Polish spaces are topological spaces that have a lot of nice properties; they are Hausdorff, they are separable, they have a metric, and they are complete, just to name a few of their nice properties. Even in these nice objects we can create pathological objects inside of them. Here are a few ways to create inscrutable subsets of Polish spaces.

3.3.1 Non-Almost Open Subsets of Polish Spaces Are Inscrutable.

Theorem 3.18. *The class of non-almost open subsets of Polish spaces is inscrutable.*

Proof.

(a) The kernel of the homomorphism in Lemma 1.22(d) is not almost open. This shows that ZFC + "All subsets of Polish spaces are almost open" is inconsistent.

(b) Next we show that ZF + DC + "All subsets of Polish spaces are almost open" is equiconsistent with ZFC.

($\Rightarrow$) The consistency of ZF + DC + "All subsets of Polish spaces are almost open" implies the consistency of ZF which implies the consistency of ZFC by [17, P. 53].

($\Leftarrow$) Assume that ZFC is consistent. Shelah showed that the consistency of ZFC implies the consistency of ZF + DC + "Every subset of $\mathbb{R}$ is almost open" [3, p. 43].

Suppose P is a Polish space in a model of ZF + DC + "Every subset of $\mathbb{R}$ is almost open" and that $A \subset P$ is a non-almost open subset of P. Theorem 2.11 implies that there is a subspace of P containing no isolated points, but also containing a non-almost open set. Call this subspace X and its non-almost open subset B.

Lemma 2.17 implies that there exists an almost homeomorphism $h \colon X \to \mathbb{R}$, while Lemma 2.3(c) implies that almost homeomorphisms between Baire spaces preserve almost openness, so $h(B)$ is a non-almost open subset of $\mathbb{R}$.

This implies that the existence of a non-almost open subset of a Polish space contradicts ZF + DC + "Every subset of $\mathbb{R}$ is almost open." So ZF + DC + "All subsets of Polish spaces are almost open" is consistent.

So the class of non-almost open subsets of Polish spaces is an inscrutable class. $\square$

3.3.2 Inscrutable Subgroups of Polish Groups.

Theorem 3.19. *The class of non-open, countably indexed subgroups of Polish groups is an inscrutable class.*

Proof. (a) First we show that ZFC + "The class of non-open, at most countably indexed subgroups of Polish groups is empty" is inconsistent.

The kernel of the homomorphism in Lemma 1.22(d) is a finitely indexed, non-open subgroup of a Polish group, so ZFC + "The class of non-open, at most countably indexed subgroups is empty" is inconsistent.

(b) Now we show that ZF + DC + "The class of non-open, at most countably indexed subgroups of a Polish group is empty" is equiconsistent with ZFC.

($\Rightarrow$) The consistency of ZF + DC + "The class of non-open, at most countably indexed subgroups of Polish groups is empty" implies the consistency of ZF which implies the consistency of ZFC by [17, P. 53].

($\Leftarrow$) Assume that ZFC is consistent. Theorem 3.18 implies the consistency of ZF + DC + "All subsets of Polish spaces are almost open". Assume that K is a non-open, at most countably indexed subgroup of a Polish group in a model of ZF + DC + "All subsets of Polish spaces are almost open."

Lemma 1.22(c) implies that K is not almost open, which contradicts "All subsets of Polish spaces are almost open." So ZF + DC + "The class of non-open, at most countably indexed subgroups of Polish groups is empty" is consistent.

So the class of non-open at most countably indexed subgroups of Polish spaces is an inscrutable class. □

3.3.3 Bases of $\mathbb{R}$ over $\mathbb{Q}$ as a Vector Space Are Inscrutable.

Lemma 3.20. *The existence of a basis for $\mathbb{R}$ over $\mathbb{Q}$ as a vector space implies the existence of a subset of $\mathbb{R}$ that is not almost open.*

Proof. Suppose that B is a basis for $\mathbb{R}$ over $\mathbb{Q}$. Pick $b \in B$, and let V be the span of $B \setminus \{b\}$. Notice that $qb + V$ is the homeomorphic image of V under $f(x) = qb + x$ for all $q \in \mathbb{Q}$, and $\{qb + V \mid q \in \mathbb{Q}\}$ is a countable partition of $\mathbb{R}$, so V cannot be meager.

We contruct the set $\{a - b \mid a, b \in V\}$. Lemma 1.22(a) states that $\{a - b \mid a, b \in V\}$ contains an open set of 0 if and only if V contains a non-meager almost open set. Since $\mathbb{Q}b \cap V = \emptyset$, there is no open set in V. So V is not almost open, which implies that B is not almost open. □

Theorem 3.21. *The class of bases of $\mathbb{R}$ over $\mathbb{Q}$ as a vector space is an inscrutable class.*

Proof. (a) First we notice that ZFC + "bases of $\mathbb{R}$ over $\mathbb{Q}$ as a vector space is empty" is inconsistent.

(b) Next we show that ZF + DC + "There does not exist a basis for $\mathbb{R}$ over $\mathbb{Q}$ as a vector space" is equiconsistent with ZFC.

($\Rightarrow$) The consistency of ZF + DC + "There does not exist a basis for $\mathbb{R}$ over $\mathbb{Q}$ as a vector space" implies the consistency of ZF which implies the consistency of ZFC by [17, P. 53].

($\Leftarrow$) Assume that ZFC is consistent. Theorem 3.18 implies the consistency of ZF + DC + "Every subset of $\mathbb{R}$ is almost open."

Suppose that B is a basis for $\mathbb{R}$ over $\mathbb{Q}$ as a vector space in ZF + DC + "Every subset of $\mathbb{R}$ is almost open."

Lemma 3.20 implies that a basis for $\mathbb{R}$ over $\mathbb{Q}$ as a vector space is not almost open.

This implies that the existence of a basis for $\mathbb{R}$ over $\mathbb{Q}$ as a vector space contradicts ZF + DC + "Every subset of $\mathbb{R}$ is almost open" So ZF + DC + "There does not exist a basis for $\mathbb{R}$ over $\mathbb{Q}$ as a vector space" is consistent.

So the class of bases of $\mathbb{R}$ over $\mathbb{Q}$ as a vector space is an inscrutable class. □

3.4 Intangible Homomorphisms

As stated in the abstract, there exist homomorphisms in ZFC that are not induced by the π_1 functor. We formally define this property here and give it a simple term with which to reference it throughout the paper (i.e. "intangible").

3.4.1 1-Skeletons, Basepoint Invariance of Kernels, and Peano Continua. While path connected topological spaces have the property that changing the basepoint of the space gives isomorphic fundamental groups, homomorphisms from one such fundamental group have no canonical correspondence to homomorphisms from another fundamental group based at a different point. We show that even though there is no cannonical correspondence, that the kernels of these homomorphisms is an invariant.

Lemma 3.22. *A loop in a simplicial complex X is homotopic to a loop contained in the 1-skeleton of X.*

Proof. Let l be a loop in X. Let δ be a simplex of maximal dimension in X. Suppose Δ has dimension greater than 1. Let s be a connected component of $(l \cap \Delta)$. If s contains the center of Δ (using barycentric coordinates), then s can be homotoped so as not to contain the center of δ as seen in the proof of Proposition 1.14 in [7, P. 35].

The projection of s to the boundary of Δ is homotopic to s, because a punctured simplex is homotopic to its boundary, so s is homotopic to its image under such a projection. By repeating this process for every connected component of the intersection of l with an $n-$simplex, we find that l is homotopic to a curve that is contained in the $(n-1)-$skeleton of X. By induction, l is homotopic to a loop that is contained in the 1-skeleton of X. $\square$

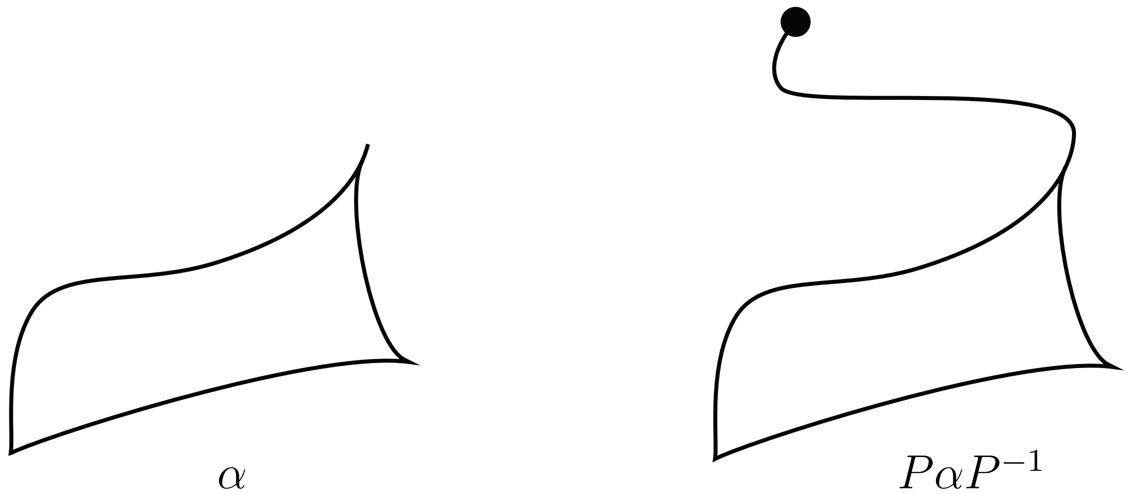
Definition 3.23 (Peano continuum)**.** A *Peano continuum* is a nonempty, compact, connected, locally connected metric space.

Definition 3.24 (Loop)**.** A *loop* of X is the a function $f\colon [0,1] \to X$ such that $f(0) = f(1)$. The *basepoint* of a loop is $f(0)$.

Definition 3.25 (Free homotopy)**.** A *free homotopy class* of X is the a function $f\colon [0,1] \times [0,1] \to X$ such that $f(\cdot, 0)$ is the loop a $f(\cdot, 1)$ is the loop b and $f(0, s) = f(1, s)$ for all s. The *basepoint* of a loop is $f(0)$. The *free homotopy class* of a loop is the set of loops which are freely homotopic to it.

Definition 3.26 (Trivial free homotopy class relative to ϕ)**.** Let C be a free homotopy class of loops of X. C is *trivial relative to* $\phi\colon \pi_1(X, x) \to G$ if there exists a loop $\alpha \in C$ based at x such that $[\alpha] \in \ker(\phi)$. We denote this property by $C \in \ker\phi$.

Figure 3.1: Conjugation by a path P



Lemma 3.27. *If $\phi' \colon \pi_1(X, x_1) \to G$ is induced by $\phi \colon \pi_1(X, x_0) \to G$ then $H \in \ker \phi$ implies $H \in \ker \phi'$. By induced we mean $\phi' = \phi \circ i_p$ where i_p is the isomorphism given by conjugation by a path P from x_0 to x_1, $[\alpha]_{\pi_1(X,x_0)} \mapsto [P\alpha P^{-1}]_{\pi_1(X,x_1)}$.*

This proves that the kernel of a homomorphism from the fundamental group of a space is invariant under choice of basepoint.

Proof. Let $H \in \ker \phi$. There exists $\alpha \in H$ based at x_0 such that $[\alpha] \in \ker \phi$. Let P be a path such that $\phi' = \phi \circ i_P$. Then $P^{-1}\alpha P \in H$ and $\phi'([P^{-1}\alpha P]) = \phi(i_P([P^{-1}\alpha P])) = \phi([PP^{-1}\alpha PP^{-1}]) = \phi([\alpha])$, which is trivial. So $H \in \ker \phi'$. □

3.4.2 Local Triviality.

Intuitively, for a homomorphism of groups $\phi : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ to be intangible, the homomorphism must require some sequence of small loops of X to be mapped to loops that do not get small in Y. Local triviality is the opposite of this condition. That is, local triviality of a space relative to a homomorphism implies that there is a continuous function inducing the homomorphism because the homomorphism allows all small loops to be mapped to small loops. We study the property formally in the lemmas in this section.

Figure 3.2: Commutative Diagram for Lemma 3.31

$$
\begin{tikzcd}
 & N(C) \arrow[dr, "\alpha"] \arrow[dd, "\pi_1"] & \\
X \arrow[ur, "\psi"] \arrow[rr, "f"] \arrow[dd, "\pi_1"'] & & Y \arrow[dd, "\pi_1"] \\
 & \pi_1(N(C)) \arrow[dr, "\sigma = \alpha_*"] & \\
\pi_1(X) \arrow[ur, "\psi_*"] \arrow[rr, "\phi"] & & \pi_1(Y)
\end{tikzcd}
$$

The notions of *2-set simple* (*rel* ϕ) and *locally trivial* (*rel* ϕ) are defined in [2, pp. 2670-2671].

Definition 3.28 (2-set simple (rel ϕ))**.** Let X be a topological space, $\phi\colon \pi_1(x) \to K$ be a homomorphism of groups, and C an open cover of X by path connected sets. The cover C is *2-set simple* (*rel* ϕ) if for every $[\gamma] \in \pi_1(X)$ and γ is freely homotopic to a curve contained in the union of two elements of C then $[\gamma] \in \ker(\phi)$.

Definition 3.29 (Locally trivial (rel ϕ))**.** Let X be a topological space and $\phi\colon \pi_1(X) \to K$ be a homomorphism of groups. The space X is *locally trivial* (*rel* ϕ) if every point $x \in X$ is contained in an open neighborhood U of x such that any homotopy class based at x which has a representative contained in U lies in $\ker(\phi)$. This property is independent of the basepoint chosen in X.

Definition 3.30 (Tangible homomorphism)**.** Let X be a Peano continuum, Y be an aspherical simplicial complex and $\phi\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an abstract homomorphism. Then ϕ is *tangible* if there exists a continuous function $f\colon X \to Y$ such that $f_* = \phi$. The homomorphism ϕ is *intangible* (as a homomorphism of groups) if no such f exists.

Theorem 3.31. *Let X be a nonempty, connected, locally path connected metric space and Y an aspherical simplicial complex. Then X is locally trivial* (*rel* ϕ) *if and only if ϕ is tangible.*

Proof. ($\Rightarrow$) Let $\phi\colon \pi_1(X) \to \pi_1(Y)$ have the property that X is locally trivial (rel ϕ). Cover X with open sets that are trivial (rel ϕ). There is a locally finite subcover of this cover. Each $x \in X$ is covered by finitely many elements of this subcover, so there exists δ_x such that the ball $B(x, \delta_x)$ is contained in the intersection of the elements of the subcover which contain x. The collection $\{B(x, \frac{\delta_x}{3}) \mid x \in X\}$ is a cover of X, so the collection of the connected components of elements of this cover is also a cover. Since X is locally path connected, elements of this cover are path connected. There exists a locally finite subcover of this cover, call it C. The union of two intersecting elements of C is contained in a set that is trivial (rel ϕ) so C is 2-set simple (rel ϕ). Let $N(C)$ denote the nerve of C.

Theorem 7.3 in [2, p. 2670] implies that ϕ factors through a map as follows:

$$\psi_*\colon \pi_1(X, x_o) \to \pi_1(N(C)) \to \pi_1(Y, y_0).$$

The proof of this theorem constructs a continuous function from X to $N(C)$ that induces ψ_*, call it $\psi\colon X \to N(C)$, so there exists $\sigma\colon \pi_1(N(C)) \to \pi_1(Y)$ such that $\phi = \sigma \circ \psi_*$. This reduces the problem of finding a continuous function $f\colon X \to Y$ that induces ϕ to finding a continuous function $h\colon N(C) \to Y$ that induces σ.

By Lemma 3.22, we know that the fundamental group of the 1-skeleton of $N(C)$ contains the fundamental group of $N(C)$. If we take an injective loop l in the 1-skeleton of $N(C)$, then we have an ordered list of vertices traversed by l, which can be mapped to the ordered list of points received by taking an injective loop in the homotopy class $\sigma([l])$. Doing this for all injective loops of the 1-skeleton of $N(C)$ gives a vertex map from $N(C)$ to Y, which extends to a continuous map from $N(C)$ to Y that induces σ.

($\Leftarrow$) Let X be a topological space, Y a locally simply connected space, and $f\colon X \to Y$ be continuous. Let $p \in X$ and U be a simply connected open set containing $f(p)$. Then any loop based at p and contained in $f^{-1}(U)$ lies in $\ker f_*$. So X is locally trivial (rel f_*). Since this works for X a general topological space and Y a general locally connected space, the theorem holds in this direction for the less general case of the hypotheses. $\square$

Definition 3.32 (The Hawaiian earring)**.** The *Hawaiian earring* is the one point compactification of a disjoint union of countable many arcs. Notice that each arc is compactified to be a copy of S^1. We will use the following conventions: *HE* will denote the Hawaiian earring, *HEG* denotes $\pi_1(HE)$, l_i denotes the i^{th} copy of S^1, f_i denotes a parameterization of l_i, c_i denotes the element of *HEG* representing the homotopy class of l_i.

Theorem 3.33. *Let X be a nonempty, locally path connected, first countable space and $\phi\colon \pi_1(X) \to G$. X is locally trivial (rel ϕ) if and only if for every continuous function $g\colon \mathrm{HE} \to X$, the element $(\phi \circ g_*)(c_i)$ is trivial for some i.*

Proof. ($\Rightarrow$) Suppose there exists a continuous function $g\colon HE \to X$ such that $(\phi \circ g_*)(c_i)$ is non-trivial for all i. By continuity of g, the sequence of loops $g(l_i)$ converges to a point, so every open set containing the point of convergence contains a loop mapped non-trivially by ϕ.

($\Leftarrow$) Let X be first countable and not locally trivial (rel ϕ). There exists $p \in X$ such that every open set containing p contains a loop α such that $\phi([\alpha])$ is not trivial. Conjugate the loop with a path connecting the basepoint of the loop with the basepoint of X. X is first countable, so there exists a sequence of nested open sets $\{U_i\}$ whose intersection is $\{p\}$. For each U_i choose a loop $\alpha_i \subset U_i$ based at p such that $\phi([\alpha_i])$ is not trivial. We now have a sequence of loops based at p, such that no loop of the sequence is mapped trivially.

We construct $g\colon HE \to X$ by mapping l_i of *HE* to α_i, with the basepoint of *HE* being mapped to $p \in X$. This gives a continuous function from *HE* to X such that $\phi(g_*(c_i)) = \phi([\alpha_i]) \neq 1$ for all i. $\square$

Corollary 3.34. *Let X be a Peano continuum and Y an aspherical simplicial complex. A homomorphism $\phi\colon \pi_1(X) \to \pi_1(Y)$ is tangible if and only if for every continuous function $g\colon HE \to X$, $(\phi \circ g_*)(c_i)$ is trivial for some i.*

Proof. This follows immediately from Theorems 3.31 and 3.33. $\square$

3.4.3 The Shelah Function.

We borrow a function that was developed by Shelah that will let us build a non-almost open set in $\mathcal{K}$ from an intangible homomorphism.

Definition 3.35 (Shelah function)**.** We define *the Shelah function* from the Cantor group to the Hawaiian earring group, $S_h\colon \mathbb{Z}_2^{\mathbb{N}} \to \mathit{HEG}$, by

$$a = (a_1, a_2, \cdots) \mapsto S_h(a) = [\alpha].$$

Where α is the loop given by

$$\alpha(x) = \begin{cases} f_i\big(2^i(x - 1 + \frac{1}{2^i})\big), & x \in \big[1 - \frac{1}{2}^i, 1 - \frac{1}{2}^{i+1}\big] \text{ and } a_i = 1 \\ x_0, & x \in \big[1 - \frac{1}{2}^i, 1 - \frac{1}{2}^{i+1}\big] \text{ and } a_i = 0 \end{cases}$$

where f_i is a parameterization of the i^{th} copy of S^1 in the Hawaiian earring and x_0 is the basepoint of the Hawaiian earring. This loop α is the loop that "goes around" c_i if $a_i = 1$ and doesn't "go around" c_i if $a_i = 0$.

Definition 3.36 (Weakly locally path connected)**.** A topological space X is *weakly locally path connected* if for every point $x \in X$ and every open set U such that $x \in U$ there exists an open set V such that $x \in V \subset U$ and for all $v \in V$ there exists a path from x to v in U.

Lemma 3.37. *[4] If X is a weakly locally path connected compact metric space and $\pi_1(X)$ is not finitely generated, then there exists a point $x \in X$ such that every neighborhood U of x contains a loop at x (i.e. an $f \in \pi_1(X, x)$) which is essential in X (i.e. $[f]$ is not the unit in the homotopy group).*

Lemma 3.38. *Suppose that $\phi\colon \mathrm{HEG} \to G$ is a homomorphism that maps all of the base loop equivalence classes non-trivially. Suppose also that $a, b \in \mathbb{Z}_2^{\mathbb{N}}$, where $a = (a_1, a_2, \cdots)$ and $b = (b_1, b_2, \cdots)$, such that $\{i \mid a_i \neq b_i\}$ is a singleton set. Then $(\phi \circ S_h)(a) \neq (\phi \circ S_h)(b)$.*

Proof. Suppose that $a, b \in \mathbb{Z}^{\mathbb{N}}$ such that $\{i \mid a_i \neq b_i\} = \{j\}$. Suppose also that $a = uv$ and $b = ue_jv$, where $u_i = 0$ for all $i \geq j$ and $v_i = 0$ for all $i \leq j$. Then

$$\phi(S_h(a))[\phi(S_h(b))]^{-1} = \phi(S_h(u)S_h(v))[\phi(S_h(u)S_h(e_i)S_h(v))]^{-1}$$

$$= \phi(S_h(u))\phi(S_h(v))[\phi(S_h(v))]^{-1}[\phi(S_h(e_i))]^{-1}[\phi(S_h(u))]^{-1}$$

$$= \phi(S_h(u))[\phi(S_h(e_i))]^{-1}[\phi(S_h(u))]^{-1}$$

By assumption, each e_i is mapped non-trivially, so $(\phi \circ S_h)(a) \neq (\phi \circ S_h)(b)$. □

Lemma 3.39. *Suppose that ϕ: HEG $\to G$ is a homomorphism that maps all of the base loop equivalence classes nontrivially. No e_j is contained in $\{ab^{-1} \mid a, b \in (\phi \circ S_h)^{-1}(g)\}$ for any g.*

Proof. Suppose that $e_j \in \{ab^{-1} \mid a, b \in (\phi \circ S_h)^{-1}(g)\}$ for some $g \in G$. Then there exist $a, b \in (\phi \circ S_h)^{-1}(g)$ such that $ab^{-1} = ab = e_j$. This implies that $\{i \mid a_i \neq b_i\} = \{j\}$. Lemma 3.38 implies that a and b cannot be in the same point preimage under $(\phi \circ S_h)$, which is a contradiction. So e_j cannot be in $\{ab^{-1} \mid a, b \in (\phi \circ S_h)^{-1}(g)\}$ for any $g \in G$. □

Lemma 3.40. *Suppose that we are working in a model of ZF + DC + "Every subset of a Polish space is almost open." Then every homormophism from HEG to a countable group G is tangible.*

Proof. Since there are countably many elements of G, there are at most countably many point preimages under $(\phi \circ S_h)$, so at least one of these point preimages is non-meager, call it $(\phi \circ S_h)^{-1}(g)$. Lemma 3.39 implies that no e_i is in $\{ab^{-1} \mid a, b \in (\phi \circ S_h)^{-1}(g)\}$ so $\{ab^{-1} \mid a, b \in (\phi \circ S_h)^{-1}(g)\}$ does not contain an open neighborhood of the identity. Since $(\phi \circ S_h)^{-1}(g)$ is non-meager in $\mathbb{Z}_2^{\mathbb{N}}$ and $\{ab^{-1} \mid a, b \in (\phi \circ S_h)^{-1}(g)\}$ does not contain an open neighborhood of the identity, Lemma (a) implies that $(\phi \circ S_h)^{-1}(g)$ is not almost open. □

3.4.4 Intangible Homomorphisms Are Inscrutable.

Theorem 3.41. *The class of intangible homomorphisms is an inscrutable class.*

Proof.

(a) First we show that ZFC + "The class of intangible homomorphisms is empty" is inconsistent.

Conner and Spencer [1, p. 225] use non-principle ultrafilters to construct a homomorphism from *HEG* to the fundamental group of a topological space that cannot be induced by a continuous function. In ZFC there exist non-principle ultrafilters. So ZFC + "The class of intangible homomorphisms is empty" is inconsistent.

(b) Next we show that ZF + DC + "every homomorphism is tangible" is equiconsistent with ZFC.

($\Rightarrow$) The consistency of ZF + DC + "every homomorphism is tangible" implies the consistency of ZF which implies the consistency of ZFC by [17, P. 53].

($\Leftarrow$) Assume that ZFC is consistent. Theorem 3.18 implies the consistency of ZF + DC + "All subsets of Polish spaces are almost open."

Let X be a Peano continuum in ZF + DC + "All subsets of Polish spaces are almost open" and suppose $\phi\colon \pi_1(X) \to G$ is an intangible homomorphism. Corollary 3.34 implies that there exists a continuous $g\colon HE \to X$ such that $(\phi \circ g_*)(l_i)$ is non-trivial for all i. Lemma 3.40 implies that there exists a subset of $\mathbb{Z}_2^{\mathbb{N}}$ with the usual Cantor Space topology that is not almost open.

This implies that the existence of a discontinuous homomorphism contradicts ZF + DC + "All subsets of Polish spaces are almost open." So ZF + DC + "no homomorphism is discontinuous" is consistent.

So the class of discontinuous homomorphisms is an inscrutable class. $\square$

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