Methods for Creating Rigid Foldability in Origami-Inspired Deployable Mechanisms

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Methods for Creating Rigid Foldability in Origami-Inspired Deployable Mechanisms

Alden Daniel Yellowhorse

A dissertation submitted to the faculty of Brigham Young University in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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ABSTRACT

Methods for Creating Rigid Foldability in Origami-Inspired Deployable Mechanisms

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Because origami has proved to be a tremendously rich source of inspiration in engineering, interest in solving some of the challenges that affect origami-inspired design has been significant. One such challenge involves ensuring that origami-inspired mechanisms are rigid-foldable or capable of moving without requiring links to bend or distort. Because rigid-foldability is essential in mechanisms that are constructed using rigid materials, access to methods of engineering this characteristic are highly desirable.

This research addresses this need by developing methods for the design of origami-inspired mechanisms that are rigid-foldable. Methods for modifying crease patterns to achieve this are described and compared. Methods for achieving rigid-foldability using thick materials are also developed. Proofs of a process for generating new variations of existing thick-origami models are developed and demonstrated on multiple models. The possibility of using compliant panels to create rigid-foldability is also studied.

Because of the relationship between mechanism stiffness and rigid-foldability, means of managing the pattern stiffness are also examined. The design of compliant, deployable stiffeners is studied to permit a comparison of different stiffener types. This comparison is used to identify dominant configurations that are most advantageous for a deployable mechanism. The use of thick-origami models are also considered. The geometry of two varieties of a cantilever tube are optimized to support a cantilever beam.

Keywords: origami, deployable stiffeners, origami design, origami-inspired, spacecraft deployable
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NOMENCLATURE

Variables used in Chapter 2
\(\alpha\) The angle between two adjacent creases in an origami vertex
\(E\) Number of edges in a network of spherical joints
\(J\) Number of joints or creases in a panel
\(M\) Mobility or number of degrees of freedom in a mechanism
\(\Delta M\) Change in mobility for a mechanism after alteration
\(m\) The mountain or valley assignment coded as 1 or -1, respectively
\(N\) Number of links in a pattern
\(n\) Number of vertices bordering a panel
\((R_x)_j\) The sum of all the \(x\)-components of the crease angular velocities for the \(j\)-th crease
\((R_y)_j\) The sum of all the \(y\)-components of the crease angular velocities for the \(j\)-th crease
\((R_d)_j\) The residual directed area for the \(j\)-th vertex
\(SS\) A correction term accounting for mechanism singularities
\(s\) The number of consecutive split creases
\(V\) Number of spherical joints in a panel
\(\vec{\omega}_i\) The relative angular velocity vector between to panels at the \(i\)-th crease
\(\theta_i\) The angular orientation of the \(i\)-th crease when the vertex is fully flat
\(x_i\) Order or number of creases in the \(i\)-th vertex

Variables used in Chapter 3
\(a_i\) The panel thickness
\(\alpha_i\) The axial twist in a Denavit-Hartenburg transformation
\(b_i\) The \(i\)-th panel thickness
\(c_j\) The \(j\)-th panel thickness
\(C(...)\cos(...)\)
\(D\) The sum of all twist angles \(\alpha_i\) in a spatial linkage
\(d_i\) The axial shift in a Denavit-Hartenburg transformation
\(d^i\) A vector written with respect to frame \(i\)
\(h\) A point of solid material in a panel of a spatial linkage
\(\phi_i\) The angle between the \(i\)-th pair of adjacent lines in the zig-zag profile that defines a chain of linkage vertices
\(i\) One of two indices defining the location of a Bennett linkage in a tessellation
\(j\) One of two indices defining the location of a Bennett linkage in a tessellation
\(k_{ji}\) The ratio of \(a_2\) and \(a_1\).
\(L_i\) Length of the \(i\)-th line in the zig-zag profile of a single chain of Linkage III vertices
\(m, n, s, t\) Arbitrary non-negative integers defining a linkage location in a tessellation
\(\mu_{ji}\) A characteristic number describing a Bennett linkage at coordinates \(ji\) that depends on its particular \(\alpha_1\) and \(\alpha_2\)
\(P\) The set of all points defining a panel in a spatial linkage
\(p\) An arbitrary constant
\(q\) An arbitrary constant
\(R\) A value describing the curved boundary of a panel
\(S(...)\sin(...)\)
A Denavit-Hartenburg transformation accepting the parameters $\theta_i, a_i, d_i$ and $\alpha_i$.

$T[i]$ A transpose of the $i$-th hinge and its adjacent links.

$\theta_i$ The twist angle in a Denavit-Hartenburg transformation.

$u$ The hinge number.

$V[u; u_j]$ A split vertex with links inserted at hinges $i$ and $j$.

$\nu$ An arbitrary constant.

$Z[u]$ A reduction of the thickness parameter $a_i$ to zero for the link between the $u$-th and $(u+1)$-th hinges.

$\hat{z}_{nm}$ The vector $\hat{z}_n$ expressed in the $n$-th frame.

$\hat{z}_{nm}$ The vector $\hat{z}_n$ expressed in the $m$-th frame.

$0$ A three-dimensional zero vector.

**Variables used in Chapter 4**

$c_i$ Constant used to create a polynomial approximation of a torsional spring.

$D$ Flexural rigidity.

$\Delta$ The angular twist of a torsional spring from its unstressed state.

$\delta$ Distance between two points at a unit distance from the origin.

$E$ Modulus of elasticity.

$\varepsilon$ Strain.

$\eta$ The second derivative of $\phi$.

$F$ The measured force perpendicular to the arm.

$\phi$ Angular altitude in spherical coordinates.

$\gamma$ Angle of the surface normal at a boundary with respect to the horizontal.

$h$ Panel thickness.

$\kappa$ Curvature.

$l_0$ Center-line length of a slab in bending.

$M$ Internal moment.

$N$ A collection of all variables that only depend on geometry.

$R$ Radius of curvature of a beam in bending.

$r$ Radial position in spherical coordinates.

$\rho$ Radius of curvature.

$\rho_0$ Radius of curvature at a unit distance from the origin.

$S$ Points describing a conical surface.

$\sigma$ Stress.

$T$ Torque from an equivalent torsional spring.

$\theta$ Angular azimuth in spherical coordinates.

$V$ Stored elastic energy in a thin panel.

$\nu$ Poisson’s ratio.

$w$ Beam width.

$\vec{w}$ A load applied to the panel edge.

**Variables used in Chapter 5**

$\alpha$ Panel sector angles.

$A$ Panel area.

$a$ Distance of force in four-point test from outer edge.

$B$ The crease whose angle is used to measure partial actuation in the test prototype.
Width of a panel cross-section
Foam thickness
Vertical coordinate of the section centroid
The area cost
The ratio between the deployed area and the area of one face of a cube of equivalent volume
Ratio between the density of a foam and its constituent material
Width of the partially-actuated prototype
The strain at the initiation of densification
Young’s Modulus
Young’s Modulus of the foam core surface
Young’s Modulus of the foam core center material
Total force applied to test bending stiffness
Parametric parameter \( h/w \)
The fold angle at a crease
Maximum fraction of material removed from crease
Minimum fraction of material removed from crease
Height of the yielded stiffener
Area moment of inertia
Effective spring constant
The ratio of \( E_f/E_s \)
Largest prototype dimension
Fold angle multiplier
Ratio between face and core material densities \( \rho_f/\rho_s \) for the expanded stiffener
The ratio between the current mass and the initial mass
Safety factor
Density of the core
Density of the face material
Density of the material in the core
Parameter describing the cost of a stiffener size
Ratio of \( b/t \)
First moment of the area
Ratio \( C/t \)
Ratio of \( I \) for the stiffened un-stiffened version
Unit-less ratio of \( w/t \)
Constant collapse stress for an elastic foam
Maximum acceptable shear stress
crease length
The tilt angle of the panel cross-section in a partially-actuated vertex
Thickness of the wall of a foam core panel
Thickness of a panel cross-section
Shear force at a cross-section
Total width of a single yielded stiffener unit
Unitless parameter describing size of a generic stiffener size relative to a specific panel dimension
Variables used in Chapter 6

- $a$: Parameter governing the proportions of an Euler Spiral
- $b$: Half the stiffener width
- $C$: Constant of integration
- $C(...)$: Fresnel cosine integral
- $c_{a,b}$: Location of the cross-section centroid relative to the panel
- $d$: Height of the stiffener
- $E$: Modulus of elasticity
- $F$: One of two forces applied in a four-point bend test
- $\bar{F}$: Force per unit length applied to the edge of a stiffener
- $h$: Test panel length
- $I$: Area moment of inertia
- $k$: Predicted spring constant based on test geometry
- $\kappa$: Curvature
- $L$: Length of an Euler Spiral
- $l$: Stiffener length perpendicular to its profile
- $\lambda$: Distance of a load in the four-point bend test from its nearest support
- $M$: Bending moment
- $\bar{M}$: Bending moment per unit length
- $\mu$: Constant of proportionality
- $N$: Factor of safety
- $n$: Number of layers of panels in a folded device
- $P$: Ratio of two internal bending moments
- $p$: Non-dimensional variable
- $q$: Non-dimensional variable
- $r$: Non-dimensional variable
- $S(...)$: Fresnel sine integral
- $s$: Arc length position
- $\sigma_y$: Yield stress
- $T$: Panel thickness
- $t$: Stiffener thickness
- $\theta$: Slope of a curve with respect to the $x$-axis in radians
- $u$: Non-dimensional variable
- $v$: Non-dimensional variable
- $w$: Panel width per stiffener (distance between stiffeners)

Variables used in Chapter 7

- $\alpha$: Fold angle between two adjacent panels in a segment
- $C$: Constant used to simplify formulas
- $d$: Total length of tube during deployment
- $F$: Force applied to tube
- $H$: Side panel height
- $L$: Side length of tube panels
- $\lambda$: Total tube length
- $m_1$: Optimized panel mass
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>Angle between two adjacent sides in parallelogram panels</td>
</tr>
<tr>
<td>$n$</td>
<td>Number of tube segments</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Material density</td>
</tr>
<tr>
<td>$T$</td>
<td>Panel thickness</td>
</tr>
<tr>
<td>$t$</td>
<td>Thickness of hollow panel walls</td>
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<tr>
<td>$V$</td>
<td>Tube material volume</td>
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<tr>
<td>$W$</td>
<td>Lower panel width</td>
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<tr>
<td>$w$</td>
<td>Ratio of lower panel width to tab width</td>
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<tr>
<td>$\omega$</td>
<td>Angle at which force is applied</td>
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CHAPTER 1. INTRODUCTION

1.1 Problem Statement

Rigid foldability is a property of origami crease patterns that has received significant attention because of its importance in extending origami concepts to materials other than paper. Rigid foldability is defined as the ability of an origami pattern to fold without requiring the panels of the pattern to bend or stretch [2]. This ability to move without requiring panel flexibility can provide significant advantages. Some of these include a wider range of materials for panel construction and lower actuation force [3]. Because of the significance of this problem, the purpose of this research is to develop new methods for designing rigid-foldable origami mechanisms.

Origami models can be divided into multiple categories as shown in Figure 1.1. They can be either rigid-foldable (RF) or not rigid-foldable (not RF). Rigid-foldable models can either be thick models or of zero-thickness (thin). One way of classifying models that are not rigid-foldable is by the amount of deformation they experience during folding. This deformation may either be large or small.

The process of designing rigid-foldable mechanisms is complicated by multiple challenges. Many of these challenges derive from the difficulty associated with switching categories as shown in Figure 1.1. Switching categories is often desirable because certain models may have desirable characteristics, but belong to the wrong category for an application. One example is converting non-rigid-foldable models to rigid-foldable models. Doing so requires developing design methods that are compatible with the strict kinematic requirements of rigid-foldability. The exact forms of these relationships have been defined [4] for many mechanisms in origami-inspired devices. However, procedures for utilizing these relationships to design mechanisms with specific characteristics are often unclear. To address this need, this research will develop methods to not only modify existing patterns and create rigid-foldability but also generate new rigid-foldable mechanisms.
Figure 1.1: Different categories that can define a specific origami model.

A second goal of this research includes developing design methods that consider both mechanism kinematics and deformation due to finite stiffness. This will support transitions from large-deformation to small-deformation origami models. Specific problems that will be studied include modeling the effect of flexible panels in a pattern as well as managing global bending stiffness. One reason panel flexibility will be studied is that it may be used to allow patterns that are not strictly rigid-foldable (pseudo-rigid-foldable) to fold because of the additional degrees of freedom that it provides. Developing methods for controlling pattern bending stiffness will complement this work by preventing a pattern from becoming so flexible that it cannot fulfill its intended function.

1.2 Background

Origami has been shown to be inspirational for engineering applications in a variety of different fields. Some practical examples of spacecraft mechanisms derived from origami patterns include deployable antennas [5], solar sails [6] and solar panel arrays [7]. One primary advantage of these designs is that large deployment ratios or large deployed-to-stowed size ratios can be obtained. Higher deployment ratios are significant for spacecraft because they increase the mass efficiency of the deployable [8]. The folding process is also beneficial because of its ability to make large, unwieldy objects more manageable. An origami-inspired shelter [9], surgical equipment [10], robotics [11], automotive airbags [12] and air filter [13] represent terrestrial applications that benefit from this.

Two major categories of origami mechanisms have been examined by research conducted on these applications of origami. The first category consists of research focused on design prob-
lems for zero-thickness or thin origami in Figure 1.1. It includes work on rigid-foldability as well as problems involving mechanism compliance. The second group addresses problems in the design of thick origami as shown in Figure 1.1. This body of work includes research on ensuring the rigid-foldability of these mechanisms as well as other special considerations such as dealing with self-intersection.

1.2.1 Rigid-Foldability

An important assumption made in the study of rigid-foldable origami is that it behaves like a mechanism composed of rigid links and revolute joints [14, 15]. This assumption has permitted analyses of origami patterns for shopping bags [3] and foldable cylinders [16]. In certain cases, these analyses have shown that the specific patterns are rigid-foldable and provided means of simulating the motion of the pattern. However, these analyses are complex [17] because of the complicated geometry needed to describe the motion of origami-inspired mechanisms. In addition, a high level of mathematics is often required to successfully analyze them.

In many of these design cases, an important consideration is the number of degrees of freedom that exist in the pattern. Because origami mechanisms are often composed of many interconnected spherical mechanisms [18], the number of constraints frequently exceeds the number of degrees of freedoms. Consequently, it is easy for designed patterns to become over-constrained [19] and incapable of rigid-foldable motion. Although formulas such as Grübler’s Equation are capable of estimating the number of degrees of freedom or mobility of such patterns, their accuracy is not guaranteed and mechanisms with more mobility are possible due to kinematic singularities [20].

This uncertainty increases the difficulty of successfully designing rigid-foldable origami mechanisms.

1.2.2 Zero-Thickness Origami

Because of the importance of pattern mobility, many methods for designing zero-thickness, rigid-foldable mechanisms focus on correctly determining and controlling the degrees of freedom. Kazuko and Díaz [21] have discussed adding a large number of creases between known vertices and then steadily eliminating them while preserving foldability. Wu and You [3] demonstrated a
more experimental approach to adding creases to a pattern. Tachi and Miura [16] have studied the use of tessellations as a method for obtaining rigid foldable motion. Tachi [22] also suggested the possibility of triangulating panels with more than three corners. Research on the calculation of mechanism mobility is also important in understanding this modification process. Their work also discussed the use of numeric simulation to model the folding behavior of a pattern.

Research has also focused on correcting the inaccuracies that can arise in the Chebychev-Grübler-Kutzbach Criterion used to study mobility in multi-link mechanisms [20]. These inaccuracies result from special geometric arrangements in a crease pattern [23] and have been shown to be predictable with differing amounts of success [23, 24]. Screw loop equations have also been used to determine the mobility of mechanisms and sub-chains in linkages [25–29]. Nevertheless, predicting the exact number of degrees of freedom in a mechanism remains difficult and can often only be assured through a detailed analysis of the mechanism in question.

1.2.3 Thick Origami

A significant assumption made by these methods is that the thickness of the material in the origami mechanism is thin enough that it can be neglected. While this is true for thin materials where few layers overlap in the folded state, finite thickness can significantly complicate the design of rigid-foldable origami. Attempts to overcome this difficulty have resulted in several major methods that achieve rigid-foldability while still permitting finite-thickness.

One significant set of methods for accommodating thickness relies on using special panel geometry with the zero-thickness models. One example is the tapered-panel method described by Tachi [30], which involves trimming panels to prevent self-intersection. While the resulting mechanism can no longer fold flat, it still allows thick materials to be used. When it is necessary for the mechanism to fold flat, additional research [31] has shown that panels can simply be offset in their folded state. The mechanism is still equivalent to the zero-thickness model because extensions link the offset panels to the appropriate hinge locations.

Another group of methods for designing thick origami attempt to modify the hinges. A modification discussed by Zirbel et al. [7] involved inserting flexible spacing material between rigid panels. This material is intended to flex during the folding process and accommodate thick panels even though the crease pattern made an assumption of zero-thickness. Research has also been
performed on the possibility of splitting creases in a pattern [32] and avoiding a zero-thickness, flat-folded state. The split creases in this method are advantageous because they allow the affected creases to only fold by 90 degrees instead of 180.

Thickness has also been successfully included in origami-inspired mechanisms by using offset hinges [30] in linkage-based mechanisms. A correspondence between different zero-thickness origami mechanisms and specific spatial linkages was discussed by Chen, Peng and You [33] and several example mechanism were shown. The over-constrained spatial linkages used in these devices was shown to effectively include panels of substantial thickness as well as retain a positive number of degrees of freedom. Work has also been done on designing networks of linkage-based mechanisms that are deployable [34–36]. Although these mechanisms were realized as deployable trusses, the concepts used have application as origami-inspired mechanisms.

1.2.4 Compliance of Origami Structures

In addition to studying problems on rigid-foldability, research has also been done on the influence of material deformation on origami design. Energy absorbing structures [37,38] represent one application where deformation characteristics have been important. The bending stiffness of deployed origami tessellations has also been studied [39]. Origami deformation behavior was also studied in the context of designing deployable shelters [40]. In this work, the authors optimized the thermal characteristics of the shelter while ensuring that structural constraints on its design were met. Research has also been done on understanding the deformation behavior of deployable origami tubes [41]. Multiple deformation and collapse modes were analyzed by the authors to enhance understanding of its mechanical characteristics.

1.3 Research Objectives

Research conducted for this dissertation focused on developing methods for the design of both rigid-foldable and pseudo-rigid-foldable mechanisms. These methods will support design by directing transfers of origami models between the categories shown in Figure 1.1. The work on this topic is divided into three key challenges that form basic categories for the tasks pursued as part of the dissertation research. The first challenge is to develop methods for altering the crease pattern
to create rigid-foldability (not RF to RF). The second involves modeling pseudo-rigid-foldable mechanisms with compliant sections (small deformation). Finally, methods were developed to manage the bending stiffness of rigid- and pseudo-rigid-foldable mechanisms (large deformation to small deformation). In all three cases, the methods that are developed enable engineers to address common problems in the design of rigid-foldable, origami-inspired mechanisms.

Research on a method for altering a crease pattern to create rigid-foldability was directed towards multiple smaller goals. One of these goals was to quantify the effect of different modifications to the crease pattern. Another was to compare their usefulness and determine how other considerations like preserving flat-foldability influence their application. The results of this work are described in Chapter 2 and were published in the *Journal of Mechanisms and Robotics* [42].

Additional research on creating rigid-foldability was performed to develop methods for generating new mechanisms from existing models. This was done using observations of some existing, thick-origami models. One objective of this research was to not only find new methods, but also mathematically prove their effectiveness. Other goals of this work included demonstrating the techniques and using the resulting models to design thick-origami tessellations. This research is planned for future publication.

Efforts on modeling compliant panels in origami-inspired mechanisms were focused on three major objectives. First, a mathematical model of the force-deflection behavior of the compliant panels was created. Next, this model was experimentally verified. Finally, a pseudo-rigid-body model was created for a general compliant panel that approximated its behavior. This work is detailed in Chapter 4 and was published in the *Proceedings of the International Design and Engineering Technical Conference* [43].

Work on managing the bending stiffness of rigid-foldable origami was also divided into three goals. The first involved modeling the effect of a slight pattern actuation on the stiffness of a crease pattern. Another important goal included modeling the behavior of additional deployable stiffeners on a crease pattern. Experimentally confirming the predictions of these models was a final goal of this phase of research. The results of this research are described in Chapters 5 and 6. The material in Chapter 5 has been accepted for publication in the *Proceedings of the International Design and Engineering Technical Conference* and the material in Chapter 6 is published in the *Journal of Mechanics-Based Design of Structures and Machines* [44].
CHAPTER 2. CREATING RIGID FOLDABILITY TO ENABLE MOBILITY OF ORIGAMI-INSPIRED MECHANISMS

2.1 Introduction

Rigid foldability is important to ensure mobility when an origami pattern is to be realized in rigid materials and represents a common design objective for origami-inspired engineering. If an origami crease pattern is viewed as a rigid-link mechanism, then it will be capable of motion unless it is so over-constrained that it becomes a structure [19]. However, because problems affecting rigid-foldability may not be obvious due to complex folding motion, a designer may need to revise an initial design to obtain a rigidly foldable pattern with the desired characteristics [17].

The process of making alterations to this pattern can be divided into three principal tasks:

1. Determine what type of crease pattern alterations will be made
2. Determine where these alterations will occur
3. Determine how many alterations will be made

In this work, methods for completing each of these tasks is developed. Multiple crease pattern alterations are considered and criteria are derived to describe their relative value. Techniques for selecting appropriate locations for these alterations are also discussed. A unique process for estimating how many alterations to apply is also described. An example problem is provided to illustrate the use of these methods.

2.2 Method for Creating Rigid Foldability

Decisions regarding the type and location of alterations made to a pattern to make it rigidly foldable, and thus ensure the corresponding rigid-link mechanism has mobility, are influenced by the particular characteristics of each type of alteration as well as its effect on the functionality of the
final design. If the mobility of a mechanism is less than or equal to zero, then the mechanism cannot undergo rigid-body motion and so cannot be rigid foldable [23]. Because of this, the mobility characteristics of a mechanism will be a key factor influencing design changes. For example, a choice may increase the mobility of the crease pattern with relatively few modifications but may compromise other important characteristics of the design.

2.2.1 Methods for Increasing Mobility

Several options exist when more mobility needs to be added to a crease pattern. One solution is to add more creases by splitting an appropriate number of panels, as suggested by Tachi [22]. Other options include removing a panel from an origami mechanism and splitting creases. Methods like these are important when considering devices such as the one-piece mechanisms described by Winder et al. [45] that are manufacturable in a flat state and do not require assembly. These approaches are each described next.

Adding Creases

The option of adding creases to a pattern to increase mobility is illustrated by Figure 2.1. In this example, the extra crease increases both the number of links and number of joints by one. An extra crease will also increase the order of the vertices that it intersects. This is because the order of a vertex is determined by the number of creases that intersect to create it. If a series of \( n \) vertices of order \( x_1, x_2, x_3, \ldots \) form the corners of a given panel where no vertex lies on the edge of the pattern, then the number of links in the mechanism is given by
\[ N = \sum_{i=1}^{n} (x_i - 2) + 1 \]  

(2.1)

where \( x_n \) is the order of the \( n \)th vertex. The number of joints or creases, \( J \), is given by

\[ J = \sum_{i=1}^{n} (x_i - 1) \]  

(2.2)

These results permit the calculation of the mobility, \( M \), of a spherical mechanism as

\[ M = 3N - 3 - 2J \]  

(2.3)

where \( N \) and \( J \) are the number of links and joints found previously [46]. If the origami mechanism is modeled in terms of spherical joints and links as shown in Figure 2.2, the mobility can also be determined using

\[ M = 3V - E - 6 \]  

(2.4)

as suggested by Wampler et al. [23] Here, \( M \) is the system mobility, \( V \) represents the number of vertices in the system and \( E \) represents the number of edges. Adding a crease to a system modeled by Equation 2.3 between any two vertices will theoretically increase the mobility of the system by

\[ \Delta M = [3(N + 1) - 3 - 2(J + 1)] - (3N - 3 - 2J) = 1 \]  

(2.5)

Similar results occur whether the panel is located at the edge of the pattern, as shown in Figure 2.1c, or in the interior as shown in Figure 2.1b. If the mechanism is modeled using Equation 2.4, the same result is obtained for both edge and interior creases.

For interior creases, if no new vertices are created, then the number of added creases is limited by the number and order of polygon panels that are not triangles because only non-triangular panels can be divided with vertex-to-vertex creases [47]. Panels at the edge of a crease pattern are not subject to this constraint.
Figure 2.2: (a)(c) An arbitrary degree-4 vertex is shown modeled with rods and ball joints and (b)(d) its equivalent configuration when one panel has been removed.

**Removing Panels**

Another possibility for adding mobility is to remove a panel completely from the origami pattern, as shown in Figures 2.2. Figures 2.2a and 2.2c show an arbitrary degree-4 vertex modeled as a series of triangular trusses formed from rods and spherical joints. Eliminating one link is equivalent to removing a panel from a single spherical mechanism as shown in Figures 2.2b and 2.2d. Such a change would increase its mobility by one if it were modeled according to the equations proposed by Wampler et al. [23] While self-intersection or collision between panels remains a potential problem after panel removal, the constraints contributed by these vertices are less strict.

For a panel enclosed by \( n \) sides where no side lies on the edge of the pattern, the number of degrees of freedom can be found by Equation 2.4. For it to be rigid, the number of edges this panel must include is

\[
E = 3V - 6 = 3n - 6
\]  

(2.6)

For example, a four-sided panel would require \( 3(4) - 6 \) or 6 edges or bars to make it completely rigid. If this panel is removed, the change in mobility for this mechanism is similarly given by

\[
\Delta M = [3V - (n) - 6] - [3V - (3n - 6) - 6] = 2n - 6
\]  

(2.7)

If this situation is studied using Equation 2.3, a different formula is found with

\[
\Delta M = [3(N - 1) - 3 - 2(J - n)] - (3N - 3 - 2J) = 2n - 3
\]  

(2.8)
However, since removing a triangular interior panel should not increase the mobility of a crease pattern, only Equation 2.7 can be considered correct. When only one side of the panel touches the edge of the pattern, then the change in mobility is

\[ \Delta M = -\Delta E = [3n - 6] - [n - 1] = 2n - 5 \]  (2.9)

In this case, both Equations 2.3 and 2.4 agree in their results. From these equations, it can be seen that removing any panel will be beneficial in proportion to the number of sides it possesses. It is also apparent that the location of the removed panel is important. A panel removed from the edge of the pattern provides one more degree of mobility than a similar panel removed from its interior.

**Splitting Creases**

An alternative to removing panels is to partially cut a panel away from the rest of the pattern at its creases to reduce the number of constraints while still allowing all panels to be retained. Making this change to the mechanism in Figures 2.3a and 2.3c produces the system shown in Figures 2.3b and 2.3d. Equation 2.4 predicts an increase in the instantaneous mobility of a single vertex by two.

Figure 2.4 shows a more general case of a pattern where crease splitting could take place. For a group of \( n \) vertices in a crease pattern, if one crease connecting a vertex to the edge of the pattern is split, then the mobility of the system also increases by two as

\[ \Delta M = [3N - 3 - 2(J - 1)] - (3N - 3 - 2J) = 2 \]  (2.10)
Figure 2.4: (a) An arbitrary crease pattern and (b) where one crease has been split.

Figure 2.5: A situation where splitting a crease does not increase mobility.

according to Equation 2.3. The same result is obtained if the system is modeled using Equation 2.4. This result does not hold when a crease not touching the edge of the pattern is split. An example is shown in Figure 2.5. Splitting crease $a$ would not prevent the rigidity of the two adjacent panels from constraining the motion of the whole system. A split edge crease has also lost some constraint on one of its terminating vertices and so the two split edges are capable of moving in opposite directions or farther apart if necessary. This is not possible for the split internal crease. Because the vertices at both ends of the crease are still intact, they will prevent the split edges from moving relative to each other.

While splitting a single internal crease will not increase the mobility of a crease pattern, splitting multiple creases can be helpful. Figure 2.6 shows an arbitrary crease pattern where two connected creases are cut. Figure 2.6a shows the creases that are cut and Figure 2.6b shows an equivalent system where the separated section is removed. This section is cut away because it should no longer constrain the motion of the model through its hinges. Self-intersection may still occur.
More generally, if $s$ consecutive creases are split on the boundary of an internal panel of $n$ sides, then the amount of additional mobility is obtained by combining Equations 2.5 and 2.7 and is given by

$$
\Delta M = \Delta M_{AddedCrease} + \Delta M_{RemovedPanel} = 1 + [2(s + 1) - 6] = 2s - 3
$$

(2.11)

since this alteration is equivalent to adding a crease to the internal panel and then removing the new panel that is only connected along one hinge. When using this method, $s$ must be smaller than $n$ and greater than one. This relation confirms that the increase in mobility of a crease pattern is beneficial in proportion to the number of creases split.

### 2.2.2 Methods for Determining Alteration Placement

Where to apply alterations to a crease pattern depends on which portions of the crease pattern are over-constrained. Depending on the number and order of vertices in a particular region of the pattern, a section may not be rigidly foldable.

One way that these over-constrained regions may be identified is by using Equations 2.3 and 2.4 to make a first estimate of the mobility of either the entire crease pattern or a cluster of included vertices. Searching for regions of over-constrained vertices would permit modifications to the crease pattern to be made in areas where they would be most likely to improve the mobility of the system.

There are cases where the results of the mobility equation underestimate mechanism mobility [48]. The crease pattern described in Figure 2.7 is an example. For this map crease pattern,
Figure 2.7: This map-fold crease pattern is calculated to have a negative degree of freedom, but symmetry allows mobility by folds along the vertical and horizontal creases.

Figure 2.8: (a) A pattern with zero degrees of freedom where (b) a portion of the pattern is rigid-foldable. The mobility is -3 as calculated by Equation 2.3. This is due to the presence of 24 creases/joints and 16 panels/links. However, due to symmetry such a fold pattern can rigidly fold by folding just along the vertical or horizontal creases.

An over-constrained crease pattern is shown in Figure 2.8. Here, the dashed and solid lines represent mountain and valley fold, respectively. This particular model is made of two particular types of flat-foldable square twists paired side by side [49]. Because the pattern can be modeled as a set of spherical mechanisms, the mobility can be calculated through Equation 2.3 as $M = 3(17) - 3 - 2(24) = 0$. While the complete pattern has a mobility of zero, only the twist on the right can rigidly fold. The twist on the left will remain flat during the folding process. Adding creases to the pattern that is already foldable will not influence the over-constrained state of the other twist. Because different areas of the model are constrained by different amounts, it would be most advantageous to apply changes to the most heavily constrained regions to ensure that enough flexibility is given to the system to enable it to fold rigidly.

Another method that can be used to identify regions of low instantaneous mobility is through using a numeric method developed by Tachi [22], called the diagram method. It consists of two basic conditions affecting angular velocity magnitudes and directed area. The first
condition specifies that the relative angular velocity vectors of each crease should sum to zero for any vertex in a pattern. The second condition requires the directed area of this vector sum to be zero. The first condition is implemented by calculating the sum of the $x$ and $y$ components of these vectors as $R_x$ and $R_y$. They are calculated as

\begin{align}
(R_x)_j &= \left| \sum_{i=1}^{n} \omega_i \cos \theta_i \right| \\
(R_y)_j &= \left| \sum_{i=1}^{n} \omega_i \sin \theta_i \right|
\end{align}

where for each of $n$ creases in a vertex, $\omega_i$ is the angular velocity of the crease and $\theta_i$ is its angular position measured from some reference. The horizontal and vertical components of the relative angular velocity components are summed to obtain a residual quantity that should ideally be zero to describe a rigidly foldable system. These equations give residuals for the $i$ creases of the $j$th vertex.

The second condition is implemented by calculating the squared directed area, $R_d$, of the polygon bounded by the angular velocity vectors. For the $j$th vertex this is given by

\begin{equation}
(R_d)_j = \left( \sum_{i=2}^{n-1} \frac{1}{2} \left( \left( \sum_{k=1}^{i-1} \vec{\omega}_k \right) \times \vec{\omega}_i \right) \right)^2
\end{equation}

The directed area of a triangular shape bounded by vectors $\vec{a}$, $\vec{b}$ and their difference is given by

\begin{equation}
A = \frac{1}{2} \| \vec{a} \times \vec{b} \|
\end{equation}

The total directed area of a polygon with $n$ sides can be found by finding the directed area of all $n$ triangles formed by a vector $\vec{\omega}_i$ along the $i$th side and a vector extending from the origin to the base of $\vec{\omega}_i$. As shown in Figure 2.9, the directed area of each colored triangle in Figure 2.9a is found by substituting the vectors $\vec{p}$ and $\vec{\omega}_i$ for $\vec{a}$ and $\vec{b}$. The sign of the area computed for each triangle will be different depending on the relative directions of $\vec{p}$ and $\vec{\omega}_i$ as shown in Figure 2.9b. The triangle with $\vec{p}$ will produce a positive value while the triangle bordered by $\vec{q}$ will return a
Figure 2.9: An arbitrary polygon whose directed area is being calculated by summing the directed areas of its side triangles, where (a) each side forms a triangle with the origin and (b) the polygon’s sides are shown as $\omega_i$ and their directed areas.

negative result. If one corner of the polygon is located at the origin, then the two sides adjacent to that corner will have directed areas of zero. The vector $\vec{p}_i$ for all other sides of the polygon is given by

$$\vec{p}_i = \sum_{k=1}^{i-1} \vec{\omega}_k$$  \hspace{1cm} (2.16)$$

Because the two vectors extending from and terminating at the origin will generate a triangle with zero directed area, these triangles are excluded from the calculation. The sum of the remaining triangles is found by Equation 2.14 and returns the total directed area of the polygon [50].

This total directed area should also be zero for a rigidly foldable vertex. For a crease pattern containing multiple vertices, each vertex must satisfy this condition and consistency must be maintained throughout the pattern as shown by Tachi [22]. However, for this analysis it is equated to a residual variable in Equation 2.14.

A third set of conditions that was added to this system constrains all the angular velocities in the model to be either positive or negative. These constraints are implemented by calculating a difference, $R_{a}$, between $\omega_i$ and an arbitrary positive or negative number. This difference is given by

$$(R_{a})_j = \sum_{i=1}^{n} (\omega_i + (m)a_i^2)^2$$  \hspace{1cm} (2.17)$$

Here, $m = -1$ if the crease folds as a valley and $m = 1$ if it folds as a mountain. The variable $a_i$ is found through optimization but whose sign is controlled by $m$. The total residual $R_{total}$ is
calculated for the entire crease pattern by

$$R_{total} = \sum_{j=1}^{p} [(R_x)_j + (R_y)_j + (R_d)_j + (R_a)_j]$$  \hspace{1cm} (2.18)

Here, all the residuals from each equation for each of $p$ vertices are summed for the entire crease pattern.

Setting all $\omega_i$ in the model to either 1 or -1 rad/s, depending on their mountain-valley assignment, will help in converging to a solution. The total residual is minimized using a GRG non-linear solver. A certain number of $\omega_i$ are chosen as inputs to determine the motion and all other $\omega_i$ and $a_i$ in the pattern are modified by the solver to minimize the total residual.

Using this model permits the angular velocity of the creases in a pattern to be calculated given an appropriate number of input angular velocities. If the model is only partially rigidly foldable, then the angular velocities of the affected creases should be approximately zero. This reveals any over-constrained regions.

### 2.2.3 Determining Number of Modifications

Knowing the number of alterations required for rigid foldability is helpful in reducing the complexity of the final model. The first step is to estimate the degree of over-constraint in a system using Equations 2.3 and 2.4. These equations can be applied iteratively on all combinations of successively larger groupings of adjacent vertices. While such a process helps identify regions of over-constraint, it is inaccurate in some cases because of singularities that may be present in the system. It can only provide an estimate of the true instantaneous mobility of a crease pattern [51, 52].

A method for correcting the inaccuracies of Equations 2.3 and 2.4 is presented by Shai [24]. This method permits an accurate calculation of mobility for a system by taking into account any existing states of self-stress which increase the mobility. Shai defines self-stress as “inner forces that exist in the mechanical elements satisfying the force equilibrium around each joint.” This correction adds another term to Equation 2.3 resulting in
Figure 2.10: A square twist crease pattern that has a mobility of zero but actuates with one degree of freedom.

\[ M = 3N - 3 - 2J + SS \]  \hspace{1cm} (2.19)

where SS is the correction term. It is obtained by determining the number of independent forces that can act along the creases of a pattern with any number of vertices. The validity of this method can be confirmed by applying it to a crease pattern such as the square twist shown in Figure 2.10. While the mobility of this crease pattern is zero, it contains one state of self-stress and so the correction factor to Equation 2.3 becomes one. This modification allows the equation to return a mobility of one degree of freedom for the system.

### 2.2.4 Designing for Flat-Foldability and Rigid-Foldability

If flat foldability is a design requirement, then special consideration is needed to maintain flat-foldability when determining the number and types of modifications to make to a crease pattern. Due to Kawasaki’s Theorem, only an even number of creases can be added to the pattern if this method is used to maintain flat-foldability and enhance the system mobility. The creases that are added must also respect Maekawa’s Theorem which requires that the number of mountain and valley folds in a pattern differ by two [53, 54]. Neither removing panels nor splitting creases should negatively impact the local flat-foldability of a structure because none of the vertex panel dimensions are changed.

In addition to constraining the number of creases added to a pattern, flat-foldability conditions also prevent their location and orientation in the pattern from being completely arbitrary. One constraint that must be respected is that all creases added to a vertex cannot be located within
Figure 2.11: (a) An arbitrary crease pattern where creases have been added to one panel and to adjacent panels.

the same panel. This requirement stems from another part of Kawasaki’s Theorem stating that the alternating sum of angles around a vertex must be zero [53]. Figure 2.11a illustrates this situation. If each of \( n \) panel angles in a vertex is given by \( \theta_i \), then for a flat foldable panel

\[
0 = \sum_{i=1}^{n} (-1)^i \theta_i = \sum_{i=1}^{k-1} (-1)^i \theta_i + (-1)^k \theta_k + \sum_{i=k+1}^{n} (-1)^i \theta_i \tag{2.20}
\]

If the \( k \)th panel is split by an even number of creases, \( a \), then

\[
0 = \sum_{i=1}^{k-1} (-1)^i \theta_i + \sum_{i=k}^{k+a-1} (-1)^i \alpha_i + \sum_{i=k+a}^{n} (-1)^i \theta_i \tag{2.21}
\]

where \( \alpha_i \) represent the angles of each panel created by the crease additions, as illustrated in Figure 2.11. Subtracting Equations 2.20 from 2.21 yields

\[
0 = (-1)^{k+1} \theta_k + \sum_{i=k}^{k+a-1} (-1)^i \alpha_i \tag{2.22}
\]

because the last terms in each equation cancel due to \( a \) being even. Because the sum of all \( \alpha \) is equal to \( \theta_k \), Equation 2.22 cannot be satisfied.

If two adjacent panels are split with \( a \) and \( b \) creases as shown in Figure 2.11b, then the process shown before yields

\[
0 = \theta_k - \theta_{k+1} + \sum_{j=1}^{a} (-1)^j \alpha_j + \sum_{j=1}^{b} (-1)^{a+j} \beta_j \tag{2.23}
\]
If the angle requirements established by this equation are respected by any added creases, then local flat-foldability will be retained. Because this condition places strict constraints on the positioning of added creases in a vertex, a simple solution is to only add creases to the edge panels of a pattern where both ends of the crease will not touch an interior panel.

### 2.3 Examples

The approach described above can be used to modify existing non-rigid origami to become rigidly foldable. For example, the sink fold shown in Figure 2.12 is non-rigid and is a candidate for modification to make it rigid foldable. Here the mountain folds are shown in red and the valley folds are shown in blue.

Equation 2.3 gives the mobility of this pattern as -3. There are no states of self-stress in the system and this value is accepted without correction. Obtaining a mobility greater than zero will require the addition of four degrees of freedom to the model by one of the methods discussed earlier.
If adding creases is the only modification that will be made to this system, then four creases will be required because they each contribute one degree of freedom, as shown in Equation 2.5. If the vertices shown in Figure 2.13 are isolated and studied, it can be seen that according to Equation 2.3, their combined mobility is zero. By this same measure the remaining two vertices have a combined mobility of 1. This indicates that the circled section can be viewed as an over-constrained region. Because the pattern is symmetric, the same result can be obtained by considering a mirror image of the selection described. Because this selection and its mirror share a set of vertices in the center, this would be a reasonable place to make modifications.

Since the central vertex of the vertices identified is only bordered by triangular panels, creases will only be added to the circled vertices shown in Figure 2.14. While flat-foldability may not be desired, these alterations will be made to preserve it. If the outside quadrilateral panels are split by one crease per panel, then Equation 2.23 can be satisfied if $a = b = 1$ and $\alpha_i = \beta_i = 45^\circ$.

Making this modification was shown to permit rigid folding motion to take place as verified by the Rigid Foldability Simulator developed by Tachi [22]. The same results can be obtained through analysis using the method described earlier, (the relative magnitudes of the crease angular
velocities can be determined and used to verify that the pattern has a solution for folding. The images in Figure 2.15 show folding that the altered crease pattern can now experience.

Edge creases could also be split to increase the mobility of the mechanism. Two creases would need to be split to achieve an increase of four degrees of freedom as shown by Equation 2.10. Both of these alterations could be made on creases surrounding the two over-constrained vertices identified earlier. If they are made on the two creases shown in Figure 2.16, then the model will be able to fold as shown in Figure 2.17. Further alterations such as adding creases or cutting away portions of panels are needed to avoid self-intersection during the folding motion.

Lastly, the pattern can also be made mobile by removing panels from the pattern. According to Equation 2.9, removing two quadrilateral panels from the edge of the pattern will provide six added degrees of freedom and permit movement. The panels chosen for removal are adjacent to the over-constrained vertices shown previously and are shown in Figure 2.18. Figure 2.19 shows the folding motion of the resulting model.
Figure 2.18: Panels removed to permit rigid folding motion.

Figure 2.19: Sequence of folded states of the sink fold crease pattern with missing panels.

2.4 Conclusions

Because the ability of a crease pattern to fold rigidly is not always easily predictable, a designer may be required to find ways to modify a crease pattern to permit rigid motion of a mechanism. This chapter investigated three types of alterations that can be made to a crease pattern to give it additional mobility where needed: adding creases, removing panels and splitting creases. These modifications were used to develop a process that can successfully convert non-rigid-foldable origami to similar, rigid-foldable models. This process is outlined below with the steps developed in this work.

1. Calculate pattern mobility $M$

2. Choose 1 or more modification types

3. Determine number of modifications

4. Find locations for modifications
Equations were derived that predict the change in mobility produced by each of these modifications. This will allow comparisons of their relative usefulness to be made. Techniques for choosing suitable locations for these alterations were also discussed. A unique strategy for using the current mechanism mobility to estimate the number of modifications was also developed. In situations where flat-foldability must be preserved, formulas were developed to guide modification design. A specific crease pattern was studied and used as an example of the application of these ideas. This example shows that the modification procedure presented in this work can successfully create rigid-foldability in mechanisms that do not possess this quality.

This result is significant because it allows engineers to utilize models in design that would otherwise be unusable because of their lack of rigid-foldability. Although the complexity of a pattern will necessarily increase because of these modifications, other desirable properties may be sufficiently valuable to offset this cost.

While these methods are useful, they are subject to some limitations. The means of calculating mobility suffers from the same issues as those for the Chebychev-Grübler-Kutzbach Criterion. This may affect estimates of the number of alterations needed as well as their location. Another limitation of these methods is that they rely on the instantaneous mobility and so cannot guarantee general rigid foldability.

Nevertheless, the successful use of the methods described in this chapter represents evidence that they can be used to design origami crease patterns that are rigidly foldable while minimizing the number of alterations. This is an important step in converting origami models to a form that permits engineering systems in materials other than paper.
CHAPTER 3. CREATING LINKAGE PERMUTATIONS TO PREVENT SELF-INTERSECTION AND ENABLE DEPLOYABLE NETWORKS OF THICK-ORIGAMI

An important part of designing thick origami mechanisms is preventing distortions during the folding process. While it is possible to design thick origami that is rigid-foldable [16], the thickness of the material complicates the process because it often requires the modification of the crease pattern to retain mobility. Linkage-based origami has been demonstrated as one way to address this [33,55] through a variety of different linkage-based models [56]. Deployable networks of linkages have also been investigated [29, 34, 57–59]. Both Tachi [4] and Lang and Howell [60] studied problems in the design of zero-thickness, rigid-foldable, quadrilateral meshes. This work has been supplemented by research on thick tessellations [61–65] and 3D, thin tessellations [41, 66]. Nevertheless, there is still much to learn about how to modify these mechanisms for use in materials other than paper. Processes for connecting origami-inspired linkages into networks that can be used in larger deployable mechanisms would also be helpful.

The objective of this chapter is to enable translation of origami to thick materials by developing methods that generate permutations of spatial linkages with desirable characteristics. These characteristics include no self-intersection, maintaining the same number of degrees of freedom as the root linkage, and the existence of a simple kinematic model, which is particularly important for mechanisms based on spatial linkages. Hinge transposes, thickness shifts, and splitting vertices are proposed as ways to generate linkage permutations with these characteristics. These methods will help create modified spatial linkages that are suitable for use in linkage tessellations.

3.1 Results

The unique kinematics of overconstrained linkages allow them to be implemented as origami-inspired mechanisms because their hinge axes are not parallel. This is illustrated through the Bennett linkage with the joints and thicknesses labeled in Figure 3.1a and the sector angles noted
Figure 3.1: The Bennett linkage described with (a) thicknesses $a_i$ and hinges indicated by numbers 1 through 4 and (b) angles $\alpha_i$ and $\theta_i$. A zero-thickness equivalent (c) where $a_i = 0$ and two frames $m$ and $n$ are used to define a panel (d) where $m$ and $n$ are related by a homogeneous transformation $T$.

in Figure 3.1b. Here, it can be seen that joint axes 1, 2, 3 and 4 are neither parallel nor co-planar. The similarity between this linkage and an origami vertex can be seen as $a_i = d_i = 0$ for a zero-thickness vertex in Figure 3.1c. Multiple variations of this and other linkages can be obtained through three different methods presented in this section. If we represent the linkage using the Denavit-Hartenburg (DH) convention, it can be shown that these methods do not prevent the mechanism from moving.

The DH convention [67] uses four parameters ($\theta$, $a$, $d$ and $\alpha$) in a homogeneous transformation matrix $T_i(\theta_i, a_i, d_i, \alpha_i)$ to describe each link. $T_i$ is defined in Appendix A and represents a set of consecutive, specific rotations and translations that relate frames attached to two adjacent links. Because it must be true that the product of all $T_i$ in a linkage loop must be the identity matrix, a formal definition of a linkage can be made for this work.

**Definition 1.** A mechanism consisting of a closed loop of $n$ links is defined as a hoop linkage if and only if

$$T_1T_2...T_n = \prod_{i=1}^{n} T_i = I \quad (3.1)$$

where each $T_i$ relates two frames rigidly fixed to adjacent links. Only $\theta_i$ in each $T_i$ is allowed to vary.

This definition has been used to derive formulas governing the behavior of multiple hoop linkages such as the Bennett linkage, as listed in Appendix A.
Figure 3.2: A Bennett linkage with self-intersection (a), a transposed panel (b) and the same linkage with a transpose of the fourth hinge (c) that produces a developable version of the mechanism. The unchanged links are in grey.

In origami, the panel sector angle, $\alpha$, and the panel thickness, $a_i$, as shown in Figure 3.1 are key variables describing the vertex geometry. They are naturally described by the twist parameter $a_i$ and linear offset $a_i$ in the DH representation. Using this transform and the frames shown in Figure 3.1d, we define an origami panel that depends on the transform and has a simple shape.

**Definition 2.** Let two hinge axes $z_m$ and $z_n$ be related by a DH transformation $T$ defined by the parameters $\theta, a, d, \alpha$. Also, let $\hat{z}_m$, $\hat{x}_m$ and $\hat{z}_n$ represent unit vectors aligned with the $z$- and $x$-axes of frame $m$ and the $z$-axis of frame $n$, respectively. The panel $P$ is defined by the set of all points satisfying

$$P = \{h \in \mathbb{R}^3 | h = p\hat{z}_m + q\hat{z}_n + v\hat{x}_m\} \quad (3.2)$$

with respect to frame $m$ where $0 < p < R$, $0 < q < R$, $0 < v < a$, and $R$ is a scalar that defines the curved boundary of the panel shown in Figure 3.1d.

### 3.1.1 Hinge Transpose

Here, we introduce the concept of a hinge transpose. It will be valuable in preventing self-intersection of links as shown in Figure 3.2a, enabling new configurations of mechanisms. This transpose takes place when the twist angles $\alpha$ of the two links adjacent to the hinge are replaced by their supplements (e.g. $\alpha - \pi$) as shown in Figure 3.2b. This transpose is shown in Figure 3.2c. While hinges 1, 2, and 3 remain in the same position, the fourth hinge changes direction. A hinge transpose can also be more formally defined below.
Definition 3. Given a hoop linkage $B$ composed of $n$ homogeneous transformations $T_i$, a hinge transpose of a joint between the $i$-th and $(i+1)$-th link consists of replacing transforms $i$ and $i+1$ with transforms $T'_i$ and $T'_{i+1}$ where $T_i \neq T'_i$ and $T_{i+1} \neq T'_{i+1}$.

This definition leads to the following theorem and corollary that such a transformation will not prevent the resulting mechanism from moving. The fact that multiple hinge transposes also do not prevent movement of the mechanism can be proved through variable substitutions, as shown in Appendix A.

Theorem 1. If we define a specific hinge transpose which exchanges $T_i = T_i(\theta_i, a_i, d_i, \alpha_i)$ and $T_{i+1} = T_{i+1}(\theta_{i+1}, a_{i+1}, d_{i+1}, \alpha_{i+1})$ for $T'_i$ and $T'_{i+1}$ and where

$$T'_i = T_i'(\theta_i, a_i, d_i, -(\pi - \alpha_i)) \quad (3.3)$$
$$T'_{i+1} = T_{i+1}'(\theta_{i+1}, a_{i+1}, d_{i+1}, -(\pi - \alpha_{i+1})) \quad (3.4)$$

then $T_i T_{i+1} = T'_i T'_{i+1}$

Corollary 1.1. Let $B$ be a hoop linkage as described in Definition 1. If a hinge is transposed as defined in Theorem 1, the new mechanism $B'$ will also be a hoop linkage. Also, it will continue to be capable of motion.

Theorem 1 and Corollary 1.1 have important consequences. Together, they describe a simple process for modifying the geometry of an origami-inspired linkage without affecting its motion. One application of this modification would be to reduce self-intersection.

3.1.2 Thickness Shift

Here, we introduce another modification method called thickness shift which requires that $a_i = 0$ for a certain subset of the transforms $T_1, T_2, \ldots, T_n$. This operation is illustrated in Figure 3.3. If the thickness shift is performed on all the links, then we obtain the spherical mechanism shown in Figure 3.3b. However, if it is only applied to the middle link, then Figure 3.3c shows the resulting mechanism. While the middle link may be drawn with no thickness, it may also include the material thickness above or below the panel as illustrated in Figure 3.3d and 3.3e. Unlike the hinge
Figure 3.3: Three links where (a) thicknesses are non-zero, (b) $a_1 = a_2 = a_3 = 0$ (spherical mechanism), (c) $a_2 = 0$, (d) $a_2 = 0$ and the panel thickness is above and (e) $a_2 = 0$ and the panel thickness is below.

transpose, this type of transformation does not necessarily result in an equivalent linkage. For a hoop linkage, changing the thickness parameter $a$ to zero must respect any additional constraints enforced by the loop. For the Bennett linkage, these requirements are shown in Appendix A in Equations 2, 3 and 7. For this particular linkage, these equations can only be satisfied if all thicknesses are zero. Consequently, a thickness shift transformation can only be applied to either all or none of the links. Similar rules will apply to other linkage types.

3.1.3 Split Vertex

A final modification that we consider is the split vertex technique [32,68]. In this technique, two links representing a linear offset are included at two arbitrary, non-consecutive points in a hoop linkage consisting of $n$ links. This transformation is shown in Figure 3.4. This modification is possible as long as a vector from 3 to 3’ and 1 to 1’ are identical and hinge 3 is parallel to 3’ and hinge 1 is parallel to 1’. The fact that this modification also preserves the ability of the hoop linkage to move can also be proven (see the corresponding proof in Appendix A).

**Theorem 2.** Let $B$ be a hoop linkage as described in Definition 1, $d^i$ represent a translation with respect to frame $i$ and $T_i$ and $T_j$ be homogeneous transformations where
For the homogeneous transformations describing $B$, $T_1 T_2 ... T_n = T_1 T_2 ... T_{i-1} T_i T_{i+1} ... T_n T_j = I$ and the new set of links is also a hoop linkage as described by Definition 1.

This result is significant because it shows that a split vertex transformation can be applied to any non-consecutive hinges in a hoop linkage provided the offset links $T_i$ and $T_j$ have certain properties: they must both be linear translations described by vector $d$ and the hinges that they connect to must be parallel because $T_i$ and $T_j$ include the identity rotation $I$.

3.1.4 Linkage Permutations

The hinge transpose, thickness shift and split-vertex methods can be used to generate variations of a particular hoop linkage. If we designate a hinge transpose by $T[u_i]$, a thickness shift by $Z[u_z]$ and a vertex split by $V[u_1, u_2]$, then recording alterations can be simplified. In this notation, $u_i$ is the hinge that is transposed, $u_z$ is the smallest hinge number adjacent to the thickness shift and $u_1$ and $u_2$ are the lowest-numbered hinges adjacent to the offset links in the split mechanism. If we list the left-most operation occurring first, a table can be created that connects every variation to a fundamental set of linkages. This information is listed in Table 3.1. For $n = 4$, the Bennett
linkage described in Figure 3.2a is used as a starting point. For \( n = 6 \), a vertex in Figure 3.5c that was described by Chen et al. [33] is the starting point.

Different linkage variations created by the hinge transpose and thickness shift methods can also be combined to form more complex mechanisms. An example of a model using Linkages I and III is shown in Figure A1. The hinges for this model are shown in Figure A2. In this combined form, the new mechanism acts like an extendable tube.

3.1.5 Tessellation of Linkage-Based Origami

The ability of linkage-based origami to tessellate in two dimensions is an important characteristic that will influence its suitability for engineering applications such as large deployables. Such a tessellation is possible using developable vertices [4], but complicated geometry is required to prevent self-intersection in the folded state for thick models. Tessellation is also possible using

<table>
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<th>Figure</th>
<th>Transformations</th>
<th>( \Sigma \alpha )</th>
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<td>4</td>
<td>3.2c</td>
<td>( T[4] )</td>
<td>D</td>
</tr>
<tr>
<td>Linkage II</td>
<td>4</td>
<td>3.5a</td>
<td>( T[4], T[1], Z[all] )</td>
<td>U</td>
</tr>
<tr>
<td>Linkage III</td>
<td>4</td>
<td>3.5b</td>
<td>( T[4], T[1] )</td>
<td>O</td>
</tr>
<tr>
<td>Linkage IV</td>
<td>6</td>
<td>3.5d</td>
<td>( T[1] )</td>
<td>O</td>
</tr>
<tr>
<td>Linkage V</td>
<td>6</td>
<td>3.5e</td>
<td>( T[1], T[4], Z[2], Z[5] )</td>
<td>O</td>
</tr>
<tr>
<td>Linkage VI</td>
<td>6</td>
<td>3.5f</td>
<td>( T[1], T[4] )</td>
<td>O</td>
</tr>
</tbody>
</table>
the thick, Eggbox tessellation shown in Figure 3.6. The combination of Linkage II and III allows the entire pattern to fold without self-intersection. While this pattern is useful in its current state, understanding how the geometric constraints in the mechanism influence its tailorability enable its deployment to be tailored to specific applications.

Connecting linkages in a network and ensuring that the network is rigid-foldable is similar to ensuring rigid-foldability in zero-thickness origami networks because a similar set of constraints apply. One of these is that fold angles must be consistent when traced in loops containing multiple vertices and creases [1]. Another constraint results from the need for panel thicknesses to be compatible throughout the pattern. While it is possible to allow discontinuities in the panel thickness, this will not be considered here. With this information, a set of equations describing this specific tessellation can be found.

While criteria for designing networks of rigid-foldable linkages have not been mathematically proved, it will be assumed that satisfying three different constraint types throughout the pattern will be sufficient. These constraints are: Thickness Compatibility, Fold-Angle Compatibility and Sector-Angle Compatibility.

Thickness Compatibility

One constraint affecting Linkage III is that opposite panels must have the same thickness, as shown in Figure 3.7a. In this figure, opposite panels around the vertices marked with dots have the same thickness $a$ or $b$. Extending this constraint across the pattern requires that panels along diagonals be of the same thickness $a_i$ or $b_j$ as shown in Figure 3.7b.
A relationship based on specific thickness ratios determined by the twist angles provides an additional loop thickness constraint. Specifically, this relation requires that

\[
\frac{a}{b} = \frac{\sin \alpha_1}{\sin \alpha_2} = \frac{1}{k} \tag{3.7}
\]

where \( k \) is the thickness ratio \( b/a \). The loop constraints created by Equation 3.7 become clear when we consider that a line of constant thickness \( a_i \) is related to perpendicular lines of constant thickness \( b_j \) by the relation \( b_j = k_{ji}a_i \) where \( k_{ji} \) is a thickness ratio calculated for a particular vertex at coordinates \( j \) and \( i \) as shown in Figure 3.7c. If we relate a given thickness line \( a_0 \) to its perpendicular at coordinates \( (j, i) = (0, 0) \), then we have \( b_0 = k_{00}a_0 \). If this thickness is related to a thickness line parallel to \( a_0 \) at some coordinate \( (0, n) \), then we have

\[
a_n = \frac{b_0}{k_{0n}} = \frac{k_{00}}{k_{0n}} a_0 \tag{3.8}
\]
Similarly, this thickness line can be related to its perpendicular line at coordinates \((m,n)\). This gives
\[
b_m = k_{mn}a_n = k_{mn} \left( \frac{k_{00}}{k_{0n}}a_0 \right) = \frac{k_{mn}k_{00}}{k_{0n}}a_0 \tag{3.9}
\]

Finally, this thickness can be related to the original thickness line, \(a_0\). This gives
\[
a_0 = \frac{b_m}{k_{m0}} = \frac{k_{mn}k_{00}}{k_{m0}k_{0n}}a_0 \tag{3.10}
\]

Assuming that \(a_0 \neq 0\) allows it to be eliminated from both sides of Equation 3.10, resulting in
\[
k_{mn} = \frac{k_{m0}k_{0n}}{k_{00}} \tag{3.11}
\]

This equation must hold for any loop of Linkage III vertices. However, listing all possible loops may be difficult if the pattern is large. The number of equations can be reduced if we recognize that knowing that loop \((k_{00}, k_{0n}, k_{sn}, k_{s0})\) and \((k_{00}, k_{0n}, k_{tn}, k_{t0})\) are satisfied guarantees that loop \((k_{s0}, k_{sn}, k_{tn}, k_{t0})\) is also satisfied. This situation is depicted in Figure 3.7d. Here, we assume that \(0 < i < m\). This can be shown by expressing thickness consistency using Equation 3.11 for both loops. This yields
\[
k_{00} = \frac{k_{s0}k_{0n}}{k_{sn}} \tag{3.12}
\]
\[
k_{tn}k_{00} = k_{t0}k_{0n} \tag{3.13}
\]

Substituting Equation 3.12 into Equation 3.13 and eliminating \(k_{00}\) gives \(k_{tn}k_{s0} = k_{t0}k_{sn}\) which matches the condition for the third loop \((k_{s0}, k_{sn}, k_{tn}, k_{t0})\). Generally, consistency of all interior loops can be shown by guaranteeing that all loops \((k_{00}, k_{0i}, k_{ji}, k_{j0})\) are compatible for all \(j \in \{1,2,3...m\}\) and \(i \in \{1,2,3...n\}\).

Equation 3.13 is significant because it suggests that specifying \(k_{00}, k_{m0}\) and \(k_{0n}\) will determine all \(k_{mn}\) in the interior of the pattern. In the context of pattern design, this indicates that setting the vertex panel angles \(\alpha_1\) and \(\alpha_2\) for all the Bennett linkages along two edges of a rectangular pattern should specify all \(k_i\) elsewhere in the pattern. While the panel angles for these interior vertices
will not be completely determined, thickness consistency loops will provide additional constraints on the pattern geometry.

**Fold-Angle Compatibility**

A second set of constraints derives from maintaining compatible fold angles between all vertices in the tessellation because they share hinges. Making the substitutions required by Table 3.1 in Equations 8 through 10 in Appendix A shows that \(2\pi = -\theta_1 + \theta_3, 2\pi = \theta_2 - \theta_4\). These relations are equivalent to \(\theta_1 = \theta_3\) and \(\theta_2 = \theta_4\) if we assume that \(\theta_i = \theta_i + 2\pi\). If opposite fold angles in a vertex are equal, then the opposite-angle chains shown in Figure 3.7e should all be of equal angles. These chains are indicated by uniform line types and angles \(\theta_i, \theta_{i+1}, \theta_{i+2}\). The relationship between the angles in Linkages II and III can be found by making the appropriate substitutions in Equation 10 in Appendix A, giving

\[
\tan \frac{\theta_i}{2} \tan \frac{\theta_j}{2} = \frac{\cos \frac{1}{2}(\alpha_1 - \alpha_2)}{\cos \frac{1}{2}(\alpha_1 + \alpha_2)} = \mu_{ji}
\]

(3.14)

Like Evans, et al. [1] we define the constant \(\mu\) to simplify calculations. Here, \(\mu_{ji}\) is calculated for a vertex with coordinates \(i\) and \(j\) similar to those shown in Figure 3.7c. Like constraints on thickness, perpendicular lines of constant fold angle can be related in loops between four vertices \((0,0), (0,n), (m,n)\) and \((m,0)\). This relationship can be expressed as

\[
\mu_{mn} = \frac{\mu_{m0}\mu_{0n}}{\mu_{00}}
\]

(3.15)

Because this equation has the same form as Equation 3.11, it can be generalized in the same way. When combined, these two equations fill a similar roll as those in [4] and manage constraint in this linkage tessellation.

**Sector-Angle Compatibility**

The final set of constraints guarantee that the panels of the linkage tessellation are quadrilaterals and is written as

\[
2\pi = \alpha_{ji} + \alpha_{j+1,i} + \alpha_{j+1,i+1} + \alpha_{j,i+1}
\]

(3.16)
Figure 3.8: A model of a linkage tessellation (a) with a sinusoidal profile and one of its stowed states (b) similar to the original eggbox.

However, in its current form this equation cannot be easily integrated with the other equations. An assumption that will facilitate integration is that panel thickness is uniform throughout the pattern. This assumption will be desirable for manufacturing considerations but will also simplify the resulting equations. If \( a/b = 1 \), then \( \sin \alpha_1 = \sin \alpha_2 \). Satisfying this equality requires that either \( \alpha_1 = \alpha_2 \) or \( \alpha_2 = \pi - \alpha_1 \). The latter is impossible for non-developable vertices since for the angle sum \( \Sigma \alpha = 2\alpha_1 + 2(\pi - \alpha_2) \neq 2\pi \). If \( \alpha_1 = \alpha_2 \), then Equation 3.15 evaluated for a single panel reduces to \( \sec \alpha_{j+1,i+1} \sec \alpha_{ji} = \sec \alpha_{j+1,i} \sec \alpha_{j,i+1} \). Equation 3.16 can be used to eliminate \( \alpha_{j+1,i+1} \) and show that

\[
\cos(\alpha_{ji} + \alpha_{j+1,i} + \alpha_{j,i+1}) \cos \alpha_{ji} = \cos \alpha_{j+1,i} \cos \alpha_{j,i+1}
\]

\[
\sin(\alpha_{ji} + \alpha_{j+1,i}) \sin(\alpha_{ji} + \alpha_{j,i+1}) = 0
\]  

(3.17)

Equation 3.17 can only be satisfied if either \( \alpha_{ji} + \alpha_{j,i+1} = 0, \pi \) or \( \alpha_{ji} + \alpha_{j+1,i} = 0, \pi \). Because \( \alpha_{ji} > 0 \), the angle sums must equal \( \pi \). These relations effectively eliminate control over either all the vertices \((0,n)\) or \((m,0)\). Once the geometry of either the horizontal or vertical row of vertices are set, the perpendicular direction is also defined.

### 3.2 Discussion

Linkage-based origami-inspired mechanisms have the potential to solve difficult engineering problems because of their unique ability to realize rigid-foldable mechanisms as well as incorporate materials with finite thickness. These two traits can be valuable in applications such as
deployable equipment for spacecraft. However, important challenges have complicated the design of these mechanisms.

This work demonstrates a method for generating mechanism variations that can result in new thick-origami designs. This method can be summarized with steps developed in this chapter.

1. Choose linkage

2. Construct variations with any combination of transforms

(a) Transpose
   i. Select hinge and alter
   ii. Repeat as needed

(b) Thickness shift
   i. Select panel and eliminate thickness
   ii. Eliminate thicknesses of panels to satisfy linkage equations
   iii. Choose side for thickness
   iv. Repeat as needed

(c) Vertex split
   i. Select two creases
   ii. Choose offset vector $\vec{d}$
   iii. repeat as needed
A mathematical basis for generating different variations of a hoop linkage was proved and shown to produce legitimate variations of the linkage. This was demonstrated by creating six variations of linkage-based origami vertices based on the Bennett and Bricard linkages. Equations describing the kinematics of these mechanisms were also derived to characterize the deployment of the linkage tessellation.

Another challenge facing linkage-based origami design is ensuring that they are rigid-foldable. While relations for designing thin, quadrilateral-mesh origami have been developed [4], designing thick versions has proven challenging. This chapter demonstrated that in some cases it is acceptable to require only these three classes of constraints to design rigid-foldable networks of linkage-based vertices. For patterns of quadrilateral panels and fourth-order vertices like the linkage tessellation, a method for ensuring that these constraints were met throughout the pattern was derived and demonstrated to allow design of the deployed mechanism shape. This method was used to design the mechanisms shown in Figure 3.8. Details on its construction are provided in Appendix A.

These results are significant because they demonstrate a method for expanding the limited number of thick origami models and utilizing them in the design of thick tessellations. With larger numbers of thick, rigid-foldable models, more opportunities for finding mechanisms with suitable characteristics will be available to designers. This work also shows that these thick-origami vertices can be incorporated into quadrilateral-mesh tessellations with significant flexibility in their geometry.

A potential application for this method is the design of deployable, curved structures such as rover wheels [69]. Deployable wheels for rovers are desirable because space is limited in a lander and reducing wheel size can save significant amounts of volume [70]. Another advantage of wheel deployment is the increased cross-country ability that larger wheels can provide [71]. A wheel approximated by a linkage tessellation of constant curvature is shown in Figure 3.9. Details on its construction are provided in Appendix A. This particular model is capable of folding a 15 centimeter diameter wheel into approximately one sixth the original volume. Because of the method used, the design also has significant design flexibility. If wider wheels are desired, additional rows can be added. If the wheel profile needs to be more or less jagged, the vertex geometry can be adjusted and vertices can be added or removed.
Addressing the challenges of accommodating thickness and ensuring rigid-foldability will enable novel deployable mechanisms because of the solution simplicity. It is hoped that the increased accessibility of methods for developing variations of overconstrained linkages will enable knowledge in origami to be exploited to create systems not previously possible.
CHAPTER 4. A PSEUDO-RIGID BODY MODEL FOR COMPLIANT EDGE PANELS IN ORIGAMI-INSPIRED MECHANISMS

4.1 Introduction

Certain applications for origami-inspired mechanisms may call for a crease pattern geometry that is not perfectly rigid-foldable. In such a case, understanding the effect that this has on the actuation of the mechanism will be important for such tasks as designing the panels in the mechanism and the devices to actuate the mechanism. Allowing a panel to flex would enable the pattern to move as if the compliant panel had been removed. The effect of this panel removal can be predicted using methods described in Chapter 2. However, a compliant panel would also increase the required actuation force and internal stresses. The desirability of this trade-off will depend on the specific application.

The introduction of this compliant panel could occur in a variety of ways. For example, a panel at the edge of the pattern could be made compliant. The compliant panel could also be included at an interior location. However, this panel may also require other alterations such as slits to allow enough flexibility in the panel to prevent tearing or yielding.

In this chapter, methods will be discussed for modeling the force and deflection characteristics of flexible panels to enable designers to accurately estimate the characteristics of compliant panels in origami-inspired design. Analytical models will first be developed for different loading conditions. Finite element models and experimental data will also be collected to verify the analytical results. Lastly, these results will be used to create a pseudo-rigid-body model that can accurately reflect the force and deflection characteristics of these panels without requiring a difficult analysis.

Developing this pseudo-rigid-body model is significant because it will permit designers to use a simple model to describe the complex motion of a deflecting panel [72] in a mechanism that is not rigid-foldable. Typically, determining the forces needed to actuate a non-rigid-foldable mech-
anism would require the construction of complicated computer models or prototypes. However, a pseudo-rigid-body model would allow a simple, rigid-body mechanism to replace the deflecting panel in large origami models. This substitution would facilitate optimization of compliant panel characteristics and location.

4.2 Background

Significant amounts of research have been done on the use of pseudo-rigid-body models for designing compliant mechanisms. Roach and Howell [73] describe the design of a compliant, over-running clutch and the use of this type of model in its design. Another application is discussed by Chen et al. [74] In this article, the use of the pseudo-rigid-body model in the design of tri-stable mechanisms is described. Lusk and Howell [75] also discuss the design of a spherical micromechanism using this approximating model.

Several authors have also discussed the possibility of creating pseudo-rigid-body models for compliant panels. One important application of this model to membranes is discussed by Bandopadhya and Njuguna [76]. Their work illustrates how a pseudo-rigid-body model can be used to approximate the bending behavior of a thin membrane. Sönmez and Tutum [77] also provide examples of the use of a pseudo-rigid-body model in a buckling problem. The use of the model simplified their analysis and provided a more effective solution. A dynamic application of this model was made by Casals-Terre and Shkel [78]. Their work focused on the design and actuation of a compliant switch. The effects of vibration on the switch were also investigated.

4.3 Energy Method for Deflection Characteristics

One way the force-deflection characteristics of a compliant panel in an origami vertex can be found is through its deformation energy. Given the energy in the panel as a function of displacement, the force at that displacement can be calculated using Castigliano’s first theorem. Applying this technique to compliant panels in origami requires an expression for the strain energy in the panel.

If the thickness of the panel is small in comparison with its other dimensions, then panel motion can be assumed to be largely due to bending [79, 80]. Furthermore, if the forces acting
on the panel are small, the deformed shape of the panel can be approximated by a developable surface [81]. A developable surface is defined as a surface where no stretching occurs in the plane of the material during motion and so approximately represents the motion of a compliant origami panel [82].

If axial stretching is small in comparison with bending strains, then the majority of energy stored in the panel will be due to bending. Solomon et al. summarizes this by stating that “for sufficiently thin plates and appropriate boundary/loading conditions, the plate may be treated as in-extensible, its shape governed by the stored bending energy and boundary conditions” [83]. If the surface is developable, one of the principal curvatures at any point will be zero [84,85] and the elastic energy stored in the plate will be proportional to the remaining principal curvature. This energy, $V$, is shown by Starostin and van der Heijden [86] to depend on the flexural rigidity, $D$, and the curvature $\kappa$ of the panel as

$$V = \frac{1}{2} D \int_A \kappa^2 dA$$

(4.1)

where

$$D = \frac{E h^3}{12 (1 - \nu^2)}$$

(4.2)

Here, $h$ is the thickness of the panel, $E$ is the Young’s Modulus and $\nu$ is Poisson’s Ratio. The integration is taken over the area of the surface. If the adjacent rigid panels force the compliant panel edges to continue to point towards the origami vertex, ruling lines drawn from the vertex to the panel’s edge will remain straight if the outer edge is assumed to be unrestrained.

If a conical ruled surface is given by

$$S = r \tilde{\phi}(\theta) = \begin{bmatrix} r & \theta & \phi(\theta) \end{bmatrix}$$

(4.3)

where $\tilde{\phi}(\theta)$ is a vector pointing in the direction of $\phi$ and $\theta$ and $r$ is its length, then its only non-zero principal curvature can be found by placing a Cartesian coordinate frame at some point $P$ on the
surface. If the x-axis points toward the point \( r = 0 \) then the curvature at \( P \) is given by

\[
\kappa = \frac{\partial^2 z}{\partial y^2} \cdot \frac{1}{\left[1 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{3/2}} \quad (4.4)
\]

If the derivatives in Equation 4.4 are converted to spherical coordinates, we have

\[
\frac{\partial z}{\partial y} = \sin \theta \sin \phi \frac{\partial z}{\partial r} + \frac{\cos \theta}{r \sin \phi} \frac{\partial z}{\partial \theta} + \frac{\sin \theta \cos \phi}{r} \frac{\partial z}{\partial \phi} = -\frac{\partial \phi}{\partial \theta} \quad (4.5)
\]

With reference to this new frame, \( \theta = 0, \phi = 90^\circ, \frac{dz}{dr} = 0 \) and \( z = r \cos \phi (\theta) \). Similarly,

\[
\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left( -\frac{\partial \phi}{\partial \theta} \right) = -\sin \theta \sin \phi \frac{\partial^2 \phi}{\partial \theta \partial r} - \frac{\cos \theta}{r \sin \phi} \frac{\partial^2 \phi}{\partial \theta^2} - \frac{\sin \theta \cos \phi}{r} \frac{\partial^2 \phi}{\partial \phi \partial \theta} = -\frac{1}{r} \frac{\partial^2 \phi}{\partial \theta^2} \quad (4.6)
\]

using the same specific conditions used before. Substituting Equations 4.5 and 4.6 into Equation 4.4 yields

\[
\kappa = \frac{1}{r} \frac{-\frac{\partial^2 \phi}{\partial \theta^2}}{\left[1 + \left(\frac{\partial \phi}{\partial \theta}\right)^2\right]^{3/2}} = \frac{1}{r} \frac{-\phi''(\theta)}{\left[1 + (\phi'(\theta))^2\right]^{3/2}} \quad (4.7)
\]

which suggests that the curvature is inversely proportional to the distance from the cone vertex since neither \( \phi'(\theta) \) nor \( \phi''(\theta) \) is a function of \( r \). The radius of curvature can then be written as

\[
\rho = \rho_0 r = \rho_0(\theta) r \quad (4.8)
\]

where \( r \) is the distance from the pattern vertex and \( \rho_0 \) is the radius of curvature of the surface at a unit distance from the pattern vertex. Inverting \( \rho \) will give the needed expression for \( \kappa(r, \theta) \). \( \kappa \) as a function of \( \theta \) can be written as

\[
\kappa(r, \theta) = \frac{1}{\rho_0(\theta) r} \quad (4.9)
\]

Substituting this expression into the Equation 4.1 yields

\[
V = \frac{1}{2} D \int_{\phi_1}^{\phi_2} \int_{r_1}^{r_2} \left( \frac{1}{\rho_0(\theta) r} \right)^2 r dr d\theta = \frac{1}{2} D \int_{\phi_1}^{\phi_2} \frac{1}{\rho_0^2(\theta)} \ln \frac{r_2(\theta)}{r_1(\theta)} d\theta \quad (4.10)
\]
because both $r_1$ and $r_2$ can be functions of $\theta$. The exact value of $r_1$ depends on the geometry of the panel involved but can be determined from relations given by Mora et al. [87]. However, if for simplicity the panel is assumed to have inner and outer circular edges the expression reduces to

$$V = \frac{1}{2} D \ln \frac{r_2}{r_1} \int_{\phi_1}^{\phi_2} \frac{1}{\rho_0^2(\theta)} d\theta$$

(4.11)

### 4.3.1 Displacement-Based Treatment

Further work to determine the energy stored in a conical origami panel depends on the nature of the loading applied to the panel to cause the deflection. This section evaluates the case where a panel with constant $r_1$ and $r_2$ is loaded by a specified displacement where the angle between the edges $\phi_1$ and $\phi_2$ is reduced. In this condition, the panel will buckle as shown in Fig. 4.1. Because the shape it assumes will minimize the bending energy stored in the panel, stretching will be reduced since it requires more energy than bending [87]. If the surface is assumed to be developable, one of the principal curvatures at any point on the surface will be zero. The other principal curvature can vary but must enable the surface to extend between the two edges $\phi_1$ and $\phi_2$. However, because the lowest energy configuration for a bent developable panel is one that includes constant curvature the panel will form a sector of a circular cone [81]. If the remaining principal curvature is assumed to be constant, then Equation 4.11 can be computed. This yields
\[ V = \frac{D(\phi_2 - \phi_1)}{2\rho_0^2} \ln \frac{r_2}{r_1} \quad (4.12) \]

The value of \( \rho_0 \) will depend on the deflection that the panel experiences. This deflection can be described in terms of the angle \( \gamma \) between the surface tangent plane at edge \( \phi_1 \) and the plane including edges \( \phi_1 \) and \( \phi_2 \).

Let \( \delta \) represent the distance between point A and B where A and B are both a unit distance from the center of rotation of the panel shown in Fig. 4.1. If \( \theta_1 \) represents the initial angular distance between \( \phi_1 \) and \( \phi_2 \), then \( \delta \) is given by

\[ \delta = 2 \sin \frac{\theta_1}{2} \quad (4.13) \]

If \( \theta_2 \) is the angular distance between A and B once deformation has occurred, then we can write

\[ \theta_2 = 2 \arcsin \frac{\delta}{2} \quad (4.14) \]

The vector position of A is given by

\[ \vec{a} = \frac{\delta}{2} \hat{i} + \sqrt{1 - \frac{\delta^2}{4}} \hat{j} \quad (4.15) \]

A vector of length \( l \) normal to the surface at A is given by

\[ \vec{N} = \left( -l \sin \gamma \cos \frac{\theta_2}{2} \right) \hat{i} + \left( l \sin \gamma \sin \frac{\theta_2}{2} \right) \hat{j} - (l \cos \gamma) \hat{k} \quad (4.16) \]

Since \( l \) can be found to be

\[ l = \frac{r}{\sqrt{1 - r^2}} \quad (4.17) \]

we can substitute Equation 4.13 and Equation 4.14 into Equation 4.16. The result must have an x-component equal to zero since it extends to the plane of symmetry of the panel. Setting this
component of $\vec{a}$ to zero yields

$$\sin \frac{\theta_1}{2r} \sqrt{1 - r^2 \cos^2 \gamma} = \sin \gamma$$ \hspace{1cm} (4.18)

This relationship allows $r$ to be plotted in terms of $\gamma$ as shown in Fig. 4.2. If the radius of the cone is known, then the curvature $\kappa_0$ of the surface at A can be calculated by

$$\kappa_0 = \frac{1}{r \sqrt{1 - r^2}} = \sqrt{\frac{1 - r^2}{r^2}}$$ \hspace{1cm} (4.19)

Once the curvature is known, the bending energy in the surface can be plotted in terms of $\gamma$ using Equation 4.12, as shown in Fig. 4.3. Differentiating this plot with respect to $\gamma$ gives the torque required to deflect the panel, as plotted in Fig. 4.4. While these equations are capable
of generating a plot of torque with respect to deflection, the relationships are implicit and more difficult to manipulate and solve. In addition, these relationships assume constant \( r_1 \) and \( r_2 \) and so only apply to panels with circular edges.

### 4.3.2 Force-Based Treatment

An alternate model of the force-deflection behavior of a compliant panel can be found if the force applied is known instead of the displacement. However, relating the displacement to the force requires a relationship between load and bending deformation.

If the panel is deflected by two known collinear forces instead of known displacements as studied previously, then the problem of finding the deflection characteristics of the panel can be approached using a similar method. First, applying two collinear forces at A and B will produce internal moments throughout the panel that will vary depending on the distance of the section from A or B. If the edges AC and BC are assumed to be rigid, then the panel will take a developable shape during deflection because this will be the lowest energy mode of buckling [87]. Because the adjacent panels in the vertex will constrain the edges \( \phi_1 \) and \( \phi_2 \) to point towards the sector center of rotation, C, the deflected surface will be a form of cone. For a cut \( CE \) along a ruling line of the cone, the force at A can be converted into moments and loads described about a local coordinate system E aligned with the tangent plane of the surface at that point. This cut is shown in Fig. 4.5 and Fig. 4.6.

For a thin plate in bending where the thin dimension lies in the y-direction, one face of the plate will be tensed and the other compressed if the moment vector lies in the x-z plane. However,
if the plate is assumed to deflect as a ruled surface, then the strain in the x-z plane will be zero. Hooke’s law yields

\[
\varepsilon_x = 0 = \frac{1}{E} (\sigma_x - \nu(\sigma_y + \sigma_z))
\]

(4.20)

\[
\sigma_x = \nu\sigma_z
\]

(4.21)

if \(\sigma_y\) is assumed to be small and the moment is applied along the x-axis. The strain in the z-
direction can then be written as

\[ \varepsilon_z = \frac{1}{E} (\sigma_z - \nu(\sigma_x + \sigma_y)) \]  
\[ \sigma_z = \frac{E}{1 - \nu^2} \varepsilon_z \]  

This relation can be equated to an expression of moment and written as

\[ \sigma_z = \frac{M y}{I} = \frac{E}{1 - \nu^2} \varepsilon_z \]  
\[ M = \frac{E w h^3}{12(1 - \nu^2)} \frac{\varepsilon_z}{y} \]

where \( w \) is the width of the beam, \( h \) is its thickness and \( y \) is a certain distance from the neutral axis. For a short segment of beam, the strain and radius of curvature can be related by

\[ \frac{\Delta l}{y} = \frac{l_0}{R} = l_o \kappa \]  
\[ \kappa = \frac{\Delta l}{l_0 y} = \frac{\varepsilon}{y} \]

as shown in Fig. 4.7 where \( \kappa \) is the curvature of the beam.

Substituting this expression into Equation 4.25 yields

\[ M = \frac{E w h^3}{12(1 - \nu^2)} \kappa \]
Figure 4.8: The buckled configuration of a thin segment of a thick plate loaded at its edge.

This expression can now be used to find the internal moment along some ruling line in a conical surface. However, because Equation 4.28 was derived assuming constant curvature, integration will be required. For a thin strip of the panel as shown in Fig. 4.8, the curvature can be assumed to be constant since $dr$ is small. The moment at point $E$ is given by

$$dM = \frac{Eh^3 dr}{12(1-v^2)} \kappa(r) = \frac{Eh^3 dr}{12(1-v^2)} \frac{\kappa_0(\theta)}{r}$$

(4.29)

Summing these moments from $r_1$ to $r_2$ (inner to outer edge) yields the total internal moment $M$.

$$M = \int_{r_1}^{r_2} \frac{Eh^3 dr}{12(1-v^2)} \frac{\kappa_0(\theta)}{r} = \frac{Eh^3 \kappa_0(\theta)}{12(1-v^2)} \int_{r_1}^{r_2} \frac{dr}{r}$$

(4.30)

$$= \frac{Eh^3 \kappa_0(\theta)}{12(1-v^2)} \ln \frac{r_2}{r_1}$$

(4.31)

where $\kappa_0(\theta)$ is the curvature at a unit radius from the origin.

If the panel is cut along ruling line $CE$, then the internal moment at the cut due to a load along the edge can be found. Let the load $\vec{w}$ be a force a unit distance from $C$ acting perpendicular to the edge $\phi_1$ in the plane of $\phi_1$ and $\phi_2$. In this situation, the load $\vec{w}$ will produce reaction moments and shears at the centroid of the cut. Assuming shear forces are negligible, only the moments produced will be significant. This moment will have three components relative to the axis at $E$ as shown in Fig. 4.9. The component perpendicular to the surface will produce negligible deflection because the stiffness of the panel resisting this moment is large by comparison. The component acting perpendicular to the cut face will not be considered because it will cause twisting, which
Figure 4.9: The axis used to study moments at the cut $CE$.

will not occur if the panel is assumed to be developable. Therefore, only the component acting along the line $CE$ will be considered.

If the position of $A$ is defined by $\vec{A}$, the centroid of the cut $\vec{E}$ and the load $\vec{w}$ are given by

$$\vec{A} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\vec{E} = \begin{bmatrix} \cos \phi \cos \theta & -\cos \phi \sin \theta & \sin \phi \end{bmatrix} \frac{r_2(\theta) + r_1(\theta)}{2}$$

and

$$\vec{w} = \begin{bmatrix} 0 & -w & 0 \end{bmatrix}$$

where these vectors are written in a Cartesian frame. Here, the x-axis points along AC and positive $z$ is directed upwards. If the origin is located at the center of rotation of the sector, $C$, then the moment $\vec{M}$ that the force at $A$ produces about a ruling line of the surface is given by

$$\vec{M} = \vec{w} \times (\vec{E} - \vec{A}) = \vec{w} \times \left( \frac{r_2(\theta) + r_1(\theta)}{2} \vec{e} - \vec{A} \right)$$

where $\vec{e}$ is a unit vector along $\vec{E}$. The component of $\vec{M}$ along the cut line is given by

$$M_{CE} = \vec{e} \cdot \vec{w} \times \left( \frac{r_2(\theta) + r_1(\theta)}{2} \vec{e} - \vec{A} \right) = -w \sin \phi \quad \text{or} \quad w \sin \phi$$
since the sign comes from the component of \( \vec{M} \) being in the opposite direction of \( \vec{e} \). Equating this expression to Equation 4.31 yields

\[
w \sin \phi(\theta) = \frac{Eh^3 \kappa_0(\theta)}{12(1 - v^2) \ln \frac{r_2(\theta)}{r_1(\theta)}} (4.37)\]

\[
w \sin \phi(\theta) = -N^{-1} \kappa_0(\theta) \quad \text{where} \quad N = \frac{12(v^2 - 1)}{Eh^3 \ln \frac{r_2(\theta)}{r_1(\theta)}} (4.38)\]

Substituting Equation 4.7 into this relation for \( \kappa_0(\theta) \) where \( r = 1 \) yields

\[
Nw \sin \phi(\theta) = \frac{\phi''(\theta)}{\left[1 + (\phi'(\theta))^2\right]^{3/2}} (4.40)\]

Converting this equation to state-space form yields

\[
\eta'(\theta) = Nw \sin \phi(\theta) \left[1 + \eta^2(\theta)\right]^{1/2} (4.41)\]

\[
\phi'(\theta) = \eta(\theta) (4.42)\]

These equations can be solved to determine the deflected shape of the panel given its geometry and material properties contained in \( N \). Because \( N \) contains \( r \) as a function of \( \theta \), this relationship can describe panels of arbitrary profile. For example, if we consider a triangular panel with sides of length one and two and an included angle of 90° then the shape of this panel can be found for a certain initial deflection \( \eta(0) \) at the edge. If the panel is constructed of steel with \( h = 0.001 \text{m} \), \( E = 207e9 \text{Pa} \), \( \eta = 0.45 \) and \( \eta = \tan(85^\circ) \), then we obtain the shape shown in Fig. 4.10.

### 4.4 Pseudo-Rigid-Body Model

This method for obtaining the deflected shape of an origami panel is limited in its usefulness because extracting the force-deflection behavior of the panel is not straightforward. To make the process of calculating the force-deflection behavior of these panels simpler, a pseudo-rigid-body model similar to others presented for compliant mechanisms [88] will be created.
Figure 4.10: A sample curve obtained from a triangular panel with sides of length one and two and including a right angle.

Figure 4.11: An image of the pseudo-rigid-body model used to approximate the behavior of a compliant panel buckling under side loads.

The purpose of this model is to approximate the behavior of the deflecting panel by the deflection of a pair of pseudo-rigid panels connected by a hinge. An image of the proposed model is shown in Fig. 4.11. The combined sector angles of the two panels will equal the total sector angle of the original panel. A torsional spring will be placed at the joint between the two pseudo-rigid panels to simulate the deflection characteristics of the panel. The torque-deflection behavior of the spring will be described by a second-order polynomial to improve the fidelity of the model. The formula used to calculate the spring torque is given by
\[ T = c_2 \Delta^2 + c_1 \Delta + c_0 \]  

(4.43)

where \( \Delta \) is the change in the angle between the two pseudo-rigid panels. \( \Delta \) is zero when the angle is 180°. The two pseudo-rigid panel angles \( \lambda \) and \( \mu \) and the three constants \( c_0, c_1 \) and \( c_2 \) can be found through a least-squares fit with data obtained from Equation 4.41 and Equation 4.42.

Because the results obtained from solving these two differential equations will have to be applicable to a variety of panel geometries and panel materials, they will be solved for a predetermined panel thickness and material and a specific set of panel shapes. If the deflection characteristics of a panel of different material or thickness is desired, then the value of \( N \) will be different. However, if the value of \( w \) is adjusted appropriately, the combined value of \( Nw \) will remain the same and the solution curve will be identical. This is allowable because if the panel is made thicker, for example, then the load at the edge will be larger to achieve the same deflected shape.

The shape of the panel cannot be dealt with as easily as the material properties or the thickness because the terms \( r_1 \) and \( r_2 \) vary with \( \theta \) depending on the edge shape. Consequently, Equation 4.41 and Equation 4.42 will need to be solved for a series of different shapes and the resulting constants \( \lambda, \mu, c_0, c_1 \) and \( c_2 \) tabulated. To reduce the number of shapes that will be analyzed, only triangular shapes described by an angle \( \alpha \) and a side length ratio \( s/t \), as shown in Fig. 4.12 are considered. For all the panels studied, \( s = 1 \text{ m} \) since only the ratio between \( s \) and \( t \) is important. Tables of constants for this pseudo-rigid-body model are included in Appendix B.

While this model can predict deflection behavior in certain circumstances, there are important limitations on its applicability. Its derivation assumes that the panels are deflecting elastically and that no yielding occurs. The model also assumes that deflections do not drive the original panel beyond the point where the fixed edges of the panel become vertical. Another assumption made is that the loads are applied slowly enough that panel mass is insignificant.

### 4.5 Finite Element Model

A finite element model of the deflection behavior of the panels was also constructed to validate the analytical model and its accompanying pseudo-rigid-body model. This model was developed in ANSYS and was used to simulate the behavior of a few specific panel geometries.
made from polypropylene. An area as shown in Fig. 4.12 was meshed using plate elements with the properties shown in Table 4.1. Nodes on side AC were fixed in all translational degrees of freedom and displacement loads were applied to nodes in side BC such that it rotated about C and did not distort. A small bias load of 1 N was distributed over the interior nodes and directed out of plane so that the panel would buckle rather than compress. A convergence study was not performed. The results of these analyses are shown along with the experimental results obtained in the next section.

### 4.6 Experimental Results

To validate the results obtained through the analytical methods described in Section 4.3.2, data on the force-deflection behavior of triangular panels of different shapes were collected and compared to the analytical results. A test fixture was designed and built to accomplish this. An image of the device used is shown in Fig. 4.13. The tester consists of two arms joined with a hinge allowing them to close like a pincer and buckle a triangular plate held by the teeth along each arm. The force required to deflect the panel is measured by connecting a force sensor to a

<table>
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Figure 4.13: The deflection tester used to determine the relationship between the change in sector angle and the edge load \( w \).

cable extending from the tip of one arm to the other. Because this force does not act perpendicular to the arms, all the force measurements taken must be modified to find the load \( w \) determined by the equations in Section 4.3.2. If the angle between the two arms \( \alpha \) is known, then the load perpendicular to the arm whose component lies along the cable is given by Equation 4.44. Here, \( T \) is the measured force, \( \alpha \) is the angle between the arms and \( F \) is the force normal to the arms of the tester whose component is \( T \). The fact that the cables are not attached a unit distance from the axis of rotation of the tester must also be accounted for when determining the load \( w \).

\[
F = \frac{T}{\cos \frac{\alpha}{2}} \quad (4.44)
\]

Measurements of the deflection-angle characteristics of polypropylene sheets of different shapes were made using this tester. The plot in Fig. 4.14 describes the deflection characteristics predicted and observed for a panel of polypropylene with a thickness of 1.27 mm and properties shown in Table 4.1. All error bars show 95% confidence intervals for four samples. The panel used had shape values of \( s/t = 1 \) and \( \alpha = 90^\circ \). The measured values are shown by the red dotted line and the predicted values are shown with the blue solid line. Figure 4.15 shows a similar plot for a panel of the same material and thickness but where \( \alpha = 30^\circ \) and FEA results are included for comparison.
Figure 4.14: The analytically and experimentally derived load-deflection curves for a panel with a side ratio of $s/t = 1$ and a sector angle of $\alpha = 90^\circ$.

Figure 4.15: The analytically and experimentally derived load-deflection curves for a panel with a side ratio of $s/t = 1$ and a sector angle of $\alpha = 30^\circ$.

Similar measurements were made on panels made from the same sheet material but where $s/t = 1.5$. The response of panels where $\alpha = 34^\circ$ and $\alpha = 90^\circ$ are shown in Fig. 4.16 and Fig. 4.17. This new $\alpha$ was computed after the test panels were cut and so was not matched exactly to previous panels. In all cases, the panel appears to require a certain buckling load before motion occurs. After buckling the force-deflection relationship is relatively flat.

These experimental results roughly correspond with the estimates of force given by Equation 4.41 and Equation 4.42. While the magnitudes of force predicted to deflect the panel are relatively accurate, there are some differences. These differences include somewhat different values for the onset of buckling and some spread in the steepness of the curve after buckling. This
Figure 4.16: The analytically and experimentally derived load-deflection curves for a panel with a side ratio of $s/t = 1.5$ and a sector angle of $\alpha = 34^\circ$.

Figure 4.17: The analytically and experimentally derived load-deflection curves for a panel with a side ratio of $s/t = 1.5$ and a sector angle of $\alpha = 90^\circ$.

may be due to the fact that the panels did not seat perfectly in the jaws of the tester. Some amount of force had to be exerted before the panel was completely held by all the teeth. Another factor that may have contributed was the need to quickly take a reading before the panels relaxed too much. It was observed that after the panel was deflected a certain distance, the force required to keep it in that position diminished over time. A force reading was taken immediately after deflecting the panel a given distance to collect the data shown. Finally, the fact that the panel was assumed to be very thin during the derivation of the analytical equations will contribute to any experimental error observed since this cannot be completely satisfied.
Some differences also exist between the analytical and finite-element simulation results. While both sets of data have approximately the same slope, they are offset by a constant value with the analytical results always being larger. This difference may be due to the bias load of 1 N applied to initiate the buckling. Such a load would diminish the force required to buckle the panel.

4.7 Conclusion

This chapter presents analytical models of the deflection behavior of origami panels subjected to two different loading conditions. The models presented in Sections 4.3.1 and 4.3.2 apply to displacement and force loading conditions, respectively. Some key assumptions in these models are that the hinged edges of the panels are rigid and that they deflect as developable surfaces. Experimental data on the force-deflection behavior of polypropylene panels is also collected to corroborate the results obtained for the force-based loading conditions in Section 4.3.2. The force-deflection equations were then used to create a pseudo-rigid-body model to simplify the analysis of compliant panels. Corresponding tables of constants for this model are included in Appendix B.

The combination of analytical formulas and experimental results presented in this chapter suggest that bending in triangular origami panels can be modeled in a relatively simple way. Using a pseudo-rigid-body model to approximate the force-deflection behavior of a compliant origami panel has the potential to reduce the complexity that often accompanies dealing with finite-element models or experimental results developed to predict mechanism behavior [72]. This could be particularly useful when optimizing the location of the compliant panel, for example. It will also facilitate the design of pseudo-rigid-foldable origami by allowing panel deformation to be incorporated into the design analyses.

The pseudo-rigid-body model developed for compliant panels has the potential to simplify a force-deflection analysis of origami. Consequently, the design process of any mechanisms incorporating compliant origami may also be benefited. The design of rigid-foldable origami may also benefit from the ability to include compliant panels providing extra degrees of freedom. While approximations made by this model may cause some inaccuracies in the model results, it represents a possible method for simplifying the design of compliant origami.
CHAPTER 5. THREE APPROACHES FOR MANAGING STIFFNESS IN ORIGAMI-INSPIRED MECHANISMS

5.1 Introduction

One concern that applies specifically to large, light-weight structures is their ability to maintain their designed shape under their loads. This designed deployed shape could be specified by a number of factors. For an antenna, the shape would be governed by the signal wavelength it was designed to transmit or receive. The shape of a solar collector or optical array also needs to be closely controlled to achieve their function.

Nevertheless, controlling the shape of the array can be particularly challenging when dealing with origami panels deployed over a large area. Origami-inspired mechanisms often consist of a collection of thin panels connected with hinges that are folded into a small package for transportation. When ready to deploy, these folded mechanisms are generally deployed into a large, relatively flat and thin configuration [89]. While the packing efficiency is high, the resulting deployed configuration is vulnerable to shape distortion because of its low bending stiffness.

In a space application, this distortion could be introduced by a variety of factors, including the manufacturing tolerances and materials used to construct the array [90]. Another important factor is the load placed on the array by reaction wheels used to control the spacecraft and other mechanisms such as cryo-coolers [91, 92]. Thermal distortions can also influence the deployed configuration of the structure [93]. These factors make it essential to design deployable structures with enough stiffness to resist applied loads and maintain their desired configuration.

This chapter describes three methods of changing the deployable geometry and adjusting the stiffness of a large, origami-inspired deployed structure. This will allow origami models to transition from the large-deformation category to the small-deformation category in Figure 1.1. Folding methods are used so that the advantages of origami systems are retained and to maintain compatibility with the methods used to construct the origami system. Procedures for assessing
the potential change in stiffness produced by this method are examined. One method involves partially folding the entire pattern in order to enhance stiffness. The use of a rib that is yielded into place along a crease is also examined. Finally, the use of panels with expandable, foam cores that increase stiffness is proposed. In all cases, the positioning and proportions of the stiffeners are considered to facilitate the optimal design of these components.

5.2 Background

Understanding the stiffness of a deployed structure is a fundamental problem that has been investigated by several researchers [44, 94, 95]. Tan and Pellegrino [96] extensively studied the properties of a stiffener located at the edge of a flexible dish. Their work investigated specific problems with this stiffener such as flexibility, vibration and stiffness. Research on the use of stiffeners at the hinge of a deployable solar array has also been conducted by Jorgensen et al. [97] Their work focused on the dynamics of the deployment process but still illustrated the use of such stiffeners to obtain specific behavior in their deployable mechanism.

Research performed by Schenk and Guest [98] specifically focused on the modeling of stiffness in textured origami sheets. The difference in the stiffness of these patterns in different modes of deformation was considered to give insight into origami mechanism design. A similar analysis of the stiffness of different modes of deformation was performed by Filipov et al. [41] while specifically examining variations of origami zippers. Their research highlighted the ability of these origami mechanisms to provide both a compliant, deployable mode and a stiff load-bearing mode.

Work has also been performed on characterizing origami tessellations as a meta-material and computing its mechanical characteristics. For example, Cheung et al. [99] discuss the possibility of using cellular origami tessellations to enable the construction of light, strong structures that can retain deployability. Their work discusses both the kinematics and mechanics of the resulting structures and its capacity to resist loads as well as permit motion. Similarly, Wonoto et al. [39] investigate the ability of simple origami tessellations to support their own weight when deployed. They also conduct an optimization of the geometry to meet this structural need.
5.3 Methods for Managing Pattern Stiffness

When constructing an origami-inspired deployable mechanism, many factors need to be considered before decisions about its design can be made, including the inherent stiffness of the deployed pattern and its complexity. The complexity and usefulness of additional stiffeners would also be a factor. In this section, different methods for managing the stiffness of an origami pattern are discussed and their relative advantages and disadvantages are listed to facilitate their design.

5.3.1 Partially Folded Patterns

The first method for managing pattern stiffness is partially folding a pattern as shown in Figure 5.1. Making this alteration to the pattern can be advantageous because it requires no modification to the device design and so can be implemented at a low cost. However, it may necessitate other changes if the original surface was required to be perfectly flat. Nevertheless, if this partial folding is minimized, any negative side effects may reduce to acceptable levels. To quantify how much folding is necessary to sufficiently increase stiffness, the kinematics of individual origami vertices must be modeled.

It has been demonstrated [1] that the fourth-order origami vertex shown in Figure 5.2a can be modeled using the fold-angle multiplier

$$\mu = \frac{\sin \left[ \frac{1}{2} (\alpha_1 + \alpha_2) \right]}{\sin \left[ \frac{1}{2} (\alpha_1 - \alpha_2) \right]}$$  \hspace{1cm} (5.1)

and the relationship between major and minor fold angles

$$\tan \left( \frac{1}{2} \gamma_2 \right) = \mu \tan \left( \frac{1}{2} \gamma_1 \right)$$  \hspace{1cm} (5.2)
Figure 5.2: (a) A flat vertex and a (b) partially folded version [1].

Here, $\alpha_1$ and $\alpha_2$ are the sector angles of two panels not divided by the minor crease line, and $\gamma_1$ is the dihedral angle between the panel containing $\alpha_1$ and $\alpha_2$. The dihedral angle $\gamma_2$ is the angle between the panels containing $\alpha_1$ and $\alpha_2$. The angles $\gamma_1$ and $\gamma_2$ are defined as zero when the pattern is flat and $\pi$ when it is fully folded. Opposite dihedral angles are equal, or $\gamma_1 = \gamma_3$ and $\gamma_2 = \gamma_4$.

If the folded angles $\gamma_1$ and $\gamma_2$ are limited to small angles where $\gamma \leq 0.2$ and $\tan x \approx x$ then we have

$$\gamma_2 \approx \mu \gamma_1$$

which demonstrates that at small dihedral angles, the major and minor dihedral angles $\gamma_1$ and $\gamma_2$ are related linearly. More generally, all the dihedral angles in a pattern composed of fourth-order vertices are linearly related. This is important because it permits the characterization of the area moment of inertia of any cross-section perpendicular to the face of the origami model.

If a cut is made using a plane perpendicular to the face of the origami, then the resulting cross-section will be composed of a certain number of connected segments. These segments will be approximately rectangular if the thickness of the panels is much less than its length. If these
segments are tilted at an angle $\theta$, then the area moment of inertia of the segment is given by

$$I = \frac{bt}{12} (t^2 \cos^2 \theta + b^2 \sin^2 \theta)$$  \hspace{1cm} (5.4)

where $b$ is the length of the segment, $t$ is its thickness and $\theta$ is its tilt angle as shown in Figure 5.3. The tilt angle is zero when the segment is completely horizontal. If we make the substitution $b = pt$, then we have

$$I = \frac{t^4}{12} p (\cos^2 \theta + p^2 \sin^2 \theta)$$  \hspace{1cm} (5.5)

where $p$ is the ratio of the width ($b$) to the thickness ($t$). If we find the ratio of a tilted segment’s area moment of inertia to its inertia when flat, $R$, we find that this ratio is

$$R = \cos^2 \theta + p^2 \sin^2 \theta$$  \hspace{1cm} (5.6)

This relationship shows that for large values of $p$ (i.e., the segment is very thin relative to its thickness), the increase in inertia is large. The potential gains in $I$ is illustrated by considering that if the maximum tolerated actuation angle is $\theta = 0.05$ radians, a width to thickness ratio of twenty ($p = 20$) would double the moment of inertia. If it were forty, the inertia ratio would quintuple.

### 5.3.2 Yielded Panel Stiffeners

Another possible mechanism for stiffening a panel is to add a plate that can be yielded at its crease and deployed to enhance stiffness. This concept is shown in Figure 5.4. Employing this type of stiffener promises some important advantages. First, the stiffener could be stamped out of the material and so would be relatively simple to add to a design. Such a stiffener would also
automatically lock into place following deployment. While deployment would be an issue due to the nature of the crease in the stiffener, many options exist for driving the motion [65, 100]. The gains in stiffness will be significant as long as the panels in the deployable are thin.

In this section, we will analyze a yielded stiffener and its ability to enhance the bending stiffness of an origami panel. A diagram of the cross-section of the stiffener in Figure 5.4 is shown in Figure 5.5. With the given geometry, it is possible to calculate the centroid of the section with respect to \( y \). If the panels are assumed to be thin, then the centroid for the cross-section in Figure 5.5 is located at

\[
c = \frac{h^2}{2w}
\]  

(5.7)

respectively where \( h, t \) and \( w \) are described in Figure 5.5. Knowing the centroid allows us to calculate the area moment of inertia of the cross-section. This inertia is

\[
I = \frac{1}{3}th^3 - \frac{1}{4}th^4 + \frac{1}{12}t^3w - \frac{1}{12}t^3h
\]  

(5.8)
using the parameters shown in Figure 5.5. If this inertia is divided by the area moment of inertia of the unmodified panel of width \( w \) and thickness \( t \), then we have

\[
R = \frac{I}{I_0} = 4r^2f^3 - 3r^2f^4 + 1 - f
\]  

(5.9)

if we make the parametric substitutions \( f = h/w \) and \( r = w/t \). These equations will be accurate until buckling occurs.

**Stiffener Crease Perforations**

Successfully implementing this stiffener type requires some method of deploying it and then locking it into place. One approach for accomplishing this is to perforate the crease about which the stiffener bends. This would provide a predictable bend location as well as a suitable restraining force once the stiffener is fully deployed. An image of the suggested perforations are shown in Figure 5.6.

A side effect of this perforation is that it predisposes the stiffener to failure in shear at the crease due to bending. The shear at the crease line is given by

\[
\tau = -\frac{VQ}{It}
\]  

(5.10)
where \( Q \) is derived for the specific geometry shown in Figure 5.4 and is given by

\[
Q = \frac{(w-h)th^2}{2w} \tag{5.11}
\]

This derivation is performed using the method described in [101]. Substituting these values for \( Q \) into Equation 5.10 yields

\[
\tau = 6 \frac{h^2(-w+2h)V}{t(-4h^3w+3h^4-t^2w^2+t^2wh)} \tag{5.12}
\]

\[
= 6 \frac{Vr(-1+2f)f^2}{t^2(-4r^2f^3+3r^2f^4-1+f)} \tag{5.13}
\]

where \( V \) is the shear load applied to the end of the beam. Since this shear stress is the stress felt by the crease if it were fully dense, it must be increased when material is removed. If the maximum tolerable shear stress is \( S \), then the maximum fraction of material that can be removed, \( H_{max} \), is given by

\[
H_{max} = 1 - \frac{\tau}{S} \tag{5.14}
\]

**Material Removal to Permit Deployment**

While \( H_{max} \) provides a maximum amount of material that can be removed, the minimum of this quantity should also be determined \((H_{min})\). Enough material should be removed so that the stiffener is loaded elastically while the crease is loaded plastically during deployment. If the crease is loaded elastically with a factor of safety \( N \), then the maximum tolerable moment, \( M_e \), is

\[
M_e = \frac{\sigma_yst^2}{6N} \tag{5.15}
\]

where \( \sigma_y \) is the yield strength of the material and \( s \) is the crease length. If material is removed from the cross-section and only a fraction of the original cross-sectional area \( st(1-H_{min}) \) remains, then the plastic moment for the crease is

\[
M_p = \frac{(1-H_{min})st^2}{4}\sigma_y \tag{5.16}
\]
Equating these two moments and solving for $H_{min}$ yields

$$H_{min} = 1 - \frac{2}{3N}$$

Therefore, if $N = 2$ then $2/3$ of the crease material should be removed. In this case, the crease will yield plastically while the stiffener will have a factor of safety of two against yielding.

### 5.3.3 Expanded Stiffener

A third option explored here for enhancing panel stiffness involves using an expanded stiffener as shown in Figure 5.7. In this device, the panel faces are made of a strong, dense material and the core (orange) is made of a low-density foam. This combination of materials has been shown to be very strong and stiff [102] compared to its weight and has been used widely in a variety of structures. However, the thickness of these sandwich structures can still be too large to effectively employ in origami-inspired mechanisms because of the large number of layers that can be stacked in its folded form. This volume can be reduced if the foam core can be compressed elastically in the stowed state as shown in Figure 5.7b.

Two distinct cases of this method are described. In the first, it is assumed that the weight of the expanded stiffener must equal that of the panel being replaced. In this scenario, as the amount of foam in the stiffener is increased, the thickness of the walls will decrease. The second case will consist of a panel with walls that are half the thickness of the original panel. This problem variation is considered because the panel walls may already be at a minimum thickness due to other constraints such as raw material dimensions. Consequently, the weight of this panel variation will
increase as the amount of foam is increased. In both cases, the area moment of inertia gain as well as other important characteristics can be found.

For a foam-core sandwich panel with the dimensions shown in Figure 5.7a, the panel rigidity, $EI$, modulus of elasticity of the foam core, $E_c$, and panel mass, $m$, are given by [99]

$$EI = \frac{E_f b T^3}{6} + \frac{E_c b C^3}{12} + \frac{E_f b T (C + T)^2}{2}$$ (5.18)

$$E_c = E_s \left(\frac{\rho_c}{\rho_s}\right)^2$$ (5.19)

$$m = b C L \rho_c + 2 b L T \rho_f$$ (5.20)

where $b$ is the panel width, $T$ is the thickness of the sandwich face, $C$ is the thickness of the foam core, $\rho_c$ is the density of the foam core, $\rho_s$ is the density of the foam material, $\rho_f$ is the density of the face material, $E_f$ is the Young’s Modulus of the face, $E_c$ is the modulus of the foam core and $E_s$ is the modulus of the core material. If we use Equation (5.19) to eliminate $E_c$ in Equation (5.18) and make the substitution $\delta = \rho_c / \rho_s$, then the panel rigidity can be expressed as

$$EI = \frac{1}{12} b C^3 \delta^2 E_s + \frac{1}{2} b C^2 E_f T + b C E_f T^2 + \frac{2}{3} b E_f T^3$$ (5.21)

The area moment of inertia can be found if we compare the sandwich panel to another made solely of the face material. $E = E_f$ in this case and the area moment of inertia is

$$I = \frac{b C^3 \delta^2 E_s}{12 E_f} + \frac{1}{2} b C^2 T + b C T^2 + \frac{2 b T^3}{3}$$ (5.22)

If the initial solid panel has a thickness of $t$, then its area moment of inertia is

$$I_0 = \frac{1}{12} b t^3$$ (5.23)

However, before a gain $R$ can be calculated, a relationship between $T$ and $t$ must be found. If the mass of the initial panel and the sandwich panel are equal, then we can use Equation (5.20)
to write the following relationship and solve for \( t \).

\[
m(b, L, C, T)|_{C=0,T=t/2} = m(b, L, C, T)
\]

\[
t = \frac{C\delta \rho_c + 2T \rho_f}{\rho_f}
\]

(5.24)

Dividing Equation (5.22) by Equation (5.23) and substituting Equation (5.24) to eliminate \( t \) yields

\[
R_a = \frac{\rho_f^3 (C^3 E_s \rho_c^2 + 2E_f T \left(3C^2 + 6CT + 4T^2\right) \rho_f^2)}{E_f \rho_s^2 (C \rho_c + 2T \rho_f)^3}
\]

(5.25)

\[
= \frac{\eta^3 (8\lambda + \delta^2 q^3 + 6\lambda q^2 + 12\lambda q)}{\lambda (2\eta + \delta q)^3}
\]

(5.26)

if we make the parametric substitutions \( \eta = \rho_f/\rho_s \), \( \lambda = E_f/E_s \), \( \delta = \rho_c/\rho_s \) and \( q = C/T \).

A similar derivation and result can be obtained if we consider the case where a minimum panel thickness has already been reached. If this minimum face panel thickness is half the initial panel thickness \( t \), then \( T = t/2 \) and the mass of this panel will be

\[
m_w = bL \left(C \rho_c + t \rho_f\right)
\]

(5.27)

The gain in area moment of inertia can be determined by making these substitutions and dividing by \( I_0 \).

\[
R_b = \frac{1}{8} \left(\frac{C^3 E_s \rho_c^2}{E_f T^2 \rho_s^2} + \frac{6C^2}{T^2} + \frac{12C}{T} + 8\right)
\]

(5.28)

\[
= \frac{1}{8} \left(\frac{\delta^2 q^3}{\lambda} + 6q^2 + 12q + 8\right)
\]

(5.29)

Collapse Behavior

An important requirement for the expanded stiffener to function efficiently is for it to compress substantially and to use significantly less space when stowed. It must also be capable of undergoing this compression without requiring too much force. A minimal amount of collapse will diminish the potential benefits of implementing this system and requiring excessive force to compress the core may also cause problems for the storage and deployment mechanisms. These
Figure 5.8: The stress-strain diagram of an elastic foam.

two constraints of core compression can be modeled and optimized to obtain a beneficial combination for a particular set of stiffener materials and design parameters.

The load required to collapse the core can be found through determining the collapse stress $\sigma_{el}$ shown in Figure 5.8. When this stress is reached, open-cell foams will collapse with an approximately constant stress given by [103]

$$\frac{\sigma_{el}}{E_s} = 0.03\delta^2(1 + \delta^{1/2})^2$$  \hspace{1cm} (5.30)

and will enter the densification region at a strain of [104]

$$\varepsilon_{cd} = 1 - \frac{\delta}{0.3}$$  \hspace{1cm} (5.31)

Knowledge of this maximum collapse strain allows us to find $C$ for the stowed panel:

$$C_{stowed} = (1 - \varepsilon_{cd})C = 3.33\delta C$$  \hspace{1cm} (5.32)

### 5.4 Experimental Results

Confidence in the theoretical relations for two stiffener types was obtained through experiments and finite-element simulations. Because of the unique geometry of each stiffener type, multiple fixtures were needed. All finite element simulations were performed on models constructed with shell elements in ANSYS. Convergence studies were performed in each case to verify that convergence had occurred.
5.4.1 Partially-Actuated Stiffener

Testing the stiffness of a partially-actuated pattern was accomplished by constructing the prototype shown in Figure 5.9 using the fixture shown in Figure 5.10. Prototype dimensions are listed in Table 5.1. The material was sheet polypropylene and the panels adjacent to crease B were tilted to the $\theta$ value in the table. It was constructed with small Nylon hinges used for model aircraft stabilizers. The stiffness was measured by loading the vertex as a cantilevered beam in the fixture shown in Figure 5.10a. This fixture consisted of an upper (a) and lower (b) grip, two supporting columns (c) and a base (d) that can be secured to the workbench. The test specimen (1), Kistler 9212 load cell/deflector (2) and test fixture (3) are shown in Figure 5.10b.

Using the cantilever setup shown, three tests of a partially-actuated vertex were conducted. The actuation angle reported in Table 5.1 should result in an increase in stiffness of $R = 18.4$ according to Equation 5.6. A similar value was found through an finite element simulations shown in Figure 5.11. The vertex was calculated to have a stiffness gain of $R = 15.0$. An element size of

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<td>$d$</td>
<td>15.2</td>
<td>cm</td>
</tr>
<tr>
<td>$t$</td>
<td>0.159</td>
<td>cm</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.0875</td>
<td>rad</td>
</tr>
</tbody>
</table>
1.59 millimeters was used. The results from the tests of the actuated models showed that the stiffness increased by $R = 15.8 \pm 3.8$. The confidence interval resulted from experiments on multiple samples. This differs by 16.9% from the analytical results and 4.7% from the FEA result. Both results are also within the 95% confidence intervals of the $R$ value, which is reasonable considering the unmodeled flexibility of the frame, nonlinearity in the material properties, and other variables.

Other important assumptions made by the analysis include neglecting buckling in the panel. Such non-linear effects would alter the slight actuation angles and significantly alter the stiffness.
Another assumption is that the hinges do not add significant amounts of flexibility. When this is not correct, parasitic motion at the hinges can alter the results. This method can be attractive because the only requirement for it to function is that the pattern cannot completely unfold. Consequently, implementing it is relatively straightforward. A combination of appropriate hard-stops and self-locking hinges would be one method of ensuring the appropriate partially folded final state.

### 5.4.2 Yielded Stiffener

The predictions of stiffness increase for Equation (5.9) were confirmed experimentally using a four-point bend test illustrated in Figure 5.12. The prototypes tested in this fixture were made of polypropylene sheets with properties shown in Table 5.2. For a four-point test shown in Figure 5.12, the deflection of the panel is given by

$$y = \frac{Fx}{6EI} \left( x^2 + 3a^2 - 3La \right)$$  \hspace{1cm} (5.33)

**Table 5.2:** Parameters used to define the yielded stiffener prototype and test fixture

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>1.5</td>
<td>GPa</td>
</tr>
<tr>
<td>$v$</td>
<td>0.42</td>
<td>none</td>
</tr>
<tr>
<td>$t$</td>
<td>0.159</td>
<td>cm</td>
</tr>
<tr>
<td>$L$</td>
<td>59.7</td>
<td>cm</td>
</tr>
<tr>
<td>$w$</td>
<td>10.8</td>
<td>cm</td>
</tr>
<tr>
<td>$h$</td>
<td>2.54</td>
<td>cm</td>
</tr>
<tr>
<td>$a$</td>
<td>15.2</td>
<td>cm</td>
</tr>
</tbody>
</table>
where $F$ is the force load at one of the two load teeth, $E$ is Young’s Modulus, $I$ is the area moment of inertia of the panel cross-section, $x$ is the distance of a point from the outermost load point in the test and $a$ is shown in Figure 5.12. Solving this equation for $I$ where $k = F/y$ and $x = a$ yields

$$I = \frac{ka(4a^2 - 3La)}{6E}$$  \hspace{1cm} (5.34)

Using Equation (5.34), the mean stiffness gain for three samples was calculated to be $R_a = 191 \pm 16$. The confidence interval resulted from experiments on multiple samples. This value compares favorably with the analytical prediction of $R_a = 198$ from Equation (5.9). These results were supplemented by a finite-element simulation shown in Figure 5.13. The element size was 3.28 millimeters. This simulation predicted a gain of $R_a = 232$. While there is an error of 17.0% between this value and the analytical results, this difference may result from the unmodeled effect of the crease used to deploy the stiffener. Any deviation of the stiffener from vertical during the loading process could also have influenced the results through a lateral torsional buckling mode. The analytical and experimental results agree closely and the FEA results are not widely different, building confidence in the model for the yielded stiffener.

### 5.5 Comparison of Stiffeners

One important question that arises about these methods is how they compare and what type of considerations will influence the selection of one method over another. Because the gain in the moment area of inertia for each stiffener depends on several parametric variables, a good comparison can only be made if a range of input values are considered. This can be done by
graphing the moment of inertia gain, $R$, versus a certain cost variable $P$. Both $R$ and $P$ depend on
the geometry and so are plotted parametrically.

One potential cost for each stiffener is the deployment ratio. Because each stiffener type
either increases the volume of the panels or diminishes the available panel area, their use will
decrease the final deployed area for a constant stored volume. Because many applications for
deployable mechanisms seek to maximize deployed area, this may be an important design cost.
For this comparison, the deployment ratio is defined as

$$
\Delta_r = \frac{A_{\text{deployed}}}{V_{\text{stowed}}^{2/3}}
$$

(5.35)

where $A_{\text{deployed}}$ is the area of a deployed pattern and $V_{\text{stowed}}$ is the volume of the stored pattern.
Because our objective is to compare stiffeners, the percent change in deployment ratio between a
single, unit-square panel of thickness $t$ and its stiffened counterpart will be defined as the area cost.
The area cost can be written as

$$
\Delta = \frac{\Delta_r - \Delta_r^s}{\Delta_r} = \frac{A_{\text{deployed}} - A_{\text{deployed}}^s}{A_{\text{deployed}}}
$$

(5.36)

where $\Delta_r^s$ is the deployment ratio for the stiffened panel.

Another cost to consider is the increase in mass produced by each stiffener. This cost is
represented by $M$ and is defined as the percent change in panel mass between the stiffened and
initial versions. Both the initial and final panels must have equivalent area.

### 5.5.1 Stiffener Area and Mass Costs

Before any effective comparison can be made, specific area costs described in Equation
(5.36) must be found for all three stiffeners. For a partially-actuated pattern, its viewable area can
be expressed as $A = A_0 \cos \theta$ where $A_0$ is the real panel area and $\theta$ is the angle between the panel
angle and a line from the base of the normal to the viewer. Because the stowed volume remains
unchanged between stowed and deployed states, the area cost for partial actuation is given by

$$
\Delta_{PA} = 1 - \cos \theta
$$

(5.37)
Similar area costs can be found for the yielded and expanded stiffeners and are given by Equations (5.38) and (5.39).

\[
\Delta_Y = \frac{h}{w} = f \tag{5.38}
\]

\[
\Delta_E = 1 - \frac{0.629961\left(\frac{\delta q}{\eta} + 2\right)^{2/3}}{(1.665\delta q + 1)^{2/3}} \tag{5.39}
\]

The result for the expanded stiffener was found after determining that the deployment ratios for the initial and stiffened version are

\[
\Delta_0^r = \frac{bL}{(bLt)^{2/3}} \tag{5.40}
\]

and

\[
\Delta_s^r = \frac{0.629961bL}{(bL(1.665C\delta + T))^{2/3}} \tag{5.41}
\]

A similar set of mass costs can be found for the three stiffeners. Because the panels are constrained to have constant area, adding each stiffener will lead to increased mass. Partial actuation will diminish the viewable area and so more panel area will be required for increasing amounts of actuation. The yielded stiffener will also diminish panel area and so will require larger panels and increased mass. Although the expanded stiffener does not reduce the area, the constant wall case will increase the mass through the addition of the core material.

For partial-actuation, the initial area \(A_0\) required to achieve a constant viewable area \(A\) is given by \(A = A_0\cos \theta\). If we assume that area is proportional to mass, then this relation can be used to determine the percent mass increase or mass cost for this design as

\[
M_{PA} = \frac{m^r - m^0}{m^0} = \frac{1 - \cos \theta}{\cos \theta} \tag{5.42}
\]

| Table 5.3: The equation number used for each parametric parameter (\(\xi\)), its cost variable (\(\Delta\) or \(M\)) and stiffener type |
|------------------|---|---|---|
| Partially-actuated | Eq. 5.6 | Eq. 5.6 | \(\theta\) |
| Yielded stiffener | Eq. 5.9 | Eq. 5.9 | \(f\) |
| Expanded stiffener | Eq. 5.26 | Eq. 5.29 | \(q\) |
where $M$ is the mass cost, $m^s$ is the mass of a unit square panel with stiffeners and $m^0$ is the mass without.

For the yielded stiffener shown in Figure 5.4, the mass cost is

$$M_Y = \frac{h}{w} = f$$  \hspace{1cm} (5.43)

The mass cost for the expanded stiffener can be found if we set the stiffened mass to Equation (5.20) where $T = t/2$ (half the initial panel thickness) and the initial mass is found when $T = t/2$ and $C = 0$. Computing the mass cost as described previously gives

$$M_E = \frac{\delta q}{2\eta}$$  \hspace{1cm} (5.44)

A comparison of these stiffeners can be made if we designate one variable in each equation as the independent variable $\xi$ and make all other values constant. The variable chosen as $\xi$ for each stiffener is shown in Table 5.3. This parametric variable $\xi$ can then be used to generate plots of the functions $(R_a(\xi), \Delta(\xi))$ and $(R_b(\xi), M(\xi))$ for each stiffener. Plots of the stiffener area costs $\Delta$ versus their gain $R$ are shown for the three stiffeners in Figure 5.14. A similar plot of their performance versus their mass cost is shown in Figure 5.15. In both plots, it can be seen that partial actuation demonstrates the largest increases in $R$ for low costs in mass and area. Similarly, the expanded stiffener performance is the second best in both cases while the yielded stiffener is the most costly.

Figure 5.14: Plots of the parametric inertia gain $R(\xi)$ and $\Delta(\xi)$ with respect to the variable $\xi$. 

![Figure 5.14: Plots of the parametric inertia gain $R(\xi)$ and $\Delta(\xi)$ with respect to the variable $\xi$.](image-url)
5.6 Conclusion

Managing the stiffness of an origami-inspired mechanism is an important capability to ensure that lightweight deployables can adequately fulfill their functions. In this chapter, three types of deployable stiffeners to control pattern stiffness were investigated. Analytical equations modeling the increase in moment of inertia were derived for each stiffener type. The predictions of these equations were later verified experimentally and through finite-element simulations.

The equations expressing the increase in inertia were written parametrically with respect to a characteristic variable $\xi$ and then plotted against two specific design costs. This enabled a comparison of their performance. This particular comparison showed that partial actuation is most efficient, followed by expanded stiffeners. While this comparison only considered three stiffener types and two costs, the method can be applied to other stiffeners and costs to facilitate design selection.

These results are significant because they have the potential to manage deformation in large, thin, origami-inspired mechanisms. Mechanisms that would experience large deformation and be unusable can be converted into small-deformation devices as described in Figure 1.1. The resulting mechanisms can then more closely approximate rigid-foldable motion in spite of applied loads and achieve the benefits that rigid-foldable motion provides.
CHAPTER 6. DEPLOYABLE LENTICULAR STIFFENERS FOR ORIGAMI-INSPIRED MECHANISMS

6.1 Introduction

Origami-inspired mechanisms have demonstrated potential in solving important problems that exist in the design of spacecraft. However, as deployment ratios increase, the stiffness of the pattern may be compromised. Potential problems include vibrations [97] resulting from the deployment process or originating from steady-state sources on the spacecraft such as reaction wheels [91, 92].

This chapter proposes the use of deployable stiffeners based on the Euler Spiral [96] as a solution with properties that are particularly well suited to origami-inspired, space-based deployables. They are referred to as lenticular stiffeners. The Euler Spiral is defined as a curve whose curvature is a linear function of its arc length [105]. Some current applications of the spiral include its use in the design of transitions in railways and roads [106]. Computer vision has been another application of this curve type [107]. However, when it was originally defined by Euler in 1744, the Spiral was intended to solve the inverse of the elastica problem [108]. This problem involved determining what curved cantilever beam profile will deflect to a horizontal beam when loaded with a weight at its end [108]. The linearly varying curvature of the spiral was shown to be exactly counteracted by the linearly varying moment in a loaded cantilever beam.

Because stiffeners with this profile can collapse to a flat state, such a solution can provide increased stiffness without compromising the deployment ratio of the mechanism. In addition, the fact that these stiffeners deploy using stored strain energy eliminates the need for a dedicated deployment mechanism. Because it can more easily take advantage of geometry to increase stiffness, mass may also be conserved. A potential drawback may be the increased complexity of the final design. However, with some designs this complexity may be reduced to provide an effective solution to low rigidity.
In this chapter, deployable lenticular stiffeners are studied and different aspects of their design are discussed. Their design is studied to support the transition of origami from large-deformation to small-deformation as shown in Figure 1.1. Tape springs are a subset of lenticular stiffeners [109, 110] due to their curved and deployable nature. One contribution of this work is an analytical derivation of the increase in stiffness provided by the deployable. Both an exact and approximate expression for the area moment of inertia are described for the Euler Spiral profile. Next, these results are verified using finite element analysis and experiments. Deployment is also considered and relations needed to design their cross-section and predict their deployment are derived. These relations include formulas for acceptable dimensions of the stiffener given material constraints. Lastly, different configurations are examined to provide guidance on implementation throughout the deployable mechanism. Experiments are conducted to verify the models of these configurations. The concepts discussed related to designing deployable stiffeners are then implemented in the design of a CubeSat antenna.

6.2 Background

Stiffness is a fundamental problem in the design of deployable mechanisms and significant research exist on the topic [111–114]. Puig et al. [115] discusses variations of expandable telescope reflectors and the challenges created by a lack of stiffness. The importance of deployable mechanism stiffness during spacecraft maneuvers has also been discussed by Ross et al. [116].

In addition to examining the stiffness of deployable mechanisms, researchers have also examined different options for expandable stiffeners. Sokolowski [70], for example, discussed the possibility of using cold-hibernated elastic memory structures to deploy stiffeners from a stowed state. Similar compliant, deployable stiffeners are discussed by other authors [117]. The ability that compliant, lenticular hinges have to actuate deployment and provide stiffness was also studied by Mobrem and Adams [109]. Research by Arya, Lee and Pellegrino on solar collectors examined the potential of compliant TRAC longerons to stiffen these structures [118].

Problems related to propelling the deployment process have also been discussed [119,120]. One solution that has been suggested for actuating deployables is through biomimicry of elastic propulsion [121]. The use of the elastic properties of SMA as a deployment mechanism has also been investigated [122, 123]. In these situations, the advantage of using the material in the stiff-
ener to perform the actuation is to eliminate the need for a separate mechanism to deploy the mechanism. The impact of the stiffness on deployment dynamics has also been examined by Ghaleh and Malaek [124].

### 6.3 Lenticular Stiffener

A stiffener that might be considered for increasing the rigidity of a deployable panel is a curved beam, as depicted in Figure 6.1. The version in Figure 6.1a is made by cutting the two arms of the stiffener from the panel and then curving them upwards. The version in Figure 6.1b is made by adding the beam to the panel. This deployable stiffener will enhance the bending stiffness of a panel in its deployed state but will also be capable of compressing flat in a stowed state to permit tight compaction of the mechanism panels. This capability will permit the resulting origami to compact to smaller sizes but retain an increased level of stiffness.

#### 6.3.1 Cross-Section for Lenticular Stiffener

An important requirement for lenticular stiffeners is that adjacent panels must flatten the stiffeners when panels are stacked on top of each other in their folded state. Stiffeners in this situation will be loaded as shown in Figure 6.2. If we describe the cross-section of the stiffener with a curve $y(x)$ written in the frame shown in Figure 6.2, then we can find its deflected shape. Here, it can be seen that the moment in the stiffener at point $(x, y(x))$ is proportional to the distance from the edge at $x = b$. Once a large enough force $F$ is reached at this edge, the curvature in the stiffener will be completely corrected at the base $(x = 0)$ where the stiffener connects to the panel. Due to the linear variation of the bending moment in the stiffener, the curvature will be progressively less affected as distance from $x = b$ decreases.
To counteract this effect, the curvature of the stiffener must vary linearly with $x$. Curves of this type are called Euler Spirals [105]. Generally, a point on an Euler Spiral $(x,y)$ at an arc-length $L$ from the origin is given by the relation

$$x = \frac{1}{a} \int_0^L \cos s^2 \, ds \quad (6.1)$$

$$y = \frac{1}{a} \int_0^L \sin s^2 \, ds \quad (6.2)$$

where $a$ is a parameter governing the proportions of the spiral, $s$ is the arc-length of the spiral from the origin and $x$ is allowed to vary from 0 to a value $b$ determined by $L$. This derivation assumes that the curvature at a point on the spiral is proportional to the curve length between itself and the origin where $s = 0$. For the stiffener cross-section, the curvature must vary linearly from $\kappa = 0$ at $x = b$ to $\kappa = \kappa_0$ at $x = 0$. This version of the Euler Spiral is not appropriate for a deployable beam since the curvature is zero at the origin. For this curve to be flattened as described, a force would have to be applied at an odd angle at the origin. The curve must be rederived to position the point of maximum curvature at the origin to facilitate future calculations.

For the curvature to be its largest at the origin, it must be true that

$$\kappa \propto (L - s) \quad (6.3)$$

where $L$ is the length of the curve on $0 < x < b$ and $s$ is the arc length between the origin and a point on the curve (e.g. $s = L$ at $x = b$). From this equation, it can be observed that $\kappa/(L - s)$ is
constant. From the definition of curvature, we know that

\[
\frac{d\theta}{ds} = \kappa = (L - s)\mu
\]  \hspace{1cm} (6.4)

where \( \theta \) is the slope of the curve in radians and \( \mu \) is a constant of proportionality given by

\[
\mu = \frac{\kappa}{L - s} = \frac{\kappa_0}{L}
\]  \hspace{1cm} (6.5)

Integrating Equation 6.4 yields

\[
\theta(s) = \int (L - s)\mu \, ds = \frac{\kappa_0}{L} \left( Ls - \frac{1}{2} s^2 \right) + C = \frac{\kappa_0}{2L} (2Ls - s^2)
\]  \hspace{1cm} (6.6)

\( C = 0 \) in this case because of the boundary condition \( \theta(s = 0) = 0 \). For any curve,

\[
\begin{align*}
x(s) &= \int_0^s \cos \theta(s) \, ds = \int_0^s \cos \left( \frac{\kappa_0}{2L} (2Ls - s^2) \right) \, ds \hspace{1cm} (6.7) \\
y(s) &= \int_0^s \sin \theta(s) \, ds = \int_0^s \sin \left( \frac{\kappa_0}{2L} (2Ls - s^2) \right) \, ds \hspace{1cm} (6.8)
\end{align*}
\]

Although it may seem that Equations 6.7 and 6.8 are not equal to Equations 6.1 and 6.2, they are in fact equivalent. These quadratics of \( s \) can be made equal with an appropriate linear coordinate system transformation and scaling.

This parametric function \((x(s), y(s))\) can be non-dimensionalized by dividing both equations by \( L \) and making the substitutions \( p = s/L \) and \( q = \kappa_0L \). With these substitutions, we have

\[
\begin{align*}
\frac{x(s)}{L} &= -\frac{\sqrt{\pi} \cos \left( \frac{q}{2} \right) C \left( \frac{(1-p)\sqrt{q}}{\sqrt{\pi}} \right)}{\sqrt{q}} + \frac{\sqrt{\pi} C \left( \frac{\sqrt{q}}{\sqrt{\pi}} \right) \cos \left( \frac{q}{2} \right)}{\sqrt{q}} - \\
\frac{\sqrt{\pi} \sin \left( \frac{q}{2} \right) S \left( \frac{(1-p)\sqrt{q}}{\sqrt{\pi}} \right)}{\sqrt{q}} + \frac{\sqrt{\pi} S \left( \frac{\sqrt{q}}{\sqrt{\pi}} \right) \sin \left( \frac{q}{2} \right)}{\sqrt{q}}
\end{align*}
\]  \hspace{1cm} (6.9)
Figure 6.3: The profile of a stiffener that can be flattened with a force on one edge.

\[
\frac{y(s)}{L} = -\sqrt{\pi} \frac{\sin\left(\frac{q}{2}\right) C\left(\frac{(1-p)\sqrt{q}}{\sqrt{\pi}}\right)}{\sqrt{q}} + \sqrt{\pi} \frac{C\left(\frac{\sqrt{q}}{\sqrt{\pi}}\right) \sin\left(\frac{q}{2}\right)}{\sqrt{q}} + \sqrt{\pi} \frac{\cos\left(\frac{q}{2}\right) S\left(\frac{(1-p)\sqrt{q}}{\sqrt{\pi}}\right)}{\sqrt{q}} - \sqrt{\pi} \frac{S\left(\frac{\sqrt{q}}{\sqrt{\pi}}\right) \cos\left(\frac{q}{2}\right)}{\sqrt{q}}
\]

(6.10)

where \(S(\ldots)\) and \(C(\ldots)\) are Fresnel integrals [125]. Plotting this curve where \(\kappa_0 = 2\) and \(L = 1\) gives the cross-section shown in Figure 6.3. Visually, it can be confirmed that the curvature is greatest at the junction between the stiffener and the panel and smallest at the free edge. This cross-section will permit the stiffener to flatten completely with a vertical load applied at the free edge. This ability to flatten completely will be essential in applications where multiple panels will be stacked on top of each other, such as might occur in origami designs with large deployment ratios.

When stiffeners must be compressed to their flat state, an important variable will be the amount of force needed at the edge per unit length. This quantity can be found by considering that moment is proportional to curvature for very thin elastic beams. To completely flatten these stiffeners, the largest curvature \(\kappa_0\) must be eliminated. The required force per unit length, \(\bar{F}\), is given by

\[
\kappa_0 = \frac{M}{EI} = \frac{(\bar{F}l) L}{E \frac{1}{12} l t^3} = \frac{12 \bar{F} L}{E t^3}
\]

\[
\bar{F} = \frac{\kappa_0 E t^3}{12L}
\]

(6.11)

where \(L\) is the total arc length of the spiral, \(t\) is the thickness of the material forming the curve, \(l\) is the length of the stiffener in the \(z\)-direction as shown in Figure 6.4 and \(E\) is the Young’s Modulus of the material. The maximum force per length that can be tolerated by the material must also be
Figure 6.4: The dimensioned lenticular stiffener cross-section.

considered. If \( N \) is a safety factor for the stiffener, \( t \) is its thickness and \( \sigma_y \) is its yield stress, then the force per unit length, \( \bar{F} \), in the base of the stiffener \((x = 0)\) is given by

\[
\bar{F} = \frac{\sigma_y t^2}{6NL}
\]  

(6.12)

where \( L \) is the length of the Spiral on \( 0 < x < b \). Substituting Equation 6.11 into Equation 6.12 and eliminating \( \bar{F} \) yields

\[
\kappa_0 = \frac{2\sigma_y}{NEt}
\]  

(6.13)

which gives the maximum stiffener base curvature that can be completely flattened with yield strength constraints. Knowing the maximum force per length and base curvature will be critical for ensuring that the stiffener does not yield during the collapsing process.

6.3.2 Lenticular Stiffener Inertia Gain

Another important objective of this chapter is to develop a simple expression for the area moment of inertia of the curved stiffener. The variables shown in Figure 6.4 will be used to derive expressions for the increase in stiffness. A relatively compact expression for the change in area moment of inertia can be found if a basic set of assumptions are made about the stiffener geometry. One of these assumptions is that the cross-section of the stiffener is approximately parabolic and can be expressed by \( y = x^2d/b^2 \) where \( d \) and \( b \) are shown in Figure 6.4. This assumption introduces some error since it can only approximate the Euler Spiral. However, it can be shown that the deviation is minimal in the case studied here.
The difference between the approximation and the Spiral can be found at any arc length \( s \) and will depend on the specific approximation scheme utilized. The Euler Spiral can be approximated by using the first two terms of a Taylor series expansion of sine and cosine. Equation 6.7 and 6.8 can be expressed as

\[
\frac{x(s)}{L} = \frac{1}{L} \int_0^s \cos(\theta(s)) \, ds = \frac{1}{L} \int_0^s \left(1 - \frac{\theta(s)^2}{2}\right) \, ds \\
= -\frac{1}{40} p^5 q^2 + \frac{p^3 q^2}{8} - \frac{p^3 q^2}{6} + p
\]

\( (6.14) \)

\[
\frac{y(s)}{L} = \frac{1}{L} \int_0^s \sin(\theta(s)) \, ds = \frac{1}{L} \int_0^L \left(\theta - \frac{\theta(s)^3}{6}\right) \, ds \\
= \frac{p^7 q^3}{336} - \frac{p^6 q^3}{48} + \frac{p^5 q^3}{20} - \frac{p^4 q^3}{24} - \frac{p^3 q^2}{6} + \frac{p^2 q}{2}
\]

\( (6.15) \)

where \( p = s/L \) and \( q = \kappa_0 L \). These expressions are given in their non-dimensional form after substituting Equation 6.6. The difference between these approximate coordinates and the exact values can be plotted versus \( p \) and \( q \). A plot of the shape error \( y/L - (y/L)_{\text{Taylor}} \) resulting from Equations 6.14 and 6.15 is shown in Figure 6.5. As can be seen, the error does not exceed 0.3% in magnitude. However, because these equations are not parabolic, further approximation will be required. If we only consider the last terms from Equations 6.14 and 6.15, then the non-parametric profile is given by

\[
\frac{y(s)}{L} = \frac{d}{b^2} \left( \frac{x(s)}{L} \right)^2 \\
= -\frac{15q(q^2 - 35)}{7(q^2 - 15)^2} \left( \frac{x(s)}{L} \right)^2 \text{ where} \\
0 < \left( \frac{x(s)}{L} \right)^2 < 1 - \frac{q^2}{15}
\]

\( (6.16) \)

\( (6.17) \)

where \( d \) and \( b \) are found by evaluating the Equations 6.14 and 6.15 at \( t = 1 \). The shape error is given by Figure 6.6. It can be seen that with this additional simplification, the shape error increases to at most 2%. Errors of this size should not significantly impact the stiffness of the profile and so a parabolic assumption will be appropriate.
Figure 6.5: The non-dimensional error in $\Delta y/L$ between the Taylor Series approximation and exact solutions for the Euler Spiral where $p = s/L$ and $q = \kappa_0 L$.

Figure 6.6: The non-dimensional error in $\Delta y/L$ between the parabolic approximation and exact solutions for the Euler Spiral where $p = s/L$ and $q = \kappa_0 L$.

A second assumption is that the thickness $t$ is small in comparison with the other dimensions. With these assumptions, the centroid of the sections from Figure 6.1a and Figure 6.1b with respect to $y$ are located at

$$c_a = \frac{1}{3} d \frac{\sqrt{b^2 + 4d^2 + b}}{w}$$  \hspace{1cm} (6.18)

$$c_b = \frac{1}{3} d \frac{\sqrt{b^2 + 4d^2 + b}}{w + \sqrt{b^2 + 4d^2 + b}}$$  \hspace{1cm} (6.19)
respectively, where \( b \) is half of the stiffener width, \( t \) is the stiffener and panel thickness, \( d \) is the stiffener height and \( w \) is the panel width. With these centroids, the area moment of inertia of the section can now be calculated. Using a relation for the area moment of inertia of a parabola given by Kollar and Dulacska [126], this inertia can be expressed as

\[
I_a = -\frac{d^2}{45w^2} \left( (-5tw^2 + 20(-tb^2 + Td^2 - td^2) +
20tbw - 10Tbw) \sqrt{b^2 + 4d^2 + 10Twb^2 + 13tbw^2 +
20(Td^2w - Tb^3 - twb^2 + b^3) + 60(td^2b - Td^2b) -
40td^2w) \right)
\]  
(6.20)

\[
I_b = \frac{d^2}{45} \left( \frac{w + \sqrt{b^2 + 4d^2 + b^2}}{b^2 + 4d^2 + b^2} \right)^2 (13tbw^2 + 16tb^2w +
(16tbw + 16tb^2 + 5tw^2 + 10wTb) \sqrt{b^2 + 4d^2 +
16tb^3 + 32td^2b + 10wTb^2 + 20wTd^2) \right)
\]  
(6.21)

where \( b, d, t, T \) and \( w \) are defined as in Figure 6.4. If we take the ratio of this inertia and the area moment of inertia of the panel alone, then we find that

\[
R_a = -\frac{4}{15} u^2v^3r^2 \left( (20(-v^2 + qv^2 + qu^2v^2 - u^2v^2 + v) -
5 - 10qv) \sqrt{1 + 4u^2 - 13 + 20v - 10qv + 40u^2v -
60u^2v^2 - 20qu^2v + 60qu^2v^2 - 20v^2 + 20qv^2) \right)
\]  
(6.22)

\[
R_b = \frac{4u^2v^3r^2}{15 \left( 1 + v \sqrt{1 + 4u^2 + v} \right)^2} \left( 13 + 16v + 16v^2 + 32u^2v^2 - (5 + 16v + 16v^2 + 10qv) \sqrt{1 + 4u^2 + 10qv + 20qu^2v} \right)
\]  
(6.23)

where the parametric substitutions \( q = T/t, r = w/t, v = b/w \) and \( u = d/b \) were made.

6.3.3 Test of Physical Prototype

This result was also verified experimentally using the four-point fixture sketched in Figure 6.8. This tester is composed of a lower box (1) to support the test specimen, a loading plate (2) with
two sets of loading teeth, a beam (3) that is used to apply force to the loading plate and specimen and a load cell (4). For a four-point test, the deflection of the panel is given by

$$y = \frac{Fx}{6EI} \left( x^2 + 3\lambda^2 - 3h\lambda \right)$$

(6.24)

where $F$ is the force load at one of the two load teeth, $h$ is the length of the panel, $E$ is Young’s Modulus, $I$ is the area moment of inertia of the panel cross-section, $x$ is the distance of a point from the outermost load point in the test and $\lambda$ is the distance of either load tooth from the outermost load point as shown in Figure 6.7. Solving this equation for $k = F/y$ where $x = \lambda$ yields

$$k = \frac{6EI}{\lambda^2(4\lambda - 3h)}$$

(6.25)

The prototype used two steel lenticular stiffeners in the second configuration with the dimensions listed in Table 6.1. Both stiffeners were made from sections of commercially available tape measure blades with the corresponding dimensions. Tape measure blades were used because the one-inch versions obtained were measured to closely approximate the desired parabolic profile.
The two stiffeners were attached to a steel sheet using five 0.3175 cm (1/8 inch) diameter, 0.3175 cm (1/8 inch) grip pop rivets in each.

The stiffness was measured by finding the slope of a linear fit to the elastic portion of the force-deflection curve. Using the fixture described and three samples, the stiffness was predicted and measured to be 718 N/m and 726 ± 79.1 N/m, respectively. Test repeatability appeared to be good since differences in measured stiffness for individual samples was slight. The predicted value is within the uncertainty range of the measurement.

6.3.4 Verification using Finite-Element Simulation

These results were confirmed in a finite-element simulation shown in Figure 6.8d. The fixed ends were implemented as in Figure 6.8c using pinned constraints at the plate end nodes. Loads were applied at two specific nodes at the two locations. A mesh size of 2 mm was used in the simulation with a linear-elastic material model and no geometric nonlinearities. Convergence was verified by varying element size and selecting a reasonably small value. The stiffener was modeled using ANSYS thin shell elements and the corresponding stiffness prediction was 743 N/m. This result is 3.5% larger than the analytical results and 2.3% larger than the average experimental result. The fact that these measurements are close and that the analytical and finite-element results lie within the error of the experimental values support the accuracy of the analytical model previously constructed.

Table 6.1: Parameters used to define the lenticular stiffener prototype

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>h</td>
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<td>cm</td>
</tr>
<tr>
<td>t</td>
<td>0.1778</td>
<td>mm</td>
</tr>
<tr>
<td>T</td>
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<td>GPa</td>
</tr>
<tr>
<td>σy</td>
<td>535</td>
<td>MPa</td>
</tr>
</tbody>
</table>
6.4 Configuration of Lenticular Stiffeners

A problem related to designing curved stiffeners is configuring them so that they can be stowed properly without damaging adjacent structures. This problem derives from the need for adjacent panels to be stiff enough to collapse curved stiffeners. In this situation, the stiffeners must be compliant enough to collapse without overstressing the flat panel. To prevent overstressing, we considered the two different configurations shown in Figure 6.9. If the stiffeners are arranged as shown in these figures, then the shear and moment functions can be found with respect to \( x \).

If we model the two loading situations as shown in Figure 6.10, then the moment functions for the first and second cases are given by

\[
M_1 = \begin{cases} 
F(b-x) - \frac{Fb^2}{w} & x < b \\
\frac{Fb(w-b)}{w} - Fb & x \geq b 
\end{cases} 
\]  

(6.26)
\[ M_2 = \begin{cases} -Fx + \frac{F(w^2 - 4b^2)}{4w} & x < \frac{w}{2} - b \\ Fb - \frac{Fw}{2} + \frac{F(w^2 - 4b^2)}{4w} & x \geq \frac{w}{2} - b \end{cases} \]  

(6.27)

where \( F \) is the force applied by one edge of the stiffener, \( b \) is half the width of the stiffener and \( w \) is the distance between two stiffeners. Because the moment is constant for \( A < x < B \) in either case, the moment will be greatest at \( x = 0 \) or at \( A \). Taking the ratio between these moments for both cases yields

\[
P_1 = \frac{M_{1,x=0}}{M_{1,x=A}} = \frac{w}{b} - 1
\]

(6.28)

\[
P_2 = \frac{M_{2,x=0}}{M_{2,x=A}} = \frac{w + 2b}{w - 2b}
\]

(6.29)

From these equations, it can be seen that as long as \( w > 2b \), then \( P_1 > 1 \) and \( P_2 > 1 \). Therefore, the moment at \( x = 0 \) will always be the largest. A ratio between the two moments at \( x = 0 \) can also be taken to determine which case produces the most moment. This ratio is given by

\[
P_{21} = \frac{M_{2,x=0}}{M_{1,x=0}} = \frac{r^2 - 1}{r(r - 2)}
\]

(6.30)

where \( r = 2b/w \). From this ratio, it can be seen that \( P_{21} > 1 \) when \( r < 0.5 \). Therefore, the maximum moment will always be higher for the second configuration when the stiffener spacing is above this threshold. For an equal number of stiffeners per unit area, the first configuration will have lower stress.

This result also applies to individual stiffeners or stiffeners at the edge of a panel. In either situation, the loading conditions affecting the panel would change. If we consider a case where the panel terminates at \( x = w/2 \), no moments are possible in the region \( A < x < w/2 \). The moment function \( M_i(x) \) also simplifies in the region \( 0 < x < A \). It can be shown that without an adjacent stiffener, the moment relations are

\[
M_1 = \begin{cases} F(b - x) & x < b \\ 0 & x \geq b \end{cases}
\]

(6.31)
\[ M_2 = \begin{cases} 
F\left(\frac{w}{2} - b - x\right) & x < \frac{w}{2} - b \\
0 & x \geq \frac{w}{2} - b 
\end{cases} \]  
(6.32)

In both cases, the moment is greatest at \( x = 0 \) as in the previous case. Also, the first loading scenario is still the most desirable because

\[ P_{21} = \frac{w}{2b} - 1 = \frac{1}{r} - 1 \]  
(6.33)

which is greater than one when \( r = 2b/w < 0.5 \), as shown previously.

### 6.4.1 Collapse Behavior of Multiple Layers

The relations describing the collapse of multiple layers of stiffeners were investigated experimentally to support their applicability. This was done by measuring the force-deflection behavior of a stack of panel samples during the storage process. From Equation 6.11, it can be observed that the spring constant of the stiffener during collapse is approximately given by

\[ k = \frac{2Fl}{d} = \frac{\kappa_0 E l^3}{6Ld} \]  
(6.34)

In a situation where multiple panels are stacked and compressed, the panels will also deflect with a certain spring constant that will influence the total stiffness of the folded stack. For a stiffener without an adjacent stiffener, we can assume that the region \( 0 < x < b \) deflects as a simply-supported beam with a load at its center. The stiffness of this beam is given by

\[ k = \frac{IT^3E}{2b^3} \]  
(6.35)

For a stack of \( n \) panels, the total stiffness is

\[ k_{total} = \frac{\kappa_0 l t^3E}{2b^3 \kappa_0 (n-1) + 6dLn} \]  
(6.36)

where all variables are as previously defined. This constant was both calculated and measured for three sets of different sample layers by finding the slope of linear fits to the elastic portion of the
force-deflection curve. Each sample layer consisted of a sheet with a single stiffener constructed with the dimensions listed in Table 6.2. The predicted and measured values of the stiffness for different numbers of layers is shown in Table 6.3. Figure 6.11 shows an image of one of these layers (2) and the fixture (1) used to compress them. Force measurements in this test were taken with a Kistler 9212 load cell (3). The agreement between these values suggest that the relations describing moment in the stiffener are accurate. None of the test samples manifested signs of yielding, which also supports the predictions made by Equations 6.26 and 6.31.

Understanding the relative behavior of these two cases is important because it permits the selection of the most advantageous stiffener configuration. It also allows the appropriate base curvature, $\kappa_0$, to be chosen which will maximize stiffness. Dividing both sides of Equation 6.26 by stiffener length will yield an equation in terms of force and moment per unit length ($F$ and $M$).

![Figure 6.11: Images of (a) one of the samples used in the compression tests and the (b) fixture used to compress multiple stiffened layers.](a)(b)

Table 6.2: Parameters used to define the collapse test prototype

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<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
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<tr>
<td>$t$</td>
<td>0.1778</td>
<td>mm</td>
</tr>
<tr>
<td>$T$</td>
<td>0.1778</td>
<td>mm</td>
</tr>
<tr>
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<td>mm</td>
</tr>
<tr>
<td>$E$</td>
<td>207</td>
<td>GPa</td>
</tr>
<tr>
<td>$\sigma_y$</td>
<td>535</td>
<td>MPa</td>
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</tbody>
</table>
Substituting Equation 6.12 into Equation 6.26 and eliminating $F$ will permit determination of the maximum moment in the panel for a given stiffener design.

However, one key unknown in Equation 6.12 is the stiffener curve length $L$. Determining this length given $b$ and $d$ is difficult for an Euler Spiral but is more manageable for the parabolic approximation. It can be shown that the arc length of the parabolic approximation is

$$L = \frac{1}{2} \sqrt{b^2 + 4d^2} + \frac{b^2 \sinh^{-1} \left( \frac{2d}{b} \right)}{4d} \quad (6.37)$$

The moment-per-length in the panel is

$$\bar{M}_p = -\frac{2bdt^2(b - w)\sigma_y}{3Nw \left( 2d\sqrt{b^2 + 4d^2} + b^2 \sinh^{-1} \left( \frac{2d}{b} \right) \right)} \quad (6.38)$$

$$= -\frac{2t^2u(v - 1)\sigma_y}{3N \left( 2\sqrt{4u^2 + 1u + \sinh^{-1}(2u)} \right)} \quad (6.39)$$

where $u = d/b$ and $v = b/w$ as defined previously. This moment must be taken into account when designing the panels to prevent failure in these components during stowage. For the lenticular stiffeners used in the prototype discussed previously, the panel thickness needed to prevent failure can be calculated if the moment in the panel is known to be

$$\bar{M} = \frac{\sigma_y T^2}{6N} \quad (6.40)$$

where $T$ is the thickness of the panel and $N$ is the safety factor. This minimum thickness can be found in terms of the geometry of the stiffener by equating Equation 6.40 to Equation 6.38. This

<table>
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<tr>
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<th>$n = 2$</th>
<th>$n = 3$</th>
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<tbody>
<tr>
<td>Analytical (kN/m)</td>
<td>2.61</td>
<td>1.67</td>
</tr>
<tr>
<td>Experimental (kN/m)</td>
<td>2.56 ± 0.236</td>
<td>1.69 ± 0.187</td>
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thickness $T$ is given by

$$T = \sqrt{\frac{4bd^2(w - b)}{w \left(2d\sqrt{b^2 + 4d^2 + b^2 \sinh^{-1} \left(\frac{2d}{b}\right)}\right)}}$$

(6.41)

Plotting this function $T$ with respect to $w$ using all other prototype parameters defined in Table 6.1 produces the plot shown in Figure 6.12. As shown by the figure, even large panel widths will at most require a panel thickness of 0.08 mm. Since a panel thickness equivalent to the stiffener thickness was assumed in the analysis, the panels will be sufficiently thick to prevent bending failure.

6.5 Deployable Mechanism Implementation

An example of how the principles described previously can be applied to a specific problem is the design of deployable antennas for CubeSats. This application has the need to store large deployables in severely limited spaces. Because the CubeSat is limited to a volume less than 10 cm x 10 cm x 10 cm, other needed hardware such as batteries and control equipment quickly consume available space. The antenna requirements of the CubeSat exaggerate these problems. Because of a need for CubeSat antennas to communicate at UHF frequencies, designers are under great pressure to increase antenna sizes [127]. Increasing antenna size can also be beneficial because of the relationship between effective aperture and received and transmitted power. A larger effective CubeSat aperture can reduce the needed transmission power [128]. Deployable, lenticular stiffen-
Figure 6.13: A demonstration of the use of deployable stiffeners to reduce the stowed volume of an origami-inspired deployable mechanism. (a) The stowed antenna configuration, (b) the deployed antenna configuration, and (c) a version with the stiffener integrated into the panel.

Stiffeners would allow the antenna to be larger while still allowing it to stow in a small volume. This is demonstrated by the concept for a deployable antenna shown in Figure 6.13.

The images in Figure 6.13a and 6.13b show the folded and deployed states of the array. Stiffness is enhanced with a single, main stiffener extending from the chassis to the outermost panel and two branching stiffeners supporting panels on either side. A key advantage of this system is that the deployed antenna area of 316 cm$^2$ fits into a 88.9 mm x 88.9 mm x 4 mm volume. This represents 3.16% of the total CubeSat volume while still providing a significant deployed area. The image in Figure 6.13c shows a version where the stiffener and panel were made in one piece as in Figure 6.1a. While it must be noted that the exact size of the stored deployable will depend on the nature of the panels that need to be folded, the use of deployable stiffeners will permit the design of larger patterns without substantially increasing the storage volume or compromising rigidity.
6.6 Conclusion

Investigating the use of deployable stiffeners to support light-weight deployables has the potential to benefit these designs in several ways. Many of the challenges that confront designers of these devices relate to their deployed stiffness. An example of this is maintaining a specific shape in spacecraft deployables in spite of maneuvering. Deployable, curved stiffeners described in this work represent a potential solution to this problem. Expressions for the increase in area moment of inertia for these panels are developed in this chapter and their predictions are verified through FEA analysis and experiments.

Another issue that must be addressed with any deployable stiffener is the actuation system needed to deploy it. In the case of compliant, lenticular stiffeners, the stored strain energy in the profile provides the deployment energy and actuation system. Relations describing the achievable characteristics of the stiffener given specific material properties were derived in this work to clarify this process. Stiffener collapse behavior was also modeled mathematically and experimentally verified. Using strain energy to deploy the stiffeners can significantly simplify the design. It also provides large increases in the rigidity of the panels. This can be useful for origami-inspired mechanisms because of the tendency that these devices have for expanding to a large, flat state.

The research identifies a method that can be used to solve problems of stiffness in these situations and increase the utility of origami-inspired design. Managing stiffness can be particularly beneficial by reducing deformation and allowing the model to more closely approximate rigid-foldable motion. The equations derived here can be used to identify the increased stiffness provided by these stiffeners. Configurations are proposed for multiple stiffeners and the collapse behavior of multiple layers containing stiffeners is analyzed. The resulting methods are helpful in designing origami-based deployable systems.
CHAPTER 7. OPTIMIZATION OF ORIGAMI-BASED TUBES FOR LIGHTWEIGHT DEPLOYABLE STRUCTURES

7.1 Introduction

Origami tubes are a type of origami pattern which have been shown to have potential in structural applications [41]. These tubes are rigid-foldable and can be constructed from thick materials [129, 130]. Due to the number of geometric variables that are involved in the design of origami tubes, the mechanical properties of these tubes can vary greatly [41]. Utilizing optimization methods, this work seeks to illustrate the design of rigid-foldable mechanisms using thick materials. Specific goals will include evaluating the variables involved in the design of thick origami tubes and identifying parameters that lead to optimal tube designs for structural purposes.

7.2 Background

Many precedents exist for using origami-inspired mechanisms in the design of deployable structures. Examples of possible applications [131] include masts capable of coiling into a compact, stowed configuration as well as flexible sheets deployed by the STEM boom [132]. Other examples include deployable reflectors as explained by You and Pellegrino [133].

Optimization has played an important role in the design of origami patterns [21, 134], as well as in origami-inspired systems. Its techniques have been used in the design of deployable shelters [39, 40], deployable bridges [135, 136], and energy absorbing crash boxes [37, 38].

While the optimization of origami-tube-based deployable structures is a relatively novel concept in the literature, the potential of origami tubes to serve as a structure has been investigated by several authors. Specifically, Filipov et al. describe an analysis of the stiffness of a thin-walled zipper model [41]. Their analysis focuses on the difference in deployment stiffness versus load-carrying stiffness and the size of this difference in origami tubes in general. Work was also conducted by Ma and You [137] on beams constructed with origami tube patterns. Their research
explored a series of variations of a particular origami beam model in pursuit of a light-weight, energy-absorbing automobile crash beam.

The origami tubes explored in this study are derived from the Miura-ori origami pattern. Although there are many geometric variations of origami tubes [129], in this chapter only 4-sided tubes are evaluated. A 4-sided tube is created by connecting a single sheet Miura-ori zig-zag pattern with its mirrored counterpart as is shown in Fig. 7.1. A thick rigid-foldable origami version of the tube was chosen because most engineering applications require the use of thick rigid materials. Thick rigid-foldable tubes introduced by Tachi [130] utilize offset axis techniques to accommodate thickness at the folding vertices.

This chapter builds upon the work presented in [41] by modifying the design to accommodate thick materials and support loads. Some specific objectives of the chapter are the optimization of origami-tube-based structures that are lightweight, stiff and strong. A version using cutouts is also investigated in order to reduce weight. The thick-tube designs presented in [130] are used as the basis of the thick-tube mechanisms. By evaluating thick-tube designs, this work seeks to develop robust tube mechanisms for applications in deployable structures.

7.3 Design of Thick Rigid-Foldable Tubes

For this work both a symmetric version and non-symmetric version of the thick 4-sided tube were evaluated. These tubes, as seen in Fig. 7.2, accommodate thickness through the use of offsets
at certain rotational axes. The symmetric pattern is comprised of identical parallelogram panels and the non-symmetric pattern is comprised of square and parallelogram panels. Another difference between these two models is that each section of the symmetric model (which is comprised of four panels) can be adjoined to an adjacent section of four panels utilizing hinges along all four edges of the section. The non-symmetric version, however, due to its folding kinematics can only join each section comprised of four panels to its adjacent section along the edges of the square panels. Therefore the symmetric version of the thickened tube is able to utilize four hinges to connect sections whereas the non-symmetric version is only able to utilize two.

7.3.1 Design considerations for tube-based deployable mechanisms

In the engineering design of a tube-based deployable mechanism, many design variables must be considered. Some of these variables include, but are not limited to the following:

Panel geometry. Origami tubes are made up of a series of quadrilateral panels. These panels joined together in a ring to form a single section of an origami tube. For a tube to achieve
mobility at least four panels must make up a section. More than four panels may be used and such structures have been examined in [99] but in this chapter we narrow our scope to four sided tubes.

Traditional mirrored Miura-ori based tubes are constructed such that all four of the panels in a tube section share the same geometry. This type of tube preserves symmetry in its construction. Tubes may be also constructed where two pairs of panels share the same geometry or where all four panels have different geometry. If all four panels have different geometry than the tube will have curvature to it as it is deployed. In this chapter we examine the traditional symmetric tube where all four panels share the same geometry and non-symmetric tubes where two pairs of panels share the same geometry and curvature is not induced in the overall structure.

**Mechanism geometry.** In addition to the individual unit or cell geometry being varied, the overall mechanism geometry can also vary. This can change by varying the parameters of one cell to the next which can also create curvature in the overall mechanism. Cells can also be stacked side by side to create multi-cell deployable structures or patterns can be stacked in opposing directions to create "zipper" constructions [41]. This chapter considers single-cell tube designs where the parameters of each section are consistent throughout the tube.

**Materials.** A wide variety of materials may be used for the tubes ranging from metals, composites, wood, and plastics. This work is general enough to apply to a range of materials.

**Thickness.** Because paper is thin, thickness in origami design algorithms is typically ignored. Adding thickness to a origami pattern creates interference that prevents it from folding. Several thickness accommodation methods have been developed to address the issue of interference [31,33,138,139]. Many of these different thickness accommodation techniques could be used in the design of a thick origami tube and each design would be somewhat different from the next. For the thick-tube designs discussed in this chapter, offset-axis thick tube designs as presented in [130] are used.

**Hinges.** One aspect of origami-inspired systems that is often overlooked is the hinge mechanism. In traditional origami where paper is used, the paper serves both as the structural panel and hinge mechanism. This is made possible by paper’s unique property where its stiffness is reduced when creased. In folding mechanisms incorporating a material that cannot be simply folded, an alternative hinge method is required. Hinge methods that have been developed for origami based systems include membrane hinges [7], compliant hinges [139], traditional pin-joint hinges, and
rolling contact hinges [138]. The tube designs considered in this chapter assume traditional pin-joint hinges due to their load-carrying capability and an implementation of a tube constructed with these hinges is presented in Section 7.8. Because the tubes presented in this chapter are single degree-of-freedom mechanisms, some sort of locking mechanism must be implemented for the tubes to carry a structural load. In this work it is assumed that some sort of actuator would exist on one end of the tube that controls the position of at least two panels relative to each other thereby controlling the position of the tube as well as creating a structure to carry a load.

**Cutouts.** In traditional origami tubes the edges of each panel extend along each fold from vertex to vertex to create a structure that has no holes in it except for the openings at each end of the tube. In engineering practice such a sealed tube may be desirable in some applications but may not be necessary in other applications. As the application that we are considering is a structural application we consider introducing cut-outs both at the vertices as well in the middle of the panels as a method of decreasing weight.

### 7.4 Structural Analysis of Symmetric Tubes

If the tube is intended to be used as a deployable light-weight cantilever boom, then a set of dimensions for an optimal tube should exist for a certain load situation. The main objective of this work is to determine geometric parameters for a minimum mass deployable origami tube for a specific loading condition. One of the requirements of this origami tube is that it must not yield under the specified design load. To evaluate the stress experienced under a design load, a parametric finite element model of a 4-sided origami tube was constructed using the commercial code ANSYS. To avoid stress concentrations that could form on the offset volumes, a simplified model that did not incorporate offsets was used. This was done to simplify the optimization routine, however, because the offsets are required for the tubes to fold, the effect of these offsets would need to be evaluated in a later analysis when designing a fully functional folding tube. This model was constructed using volumes to represent the thick panels and was meshed with 1.0 cm shell elements as shown in Fig. 7.3. The first four panels were constrained in all directions on the YZ plane and the edges on the opposite end of the tube were loaded with a force. Each of the panels were connected with hinge constraints such that no moments are transferred.
A preliminary study was performed on the symmetric model to explore its behavior under certain loading conditions throughout its deployment range. The parameters shown in Fig. 7.4 are the parameters used to define the geometry of the tube. To understand the structural behaviour of the thickened tube, the maximum stress was evaluated for the tubes at varying deployment lengths. Three different panel configurations were evaluated throughout the deployment range at different values of $\mu$ shown in Fig. 7.4. The variable $\mu$ was varied between $80^\circ$, $60^\circ$, and $40^\circ$. Each configuration in this study utilized three sections ($n = 3$) and was given a final deployment length of $d = 1$ meter and a panel thickness of $T = 1$mm. The other parameters in the design were kept constant and are listed in Table 7.1. For each of the configurations the yielding safety factor was evaluated with a basic cantilever end loading condition which corresponds to the $y$-direction, as
Figure 7.5: Maximum stress in the symmetric tube model during deployment. The three different curves represent the maximum stress behavior for models with three different values of $\mu$.

shown in Fig. 7.3. The maximum stress safety factor from loading in the y-direction is shown in Fig. 7.5. From these results we can see that a design with a smaller angle $\mu$ will have lower stress during its initial deployment than one with a larger angle $\mu$. However in its final deployment, a tube with a larger angle $\mu$ has a higher factor of safety which corresponds to a lower stress. It is interesting to note that as each tube is deployed the stress in the fully deployed state is much less than in the fully folded state. This is opposite of some deployment mechanisms such as telescoping tubes. This phenomenon is due to the fact that as the tube deploys, its cross-sectional moment of inertia increases as it goes from a flat zig-zag shape to a larger tube shape as can be seen in Fig. 7.1.

This initial analysis demonstrates the trade-offs of the some of the parameters. We can see that in general smaller angles of $\mu$ correlate to smaller stress values during the initial deployment of a beam. Depending on the application different designs may be desirable. For example, using this mechanism as a positioning mechanism requires that the mechanism is robust throughout its entire deployment. In this case a design with a smaller $\mu$ would be preferred. If the application of the tube is to be deployed once and maintain the deployed state, a tube with a larger angle of $\mu$ can have a more efficient final shape. Because of the many design variables that can influence the structural performance of the tube mechanism, an optimization routine was constructed to determine designs that achieve maximum performance.
7.5 Symmetric Tube Optimization

7.5.1 Optimization setup

The objective of the optimization is to minimize the mass of an origami-based-tube mechanism for a given load requirement at a unit distance. The mass of the origami tube is calculated by the sum of the volumes of all of the panels multiplied by the density of the material. The volume for this model is given by:

\[ V = 8nL\sin(\mu)T^2 + 16nLtT \]

where \( L, \mu \) and \( T \) correspond to the dimensions shown in Fig. 7.4 and where \( t \) is the thickness of the walls of the hollow panels used in the shell model. The variable \( n \) represents the number of segments used in the model. Because this variable is discrete, optimization was repeated for multiple values of \( n \). Only small value of \( n \) were used because it is assumed that more segments would be worse. The figures show a model with three segments. The material chosen for this study was steel with properties and mesh size listed in Table 7.1. The other parameters were kept consistent from the study described in Section 4. Panel thickness \( t \), deployment angle \( \alpha \), and panel angle \( \mu \) were the design variables. A deployment distance \( d \) of one meter was fixed. The constraints of the optimization were such that the maximum stress in the structure should not exceed the yield strength of the material (Table 7.1). A 2000 N force in the positive Z direction was chosen as an end loading condition on the beam. A lower thickness limit of 0.2 mm was also put in place to ensure the final design was feasible. The model was meshed with shell elements and the nodes along the rim of one open end were constrained in all degrees of translation.

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<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus</td>
<td>200 × 10⁹</td>
<td>Pa</td>
</tr>
<tr>
<td>Yield Strength</td>
<td>172 × 10⁶</td>
<td>Pa</td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
<td>0.30</td>
<td>-</td>
</tr>
<tr>
<td>( d )</td>
<td>1</td>
<td>m</td>
</tr>
<tr>
<td>( F )</td>
<td>2000</td>
<td>N</td>
</tr>
<tr>
<td>( \rho )</td>
<td>8000</td>
<td>kg/m³</td>
</tr>
<tr>
<td>Mesh size</td>
<td>0.01</td>
<td>m</td>
</tr>
</tbody>
</table>
load in the Z direction was distributed evenly over all the nodes along the rim of the other open end. Hinges were simulated by coupling translational degrees of freedom at the panel edges. This optimization problem is defined as follows:

$$\text{minimize} \quad V$$

$$\text{with respect to} \quad t, \mu, \alpha, n$$

$$\text{subject to} \quad 0.2 \text{mm} < t$$

$$0 < n$$

$$40^\circ \leq \mu \leq 80^\circ$$

$$0 \leq \alpha \leq \mu$$

$$|\sigma_{\text{max}}| \leq \sigma_{\text{yield}}$$

A SLSQP gradient-based optimizer was chosen to evaluate the optimization function. The SLSQP function included in the python environment SciPy.minimize was used. A python program was created to modify the variables in an ANSYS batch file. This program called the ANSYS batch solver to solve the finite element model and the maximum stress of the structure was read back into the program.

### 7.5.2 Optimization Results

The optimization routine reduced the total mass of the beam from the starting mass of 1.48 kg to an optimal mass of 0.7366 kg. A convergence study was not performed for this analysis. The resulting design is seen in Fig. 7.6. The variables describing this model are shown in Table 7.2. From the stress profile shown in Fig. 7.6 we can see that the stress is concentrated along the edge of the panels. This is expected due to the hinge constraints placed on each of the vertices of the structure.

### 7.6 Structural Analysis of Non-Symmetric Tubes

The next model that was optimized was the non-symmetric origami-tube model shown in Fig. 7.7. This model differs from the last model in that all of its panels are not the same shape. The
panels on the top and bottom of the model shown in the figure are square whereas the side panels are parallelograms with an acute angle of $\mu$. A key difference between this and the symmetric models is that each segment of the tube is connected to the next with only two hinges rather than four hinges. Other alterations made were the inclusion of more design variables in this analysis. Solving this problem required several parameters and nine design variables. Aside from these changes, the analysis was performed in almost the same way. Steel material properties were also used to be consistent with the models discussed previously. The geometry was modeled using volumes and then all the surfaces of the resulting volumes were meshed. The mesh size is 0.01 m and plate elements were used. Like the previous model, the nodes around the edge of the opening of one end of the tube were fixed in three translational degrees of freedom. The nodes around the rim of the other opening at the other end were equally loaded according to the appropriate loading scenario. The hinges between panels were only constrained to transmit forces and not moments.

### 7.7 Non-Symmetric Tube Optimization Methodology

Developing optimized versions of this non-symmetric origami tube requires a function definition, optimization procedures and design variable constraints. While the function chosen does not have a large number of design variables, solving this problem remains difficult because of the complex interactions between these variables. However, the continuous nature of the objective function simplifies the optimization procedure because it permits the use of gradient-based methods.

<table>
<thead>
<tr>
<th>Design Variable</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>0.7366</td>
<td>kg</td>
</tr>
<tr>
<td>$t$</td>
<td>0.498</td>
<td>mm</td>
</tr>
<tr>
<td>$\mu$</td>
<td>79.9$^\circ$</td>
<td>deg</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>37.9$^\circ$</td>
<td>deg</td>
</tr>
<tr>
<td>$n$</td>
<td>3</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 7.2: Variable values obtained in the symmetric tube optimization with a cantilever load.
7.7.1 Function Definitions

The objective of this optimization is to minimize the mass of the second beam model while supporting a specified load. Two optimal beams will be found. One will support a cantilever load located at a unit distance from the wall. The other will support a column load the same distance from the wall. The design variables in this case are all the dimensions needed to define the geometry of the tube as shown in Fig. 7.7. Because the total beam length is fixed, the optimization will be repeated for different values of $n$ or discrete numbers of tube segments. The complete function to be optimized is

$$m_2 = f(H, W, T, t, \mu, \alpha, \omega, w, n)$$ (7.2)
where all other geometric dimensions and materials properties are held constant. Here, \( m_2 \) is the mass of the beam, \( H \) is the side wall height, \( W \) is the top wall width, \( T \) is the hinge offset, \( t \) is the thickness of the wall of the hollow panels, \( \mu \) is the side panel slant angle, \( \alpha \) is the deployment angle, \( \omega \) is the load angle measured about the tube longitudinal axis, \( w \) is the ratio of the hinge width to the total width \( W \) and \( n \) is the number of tube segments. Gradients of the mass with respect to the design variables are also calculated. During the optimization process, all input variables except for \( n \) and output masses are assumed to be continuous functions. This will allow gradient-based optimization to be performed for each value of \( n \). The complete function and its gradients are

\[
V = \frac{4H\lambda t \sin(\alpha) \sin(\mu)}{C} + \frac{4\lambda t W \sin(\alpha)}{C} - \frac{4HntT \sin(\mu) \cos(\mu)}{C} - \frac{4ntT^2 \cos(\mu)}{C} - \frac{4ntW \cos(\mu)}{C} + 2Hnt + nt^2 + 2ntW + 2ntW \tag{7.3}
\]

where \( V \) is the volume of material in the beam (the objective), \( \lambda \) is the distance of the load from the wall and

\[
C = \sqrt{\sin^2(\alpha) - \cos^2(\mu)} \tag{7.4}
\]

and

\[
\frac{\partial V}{\partial H} = 2nt \left( \frac{2\sin(\mu) \left( \frac{\lambda \sin(\alpha)}{n} - T \cos(\mu) \right)}{C} + T \right) \tag{7.5}
\]
\[ \frac{\partial V}{\partial W} = 2nt \left( 2 \left( \frac{\lambda \sin(\alpha)}{n} - T \cos(\mu) \right) \frac{C}{C} + Tw + T \right) \quad (7.6) \]

\[ \frac{\partial V}{\partial T} = \frac{4\lambda t \sin(\alpha)}{C} - \frac{4Hnt \sin(\mu) \cos(\mu)}{C} - \frac{8ntT \cos(\mu)}{C} \]
\[ - \frac{4ntW \cos(\mu)}{C} + 2Hnt + 2nt + 2ntwW + 2ntW \quad (7.7) \]

\[ \frac{\partial V}{\partial t} = \frac{4H \lambda \sin(\alpha) \sin(\mu)}{C} + \frac{4\lambda t \sin(\alpha)}{C} + \frac{4\lambda W \sin(\alpha)}{C} - \frac{4Hnt \sin(\mu) \cos(\mu)}{C} - \frac{4nT^2 \cos(\mu)}{C} - \frac{4ntW \cos(\mu)}{C} + 2Hnt + nT^2 + 2ntwW + 2ntW \quad (7.8) \]

\[ \frac{\partial V}{\partial \mu} = -\frac{4H \lambda t \sin(\alpha) \cos^3(\mu)}{C^3} - \frac{2H \lambda t \sin(\alpha) \cos(2\alpha) \cos(\mu)}{C^3} + \frac{2H \lambda t \sin(\alpha) \cos(2\mu) \cos(\mu)}{C^3} + \frac{4Hnt \sin^2(\alpha) \sin^2(\mu)}{C^3} - \frac{4Hnt \sin^2(\alpha) \cos^2(\mu)}{C^3} - \frac{4\lambda t \sin(\alpha) \sin(\mu) \cos(\mu)}{C^3} + \frac{4ntW \sin^2(\alpha) \sin(\mu)}{C^3} + \frac{4ntW \sin^2(\alpha) \cos(\mu)}{C^3} + \frac{4Hnt \cos^4(\mu)}{C^3} \quad (7.9) \]

\[ \frac{\partial V}{\partial \alpha} = -\frac{4H \lambda t \cos(\alpha) \sin(\mu) \cos^2(\mu)}{C^3} + \frac{4Hnt \sin(\alpha) \cos(\alpha) \sin(\mu) \cos(\mu)}{C^3} - \frac{4\lambda t \cos(\alpha) \cos^2(\mu)}{C^3} - \frac{4\lambda t \cos(\alpha) \cos^2(\mu)}{C^3} + \frac{4ntT \cos(\alpha) \cos^2(\mu)}{C^3} + \frac{4ntW \sin(\alpha) \cos(\alpha) \cos(\mu)}{C^3} \quad (7.10) \]

\[ \frac{\partial V}{\partial W} = 2ntW \quad (7.11) \]
7.7.2 Optimization Constraints

Several constraints will also need to be applied to all of the design variables to narrow the search space and prevent failures from occurring during the FEA analysis. All of the dimensions needed to define the structure geometry will be constrained as shown in the optimization definition in Section 7.7.3. In the case of the BFGS optimizer and the genetic algorithm, a quadratic penalty was used to implement any pertinent constraints. The stresses in the structure will also be limited to be below the yield stress given in Table 7.3.

7.7.3 Optimization Routine

Multiple optimization methods were used, including gradient-based methods like BFGS and SLSQP, as well as gradient-free genetic algorithms. With respect to convergence, both the gradient-based solvers were run until convergence was achieved with a gradient magnitude less than $10^{-6}$. The genetic algorithm was run for one hundred generations with a population of sixteen times the number of input variables. The complete optimization problem is

\[
\begin{align*}
\text{minimize} & \quad V \\
\text{with respect to} & \quad H, W, T, t, \mu, \alpha, \omega, w, n \\
\text{subject to} & \quad 0.01m < H, W < 0.8m \\
& \quad 0.004m < T \\
& \quad 0.001m < t \\
& \quad 45^\circ \leq \mu \leq 89^\circ \\
& \quad 50^\circ \leq \alpha \leq 90^\circ \\
& \quad 0.25 \leq w \leq 0.8 \\
& \quad |\sigma_{\text{max}}| \leq \sigma_{\text{yield}} \\
& \quad 0 < \omega, n
\end{align*}
\]

where separate optimizations are run for a cantilever load scenario and a column load scenario. The load magnitude $F$ and distance from the wall $\lambda$ are the same in both situations.
7.7.4 Optimization Results

Of the different optimization methods attempted, the BFGS algorithm produced the lightest model. This is unusual but could be due to scaling or noise in the non-linear, finite-element simulation. Performing this optimization using the parameters in Table 7.3 yielded the results shown in Table 7.4. A convergence study was not performed for these results. Table 7.4 shows both the final design variable values and the final mass of the structure for a beam loaded as a cantilever. A similar set of results are shown in Table 7.5 where a column load of magnitude $F$ is applied. These results were obtained using the same set of parameters. An important limitation of these values is that they do not take the possibility of buckling into account.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus</td>
<td>$200 \times 10^9$ Pa</td>
<td></td>
<td>$\lambda$</td>
<td>1 m</td>
<td></td>
</tr>
<tr>
<td>Yield Strength</td>
<td>$172 \times 10^6$ Pa</td>
<td></td>
<td>$\rho$</td>
<td>8000 kg/m$^3$</td>
<td></td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
<td>0.30</td>
<td>-</td>
<td>Mesh size</td>
<td>0.01 m</td>
<td></td>
</tr>
<tr>
<td>$F$</td>
<td>2000 N</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7.3: Parameters used in the optimization of the non-symmetric tube

<table>
<thead>
<tr>
<th>Design Variable</th>
<th>Value</th>
<th>Units</th>
<th>Design Variable</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_2$</td>
<td>1.48</td>
<td>kg</td>
<td>$\mu$</td>
<td>74.5°</td>
<td>deg</td>
</tr>
<tr>
<td>$H$</td>
<td>0.270</td>
<td>m</td>
<td>$\alpha$</td>
<td>90.3°</td>
<td>deg</td>
</tr>
<tr>
<td>$W$</td>
<td>0.0175</td>
<td>m</td>
<td>$\omega$</td>
<td>0.00°</td>
<td>deg</td>
</tr>
<tr>
<td>$T$</td>
<td>0.0549</td>
<td>m</td>
<td>$w$</td>
<td>0.702</td>
<td>-</td>
</tr>
<tr>
<td>$t$</td>
<td>0.00136</td>
<td>m</td>
<td>$n$</td>
<td>1</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 7.4: Values obtained in the non-symmetric tube optimization with a cantilever load
These results can be compared to a solid tube with a square cross-section to gain a sense of how efficient these origami tubes are when supporting loads. It can be shown that if a rectangular tube with dimensions shown in Figure 7.8 is loaded as a cantilever beam with a force $F$, then the safety factor for this tube is given by

$$SF = \frac{\sigma}{\sigma_y} = \frac{2H\sigma_y t(H + 3W)}{3FL} = 6.79$$

where $\sigma_y$ is the yield strength of the material and the other variables are as previously defined. This equation was derived assuming that the thickness of the walls was small in comparison with the other dimensions. The thickness of the comparable beam is twice the thickness, $t$, because the panels in the simulation were hollow with a thickness of $t$. This allows the quantity of material to remain roughly the same. The value of 6.79 was calculated using variable values from the optimized beam. This result indicates that with roughly the same geometry and amount of material, the solid tube would enjoy a factor of safety of 6.79 while the deployable tube would be loaded at its yield strength.

### 7.7.5 Symmetric Tube with Cutouts

An observation from the optimization results was that the high stress areas in the tube-based structure were generally located at the edges of the panels. To achieve a tube design with less mass, a study was performed to evaluate the effects of removing material from the panels. To determine areas where material should be removed, a preliminary optimization was performed using a truss representation of the panel-based tube. In this model each panel was represented by four links for the outside edges and two links which connect opposing panel corners. To keep the geometry of

<table>
<thead>
<tr>
<th>Design Variable</th>
<th>Value</th>
<th>Units</th>
<th>Design Variable</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_2$</td>
<td>0.197</td>
<td>kg</td>
<td>$\mu$</td>
<td>89.0°</td>
<td>deg</td>
</tr>
<tr>
<td>$H$</td>
<td>0.0205</td>
<td>m</td>
<td>$\alpha$</td>
<td>90.0°</td>
<td>deg</td>
</tr>
<tr>
<td>$W$</td>
<td>0.0214</td>
<td>m</td>
<td>$w$</td>
<td>0.615</td>
<td>-</td>
</tr>
<tr>
<td>$T$</td>
<td>0.0207</td>
<td>m</td>
<td>$n$</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>$t$</td>
<td>0.000971</td>
<td>m</td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>
each panel consistent in the design, the edge links were all kept the same size and the interior links were also sized the same. The overall length of the beam, panel angles, and number of sections were kept consistent with the optimized results of the panel based optimization. An optimization of the truss model was run where the interior link sizes and exterior link sizes were varied with the objective of minimizing mass without yielding. The results of this optimization are shown in Fig. 7.9 and revealed that a truss design with interior links smaller than exterior links is preferred. The truss model held the same load as the panel-based model, but only weighed 0.21 kg.

Based upon the results of the optimized truss model, a panel-based model was created with cutouts similar to those created by Song, Chen and Lu [140]. These cutouts are shown in Fig. 7.10. The cutout has the same shape as the panel and is placed in the center. The distance from the edge of the panel to the edge of the cutout was introduced as a new variable ($W$) in this design. This modified tube with cutouts was optimized again using $W$ and $t$ as design variables. The exterior panel dimensions were kept consistent with the results of the optimized full panel design. The optimized result of the design with cutouts is shown in Fig. 7.11. The results of this optimization are listed in Table 7.6. The panel model with cutouts reduced the mass of the structure from 0.7366 kg to 0.518 kg. To accommodate for the cutouts in the design, the thickness also increased from
0.498 mm to 1.2 mm to meet the yield-stress constraint. We can see from the stress distribution of this model as shown in Fig. 7.11 that the stress is distributed more evenly than previous designs.

The panel model with cutouts offers advantages over the basic panel model and the truss-based model. The obvious advantage over the panel model is the weight savings. An advantage that it has over the truss-based model is that it can fold flat similar to the panel model whereas the truss-based model is a structure and is not collapsible. The panel model with cutouts therefore

<table>
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<tr>
<th>Design Variable</th>
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<tbody>
<tr>
<td>$m_1$</td>
<td>0.518</td>
<td>kg</td>
</tr>
<tr>
<td>$t$</td>
<td>1.2</td>
<td>mm</td>
</tr>
<tr>
<td>$W$</td>
<td>0.0085</td>
<td>m</td>
</tr>
</tbody>
</table>
takes the strengths of the basic panel model and the truss-based model and combines them into one design.

### 7.8 Implementation of tube-based mechanisms

The origami tube-based mechanism presented was optimized to hold an end load with minimal mass. The optimization-based design exploration showed that a symmetric tube with cut-outs proved to be the best design of all of those explored in this study. To demonstrate the feasibility of this type of mechanism a prototype of this design was constructed. This prototype, shown in Fig. 7.12, is constructed of four sections. Each panel has the same geometry and is
constructed from an aluminum-polyethylene sandwich panel that has a high stiffness-to-weight ratio. The panels are connected using steel piano hinges which are attached to panels using rivets.

Despite being a highly over-constrained mechanism where tolerance stack up could cause the mechanism to bind while folding, the prototype achieves a smooth folding motion with little resistance from the hinges. The symmetric design of the prototype simplifies the manufacturing of the mechanism as every panel is identical and only one shape needs to be cut out several times. The prototype demonstrates the feasibility for tube-based mechanisms to be used in applications requiring a robust deployable mechanism.
7.9 Conclusion

This work has presented the evaluation of thick, rigidly-foldable origami tubes for structural purposes. Considerations for designing origami tubes were discussed and both symmetric and non-symmetric thick tubes were optimized for a minimum mass. Lightweight designs for all configurations were achieved and a symmetric tube with cutouts achieved the lightest design which is still deployable. Each of the tube configurations offer different strengths. The non-symmetric configuration offers the most flexibility in design as it can accommodate more geometries than the symmetric model. The symmetric model can offer a more robust design as it utilizes hinges at every axis whereas the non-symmetric has only two hinges connecting separate sections together. The extra hinges in the symmetric design allow it to be more robust in handling loading from various directions.

The optimization of the tube geometry as a cantilever beam provides an example of a beneficial application of rigid-foldable, thick origami. The ability of thick origami to provide both rigid-foldable motion as well as stiffness is a significant advantage for mechanisms such as those discussed in this work. Minimizing the weight of a thick-origami mechanism is one example of a consideration that will utilize these advantages to better address engineering problems.
CHAPTER 8. CONCLUSIONS AND RECOMMENDATIONS

8.1 Summary

Rigid-foldable mechanisms possess many characteristics that are valuable to engineers but also present unique challenges to those attempting to design them. These challenges often relate to converting a useful origami pattern from one category to another as shown in Figure 1.1. One of these challenges is converting non-rigid-foldable models to rigid-foldable equivalents. Another significant obstacle to their design is ensuring that a rigid-foldable mechanism has sufficient mechanical rigidity to not impede movement and deploy to a precise shape. This work seeks to address these challenges by developing methods for designing rigid- and pseudo-rigid-foldable mechanisms. Each chapter in this work describes specific problems associated with designing rigid-foldability and possible solutions.

In Chapter 2, a technique for converting inflexible origami patterns into rigid-foldable mechanisms is described. This technique is made possible by development of equations for predicting the change in mobility produced by three different crease pattern alterations. This enables the number of modifications needed to be estimated first. It was demonstrated on a crease pattern to showcase its usefulness.

Chapter 3 discusses methods for generating permutations of an origami vertex and using those permutations to design thick-origami mechanisms and tessellations of thick vertices. This will facilitate the transition of origami models from thin to thick materials. Proofs are written for each of these methods. Multiple new mechanisms are created from existing models to demonstrate its capabilities. Two of these new mechanisms are used in the design of a singly-curved linkage tessellation.

Another technique for modifying an origami crease pattern to make it pseudo-rigid-foldable is presented in Chapter 4. The possibility of using compliant panels to allow movement is suggested and formulas are developed to calculate the actuation effort of the resulting mechanism.
This technique supports the design of small-deformation origami where some strain is acceptable. These results are compared with experimental results.

Chapter 5 discusses the management of stiffness of rigid-foldable mechanisms through deployable stiffeners. This is done to provide methods for transitioning from large- to small-deformation origami. Three different deployable stiffeners are considered and compared parametrically and one is found to be dominant.

The problem of controlling mechanism stiffness is also addressed in Chapter 6 by considering a compliant, deployable stiffener. Formulas are developed predicting its stiffening characteristics and experimental confirmation for these formulas are found.

A specific design problem utilizing a rigid-foldable, thick-origami mechanism is described in Chapter 7 to illustrate an application for these methods. The geometry and kinematics of an origami tube are modeled and optimized to carry a cantilever load. A second variation of this tube is also modeled and optimized. Utilizing and optimizing this model demonstrates some of the advantages in stiffness that thick-origami can provide.

Together, the methods discussed in these chapters provide means of mitigating the problems of finding useful origami patterns that also appear in the correct category as shown in Figure 1.1. Previously, successfully using a pattern in an undesirable category could involve serious disadvantages such as material limitations. However, the methods developed in this work allow such patterns to be transitioned to other categories in an organized way.

8.2 Relevance

Developing new techniques for designing rigid- and pseudo-rigid-foldable mechanisms will be influential because of the number of mechanisms and applications that require these characteristics. Spacecraft mechanisms, for example, are often constructed with rigid components and therefore must always be designed using rigid-foldable mechanisms. This research also addresses issues relating to the stiffness of the mechanism. Because of the importance of reducing mass in spacecraft applications, deployable mechanisms on spacecraft are often large, light-weight structures that suffer from low bending stiffness. Techniques that manage deployable stiffness will help ensure that these devices can perform their intended function while remaining rigid.
8.3 Contributions

Research conducted during my PhD program resulted in the following publications:


One additional paper has been proposed for publication.

• Yellowhorse, A., and Howell L. (Submitted) ”Creating Linkage Permutations to Prevent Self-Intersection and Enable Deployable Networks of Thick-Origami.” Proceeding of the National Academy of Sciences, submitted May 9 (included as Chapter 3)
8.4 Conclusions

This work on designing rigid-foldable mechanisms resulted in multiple significant conclusions. One of these is that multiple techniques exist for designing mechanisms that are rigid-foldable. These techniques include methods for altering existing patterns to create rigid-foldability and generating novel variations of existing mechanisms. Another important conclusion is that multiple strategies exist for managing the stiffness of pseudo-rigid-foldable mechanisms. Other specific conclusions made are listed below:

- Origami-inspired mechanisms can be efficiently made rigid-foldable through modifications to their crease pattern
  - The relative value of different modifications in changing mobility can be compared
  - The number of these modifications can be estimated before construction
- Additional rigid-foldable mechanisms can be generated by modifying existing models.
  - Mechanisms generated in this way are guaranteed to be rigid-foldable
  - Simple variable substitution can quickly generate their governing equations
  - Permutations of thick-origami vertices can also be generated
  - Vertex permutations can be used to design thick-origami tessellations
- Compliant panels can create pseudo-rigid-foldability and their actuation effort can be found
- Stiffeners that create rigidity in rigid-foldable mechanisms can be compared parametrically
- Compliant, deployable stiffeners can improve rigidity without impacting deployment ratio

Together, these conclusions address important problems that arise when designing rigid-foldable mechanisms. These problems affect both the early design decisions such as crease pattern selection as well as later difficulties like managing mechanism stiffness. Although additional research should be done on this topic, these results will help simplify the process of designing rigid-foldable mechanisms.
8.5 Recommendations

Although significant progress has resulted from the research conducted in this work, more research must be done in a variety of areas. More research on rigid-foldability must be done to better understand the wide range of problems that can arise when attempting to design mechanisms with this property. One important problem that remains unsolved includes establishing a simple set of necessary and sufficient conditions for rigid-foldability. Because these criteria have not been discovered, analyzing and designing origami mechanisms remains a complicated process. Possible solutions to this problem may include investigating better matrix methods for detecting origami singularities. Such a method could improve either the simplicity or the accuracy of current approaches.

Another important problem that must be solved includes developing more efficient methods for generating tessellations of rigid-foldable vertices. Currently, these tessellations must often be created based on an existing paper model. This limits potential to design new tessellation types. Better means of accomplishing this will provide more options to designers. One potential method for approaching this problem includes developing systematic techniques for assembling vertices into units that can be tessellated. Only a small number of vertex combinations can form useful tessellating units and a method for identifying these combinations would significantly increase options for tessellation design.

Significant opportunities also exist for solving problems that include stiffness in the design of rigid-foldable, origami-inspired mechanisms. One important need in this area is for better means of stiffening and controlling creases in origami. Creases must be flexible by definition to allow movement of the mechanism. However, this flexibility frequently leads to large amounts of undesirable parasitic motion. This parasitic motion can degrade the stiffness of the pattern and impact its ability to deploy precisely. Compliant hinges with reconfigurable geometry may be one solution to this issue. Such a device could remain thin and bend easily during actuation and swell later to increase its stiffness. Other needs include considering other stiffeners that may be used in deployed mechanisms. Comparing new stiffer types with those in this work would further expand options for engineers attempting to design light-weight, origami-inspired deployables.
REFERENCES


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A.1 Homogeneous Transformation

The homogeneous transformation $T_i$ is given by

$$T_i = \begin{pmatrix} R & d \\ 0 & 1 \end{pmatrix}$$

where $R$ is a $3 \times 3$ rotation matrix, $d$ is a $3 \times 1$ translation vector, $C(...) = \cos(...) \text{ and } S(...) = \sin(...)$. 

A.2 Bennett Linkage Relationships

It has been shown [141] that Definition 1 results in the following equations for a mechanism with four links:

$$a_1 = a_3 \quad (A.2)$$
$$a_2 = a_4 \quad (A.3)$$
$$\alpha_1 = \alpha_3 \quad (A.4)$$
$$\alpha_2 = \alpha_4 \quad (A.5)$$
$$d_1 = d_2 = d_3 = d_4 = 0 \quad (A.6)$$
$$\frac{a_1}{a_2} = \frac{\sin \alpha_1}{\sin \alpha_2} \quad (A.7)$$
\[ 2\pi = \theta_1 + \theta_3 \quad (A.8) \]
\[ 2\pi = \theta_2 + \theta_4 \quad (A.9) \]
\[ \tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} = \frac{\sin \left( \frac{1}{2}(\alpha_2 + \alpha_1) \right)}{\sin \left( \frac{1}{2}(\alpha_2 - \alpha_1) \right)} \quad (A.10) \]

where \( \theta_i, a_i, d_i, \) and \( \alpha_i \) correspond to the DH parameters defined previously. An interesting observation that can be made about Equation A.10 is its similarity to the fold angle multiplier \( \mu \) defined for rigid-foldable origami [1].

### A.3 Proofs

#### A.3.1 Proof of Theorem 1

**Proof:** Showing equivalence is possible by making the substitutions below:

\[ \alpha_j = - (\pi - \alpha'_j) \quad (A.11) \]
\[ \alpha_k = - (\pi - \alpha'_k) \quad (A.12) \]
\[ \theta_k = - \theta'_k \quad (A.13) \]
\[ d_k = - d'_k \quad (A.14) \]

and the transformations become

\[ T'_{1j} = \begin{pmatrix}
C(\theta_j) & C(\alpha_j)S(\theta_j) & -S(\alpha_j)S(\theta_j) & C(\theta_j)a_j \\
S(\theta_j) & -C(\alpha_j)C(\theta_j) & C(\alpha_j)S(\theta_j) & S(\theta_j)a_j \\
0 & -S(\alpha_j) & C(\alpha_j) & d_j \\
0 & 0 & 0 & 1
\end{pmatrix} \quad (A.15) \]

\[ T'_{2k} = \begin{pmatrix}
C(\theta_k) & -C(\alpha_k)S(\theta_k) & S(\alpha_k)S(\theta_k) & C(\theta_k)a_k \\
-S(\theta_k) & -C(\alpha_k)C(\theta_k) & C(\alpha_k)S(\theta_k) & -S(\theta_k)a_k \\
0 & -S(\alpha_k) & -C(\alpha_k) & -d_k \\
0 & 0 & 0 & 1
\end{pmatrix} \quad (A.16) \]
Taking the difference between the two pairs of transformations and simplifying yields

\[ T'_1 T'_2 - T_1 T_2 = 0 \quad (A.17) \]

Therefore, \( T'_1 T'_2 = T_1 T_2 \).

**A.3.2 Proof of Corollary 1.1**

*Proof:* By Definition 1, a closed linkage is a linkage if \( T_1 T_2 \ldots T_n = I \). Also, Theorem 1 guarantees that \( T_i T_{i+1} = T'_i T'_{i+1} \). Without loss of generality, making this substitution in the linkage relation yields \( T'_1 T'_2 T_3 \ldots T_n = I \). Because \( T'_1 \) and \( T'_2 \) are also homogeneous transformations, the modified linkage is still a linkage.

**A.3.3 Proof of Theorem 2**

*Proof:* Because the product of any number of homogeneous transformations is also a homogeneous transformation, we can write \( T_1 T_2 \ldots T_{i-1} T_i T_{i+1} \ldots T_n T_j = T_a T_b T_j \). Evaluating the product \( T_i T_b T_j \) gives

\[
\begin{pmatrix}
I & d^i \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
R_b & d_b^{i+1} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
I & -d^j \\
0 & 1
\end{pmatrix}
= \\
\begin{pmatrix}
R_b & -R_b d^j + d_b + d^i \\
0 & 1
\end{pmatrix}
\quad (A.18)
\]

Because \( R_b \) relates the frame at \( j \) to the frame at \( i \), we know that \( d^j = R_b d^i \). Substituting this relation into Equation A.18 shows that \( T_i T_b T_j = T_b \). Therefore, \( T_1 T_2 \ldots T_n = T_1 T_2 \ldots T_{i-1} T_i T_{i+1} \ldots T_n T_j = I \).

Also, because the new product of transformations is equal to the identity transformation, it is also a linkage.

**A.4 Other Multi-Vertex Mechanisms**

Here, we show other models constructed with Linkage I and II vertices.
Figure A.1: Foam-board prototype of a non-developable-thick tube. Its motion is further illustrated in Video S2.

Figure A.2: The two different linkages used in the symmetric-thick-origami tube. Left: Linkage III. Center: Linkage I. Right: Linkage III.

A.5 Prototype Geometry

The prototypes in Figures 3.8 and 3.9 were designed using several specific formulas derived using the methods presented in the paper. Equations 11, 15 and 17 can be used to design linkage tessellations with controllable shape by specifying the number of available degrees of freedom in a given model. One procedure for setting these degrees of freedom is by specifying a zig-zag pattern as shown in Figure A.4a that sufficiently approximates a desired curve. Because this curve approximation sets vertex geometry along a row of vertices with coordinates \((0, n)\), the relationships defined above specify the geometry of the remaining pattern. If we specify the individual segment lengths \(L_1, L_2, \ldots, L_n\), the internal angles \(\phi_1, \phi_2, \ldots, \phi_n\), the starting fold angle \(\theta_i\) and assume that \(\alpha_1 = \alpha_2\), the remaining parameters can be found. Given \(\theta_i\) and all \(L_i\)’s and \(\phi_i\)’s, we have

\[
\tan \frac{\theta_i}{2} \tan \frac{\theta_j}{2} = \mu_{ji} = \sec \alpha_{ji} \tag{A.19}
\]

The spherical law of cosines can be used to relate \(\alpha_{ji}, \theta_j\) and \(\phi_i\) in Figure A.4, giving

\[
\cos \phi_i = \cos^2 \alpha_{ji} + \sin^2 \alpha_{ji} \cos \theta_j
\]

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Figure A.3: Eggbox inspired non-developable tube comprised of excess-vertex Bennett linkages and deficient-vertex spherical linkages.

(a) Both Linkages
(b) Linkage II
(c) Linkage III

Figure A.4: (a) A zig-zag profile formed by the ridge of the pattern described by angles $\phi_i$ and lengths $L_i$ and (b) (c) angular dimensions for both Linkage II an III vertices

\[
\cos \theta_j = -\frac{1}{2} (1 + \cos \theta_i) + \cos \phi_i \sin^2 \frac{\theta_i}{2} \quad (A.20)
\]

If $\theta_i$ and $\phi_i$ are known, then $\theta_j$ can be determined from Equation A.20. This can in turn be used to calculate $\alpha_{ji}$ from Equation A.19. The exact values of $L_i$ can be set freely. Because
the geometry expressed in Figure A.4 applies to both over- and under-developed vertices, this procedure can be used to design both types of vertices.

### A.5.1 Model Construction

The model in Figure 3.8 was constructed using vertices with angles shown in Table A.1. The model in Figure 3.9 was fabricated using vertices with the vertex angles shown in Table A.2.

#### Table A.1: The panel sector angles in degrees for each vertex in the prototype in Figure 3.8 are listed for each row \( i \) and column \( j \) location of the vertex

<table>
<thead>
<tr>
<th>( j = )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = 1 )</td>
<td>120°</td>
<td>60°</td>
<td>120°</td>
<td>60°</td>
<td>110°</td>
<td>60°</td>
</tr>
<tr>
<td>( i = 2 )</td>
<td>60°</td>
<td>120°</td>
<td>60°</td>
<td>120°</td>
<td>80°</td>
<td>120°</td>
</tr>
<tr>
<td>( i = 3 )</td>
<td>120°</td>
<td>60°</td>
<td>120°</td>
<td>60°</td>
<td>110°</td>
<td>60°</td>
</tr>
</tbody>
</table>

#### Table A.2: The panel sector angles in degrees for each vertex in the prototype in Figure 3.9 are listed for each row \( i \) and column \( j \) location of the vertex

<table>
<thead>
<tr>
<th>( j = )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>17</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = 1 )</td>
<td>100°</td>
<td>70°</td>
<td>100°</td>
<td>...</td>
<td>100°</td>
<td>70°</td>
</tr>
</tbody>
</table>

### A.6 Vertex Developability

Here, we define developability and related terms. These new mechanism configurations can be categorized based on the sum of all the panel angles, \( D \), where

\[
D = \sum_{i}^{n} |\alpha_i| \tag{A.21}
\]
and $\alpha_i$ are the panel angles for a given mechanism. If $D < 2\pi$, then the vertex is defined as under-developed. If $D = 2\pi$, then the vertex is developable. If $D > 2\pi$ then the vertex is defined as over-developed.
APPENDIX B. PSEUDO-RIGID-BODY MODEL CONSTANTS

The following tables contain values of the constants needed for the pseudo-rigid-body model discussed previously. Each table applies to triangle panels with a particular vertex angle \( \alpha \). Within each table, different sets of constants are listed with respect to a particular side ratio.
Table B.1: A panel where $\alpha = 30^\circ$

<table>
<thead>
<tr>
<th>$s/t$</th>
<th>$\lambda$ (rad)</th>
<th>$\mu$ (rad)</th>
<th>$c_2$</th>
<th>$c_1$</th>
<th>$c_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.500</td>
<td>0.182</td>
<td>0.342</td>
<td>0.543</td>
<td>19.853</td>
<td>0.077</td>
</tr>
<tr>
<td>0.600</td>
<td>0.178</td>
<td>0.345</td>
<td>-0.300</td>
<td>21.879</td>
<td>-0.151</td>
</tr>
<tr>
<td>0.700</td>
<td>0.178</td>
<td>0.346</td>
<td>-0.585</td>
<td>23.016</td>
<td>-0.178</td>
</tr>
<tr>
<td>0.800</td>
<td>0.182</td>
<td>0.341</td>
<td>-0.521</td>
<td>23.962</td>
<td>-0.129</td>
</tr>
<tr>
<td>0.900</td>
<td>0.210</td>
<td>0.314</td>
<td>0.678</td>
<td>24.878</td>
<td>0.124</td>
</tr>
<tr>
<td>1.000</td>
<td>0.203</td>
<td>0.321</td>
<td>0.286</td>
<td>25.487</td>
<td>0.061</td>
</tr>
<tr>
<td>1.100</td>
<td>0.207</td>
<td>0.317</td>
<td>0.157</td>
<td>26.357</td>
<td>0.020</td>
</tr>
<tr>
<td>1.200</td>
<td>0.213</td>
<td>0.311</td>
<td>0.276</td>
<td>26.967</td>
<td>0.044</td>
</tr>
<tr>
<td>1.300</td>
<td>0.219</td>
<td>0.305</td>
<td>0.280</td>
<td>27.611</td>
<td>0.040</td>
</tr>
<tr>
<td>1.400</td>
<td>0.207</td>
<td>0.316</td>
<td>-0.129</td>
<td>27.677</td>
<td>0.024</td>
</tr>
<tr>
<td>1.500</td>
<td>0.216</td>
<td>0.308</td>
<td>0.149</td>
<td>28.058</td>
<td>0.090</td>
</tr>
<tr>
<td>1.600</td>
<td>0.210</td>
<td>0.314</td>
<td>-0.276</td>
<td>28.477</td>
<td>0.013</td>
</tr>
<tr>
<td>1.700</td>
<td>0.206</td>
<td>0.318</td>
<td>-0.543</td>
<td>28.781</td>
<td>-0.038</td>
</tr>
<tr>
<td>1.800</td>
<td>0.219</td>
<td>0.305</td>
<td>-0.079</td>
<td>29.104</td>
<td>0.069</td>
</tr>
<tr>
<td>1.900</td>
<td>0.210</td>
<td>0.314</td>
<td>-0.538</td>
<td>29.359</td>
<td>-0.021</td>
</tr>
<tr>
<td>2.000</td>
<td>0.215</td>
<td>0.309</td>
<td>-0.426</td>
<td>29.692</td>
<td>0.001</td>
</tr>
</tbody>
</table>

Table B.2: A panel where $\alpha = 40^\circ$

<table>
<thead>
<tr>
<th>$s/t$</th>
<th>$\lambda$ (rad)</th>
<th>$\mu$ (rad)</th>
<th>$c_2$</th>
<th>$c_1$</th>
<th>$c_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.500</td>
<td>0.242</td>
<td>0.456</td>
<td>0.222</td>
<td>15.022</td>
<td>0.044</td>
</tr>
<tr>
<td>0.600</td>
<td>0.236</td>
<td>0.462</td>
<td>-0.430</td>
<td>16.480</td>
<td>-0.112</td>
</tr>
<tr>
<td>0.700</td>
<td>0.246</td>
<td>0.452</td>
<td>-0.293</td>
<td>17.339</td>
<td>-0.054</td>
</tr>
<tr>
<td>0.800</td>
<td>0.240</td>
<td>0.458</td>
<td>-0.714</td>
<td>18.181</td>
<td>-0.148</td>
</tr>
<tr>
<td>0.900</td>
<td>0.271</td>
<td>0.427</td>
<td>0.026</td>
<td>18.918</td>
<td>0.022</td>
</tr>
<tr>
<td>1.000</td>
<td>0.259</td>
<td>0.439</td>
<td>-0.403</td>
<td>19.471</td>
<td>-0.069</td>
</tr>
<tr>
<td>1.100</td>
<td>0.272</td>
<td>0.426</td>
<td>-0.206</td>
<td>20.020</td>
<td>-0.023</td>
</tr>
<tr>
<td>1.200</td>
<td>0.292</td>
<td>0.406</td>
<td>0.091</td>
<td>20.573</td>
<td>0.040</td>
</tr>
<tr>
<td>1.300</td>
<td>0.276</td>
<td>0.422</td>
<td>-0.278</td>
<td>20.724</td>
<td>0.008</td>
</tr>
<tr>
<td>1.400</td>
<td>0.266</td>
<td>0.432</td>
<td>-0.582</td>
<td>20.943</td>
<td>-0.038</td>
</tr>
<tr>
<td>1.500</td>
<td>0.277</td>
<td>0.421</td>
<td>-0.450</td>
<td>21.352</td>
<td>-0.005</td>
</tr>
<tr>
<td>1.600</td>
<td>0.275</td>
<td>0.423</td>
<td>-0.567</td>
<td>21.626</td>
<td>-0.031</td>
</tr>
<tr>
<td>1.700</td>
<td>0.278</td>
<td>0.420</td>
<td>-0.626</td>
<td>21.959</td>
<td>-0.041</td>
</tr>
<tr>
<td>1.800</td>
<td>0.287</td>
<td>0.411</td>
<td>-0.500</td>
<td>22.263</td>
<td>-0.017</td>
</tr>
<tr>
<td>1.900</td>
<td>0.286</td>
<td>0.412</td>
<td>-0.574</td>
<td>22.432</td>
<td>-0.021</td>
</tr>
<tr>
<td>2.000</td>
<td>0.282</td>
<td>0.416</td>
<td>-0.716</td>
<td>22.583</td>
<td>-0.042</td>
</tr>
</tbody>
</table>
Table B.3: A panel where $\alpha = 50^\circ$

<table>
<thead>
<tr>
<th>$s/t$</th>
<th>$\lambda$ (rad)</th>
<th>$\mu$ (rad)</th>
<th>$c_2$</th>
<th>$c_1$</th>
<th>$c_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.500</td>
<td>0.296</td>
<td>0.576</td>
<td>-0.208</td>
<td>12.287</td>
<td>-0.036</td>
</tr>
<tr>
<td>0.600</td>
<td>0.310</td>
<td>0.563</td>
<td>-0.137</td>
<td>13.184</td>
<td>0.009</td>
</tr>
<tr>
<td>0.700</td>
<td>0.307</td>
<td>0.566</td>
<td>-0.394</td>
<td>13.973</td>
<td>-0.040</td>
</tr>
<tr>
<td>0.800</td>
<td>0.298</td>
<td>0.575</td>
<td>-0.791</td>
<td>14.696</td>
<td>-0.134</td>
</tr>
<tr>
<td>0.900</td>
<td>0.334</td>
<td>0.539</td>
<td>-0.272</td>
<td>15.360</td>
<td>-0.012</td>
</tr>
<tr>
<td>1.000</td>
<td>0.322</td>
<td>0.551</td>
<td>-0.628</td>
<td>15.850</td>
<td>-0.088</td>
</tr>
<tr>
<td>1.100</td>
<td>0.339</td>
<td>0.534</td>
<td>-0.347</td>
<td>16.089</td>
<td>0.018</td>
</tr>
<tr>
<td>1.200</td>
<td>0.337</td>
<td>0.535</td>
<td>-0.530</td>
<td>16.503</td>
<td>-0.015</td>
</tr>
<tr>
<td>1.300</td>
<td>0.330</td>
<td>0.543</td>
<td>-0.699</td>
<td>16.751</td>
<td>-0.045</td>
</tr>
<tr>
<td>1.400</td>
<td>0.334</td>
<td>0.539</td>
<td>-0.818</td>
<td>17.188</td>
<td>-0.075</td>
</tr>
<tr>
<td>1.500</td>
<td>0.333</td>
<td>0.540</td>
<td>-0.828</td>
<td>17.314</td>
<td>-0.060</td>
</tr>
<tr>
<td>1.600</td>
<td>0.346</td>
<td>0.527</td>
<td>-0.688</td>
<td>17.598</td>
<td>-0.028</td>
</tr>
<tr>
<td>1.700</td>
<td>0.352</td>
<td>0.521</td>
<td>-0.766</td>
<td>17.968</td>
<td>-0.051</td>
</tr>
<tr>
<td>1.800</td>
<td>0.350</td>
<td>0.523</td>
<td>-0.816</td>
<td>18.117</td>
<td>-0.059</td>
</tr>
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<td>1.900</td>
<td>0.342</td>
<td>0.531</td>
<td>-0.982</td>
<td>18.258</td>
<td>-0.092</td>
</tr>
<tr>
<td>2.000</td>
<td>0.344</td>
<td>0.529</td>
<td>-0.944</td>
<td>18.342</td>
<td>-0.073</td>
</tr>
</tbody>
</table>

Table B.4: A panel where $\alpha = 60^\circ$

<table>
<thead>
<tr>
<th>$s/t$</th>
<th>$\lambda$ (rad)</th>
<th>$\mu$ (rad)</th>
<th>$c_2$</th>
<th>$c_1$</th>
<th>$c_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.500</td>
<td>0.372</td>
<td>0.675</td>
<td>-0.057</td>
<td>10.212</td>
<td>0.053</td>
</tr>
<tr>
<td>0.600</td>
<td>0.358</td>
<td>0.690</td>
<td>-0.491</td>
<td>11.078</td>
<td>-0.032</td>
</tr>
<tr>
<td>0.700</td>
<td>0.359</td>
<td>0.688</td>
<td>-0.685</td>
<td>11.838</td>
<td>-0.079</td>
</tr>
<tr>
<td>0.800</td>
<td>0.349</td>
<td>0.699</td>
<td>-1.048</td>
<td>12.466</td>
<td>-0.163</td>
</tr>
<tr>
<td>0.900</td>
<td>0.382</td>
<td>0.665</td>
<td>-0.680</td>
<td>12.912</td>
<td>-0.042</td>
</tr>
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<td>1.000</td>
<td>0.353</td>
<td>0.694</td>
<td>-1.126</td>
<td>13.164</td>
<td>-0.138</td>
</tr>
<tr>
<td>1.100</td>
<td>0.369</td>
<td>0.678</td>
<td>-1.016</td>
<td>13.545</td>
<td>-0.095</td>
</tr>
<tr>
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<td>0.389</td>
<td>0.658</td>
<td>-0.845</td>
<td>13.925</td>
<td>-0.051</td>
</tr>
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<td>0.393</td>
<td>0.654</td>
<td>-0.962</td>
<td>14.330</td>
<td>-0.080</td>
</tr>
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<td>1.400</td>
<td>0.389</td>
<td>0.659</td>
<td>-1.055</td>
<td>14.521</td>
<td>-0.097</td>
</tr>
<tr>
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<td>0.404</td>
<td>0.644</td>
<td>-0.987</td>
<td>14.831</td>
<td>-0.077</td>
</tr>
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<td>0.411</td>
<td>0.636</td>
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<td>15.097</td>
<td>-0.079</td>
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<td>1.700</td>
<td>0.414</td>
<td>0.633</td>
<td>-0.954</td>
<td>15.188</td>
<td>-0.061</td>
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<td>0.640</td>
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</tr>
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<td>-0.916</td>
<td>15.672</td>
<td>-0.055</td>
</tr>
<tr>
<td>2.000</td>
<td>0.404</td>
<td>0.643</td>
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<td>15.540</td>
<td>-0.083</td>
</tr>
</tbody>
</table>
Table B.5: A panel where $\alpha = 70^\circ$

<table>
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<th>$s/t$</th>
<th>$\lambda \ (rad)$</th>
<th>$\mu \ (rad)$</th>
<th>$c_2$</th>
<th>$c_1$</th>
<th>$c_0$</th>
</tr>
</thead>
<tbody>
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<td>0.799</td>
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<td>8.868</td>
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</tr>
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<td>0.804</td>
<td>-0.628</td>
<td>9.660</td>
<td>-0.037</td>
</tr>
<tr>
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<td>0.803</td>
<td>-0.811</td>
<td>10.273</td>
<td>-0.057</td>
</tr>
<tr>
<td>0.800</td>
<td>0.430</td>
<td>0.792</td>
<td>-0.801</td>
<td>10.733</td>
<td>-0.037</td>
</tr>
<tr>
<td>0.900</td>
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<td>0.823</td>
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</tr>
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<td>-1.222</td>
<td>11.747</td>
<td>-0.111</td>
</tr>
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<td>0.814</td>
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<td>12.761</td>
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<td>-0.072</td>
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<td>-1.385</td>
<td>13.262</td>
<td>-0.081</td>
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<td>0.781</td>
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<td>13.336</td>
<td>-0.095</td>
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</table>

Table B.6: A panel where $\alpha = 80^\circ$

<table>
<thead>
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<th>$s/t$</th>
<th>$\lambda \ (rad)$</th>
<th>$\mu \ (rad)$</th>
<th>$c_2$</th>
<th>$c_1$</th>
<th>$c_0$</th>
</tr>
</thead>
<tbody>
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<td>0.500</td>
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<td>0.934</td>
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<td>7.874</td>
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<td>8.451</td>
<td>-0.012</td>
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<td>9.881</td>
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</tr>
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<td>10.059</td>
<td>-0.147</td>
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</tr>
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<td>0.942</td>
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<td>0.441</td>
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<td>10.937</td>
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<td>0.449</td>
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</table>
Table B.7: A panel where $\alpha = 90^\circ$

<table>
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<th>$\lambda$ (rad)</th>
<th>$\mu$ (rad)</th>
<th>$c_2$</th>
<th>$c_1$</th>
<th>$c_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.500</td>
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<td>-0.742</td>
<td>6.949</td>
<td>0.011</td>
</tr>
<tr>
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<td>7.587</td>
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<tr>
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<tr>
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<tr>
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</table>