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A Matched Payout Model for Investment, Consumption, and Insurance with a Risky Annuity Income

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ABSTRACT

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We introduce a new insurance instrument allowing retirees to hedge against risk of mortality and risk of default. At retirement, the retiree is allowed to purchase an annuity that provides a defaultable income stream over his lifetime. The time of mortality and time of default are both uncertain, but are accompanied by determined hazard rates. The retiree will make consumption and investment choices throughout his lifetime, which have certain restrictions: the retiree can never enter a bankruptcy state (negative total wealth), and the investment choices are made in a risk-free financial instrument (such as a treasury bill or bond) and a risky instrument (such as commodities or stock). The retiree also makes insurance premium payments which hedge against mortality and default risks simultaneously. This new form of insurance is one which can be implemented by financial institutions as a means for retirees to protect their illiquid assets. In doing so, we calculate the optimal annuity rate a retiree should purchase to maximize his utility of consumption and bequest.

Throughout the paper, we develop stochastic control models for a retiree's optimal investment and consumption policies over an uncertain planning horizon in several models which may or may not allow for insurance purchases. We find exact solutions to several models, and apply dynamic programming and the logarithmic transformation to other models to find numerical solutions when constraints are needed. We also analyze the effects of loading on insurance, analyzing the effects of more expensive insurance on the retiree's control policies and value functions. In particular, we will consider the model in which the retiree can purchase life insurance and credit default insurance (in the form of a credit default swap, or CDS) separately to hedge against life events. CDS's do not exist for annuities, but we extend this model by incorporating life insurance and the CDS into a single entity, which can be a viable, and realistic, option to hedge against risk. This model is beneficial in providing a solution to the annuity problem by showing that minimal annuity purchase is optimal.

Keywords: annuity, annuity puzzle, life insurance, consumption, investment, credit default swap, random endowment, matched payout

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Chapter 1. Introduction

When a retiree leaves the workforce, they have many questions they need to consider asking. Where will I retire? How will I spoil my grandchildren? How do I make sure I don't run out of money before I die? How do I leave my family financially secure when I die? All of these are good questions, but the last two are of particular interest. A retiree wants to enjoy their retirement, not stress about their financial stability. They don't want to live beyond their resources, and on the other hand they may not want a stockpile of cash left over when they die. The retiree wants to maximize the enjoyment, or utility, they get out of their hard earned money and investments. How do they do that?

A life annuity is a great financial instrument that provides an extra flow of income to retirees, in which one puts in a lump sum of money at retirement and over time collects a steady income until their death. Historically, annuities have been around since the time of the Romans, with soldiers making a lump-sum payment in return for lifetime payments made once a year, known as "annua." It was also a popular tool in 18$^{\text{th}}$ century England, where members of higher society could purchase annuities as a way to prevent their fall in status or grace. Other examples of the use of annuities over the years can be found in [1].

Today, this instrument still exists, and in recent years has become a more popular tool to use as part of retirement portfolios. It helps hedge against retirement risks and brings with it many pros and cons. The pros include, and are not limited to, restricting the retiree from burning through all their resources early on and then having nothing left for the last years of their life; paying a fixed income for the lifetime of the retiree, no matter how long they live, so they will be financially secure at all times; providing means for the retiree to be left with something to bequeath to their heirs. Cons include the fact that if the retiree puts all their money into an annuity, it is locked in and they may not be able to pay big medical expenses should they come up unexpectedly. Also, if the retiree dies early, the money does not get pulled out for posterity.

In past literature regarding life annuities, such as [2, 3], some have suggested that a retiree annuitizing (placing money into an annuity) all or most of their savings is optimal in terms of utility. This is where the annuity problem comes in, and is mentioned in [2]: If the literature says annuitizing most, if not all, of one's savings at the time of retirement is optimal, why are more people not doing it? Some have suggested that people don't annuitize more because they desire to leave money for their posterity at time of death - if the retiree puts everything into an annuity, there is no money returned at time of death, regardless of whether the retiree only lived another two months or 20 years. Others suggest that sharp health decline could be another factor. Potential factors that play into the discrepancy between optimal levels of annuitization and actual levels of annuitization are cited in [2, 4, 5].

One of the largest factors we consider is the possibility of default of the annuity, a fact passed over by most early papers on the topic. Annuities companies are just that, companies; they come and go like any business, and have the potential to go bankrupt. In that situation, the retiree must consider whether or not it's worth fully annuitizing if they face the possibility of losing everything.

In early work done on the optimization problem, solutions to optimal investment and consumption rules were found with varying degrees of success. These solutions were centered on optimizing the individual's utility on consumption, namely

$$E\left[\int_0^T U(t, c(t))dt\right],$$

where T, representing death, is fixed or some random variable in $[0, \overline{T}]$, and U is the utility for consumption, $c(t)$. Hakansson considered the arbitrary lifetime model in [6], but only in the discrete time case. Solutions were found in [2] and [3], but under strict conditions, and were mainly focused on showing conditions under which full annuitization is optimal. Even Merton only found solutions for models with no bequest motive in [7, 8]. These papers were simply stepping stones to more detailed work down the road.

Merton [8], Yaari [3], and Fischer [9] all considered an investor with life insurance with varying success, even in the case with a bequest motive, but never found a closed-form solution. Richard [10] found a solution by considering an investor with an arbitrary lifetime in a continuous-time model, with income, life insurance and a bequest motive. To deal with the added bequest motive, Richard [10] attempted to optimize

$$E\left[\int_0^T U(t, c(t))dt + B(T, Z(T))\right],$$

where B is the utility for bequest and $Z(T)$ is the liquid wealth after life insurance pays out at time of death. However, Richard had to add the constraint that the individual could not purchase life insurance at time $\overline{T}$, which had problems of its own. Another problem Richard faced was assuming the market was complete, meaning the individual could buy and sell life insurance; this is not how insurance markets operate. One other restriction evident in those papers is how they treated the process as a life cycle model, in which individuals tend to consume equally over time.

Ye improves on these models in three ways in [11]. First, he drops boundedness of the random terminating time he called τ, thus letting $\overline{T} \to \infty$. The planning horizon was still taken to be some fixed $T < \infty$. Second, he drops the life cycle model and considers an intertemporal model, which allows the individual to consume more earlier or later depending on their preferences. This intertemporal model takes the form of a discounting factor attached to the consumption and bequest utility functions. This allowed him to find exact solutions in a complete market. Lastly, he dropped the complete market assumption, allowing him to find numerical solutions in an incomplete market, avoiding unnecessary restrictions on the individual (such as no life insurance purchase in the last moment of the individual's life). The work by Ye in [11] and Ye and Pliska in [12] made great advances in the economic theory behind optimal rules for consumption, investment, and life insurance purchases.

Up to this point, these papers referred to a consumer living on an income which terminated at death. Despite efforts to solve the problem of finding optimal consumption, investment, and life insurance rules, these papers never got around to answering the question of why people don't annuitize more of their wealth at retirement. Babbel and Merrill [4] considered the case of a defaultable annuity, but restricted investments and only allowed the individual to gain a certain amount of his annuity back in case of insolvency. Taylor [13] took the work done in [10] and [11] and turned it into a risky annuities problem with insurance to cover default risk. To make the consumption/investment/insurance model more realistic, Taylor [13] expanded on the randomly terminating income by allowing it to terminate due to default, in addition to death. This new default of income required a second insurance premium, known as a credit default swap (CDS), or insurance designed to hedge against the annuity company's insolvency.

Taylor needed to solve two problems. He needed to optimize the retiree's expected utility for bequest and consumption over the retiree's lifetime after default, described by

$$D(t,x) = \sup_{(c,\pi,p_M)} E\left[\int_t^{T\wedge\tau_M} U(s,c(s))ds + B(\tau_M, Z(\tau_M))\right],$$

where T is the planning horizon, τ_M is the random time of death, and p_M is the premium paid for life insurance. With this, he then needed to optimize the retiree's expected utility for bequest and consumption from time $t=0$ to $t = T\wedge\tau_M := \min(T,\tau_M)$, defined by

$$V(t,x) = \sup_{(c,\pi,p_M,p_D)} E\left[\int_t^{T\wedge\tau_M} U(s,c(s))ds + B(\tau_M, Z(\tau_M))\right],$$

where p_D is the premium paid for default insurance. In the value functions, the control variables $p_M(t)$ and $p_D(t)$ are involved in the associated wealth processes, in which the individual buys insurance to hedge against certain risks. Thus, Taylor answers the following question: If an individual retires with wealth w, how much should they invest into an annuity which pays at a rate $\alpha(t)$ and costs $A(\alpha)$, in order to maximize $V(t, w - A(\alpha))$? He solves for

$$\alpha^* = \arg\max_{\alpha} \left\{ V(0, w - A(\alpha)) \middle| 0 \le A(\alpha) \le w \right\}.$$

The drawback to Taylor's model is that the CDS used to hedge against annuity insolvency doesn't exist. There is no tool for retirees to use to hedge against default risk. In this paper, we extend the results of Taylor [13] in finding out why people might not annuitize optimally, and propose a method in which financial institutions may realistically incentivize higher annuitization through a potential new insurance instrument. We will consider the impact that different annuity company default rates (probabilities of default) will have on optimal annuity purchases in an actuarially fair market and in one that is not fair (loaded). We will be looking to answer the question, "What is the optimal amount to annuitize in the presence of annuity insolvency risk?"

In order to achieve these results, we will consider four (4) different financial models.

- In Chapter 2 we consider a basic model, which is simply concerned with a risk-free annuity with no life insurance purchase. We make the assumption of no bequest motive, but allow borrowing, to come up with necessary solutions. We refer to this as the Basic Model.

- Chapter 3 contains the model presented by Richard: the annuity is again risk free, but we are allowed to purchase term (instantaneous) life insurance. We will apply the results found by Taylor for this simpler model to find results on annuity optimization. We refer to this as the Richard Model.

- The model in Chapter 4 is Taylor's model, in which the annuity becomes risky, so we consider both life and default insurance payments. We will show that even in face of annuity insolvency, the retiree may still annuitize a large portion of wealth. This is referred to as the Taylor Model.

- The final model in Chapter 7 is our own result: we consider a risky annuity and allow for life and default insurance payments, but add a restriction to the insurance payouts in a way that allows the retiree to purchase a single insurance policy to cover death and default simultaneously. This will be a realistic model that can be implemented by insurance companies to protect against annuity default. This model will show that minimal annuitization is optimal. We call this the Matched Payout Model.

In these chapters we consider the case in which a retiree can short-sell their insurance purchases (or borrow against insurance payouts, in a complete market). This case is not realistic and is seen nowhere in the real world, but it allows us to compute an exact solution for the first three models.

Chapter 5 is dedicated to extending Ye's MCALT (Markov Chain Approximation with the Log Transform) method to the above models to work out numerical solutions for the case where short-selling insurance is not allowed (incomplete market, not all control choices are available). We also apply insurance loading to see what happens to the retiree's value function when insurance becomes difficult to maintain. From all these calculations, we will find out quantitatively why people don't annuitize more in retirement, thus providing a possible solution to the annuity puzzle.

Chapter 6 looks at how insurance purchases can be viewed in terms of borrowing against illiquid wealth. One issue with Richard's approach is that term life insurance is not standard practice. We consider turning these insurance payments into payments on loans borrowed against illiquid wealth (expected present discounted value of wealth from income). In doing so, we see that while the Taylor model is infeasible due to multiple insurance payouts, our Matched Payout Model is marketable due to its single payout setup which provides a reliable source of illiquid wealth to borrow against.

Chapter 8 concludes with an overview of the results and future problems that can be explored.

Chapter 2. Basic Model

In the Basic Model, we consider a retiree who purchases an annuity which cannot default and does not purchase life insurance to guard against premature loss of life. We also assume the individual has no bequest motive, yet can borrow against future income. The work presented here follows closely the work by Merton in [8], with the exception that the annuity holder is assumed to have an uncertain lifetime. Over the course of the retiree's lifetime, they will continually have choices to make regarding how much of their savings to consume, how much to invest in a risky security, and how much to invest in a risk-free security. We will provide a base case for the value function in preparation to show how newer models have improved on said value function. The following sections will build up the background necessary to defining and solving the retiree's value function. Following chapters will have similar structure and build on the results of this introductory model.

2.1 The Financial Market

Let $\mathcal{W}(t)$ be a standard 1-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. T is the fixed planning horizon, which we interpret as the upper limit of the retiree's life span since retirement (with retirement at $t = 0$), so $T < \infty$.

We want to represent all the information we can at time t, which is affected by the Brownian motion, $\mathcal{W}(t)$. Our definitions are taken from Ye [11]. For $t \in [0, T]$, consider the set $\{\mathcal{W}(s), s \leq t\}$ and its σ-algebra $\mathcal{F}^{\mathcal{W}}(t) = \sigma\{\mathcal{W}(s), s \leq t\}$. This is a filtration, meaning if $i \leq j$, then $\mathcal{F}^{\mathcal{W}}(i) \subseteq \mathcal{F}^{\mathcal{W}}(j)$. Next, let $\mathcal{N}$ be the subsets of $\mathcal{F}^{\mathcal{W}}(t)$ with measure 0 according to the measure P. We call $\mathcal{F}_t = \sigma\{\mathcal{F}^{\mathcal{W}}(t) \cup \mathcal{N}\}$ the P-augmented filtration (in the sense that if $i \leq j$, then $\mathcal{F}_i \subseteq \mathcal{F}_j$) of $\mathcal{F}^{\mathcal{W}}(t)$. Finally, let $\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}$ be the filtration. This set contains all the information we must consider up to any time $t \in [0, T]$.

A security is some negotiable financial instrument that represents some type of financial value, such as stocks (representing an ownership position in a publicly traded corporation), bonds (representing a creditor relationship with a government body or corporation), and options (rights to ownership). In the financial market, assume there is both a risk-free security (e.g. government bills and bonds) and a risky security (e.g. stocks, options, commodities). The following definitions are similar to those given [8, 11, 13]. A further discussion can be found in [14, 15].

Assume the risk-free security has price $S_0(t)$ at time t and satisfies the differential equation

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt, \tag{2.1}$$

where $S_0(0) = s_0$ is a given initial value and $r : [0, T] \to \mathbb{R}^+$ is a continuous money market rate process.

Define the risky security as having price $S_1(t)$ with initial condition $S_1(0) = s_1$ and evolution given according to the linear stochastic differential equation

$$\frac{dS_1(t)}{S_1(t)} = \mu(t)dt + \sigma(t)d\mathcal{W}(t). \tag{2.2}$$

Here, $\mu : [0, T] \to \mathbb{R}$ is a continuous mean rate of return and $\sigma : [0, T] \to \mathbb{R}$ is a continuous volatility function satisfying $\sigma^2 \geq k$ for all $t \in [0, T]$ and some $k \in \mathbb{R}^+$.

2.2 The Retiree's Lifetime

The retiree's lifetime is unpredictable, assuming otherwise would make this whole problem trivial. Assume the individual retires at time $t = 0$ and has an uncertain lifetime denoted by τ, which will be a non-negative random variable taken from $(\Omega, \mathcal{F}, P)$. Let $f(t)$ be the probability density function for the probability distribution of τ, and assume τ is independent of the filtration $\mathbb{F}$ as described in Section 2.1. Consider the distribution function

$$F(t) = P(\tau < t) = \int_0^t f(s)ds, \tag{2.3}$$

which represents the probability that the retiree will have died before time t. Similarly consider

$$\overline{F}(t) = P(\tau \geq t) = 1 - F(t), \tag{2.4}$$

which is called the survivor function in [16]. It is the probability that the retiree is alive at time t. Next define the hazard function, or the instantaneous death rate for the individual surviving till time t, as

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{P(t \leq \tau < t + \Delta t \big| \tau \geq t)}{\Delta t}. \tag{2.5}$$

This represents the probability the retiree survives to time t, but then dies immediately afterwards. The conditional probability in the numerator of (2.5) can be rewritten as

$$\begin{aligned} P(t \leq \tau < t + \Delta t \big| \tau \geq t) =& \frac{P(t \leq \tau < t + \Delta t)}{P(\tau \geq t)} \\ =& \frac{P(\tau < t + \Delta t) - P(\tau < t)}{P(\tau \geq t)} \\ =& \frac{F(t + \Delta t) - F(t)}{\overline{F}(t)}. \end{aligned}$$

This gives

$$\begin{aligned} \lambda(t) &= \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\overline{F}(t)} \cdot \frac{1}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t} \cdot \frac{1}{\overline{F}(t)} \\ &= \frac{f(t)}{\overline{F}(t)}. \end{aligned} \tag{2.6}$$

From (2.3) and (2.4),

$$\begin{aligned} \frac{d}{dt} F(t) =& f(t) \\ \frac{d}{dt} \overline{F}(t) =& \frac{d}{dt}(1 - F(t)) = -f(t), \end{aligned}$$

and so by the chain rule,

$$\lambda(t) = -\frac{d}{dt}(\ln \overline{F}(t)).$$

Integrating and solving for the survivor function $\overline{F}(t)$ gives

$$\overline{F}(t) = \exp\left(- \int_0^t \lambda(s)ds \right). \tag{2.7}$$

From (2.6) and (2.7),

$$f(t) = \lambda(t)\overline{F}(t) = \lambda(t) \exp\left(- \int_0^t \lambda(s)ds \right). \tag{2.8}$$

Thus, there is a relation between the hazard function $\lambda(t)$ and the density function $f(t)$. Throughout this paper we will always assume the hazard function $\lambda(t)$ is given, where $\lambda : [0, \infty] \to \mathbb{R}^+$ is a continuous, deterministic function satisfying

$$\int_0^\infty \lambda(t)dt = \infty.$$

Denote the conditional probability density for death at time s conditional upon the individual being alive at time $t \leq s$ by $f(s|t)$. Formulaically,

$$\begin{aligned} f(s|t) &= \lim_{\Delta s \to 0} \frac{P(s \leq \tau < s + \Delta s \big| t \leq \tau)}{\Delta s} \\ &= \lim_{\Delta s \to 0} \frac{P(s \leq \tau < s + \Delta s)}{P(t \leq \tau)} \cdot \frac{1}{\Delta s} \\ &= \lim_{\Delta s \to 0} \frac{P(\tau < s + \Delta s) - P(\tau < s)}{\Delta s} \cdot \frac{1}{\overline{F}(t)} \\ &= \lim_{\Delta s \to 0} \frac{F(s + \Delta s) - F(s)}{\Delta s} \cdot \frac{1}{\overline{F}(t)} \\ &= \frac{f(s)}{\overline{F}(t)}. \end{aligned}$$

Substituting (2.7) and (2.8) gives

$$f(s|t) = \frac{f(s)}{\overline{F}(t)} = \lambda(s)\exp\Big(-\int_t^s \lambda(u)du\Big). \tag{2.9}$$

Let $\overline{F}(s|t)$ be the conditional probability for the wage earner to be alive at time s conditional on surviving to time $t \le s$. This gives

$$\overline{F}(s|t) = P(\tau \ge s \big| \tau \ge t) = \frac{P(\tau \ge s)}{P(\tau \ge t)} = \frac{\overline{F}(s)}{\overline{F}(t)} = \exp\Big(-\int_t^s \lambda(u)du\Big). \tag{2.10}$$

2.3 Wealth and Maximum Utility

The problem we seek to solve is how to maximize the retiree's utility of wealth through consumption and investment. We define the control variables with their rules below. These variables all affect the retiree's wealth process, which is also defined. Then, having established the wealth process and control variables, we will set up the retiree's value function, which represents his total utility over consumption and bequest.

2.3.1 Control Variables and Wealth Process.

We define the following.

$W(t)$ = liquid wealth at time t.

$w = W(0)$ = initial wealth at retirement.

$\alpha(t)$ = annuity income rate (dollars per year) at time t.

$c(t)$ = consumption rate (dollars per year) at time t.

$\pi(t)$ = risky security investment (dollars per year) at time t.

$\pi_0(t) = W(t) - \pi(t)$ = risk-free security investment (dollars per year) at time t.

$\mathcal{A}(w)$ = the set of all admissible pairs (c, π) with starting wealth w.

Note that the annuity only pays during the period $t \in [0, T \wedge \tau]$, where $T \wedge \tau := \min(T, \tau)$. It terminates at the individual's death τ or time horizon T, whichever occurs first. Here,

$\alpha : [0, T] \to \mathbb{R}^+$ satisfies

$$\int_0^T \alpha(s)ds < \infty.$$

At any given time $t \in [0, T \wedge \tau]$, the individual will either invest their current wealth in the risk-free security or the risky security so we assume $\pi_0(t) = W(t) - \pi(t)$. Given the investment processes $\pi(t)$, consumption process $c(t)$, and income rate $\alpha(t)$, the wealth process $W(t)$ for $t \in [0, \min\{T, \tau\}]$ is defined by

$$W(t) = w - \int_0^t c(s)ds + \int_0^t \alpha(s)ds + \int_0^t \frac{W(s) - \pi(s)}{S_0(s)} dS_0(s) + \int_0^t \frac{\pi(s)}{S_1(s)} dS_1(s). \tag{2.11}$$

Substituting (2.1) and (2.2) into (2.11) gives

$$W(t) = w - \int_0^t c(s)ds + \int_0^t \alpha(s)ds + \int_0^t (W(s) - \pi(s))r(s)ds + \int_0^t \pi(s)(\mu(s)ds + \sigma(s)d\mathcal{W}(s)).$$

Taking derivatives gives the stochastic differential equation

$$\begin{aligned} dW(t) =& - c(t)dt + \alpha(t)dt + (W(t) - \pi(t))r(t)dt + \pi(t)(\mu(t)dt + \sigma(t)d\mathcal{W}(t)) \\ =& \big(r(t)W(t) - c(t) + \alpha(t) + \pi(t)(\mu(t) - r(t))\big)dt + \pi(t)\sigma(t)d\mathcal{W}(t). \end{aligned} \tag{2.12}$$

A pair (c, π) is admissible (similar to Taylor's conditions in [13]) if it satisfies the following conditions:

- $\int_0^T \big(|c(s)| + |\pi(s)| + \pi(s)^2\big)ds < \infty$, and
- $W(t) \geq 0$ for all $t \in [0, T]$.

Thus a pair (c, π) is admissible if the controls are bounded and the retiree never enters a bankruptcy condition.

The retiree desires to choose consumption and portfolio investment strategies that maximize their expected utility from consumption and from their legacy, which is their terminal wealth at time $t = \tau$. If the retiree dies at the time horizon T, then they can leave behind $W(T)$ wealth for their posterity, which may or may not be desirable and is reflected in the utility function. If the retiree dies early, at time $t = \tau$, then they leave behind $W(\tau)$ wealth for posterity.

2.3.2 Utility and Value Function.

Let $U(t, c(t))$ be the utility function for consumption, which is assumed to be strictly concave in $c(t)$. Let $B(t, w(t))$ be the utility function for legacy if the retiree dies at time t, which is also assumed to be strictly concave in $w(t)$. $B(t, w(t))$ is assumed to be 0 in this model, but it is an important feature for later chapters, so is included throughout the model.

The maximum expected utility, where the retiree starts with liquid wealth w at time $t = 0$, can be expressed as

$$V(w) = \sup_{(c,\pi)\in\mathcal{A}(w)} E\left[\int_0^{T\wedge\tau} U(s, c(s))ds + B(\tau, W(\tau))\mathbb{1}_{\{\tau<T\}} + B(T, W(T))\mathbb{1}_{\{\tau\geq T\}}\right]. \quad (2.13)$$

We will be interested later in finding the optimal control choices starting at time t, not necessarily $t = 0$, with starting wealth $w = W(t)$. Define $\mathcal{A}(t, w)$ similar to $\mathcal{A}(w)$, with the exception that t is the starting time and w is the starting wealth at time t. This generalizes (2.13) to

$$V(t, w) = \sup_{(c,\pi)\in\mathcal{A}(t,w)} E\left[\int_t^{T\wedge\tau} U(s, c(s))ds + B(\tau, W(\tau))\mathbb{1}_{\{\tau<T\}} + B(T, W(T))\mathbb{1}_{\{\tau\geq T\}}\right]. \quad (2.14)$$

In future equations, the control space is assumed to be $\mathcal{A}(t, w)$, unless otherwise noted, so the supremum notation will refer simply to the admissible control pair (c, π).

2.4 Stochastic Dynamic Programming

In this section, the stochastic dynamic programming technique will be used to set up the optimality principle, derive the Hamilton-Jacobi-Bellman (HJB) equation, and then derive the optimal feedback control for the HJB equation. Similar steps were done by Ye in [11] and will be referred to again in subsequent models.

To begin, restate (2.14) in a dynamic programming form. For any $(c, \pi) \in \mathcal{A}(t, w)$, define

$$J(t, w; c, \pi) = E\left[\int_t^{T\wedge\tau} U(s, c(s))ds + B(\tau, W(\tau))\mathbb{1}_{\{\tau<T\}} + B(T, W(T))\mathbb{1}_{\{\tau\geq T\}}\middle|\tau \geq t, \mathcal{F}_t\right]. \tag{2.15}$$

Equation (2.14) shows

$$V(t, w) = \sup_{(c,\pi)} J(t, w; c, \pi).$$

Note τ is independent of the filtration $\mathbb{F}$, so $J(t, w; c, \pi)$ can be rewritten as stated in the following lemma.

Lemma 2.1. *Suppose that $U(\cdot, \cdot)$ is a non-negative or non-positive function. If the time of death, τ, is independent of the filtration, $\mathbb{F}$, then*

$$J(t, w; c, \pi) = E\left[\int_t^T \big(f(s|t)B(s, W(s)) + \overline{F}(s|t)U(s, c(s))\big)ds + \overline{F}(T|t)B(T, W(T))\middle|\mathcal{F}_t\right], \tag{2.16}$$

where $f(s|t)$ is given by (2.9) *and $\overline{F}(s|t)$ is given by* (2.10).

Proof. See Appendix A.1. □

From Lemma 2.1, the retiree will act as if they will live until time T, even if they face unpredictable death. However, there will be a subjective rate of preference equal to the "force of mortality" on their consumption and terminal wealth. Essentially, the optimization problem with a random terminal time has been converted to a problem with fixed terminal time. The dynamic programming principle then follows.

Lemma 2.2 (Dynamic Programming Principle)**.** *For $0 \leq t < s < T$,*

$$\begin{aligned} V(t, w) = \sup_{(c,\pi)} E\bigg[&\exp\left(-\int_t^s \lambda(v)dv\right)V(s, W(s)) \\ &+ \int_t^s \big(f(u|t)B(u, W(u)) + \overline{F}(u|t)U(u, c(u))\big)du\bigg|\mathcal{F}_t\bigg]. \end{aligned}$$

Proof. See Appendix A.2. □

This immediately lends itself to the formulation of the HJB equation for the Basic Model.

Theorem 2.3 (Dynamic Programming Equation)**.** *Suppose the value function $V(t,w)$ is smooth. Then it satisfies the following:*

$$\begin{cases} V_t(t,w) - \lambda(t)V(t,w) + \sup\limits_{(c,\pi)} \Psi(t,w;c,\pi) = 0 \\ V(T,w) = B(T,w), \end{cases} \tag{2.17}$$

where

$$\begin{aligned} \Psi(t,w;c,\pi) = &\big(r(t)w - c(t) + \alpha(t) + \pi(t)(\mu(t) - r(t))\big)V_w(t,w) \\ &+ \tfrac{1}{2}\pi^2(t)\sigma^2(t)V_{ww}(t,w) + \lambda(t)B(t,w) + U(t,c(t)). \end{aligned}$$

Proof. See Appendix A.3. □

A verification theorem similar to Theorem 3.3.2 in [11] can similarly be achieved. It is given below, with proof given in the appendix.

Theorem 2.4 (Verification Theorem)**.** *Let a smooth function $V(t,w)$ be a solution of the HJB equation* (2.17)*. Then*

$$V(t,w) \geq J(t,w;c,\pi), \quad \forall (c,\pi) \in \mathcal{A}(t,w), (t,w) \in [0,T] \times \mathbb{R}. \tag{2.18}$$

Furthermore, an admissible tuple (c^*,π^*), with corresponding wealth W^*, is optimal $\iff$

$$\begin{aligned} V_t(s,W^*(s)) - \lambda(s)V(s,W^*(s)) + \sup_{(c,\pi)\in\mathcal{A}(s,W^*(S))} &\Psi(s,W^*(s);c,\pi) = 0 \\ &a.e.\ \ s \in [t,T], P - a.s. \end{aligned} \tag{2.19}$$

Proof. See Appendix A.4. □

According to Theorem 2.4, if an admissible pair (c^*, π^*) is optimal, then (2.19), along with the definition of $\Psi(t, w; c, \pi)$ in Theorem 2.3 and the fact that $c(t)$ and $\pi(t)$ occur independently, gives

$$\begin{aligned}
0 =& V_t(t,w) - \lambda(t)V(t,w) + \Psi(t,w;c^*,\pi^*) \\
=& V_t(t,w) - \lambda(t)V(t,w) + \sup_{(c,\pi)} \Psi(t,w;c,\pi) \\
=& V_t(t,w) - \lambda(t)V(t,w) + \sup_{(c,\pi)} \Big\{ \big(r(t)w - c(t) + \alpha(t) + \pi(t)(\mu(t) - r(t))\big) V_w(t,w) \\
& + \tfrac{1}{2}\pi^2(t)\sigma^2(t)V_{ww}(t,w) + \lambda(t)B(t,w) + U(t,c(t)) \Big\} \qquad (2.20) \\
=& V_t(t,w) - \lambda(t)V(t,w) + (r(t)w + \alpha(t))V_w(t,w) + \lambda(t)B(t,w) \\
& + \sup_{c} \left\{ U(t,c(t)) - c(t)V_w(t,w) \right\} \\
& + \sup_{\pi} \left\{ \tfrac{1}{2}\pi^2(t)\sigma^2(t)V_{ww}(t,w) + \pi(t)(\mu(t) - r(t))V_w(t,w) \right\}.
\end{aligned}$$

Note this is the same optimality equation given in Example 3 in Chapter 8 of [8].

Now consider the Euler-Lagrange equations to get first order conditions for the supremum in (2.20). Taking derivatives with respect to c and π, respectively,

$$\Psi_c(t,w;c^*,\pi^*) = 0 = -V_w(t,w) + U_c(t,c^*), \qquad (2.21)$$

$$\Psi_\pi(t,w;c^*,\pi^*) = 0 = (\mu(t) - r(t))V_w(t,w) + \pi^*\sigma^2(t)V_{ww}(t,w). \qquad (2.22)$$

Next consider the set of sufficient conditions for the supremum by looking at the second derivatives with respect to c and π.

$$\begin{aligned}
\Psi_{cc}(t,w;c^*,\pi^*) =& U_{cc}(t,c^*) < 0, \\
\Psi_{\pi\pi}(t,w;c^*,\pi^*) =& \sigma^2(t)V_{ww}(t,w) < 0.
\end{aligned}$$

Since the utility functions $U(t,c)$ is strictly concave in c, the first condition is automatically satisfied. Thus, a sufficient condition for the supremum is to ensure $\Psi_{\pi\pi}(t,w;c,\pi) < 0$, or equivalently $V_{ww}(t,w) < 0$. Suppose to the contrary that $V_{ww}(t,w) \geq 0$. Then letting $\pi \to \infty$ in (2.20) gives

$$\sup_{(c,\pi)} \Psi(t,w;c,\pi) = \infty.$$

The HJB equation (2.17) would then imply either $V_t(t,w)$ or $V(t,w)$ must be infinity, which contradicts the smoothness of $V(t,w)$.

2.5 Constant Relative Risk Aversion

The retiree's consumption and bequest are modelled using utility functions similar to isoelastic (power) functions. Let

$$U(t,c) = \frac{e^{-\rho t}}{\gamma} c^{\gamma}, \tag{2.23}$$

$$B(t,w) = \frac{e^{-\rho t}}{\gamma} w^{\gamma}. \tag{2.24}$$

These functions are concave in c and w, respectively. The variable ρ is the utility discount rate, meaning what is consumed or bequeathed is worth more in terms of value now than in the future. Assume $\rho > 0$. It represents the patience of the retiree in their consumption choices; a larger value would indicate the retiree has less patience to save money and gets greater utility from early consumption. Since we will be considering small values for ρ, [17] says the retiree will likely keep consumption small throughout their lifetime, and as $t \to T$, they will consume more.

The retiree's risk aversion is represented by γ, with $\gamma < 1$ and $\gamma \neq 0$. In this paper, we consider an individual who is risk averse, with $\gamma < 0$. According to [18], a retiree is more risk averse if they are willing to trade their risk for smaller amounts of cash, called their cash equivalent. It is shown in [19] that a utility has constant relative risk aversion if, given utility function $U(w)$,

$$\gamma = -w\frac{U''(w)}{U'(w)}.$$

Constant relative risk aversion (CRRA) means the utility function scales with wealth, and the retiree would consume or bequeath the same proportion of wealth, regardless of scale. Equations (2.23) and (2.24) are examples of CRRA utility functions, and we assume the retiree has the same constant relative risk aversion γ for consumption and bequest.

2.6 Exact Solution

We work out an exact solution to (2.20), similar to that found by Ye in [11]. By assuming zero bequest motive, we can find this exact solution. Thus this section is provided merely as motivation for the exact solutions in subsequent models.

Using the utility function (2.23) in conjunction with the first order conditional (2.21) gives

$$\begin{aligned} U_c(t, c^*) - V_w(t, w) &= 0 \\ e^{-\rho t}(c^*)^{\gamma-1} - V_w(t, w) &= 0. \end{aligned}$$

Rearranging in terms of c^* gives

$$c^* = \left(V_w(t, w)e^{\rho t}\right)^{1/(\gamma-1)}. \tag{2.25}$$

Similarly rearranging (2.22) for π^* gives

$$\pi^* = -\left(\frac{\mu(t) - r(t)}{\sigma^2(t)}\right)\frac{V_w(t, w)}{V_{ww}(t, w)}. \tag{2.26}$$

Plugging the optimal control policies (2.25) and (2.26) into (2.17) gives the system

$$\begin{cases} 0 = \frac{1-\gamma}{\gamma}\left(e^{\rho t/\gamma}V_w(t, w)\right)^{\gamma/(\gamma-1)} + \lambda(t)B(t, w) + V_t(t, w) - \lambda(t)V(t, w) \\ \quad +(\alpha(t) + r(t)w)V_w(t, w) - \frac{1}{2}\left(\frac{\mu(t)-r(t)}{\sigma(t)}\right)^2 \frac{V_w^2(t,w)}{V_{ww}(t,w)} \\ V(T, w) = B(T, w), \end{cases} \tag{2.27}$$

where the terminal condition comes from (2.17). The derivation of (2.27) can be found in Appendix A.5. Based off the utility functions (2.23) and (2.24), we make a guess at the solution

$$V(t,w) = \frac{a(t)}{\gamma}(w + I(t))^{\gamma}, \tag{2.28}$$

where $a(t)$ and $I(t)$ are functions which we will determine below. $I(t)$ can be considered as the retiree's illiquid wealth from income. Taking partial derivatives of (2.28) gives the following:

$$\begin{aligned}
V_w(t,w) &= a(t)(w + I(t))^{\gamma-1}, \\
V_{ww}(t,w) &= a(t)(\gamma - 1)(w + I(t))^{\gamma-2}, \\
V_t(t,w) &= a(t)(w + I(t))^{\gamma-1} I'(t) + \frac{a'(t)}{\gamma}(w + I(t))^{\gamma}, \\
&= \frac{a'(t)}{a(t)} V(t,w) + I'(t) V_w(t,w), \\
\frac{V_w^2(t,w)}{V_{ww}(t,w)} &= \frac{\gamma}{\gamma - 1} V(t,w), \\
V_w(t,w)^{\gamma/(\gamma-1)} &= a(t)^{1/(\gamma-1)} \gamma V(t,w), \\
w V_w(t,w) &= \gamma V(t,w) - I(t) V_w(t,w).
\end{aligned} \tag{2.29}$$

Plugging these into (2.27) gives

$$\begin{aligned}
0 =& \left(\frac{1-\gamma}{\gamma}\right)(e^{\rho t})^{1/(\gamma-1)} V_w(t,w)^{\gamma/(\gamma-1)} + \lambda(t) B(t,w) \\
&+ a(t)(w + I(t))^{\gamma-1} I'(t) + \frac{a'(t)}{\gamma}(w + I(t))^{\gamma} - \lambda(t) V(t,w) \\
&+ (\alpha(t) + r(t) w) V_w(t,w) - \frac{1}{2}\left(\frac{\mu(t) - r(t)}{\sigma(t)}\right)^2 \left(\frac{\gamma}{\gamma - 1}\right) V(t,w) \\
=& \left[\frac{a'(t)}{a(t)} + (1-\gamma)(e^{\rho t})^{1/(\gamma-1)} a(t)^{1/(\gamma-1)} - \left(\lambda(t) - r(t)\gamma + \tfrac{1}{2}\left(\tfrac{\gamma}{\gamma-1}\right)\left(\tfrac{\mu(t)-r(t)}{\sigma(t)}\right)^2\right)\right] V(t,w) \\
&+ \lambda(t) B(t,w) + \left[I'(t) - r(t) I(t) + \alpha(t)\right] V_w(t,w).
\end{aligned} \tag{2.30}$$

In order to find an exact solution from (2.30), we must assume something about $B(t, w)$. Hence, our assumption of zero bequest motive, and so $B(t, w) = 0$. This implies the terminal condition to (2.27) becomes

$$V(T, w) = \frac{a(T)}{\gamma}(w + I(T))^{\gamma} = 0.$$

This implies $a(T) = 0$. We also assume at time T the retiree has no income and so $I(T) = 0$. Consider first the $V_w(t, w)$ term. Solve

$$0 = I'(t) - r(t)I(t) + \alpha(t),$$

subject to $I(T) = 0$, by way of integrating factor. This gives

$$I(t) = \int_t^T \alpha(s) \exp\left(- \int_t^s r(v)\, dv \right) ds. \tag{2.31}$$

Now consider the $V(t, w)$ term. This is done similarly to Ye in [11]. To solve

$$0 = \frac{a'(t)}{a(t)} + (1-\gamma)(e^{\rho t})^{1/(\gamma-1)} a(t)^{1/(\gamma-1)} - \left(\lambda(t) - r(t)\gamma + \tfrac{1}{2}\left(\tfrac{\gamma}{\gamma-1}\right)\left(\tfrac{\mu(t)-r(t)}{\sigma(t)}\right)^2\right),$$

subject to $a(T) = 0$, make the substitution

$$a(t) = e^{-\rho t} g(t)^{1-\gamma}.$$

This gives the simpler ODE

$$0 = g'(t) - H(t)g(t) + 1, \quad g(T) = 0,$$

where

$$H(t) = \frac{\lambda(t) + \rho - r(t)\gamma}{1-\gamma} - \frac{\gamma}{2}\left(\frac{\mu(t) - r(t)}{(1-\gamma)\sigma(t)}\right)^2.$$

Again solving this by integrating factor gives

$$g(t) = \int_t^T \exp\left(- \int_t^s H(v)dv \right) ds.$$

In the following section we will define the parameters for solving this problem.

2.7 Model Parameters

In order to find optimal annuitization, we must determine real world parameters for the financial market, CRRA utility functions, and mortality probabilities. The parameters will be discussed below.

2.7.1 Financial Market.

When considering the financial market, we must find a reasonable risky and risk-free asset that anyone could have access to for investing. One risk-free asset is the US Treasury Inflation Protected Security (TIPS), which is a type of security that grows with inflation and avoids any negative side effects of deflation. According to the US Treasury, recent yields have only been around 1%. We will let $r = .01$.

For a standard risky asset, the market mean rate of return μ is found by calculating the equity risk premium $\mu - r$. The equity risk premium is equal to the slope of the security market line (SML), which historically has been between 3.5 and 5.5. According to KPMG Advisory N.V. in [20], as of June, 2018, the historic equity risk premium is 5.5%. We will be conservative and let $\mu - r = .05$, and thus $\mu = .06$.

Lastly, a standard tool for calculating the volatility function σ for the risky market is the Chicago Board of Exchange Volatility Index. As of November, 2018, the volatility index was around 20, so $\sigma = .20$.

2.7.2 CRRA Utility Functions.

As discussed already, the utility functions in (2.23) and (2.24) are given by

$$U(t,c) = \frac{e^{-\rho t}}{\gamma} c^{\gamma},$$
$$B(t,w) = \frac{e^{-\rho t}}{\gamma} w^{\gamma}.$$

Recall the constant relative risk aversion parameter γ represents the retiree's aversion to taking risks in investing; the smaller the value, the more averse to risk the retiree. We will set $\gamma = -3$, with a very risk averse retiree.

The utility discount rate, ρ, represents the patience of the retiree. Assume $\rho = .03$, similar to [11] and [13].

2.7.3 Mortality Probability.

The hazard rate $\lambda(t)$ is given in [4] using the Gompertz equation

$$\lambda(t) = \frac{1}{d} \exp\left(\frac{R+t-c}{d}\right),$$

where $R = 65$ is the retirement age, and c and d are tractable constants. The constants are calculated against the basic annuity 2000 period mortality tables and found to be $c = 87.98$ and $d = 11.19$ in [4].

2.7.4 Summary of Parameters.

Table 2.1 contains the financial market parameters described above.

r	μ	σ	γ	ρ	T	$\lambda(t)$
.01	.06	.20	-3	.03	40	$\frac{1}{d}\exp\left(\frac{R+t-c}{d}\right)$

Table 2.1: Parameters used in the Basic Model.

2.8 Annuity Pricing

Unique to this paper is the application of annuities to Richard's model. Suppose the retiree has $w = \$500{,}000$ at the time of retirement, and they wish to put some amount, A, into an annuity. For this model, we match the cost of the annuity with the expected present value of future income, $I(t)$, given in (2.31). Then

$$A(\alpha) = \int_0^T \alpha(s) \exp\left(- \int_0^s r(v)\, dv \right) ds. \tag{2.32}$$

If $\alpha(s) = \alpha$ is taken to be a constant payment in the duration of the retiree's lifetime, then simplifying (2.32) gives

$$\alpha_{\max} = \frac{w}{\int_0^T \exp\left(-\int_0^s r(v)dv\right) ds}.$$

The retiree's liquid wealth is found by subtracting the cost of the annuity from the initial wealth, so $l_0 = w - A(\alpha)$. They wish to find

$$\alpha^* = \underset{\alpha \in [0, \alpha_{\max}]}{\arg\max}\, V(0, l_0). \tag{2.33}$$

The optimal level of annuitization as a ratio of wealth then is given by

$$\theta^* = \frac{A(\alpha^*)}{w}.$$

2.9 Computations and Results

We ran (2.33) in Python 3.7 on a 4×2.80GHz Intel Core i7-2640M processor. We chose an α-step size of 3% of $\alpha_{\max}$. Run time was approximately 5 minutes, although this could be improved by parallelizing over α. Figure 2.1 shows the results.

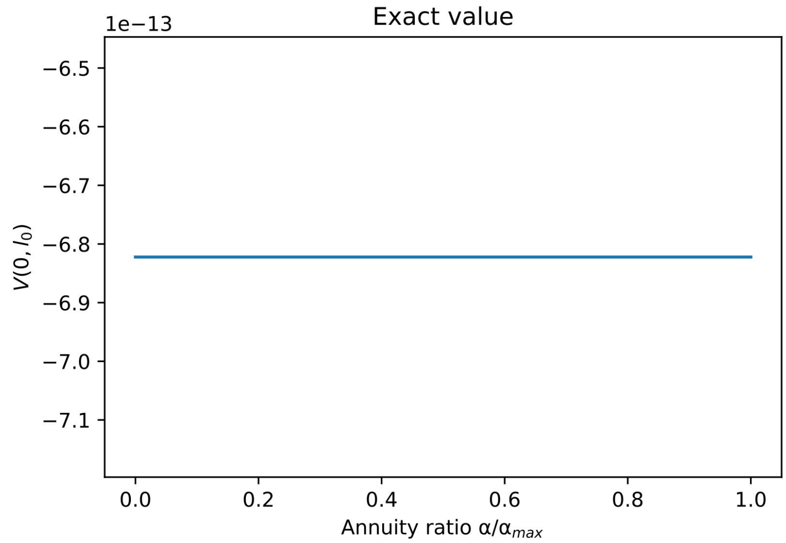

Figure 2.1: Exact value function for several annuity payment ratios α/α_{max} in the Basic Model. There is no optimal α^* to annuitize, since the retiree is able to consume as much as desired with no regard to final wealth.

There is no optimal wealth $A(\alpha)$ to annuitize. This is a result of the retiree's ability to consume beyond his means. Figure 2.2 is the graph of consumption, $c(t)$, as a ratio of total wealth $w + I(t)$ found using (2.25); we call this $\widehat{c}(t)$. Notice that as $t \to T$, consumption exceeds total wealth. This requires us to constrain the system, which we will look at in Chapter 5.

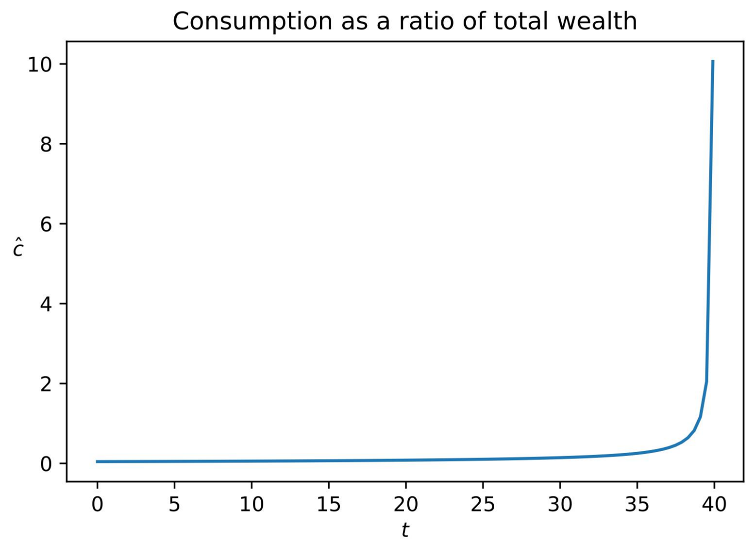

Figure 2.2: Consumption $c(t)$ as a ratio of total wealth $w + I(t)$. Consumption becomes unbounded as $t \to T$.

Chapter 3. Richard Model

Now consider Richard's model in [10]. In this model, the retiree is assumed to have a non-defaultable annuity, and he may hedge against mortality risk by purchasing life insurance. The life insurance will be term insurance, where the term is taken to be infinitesimally small.

The notation used here follows Taylor's notation in [13]. Richard was interested in finding the optimal process for an individual making consumption, portfolio, and life insurance purchase choices with an uncertain lifetime. Ye added to it by considering the same problem in [11], but introducing the MCALT method to improve the numerical solution, which will be looked at in Chapter 5. These results will be reproduced, using the revised financial market parameters found in Table 2.1. Many of the equations in the Basic Model above are a direct result and simplification of equations in [10] and [11], so most proofs will be skipped except in cases where they provide context and clarity to the problem.

3.1 The Financial Market

The financial market for the Richard Model is the same as that for the Basic Model in Section 2.1. We do not repeat the discussion here.

3.2 The Retiree's Lifetime

The retiree has an uncertain lifetime, just as in the Basic Model. The equations for the Richard Model are the same as those in Section 2.2 for the Basic Model.

3.3 Wealth and Maximum Utility

The basic definitions from Section 2.3 still hold, and we also include the life insurance premium control variable and insurance premium-payout ratio. The new definitions and value functions used in the Richard Model are defined below.

3.3.1 Control Variables and Wealth Process.

We define the new variables used in the Richard Model.

$p(t)$ = premium rate of life insurance purchased (dollars per year) at time t.

$\eta(t)$ = life insurance premium-payout ratio.

$\mathcal{A}(t, w)$ = the set of all admissible tuples (c, π, p) with starting wealth w at time t.

Life insurance is assumed to be purchased continuously; this is term insurance where the term is assumed to be infinitesimally small. This is not how the actual market works, but for finding exact and numerical solutions it suits our purposes. At the retiree's time of death, $t \in [0, T]$, where the retiree has purchased life insurance at rate $p(t)$, the insurance company is obligated to pay the insurance amount of $p(t)/\eta(t)$ dollars, leaving the retiree with wealth $W(t) + p(t)/\eta(t)$.

The purpose of life insurance is to allow the retiree to ensure some life-end dollar amount with can then be bequeathed to posterity. If the retiree is well off, with large $W(t)$, they may short-sell life insurance to use as a second income; if they are not, then either $p(t) = 0$ or is positive. Throughout the chapter, we consider the case where short-selling is allowed.

The old wealth process (2.11) becomes

$$W(t) = w - \int_0^t c(s)ds - \int_0^t p(s)ds + \int_0^t \alpha(s)ds + \int_0^t \frac{W(s) - \pi(s)}{S_0(s)} dS_0(s) + \int_0^t \frac{\pi(s)}{S_1(s)} dS_1(s), \tag{3.1}$$

and the SDE (2.12) becomes

$$dW(t) = \big(r(t)W(t) - c(t) - p(t) + \alpha(t) + \pi(t)(\mu(t) - r(t))\big)dt + \pi(t)\sigma(t)d\mathcal{W}(t). \tag{3.2}$$

With the introduction of life insurance purchases, the dynamics of the final wealth at time of death $t = \tau$ changes. If the retiree dies at time $t \in [0, T]$, insurance pays $p(t)/\eta(t)$.

This means the retiree's wealth at time of death is given by

$$\zeta(t) = W(t) + \frac{p(t)}{\eta(t)}. \tag{3.3}$$

If $\tau \geq T$, then the retiree's annuity payments end at the planning horizon T, and the retiree has no need of life insurance to protect against any future loss. Thus $p(T) = 0$ and the retiree's final wealth at the planning horizon T would be

$$\zeta(T) = W(T) + \frac{p(T)}{\eta(T)} = W(T).$$

Supposing the retiree starts with wealth w at time t, the introduction of life insurance purchases $p(t)$ means their decision process is governed by a choice of triples (c, π, p).

This gives us updated conditions on control variable admissibility. Now a tuple (c, π, p) is admissible if it satisfies the following conditions:

- $\int_t^T (|c(s)| + |\pi(s)| + |p(s)| + \pi(s)^2) ds < \infty$, and
- $W(t) + \frac{p(t)}{\eta(t)} \geq 0$ for all $t \in [0, T]$.

Given (3.3), maximizing the expected utility over $p(t)$ is equivalent to maximizing over the end wealth, $\zeta(t)$, were the retiree to die in that instant. We will want $p(t)$ in terms of $\zeta(t)$, which (3.3) gives as

$$p(t) = \eta(t)\big(\zeta(t) - W(t)\big).$$

The calculations done hereafter will be easier in terms of $\zeta(t)$ rather than $p(t)$, so the decision process will be given by $(c, \pi, \zeta) \in \mathcal{A}(t, w)$.

Now write (3.2) as

$$\begin{aligned} dW(t) =& \big((r(t) + \eta(t))W(t) - c(t) + \alpha(t) - \eta(t)\zeta(t) + \pi(t)(\mu(t) - r(t))\big)dt \\ &+ \pi(t)\sigma(t)d\mathcal{W}(t). \end{aligned} \tag{3.4}$$

3.3.2 Utility and Value Function.

Using the same utility functions in Chapter 2, the maximum expected utility (2.14) now depends on the optimal life insurance purchase $\zeta(t)$, as well as consumption $c(t)$ and investment $\pi(t)$ choices, and becomes

$$V(t,w) = \sup_{(c,\pi,\zeta)\in\mathcal{A}(t,w)} E\left[\int_t^{T\wedge\tau} U(s,c(s))ds + B(\tau,\zeta(\tau))\mathbb{1}_{\{\tau<T\}} + B(T,W(T))\mathbb{1}_{\{\tau\geq T\}}\right]. \tag{3.5}$$

In future equations, the control space is assumed to be $\mathcal{A}(t,w)$, so the supremum notation will refer simply to (c,π,ζ).

3.4 Stochastic Dynamic Programming

As in Section 2.4, the stochastic dynamic programming technique will be used to set up the optimality principle, derive the HJB equation, and then derive the optimal feedback control for the HJB equation. Redefine (2.15) as

$$J(t,w;c,\pi,\zeta) = E\left[\int_t^{T\wedge\tau} U(s,c(s))ds + B(\tau,\zeta(\tau))\mathbb{1}_{\{\tau<T\}} + B(T,W(T))\mathbb{1}_{\{\tau\geq T\}}\middle|\tau\geq t,\mathcal{F}_t\right].$$

This gives

$$V(t,w) = \sup_{(c,\pi,\zeta)} J(t,w;c,\pi,\zeta).$$

Similar to Lemma 2.1, we write $J(t,w;c,\pi,\zeta)$ as an integral with fixed upper bound T and weighted utility.

Lemma 3.1. *Suppose that $U(\cdot,\cdot)$ is a non-negative or non-positive function. If the time of death, τ, is independent of the filtration, $\mathbb{F}$, then*

$$J(t,w;c,\pi,\zeta) = E\left[\int_t^T \big(f(s|t)B(s,\zeta(s)) + \overline{F}(s|t)U(s,c(s))\big)ds + \overline{F}(T|t)B(T,W(T))\middle|\mathcal{F}_t\right], \tag{3.6}$$

where $f(s|t)$ *is given by Equation* (2.9) *and* $\overline{F}(s|t)$ *is given by* (2.10)*.*

Proof. Reference Lemma 2.1 or [11, Lemma 3.3.1]. □

The dynamic programming principle in Lemma 2.2 becomes

Lemma 3.2 (Dynamic Programming Principle)**.** *For* $0 \le t < s < T$*,*

$$\begin{aligned} V(t,w) = \sup_{(c,\pi,\zeta)} E\bigg[\exp\bigg(-\int_t^s \lambda(v)dv\bigg)V(s,W(s)) \\ + \int_t^s \big(f(u|t)B(u,\zeta(u)) + \overline{F}(u|t)U(u,c(u))\big)du\bigg|\mathcal{F}_t\bigg]. \end{aligned}$$

Proof. Reference Lemma 2.2 or [11, Lemma 3.3.2]. □

The Dynamic Programming Equation, or HJB equation, in Theorem 2.3 becomes

Theorem 3.3 (Dynamic Programming Equation)**.** *Suppose the value function* $V(t,w)$ *is smooth. Then it satisfies the following:*

$$\begin{cases} V_t(t,w) - \lambda(t)V(t,w) + \sup\limits_{(c,\pi,\zeta)} \Psi(t,w;c,\pi,\zeta) = 0 \\ V(T,w) = B(T,w), \end{cases} \tag{3.7}$$

where

$$\begin{aligned} \Psi(t,w;c,\pi,\zeta) =& \big((r(t)+\eta(t))w - c(t) + \alpha(t) - \eta(t)\zeta(t) + \pi(t)(\mu(t)-r(t))\big)V_w(t,w) \\ &+ \tfrac{1}{2}\pi^2(t)\sigma^2(t)V_{ww}(t,w) + \lambda(t)B(t,\zeta(t)) + U(t,c(t)). \end{aligned}$$

Proof. Reference Theorem 2.3 or [11, Theorem 3.3.1]. □

Equation (3.7) is the HJB equation for the Richard Model. Lastly, this model has a verification theorem similar to Theorem 2.4. It is given as

Theorem 3.4 (Verification Theorem)**.** *Let a smooth function* $V(\cdot,\cdot)$ *be a solution of the HJB equation* (3.7)*. Then*

$$V(t,w) \geq J(t,w;c,\pi,\zeta), \quad \forall (c,\pi,\zeta) \in \mathcal{A}(t,w), (t,w) \in [0,T] \times \mathbb{R}.$$

Furthermore, an admissible triple (c^*,π^*,ζ^*)*, with corresponding wealth* W^**, is optimal* $\iff$

$$V_t(s,W^*(s)) - \lambda(s)V(s,W^*(s)) + \sup_{(c,\pi,\zeta)\in\mathcal{A}(s,W^*(s))} \Psi(s,W^*(s);c,\pi,\zeta) = 0$$

$$a.e. \;\; s \in [t,T], P - a.s.$$

Proof. Reference Theorem 2.4 or [11, Theorem 3.3.2]. □

Now we are able to derive the optimal consumption, portfolio, and life insurance purchase policies. According to Theorem 3.4, if the triple (c^*,π^*,ζ^*) is optimal, then the equality condition, along with the mutual independence of $c(t)$, $\pi(t)$, and $\zeta(t)$ gives

$$\begin{aligned}
0 =& V_t(t,w) - \lambda(t)V(t,w) + \Psi(t,w;c^*,\pi^*,\zeta^*) \\
=& V_t(t,w) - \lambda(t)V(t,w) + \sup_{(c,\pi,\zeta)} \Psi(t,w;c,\pi,\zeta) \\
=& V_t(t,w) - \lambda(t)V(t,w) + \big((r(t)+\eta(t))w + \alpha(t)\big)V_w(t,w) \\
&+ \sup_c \big\{U(t,c(t)) - c(t)V_w(t,w)\big\} \\
&+ \sup_\pi \big\{\tfrac{1}{2}\pi^2(t)\sigma^2(t)V_{ww}(t,w) + \pi(t)(\mu(t)-r(t))V_w(t,w)\big\} \\
&+ \sup_\zeta \big\{\lambda(t)B(t,\zeta(t)) - \eta(t)\zeta(t)V_w(t,w)\big\}.
\end{aligned} \tag{3.8}$$

The first order conditionals of (3.8) with respect to c, π, and ζ are, respectively,

$$\Psi_c(t,w;c^*,\pi^*,\zeta^*) = 0 = -V_w(t,w) + U_c(t,c^*), \tag{3.9}$$

$$\Psi_\pi(t,w;c^*,\pi^*,\zeta^*) = 0 = (\mu(t)-r(t))V_w(t,w) + \pi^*\sigma^2(t)V_{ww}(t,w), \tag{3.10}$$

$$\Psi_\zeta(t,w;c^*,\pi^*,\zeta^*) = 0 = \lambda(t)B_\zeta(t,\zeta^*) - \eta(t)V_w(t,w). \tag{3.11}$$

Sufficient conditions for the supremum of $V(t, w)$ are given by the second order conditionals

$$
\begin{aligned}
\Psi_{cc}(t, w; c^*, \pi^*, \zeta^*) =& U_{cc}(t, c^*) < 0, \\
\Psi_{\pi\pi}(t, w; c^*, \pi^*, \zeta^*) =& \sigma^2(t) V_{ww}(t, w) < 0, \\
\Psi_{\zeta\zeta}(t, w; c^*, \pi^*, \zeta^*) =& \lambda(t) B_{\zeta\zeta}(t, \zeta^*) < 0.
\end{aligned}
$$

The utility functions $U(\cdot, c(\cdot))$ and $B(\cdot, \zeta(\cdot))$ are strictly concave (in $c(\cdot)$ and $\zeta(\cdot)$ respectively) so the first and third second order conditions are satisfied immediately. Thus, a sufficient guarantee of a supremum is to ensure

$$
\Psi_{\zeta\zeta}(t, w; c^*, \pi^*, \zeta^*) < 0,
$$

or in other words,

$$
V_{ww}(t, w) < 0.
$$

As in the Basic Model, it is not possible for $V_{ww}(t, w) \geq 0$, so we are guaranteed a supremum.

3.5 Constant Relative Risk Aversion

The retiree runs under the same constant relative risk aversion, γ, as used in Section 2.5. Assume as before that $\gamma < 1$,$\gamma \neq 0$, and $\rho > 0$, where ρ is the utility discount rate. The utility functions given by (2.23) and (2.24) are updated to account for the final wealth being $\zeta(t)$ instead of $W(t)$. They are

$$
U(t, c) = \frac{e^{-\rho t}}{\gamma} c^\gamma, \tag{3.12}
$$

$$
B(t, \zeta) = \frac{e^{-\rho t}}{\gamma} \zeta^\gamma. \tag{3.13}
$$

With all this information, we would like to find an exact solution to the retiree's optimization problem. Unlike in the Basic Model, the retiree now has the option for life insurance purchase. According to [18], a retiree is willing to pay more for life insurance to cover potential loss, giving him a higher insurance premium. In the next section, we consider the hypothetical situation in which life insurance short-selling is allowed – this gives a tractable solution. In Chapter 5, we will consider the more realistic case of no short-selling, which requires the method of finite differences to find an approximate solution.

3.6 Exact Solution

As mentioned, assume short-selling of life insurance purchased is permitted. Rearranging (3.9) and (3.10) in terms of c^* and π^*, it is again the case that

$$c^* = \big(V_w(t,w)e^{\rho t}\big)^{1/(\gamma-1)} \tag{3.14}$$

and

$$\pi^* = -\left(\frac{(\mu(t)-r(t))}{\sigma^2(t)}\right)\frac{V_w(t,w)}{V_{ww}(t,w)}, \tag{3.15}$$

similar to (2.25) and (2.26). The addition of life insurance premiums requires us to solve (3.11) in terms of ζ^*. This is given by

$$\begin{aligned}\lambda(t)B_\zeta(t,\zeta^*) - \eta(t)V_w(t,w) &= 0\\ e^{-\rho t}(\zeta^*)^{\gamma-1} - \frac{\eta(t)}{\lambda(t)}V_w(t,w) &= 0.\end{aligned}$$

Finally rearranging for ζ^* gives

$$\begin{aligned}(\zeta^*)^{\gamma-1} &= \frac{\eta(t)}{\lambda(t)}V_w(t,w)e^{\rho t}\\ \zeta^* &= \left(\frac{\eta(t)}{\lambda(t)}V_w(t,w)e^{\rho t}\right)^{1/(\gamma-1)}.\end{aligned} \tag{3.16}$$

Plug (3.14)-(3.16) into (3.7) and define

$$K(t) = \frac{\lambda(t)^{1/(1-\gamma)}}{\eta(t)^{\gamma/(1-\gamma)}} + 1.$$

Including the terminal condition from (3.7), we get the following system:

$$\begin{cases} 0 = V_t(t,w) - \lambda(t)V(t,w) + \big(\alpha(t) + (\eta(t) + r(t))w\big)V_w(t,w) \\ \qquad -\frac{1}{2}\left(\frac{\mu(t)-r(t)}{\sigma(t)}\right)^2 \frac{V_w^2(t,w)}{V_w(t,w)} + \frac{1-\gamma}{\gamma}e^{\rho t/(\gamma-1)}K(t)V_w(t,w)^{\gamma/(\gamma-1)} \\ V(T,w) = \frac{e^{-\rho T}}{\gamma}w^\gamma. \end{cases} \tag{3.17}$$

The derivation of (3.17) can be found in Appendix A.6. To solve (3.17) assume the solution is of the form

$$V(t,w) = \frac{a(t)}{\gamma}(w + I(t))^\gamma, \tag{3.18}$$

just as in the Basic Model. This gives the relationships defined in (2.29).

Plugging (2.29) and (3.18) into (3.17) gives the differential equation

$$\begin{aligned} 0 =& \frac{a'(t)}{a(t)}V(t,w) + I'(t)V_w(t,w) - \lambda(t)V(t,w) + \alpha(t)V_w(t,w) \\ &+ (\eta(t) + r(t))(\gamma V(t,w) - I(t)V_w(t,w)) - \frac{1}{2}\left(\frac{\mu(t)-r(t)}{\sigma(t)}\right)^2 \frac{\gamma}{\gamma-1}V(t,w) \\ &+ \frac{1-\gamma}{\gamma}e^{\rho t/(\gamma-1)}K(t)a(t)^{1/(\gamma-1)}\gamma V(t,w) \\ =& \Bigg[\frac{a'(t)}{a(t)} - \lambda(t) + (\eta(t)+r(t))\gamma - \frac{1}{2}\left(\frac{\mu(t)-r(t)}{\sigma(t)}\right)^2\frac{\gamma}{\gamma-1} \\ &+ (1-\gamma)e^{\rho t/(\gamma-1)}K(t)a(t)^{1/(\gamma-1)}\Bigg]V(t,w) \\ &+ \big[I'(t) - I(t)(r(t)+\eta(t)) + \alpha(t)\big]V_w(t,w). \end{aligned}$$

The terminal condition in (3.17) together with the ansatz (3.18) gives terminal conditions for $a(t)$ and $I(t)$, namely

$$\begin{aligned} a(T) =& e^{-\rho T}, \\ I(T) =& 0. \end{aligned}$$

This gives two differential equations,

$$\begin{cases} 0 = I'(t) - I(t)(r(t) + \eta(t)) + \alpha(t) \\ I(T) = 0, \end{cases} \tag{3.19}$$

and

$$\begin{cases} 0 = \frac{a'(t)}{a(t)} - \lambda(t) + (\eta(t) + r(t))\gamma - \frac{1}{2}\left(\frac{\mu(t)-r(t)}{\sigma(t)}\right)^2 \frac{\gamma}{\gamma-1} \\ \quad +(1-\gamma)e^{\rho t/(\gamma-1)} K(t) a(t)^{1/(\gamma-1)} \\ a(T) = e^{-\rho T}. \end{cases} \tag{3.20}$$

Solving (3.19) using the integrating factor method gives

$$I(t) = \int_t^T \alpha(s) \exp\left(-\int_t^s (r(v) + \eta(v)) dv\right) ds. \tag{3.21}$$

$I(t)$ is the expected discounted present day value of the retiree's income from the annuity, and can be thought of as the retiree's illiquid wealth at time t. In order to solve (3.20), assume $a(t)$ takes the form

$$a(t) = e^{-\rho t} g(t)^{1-\gamma}, \tag{3.22}$$

with

$$a'(t) = -\rho e^{-\rho t} g(t)^{1-\gamma} + (1-\gamma) e^{-\rho t} g(t)^{-\gamma} g'(t).$$

Substituting $a(t)$ into (3.20) and multiplying everything by $a(t)$ gives

$$0 = g'(t) - H(t)g(t) + K(t), \tag{3.23}$$

where

$$H(t) = \frac{\rho + \lambda(t)}{1-\gamma} - \frac{(r(t)+\eta(t))\gamma}{1-\gamma} - \frac{\gamma}{2}\left(\frac{\mu(t)-r(t)}{(1-\gamma)\sigma(t)}\right)^2.$$

The terminal condition for (3.23) comes from (3.22) and the terminal condition of $a(t)$, namely

$$g(t) = 1.$$

Solving (3.23) using an integrating factor gives

$$g(t) = \exp\left(-\int_t^T H(v)dv\right) + \int_t^T \exp\left(-\int_t^s H(v)dv\right)K(s)ds.$$

Plugging (3.18), (2.29), and (3.22) in (3.14)-(3.16) and doing some algebra gives

$$\begin{aligned}
c^*(t) =& \frac{1}{g(t)}(w + I(t)),\\
\pi^*(t) =& \left(\frac{\mu(t)-r(t)}{(1-\gamma)\sigma^2(t)}\right)(w+I(t)),\\
\zeta^*(t) =& \left(\frac{\eta(t)}{\lambda(t)}\right)^{1/(\gamma-1)} \frac{1}{g(t)}(w+I(t)).
\end{aligned}$$

In the next section we establish the model parameters, similar to the Basic Model. Then we will give an exact solution to the retiree's annuitization problem.

3.7 Model Parameters

The parameters in this section will be those used in Section 2.7. Refer there for financial market, utility function, and mortality probability parameters. The new parameter to consider in this section is the payout ratio $\eta(t)$ associated with the insurance premium $p(t)$ given below.

3.7.1 Life Insurance.

For insurance companies, the expected profit at time t is

$$P(t) = p(t) - \frac{p(t)}{\eta(t)}\lambda(t),$$

since the insurance company always receives the insurance premium, and pays out $\frac{p(t)}{\eta(t)}$ with probability equal to the mortality probability $\lambda(t)$. Thus, for the insurance company to keep making money, they must set $P(t) \geq 0$, i.e.

$$\eta(t) \geq \lambda(t).$$

In an actuarially fair market, the insurance company would set $\eta(t) = \lambda(t)$, which we do for most calculations. We will consider $\eta(t) > \lambda(t)$ in actuarially unfair models by setting $\eta(t) = (1+\epsilon)\lambda(t)$ for some small ϵ.

3.7.2 Summary of Parameters.

Below we list all the parameters used in Richard's problem.

r	μ	σ	γ	ρ	T	$\lambda(t)$	$\eta(t)$
.01	.06	.20	-3	.03	40	$\frac{1}{d}\exp\left(\frac{R+t-c}{d}\right)$	$\frac{1}{d}\exp\left(\frac{R+t-c}{d}\right)$

Table 3.1: Parameters used in the Richard Model.

R, c, and d are defined as in Section 2.7.3.

3.8 Annuity Pricing

Assume as before the retiree has $w = \$500{,}000$ at retirement, $t = 0$. The price of an annuity paying at rate $\alpha(t)$ matches the expected present discounted value of future wealth, $I(t)$, in an actuarially fair market. Thus we can replace $\eta(t)$ with $\lambda(t)$ and get the annuity cost

$$A(\alpha) = \int_0^T \alpha(s) \exp\left(-\int_0^s (r(v) + \lambda(v))dv\right) ds, \tag{3.24}$$

This differs from the Basic Model, as $I(t)$ is further discounted by $\lambda(t)$.

If $\alpha(t) = \alpha$ is taken to be constant, then rearranging gives

$$\alpha_{\max} = \frac{w}{\int_0^T \exp\left(-\int_0^s (r(v) + \lambda(v))dv\right) ds}. \tag{3.25}$$

In an actuarially fair market, we can replace $\lambda(t)$ with $\eta(t)$. The retiree's liquid wealth after buying the annuity at time $t = 0$ is given by $l_0 = w - A(\alpha)$. The retiree then wishes to find

$$\alpha^* = \underset{\alpha \in [0, \alpha_{\max}]}{\arg\max}\, V(0, l_0). \tag{3.26}$$

The optimal level of annuitization as a ratio of wealth is still

$$\theta^* = \frac{A(\alpha^*)}{w}.$$

3.9 Computations and Results

In this section we will first run through the exact solution and collect some baseline results. It is run on the same machine as in Section 2.9, with the same run-time . Using parameters given by Table 3.1 and initial wealth $w = \$500{,}000$, recall the exact value function given in (3.18) is

$$V(t, w) = \frac{a(t)}{\gamma}(w + I(t))^\gamma,$$

where $a(t)$ and $I(t)$ are given by (3.22) and (3.21) respectively. The retiree must purchase an annuity policy with payments at rate α, whose cost is given by (3.24). Thus the retiree's initial liquid wealth is actually given by l_0. Notice now that $A(\alpha) = I(0)$. Then

$$V(0, l_0) = \tfrac{a(0)}{\gamma}(w - A(\alpha) + I(0))^\gamma = \tfrac{a(0)}{\gamma} w^\gamma.$$

Now (3.26) gives

$$\alpha^* = \underset{\alpha \in [0, \alpha_{\max}]}{\arg\max} \tfrac{a(0)}{\gamma} w^\gamma.$$

Since $a(0)$ is independent of the annuity income, α, this has no max. Thus, the way this problem is defined, there is no single optimal solution. This result is reasonable, since the retiree is able to short-sell life insurance, giving the retiree an income similar to the annuity, and his total wealth is preserved. This idea was discussed in detail in [4] and [10].

The graph of the value function over several choices of $\alpha \in [0, \alpha_{\max}]$ is given in Figure 3.1. Although simple, it is very informative.

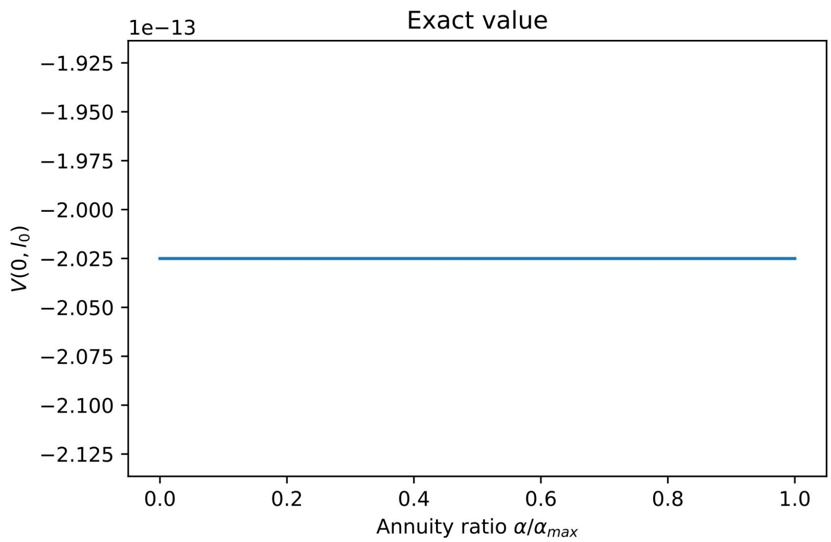

Figure 3.1: Exact value function for several annuity payment ratios α/α_{max} in the Richard Model. There is no optimal α^* to annuitize, since the retiree is able to short-sell life insurance and maintain his total wealth.

Chapter 4. Taylor Model

In this chapter, consider a retiree with fixed wealth at time of retirement who can invest in an annuity which pays a continuous income until some undetermined time, at which point the annuity defaults. As in the Richard Model, the consumer is able to invest in various financial instruments, make consumption choices, and purchase life insurance to hedge against an uncertain terminal date. The difference between this new model and the previous models is the annuity income may terminate before the planning horizon, forcing the retiree to make consumption, investment, and life insurance purchase choices without outside income. This model will deal with two random times, τ_D and τ_M, representing the default time of the income and terminal time of the retiree's mortality, respectively.

The addition of a terminating income poses great risk to the retiree. To mitigate the effects of this default possibility, the retiree is able to purchase default-contingent insurance, which we call default insurance. This comes in the form of a credit default swap, or CDS. In a CDS, the retiree pays a third party company some premium such that when the original company responsible for paying the annuity defaults on payments, the third party buys the remainder of the amount owed and keeps making payments to the retiree. As in [13], we will consider the case where the CDS makes a single lump sum payment to the retiree at time of default, similar to how life insurance works. This type of insurance does not readily exist in today's markets, making this an unmarketable model. We will see in Chapter 8.2 how to make this new insurance strategy marketable.

The change in income due to default risk changes the dynamics of the retiree's total wealth, and a single wealth process is no longer possible with the additional default probability. Two control problems need to be set up: one control problem associated with the retiree's wealth process before default (if default occurs before the retiree's random terminal time), and the other control problem associated with the retiree's wealth process after default, in which the retiree must continue to make consumption, investment, and life insurance purchase choices until the planning horizon.

Much of the setup for this model follows from previous models, and the reader will be referred to those sections for review. We will discuss the probability variables governing death and default, then lay out the new wealth processes, with their accompanying HJB equations. Many of the formulas and theorems used by Taylor are simple extensions of those done in Chapters 2 and 3, so will simply be referred to. Where necessary, additional work will be given to explain pertinent information.

4.1 The Financial Market

The financial market for the Taylor model is the same as the previous models. Refer to Sections 2.1 and 3.1 for details.

4.2 The Retiree's Lifetime

Consider two random times τ_D and τ_M in the probability space $(\Omega, \mathcal{F}, P)$, where τ_D is the time associated with annuity default and τ_M is the time associated with the retiree's death. We assume τ_D and τ_M are are mutually independent of each other and the Brownian motion $\mathcal{W}$. The times τ_D and τ_M are unpredictable, but their probability distributions can be approximated with density functions similar to that defined in Section 2.2. Let $f_D(t)$ be the probability density function for the probability distribution of the default time, τ_D, and let $f_M(t)$ be the probability density function corresponding to the retiree's mortality τ_M. From this, the distribution functions corresponding to default and mortality, respectively, are

$$F_D(t) = P(\tau_D < t) = \int_0^t f_D(u)du,$$
$$F_M(t) = P(\tau_M < t) = \int_0^t f_M(u)du.$$

The survivor functions are given as

$$\overline{F}_D(t) = P(\tau_D \geq t) = 1 - F_D(t),$$
$$\overline{F}_M(t) = P(\tau_M \geq t) = 1 - F_M(t),$$

and represent the probabilities the annuity has not defaulted or the retiree has not died, respectively. It can be shown, similar to Section 2.2, that there are hazard functions $\lambda_D(t)$ and $\lambda_M(t)$ for default and mortality. These hazard functions allow us to rewrite the survivor functions as

$$\overline{F}_D(t) = \exp\left(- \int_0^t \lambda_D(s)ds \right),$$
$$\overline{F}_M(t) = \exp\left(- \int_0^t \lambda_M(s)ds \right),$$

and the probability density functions as

$$f_D(t) = \lambda_D(t)\overline{F}_D(t) = \lambda_D(t) \exp\left(- \int_0^t \lambda_D(s)ds \right),$$
$$f_M(t) = \lambda_M(t)\overline{F}_M(t) = \lambda_M(t) \exp\left(- \int_0^t \lambda_M(s)ds \right).$$

Next define the conditional probability density functions

$$\begin{aligned} f_D(s|t) =& \frac{f_D(s)}{\overline{F}_D(t)} = \lambda_D(s) \exp\left(- \int_t^s \lambda_D(u)du \right), \\ f_M(s|t) =& \frac{f_M(s)}{\overline{F}_M(t)} = \lambda_M(s) \exp\left(- \int_t^s \lambda_M(u)du \right), \end{aligned} \tag{4.1}$$

and conditional survivor functions

$$\begin{aligned} \overline{F}_D(s|t) =& P(\tau_D \geq s|\tau_D \geq t) = \exp\left(- \int_t^s \lambda_D(u)du \right), \\ \overline{F}_M(s|t) =& P(\tau_M \geq s|\tau_M \geq t) = \exp\left(- \int_t^s \lambda_M(u)du \right). \end{aligned} \tag{4.2}$$

Finally, using the condition of mutual independence, define the conditional distribution survivor function for the minimal random time $\tau_D \wedge \tau_M = \min(\tau_M, \tau_D)$,

$$\overline{F}(s|t) = P(\tau_M \wedge \tau_D \geq s | \tau_M \wedge \tau_D \geq t) = \overline{F}_D(s|t)\overline{F}_M(s|t).$$

4.3 Wealth and Maximum Utility

We now define the control variables and state variables that will be needed to solve the retiree's optimization problem. Many of the same control variables from the Richard Model will be used here, with the exception that the insurance premium $p(t)$ and insurance premium-payout ratio $\eta(t)$ will come in two forms for default and mortality. Similarly, the state variables are all similar except the income $\alpha(t)$ from annuity will now have two pieces for pre- and post-default payments.

4.3.1 Control Variables and Wealth Processes.

In addition to the variables used in previous models, we define the following.

$\alpha_D(t)$ = defaultable annuity income rate (dollars per year) at time t.

$\alpha_{ND}(t)$ = nondefaultable annuity income rate (dollars per year) at time t

$p_D(t)$ = premium rate of default insurance purchased (dollars per year) at time t.

$p_M(t)$ = premium rate of life insurance purchased (dollars per year) at time t.

$\eta_D(t)$ = default insurance premium-payout ratio.

$\eta_M(t)$ = life insurance premium-payout ratio.

When considering the retiree's income stream, $\alpha_{ND}(t)$ can be thought of as the nondefaultable portion of the income, either by way of financial institution stability or guarantee. We can combine the defaultable and nondefaultable portions into a single income variable

$$\alpha(t) = \begin{cases} \alpha_D(t) + \alpha_{ND}(t) & \text{if } t \leq \tau_D, \\ \alpha_{ND}(t) & \text{if } t > \tau_D. \end{cases}$$

As in the Richard Model, we will allow the retiree to short-sell life insurance and default insurance for purposes of checking for an exact solution, but ultimately we will restrict $p_D(t) \geq 0$ and $p_M(t) \geq 0$, so short-selling is not allowed. The actuarially fair payout ratio for both premiums is $\eta_D(t) = p_D(t)$ and $\eta_M(t) = p_M(t)$. To ensure the insurance companies make a profit from insurance purchases, we will also consider the cases where $\eta_D(t) \geq p_D(t)$ and $\eta_M(t) \geq p_M(t)$ in Chapter 5. Again, insurance payments are made to ensure some life-end or default-time dollar amount. This amount is bequeathed to posterity if death occurs, or is used to continue to make consumption and investment choices if default occurs.

Now consider the wealth process for the retiree. Income and investments bolster the retiree's wealth, while consumption and insurance premiums are detractive. Thus the retiree's wealth process is similar to (3.1) and is given by

$$\begin{aligned} W(t) = & w - \int_0^t c(s)ds - \int_0^t p_D(s)ds - \int_0^t p_M(s)ds + \int_0^t \alpha(s)ds \\ & + \int_0^t \frac{W(s) - \pi(s)}{S_0(s)} dS_0(s) + \int_0^t \frac{\pi(s)}{S_1(s)} dS_1(s), \end{aligned}$$

and the SDE (3.2) becomes

$$\begin{aligned} dW(t) = & \big(r(t)W(t) - c(t) - p_D(t) - p_M(t) + \alpha(t) + \pi(t)(\mu(t) - r(t))\big)dt \\ & + \pi(t)\sigma(t)d\mathcal{W}(t). \end{aligned} \tag{4.3}$$

At this point it is important to consider the two different dynamics of the retiree's wealth, both pre- and post-default. Letting $W_1(t)$ be the pre-default wealth and $W_2(t)$ be the post-default wealth, then

$$W(t) = W_1(t)\mathbb{1}_{t \leq \tau_D} + W_2(t)\mathbb{1}_{t > \tau_D}.$$

Similar to the Richard Model, define $\zeta_D(t)$ and $\zeta_M(t)$ to be the liquid wealth after payouts for the credit default swap and life insurance, respectively. They are defined as

$$\zeta_M(t) = W(t) + \frac{p_M(t)}{\eta_M(t)}, \tag{4.4}$$

$$\zeta_D(t) = W(t) + \frac{p_D(t)}{\eta_D(t)}. \tag{4.5}$$

At time T, no insurance premiums will be paid and so

$$\zeta_M(T) = W(T) + \frac{p_M(T)}{\eta_M(T)} = W(T),$$
$$\zeta_D(T) = W(T) + \frac{p_D(T)}{\eta_D(T)} = W(T).$$

Define $\mathcal{A}(t,w)$ = set of all admissible tuples $(c, \pi, \zeta_M, \zeta_D)$ pre-default or (c, π, ζ_M) post-default. A pre-default tuple $(c, \pi, \zeta_M, \zeta_D)$ is admissible if

- $\int_t^T (|c(s)| + |\pi(s)| + |p_M(s)| + |p_D(s)| + \pi(s)^2) ds < \infty$,
- $W(t) + \frac{p_M(t)}{\eta_M(t)} \geq 0$ for all $t \in [0, T]$, and
- $W(t) + \frac{p_D(t)}{\eta_D(t)} \geq 0$ for all $t \in [0, T]$.

Post-default, the retiree no longer purchases default insurance, thus a post-default tuple (c, π, ζ_M) is admissible if

- $\int_t^T (|c(s)| + |\pi(s)| + |p_M(s)| + \pi(s)^2) ds < \infty$, and
- $W(t) + \frac{p_M(t)}{\eta_M(t)} \geq 0$ for all $t \in [0, T]$.

We change the insurance premium controls into end wealth controls with

$$\begin{aligned} p_M(t) &= \eta_M(t)\big(\zeta_M(t) - W(t)\big), \\ p_D(t) &= \eta_D(t)\big(\zeta_D(t) - W(t)\big). \end{aligned} \tag{4.6}$$

Then (4.3) becomes

$$\begin{aligned} dW(t) = &\big((r(t) + \eta_M(t) + \eta_D(t))W(t) - c(t) + \alpha(t) - \eta_M(t)\zeta_M(t) - \eta_D(t)\zeta_D(t) \\ &+ \pi(t)(\mu(t) - r(t))\big)dt + \pi(t)\sigma(t)d\mathcal{W}(t). \end{aligned} \tag{4.7}$$

Pre-default, with $W(t) = W_1(t)$ and $\alpha(t) = \alpha_D(t) + \alpha_{ND}(t)$, (4.7) becomes

$$\begin{aligned} dW_1(t) =& \big((r(t) + \eta_M(t) + \eta_D(t))W_1(t) - c(t) + \alpha_D(t) + \alpha_{ND}(t) - \eta_M(t)\zeta_M(t) - \eta_D(t)\zeta_D(t) \\ &+ \pi(t)(\mu(t) - r(t))\big)dt + \pi(t)\sigma(t)d\mathcal{W}(t). \end{aligned} \tag{4.8}$$

Post-default, with $W(t) = W_2(t)$ and $\alpha(t) = \alpha_{ND}(t)$, (4.7) becomes

$$\begin{aligned} dW_2(t) =& \big((r(t) + \eta_M(t))W_2(t) - c(t) + \alpha_{ND}(t) - \eta_M(t)\zeta_M(t) + \pi(t)(\mu(t) - r(t))\big)dt \\ &+ \pi(t)\sigma(t)d\mathcal{W}(t). \end{aligned} \tag{4.9}$$

4.3.2 Utility and Value Function.

Using the same utility functions as before, the value function (3.5) becomes

$$\begin{aligned} V(t, w) = \sup_{(c,\pi,\zeta_M,\zeta_D)} E\bigg[\int_t^{T\wedge\tau_M} & U(s, c(s))ds + B(\tau_M, \zeta_M(\tau_M))\mathbb{1}_{\{\tau_M<T\}} \\ &+ B(T, W(T))\mathbb{1}_{\{\tau_M\geq T\}} \bigg]. \end{aligned}$$

[13] showed that splitting the time around τ_D gives

$$\begin{aligned} V(t, w) = \sup_{(c,\pi,\zeta_M,\zeta_D)} E\bigg[\int_t^{\tau_M\wedge T\wedge\tau_D} & U(s, c(s))ds + B(\tau_M, \zeta_M(\tau_M))\mathbb{1}_{\tau_M<T\wedge\tau_D} \\ + B(T, W(T))\mathbb{1}_{T\leq\tau_M\wedge\tau_D} &+ D(\tau_D, \zeta_D(\tau_D))\mathbb{1}_{\tau_D<T\wedge\tau_M} \bigg], \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} D(t, w) = \sup_{(c,\pi,\zeta_M)} E\bigg[\int_t^{T\wedge\tau_M} & U(s, c(s))ds + B(\tau_M, \zeta_M(\tau_M))\mathbb{1}_{\tau_M<T} \\ &+ B(T, W(T))\mathbb{1}_{\tau_M\geq T} \bigg]. \end{aligned} \tag{4.11}$$

This is the pre-default value function we will work with throughout the section. If $t > \tau_D$, then we get the post-default wealth process

$$V(t, w) = D(t, w),$$

thus we will solve (4.11) in the post-default situation.

4.4 Stochastic Dynamic Programming

Now apply the stochastic dynamic programming technique to the pre- and post-default value functions to derive the HJB equations and optimal feedback controls for those equations. In order to do this, we first must apply the expectation operator given in (4.10). We also use the fact that given the retiree is alive at time t, then for $t \leq s \leq T$,

$$E_t\left[\mathbb{1}_{s \leq \tau_D}\right] = \overline{F}_D(s|t)$$

and

$$E_t\left[\mathbb{1}_{s \leq \tau_M}\right] = \overline{F}_M(s|t),$$

where $\overline{F}_M(s|t)$ and $\overline{F}_D(s|t)$ are given by (4.2).

After applying the expectation operator, no concern is given to the terminal and default times, τ_M and τ_D, and we can simply integrate over the utility functions while applying a weighting function equivalent to the hazard functions $\lambda_M(t)$ and $\lambda_D(t)$. The new, time-independent, value functions will be given in the next sections.

4.4.1 Post-Default HJB and Control.

Consider (4.11). Define the cost functional

$$\begin{aligned} J_2(t,w;c,\pi,\zeta_M) =& E\bigg[\int_t^{T\wedge\tau_M} U(s,c(s))ds + B(\tau_M,\zeta_M(\tau_M))\mathbb{1}_{\tau_M<T} \\ &+ B(T,W_2(T))\mathbb{1}_{\tau_M\geq T}\bigg|\tau_M\geq t,\mathcal{F}_t\bigg]. \end{aligned}$$

The post-default value function becomes

$$D(t,w) = \sup_{(c,\pi,\zeta_M)} J_2(t,w;c,\pi,\zeta_M). \tag{4.12}$$

Lemma 4.1. *Suppose $U(\cdot,\cdot)$ is a non-negative or non-positive function. Suppose the terminal time τ_M is independent of the planning horizon T and filtration of the probability space. Then*

$$\begin{aligned} J_2(t,w;c,\pi,\zeta_M) =& E\bigg[\int_t^T \big(f_M(s|t)B(s,\zeta_M(s)) + \overline{F}_M(s|t)U(s,c(s))\big)ds \\ &+ \overline{F}_M(T|t)B(T,W_2(T))\bigg|\mathcal{F}_t\bigg]. \end{aligned} \tag{4.13}$$

Proof. Reference Lemma 3.3.1 in [11]. □

Now apply the dynamic programming principle, from which we can derive the HJB equation for the post-default wealth process.

Lemma 4.2 (Dynamic Programming Principle, Post-Default)**.** *For $0\leq t<s<T$,*

$$\begin{aligned} D(t,w) = \sup_{(c,\pi,\zeta_M)} & E\bigg[\overline{F}_M(s|t)D(s,W_2(s)) \\ &+ \int_t^s \big(f_M(u|t)B(u,\zeta_M(u)) + \overline{F}_M(u|t)U(u,c(u))\big)du\bigg|\mathcal{F}_t\bigg]. \end{aligned} \tag{4.14}$$

Proof. Reference Lemma 3.3.2 in [11]. □

Now derive the HJB equation for the post-default wealth process.

Theorem 4.3 (Dynamic Programming Equation, Post-Default)**.** *Suppose the value function $D(t,w)$ is smooth. Then $D(t,w)$ satisfies*

$$\begin{cases} 0 = D_t(t,W_2(t)) - \lambda_M(t)D(t,W_2(t)) + \sup\limits_{(c,\pi,\zeta_M)} \Big\{ b_2(t,W_2(t))D_w(t,W_2(t)) \\ \qquad + \frac{1}{2}\pi^2(t)\sigma^2(t)D_{ww}(t,W_2(t)) + \lambda_M(t)B(t,\zeta_M(t)) + U(t,c(t)) \Big\} \\ D(T,w) = B(T,w), \end{cases} \tag{4.15}$$

where

$$b_2(t,w) = (r(t)+\eta_M(t))w - c(t) + \alpha_{ND}(t) - \eta_M(t)\zeta_M(t) + \pi(t)(\mu(t)-r(t)).$$

Proof. See Theorem 3.3.1 in [11]. □

For $t > \tau_D$, the control policy is determined by

$$(c(t),\pi(t),\zeta_M(t)) = (c_2(t),\pi_2(t),\zeta_{M_2}(t)).$$

The control policy $(c_2^*,\pi_2^*,\zeta_{M_2}^*)$ is optimal if it satisfies (4.12) such that

$$D(t,w) = J_2(t,w;c_2^*,\pi_2^*,\zeta_{M_2}^*).$$

The following verification theorem lets us know when $(c_2^*,\pi_2^*,\zeta_{M_2}^*)$ is optimal.

Theorem 4.4 (Verification Theorem, Post-Default)**.** *Let a smooth function $D(t,w)$ be a solution to the HJB equation* (4.15)*. Then*

$$D(t,w) \geq J_2(t,w;c_2,\pi_2,\zeta_{M_2}) \tag{4.16}$$

for all valid triples $(c_2, \pi_2, \zeta_{M_2})$ *in the control space.*

Furthermore, an admissible triple $(c_2^*, \pi_2^*, \zeta_{M_2}^*)$ *with corresponding wealth* W_2^* *is optimal* $\iff$

$$0 = D_t(s, W_2^*(s)) - \lambda_M(s) D(s, W_2^*(s)) + \sup_{(c,\pi,\zeta_{M_2}) \in \mathcal{A}(s, W_2^*(s))} \Psi(s, W_2^*(s); c_2, \pi_2, \zeta_{M_2})$$

for almost every $s \in [t, T]$*, where*

$$\begin{aligned} \Psi(t, w; c_2, \pi_2, \zeta_{M_2}) =& b_2(t, w) D_w(t, w) + \tfrac{1}{2}\pi_2^2(t)\sigma^2(t) D_{ww}(t, w) + \lambda_M(t) B(t, \zeta_{M_2}(t)) \\ &+ U(t, c_2(t)). \end{aligned}$$

Proof. Reference Theorem 3.3.2 in [11]. □

Then the verification theorem above says the control policy $(c_2^*, \pi_2^*, \zeta_{M_2}^*)$ is optimal if and only if

$$\begin{aligned} 0 =& D_t(t, w) - \lambda_M(t) D(t, w) + \Psi(t, w; c_2^*, \pi_2^*, \zeta_{M_2}^*) \\ =& D_t(t, w) - \lambda_M(t) D(t, w) + \sup_{(c_2, \pi_2, \zeta_{M_2})} \Psi(t, w; c_2, \pi_2, \zeta_{M_2}) \\ =& D_t(t, w) - \lambda_M(t) D(t, w) + \big((r(t) + \eta_M(t)) w + \alpha_{ND}(t)\big) D_w(t, w) \\ &+ \sup_{c_2} \big\{ U(t, c_2(t)) - c_2(t) D_w(t, w) \big\} \\ &+ \sup_{\pi_2} \big\{ \pi_2(t)(\mu(t) - r(t)) D_w(t, w) + \tfrac{1}{2}\pi_2^2(t)\sigma^2(t) D_{ww}(t, w) \big\} \\ &+ \sup_{\zeta_{M_2}} \big\{ \lambda_M(t) B(t, \zeta_{M_2}(t)) - \eta_M(t)\zeta_{M_2}(t) D_w(t, w) \big\}. \end{aligned} \tag{4.17}$$

The first order conditions for a maximum are

$$\Psi_c(t, w; c_2^*, \pi_2^*, \zeta_{M_2}^*) = 0 = U_c(t, c_2^*) - D_w(t, w), \tag{4.18}$$

$$\Psi_\pi(t, w; c_2^*, \pi_2^*, \zeta_{M_2}^*) = 0 = (\mu(t) - r(t)) D_w(t, w) + \pi_2^* \sigma^2(t) D_{ww}(t, w), \tag{4.19}$$

$$\Psi_{\zeta_M}(t,w;c_2^*,\pi_2^*,\zeta_{M_2}^*) = 0 = \lambda_M(t)B_{\zeta_M}(t,\zeta_{M_2}^*) - \eta_M(t)D_w(t,w). \tag{4.20}$$

Sufficient conditions for a maximum come from the second order partials

$$\begin{aligned}\Psi_{cc}(t,w;c_2^*,\pi_2^*,\zeta_{M_2}^*) =& U_{cc}(t,c_2^*) < 0,\\ \Psi_{\pi\pi}(t,w;c_2^*,\pi_2^*,\zeta_{M_2}^*) =& \sigma^2(t)D_{ww}(t,w) < 0,\\ \Psi_{\zeta_M\zeta_M}(t,w;c_2^*,\pi_2^*,\zeta_{M_2}^*) =& \lambda_M(t)B_{\zeta_M\zeta_M}(t,\zeta_{M_2}^*) < 0.\end{aligned}$$

The utility functions are strictly concave, so the first and third conditions are automatically met. It remains to show then that

$$D_{ww}(t,w) < 0.$$

If to the contrary we assume $D_{ww}(t,w) \geq 0$, then (4.17) shows

$$\sup_{(c_2,\pi_2,\zeta_{M_2})} \Psi(t,w;c_2,\pi_2,\zeta_{M_2}) = \infty.$$

The HJB equation (4.15) then forces either $D_t(t,w) = -\infty$ or $D(t,w) = \infty$, in which case $D(t,w)$ is not smooth, a contradiction. Thus, $D(t,w)$ is concave and the result for maximum conditions holds.

4.4.2 Pre-Default HJB and Control.

Refer now to (4.10). Define the cost functional

$$\begin{aligned}J_1(t,w;c,\pi,\zeta_M,\zeta_D) =& E\bigg[\int_t^{\tau_M\wedge T\wedge\tau_D} U(s,c(s))ds + B(\tau_M,\zeta_M(\tau_M))\mathbb{1}_{\tau_M<T\wedge\tau_D}\\ &+ B(T,W_1(T))\mathbb{1}_{T\leq\tau_M\wedge\tau_D} + D(\tau_D,\zeta_D(\tau_D))\mathbb{1}_{\tau_D<T\wedge\tau_M}\bigg|\tau_M\wedge\tau_D \geq t,\mathcal{F}_t\bigg].\end{aligned} \tag{4.21}$$

Then

$$V(t,w) = \sup_{(c,\pi,\zeta_M,\zeta_D)} J_1(t,w;c,\pi,\zeta_M,\zeta_D). \tag{4.22}$$

Lemma 4.5. *Suppose $U(\cdot,\cdot)$ is a non-negative or non-positive function. Suppose the terminal time τ_M and default time τ_D are independent of each other, the planning horizon T, and the filtration of the probability space. Then*

$$\begin{aligned} J_1(t,w;c,\pi,\zeta_M,\zeta_D) =& E\bigg[\int_t^T \big(U(s,c(s)) + \lambda_M(s)B(s,\zeta_M(s)) \\ &+ \lambda_D(s)D(s,\zeta_D(s))\big)\overline{F}(s|t)ds + B(T,W_1(T))\overline{F}(T|t)\bigg|\mathcal{F}_t\bigg]. \end{aligned} \tag{4.23}$$

Proof. See Appendix A.7. □

Now derive the dynamic programming principle and equation similar to the post-default equation.

Lemma 4.6 (Dynamic Programming Principle, Pre-Default)**.** *For $0 \le t < s < T$,*

$$\begin{aligned} V(t,w) = \sup_{(c,\pi,\zeta_M,\zeta_D)} & E\bigg[\int_t^s \big(U(u,c(u)) + \lambda_M(u)B(u,\zeta_M(u)) \\ &+ \lambda_D(u)D(u,\zeta_D(u))\big)\overline{F}(u|t)du + \overline{F}(s|t)V(s,W_1(s))\bigg|\mathcal{F}_t\bigg]. \end{aligned} \tag{4.24}$$

Proof. The proof is similar to Lemma 3.3.2 in [11], with the addition of $\lambda_D(t)D(t,\zeta_D(t))$. □

The HJB equation then follows.

Theorem 4.7 (Dynamic Programming Equation, Pre-Default). *Suppose the value function $V(t,w)$ is smooth. Then $V(t,w)$ satisfies*

$$\begin{cases} 0 = V_t(t,W_1(t)) - (\lambda_M(t)+\lambda_D(t))V(t,W_1(t)) \\ \qquad + \sup\limits_{(c,\pi,\zeta_M,\zeta_D)} \Big\{ b_1(t,W_1(t))V_w(t,W_1(t)) + \frac{1}{2}\pi^2(t)\sigma^2(t)V_{ww}(t,W_1(t)) \\ \qquad + \lambda_M(t)B(t,\zeta_M(t)) + \lambda_D(t)D(t,\zeta_D(t)) + U(t,c(t)) \Big\} \\ V(T,w) = B(T,w), \end{cases} \tag{4.25}$$

where

$$\begin{aligned} b_1(t,w) =& \big(r(t)+\eta_M(t)+\eta_D(t)\big)w - c(t) + \alpha_D(t) + \alpha_{ND}(t) - \eta_M(t)\zeta_M(t) - \eta_D(t)\zeta_D(t) \\ & + \pi(t)(\mu(t)-r(t)). \end{aligned}$$

Proof. The proof is similar to Theorem 3.3.1 in [11]. □

The pre-default control policy is determined by

$$(c(t),\pi(t),\zeta_M(t),\zeta_D(t)) = (c_1(t),\pi_1(t),\zeta_{M_1}(t),\zeta_{D_1}(t)).$$

The control policy $(c_1^*,\pi_1^*,\zeta_{M_1}^*,\zeta_{D_1}^*)$ is optimal if it satisfies (4.22) such that

$$V(t,w) = J_1(t,w;c_1^*,\pi_1^*,\zeta_{M_1}^*,\zeta_{D_1}^*).$$

Theorem 4.8 (Verification Theorem, Pre-Default). *Let a smooth function $V(t,w)$ be a solution to the HJB equation* (4.25)*. Then*

$$V(t,w) \geq J_1(t,w;c_1,\pi_1,\zeta_{M_1},\zeta_{D_1}) \tag{4.26}$$

for all tuples $(c_1,\pi_1,\zeta_{M_1},\zeta_{D_1}) \in \mathcal{A}(t,w)$.

Furthermore, an admissible tuple $(c_1^*, \pi_1^*, \zeta_{M_1}^*, \zeta_{D_1}^*)$ *with corresponding wealth* W_1^* *is optimal* $\iff$

$$\begin{aligned}0 =& V_t(s, W_1^*(s)) - (\lambda_M(s) + \lambda_D(s))V(s, W_1^*(s)) \\ &+ \sup_{(c_1,\pi_1,\zeta_{M_1},\zeta_{D_1})\in\mathcal{A}(s,W_1^*(s))} \Psi(s, W_1^*(s); c_1, \pi_1, \zeta_{M_1}, \zeta_{D_1})\end{aligned}$$

for almost every $s \in [t, T]$, *where*

$$\begin{aligned}\Psi(t, w; c_1, \pi_1, \zeta_{M_1}, \zeta_{D_1}) =& b_1(t, w)V_w(t, w) + \tfrac{1}{2}\pi_1^2(t)\sigma^2(t)V_{ww}(t, w) \\ &+ \lambda_M(t)B(t, \zeta_{M_1}(t)) + \lambda_D(t)D(t, \zeta_{D_1}(t)) + U(t, c_1(t)).\end{aligned}$$

Proof. See Appendix A.8. □

This theorem says the control policy $(c_1^*, \pi_1^*, \zeta_{M_1}^*, \zeta_{D_1}^*)$ is optimal if and only if

$$\begin{aligned}0 =& V_t(t, w) - (\lambda_M(t) + \lambda_D(t))V(t, w) + \Psi(t, w; c_1^*, \pi_1^*, \zeta_{M_1}^*, \zeta_{D_1}^*) \\ =& V_t(t, w) - (\lambda_M(t) + \lambda_D(t))V(t, w) + \sup_{(c_1,\pi_1,\zeta_{M_1},\zeta_{D_1})} \Psi(t, w; c_1, \pi_1, \zeta_{M_1}, \zeta_{D_1}) \\ =& V_t(t, w) - (\lambda_M(t) + \lambda_D(t))V(t, w) + \big(\alpha_D(t) + \alpha_{ND}(t) \\ &+ (r(t) + \eta_M(t) + \eta_D(t))w\big)V_w(t, w) \\ &+ \sup_{c_1} \big\{U(t, c_1(t)) - c_1(t)V_w(t, w)\big\} \\ &+ \sup_{\pi_1} \big\{(\mu(t) - r(t))\pi_1(t)V_w(t, w) + \tfrac{1}{2}\pi_1^2(t)\sigma^2(t)V_{ww}(t, w)\big\} \\ &+ \sup_{\zeta_{M_1}} \big\{\lambda_M(t)B(t, \zeta_{M_1}(t)) - \eta_M(t)\zeta_{M_1}(t)V_w(t, w)\big\} \\ &+ \sup_{\zeta_{D_1}} \big\{\lambda_D(t)D(t, \zeta_{D_1}(t)) - \eta_D(t)\zeta_{D_1}(t)V_w(t, w)\big\}.\end{aligned} \tag{4.27}$$

The first order conditions for a maximum are

$$\Psi_c(t, w; c_1^*, \pi_1^*, \zeta_{M_1}^*, \zeta_{D_1}^*) = 0 = U_c(t, c_1^*) - V_w(t, w), \tag{4.28}$$

$$\Psi_\pi(t, w; c_1^*, \pi_1^*, \zeta_{M_1}^*, \zeta_{D_1}^*) = 0 = (\mu(t) - r(t))V_w(t, w) + \pi_1^*\sigma^2(t)V_{ww}(t, w), \tag{4.29}$$

$$\Psi_{\zeta_M}(t,w;c_1^*,\pi_1^*,\zeta_{M_1}^*,\zeta_{D_1}^*) = 0 = \lambda_M(t)B_{\zeta_M}(t,\zeta_{M_1}^*) - \eta_M(t)V_w(t,w), \tag{4.30}$$

$$\Psi_{\zeta_D}(t,w;c_1^*,\pi_1^*,\zeta_{M_1}^*,\zeta_{D_1}^*) = 0 = \lambda_D(t)D_{\zeta_D}(t,\zeta_{D_1}^*) - \eta_D(t)V_w(t,w). \tag{4.31}$$

Sufficient conditions for a maximum come from the second order partials

$$\begin{aligned}
\Psi_{cc}(t,w;c_1^*,\pi_1^*,\zeta_{M_1}^*,\zeta_{D_1}^*) =& U_{cc}(t,c_1^*) < 0,\\
\Psi_{\pi\pi}(t,w;c_1^*,\pi_1^*,\zeta_{M_1}^*,\zeta_{D_1}^*) =& \sigma^2(t)V_{ww}(t,w) < 0,\\
\Psi_{\zeta_M\zeta_M}(t,w;c_1^*,\pi_1^*,\zeta_{M_1}^*,\zeta_{D_1}^*) =& \lambda_M(t)B_{\zeta_M\zeta_M}(t,\zeta_{M_1}^*) < 0,\\
\Psi_{\zeta_D\zeta_D}(t,w;c_1^*,\pi_1^*,\zeta_{M_1}^*,\zeta_{D_1}^*) =& \lambda_D(t)D_{\zeta_D\zeta_D}(t,\zeta_{D_1}^*) < 0.
\end{aligned}$$

By assumption, the utility functions $B(\cdot,\cdot)$ and $U(\cdot,\cdot)$ are concave so conditions one and three hold automatically. It was shown in the post-default conditions that the utility function $D(\cdot,\cdot)$ must also be concave, so condition four is met. It remains to be shown that

$$V_{ww}(t,w) < 0.$$

If to the contrary $V_{ww}(t,w) \geq 0$, then (4.27) shows

$$\sup_{(c_1,\pi_1,\zeta_{M_1},\zeta_{D_1})} \Psi(t,w;,c_1,\pi_1,\zeta_{M_1},\zeta_{D_1}) = \infty.$$

The HJB equation (4.25) then forces either $V_t(t,w) = -\infty$ or $V(t,w) = \infty$, in which case $V(t,w)$ cannot be a smooth function, a contradiction. Therefore $V_{ww}(t,w)$ is concave and the result for maximum conditions holds.

4.4.3 Optimal Policies.

The retiree wishes to find pre-default admissible controls $(c_1^*,\pi_1^*,\zeta_{M_1}^*,\zeta_{D_1}^*)$, if they exist, such that

$$V(t,w) = J_1(t,w;c_1^*,\pi_1^*,\zeta_{M_1}^*,\zeta_{D_1}^*). \tag{4.32}$$

As part of the process, the retiree must also optimize the post-default wealth process. To do this, the retiree must find post-default admissible controls $(c_2^*, \pi_2^*, \zeta_{M_2}^*)$ such that

$$D(t,w) = J_2(t,w;c_2^*,\pi_2^*,\zeta_{M_2}^*). \tag{4.33}$$

In the following sections, we will establish the exact and numerical solutions to these problems.

4.5 Constant Relative Risk Aversion

The CRRA utility functions used in Taylor' model will be the same as those used in the Richard Model. The utility functions are

$$\begin{aligned} U(t,c) =& \frac{e^{-\rho t}}{\gamma} c^{\gamma}, \\ B(t,\zeta_M(t)) =& \frac{e^{-\rho t}}{\gamma} \zeta_M(t)^{\gamma}. \end{aligned}$$

In the next section, we look for an exact solution to the problems above by assuming short-selling of life insurance and credit default insurance is allowed, just as in the exact solution to Richard's problem. From there we consider the scenario where short-selling insurance is not allowed using the method of finite differences and Ye's MCALT application in [11].

4.6 Exact Solution

To find exact solutions to (4.32) and (4.33), consider the cases separately under the post- and pre-default scenarios. Those will be provided below. Because the pre-default scenario relies on $D(\cdot,\cdot)$, we first consider the post-default scenario.

4.6.1 Exact Solution, Post-Default.

Using the updated post-default control variables

$$\begin{aligned} c(t) =& c_2(t), \\ \pi(t) =& \pi_2(t), \\ \zeta_M(t) =& \zeta_{M_2}(t), \end{aligned}$$

rewrite (4.15) as

$$\begin{cases} 0 = D_t(t, W_2(t)) - \lambda_M(t) D(t, W_2(t)) + \sup\limits_{(c_2, \pi_2, \zeta_{M_2})} \Big\{ b_2(t, W_2(t)) D_w(t, W_2(t)) \\ \qquad + \frac{1}{2}\pi_2^2(t)\sigma^2(t) D_{ww}(t, W_2(t)) + \lambda_M(t) B(t, \zeta_{M_2}(t)) + U(t, c_2(t)) \Big\} \\ D(T, w) = B(T, w), \end{cases} \tag{4.34}$$

where

$$b_2(t, w) = (r(t) + \eta_M(t))w - c_2(t) + \alpha_{ND}(t) - \eta_M(t)\zeta_{M_2}(t) + \pi_2(t)(\mu(t) - r(t)).$$

Rearranging the optimal first order conditions (4.18) - (4.20) gives results similar to (3.14) - (3.16), namely

$$c_2^* = (D_w(t, W_2(t)) e^{\rho t})^{1/(\gamma - 1)}, \tag{4.35}$$

$$\pi_2^* = - \frac{(\mu(t) - r(t)) D_w(t, W_2(t))}{\sigma^2(t) D_{ww}(t, W_2(t))}, \tag{4.36}$$

$$\zeta_{M_2}^* = \left(\frac{\eta_M(t)}{\lambda_M(t)} e^{\rho t} D_w(t, W_2(t)) \right)^{1/(\gamma - 1)}. \tag{4.37}$$

Plug (4.35) - (4.37) into (4.34), similar to (3.17) to get the updated HJB equation

$$\begin{cases} 0 = D_t(t, W_2(t)) - \lambda_M(t) D(t, W_2(t)) \\ \qquad +\big((\eta_M(t) + r(t))W_2(t) + \alpha_{ND}(t)\big) D_w(t, W_2(t)) \\ \qquad -\frac{1}{2}\big(\frac{\mu(t)-r(t)}{\sigma(t)}\big)^2 \frac{D_w^2(t,W_2(t))}{D_{ww}(t,W_2(t))} + \frac{1-\gamma}{\gamma} e^{\rho t/(\gamma-1)} K(t) D_w(t, W_2(t))^{\gamma/(\gamma-1)} \\ D(T, w) = \frac{e^{-\rho T}}{\gamma} w^\gamma, \end{cases} \tag{4.38}$$

where

$$K(t) = \frac{\lambda_M(t)^{1/(1-\gamma)}}{\eta_M(t)^{\gamma/(1-\gamma)}} + 1.$$

To solve (4.38), assume the solution is in the form

$$D(t, w) = \frac{a(t)}{\gamma}(w + I_{ND}(t))^\gamma, \tag{4.39}$$

where $a(t)$ and $I_{ND}(t)$ are functions which will be determined later. $I_{ND}(t)$ can be thought of as the present day discounted value of future income, or illiquid wealth, from the non-defaultable income $\alpha_{ND}(t)$.

Note up to this point, we have recreated the Richard Model, with our post-default substitutions. Thus, $a(t)$ is given by (3.22) and $I_{ND}(t)$ is given by (3.21). Equation (3.21) is rewritten here as

$$I_{ND}(t) = \int_t^T \alpha_{ND}(s) \exp\bigg(- \int_t^s (r(v) + \eta_M(v)) dv \bigg) ds. \tag{4.40}$$

4.6.2 Exact Solution, Pre-Default.

Now use the updated pre-default control variables

$$c(t) = c_1(t),$$

$$\pi(t) = \pi_1(t),$$

$$\zeta_M(t) = \zeta_{M_1}(t),$$

$$\zeta_D(t) = \zeta_{D_1}(t),$$

in the pre-default HJB equation (4.25), which becomes

$$\begin{cases} 0 = V_t(t, W_1(t)) - (\lambda_M(t) + \lambda_D(t))V(t, W_1(t)) \\ \qquad + \sup\limits_{(c_1, \pi_1, \zeta_{M_1}, \zeta_{D_1})} \Big\{ b_1(t, W_1(t))V_w(t, W_1(t)) + \frac{1}{2}\pi_1^2(t)\sigma^2(t)V_{ww}(t, W_1(t)) \\ \qquad + \lambda_M(t)B(t, \zeta_{M_1}(t)) + \lambda_D(t)D(t, \zeta_{D_1}(t)) + U(t, c_1(t)) \Big\} \\ V(T, w) = B(T, w), \end{cases} \tag{4.41}$$

where

$$\begin{aligned} b_1(t, w) =& (r(t) + \eta_M(t) + \eta_D(t))w + \alpha_D(t) - c_1(t) + \alpha_{ND}(t) \\ & - \eta_M(t)\zeta_{M_1}(t) - \eta_D(t)\zeta_{D_1}(t) + \pi_1(t)(\mu(t) - r(t)). \end{aligned}$$

Rearranging the optimal first order conditions (4.28) - (4.30) gives similar results to (4.35) - (4.37), with

$$c_1^* = \big(V_w(t, W_1(t))e^{\rho t}\big)^{1/(\gamma-1)}, \tag{4.42}$$

$$\pi_1^* = -\frac{(\mu(t) - r(t))V_w(t, W_1(t))}{\sigma^2(t)V_{ww}(t, W_1(t))}, \tag{4.43}$$

$$\zeta_{M_1}^* = \left(\frac{\eta_M(t)}{\lambda_M(t)} e^{\rho t} V_w(t, W_1(t))\right)^{1/(\gamma-1)}. \tag{4.44}$$

Now do the same to (4.31) to solve for $\zeta_{D_1}^*$. In order to do this, we must use the fact that

$$D(t, \zeta_{D_1}) = \frac{a(t)}{\gamma}\big(\zeta_{D_1} + I_{ND}(t)\big)^{\gamma}$$

from (4.39). Taking the derivative with respect to ζ_{D_1} gives

$$D_{\zeta_{D_1}}(t, \zeta_{D_1}) = a(t)\big(\zeta_{D_1} + I_{ND}(t)\big)^{\gamma-1}.$$

Plugging these into (4.31) and solving for $\zeta^*_{D_1}$ gives

$$\zeta^*_{D_1} = \left(\frac{\eta_D(t)}{\lambda_D(t)}\frac{V_w(t,w)}{a(t)}\right)^{1/(\gamma-1)} - I_{ND}(t). \tag{4.45}$$

Substituting (4.42), (4.44), and (4.45) into the utility functions $U(\cdot,\cdot)$, $B(\cdot,\cdot)$, and $D(\cdot,\cdot)$ respectively, plus the fact that $a(t) = e^{-\rho t}e(t)^{1-\gamma}$, gives

$$\begin{aligned}
U(t,c_1^*) =& \frac{e^{-\rho t}}{\gamma}(c_1^*)^\gamma = \frac{1}{\gamma}(e^{\rho t})^{1/(\gamma-1)}\big(V_w(t,W_1(t))\big)^{\gamma/(\gamma-1)},\\
B(t,\zeta^*_{M_1}) =& \frac{e^{-\rho t}}{\gamma}(\zeta^*_{M_1})^\gamma = \frac{1}{\gamma}(e^{\rho t})^{1/(\gamma-1)}\left(\frac{\eta_M(t)}{\lambda_M(t)}\right)^{\gamma/(\gamma-1)}\big(V_w(t,W_1(t))\big)^{1/(\gamma-1)},\\
D(t,\zeta^*_{D_1}) =& \frac{a(t)}{\gamma}(\zeta^*_{D_1} + I_{ND}(t))^\gamma\\
=& \frac{1}{\gamma}\big(a(t)\big)^{-1/(\gamma-1)}\left(\frac{\eta_D(t)}{\lambda_D(t)}\right)^{\gamma/(\gamma-1)}\big(V_w(t,W_1(t))\big)^{\gamma/(\gamma-1)}\\
=& \frac{1}{\gamma}(e^{\rho t})^{1/(\gamma-1)}e(t)\left(\frac{\eta_D(t)}{\lambda_D(t)}\right)^{\gamma/(\gamma-1)}\big(V_w(t,W_1(t))\big)^{\gamma/(\gamma-1)}.
\end{aligned}$$

Plug these into the pre-default HJB equation (4.41) and rearrange to get the updated HJB equation

$$\begin{cases}
0 = V_t(t,W_1(t)) - (\lambda_M(t)+\lambda_D(t))V(t,W_1(t)) + \frac{1-\gamma}{\gamma}(e^{\rho t})^{1/(\gamma-1)}K_D(t)V_w(t,W_1(t))^{\gamma/(\gamma-1)}\\
\qquad + \Big[\alpha_{ND}(t) + \alpha_D(t) + \eta_D(t)I_{ND}(t) + \big(r(t)+\eta_M(t)+\eta_D(t)\big)W_1(t)\Big]V_w(t,W_1(t))\\
\qquad - \frac{1}{2}\left(\frac{\mu(t)-r(t)}{\sigma(t)}\right)^2\frac{V_w^2(t,W_1(t))}{V_{ww}(t,W_1(t))}\\
V(T,w) = \frac{e^{-\rho T}}{\gamma}w^\gamma,
\end{cases} \tag{4.46}$$

where

$$K_D(t) = 1 + \frac{\eta_M(t)^{\gamma/(\gamma-1)}}{\lambda_M(t)^{1/(\gamma-1)}} + e(t)\frac{\eta_D(t)^{\gamma/(\gamma-1)}}{\lambda_D(t)^{1/(\gamma-1)}}.$$

The derivation of (4.46) can be found in Appendix A.9. To solve (4.46), assume the solution is in the form

$$V(t,w) = \frac{a_D(t)}{\gamma}\big(w + I_D(t) + I_{ND}(t)\big)^{\gamma}, \tag{4.47}$$

where $a_D(t)$ and $I_D(t)$ are functions which will be determined shortly. $I_D(t)$ is similar to $I_{ND}(t)$ in that it is the expected present discounted value of future wealth stemming from the defaultable income $\alpha_D(t)$, and so $I_D(t)+I_{ND}(t)$ is the expected present discounted value of all future wealth. Taking derivatives of (4.47) yields the relationships

$$\begin{aligned}
V_w(t,w) &= a_D(t)\big(w + I_D(t) + I_{ND}(t)\big)^{\gamma-1},\\
V_{ww}(t,w) &= a_D(t)(\gamma-1)\big(w + I_D(t) + I_{ND}(t)\big)^{\gamma-2},\\
V_t(t,w) &= \frac{a_D'(t)}{a_D(t)}V(t,w) + (I_D(t) + I_{ND}(t))'V_w(t,w),\\
\frac{V_w^2(t,w)}{V_{ww}(t,w)} &= \frac{\gamma}{\gamma-1}V(t,w),\\
V_w(t,w)^{\gamma/(\gamma-1)} &= a_D(t)^{1/(\gamma-1)}\gamma V(t,w),\\
wV_w(t,w) &= \gamma V(t,w) - (I_D(t) + I_{ND}(t))V_w(t,w).
\end{aligned}$$

Substituting these into (4.46) gives

$$\begin{aligned}
0 =& \Bigg[\frac{a_D'(t)}{a_D(t)} - (\lambda_M(t) + \lambda_D(t)) + (1-\gamma)\big(a_D(t)\big)^{1/(\gamma-1)}(e^{\rho t})^{1/(\gamma-1)}K_D(t)\\
&+ (r(t) + \eta_M(t) + \eta_D(t))\gamma - \frac{1}{2}\left(\frac{\mu(t)-r(t)}{\sigma(t)}\right)^2\frac{\gamma}{\gamma-1}\Bigg]V(t,W_1(t))\\
&+ \Bigg[I_D'(t) + I_{ND}'(t) + \alpha_{ND}(t) + \alpha_D(t) - I_{ND}(t)(r(t) + \eta_M(t))\\
&- I_D(t)(r(t) + \eta_M(t) + \eta_D(t))\Bigg]V_w(t,W_1(t)).
\end{aligned}$$

The ansatz in (4.47), together with the terminal condition in (4.46) gives the terminal conditions

$$a_D(T) = e^{-\rho T},$$
$$I_D(T) = 0.$$

Consider the two differential equations, corresponding to the $V(\cdot,\cdot)$ and $V_w(\cdot,\cdot)$ terms. In the $V_w(\cdot,\cdot)$ term, first note from (3.19) that

$$I'_{ND}(t) + \alpha_{ND} - I_{ND}(t)(r(t) + \eta_M(t)) = 0,$$

leaving the differential equation

$$\begin{cases} 0 = I'_D(t) - I_D(t)(r(t) + \eta_M(t) + \eta_D(t)) + \alpha_D(t) \\ I_D(T) = 0. \end{cases}$$

Solving this using an integrating factor reveals

$$I_D(t) = \int_t^T \alpha_D(s) \exp\left(- \int_t^s (r(v) + \eta_M(v) + \eta_D(v))dv \right) ds. \tag{4.48}$$

Hence, the expected value from the defaultable income is further discounted by the probability of default.

Next consider the $V(\cdot,\cdot)$ term. It can be rewritten by first multiplying through by $a_D(t)$ to get

$$\begin{cases} 0 = a'_D(t) + \Big[- (\lambda_M(t) + \lambda_D(t)) + (r(t) + \eta_M(t) + \eta_D(t))\gamma \\ \qquad - \frac{1}{2}\big(\frac{\mu(t)-r(t)}{\sigma(t)}\big)^2 \frac{\gamma}{\gamma-1} \Big] a_D(t) + (1-\gamma)(e^{\rho t})^{1/(\gamma-1)} K_D(t)\big(a_D(t)\big)^{\gamma/(\gamma-1)} \\ a_D(T) = e^{-\rho T}. \end{cases} \tag{4.49}$$

In order to solve (4.49), assume $a_D(t)$ takes the form

$$a_D(t) = e^{-\rho t} g_D(t)^{1-\gamma}. \tag{4.50}$$

Substituting into (4.50) yields

$$0 = g_D'(t) - H_D(t) g_D(t) + K_D(t), \tag{4.51}$$

where

$$H_D(t) = \frac{\rho + \lambda_M(t) + \lambda_D(t)}{1-\gamma} - \frac{(r(t) + \eta_M(t) + \eta_D(t))\gamma}{1-\gamma} - \frac{\gamma}{2}\left(\frac{\mu(t) - r(t)}{\sigma(t)(1-\gamma)}\right)^2.$$

The terminal condition for $g_D(t)$ comes from the terminal condition for $a_D(t)$ and the relationship (4.50), and is given by

$$g_D(T) = 1.$$

Using an integrating factor,

$$g_D(t) = \exp\left(-\int_t^T H_D(v)dv\right) + \int_t^T \exp\left(-\int_t^s H_D(v)dv\right) K_D(s)ds.$$

Finally, plugging in (4.47) and (4.50) into (4.42) - (4.45) gives

$$\begin{aligned}
c_1^*(t) &= \frac{1}{g_D(t)}(w + I_D(t) + I_{ND}(t)), \\
\pi_1^*(t) &= \left(\frac{\mu(t) - r(t)}{(1-\gamma)\sigma^2(t)}\right)(w + I_D(t) + I_{ND}(t)), \\
\zeta_{M_1}^*(t) &= \left(\frac{\eta_M(t)}{\lambda_M(t)}\right)^{1/(\gamma-1)} \frac{1}{g_D(t)}(w + I_D(t) + I_{ND}(t)), \\
\zeta_{D_1}^*(t) &= \left(\frac{\eta_D(t)}{\lambda_D(t)}\right)^{1/(\gamma-1)} \frac{g(t)}{g_D(t)}(w + I_D(t) + I_{ND}(t))
\end{aligned}$$

Next we set up the parameters used and give numerical results for the exact solution.

4.7 Model Parameters

The parameters in this model reflect those used in Section 3.7. Here, $\lambda_M(t)$ and $\eta_M(t)$ will replace $\lambda(t)$ and $\eta(t)$ used previously. The new parameters that need to be discussed are $\lambda_D(t)$ and $\eta_D(t)$. The Municipal Bond Fairness Act of 2008, [21], includes historical corporate bond default rates by credit rating from Moody's and S&P. A AAA-rated bond (the highest rating given) showed a default rate of .6%, a AA-rated bond had a default rate of 1.5%, and a A-rated bond had a default rate of 2.9%. In light of this information, we allow default rates ranging between 0% and 3%, or

$$\lambda_D(t) \in [0, .03].$$

Following the discussion in the Richard Model, the premium-payout ratio for default insurance should be

$$\eta_D(t) \geq \lambda_D(t)$$

to allow the insurance company to make a profit. In an actuarially fair market, we will consider $\eta_D(t) = \lambda_D(t)$. In an actuarially unfair market, with $\eta_D(t) > \lambda_D(t)$, we will consider $\eta_D(t) = (1 + \epsilon)\lambda_D(t)$ for some small $\epsilon > 0$.

4.7.1 Summary of Parameters.

Below is listed all the parameters to be used in this model.

r	μ	σ	γ	ρ	T	$\lambda_M(t), \eta_M(t)$	$\lambda_D(t), \eta_D(t)$
.01	.06	.20	-3	.03	40	$\frac{1}{d}\exp\left(\frac{R+t-c}{d}\right)$	[0, .03]

Table 4.1: Parameters used in the Taylor Model.

R, c, and d are defined as in Section 2.7.3.

4.8 Annuity Pricing

The annuities company prices the annuity actuarially fair, with $\lambda_M(t) = \eta_M(t)$. We might assume the company will not take its own credit risk into account. Thus, the price of the annuity is the same as (3.24). The retiree wishes to then find (3.26).

4.9 Computations and Results

Using the parameters in Table 3.1 and given initial wealth w = \$500,000, recall the exact value function given in (4.47) is given by

$$V(t,w) = \frac{a_D(t)}{\gamma}(w + I_D(t) + I_{ND}(t))^\gamma, \tag{4.52}$$

where $a_D(t)$, $I_D(t)$ and $I_{ND}(t)$ are given by (4.50), (4.48), and (4.40), respectively. The retiree can choose to purchase an annuity with payment rate α, which policy costs $A(\alpha)$ as given in (3.24). Thus the retiree's liquid wealth is actually given by $l_0 = w - A(\alpha)$ at time $t = 0$.

If we assume $\lambda_D(t) = \eta_D(t) = 0$, then this problem proceeds exactly as in Richard's problem, as the retiree has no risk of encountering a default in his annuity payments. In this case, $a_D(t) = a(t)$, $I_D(t) = 0$, and $I_{ND}(t) = I(t)$ as given in (3.21).

If we assume $\lambda_D(t) \neq 0$ or $\eta_D(t) \neq 0$, then this is no longer the case. We add in the assumption that at the time of default, the credit default swap makes a lump sum payment and leaves no other cash flow, hence $\alpha_D = \alpha$ and $\alpha_{ND} = 0$. This forces $I_{ND}(t) = 0$ for all $t \in [0,T]$. Equations (4.48) and (3.24) show $I_D(0) < A(\alpha)$. The graphs of $V(0,l_0)$ for $\lambda_D(t) = \eta_D(t) \in [0, .01, .02, .03]$ over various values of $\alpha \in [0, \alpha_{\max}]$ are given in the following figure.

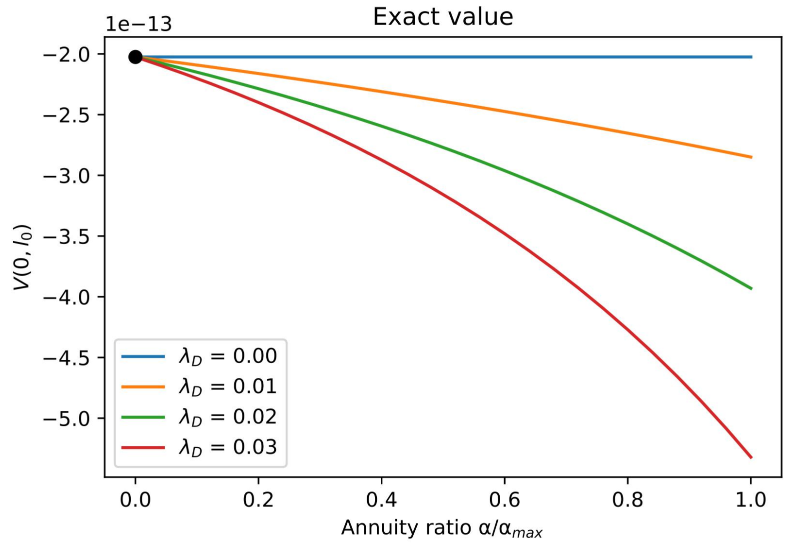

Figure 4.1: Exact value function for several annuity payment ratios α/α_{max} over different default rates $\lambda_D(t)$ in the Taylor Model. The case $\lambda_D = 0$ gives the Richard Model, matching Figure 3.1. For all other λ_D, there is risk in purchasing any amount of annuity, so the optimal annuity to purchase is always 0, which allows the retiree to short-sell insurance and improve his utility.

In every case with $\lambda_D(t) > 0$, the optimal value occurs when no annuity is purchased. This can be verified, in the case with $\alpha = \alpha_D$ and $I_{ND}(0) = 0$, by checking

$$\begin{aligned}
\frac{\partial}{\partial \alpha} V(0, l_0) =& \frac{\partial}{\partial \alpha} \frac{a_D(0)}{\gamma} \big(w - A(\alpha) + I_D(0)\big)^{\gamma} \\
=& a_D(0) \big(w - A(\alpha) + I_D(0)\big)^{\gamma - 1} \frac{\partial}{\partial \alpha} \big(- A(\alpha) + I_D(0)\big) \\
=& a_D(0) \big(w - A(\alpha) + I_D(0)\big)^{\gamma - 1} \Bigg[- \int_0^T \exp\bigg(- \int_0^s (r(v) + \eta_M(v)) dv \bigg) ds \\
&+ \int_0^T \exp\bigg(- \int_t^s (r(v) + \eta_M(v) + \eta_D(v)) dv \bigg) ds \Bigg],
\end{aligned}$$

which is necessarily negative by the last term.

This presents a problem for Taylor's risky annuity model. In the case where the retiree can short-sell insurance, he will choose to annuitize nothing and save his money. By not investing in a risky annuity, the retiree has the ability to short-sell his entire wealth. As discussed in [10], this gives him a new income in the form of an instantaneous term annuity – essentially an income that no longer has default risk. We will see in Chapter 5 that adding constraints on short-selling will motivate the retiree to annuitize a large portion of his wealth.

Chapter 5. Numerical Solutions

Up to this point, we have developed exact solutions to the Basic, Richard, and Taylor Models. While informative, they each had their own shortcomings as certain real world conditions were violated. The Basic Model allowed the retiree to consume beyond his total wealth towards the end of his life. The Richard and Taylor Models allowed the retiree to short-sell insurance, creating an instantaneous term annuity that provided a second income or removed the risk from annuity purchases.

This chapter focuses on correcting these issues by considering an incomplete market. In the Basic Model, we will restrict consumption within reasonable bounds. In the Richard and Taylor Models, we will consider restricting the retiree's insurance purchases to be non-negative. Richard had marginal success applying this restriction to his model. It was Ye who developed the necessary tools for solving the constrained problem; his method is called MCALT (Markov Chain Approximation with the Log Transform). This tool can easily be extended to the Basic and Taylor Models.

The chapter proceeds as follows. We will first define the MCALT method. We then consider each model and apply MCALT to find new HJB equations, at which point we create a discretization using finite difference schemes. We solve for the retiree's optimal annuitization level using the new discretized HJB equations and parameters found in Tables 2.1, 3.1, and (4.1).

5.1 MCALT

Ye's MCALT method is a two step process. The first we call the total wealth transformation. We introduce the total wealth variable, $X(t)$, defined as the sum of illiquid wealth, $W(t)$ and illiquid wealth, $I(t)$. This means

$$X(t) = W(t) + I(t).$$

In our models, we want to restrict the retiree from entering a bankruptcy state. We allow the retiree to borrow against illiquid wealth, so we have the restriction

$$W(t) + I(t) \geq 0.$$

For the Taylor Model, we define

$$I(t) = \begin{cases} I_{ND}(t) + I_D(t) & \text{if } t \leq \tau_D \\ I_{ND}(t) & \text{if } t > \tau_D. \end{cases}$$

This means the retiree's liquid wealth can actually go to $-I(t)$.

If $W(t) \to -I(t)$, then the retiree will have no money to consume and no money to bequeath at time of death. If $c(t) \to 0$ and $\zeta(t) \to 0$ ($\zeta_M(t)$ for the Taylor Model), then (3.6), (4.13), and (4.23) show that

$$V(t, -I(t)) = -\infty,$$
$$D(t, -I(t)) = -\infty.$$

These are known as the absorbing boundary conditions. The Basic Model also has this absorbing condition on V if we restrict $c(t) \leq W(t) + I(t)$.

These absorbing boundary conditions can pose issues if we are going to find discretized HJB equations. The total wealth transformation is accomplished (for the Basic Model, Richard Model, and pre-default Taylor Model) by setting

$$\begin{aligned} x &= w + I(t), \\ \widetilde{V}(t, x) &= V(t, w). \end{aligned} \tag{5.1}$$

We can take partial derivatives of $\widetilde{V}(t,x)$ to find

$$\begin{aligned} V_t(t,w) &= \widetilde{V}_t(t,x) + \widetilde{V}_x(t,x)\tfrac{\partial x}{\partial t}, \\ V_w(t,w) &= \widetilde{V}_x(t,x)\tfrac{\partial x}{\partial w} = \widetilde{V}_x(t,x), \\ V_{ww}(t,w) &= \widetilde{V}_{xx}(t,x)\tfrac{\partial x}{\partial w} = \widetilde{V}_{xx}(t,x). \end{aligned} \tag{5.2}$$

In the Basic Model,

$$\tfrac{\partial x}{\partial t} = I'(t) = I(t)r(t) - \alpha(t);$$

in the Richard Model,

$$\tfrac{\partial x}{\partial t} = I'(t) = I(t)(r(t) + \eta(t)) - \alpha(t);$$

and in the pre-default Taylor Model,

$$\begin{aligned} \tfrac{\partial x}{\partial t} = I'(t) =& (I_D(t) + I_{ND}(t))' \\ =& I_D(t)(r(t) + \eta_M(t) + \eta_D(t)) + I_{ND}(t)(r(t) + \eta_M(t)) - \alpha_D(t) - \alpha_{ND}(t). \end{aligned}$$

Note that since $I(T) = 0$, then $X(T) = W(T)$, and so the terminal condition remains $\widetilde{V}(T,x) = B(T,x)$.

The Taylor Model post-default total wealth transformation is defined similarly, with

$$\begin{aligned} x &= w + I(t) = w + I_{ND}(t), \\ \widetilde{D}(t,x) &= D(t,w). \end{aligned} \tag{5.3}$$

The partial fractions are also similar, with

$$\begin{aligned} D_t(t,w) &= \widetilde{D}_t(t,x) + \widetilde{D}_x(t,x)\tfrac{\partial x}{\partial t}, \\ D_w(t,w) &= \widetilde{D}_x(t,x)\tfrac{\partial x}{\partial w} = \widetilde{D}_x(t,x), \\ D_{ww}(t,w) &= \widetilde{D}_{xx}(t,x)\tfrac{\partial x}{\partial w} = \widetilde{D}_{xx}(t,x), \end{aligned} \tag{5.4}$$

where

$$\frac{\partial x}{\partial t} = I'_{ND}(t) = I_{ND}(t)(r(t) + \eta_M(t)) - \alpha_{ND}(t).$$

In all cases, making the total wealth transformation will move the singularities from $W(t) = -I(t)$ to $X(t) = 0$. It has the added benefit of removing the dependency of the HJB equations on the income $\alpha(t)$. Also, just as before, the terminal condition remains $\widetilde{D}(T, x) = B(T, x)$.

Ye [11] discussed the weakness of this first transformation, at least in its application to his model. We cannot be guaranteed to find an approximate solution as we change step sizes in the discretization. Because x is unbounded, we will see in the HJB equations related to the finite wealth transformation that it would be difficult to pick wealth- and time-step sizes small enough to guarantee convergence. Hence the necessity of the next transformation.

The second step of MCALT is to make a log transform. This transformation removes the dependencies of the HJB equations on total wealth, x. This is done in the Basic Model, Richard Model, and pre-default Taylor Model by setting

$$\begin{aligned} u &= \ln(x), \\ \widehat{V}(t, u) &= \widetilde{V}(t, x). \end{aligned} \tag{5.5}$$

The partials derivatives can be found as

$$\begin{aligned} \widetilde{V}_t(t, x) =& \widehat{V}_t(t, u), \\ \widetilde{V}_x(t, x) =& e^{-u}\widehat{V}_u(t, u), \\ \widetilde{V}_{xx}(t, x) =& e^{-2u}\big(\widehat{V}_{uu}(t, u) - \widehat{V}_u(t, u)\big) \end{aligned} \tag{5.6}$$

For the post-default Taylor Model, set

$$\begin{aligned} u &= \ln(x), \\ \widehat{D}(t, u) &= \widetilde{D}(t, x). \end{aligned} \tag{5.7}$$

The partials derivatives are

$$\begin{aligned}
\widetilde{D}_t(t,x) =& \widehat{D}_t(t,u),\\
\widetilde{D}_x(t,x) =& e^{-u}\widehat{D}_u(t,u),\\
\widetilde{D}_{xx}(t,x) =& e^{-2u}\big(\widehat{D}_{uu}(t,u) - \widehat{D}_u(t,u)\big).
\end{aligned} \tag{5.8}$$

In all cases, define new control variables as ratios of total wealth, $x = e^u$. Thus

$$\begin{aligned}
\widehat{c}(t) &= \frac{c(t)}{x} = e^{-u}c(t),\\
\widehat{\pi}(t) &= \frac{\pi(t)}{x} = e^{-u}\pi(t),\\
\widehat{\zeta}(t) &= \frac{\zeta(t)}{x} = e^{-u}\zeta(t).
\end{aligned} \tag{5.9}$$

The controls $\widehat{\zeta}_M(t)$ and $\widehat{\zeta}_D(t)$ are defined similarly for the Taylor Model.

By making this second transformation, the singularities have now moved from $x = 0$ to $u = -\infty$. Thus, in the discretized HJB equations which we derive shortly, we will only be concerned with bounded u values and do not need to worry about running into the singularity. The terminal conditions remain, namely

$$\begin{aligned}
\widehat{V}(T,u) &= B(T,x) = B(T,e^u),\\
\widehat{D}(T,u) &= B(T,x) = B(T,e^u).
\end{aligned}$$

We now proceed with applying MCALT to each of our models to find new HJB equations in terms of our new control variables, (5.9). The next section will then define the finite difference scheme that will be used.

5.1.1 Basic Model.

Recall the HJB equation in (2.17), which we write again as

$$\begin{cases} 0 = V_t(t,w) - \lambda(t)V(t,w) + \sup\limits_{(c,\pi)}\Big\{ \big[r(t)w - c(t) + \alpha(t) + \pi(t)(\mu(t) - r(t))\big]V_w(t,w) \\ \quad +\frac{1}{2}\pi^2(t)\sigma^2(t)V_{ww}(t,w) + U(t,c(t))\Big\} \\ V(T,w) = 0. \end{cases} \tag{5.10}$$

Here we set $B(t,w) = 0$, as assumed. Applying the total wealth transformation in (5.1) and (5.2) yields the HJB equation

$$\begin{cases} 0 = \widetilde{V}_t(t,x) - \lambda(t)\widetilde{V}(t,x) + \sup\limits_{(c,\pi)}\Big\{ \big[r(t)x - c(t) + \pi(t)(\mu(t) - r(t))\big]\widetilde{V}_x(t,x) \\ \quad +\frac{1}{2}\pi^2(t)\sigma^2(t)\widetilde{V}_{xx}(t,x) + U(t,c(t))\Big\} \\ \widetilde{V}(T,x) = 0. \end{cases} \tag{5.11}$$

Finishing with the log transform given in (5.5) and (5.6) gives

$$\begin{cases} 0 = \widehat{V}_t(t,u) - \lambda(t)\widehat{V}(t,u) + \sup\limits_{(\widehat{c},\widehat{\pi})}\Big\{ \big[r(t) + \widehat{\pi}(t)(\mu(t) - r(t)) - \widehat{c}(t) - \frac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)\big]\widehat{V}_u(t,u) \\ \quad +\frac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)\widehat{V}_{uu}(t,u) + U(t,e^u\widehat{c}(t))\Big\} \\ \widehat{V}(T,u) = 0. \end{cases} \tag{5.12}$$

5.1.2 Richard Model.

We next proceed with applying MCALT to the Richard Model, as was done in [11]. Recall the HJB equation given in (3.7) is

$$\begin{cases} 0 = V_t(t,w) - \lambda(t)V(t,w) + \sup\limits_{(c,\pi,\zeta)} \Big\{ \big[(r(t)+\eta(t))w - c(t) + \alpha(t) - \eta(t)\zeta(t) \\ \quad +\pi(t)(\mu(t)-r(t))\big] V_w(t,w) + \frac{1}{2}\pi^2(t)\sigma^2(t)V_{ww}(t,w) + \lambda(t)B(t,\zeta(t)) + U(t,c(t)) \Big\} \\ V(T,w) = B(T,w). \end{cases}$$

Substituting (5.1) and (5.2) into this gives the new HJB equation

$$\begin{cases} 0 = \widetilde{V}_t(t,x) - \lambda(t)\widetilde{V}(t,x) + \sup\limits_{(c,\pi,\zeta)} \Big\{ \big[(r(t)+\eta(t))x + \pi(t)(\mu(t)-r(t)) \\ \quad -(c(t)+\eta(t)\zeta)\big] \widetilde{V}_x(t,x) + \frac{1}{2}\pi^2(t)\sigma^2(t)\widetilde{V}_{xx}(t,x) + \lambda(t)B(t,\zeta(t)) + U(t,c) \Big\} \\ \widetilde{V}(T,x) = B(T,x). \end{cases} \tag{5.13}$$

Now applying the log transform, (5.5) and (5.6), gives the HJB equation

$$\begin{cases} 0 = \widehat{V}_t(t,u) - \lambda(t)\widehat{V}(t,u) + \sup\limits_{(\widehat{c},\widehat{\pi},\widehat{\zeta})} \Big\{ \big[(r(t)+\eta(t)) + \widehat{\pi}(t)(\mu(t)-r(t)) \\ \quad -(\widehat{c}(t)+\eta(t)\widehat{\zeta}(t)) - \frac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)\big] \widehat{V}_u(t,u) \\ \quad +\frac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)\widehat{V}_{uu}(t,u) + \lambda(t)B(t,e^u\widehat{\zeta}(t)) + U(t,e^u\widehat{c}(t)) \Big\} \\ \widehat{V}(T,u) = B(T,e^u). \end{cases} \tag{5.14}$$

5.1.3 Taylor Model, post-default.

Recall the HJB for the post-default Taylor Model given in (4.15) is

$$\begin{cases} 0 = D_t(t,w) - \lambda_M(t)D(t,w) + \sup\limits_{(c,\pi,\zeta_M)} \Big\{ \big[(r(t)+\eta_M(t))w \\ \quad -c(t) + \alpha_{ND}(t) - \eta_M(t)\zeta_M(t) + \pi(t)(\mu(t)-r(t))\big] D_w(t,w) \\ \quad +\frac{1}{2}\pi^2(t)\sigma^2(t)D_{ww}(t,w) + \lambda_M(t)B(t,\zeta_M(t)) + U(t,c(t)) \Big\} \\ D(T,w) = B(T,w), \end{cases}$$

where $w = W_2(t)$ represents the post-default wealth process. Using the transformations in (5.3) and (5.4), this becomes

$$\begin{cases} 0 = \widetilde{D}_t(t,x) - \lambda_M(t)\widetilde{D}(t,x) + \sup\limits_{(c,\pi,\zeta_M)} \Big\{ \big[(r(t) + \eta_M(t))x + \pi(t)(\mu(t) - r(t)) \\ \quad -(c(t) + \eta_M(t)\zeta_M(t))\big]\widetilde{D}_x(t,x) + \frac{1}{2}\pi^2(t)\sigma^2(t)\widetilde{D}_{xx}(t,x) \\ \quad +\lambda_M(t)B(t,\zeta_M(t)) + U(t,c(t))\Big\} \\ \widetilde{D}(T,x) = B(T,x), \end{cases} \tag{5.15}$$

where $x = X_2(t)$ is taken to be the post-default total wealth process.

Then applying (5.7) and (5.8) yields

$$\begin{cases} 0 = \widehat{D}_t(t,u) - \lambda_M(t)\widehat{D}(t,u) + \sup\limits_{(\widehat{c},\widehat{\pi},\widehat{\zeta}_M)} \Big\{ \big[r(t) + \eta_M(t) + \widehat{\pi}(t)(\mu(t) - r(t)) \\ \quad -(\widehat{c}(t) + \eta_M(t)\widehat{\zeta}_M(t)) - \frac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)\big]\widehat{D}_u(t,u) + \frac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)\widehat{D}_{uu}(t,u) \\ \quad +\lambda_M(t)B(t,e^u\widehat{\zeta}_M(t)) + U(t,e^u\widehat{c}(t))\Big\} \\ \widehat{D}(T,u) = B(T,e^u). \end{cases} \tag{5.16}$$

5.1.4 Taylor Model, pre-default.

The pre-default HJB equation for the Taylor Model given in (4.25) is

$$\begin{cases} 0 = V_t(t,w) - (\lambda_M(t) + \lambda_D(t))V(t,w) + \sup\limits_{(c,\pi,\zeta_M,\zeta_D)} \Big\{ \big[(r(t) + \eta_M(t) + \eta_D(t))w \\ \quad -c(t) + \alpha_D(t) + \alpha_{ND}(t) - \eta_M(t)\zeta_M(t) - \eta_D(t)\zeta_D(t) + \pi(t)(\mu(t) - r(t))\big]V_w(t,w) \\ \quad +\frac{1}{2}\pi^2(t)\sigma^2(t)V_{ww}(t,w) + \lambda_M(t)B(t,\zeta_M(t)) + \lambda_D(t)D(t,\zeta_D(t)) + U(t,c(t))\Big\} \\ V(T,w) = B(T,w), \end{cases}$$

where $w = W_1(t)$ is the pre-default wealth process. Using the transformations in (5.1) and (5.2), this becomes

$$\begin{cases} 0 = \widetilde{V}_t(t,x) - (\lambda_M(t) + \lambda_D(t))\widetilde{V}(t,x) + \sup_{(c,\pi,\zeta_M,\zeta_D)} \Big\{ \big[(r(t) + \eta_M(t) + \eta_D(t))x - c(t) \\ \quad -\eta_M(t)\zeta_M(t) - \eta_D(t)\zeta_D(t) + \pi(t)(\mu(t) - r(t))\big]\widetilde{V}_x(t,x) + \frac{1}{2}\pi^2(t)\sigma^2(t)\widetilde{V}_{xx}(t,x) \\ \quad +\lambda_M(t)B(t,\zeta_M(t)) + \lambda_D(t)\widetilde{D}(t,\zeta_D(t)) + U(t,c(t)) \Big\} \\ \widetilde{V}(T,x) = B(T,x), \end{cases} \tag{5.17}$$

where $x = X_1(t)$ is the pre-default wealth process for the Taylor Model. We needed the following fact: since $w = x - I(t)$ in the pre-default transformation,

$$D(t,\zeta_D(t)) = D\big(t, w + \tfrac{p_D(t)}{\eta_D(t)}\big) = D\big(t, x - I(t) + \tfrac{p_D(t)}{\eta_D(t)}\big).$$

Then (5.3) gives

$$D\big(t, x - I(t) + \tfrac{p_D(t)}{\eta_D(t)}\big) = \widetilde{D}\big(t, x - I(t) + I_{ND}(t) + \tfrac{p_D(t)}{\eta_D(t)}\big) = \widetilde{D}\big(t, x - I_D(t) + \tfrac{p_D(t)}{\eta_D(t)}\big).$$

By (4.5), this gives

$$D(t,\zeta_D(t)) = \widetilde{D}(t,\zeta_D(t)).$$

Applying (5.5) and (5.6) to this gives

$$\begin{cases} 0 = \widehat{V}_t(t,u) - (\lambda_M(t) + \lambda_D(t))\widehat{V}(t,u) + \sup_{(\widehat{c},\widehat{\pi},\widehat{\zeta}_M,\widehat{\zeta}_D)} \Big\{ \big[r(t) + \eta_M(t) + \eta_D(t) \\ \quad +\widehat{\pi}(t)(\mu(t) - r(t)) - \widehat{c}(t) - \eta_M(t)\widehat{\zeta}_M(t) - \eta_D(t)\widehat{\zeta}_D(t) - \frac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)\big]\widehat{V}_u(t,u) \\ \quad +\frac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)\widehat{V}_{uu}(t,u) + \lambda_M(t)B(t,e^u\widehat{\zeta}_M(t)) + \lambda_D(t)\widetilde{D}(t,\zeta_D) + U(t,e^u\widehat{c}(t)) \Big\} \\ \widehat{V}(T,u) = B(T,e^u). \end{cases} \tag{5.18}$$

At this point, $\widehat{V}(t,u)$ is dependent on $\widetilde{D}(t,\zeta_D)$. This is not practical for computing solutions in the discretization, so we want to rewrite $\widetilde{D}$ in terms of $\widehat{D}$. To do so, note (5.7) gives

$$\widetilde{D}(t,\zeta_D) = \widehat{D}(t,\ln(\zeta_D)) = \widehat{D}(t,u+\ln(e^{-u}\zeta_D)) = \widehat{D}(t,u+\ln(\widehat{\zeta}_D)).$$

Thus, we can rewrite the HJB as

$$\begin{cases} 0 = \widehat{V}_t(t,u) - (\lambda_M(t)+\lambda_D(t))\widehat{V}(t,u) + \sup\limits_{(\widehat{c},\widehat{\pi},\widehat{\zeta}_M,\widehat{\zeta}_D)} \Big\{ \big[r(t)+\eta_M(t)+\eta_D(t) \\ \quad +\widehat{\pi}(t)(\mu(t)-r(t)) - \widehat{c}(t) - \eta_M(t)\widehat{\zeta}_M(t) - \eta_D(t)\widehat{\zeta}_D(t) - \frac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)\big]\widehat{V}_u(t,u) \\ \quad +\frac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)\widehat{V}_{uu}(t,u) + \lambda_M(t)B(t,e^u\widehat{\zeta}_M(t)) + \lambda_D(t)\widehat{D}(t,u+\ln(\widehat{\zeta}_D)) + U(t,e^u\widehat{c}(t)) \Big\} \\ \widehat{V}(T,u) = B(T,e^u). \end{cases} \tag{5.19}$$

5.2 Finite Difference Scheme

At this point, we have constructed HJB equations that are independent of income, $\alpha(t)$, and total wealth, x. We proceed with the finite difference scheme that allows us to find discretized forms of the HJB equations.

In all of the HJB equations, (5.12), (5.14), (5.16), and (5.19), let the coefficient of $\widehat{V}_u(t,u)$ (or $\widehat{D}_u(t,u)$) be $\widehat{b}(t,u)$. This coefficient can be separated into positive and negative terms. Let $\widehat{b}^+(t,u)$ represent the positive term and $\widehat{b}^-(t,u)$ represent the negative term such that

$$\widehat{b}(t,u) = \widehat{b}^+(t,u) - \widehat{b}^-(t,u).$$

The finite difference scheme is given by

$$
\begin{aligned}
\widehat{V}_t(t,u) &= \frac{\widehat{V}(t+\delta,u)-\widehat{V}(t,u)}{\delta},\\
\widehat{b}^+(t,u)\widehat{V}_u(t,u) &= \widehat{b}^+(t,u)\frac{\widehat{V}(t+\delta,u+h)-\widehat{V}(t+\delta,u)}{h} := \widehat{b}^+(t,u)\widehat{V}_u^+(t,u),\\
\widehat{b}^-(t,u)\widehat{V}_u(t,u) &= \widehat{b}^-(t,u)\frac{\widehat{V}(t+\delta,u)-\widehat{V}(t+\delta,u-h)}{h} := \widehat{b}^-(t,u)\widehat{V}_u^-(t,u),\\
\widehat{V}_{uu}(t,u) &= \frac{\widehat{V}(t+\delta,u+h)+\widehat{V}(t+\delta,u-h)-2\widehat{V}(t+\delta,u)}{h^2},
\end{aligned}
\tag{5.20}
$$

and

$$
\begin{aligned}
\widehat{D}_t(t,u) &= \frac{\widehat{D}(t+\delta,u)-\widehat{D}(t,u)}{\delta},\\
\widehat{b}^+(t,u)\widehat{D}_u(t,u) &= \widehat{b}^+(t,u)\frac{\widehat{D}(t+\delta,u+h)-\widehat{D}(t+\delta,u)}{h} := \widehat{b}^+(t,u)\widehat{D}_u^+(t,u),\\
\widehat{b}^-(t,u)\widehat{D}_y(t,u) &= \widehat{b}^-(t,u)\frac{\widehat{D}(t+\delta,u)-\widehat{D}(t+\delta,u-h)}{h} := \widehat{b}^-(t,u)\widehat{D}_u^-(t,u),\\
\widehat{D}_u(t,u) &= \frac{\widehat{D}(t+\delta,u+h)+\widehat{D}(t+\delta,u-h)-2\widehat{D}(t+\delta,u)}{h^2}.
\end{aligned}
\tag{5.21}
$$

We proceed with applying the finite difference schemes to the HJB equations found earlier. The discretization of the Richard and Taylor Models can be shown to give transition probabilities, which allow the solutions to the discretized HJB equations to approximate the solutions to the HJB equations we just found as the wealth- and time-steps approach 0. Other forms for the discretized HJB equations will be given as well, allowing for easier programming of the solutions.

5.3 Computation and Results

In this section we apply the finite difference scheme from Section 5.2 to each of the models we've considered so far. We will find the constraints that are necessary for each model in an incomplete market; we restrict consumption choices in the Basic Model and insurance short-selling in the Richard and Taylor Models. We will also consider what happens when annuity costs are made actuarially unfair in the Richard and Taylor Models.

5.3.1 Basic Model.

From (5.12), let

$$\widehat{b}^{+}(t,u) = r(t) + \widehat{\pi}(t)(\mu(t) - r(t)),$$
$$\widehat{b}^{-}(t,u) = \widehat{c}(t) + \tfrac{1}{2}\widehat{\pi}^2(t)\sigma^2(t).$$

Applying (5.20) to (5.12) gives

$$\begin{aligned}
\widehat{V}(t,u) =& \tfrac{\delta}{(1+\delta h)} \sup_{(\widehat{c},\widehat{\pi})} \big\{ \tfrac{\widehat{V}(t+\delta,u)}{\delta} + \widehat{b}^{+}(t,u)\widehat{V}_u^{+}(t,u) - \widehat{b}^{-}(t,u)\widehat{V}_u^{-}(t,u) \\
&+ \tfrac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)V_{uu}(t,u) + \lambda(t)B(t,e^u) + U(t,e^u\widehat{c}(t)) \big\} \\
=& \tfrac{\delta}{(1+\delta h)} \big[\tfrac{\widehat{V}(t,u)}{\delta} + r(t)\widehat{V}_u^{+}(t,u) + \sup_{\widehat{c}} \big\{ U(t,e^u\widehat{c}(t)) - \widehat{c}(t)\widehat{V}_u^{-}(t,u) \big\} \\
&+ \sup_{\widehat{\pi}} \big\{ \widehat{\pi}(t)(\mu(t) - r(t))\widehat{V}_u^{+}(t,u) + \tfrac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)(\widehat{V}_{uu}(t,u) - \widehat{V}_u^{-}(t,u)) \big\}.
\end{aligned} \tag{5.22}$$

We can solve for optimal controls $\widehat{c}^*(t)$ and $\widehat{\pi}^*(t)$ as we did in the exact solution for (2.25) and (2.26). Since we are given $U(t,c(t))$ in (2.23), we find

$$\widehat{c}^*(t) = (\widehat{V}_u^{-}(t,u)e^{\rho t})^{1/(\gamma-1)}(e^{-u})^{\gamma/(\gamma-1)},$$
$$\widehat{\pi}^*(t) = \left(\frac{\mu(t) - r(t)}{\sigma(t)}\right)\left(\frac{\widehat{V}_u^{+}(t,u)}{\widehat{V}_u^{-}(t,u) - \widehat{V}_{uu}(t,u)}\right).$$

We want to restrict consumption so as to not exceed total wealth. This is easily accomplished by restricting

$$0 \leq \widehat{c}^*(t) \leq 1.$$

To solve (5.22), we choose $\delta = 0.01$ and $h = 0.02$. $\widehat{V}(t,u)$ must be solved backwards in time. In order to calculate $\widehat{V}(0,u)$, we take T/δ steps in time to $t = T$, each time taking a step either direction in u. This means we must calculate $\widehat{V}(T,\cdot)$ for wealth values in the range $[u - h\frac{T}{\delta}, u + h\frac{T}{\delta}$. The control variables are calculated for all of $\widehat{V}(T,\cdot)$, and then we proceed backwards in time to $\widehat{V}(t,u)$.

Since we are still solving the retiree's optimal annuity problem, we proceed as in Section 2.9. The unconstrained problem is unstable due to large consumption for t near T, but the constrained problem is graphed in

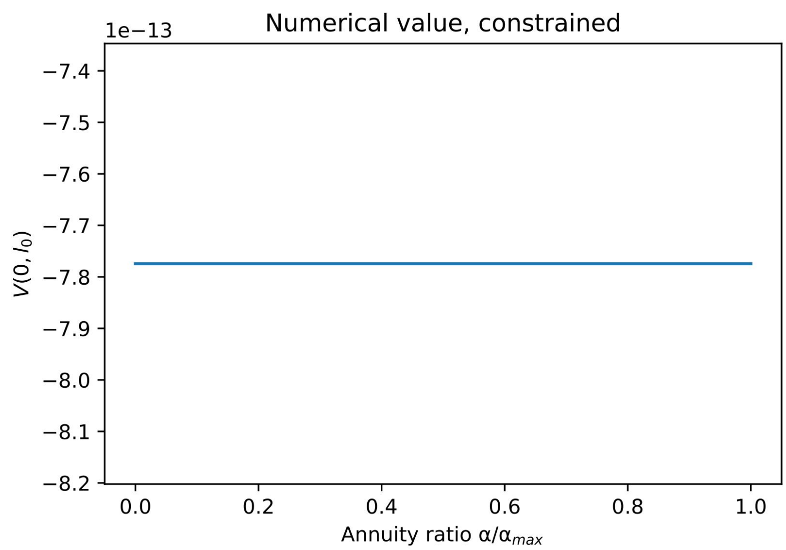

Figure 5.1: Numerical value function for several annuity payment ratios α/α_{max} in the Basic Model. There is no optimal α^* to annuitize.

By restricting consumption, the value function in Figure 5.1 decreased from that found in Figure 2.1. There is still no optimal amount to annuitize, since the retiree is still consuming against total wealth, which is constant for any annuity purchased.

There is no loading that can occur in this model (no insurance), so we next look at the Richard Model.

5.3.2 Richard Model.

From (5.14), let

$$\begin{aligned}\widehat{b}^{+}(t,u) =& r(t)+\eta(t)+\widehat{\pi}(t)(\mu(t)-r(t)),\\ \widehat{b}^{-}(t,u) =& \widehat{c}(t)+\eta(t)\widehat{\zeta}(t)+\tfrac{1}{2}\widehat{\pi}^{2}(t)\sigma^{2}(t).\end{aligned}$$

Applying (5.20) to (5.14) gives

$$\begin{aligned}\widehat{V}(t,u) =& \frac{1}{1+\delta\lambda(t)}\sup_{(\widehat{c},\widehat{\pi},\widehat{\zeta})}\Bigg\{\delta\lambda(t)B(t,e^{u}\widehat{\zeta}(t))+\delta U(t,e^{u}\widehat{c}(t))\\ &+\widehat{P}(u,u+h)\widehat{V}(t+\delta,u+h)+\widehat{P}(u,u-h)\widehat{V}(t+\delta,u-h)\\ &+\widehat{P}(u,u)\widehat{V}(t+\delta,u)\Bigg\},\end{aligned} \tag{5.23}$$

where

$$\begin{aligned}\widehat{P}(u,u+h) &= \frac{\delta}{h}\widehat{b}^{+}(t,u)+\frac{\delta\widehat{\pi}^{2}(t)\sigma^{2}(t)}{2h^{2}},\\ \widehat{P}(u,u-h) &= \frac{\delta}{h}\widehat{b}^{-}(t,u)+\frac{\delta\widehat{\pi}^{2}(t)\sigma^{2}(t)}{2h^{2}},\\ \widehat{P}(u,u) &= 1-\widehat{P}(u,u+h)-\widehat{P}(u,u-h)\end{aligned} \tag{5.24}$$

don't depend on the state u explicitly. The functions $\widehat{P}(\cdot,\cdot)$ in (5.24) can be thought of as transition probabilities in a Markov chain. By choosing small values for h and δ, we can assure that each of the transition probabilities is non-negative. Kushner and Dupuis showed in [22] that the solution to (5.14) can be approximated by the solution to (5.23) as $h \to 0$ and $\delta \to 0$.

We can make calculations easier by splitting up the supremum in (5.23). This gives the following.

$$
\begin{aligned}
\widehat{V}(t,u) =& \frac{\delta}{1+\delta\lambda(t)}\Bigg[\frac{\widehat{V}(t+\delta,u)}{\delta} + \sup_{\widehat{c}}\big\{-\widehat{c}(t)\widehat{V}_u^-(t,u) + U(t,e^u\widehat{c}(t))\big\} \\
&+ \sup_{\widehat{\pi}}\big\{\widehat{\pi}(t)(\mu(t)-r(t))\widehat{V}_u^+(t,u) + (\widehat{V}_{uu}(t,u) - \widehat{V}_u^-(t,u))\tfrac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)\big\} \\
&+ \sup_{\widehat{\zeta}}\big\{-\eta(t)\widehat{\zeta}(t)\widehat{V}_u^-(t,u) + \lambda(t)B(t,e^u\widehat{\zeta}(t))\big\} + (r(t)+\eta(t))\widehat{V}_u^+(t,u)\Bigg].
\end{aligned} \tag{5.25}
$$

Let $U(t,\cdot)$ and $B(t,\cdot)$ be the utility functions given in (3.12) and (3.13) respectively. Taking derivatives with respect to $\widehat{c}$, $\widehat{\pi}$, and $\widehat{\zeta}$ gives the following optimal control policies: For $\widehat{c}^*$:

$$
\begin{aligned}
0 =& e^{-\rho t}(\widehat{c}^* e^u)^{\gamma-1}e^u - \widehat{V}_u^-(t,u) \\
(\widehat{c}^* e^u)^{\gamma-1} =& \widehat{V}_u^-(t,u)e^{\rho t}e^{-u} \\
\widehat{c}^* e^u =& (\widehat{V}_u^-(t,u)e^{\rho t}e^{-u})^{1/(\gamma-1)} \\
\widehat{c}^* =& (\widehat{V}_u^-(t,u)e^{\rho t})^{1/(\gamma-1)}(e^{-u})^{\gamma/(\gamma-1)}.
\end{aligned} \tag{5.26}
$$

For $\widehat{\pi}^*$:

$$
\begin{aligned}
0 =& (\mu(t)-r(t))\widehat{V}_u^+(t,u) + (\widehat{V}_{uu}(t,u) - \widehat{V}_u^-(t,u))\widehat{\pi}^*\sigma^2(t) \\
\widehat{\pi}^* =& \frac{(\mu(t)-r(t))\widehat{V}_u^+(t,u)}{\sigma^2(t)(\widehat{V}_u^-(t,u) - \widehat{V}_{uu}(t,u))}.
\end{aligned} \tag{5.27}
$$

For $\widehat{\zeta}^*$:

$$
\begin{aligned}
0 =& \lambda(t)e^{-\rho t}(e^u\widehat{\zeta}^*)^{\gamma-1}e^u - \eta(t)\widehat{V}_u^-(t,u) \\
(\widehat{\zeta}^* e^u)^{\gamma-1} =& \frac{\eta(t)}{\lambda(t)}\widehat{V}_u^-(t,u)e^{\rho t}e^{-u} \\
\widehat{\zeta}^* e^u =& \left(\frac{\eta(t)}{\lambda(t)}\right)^{1/(\gamma-1)}(\widehat{V}_u^-(t,u)e^{\rho t}e^{-u})^{1/(\gamma-1)} \\
\widehat{\zeta}^* =& \left(\frac{\eta(t)}{\lambda(t)}\right)^{1/(\gamma-1)}(\widehat{V}_u^-(t,u)e^{\rho t})^{1/(\gamma-1)}(e^{-u})^{\gamma/(\gamma-1)}.
\end{aligned} \tag{5.28}
$$

As for the constrained version of the problem, we add the constraints

$$0 \leq \widehat{c}^* \leq 1,$$
$$0 \leq \widehat{\pi}^* \leq 1.$$

For $\widehat{\zeta}^*$, (3.3) shows that in the incomplete market with no insurance short-selling ($p(t) \geq 0$), we have the restriction

$$\zeta^*(t) \geq w.$$

From the MCALT method, (5.1) shows

$$\zeta^*(t) \geq x - I(t),$$

and (5.5) shows

$$\widehat{\zeta}^*(t) \geq 1 - e^{-u} I(t).$$

The discretized HJB equation can now be solved backwards in time as in the Basic Model, using the parameters provided in Section 3.7. We solve for the retiree's optimal annuity as in Section 3.9. Optimizing over α took about 5 minutes on the same setup as before, with $\delta = 0.01$ and $h = 0.02$. Figure 5.2 again shows there is no single optimal solution in the unconstrained model. Every amount results in the same value, similar to the exact solution.

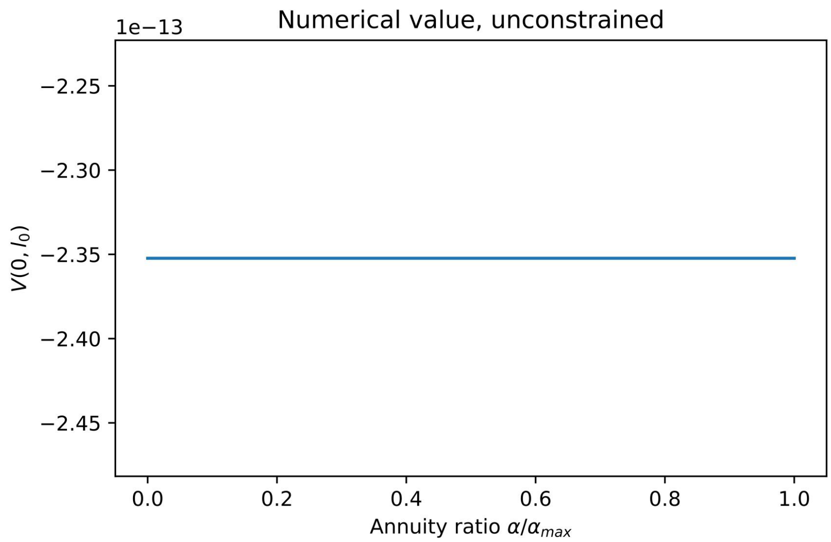

Figure 5.2: Value function for several annuity payment ratios α/α_{max} in the unconstrained Richard Model. There is no optimal annuity ratio because the retiree can short-sell life insurance and maintain his total wealth.

The restricted optimization is shown in Figure 5.3. From the graph, it is evident the individual optimizes utility by fully annuitizing. With restrictions, the retiree no longer benefits from holding onto wealth and short-selling insurance. By annuitizing more, the retiree can make better use of insurance purchases to protect against mortality risk and maintain some level of wealth for bequest.

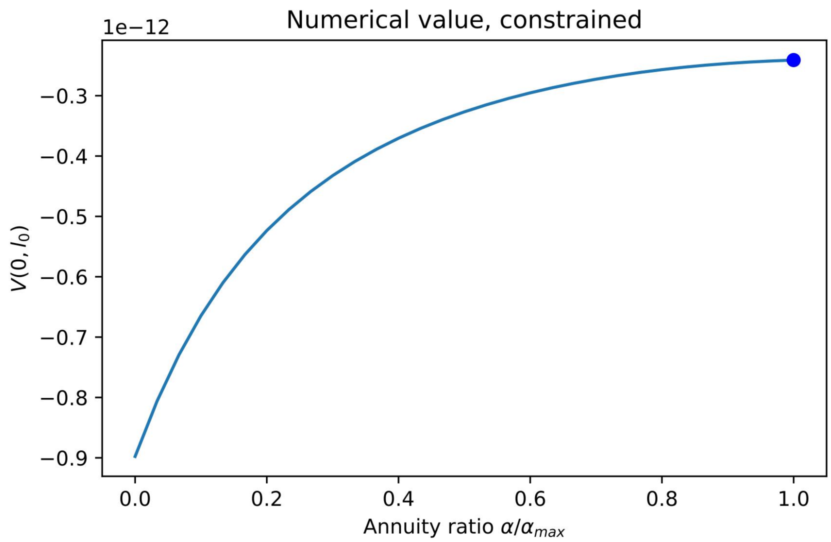

Figure 5.3: Value function for several annuity payment ratios α/α_{max} in the constrained Richard Model. The maximum occurs at $\alpha = \alpha_{max}$. This happens when short-selling is not allowed, thus the retiree loses value by holding onto any of his initial wealth at retirement.

We now conclude the discussion of this model by considering what happens when insurance purchases are no longer actuarially fair. This is done by setting $\eta(t) > \lambda(t)$. Actuarially unfair markets favor the annuities company.

Recall the cost of an annuity paying rate $\alpha(t)$ as given in (3.24) is

$$A(\alpha) = \int_0^T \alpha(s) \exp\left(-\int_0^s (r(v) + \lambda(v))dv\right) ds.$$

The expected, discounted present day value of this annuity is given by (3.21), with

$$I(t) = \int_t^T \alpha(s) \exp\left(-\int_t^s (r(v) + \eta(v))dv\right) ds.$$

With $\eta(t) > \lambda(t)$, then $A(\alpha) > I(0)$. The annuity company charges more than the annuity may be worth over time to the retiree.

We will show that letting $\eta(t) \to \infty$ gives us a solution to the annuity optimization problem where the retiree only makes consumption and investment choices, and cannot borrow against future income, but does have a bequest motive. The value for this new function will naturally be lower than the Basic Model, since we restrict borrowing, and the added bequest motive only hurts his overall consumption (he wants money left over so won't consume everything, thus decreasing utility from consumption).

This new model has the same HJB equation as (2.17). Since borrowing is not allowed, the retiree's total wealth is simply $W(t)$, and to avoid bankruptcy he must always have $W(t) \geq 0$. Applying the MCALT method to this equation, we let

$$\begin{aligned} u &= \ln(w), \\ \widehat{V}(t,u) &= V(t,w), \\ \widehat{c}(t) &= \frac{c(t)}{w} = e^{-u}c(t), \\ \widehat{\pi}(t) &= \frac{\pi(t)}{w} = e^{-u}\pi(t). \end{aligned}$$

Consider $\widehat{c}$ and $\widehat{\pi}$ as fractions of the total wealth w. Taking derivatives and plugging these into (2.17) yields the updated HJB equation

$$\begin{cases} 0 = \widehat{V}_t(t,u) - \lambda(t)\widehat{V}(t,u) + \sup\limits_{(\widehat{c},\widehat{\pi})}\Big\{ \big[(r(t) + \alpha(t)e^{-u}) + \widehat{\pi}(t)(\mu(t) - r(t)) - \widehat{c}(t) \\ \quad -\frac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)\big]\widehat{V}_u(t,u) + \frac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)\widehat{V}_{uu}(t,u) + \lambda(t)B(t,e^u) + U(t,e^u\widehat{c}(t))\Big\} \\ \widehat{V}(T,u) = B(T,e^u). \end{cases} \tag{5.29}$$

Applying the finite difference scheme (5.20) with

$$\begin{aligned} \widehat{b}^+(t,u) =& r(t) + \alpha(t)e^{-u} + \widehat{\pi}(t)(\mu(t) - r(t)), \\ \widehat{b}^-(t,u) =& \widehat{c}(t) - \frac{1}{2}\widehat{\pi}^2(t)\sigma^2(t), \end{aligned}$$

gives

$$\widehat{V}(t,u) = \frac{\delta}{(1+\delta h)} \sup_{(\widehat{c},\widehat{\pi})} \left\{ \frac{\widehat{V}(t+\delta,u)}{\delta} + \widehat{b}^{+}(t,u)\widehat{V}_u^{+}(t,u) - \widehat{b}^{-}(t,u)\widehat{V}_u^{-}(t,u) \right. \\ \left. + \frac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)V_{uu}(t,u) + \lambda(t)B(t,e^u) + U(t,e^u\widehat{c}(t)) \right\}. \tag{5.30}$$

We proceed to show that as $\eta(t) \to \infty$, then (5.14) becomes (5.30).

First, as $\eta(t) \to \infty$, then (3.16) shows that for a risk averse individual ($\gamma < 0$),

$$\widehat{\zeta}^* \to 0.$$

In $I(t)$ above, as $\eta(t) \to \infty$ then $I(t) \to 0$. The constraint on $\widehat{\zeta}^*$ in the HJB is $\widehat{\zeta}^* \geq 1 - e^{-u}I(t)$; for η large enough, we must have

$$\widehat{\zeta}^* = 1 - e^{-u}I(t).$$

Since $I(t) \to 0$, then $\widehat{\zeta}^* \to 1$. $\lambda(t)B(t, e^u\widehat{\zeta}(t))$ in (5.14) becomes $\lambda(t)B(t,e^u)$ in (5.30). Lastly,

$$\begin{aligned} \lim_{\eta\to\infty} (\eta\widehat{V}_u(t,u) - \eta\widehat{\zeta}\widehat{V}_u(t,u)) &= \lim_{\eta\to\infty} (\eta - \eta\widehat{\zeta})\widehat{V}_u(t,u) \\ &= \lim_{\eta\to\infty} (\eta - \eta(1 - e^{-u}I(t)))\widehat{V}_u(t,u) \\ &= \lim_{\eta\to\infty} \eta e^{-u}I(t)\widehat{V}_u(t,u). \end{aligned}$$

A quick calculation shows

$$\lim_{\eta\to\infty} \eta I(t) = \alpha(t),$$

so

$$\lim_{\eta\to\infty} \eta e^{-u}I(t)\widehat{V}_u(t,u) = e^{-u}\alpha(t)\widehat{V}_u(t,u),$$

which is the missing term in (5.30). Thus, (5.14) has become (5.30)

We can consider large $\eta(t)$ in the constrained numerical solution. Doing so will show that adding life insurance payouts is a great benefit to the retiree, and his value function improves in the Richard Model over the Basic Model.

We let

$$\eta(t) = (1+\epsilon)\lambda(t),$$

with $\epsilon = .25$. Figure 5.4 shows the value function over several annuity ratios. This time the maximum annuity payment is not $\alpha_{\max}$, but just short of that amount. By introducing the actuarially unfair model, the retiree's illiquid wealth decreases for large $\eta(t)$, thus full annuitization is not optimal.

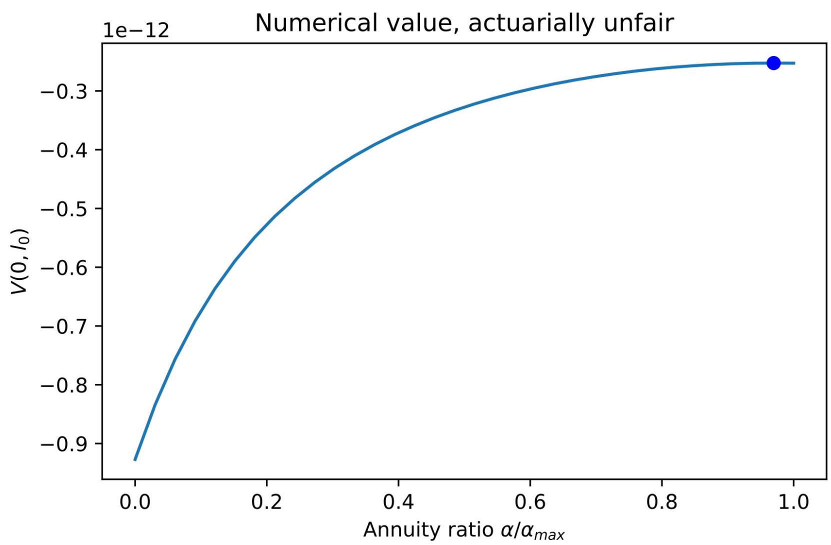

Figure 5.4: Value function for several annuity payment ratios α/α_{max} in the actuarially unfair Richard Model. The maximum does not occur at $\alpha = \alpha_{max}$, but just short of it. Such is the case for the unfair model because the annuity purchased does not maintain its discounted future value that it once did.

We list below the updated maximized value function $V(0, l_0)$ and annuity/wealth ratio, $\theta = \frac{A(\alpha^*)}{w}$, as compared to the constrained and unconstrained calculations.

Model	θ	$V(0, l_0)$
Unconstrained	—	-2.3523×10^{-13}
Constrained	1.0	-2.4118×10^{-13}
Unfair	.97	-2.5256×10^{-13}

Table 5.1: $V(0, l_0)$ for the unconstrained, constrained, and actuarially unfair Richard Model. The constrained model shows full annuitization is optimal, but as $\eta(t)$ increases in the unfair model, this is no longer the case.

From this, we do see the value function drops as $\eta(t)$ increases. This implies the Richard Model gives greater value to the retiree than a model where the retiree is unable to borrow against future wealth, an effect of the availability of life insurance. Comparing Table 5.1 to Figure 5.1, we see a drop in the value function from the constrained Richard Model to the constrained Basic Model. Once again, we see that introducing life insurance has a positive effect on the retiree's value function.

5.3.3 Taylor Model, post-default.

This section will only serve to create the discretized HJB for $\widehat{D}(t, u)$. This part of the model parallels exactly the work done for the Richard Model. This is needed for the discretized version of $\widehat{V}(t, u)$, which has a dependency on $\widehat{D}(t, u)$.

From (5.16), let

$$\begin{aligned}\widehat{b}^+(t, u) =& r(t) + \eta_M(t) + \widehat{\pi}(t)(\mu(t) - r(t)),\\ \widehat{b}^-(t, u) =& \widehat{c}(t) + \eta_M(t)\widehat{\zeta}_M(t) + \tfrac{1}{2}\widehat{\pi}^2(t)\sigma^2(t).\end{aligned}$$

Applying (5.21) to (5.16) gives

$$
\begin{aligned}
\widehat{D}(t,u) =& \frac{1}{1+\delta\lambda_M(t)} \sup_{(\widehat{c},\widehat{\pi},\widehat{\zeta}_M)} \Big\{ \delta\lambda_M(t)B(t,e^u\widehat{\zeta}_M(t)) + \delta U(t,e^u\widehat{c}(t)) \\
&+ \widehat{P}(u,u+h)\widehat{D}(t+\delta,u+h) + \widehat{P}(u,u-h)\widehat{D}(t+\delta,u-h) + \widehat{P}(u,u)\widehat{D}(t+\delta,u) \Big\},
\end{aligned}
\tag{5.31}
$$

where

$$
\begin{aligned}
\widehat{P}(u,u+h) &= \frac{\delta}{h}\widehat{b}^{+}(t,u) + \frac{\delta\widehat{\pi}^2(t)\sigma^2(t)}{2h^2}, \\
\widehat{P}(u,u-h) &= \frac{\delta}{h}\widehat{b}^{-}(t,u) + \frac{\delta\widehat{\pi}^2(t)\sigma^2(t)}{2h^2}, \\
\widehat{P}(u,u) &= 1 - \widehat{P}(u,u+h) - \widehat{P}(u,u-h).
\end{aligned}
$$

By choosing small values of h and δ, the functions $\widehat{P}(\cdot,\cdot)$ can be thought of as transition probabilities in a Markov chain which do not rely on the state u. As with the Richard Model discretization, [22] showed that the solution to (5.16) can be approximated by the solution of (5.31) by letting $h \to 0$ and $\delta \to 0$.

We can rearrange this over the supremum as

$$
\begin{aligned}
\widehat{D}(t,u) =& \frac{\delta}{1+\delta\lambda_M(t)} \Bigg[\frac{\widehat{D}(t+\delta,u)}{\delta} + \sup_{\widehat{c}} \Big\{ -\widehat{c}(t)\widehat{D}_u^{-}(t,u) + U(t,e^u\widehat{c}(t)) \Big\} \\
&+ \sup_{\widehat{\pi}} \Big\{ \widehat{\pi}(t)(\mu(t)-r(t))\widehat{D}_u^{+}(t,u) + (\widehat{D}_u(t,u) - \widehat{D}_u^{-}(t,u))\frac{1}{2}\widehat{\pi}^2(t)\sigma^2(t) \Big\} \\
&+ \sup_{\widehat{\zeta}_M} \Big\{ -\eta_M(t)\widehat{\zeta}_M(t)\widehat{D}_u^{-}(t,u) + \lambda_M(t)B(t,e^u\widehat{\zeta}_M(t)) \Big\} + (r(t)+\eta_M(t))\widehat{D}_u^{+}(t,u) \Bigg].
\end{aligned}
$$

Optimizing these yields similar results to those found in (5.26) - (5.28), namely

$$
\widehat{c}^{*} = (\widehat{D}_u^{-}(t,u)e^{\rho t})^{1/(\gamma-1)}(e^{-u})^{\gamma/(\gamma-1)}, \tag{5.32}
$$

$$
\widehat{\pi}^{*} = \frac{(\mu(t)-r(t))\widehat{D}_u^{+}(t,u)}{\sigma^2(t)(\widehat{D}_u^{-}(t,u) - \widehat{D}_{uu}(t,u))}, \tag{5.33}
$$

$$\widehat{\zeta}_M^* = \left(\frac{\eta_M(t)}{\lambda_M(t)}\right)^{1/(\gamma-1)} (\widehat{D}_u^-(t,u)e^{\rho t})^{1/(\gamma-1)}(e^{-u})^{\gamma/(\gamma-1)}. \tag{5.34}$$

When considering the incomplete market, we have the natural constraints

$$0 \le \widehat{c}^* \le 1,$$
$$0 \le \widehat{\pi}^* \le 1.$$

For $\widehat{\zeta}_M^*$, (4.4) shows that in the incomplete market with no insurance short-selling ($p_M(t) \ge 0$), we have the restriction

$$\zeta_M^*(t) \ge w.$$

From the MCALT method, (5.3) shows

$$\zeta_M^*(t) \ge x - I(t),$$

and (5.7) shows

$$\widehat{\zeta}_M^*(t) \ge 1 - e^{-u} I(t).$$

In the post-default wealth process, $I(t) = I_{ND}(t)$, so

$$\widehat{\zeta}_M^*(t) \ge 1 - e^{-u} I_{ND}(t).$$

5.3.4 Taylor Model, pre-default.

In (5.19), define

$$\widehat{b}^+(t,u) = r(t) + \eta_M(t) + \eta_D(t) + \widehat{\pi}(t)(\mu(t) - r(t)),$$
$$\widehat{b}^-(t,u) = \widehat{c}(t) + \eta_M(t)\widehat{\zeta}_M(t) + \eta_D(t)\widehat{\zeta}_D(t) + \tfrac{1}{2}\widehat{\pi}^2(t)\sigma^2(t).$$

Applying (5.20) to (5.19) gives

$$\widehat{V}(t,u) = \frac{1}{1+\delta(\lambda_M(t)+\lambda_D(t))} \sup_{(\widehat{c},\widehat{\pi},\widehat{\zeta}_M,\widehat{\zeta}_D)} \Bigg\{ \delta\lambda_M(t)B(t,e^u\widehat{\zeta}_M(t)) \\ + \delta U(t,e^u\widehat{c}(t)) + \delta\lambda_D(t)\widehat{D}(t,u+\ln(\widehat{\zeta}_D(t))) + \widehat{Q}(u,u+h)\widehat{V}(t+\delta,u+h) \\ + \widehat{Q}(u,u-h)\widehat{V}(t+\delta,u-h) + \widehat{Q}(u,u)\widehat{V}(t+\delta,u)\Bigg\}, \tag{5.35}$$

where

$$\begin{aligned} \widehat{Q}(u,u+h) &= \frac{\delta}{h}\widehat{b}^+(t,u) + \frac{\delta\widehat{\pi}^2(t)\sigma^2(t)}{2h^2}, \\ \widehat{Q}(u,u-h) &= \frac{\delta}{h}\widehat{b}^-(t,u) + \frac{\delta\widehat{\pi}^2(t)\sigma^2(t)}{2h^2}, \\ \widehat{Q}(u,u) &= 1 - \widehat{Q}(u,u+h) - \widehat{Q}(u,u-h). \end{aligned}$$

As before, for appropriate choices of h and δ, the functions $\widehat{Q}(\cdot,\cdot)$ can be thought of as transition probabilities in a Markov chain which do not rely on the state variable u. Letting $h \to 0$ and $\delta \to 0$ allows (5.35) to approximate (5.19).

Rearranging terms in (5.35) gives

$$\widehat{V}(t,u) = \frac{\delta}{1+\delta(\lambda_M(t)+\lambda_D(t))}\Bigg[\frac{\widehat{V}(t+\delta,u)}{\delta} + \sup_{\widehat{c}}\Big\{-\widehat{c}(t)\widehat{V}_u^-(t,u) + U(t,e^u\widehat{c}(t))\Big\} \\ + \sup_{\widehat{\pi}}\Big\{\widehat{\pi}(t)(\mu(t)-r(t))\widehat{V}_u^+(t,u) + (\widehat{V}_u(t,u) - \widehat{V}_u^-(t,u))\frac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)\Big\} \\ + \sup_{\widehat{\zeta}_M}\Big\{-\eta_M(t)\widehat{\zeta}_M(t)\widehat{V}_u^-(t,u) + \lambda_M(t)B(t,e^u\widehat{\zeta}_M(t))\Big\} \\ + \sup_{\widehat{\zeta}_D}\Big\{-\eta_D(t)\widehat{\zeta}_D(t)\widehat{V}_u^-(t,u) + \lambda_D(t)\widehat{D}(t,u+\ln(\widehat{\zeta}_D(t)))\Big\} \\ + (r(t)+\eta_M(t)+\eta_D(t))\widehat{V}_u^+(t,u)\Bigg].$$

Optimizing $\widehat{c}$, $\widehat{\pi}$, and $\widehat{\zeta}_M$ as before yields

$$\widehat{c}^* = (\widehat{V}_u^-(t,u)e^{\rho t})^{1/(\gamma-1)}(e^{-u})^{\gamma/(\gamma-1)}, \tag{5.36}$$

$$\widehat{\pi}^* = \frac{(\mu(t) - r(t))\widehat{V}_u^+(t,u)}{\sigma^2(t)(\widehat{V}_u^-(t,u) - \widehat{V}_u(t,u))}, \tag{5.37}$$

$$\widehat{\zeta}_M^* = \left(\frac{\eta_M(t)}{\lambda_M(t)}\right)^{1/(\gamma-1)} (\widehat{V}_u^-(t,u)e^{\rho t})^{1/(\gamma-1)}(e^{-u})^{\gamma/(\gamma-1)}. \tag{5.38}$$

Unfortunately this technique does not work for $\widehat{\zeta}_D^*$ as it relies on $\widehat{D}(\cdot,\cdot)$, which is approximated numerically. The algorithms for finding $\widehat{D}(t,y)$ and $\widehat{V}(t,y)$ only rely on a finite number of t and y values, so we solve

$$\widehat{\zeta}_D^* = \arg\max_{\widehat{\zeta}_D} \left\{ -\eta_D(t)\widehat{\zeta}_D(t)\widehat{V}_u^-(t,u) + \lambda_D(t)\widehat{D}(t, u + \ln(\widehat{\zeta}_D(t)))\right\}. \tag{5.39}$$

Equation (3.26) can now be solved numerically.

When considering constraints on the model for an incomplete market, we naturally have

$$0 \le \widehat{c}^* \le 1,$$

$$0 \le \widehat{\pi}^* \le 1.$$

The constraint on $\widehat{\zeta}_M$ is found similarly to that in the post-default setting. However, pre-default we have $I(t) = I_D(t) + I_{ND}(t)$, so

$$\widehat{\zeta}_M^*(t) \ge 1 - e^{-u}I(t).$$

The constraint on $\widehat{\zeta}_D^*$ can be found in like manner. With the restriction on short-selling ($p_D(t) \ge 0$), (4.5) shows

$$\zeta_D^*(t) \ge w.$$

For an individual with total wealth x, when default occurs the illiquid wealth drops by $I_D(t)$. This means

$$w = x - I_D(t),$$

so

$$\zeta_D^*(t) \geq x - I_D(t).$$

Then (5.5) shows

$$\widehat{\zeta}_D^*(t) \geq 1 - e^{-u} I_D(t).$$

We solved (5.35), both unconstrained and constrained, using Python 3.6 on a 64 core supercomputer with 1GB of memory per core. The run-time to calculate over all choices of $\lambda_D(t) \in [0, 0.01, 0.02, 0.03]$ and an α-step size of 3% of $\alpha_{\max}$ is about 90 minutes. We chose $\delta = 0.01$ and $h = 0.02$ as in previous models. The method we implemented is nearly 30 times faster than the process used in Taylor [13].

The following table shows the optimal fraction of initial wealth to put into an annuity for a given default rate in the unconstrained model. Recall

$$\theta = \frac{A(\alpha^*)}{w}.$$

λ_D	0	.01	.02	.03
θ	—	0.0	0.0	0.0

Table 5.2: Optimal annuitization as a ratio of wealth for the unconstrained Taylor Model. The case $\lambda_D = 0$ has no optimal amount as the value function is the same across the board. For $\lambda_D \neq 0$, the retiree is better off not annuitizing at all, and instead short-selling insurance.

This can be backed up by looking at the value function over various payment rates α, for different values of λ_D, similar to Figure 4.1.

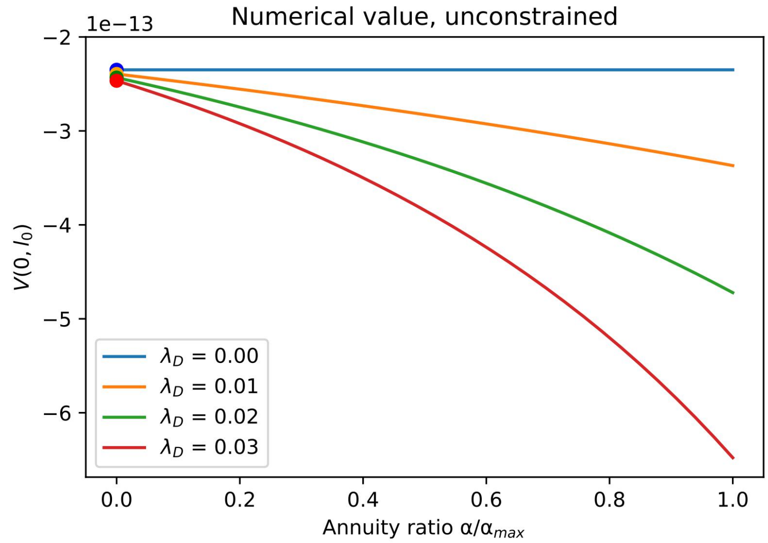

Figure 5.5: Value function for several annuity payment rates α over different default rates $\lambda_D(t)$ in the unconstrained Taylor Model.

As mentioned in Section 4.9, the retiree will choose to hold on to wealth and instead short-sell insurance, as if he were creating a risk-free annuity for himself. The unconstrained model shows the need for constraints.

Applying constraints to the algorithm is straightforward and results in the following optimal annuity ratios.

λ_D	0	.01	.02	.03
θ	1.0	0.84	0.67	0.42

Table 5.3: Optimal annuitization as a ratio of wealth for the constrained Taylor Model. By disallowing short-selling of insurance, the retiree is better off not holding onto his wealth at retirement. The $\lambda_D = 0$ case matches the Richard Model result. For $\lambda_D \neq 0$, there is still risk of default, so the retiree must balance the risk of default with his ability to purchase a CDS.

This result is expected, as the retiree no longer benefits from keeping all his money and short-selling insurance as a pseudo-income. He is now better off investing a portion into the annuity in such a way that some positive insurance purchase is beneficial. Another important result is that as the rate of default increases, the optimal annuitization ratio decreases, as one would expect with risk. This was also discussed by Taylor in [13].

Figure 5.6 shows plots of the value function for different values of λ_D over various income rates α.

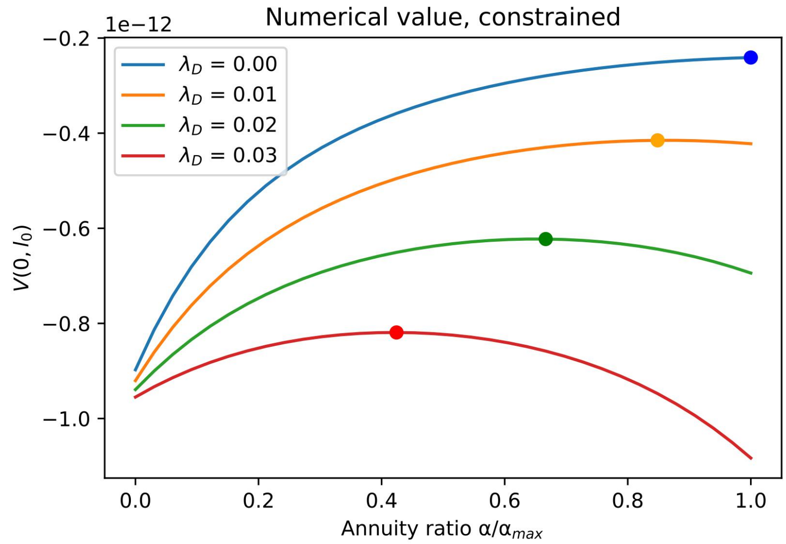

Figure 5.6: Value function for several annuity payment rates α over different default rates $\lambda_D(t)$ in the constrained Taylor Model. By restricting short-selling, the retiree must find a balance between annuity default risk and paying for a CDS to cover that risk.

For an annuity with a very small chance of default, the retiree will still likely annuitize a large portion of his wealth. This is evident in Table 5.3, where $\theta = .84$ when $\lambda_D(t) = 0.01$. Thus, despite his novel approach to risky annuities, Taylor's model still shows large annuitization is optimal for the retiree and doesn't solve the annuity puzzle.

As with the Richard Model, we now study the effects of loading on optimal annuity purchases. First consider the post-default model in (5.16). As mentioned in Section 4.3.2, in the post-default situation

$$V(t, w) = D(t, w),$$

and so

$$\widehat{V}(t,w) = \widehat{D}(t,w).$$

Thus, (5.16) is precisely (5.14). We already saw in Section 5.3.2 that as $\eta(t) \to \infty$ ($\eta_M(t) \to \infty$ for the Taylor Model), the model became a simplified model, where no insurance is purchased and borrowing does not occur.

Now consider the same for the pre-default HJB (5.19), with $\eta_M(t) \to \infty$ and $\eta_D(t) \to \infty$. Equation (5.38) shows as $\eta_M(t) \to \infty$ for a risk averse individual ($\gamma < 0$), then $\widehat{\zeta}_M(t) \to 0$.

As before, $I(t) \to 0$ as $\eta_M(t) \to \infty$. By the constraint in (5.19), for $\eta_M(t)$ large, then

$$\widehat{\zeta}_M(t) = 1 - e^{-u} I(t) \tag{5.40}$$

necessarily. Next consider $\widehat{\zeta}_D(t)$ in (5.39). Taking derivatives gives

$$\tfrac{\lambda_D(t)}{\widehat{\zeta}_D(t)} \widehat{D}_u(t, u + \ln(\widehat{\zeta}_D(t))) = \eta_D(t) \widehat{V}_u^-(t,u).$$

The right side of the equation goes to ∞. If

$$\widehat{\zeta}_D(t) \to z > 0,$$

then the smoothness of $\widehat{D}(t,u)$ would require the left side to be some finite value not going to ∞. Therefore we conclude that

$$\widehat{\zeta}_D(t) \to 0,$$

similar to $\widehat{\zeta}_M(t)$.

Since $I_D(t) \to 0$ for large $\eta_D(t)$, then the restriction in (5.19) will force

$$\widehat{\zeta}_D(t) = 1 - e^{-u} I_D(t). \tag{5.41}$$

This means

$$\widehat{\zeta}_D(t) \to 1$$

as $\eta_D(t) \to \infty$.

Plug (5.40) and (5.41) into (5.19) and take the limits $\eta_M \to \infty$ and $\eta_D \to \infty$. Most important in the limit is

$$\lim_{\substack{\eta_D \to \infty \\ \eta_M \to \infty}} \eta_M(t) + \eta_D(t) - \eta_M(t)\widehat{\zeta}_M(t) - \eta_D(t)\widehat{\zeta}_D(t) = \lim_{\substack{\eta_D \to \infty \\ \eta_M \to \infty}} e^{-u}\big(\eta_M(t)I(t) + \eta_D(t)I_D(t)\big).$$

As in Section 5.3.2, a quick calculation gives

$$\lim_{\substack{\eta_D \to \infty \\ \eta_M \to \infty}} e^{-u}\big(\eta_M(t)I(t) + \eta_D(t)I_D(t)\big) = e^{-u}\big(\alpha_{ND}(t) + \alpha_D(t)\big).$$

Then (5.19) becomes

$$0 = \widehat{V}_t(t,u) - (\lambda_M(t) + \lambda_D(t))\widehat{V}(t,u) + \sup_{(\widehat{c},\widehat{\pi})} \widehat{\Psi}(t,u;\widehat{c},\widehat{\pi}), \tag{5.42}$$

where

$$\begin{aligned} \widehat{\Psi}(t,u;\widehat{c},\widehat{\pi}) = &\big(r(t) - \widehat{c}(t) + e^{-u}(\alpha_{ND}(t) + \alpha_D(t)) + \widehat{\pi}(t)(\mu(t) - r(t)) \\ &- \tfrac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)\big)\widehat{V}_u(t,u) + \tfrac{1}{2}\widehat{\pi}^2(t)\sigma^2(t)\widehat{V}_{uu}(t,u) \\ &+ \lambda_M(t)B(t,e^u) + \lambda_D(t)\widehat{D}(t,u) + U(t,e^u\widehat{c}(t)). \end{aligned}$$

Compare this to the simple model studied briefly in the unfair Richard Model, Section 5.3.2. This HJB is similar to (5.29), with the addition of the possibility of the annuity defaulting, but no default insurance is allowed to be purchased. Thus the Richard Model reduces to one in which the retiree makes consumption and investment choices, but cannot purchase life or default insurance.

As in Section 5.3.2, we consider the solution to the updated Taylor Model HJB equation (5.19) with actuarially unfair $\eta_M(t)$ and $\eta_D(t)$. In particular we consider

$$\eta_M(t) = (1+\epsilon)\lambda_M(t),$$
$$\eta_D(t) = (1+\epsilon)\lambda_D(t),$$

with $\epsilon = .25$.

Figure 5.7 shows the graph of the actuarially unfair model, and Figure 5.8 shows a comparison of the constrained model with $\epsilon = 0$ and $\epsilon = .25$.

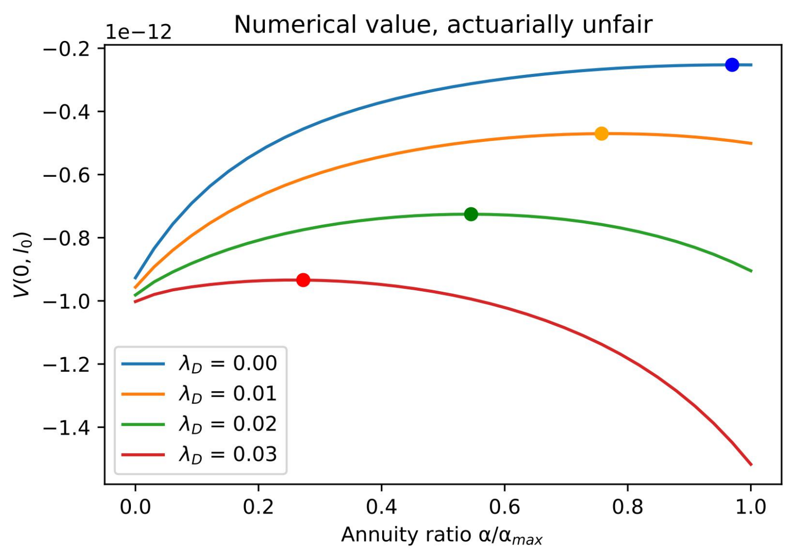

Figure 5.7: Value function for several annuity payment rates α over different default rates $\lambda_D(t)$ in the actuarially unfair Taylor Model. The optimal annuity amount is still fairly low, providing another look at the solution to the annuity puzzle.

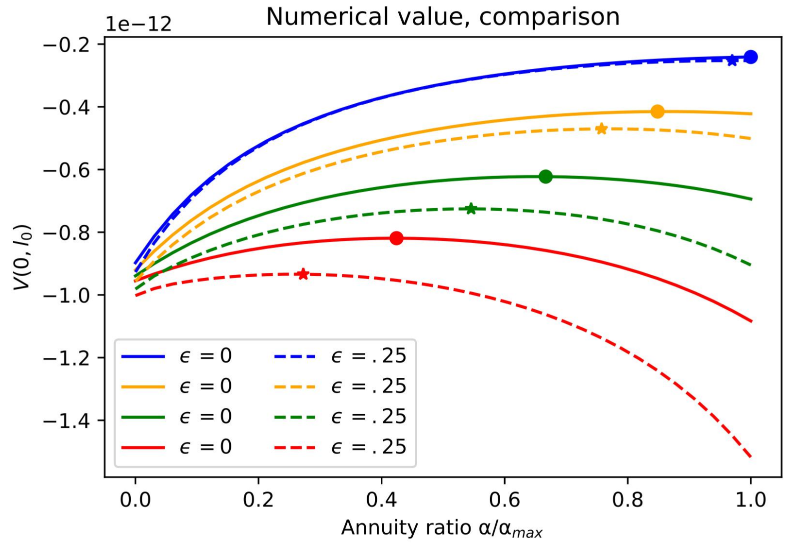

Figure 5.8: A comparison of the value function in the Taylor Model for $\epsilon = 0$ and $\epsilon = .25$. Notice the decrease in $V(0, l_0)$.

We list below the updated maximized value function $V(0, l_0)$ and annuity/wealth ratio, $\theta = \frac{A(\alpha^*)}{w} = \frac{\alpha^*}{\alpha_{max}}$, as compared to the constrained and unconstrained calculations. We break this into four comparisons, one table representing one value of $\lambda_D(t) \in [0, .03]$. From these tables, we see the value functions drop as $\eta_M(t)$ and $\eta_D(t)$ increase. Another important result is the optimal annuity ratio has decreased significantly for riskier annuities.

$\lambda_D(t) = 0.0$	θ	$V(0, l_0)$
Unconstrained	0.0	-2.3497×10^{-13}
Constrained	1.0	-2.4093×10^{-13}
Unfair	.97	-2.5250×10^{-13}

Table 5.4: $V(0, l_0)$ and θ for the unconstrained, constrained, and actuarially unfair Taylor Model with $\lambda_D(t) = 0.0$.

$\lambda_D(t) = .01$	θ	$V(0, l_0)$
Unconstrained	0.0	-2.3944×10^{-13}
Constrained	.84	-4.1508×10^{-13}
Unfair	.76	-4.7032×10^{-13}

Table 5.5: $V(0, l_0)$ and θ for the unconstrained, constrained, and actuarially unfair Taylor Model with $\lambda_D(t) = .01$.

$\lambda_D(t) = .02$	θ	$V(0, l_0)$
Unconstrained	0.0	-2.4328×10^{-13}
Constrained	.67	-6.2263×10^{-13}
Unfair	.55	-7.2547×10^{-13}

Table 5.6: $V(0, l_0)$ and θ for the unconstrained, constrained, and actuarially unfair Taylor Model with $\lambda_D(t) = .02$.

$\lambda_D(t) = .03$	θ	$V(0, l_0)$
Unconstrained	0.0	-2.4658×10^{-13}
Constrained	.42	-8.1907×10^{-13}
Unfair	.27	-9.3399×10^{-13}

Table 5.7: $V(0, l_0)$ and θ for the unconstrained, constrained, and actuarially unfair Taylor Model with $\lambda_D(t) = .03$.

5.4 Conclusions

Up to this point we have studied exact and numerical solutions to the the retiree's optimization problem, in both complete and incomplete markets. The exact and unconstrained solutions to the Richard and Taylor Models showed how short-selling insurance acts as an instantaneous term annuity. For the Richard Model, this means there is no optimal solution (see Figures 3.1 and 5.2). For the Taylor Model, this means the retiree will trade in a risky annuity for a risk-free annuity by short-selling insurance in the form of an instantaneous term annuity. In looking at the constrained solutions, we show that full or near-full annuitization is optimal, even for a risky annuity as long as the risk is low.

While Richard presented a tractable model, there is an issue with the use of an instantaneous term life insurance (and subsequently our use of it as well). That type of insurance simply does not exist, although it is nice to work with. Taylor runs into the same problem, but he uses two insurance premiums in his model. In the next chapter, we will look at how to refer to $p(t)$ not as an insurance premium, but as a payment on a loan taken out against the retiree's expected present discounted value from future wealth. Thus, we will show that financial institutions have a marketable way to allow retirees to borrow against future wealth, with little risk to both parties.

Chapter 6. Further Analysis of Richard and Taylor

The problem with Richard's model is his assumption that life insurance can be treated as instantaneous term life insurance, meaning it can be bought and traded at any time $t \in [0, T]$, and the amount can change to whatever the retiree desires. This is not how life insurance works. Instead, we will show by careful calculation that rearranging the wealth process for the Richard Model turns the insurance premium into payments on a loan that the retiree takes out against future wealth, $I(t)$. Restructuring the wealth process in this way allows for institutional implementation.

We will first discuss what this loan and payments on the loan look like. Then we will show how the wealth process in Richard's model can reflect this payment. Finally we show why this loan payment plan doesn't work for Taylor's model, and discuss how to correct the issue.

6.1 Borrowing

Normally, if an individual takes out a loan, he needs some way to pay it back. If the person dies early, they have life insurance, which can be used to pay off the amount borrowed. We are interested in changing the problem so the retiree doesn't buy life insurance to cover debts, but instead pays some rate to the lender to cover mortality risk.

As seen in (3.21),

$$I(t) = \int_t^T \alpha(s) \exp\left(-\int_t^s (r(v) + \eta(v))dv\right) ds \tag{6.1}$$

represents the retiree's expected discounted future value of wealth from an annuity income. Suppose the retiree takes out a loan, $Z(t)$, against this amount, so that his current liquid wealth is $w + Z(t)$ and he still has $I(t)$ as an expected future income. We want

$$0 \leq Z(t) \leq I(t),$$

so the retiree doesn't borrow more than he has the means to pay back. The lender also wants to make sure the loan is paid off before the retiree dies.

Over time, the loan accrues interest. The retiree must pay off this interest, while at the same time paying down the loan. Thus, in the retiree's wealth process we would need to account for this by subtracting out the interest payments, which we might call $r(t)Z(t)$, where $r(t)$ is the risk-free market rate established earlier.

Finally, there is some risk involved in this loan. If the retiree dies before the loan is paid off, loan forgiven programs would determine that the loan is forgiven and the lender does not get the money back. The lender is going to need to mitigate this loss by adding an additional charge to the loan. This charge comes in the form of the insurance payment $p(t)$ that Richard describes, where the insurance payment goes to the lender now, as a hedge against mortality risk. In the next section, we will see what this payment looks like.

6.2 Richard Borrowing

Consider the wealth process given in (3.1). We repeat it here as

$$W(t) = w - \int_0^t c(s)ds - \int_0^t p(s)ds + \int_0^t \alpha(s)ds + \int_0^t \frac{W(s) - \pi(s)}{S_0(s)} dS_0(s) + \int_0^t \frac{\pi(s)}{S_1(s)} dS_1(s),$$

The SDE given in (3.2) is

$$dW(t) = \big(r(t)W(t) - c(t) - p(t) + \alpha(t) + \pi(t)(\mu(t) - r(t))\big)dt + \pi(t)\sigma(t)d\mathcal{W}(t).$$

We define the total wealth

$$\zeta(t) = W(t) + \frac{p(t)}{\eta(t)},$$

where we can think of $\zeta(t)$ as the total liquid wealth after borrowing, and $\eta(t)$ as the risk of retiree mortality at time t. Rearranging this gives

$$p(t) = \eta(t)(\zeta(t) - W(t)) = \eta(t)Z(t),$$

where $Z(t)$ represents the amount borrowed against future income. Thus the insurance payment has turned into a payment against risk of mortality. From this we see with large $\eta(t)$ that the loan will become too expensive to maintain and the retiree may not borrow against future income. This naturally reduces risk to lenders by making loans more expensive for riskier borrowers (such as a retiree near death).

Substituting this new form of $p(t)$ into (3.2) gives (3.4), which is.

$$\begin{aligned} dW(t) =& \big((r(t) + \eta(t))W(t) - c(t) + \alpha(t) - \eta(t)\zeta(t) + \pi(t)(\mu(t) - r(t))\big)dt \\ &+ \pi(t)\sigma(t)d\mathcal{W}(t). \end{aligned}$$

By adding the term $r(t)Z(t) - r(t)Z(t)$ to this SDE, we get a new SDE

$$\begin{aligned} dW(t) =& \big(\alpha(t) - c(t) - (r(t) + \eta(t))Z(t) + r(t)\zeta(t) + \pi(t)(\mu(t) - r(t))\big)dt \\ &+ \pi(t)\sigma(t)d\mathcal{W}(t). \end{aligned} \tag{6.2}$$

Notice now the term

$$(r(t) + \eta(t))Z(t)$$

represents a payment rate on the amount borrowed, $Z(t)$, from the interest rate, $r(t)$, and the risk of the loan, $\eta(t)$. Thus the retiree's wealth process has been rewritten so that the retiree can take out a loan and make payments against interest and risk.

Another benefit of this new process is its effect on investment opportunities. In Section 5.3.2, we were interested in finding $\widehat{\pi}(t)$, representing the ratio of total wealth, $w + I(t)$, put into a risky investment. For small values of w (especially near \$0 if the retiree invests everything into an annuity) and large $I(t)$, this means the retiree is investing more money than their liquid wealth can cover. Since

$$W(t) = \pi(t) + \pi_0(t),$$

this indicates $\pi_0(t) < 0$ and the retiree is short-selling in the risk-free market. However, by allowing the retiree to borrow against future income, the last terms in (6.2) show

$$r(t)\zeta(t) + \pi(t)(\mu(t) - r(t)) = r(t)(\zeta(t) - \pi(t)) + \mu(t)\pi(t),$$

showing the retiree can use the loan to invest more in the markets.

6.3 Taylor Borrowing

Now consider the pre-default wealth process for the Taylor Model in (4.3), which is written here as

$$\begin{aligned} dW(t) =& \big(r(t)W(t) - c(t) - p_D(t) - p_M(t) + \alpha(t) + \pi(t)(\mu(t) - r(t))\big)dt \\ &+ \pi(t)\sigma(t)d\mathcal{W}(t). \end{aligned}$$

Recall the total wealth functions in (4.4) and (4.5) are

$$\begin{aligned} \zeta_M(t) &= W(t) + \frac{p_M(t)}{\eta_M(t)}, \\ \zeta_D(t) &= W(t) + \frac{p_D(t)}{\eta_D(t)}, \end{aligned}$$

and can be rearranged as

$$p_M(t) = \eta_M(t)\big(\zeta_M(t) - W(t)\big) = \eta_M(t)Z_M(t),$$
$$p_D(t) = \eta_D(t)\big(\zeta_D(t) - W(t)\big) = \eta_D(t)Z_D(t).$$

We can think of $Z_M(t) + Z_D(t)$ as the amount borrowed against future income. Z_M is borrowed against risk of death, and is bounded above by $I(t) = I_D(t) + I_{ND}(t)$. Similarly, Z_D is borrowed against risk of default, and is bounded above by $I_D(t)$. The retiree will make payments against risk of death and risk of default, with rates determined by $\eta_M(t)$ and $\eta_D(t)$, respectively. The issue arises here in that the retiree borrows against two different risks at different rates. This is not marketable, since loaning institutions don't separate loans in this way.

We proceed with the formulation of the SDE in terms of borrowing, to get a sense of how to fix the issue. Plugging (6.3) into (4.3) gives

$$\begin{aligned} dW(t) =& \big((r(t) + \eta_M(t) + \eta_D(t))W(t) - c(t) + \alpha(t) - \eta_M(t)\zeta_M(t) - \eta_D(t)\zeta_D(t) \\ &+ \pi(t)(\mu(t) - r(t))\big)dt + \pi(t)\sigma(t)d\mathcal{W}(t). \end{aligned}$$

It is easy to verify that this can be rearranged, similar to (6.2), as

$$\begin{aligned} dW(t) =& \big(\alpha(t) - c(t) - (r(t) + \eta_M(t))(\zeta_M(t) - W(t)) - (r(t) + \eta_D(t))(\zeta_D(t) - W(t)) \\ &+ r(t)(\zeta_M(t) + \zeta_D(t) - W(t)) + \pi(t)(\mu(t) - r(t))\big)dt + \pi(t)\sigma(t)d\mathcal{W}(t) \\ =& \big(\alpha(t) - c(t) - (r(t) + \eta_M(t))Z_M(t) - (r(t) + \eta_D(t))Z_D(t) \\ &+ r(t)(\zeta_M(t) + \zeta_D(t) - W(t)) + \pi(t)(\mu(t) - r(t))\big)dt + \pi(t)\sigma(t)d\mathcal{W}(t) \end{aligned} \tag{6.3}$$

For the loan corresponding to $Z_M(t)$, the wealth process has $(r(t) + \eta_M(t))Z_M(t)$ deducted, corresponding to any interest being paid back from $r(t)Z_M(t)$ and the cost of risk due to early death being paid by $\eta_M(t)Z_M(t)$. The loan corresponding to $Z_D(t)$ is paid off similarly.

In the next chapter, we consider how to change Taylor's insurance payments in a way that death and default can be covered simultaneously. In this way, there will be only one insurance payout made at time $t = \tau_M \wedge \tau_D$, and thus there will be one value the retiree can borrow against. This next model will provide a means for retirees to purchase an annuity, and then borrow against the annuity throughout their lifetime, thus leading to potentially greater annuitization than might already be seen.

In this section, we extend Taylor's results to a new scenario in which a retiree's insurance choices are reduced down to one convenient parameter, ζ. In this model, the retiree maintains his uncertain lifetime, in which he must still make consumption and investment choices. He has the option to buy an annuity with some paying rate over the course of his lifetime, just as before. The annuity is able to default, as in the Taylor Model.

Where this model diverges from the previous models is in the insurance instrument the retiree is able to purchase; instead of separate insurance premiums to cover death (life insurance) and default (credit default swap), the retiree will pay a new insurance premium, under which the insurance company will make a one-time payment at the retiree's death or the annuity's default, whichever occurs first. This type of insurance has several advantages. The retiree is guaranteed a single payout, which can be used as a means to borrow risk-free from the bank. The insurance company makes a single payout, which drives down costs and complexity. Payouts and premiums can be better regulated. Lastly, this type of instrument can be turned into a real instrument, utilized by financial institutions to help retirees hedge against default and mortality risks.

As seen in the Taylor Model in Section 4, the payout from life insurance is given by

$$\frac{p_M(t)}{\eta_M(t)},$$

and the payout from default is given by

$$\frac{p_D(t)}{\eta_D(t)},$$

where $p_M(t)$ and $p_D(t)$ are the insurance premiums of death and default, respectively, and $\eta_M(t)$ and $\eta_D(t)$ are the premium-payout ratios of death and default, respectively. Since only one insurance premium is to be paid, we require

$$\frac{p_M(t)}{\eta_M(t)} = \frac{p_D(t)}{\eta_D(t)}. \tag{7.1}$$

Using this restriction, we will show that the retiree has one expected income he can borrow against, and the lender will have a simple means to protect against loss of the loan due to death or default.

In the following sections, we consider all the same parameters and functions as used in the Taylor Model. Any significant divergences from that model will be strictly noted.

7.1 The Financial Market

The financial market will be the same as used in all other models.

7.2 The Retiree's Lifetime

The distribution functions and probability distribution functions corresponding to default and mortality are given in Section 4.2.

7.3 Wealth and Maximum Utility

The state and control variables are all previously described in Section 4.3. In this model, the retiree has income as long as the annuity does not default. However, once the annuity defaults we assume the retiree loses that income stream. Thus we let $\alpha(t) = \alpha_D(t)$ be the income stream pre-default, and assume

$$\alpha_{ND}(t) = 0.$$

We now introduce the new insurance premium control variable. Let $p(t)$ be the premium paid for this matched-payout insurance. Since (7.1) must hold, then the liquid wealth after insurance payouts, given by (4.4) and (4.5), are equivalent and will be represented by the single variable

$$\zeta(t) = W(t) + \frac{p_M(t)}{\eta_M(t)} = W(t) + \frac{p_D(t)}{\eta_D(t)}.$$

At this point it is also beneficial to find the new premium-payout ratio $\eta(t)$. To do this, let the insurance premium consist of the two premiums for life insurance and default, so

$$p(t) = p_M(t) + p_D(t). \tag{7.2}$$

From (7.1), the ratio

$$\frac{p_M(t)}{p_D(t)} = \frac{\eta_M(t)}{\eta_D(t)}$$

must hold. Then dividing by $p_M(t)$ in (7.2) yields

$$\frac{p(t)}{p_M(t)} = 1 + \frac{p_D(t)}{p_M(t)} = 1 + \frac{\eta_D(t)}{\eta_M(t)}.$$

Rearranging gives

$$p_M(t) = p(t)\frac{\eta_M(t)}{\eta_M(t) + \eta_D(t)}.$$

A similar process shows

$$p_D(t) = p(t)\frac{\eta_D(t)}{\eta_M(t) + \eta_D(t)}.$$

Notice now that

$$\frac{p_D(t)}{\eta_D(t)} = \frac{p_M(t)}{\eta_M(t)} = \frac{p(t)}{\eta_M(t) + \eta_D(t)}.$$

This last result gives the premium payout ratio

$$\eta(t) = \eta_M(t) + \eta_D(t).$$

The wealth process given by (4.3.1) can now be updated with our new control variables as

$$W(t) = w - \int_0^t c(s)ds - \int_0^t p(s)ds + \int_0^t \alpha(s)ds + \int_0^t \frac{W(s) - \pi(s)}{S_0(s)} dS_0(s) + \int_0^t \frac{\pi(s)}{S_1(s)} dS_1(s).$$

The SDE (4.3) becomes

$$dW(t) = \big(r(t)W(t) - c(t) - p(t) + \alpha(t) + \pi(t)(\mu(t) - r(t))\big)dt + \pi(t)\sigma(t)d\mathcal{W}(t).$$

As in the Taylor Model, the dynamics of the wealth process change after default time $t = \tau_D$. Thus, we let $W_1(t)$ be the pre-default wealth and $W_2(t)$ be the post-default wealth and write

$$W(t) = W_1(t)\mathbb{1}_{t \le \tau_D} + W_2(t)\mathbb{1}_{t > \tau_D}.$$

Post-default, no more insurance will be purchased. We also assume the retiree will not buy life insurance after default.

It will be easier to use the control variable

$$\zeta(t) = W(t) + \frac{p(t)}{\eta(t)} \tag{7.3}$$

instead of p(t). This, along with (4.6) shows

$$p_M(t) = (\zeta(t) - W(t))\eta_M(t),$$
$$p_D(t) = (\zeta(t) - W(t))\eta_D(t).$$

In terms of borrowing, we let $Z(t) = \zeta(t) - W(t)$ represent the amount borrowed.

Now define the admissibility of pre- and post-default controls. A pre-default tuple (c, π, ζ) is admissible if

- $\int_t^T (|c(s)| + |\pi(s)| + |p(s)| + \pi(s)^2)ds < \infty$, and

- $W(t) + \frac{p(t)}{\eta(t)} \geq 0$ for all $t \in [0, T]$.

Similarly, a post-default tuple (c, π) is admissible if

- $\int_t^T (|c(s)| + |\pi(s)| + \pi(s)^2) ds < \infty$, and
- $W(t) \geq 0$ for all $t \in [0, T]$.

With the new control variable $\zeta(t)$, we are ready for the new value function (similar to (4.10))

$$\begin{aligned} V(t, w) = \sup_{(c,\pi,\zeta)} E\bigg[\int_t^{\tau_M \wedge T \wedge \tau_D} & U(s, c(s)) ds + B(\tau_M, \zeta(\tau_M)) \mathbb{1}_{\tau_M < T \wedge \tau_D} \\ & + B(T, W(T)) \mathbb{1}_{T \leq \tau_M \wedge \tau_D} + D(\tau_D, \zeta(\tau_D)) \mathbb{1}_{\tau_D < T \wedge \tau_M} \bigg], \end{aligned} \tag{7.4}$$

where

$$\begin{aligned} D(t, w) = \sup_{(c,\pi)} E\bigg[\int_t^{T \wedge \tau_M} & U(s, c(s)) ds + B(\tau_M, W(\tau_M)) \mathbb{1}_{\tau_M < T} \\ & + B(T, W(T)) \mathbb{1}_{\tau_M \geq T} \bigg]. \end{aligned} \tag{7.5}$$

In $D(t, w)$, after annuity insolvency, we assume no further life insurance is purchased, hence final wealth is given by $W(\cdot)$ and not $\zeta_M(\cdot)$ as in (4.11).

7.3.1 Post-Default Wealth Process and Value Function.

Assume default occurs before death, so $\tau_D < \tau_M$. The wealth process is given by

$$W(t) = W_2(t),$$

and since life insurance is not purchased, the retiree will die with wealth $W_2(\tau_M)$.

As mentioned earlier, after the annuity defaults the retiree has no residual income, thus $\alpha(t) = 0$. The post-default SDE becomes

$$dW_2(t) = \big(r(t) W_2(t) - c(t) + \pi(t)(\mu(t) - r(t)) \big) dt + \pi(t) \sigma(t) d\mathcal{W}(t).$$

The value function becomes

$$
\begin{aligned}
D(t,w) = &\sup_{(c,\pi)} E\bigg[\int_t^{T\wedge\tau_M} U(s,c(s))ds + B(\tau_M, W_2(\tau_M))\mathbb{1}_{\tau_M<T} \\
&+ B(T, W_2(T))\mathbb{1}_{\tau_M\geq T}\bigg].
\end{aligned}
$$

7.3.2 Pre-Default Wealth Process and Value Function.

Pre-default, $W(t) = W_1(t)$ and the annuity still pays at a rate $\alpha(t)$. The pre-default SDE is

$$
\begin{aligned}
dW_1(t) = &\big((r(t)+\eta(t))W_1(t) - c(t) + \alpha(t) - \eta(t)\zeta(t) + \pi(t)(\mu(t) - r(t))\big)dt \\
&+ \pi(t)\sigma(t)d\mathcal{W}(t).
\end{aligned}
$$

Consider now how we can rearrange this to present it as a wealth process that accounts for paying off the loan $Z(t)$. Similar to what was done in Section 6.2, it is easy to verify that the SDE can be rewritten as

$$
dW_1(t) = \big(\alpha(t) - c(t) - (r(t)+\eta(t))Z(t) + r(t)\zeta(t) + \pi(t)(\mu(t)-r(t))\big)dt + \pi(t)\sigma(t)d\mathcal{W}(t).
$$

This shows that the model can allow for borrowing, with the retiree making payments in the form of interest from $r(t)Z(t)$ and risk payments $\eta(t)Z(t)$. The lender now has a means to loan money without getting hurt financially by risk of loss.

We now proceed with the rest of the model. The value function in (7.4) simply becomes

$$
\begin{aligned}
V(t,w) = \sup_{(c,\pi,\zeta)} E\bigg[&\int_t^{\tau_M\wedge T\wedge\tau_D} U(s,c(s))ds + B(\tau_M, \zeta(\tau_M))\mathbb{1}_{\tau_M<T\wedge\tau_D} \\
&+ B(T, W_1(T))\mathbb{1}_{T\leq\tau_M\wedge\tau_D} + D(\tau_D, \zeta(\tau_D))\mathbb{1}_{\tau_D<T\wedge\tau_M}\bigg].
\end{aligned}
$$

7.4 Stochastic Dynamic Programming

The work done in [13] clears the way for us at this time. We simply relate the necessary theorems and equations, and omit all proofs which are a direct result of everything done by Taylor.

7.4.1 Post-Default HJB and Control.

Consider (7.5). Let the post-default cost functional be given by

$$\begin{aligned} J_2(t,w;c,\pi) =& E\bigg[\int_t^{T\wedge\tau_M} U(s,c(s))ds + B(\tau_M, W_2(\tau_M))\mathbb{1}_{\tau_M<T} \\ &+ B(T,W_2(T))\mathbb{1}_{\tau_M\geq T}\bigg]. \end{aligned}$$

Then (7.5) can be rewritten as

$$D(t,w) = \sup_{(c,\pi)} J_2(t,w;c,\pi). \tag{7.6}$$

The following Lemma allows us to rewrite $J_2(t,w;c,\pi)$ as a weighted integral over the full time interval $[0,T]$. In the post-default situation, no insurance is purchased, so this will reflect Lemma 2.1 in the Basic Model, with $\alpha = 0$.

Lemma 7.1. *Suppose that $U(\cdot,\cdot)$ is a non-negative or non-positive function. If the time of death, τ_M, is independent of the filtration, then*

$$\begin{aligned} J_2(t,w;c,\pi) =& E\bigg[\int_t^T \big(f_M(s|t)B(s,W_2(s)) + \overline{F}_M(s|t)U(s,c(s))\big)ds \\ &+ \overline{F}_M(T|t)B(T,W_2(T))\bigg|\mathcal{F}_t\bigg], \end{aligned}$$

where $f_M(s|t)$ and $\overline{F}_M(s|t)$ are given by (4.1) *and* (4.2)*, respectively.*

Proof. Reference Lemma 2.1. □

The dynamic programming principle can now be applied.

Lemma 7.2 (Dynamic Programming Principle, Post-Default)**.** *For* $0 \leq t < s < T$*,*

$$\begin{aligned} D(t,w) = &\sup_{(c,\pi)} E\Bigg[\overline{F}(s|t)D(s,W_2(s)) + \int_t^s \big(f_M(u|t)B(u,W_2(u)) \\ &+ \overline{F}_M(u|t)U(u,c(u))\big)du\Bigg|\mathcal{F}_t\Bigg]. \end{aligned} \tag{7.7}$$

Proof. Reference Lemma 2.2. □

From this, construct the HJB equation for the post-default wealth process.

Theorem 7.3 (Dynamic Programming Equation, Post-Default)**.** *Suppose the value function* $D(t,w)$ *is smooth. Then* $D(t,w)$ *satisfies*

$$\begin{cases} 0 = D_t(t,W_2(t)) - \lambda_M(t)D(t,W_2(t)) + \sup\limits_{(c,\pi)}\Big\{b_2(t,W_2(t))D_w(t,W_2(t)) \\ \qquad +\frac{1}{2}\pi^2(t)\sigma^2(t)D_{ww}(t,W_2(t)) + \lambda_M(t)B(t,W_2(t)) + U(t,c(t))\Big\} \\ D(T,w) = B(T,w), \end{cases} \tag{7.8}$$

where

$$b_2(t,w) = r(t)w - c(t) + \pi(t)(\mu(t) - r(t)).$$

Proof. Reference Theorem 2.3. □

For the control policy, the retiree is not paying for insurance, so the consumption and investment policies will differ from the pre-default situation. Thus, the control policy can be

$$(c(t),\pi(t)) = (c_2(t),\pi_2(t)).$$

The control policy (c_2^*,π_2^*) is optimal if it satisfies (7.6) such that

$$D(t,w) = J_2(t,w;c_2^*,\pi_2^*).$$

The next theorem is similar to Theorem 2.4 and allows us to find the policies above.

Theorem 7.4 (Verification Theorem, Post-Default)**.** *Let a smooth function $D(t,w)$ be a solution to the HJB equation (7.8). Then*

$$D(t,w) \geq J_2(t,w;c_2,\pi_2)$$

for all valid pairs (c_2,π_2) in the control space.

Furthermore, an admissible pair (c_2^*,π_2^*) with corresponding wealth W_2^* is optimal $\iff$

$$0 = D_t(s,W_2^*(s)) - \lambda_M(s)D(s,W_2^*(s)) + \sup_{(c_2,\pi_2)} \Psi(s,W_2^*(s);c_2,\pi_2)$$

for almost every $s \in [t,T]$, where

$$\Psi(t,w;c_2,\pi_2) = b_2(t,w)D_w(t,w) + \tfrac{1}{2}\pi_2^2(t)\sigma^2(t)D_{ww}(t,w) + \lambda_M(t)B(t,w) + U(t,c_2(t)). \quad (7.9)$$

Proof. Reference Theorem 2.4. □

This says the control policy (c_2^*,π_2^*) is optimal if and only if

$$\begin{aligned}
0 =& D_t(t,w) - \lambda_M(t)D(t,w) + \Psi(t,w;c_2^*,\pi_2^*) \\
=& D_t(t,w) - \lambda_M(t)D(t,w) + \sup_{(c_2,\pi_2)} \Psi(t,w;c_2,\pi_2) \\
=& D_t(t,w) - \lambda_M(t)D(t,w) + \lambda_M(t)B(t,w) + r(t)wD_w(t,w) \qquad (7.10)\\
&+ \sup_{c_2}\big\{ - c(t)D_w(t,w) + U(t,c_2(t))\big\} \\
&+ \sup_{\pi_2}\big\{\pi_2(t)(\mu(t)-r(t))D_w(t,w) + \tfrac{1}{2}\pi_2^2(t)\sigma^2(t)D_{ww}(t,w)\big\}.
\end{aligned}$$

The first order conditions for a maximum are

$$\Psi_c(t,w;c_2^*,\pi_2^*) = 0 = U_c(t,c_2^*) - D_w(t,w), \tag{7.11}$$

$$\Psi_\pi(t,w;c_2^*,\pi_2^*) = 0 = (\mu(t)-r(t))D_w(t,w) + \pi_2^*\sigma^2(t)D_{ww}(t,w). \tag{7.12}$$

The sufficient conditions for a maximum are given by the the second order partials

$$\begin{aligned}\Psi_{cc}(t,w;c_2^*,\pi_2^*) =& U_{cc}(t,c_2^*) < 0,\\ \Psi_{\pi\pi}(t,w;c_2^*,\pi_2^*) =& \sigma^2(t)D_{ww}(t,w) < 0.\end{aligned}$$

The utility functions are strictly concave, so the first condition is automatically met. Taylor [13] showed

$$D_{ww}(t,w) < 0,$$

and so the second condition is satisfied.

7.4.2 Pre-Default HJB and Control.

Next consider (7.4). Define the cost functional

$$\begin{aligned}J_1(t,w;c,\pi,\zeta) =& E\Bigg[\int_t^{\tau_M\wedge\tau_D} U(s,c(s))ds + B(\tau_M,\zeta(\tau_M))\mathbb{1}_{\tau_M<T\wedge\tau_D}\\ &+ B(T,W_1(T))\mathbb{1}_{T\le\tau_M\wedge\tau_D} + D(\tau_D,\zeta(\tau_D))\mathbb{1}_{\tau_D<T\wedge\tau_M}\Bigg].\end{aligned}$$

Then (7.4) can be rewritten as

$$V(t,w) = \sup_{(c,\pi,\zeta)} J_1(t,w;c,\pi,\zeta). \tag{7.13}$$

The following lemmas and theorems follow the work in [11] and similar results in Section 4.4.2 above, and the proofs are omitted.

Lemma 7.5. *Suppose $U(\cdot,\cdot)$ is a non-negative or non-positive function. Suppose the terminal time τ_M and default time τ_D are independent of each other, the planning horizon T, and the filtration of the probability space. Then*

$$\begin{aligned} J_1(t,w;c,\pi,\zeta) =& E\bigg[\int_t^T \big(U(s,c(s)) + \lambda_M(s)B(s,\zeta(s)) \\ &+ \lambda_D(s)D(s,\zeta(s))\big)\overline{F}(s|t)ds + \overline{F}(T|t)B(T,W_1(T))\bigg|\mathcal{F}_t\bigg]. \end{aligned}$$

Proof. Reference Lemma 4.5. □

We derive the dynamic programming principle next.

Lemma 7.6 (Dynamic Programming Principle, Pre-Default)**.** *For $0 \le t < s < T$,*

$$\begin{aligned} V(t,w) = \sup_{(c,\pi,\zeta)} E\bigg[&\overline{F}(s|t)V(s,W_1(s)) \\ &+ \int_t^s \big(U(u,c(u)) + \lambda_M(u)B(u,\zeta(u)) + \lambda_D(u)D(u,\zeta(u))\big)\overline{F}(u|t)du\bigg|\mathcal{F}_t\bigg]. \end{aligned} \tag{7.14}$$

Proof. Reference Lemma 4.6. □

This allows us to set up the HJB equation in the following theorem.

Theorem 7.7 (Dynamic Programming Equation, Pre-Default)**.** *Suppose the value function $V(t,w)$ is smooth. Then $V(t,w)$ satisfies the following equation*

$$\begin{cases} 0 = V_t(t,W_1(t)) - (\lambda_M(t)+\lambda_D(t))V(t,W_1(t)) \\ \quad + \sup\limits_{(c,\pi,\zeta)} \Big\{ b_1(t,W_1(t))V_w(t,W_1(t)) + \frac{1}{2}\pi^2(t)\sigma^2(t)V_{ww}(t,W_1(t)) \\ \quad + \lambda_M(t)B(t,\zeta(t)) + \lambda_D(t)D(t,\zeta(t)) + U(t,c(t)) \Big\} \\ V(T,w) = B(T,w), \end{cases} \tag{7.15}$$

where

$$b_1(t, w) = (r(t) + \eta(t))w - c(t) + \alpha(t) - \eta(t)\zeta(t) + \pi(t)(\mu(t) - r(t)).$$

Proof. Reference Theorem 4.7. $\square$

Pre-default, the retiree still purchases this mixed insurance, so the control policy is determined by

$$(c(t), \pi(t), \zeta(t)) = (c_1(t), \pi_1(t), \zeta(t)).$$

The control policy $(c_1^*, \pi_1^*, \zeta^*)$ is optimal if it satisfies (7.13) such that

$$V(t, w) = J_1(t, w; c_1^*, \pi_1^*, \zeta^*).$$

The following theorem shows that this optimal policy also satisfies (7.15).

Theorem 7.8 (Verification Theorem, Pre-Default)**.** *Let a smooth function $V(t, w)$ be a solution to* (7.15)*. Then*

$$V(t, w) \geq J_1(t, w; c_1, \pi_1, \zeta)$$

for all triples (c_1, π_1, ζ) in the control space.

Furthermore, an admissible triple $(c_1^*, \pi_1^*, \zeta^*)$ with corresponding wealth W_1^* is optimal $\iff$

$$0 = V_t(s, W_1^*(s)) - (\lambda_M(s) + \lambda_D(s))V(s, W_1^*(s)) + \sup_{(c_1, \pi_1, \zeta)} \Psi(s, W_1^*(s); c_1, \pi_1, \zeta)$$

for almost every $s \in [t, T]$, where

$$\begin{aligned} \Psi(t, w; c_1, \pi_1, \zeta) = & b_1(t, w)V_w(t, w) + \tfrac{1}{2}\pi_1^2(t)\sigma^2(t)V_{ww}(t, w) \\ & + \lambda_M(t)B(t, \zeta(t)) + \lambda_D(t)D(t, \zeta(t)) + U(t, c_1(t)). \end{aligned} \tag{7.16}$$

Proof. Reference Theorem 4.8. $\square$

This says the control policy $(c_1^*, \pi_1^*, \zeta^*)$ is optimal if and only if

$$\begin{aligned}
0 =& V_t(t,w) - (\lambda_M(t) + \lambda_D(t))V(t,w) + \Psi(t,w;c_1^*,\pi_1^*,\zeta^*) \\
=& V_t(t,w) - (\lambda_M(t) + \lambda_D(t))V(t,w) + \sup_{(c_1,\pi_1,\zeta)} \Psi(t,w;c_1,\pi_1,\zeta) \\
=& V_t(t,w) - (\lambda_M(t) + \lambda_D(t))V(t,w) + \alpha(t)V_w(t,w) + (r+\eta(t))wV_w(t,w) \\
&+ \sup_{c_1}\big\{ - c_1(t)V_w(t,w) + U(t,c_1(t))\big\} \\
&+ \sup_{\pi_1}\big\{\pi_1(t)(\mu(t) - r(t))V_w(t,w) + \tfrac{1}{2}\pi_1^2(t)\sigma^2(t)V_{ww}(t,w)\big\} \\
&+ \sup_{\zeta}\big\{ - \zeta(t)\eta(t)V_w(t,w) + \lambda_M(t)B(t,\zeta(t)) + \lambda_D(t)D(t,\zeta(t))\big\}.
\end{aligned}$$

The first order conditions for a maximum are

$$\Psi_c(t,w;c_1^*,\pi_1^*,\zeta^*) = 0 = U_c(t,c_1^*) - V_w(t,w), \tag{7.17}$$

$$\Psi_\pi(t,w;c_1^*,\pi_1^*,\zeta^*) = 0 = (\mu(t) - r(t))V_w(t,w) + \pi_1^*\sigma^2(t)V_{ww}(t,w), \tag{7.18}$$

$$\Psi_\zeta(t,w;c_1^*,\pi_1^*,\zeta^*) = 0 = - \eta(t)V_w(t,w) + \lambda_M(t)B_\zeta(t,\zeta^*) + \lambda_D(t)D_\zeta(t,\zeta^*). \tag{7.19}$$

Sufficient conditions for a maximum are

$$\begin{aligned}
\Psi_{cc}(t,w;c_1^*,\pi_1^*,\zeta^*) =& U_{cc}(t,c_1^*) < 0, \\
\Psi_{\pi\pi}(t,w;c_1^*,\pi_1^*,\zeta^*) =& \sigma^2(t)V_{ww}(t,w) < 0, \\
\Psi_{\zeta\zeta}(t,w;c_1^*,\pi_1^*,\zeta^*) =& \lambda_M(t)B_{\zeta\zeta}(t,\zeta^*) + \lambda_D(t)D_{\zeta\zeta}(t,\zeta^*).
\end{aligned}$$

We know the utility functions $B(\cdot,\cdot)$ and $U(\cdot,\cdot)$ are concave by construction, so the first condition is met automatically. As shown in the Post-Default conditions, $D(\cdot,\cdot)$ must also be concave, thus the final condition is met. It remains to be shown that $V_{ww}(t,w) < 0$, but this must be true as discussed by Ye in [11].

7.5 Constant Relative Risk Aversion

The utility functions for consumption and bequest are defined as in all other models.

$$U(t,c) = \frac{e^{-\rho t}}{\gamma} c^{\gamma}, \tag{7.20}$$

$$B(t,\zeta) = \frac{e^{-\rho t}}{\gamma} \zeta^{\gamma}, \tag{7.21}$$

where $\gamma < 1$ and $\gamma \neq 0$. ρ is the utility discount rate with $\rho > 0$.

7.6 Exact Solution

As mentioned, the post-default situation is similar to the Basic Model in Chapter 2, since we assume life insurance is not purchased post-default. In this model, the retiree will still have a bequest motive, so we cannot use the results for an exact solution from the Basic Model. We will not attempt to find an exact solution for the Matched Payout Model, as it is intractable using our current tools. In the following section, however, we will see that a numerical solution does exist, as in previous models.

7.7 Numerical Solution

Similar to the Taylor Model, assume the retiree is not allowed to short-sell their new insurance product, thus forcing the restriction $p(t) \geq 0$. From (7.3), it is clear

$$\zeta^* \geq W(t).$$

As in previous models, we restrict $c(t)$ and $\pi(t)$ by the total wealth, so as to not consume or invest more than the retiree actually has.

With these constraints, the post-default HJB equation (7.8) remains unchanged, and is written again in its simplified form as

$$\begin{cases} 0 = D_t(t,w) - \lambda_M(t)D(t,w) + \sup\limits_{(c,\pi)} \Psi_D(t,w;c_2,\pi_2) \\ D(T,w) = B(T,w), \end{cases} \tag{7.22}$$

with $\Psi_D(t,w;c_2,\pi_2)$ given by (7.9).

The pre-default HJB equation (7.15), however, becomes

$$\begin{cases} 0 = V_t(t,w) - (\lambda_M(t) + \lambda_D(t))V(t,w) + \sup\limits_{(c,\pi,\zeta \geq w)} \Psi_V(t,w;c_1,\pi_1,\zeta) \\ V(T,w) = B(T,w), \end{cases} \tag{7.23}$$

with $\Psi_V(t,w;c_1,\pi_1,\zeta)$ given by (7.16).

We next consider the absorbing boundary conditions, similar to those found in Section 5.1. In the post-default situation, the retiree has no income and no insurance to cover any losses. There is a restriction on the liquid wealth going negative, giving the constraint

$$W_2(t) \geq 0.$$

Thus, as $W_2(t) \to 0$, then $c_2 \to 0$ and there is nothing for the bequest motive. Equation (7.7) then says

$$D(t,0) = -\infty.$$

Pre-default, the retiree still receives an income and buys insurance. The retiree has some amount of illiquid wealth equivalent to the discounted present day value of future wealth from $\alpha(t)$. Define this illiquid wealth as

$$I(t) = \int_t^T \alpha(s) \exp\left(- \int_t^s \big(r(v) + \eta_M(v) + \eta_D(v)\big) dv \right) ds,$$

as given by (4.48). Since the retiree has some illiquid wealth, the liquid wealth $W_1(t)$ can go negative, but satisfies the constraint

$$W_1(t) + I(t) \geq 0.$$

Thus, as $W_1(t) \to -I(t)$, then $c_1 \to 0$ and there is nothing for the bequest motive. Equation (7.14) then says

$$V(t, -I(t)) = -\infty.$$

To solve these equation numerically, we will employ the total wealth transformation and MCALT, as in previous models. Recall the purpose of those transformations is to move the singularity in the boundary absorbing conditions from $w = 0$ or $w = -I(t)$ to $u = -\infty$.

7.7.1 Total Wealth Transformation.

Since $\Psi_D(t, w; c_2, \pi_2)$ is independent of the income term, $\alpha(t)$, and the retiree has nothing to borrow against, this first transformation is not needed for the post-default HJB equation (7.22). However, $\Psi_V(t, w; c_1, \pi_1, \zeta)$ is dependent on $\alpha(t)$, so we will go forward with the total wealth transformation on the pre-default HJB equation (7.23) as detailed in Section 5.1.

Let

$$x = w + I(t),$$

and let $X(t)$ be the pre-default wealth process. The constraint on liquid plus illiquid wealth given in the absorbing boundary discussion then gives the constraint

$$X(t) \geq 0.$$

Rewrite (7.3) as

$$\zeta = X(t) - I(t) + \frac{p(t)}{\eta(t)},$$

with constraint

$$\zeta(t) \geq X(t) - I(t). \tag{7.24}$$

Let

$$\widetilde{V}(t,x) = V(t,w),$$

with partials

$$\begin{aligned} V_t(t,w) =& \widetilde{V}_t(t,x) + \widetilde{V}_x(t,x)\tfrac{dx}{dt}, \\ V_w(t,w) =& \widetilde{V}_x(t,x), \\ V_{ww}(t,w) =& \widetilde{V}_x(t,x), \end{aligned}$$

where

$$\frac{dx}{dt} = I'(t) = I(t)(r(t) + \eta(t)) - \alpha(t).$$

Substituting these into (7.23) yields

$$\begin{cases} 0 = \widetilde{V}_t(t,x) - (\lambda_M(t) + \lambda_D(t))\widetilde{V}(t,x) + \sup\limits_{(c_1,\pi_1,\zeta \geq x - I(t))} \widetilde{\Psi}_V(t,x;c_1,\pi_1,\zeta) \\ \widetilde{V}(T,x) = B(T,x) \\ \widetilde{V}(t,0) = -\infty, \end{cases} \tag{7.25}$$

where

$$\begin{aligned} \widetilde{\Psi}_V(t,x;c_1,\pi_1,\zeta) =& \big((r(t)+\eta(t))x - c_1(t) - \eta(t)\zeta(t) + \pi_1(t)(\mu(t) - r(t))\big)\widetilde{V}_x(t,x) \\ &+ \tfrac{1}{2}\pi_1^2(t)\sigma^2(t)\widetilde{V}_{xx}(t,x) + \lambda_M(t)B(t,\zeta(t)) + \lambda_D(t)D(t,\zeta(t)) + U(t,c_1(t)). \end{aligned}$$

Note the absorbing boundary condition has moved to $x = 0$.

7.7.2 MCALT. This transformation proceeds as in Section 5.1. Its usefulness is to remove the pre-default HJB equation's dependence on w and the post-default HJB equation's dependence on x (because of the total wealth transformation on the pre-default equation alone, the variable dependence is different). In both cases, it will push the singularity in the absorbing boundary condition to $u = -\infty$.

We begin with the post-default transformation. Let $U_2(t)$ be the new post-default wealth process resulting from the substitution

$$u = \ln(w).$$

Create the new value function

$$\widehat{D}(t,u) = D(t,w),$$

with partials

$$\begin{aligned} D_t(t,w) =& \widehat{D}_t(t,u), \\ D_w(t,w) =& e^{-u}\widehat{D}_u(t,u), \\ D_{ww}(t,w) =& e^{-2u}\big(\widehat{D}_{uu}(t,u) - \widehat{D}_u(t,u)\big). \end{aligned}$$

Similarly transform the control variables into

$$\begin{aligned} \widehat{c}_2(t) =& \frac{c_2(t)}{w} = c_2(t)e^{-u}, \\ \widehat{\pi}_2(t) =& \frac{\pi_2(t)}{w} = \pi_2(t)e^{-u}, \end{aligned}$$

which are considered fractions of total wealth.

Plugging these into (7.22) gives the updated HJB equation

$$\begin{cases} 0 = \widehat{D}_t(t,u) - \lambda_M(t)\widehat{D}(t,u) + \sup\limits_{(\widehat{c}_2,\widehat{\pi}_2)} \widehat{\Psi}_D(t,u;\widehat{c}_2,\widehat{\pi}_2) \\ \widehat{D}(T,u) = B(T,e^u) \\ \widehat{D}(t,-\infty) = -\infty, \end{cases} \tag{7.26}$$

where

$$\begin{aligned}\widehat{\Psi}_D(t,u;\widehat{c}_2,\widehat{\pi}_2) =&\big(r(t)-\widehat{c}_2(t)+\widehat{\pi}_2(t)(\mu(t)-r(t))-\tfrac{1}{2}\widehat{\pi}_2^2(t)\sigma^2(t)\big)\widehat{D}_u(t,u)\\ &+\tfrac{1}{2}\widehat{\pi}_2^2(t)\sigma^2(t)\widehat{D}_{uu}(t,u)+\lambda_M(t)B(t,e^u)+U(t,e^u\widehat{c}_2(t)).\end{aligned}$$

The absorbing boundary condition moved to $u=-\infty$ because $u\to-\infty$ as $w\to 0$.

We follow up now with the transformation on the pre-default HJB equation (7.25). Similar to before, let $U_1(t)$ be the new pre-default wealth process resulting in the substitution

$$u=\ln(x).$$

Use the transformation given in (5.5) and (5.6), and define the control variables

$$\begin{aligned}\widehat{c}_1(t) =&\frac{c_1(t)}{x}=e^{-u}c_1(t),\\ \widehat{\pi}_1(t) =&\frac{\pi_1(t)}{x}=e^{-u}\pi_1(t),\\ \widehat{\zeta}(t) =&\frac{\zeta(t)}{x}=e^{-u}\zeta(t),\end{aligned}$$

which are fractions of total wealth. This change in control results in the constraint (7.24) becoming

$$\widehat{\zeta}(t)\geq 1-e^{-u}I(t).$$

Plugging these into (7.25) gives the updated HJB equation

$$\begin{cases}0=\widehat{V}_t(t,u)-(\lambda_M(t)+\lambda_D(t))\widehat{V}(t,u)+\sup\limits_{(\widehat{c}_1,\widehat{\pi}_1,\widehat{\zeta}\geq 1-e^{-u}I(t))}\widehat{\Psi}_V(t,u;\widehat{c}_1,\widehat{\pi}_1,\widehat{\zeta})\\ \widehat{V}(T,y)=B(T,e^u)\\ \widehat{V}(t,-\infty)=-\infty,\end{cases}\tag{7.27}$$

where

$$\begin{aligned}\widehat{\Psi}_V(t,u;\widehat{c}_1,\widehat{\pi}_1,\widehat{\zeta}) =& \big(r(t)+\eta(t)-\widehat{c}_1(t)-\eta(t)\widehat{\zeta}(t)+\widehat{\pi}_1(t)(\mu(t)-r(t)) \\ &-\tfrac{1}{2}\widehat{\pi}_1^2(t)\sigma^2(t)\big)\widehat{V}_u(t,u)+\tfrac{1}{2}\widehat{\pi}_1^2(t)\sigma^2(t)\widehat{V}_{uu}(t,u) \\ &+\lambda_M(t)B(t,e^u\widehat{\zeta}(t))+\lambda_D(t)D(t,e^u\widehat{\zeta}(t))+U(t,e^u\widehat{c}_1(t)).\end{aligned}$$

Again, the absorbing boundary condition has moved to $u=-\infty$ since $u\to-\infty$ as $x\to-\infty$.

To make calculations easier, note as before

$$D(t,\zeta(t))=\widehat{D}(t,u+\ln(\widehat{\zeta}(t))).$$

Thus we get

$$\begin{aligned}\widehat{\Psi}_V(t,u;\widehat{c}_1,\widehat{\pi}_1,\widehat{\zeta}) =& \big(r(t)+\eta(t)-\widehat{c}_1(t)-\eta(t)\widehat{\zeta}(t)+\widehat{\pi}_1(t)(\mu(t)-r(t)) \\ &-\tfrac{1}{2}\widehat{\pi}_1^2(t)\sigma^2(t)\big)\widehat{V}_u(t,u)+\frac{1}{2}\widehat{\pi}_1^2(t)\sigma^2(t)\widehat{V}_{uu}(t,u) \\ &+\lambda_M(t)B(t,e^u\widehat{\zeta}(t))+\lambda_D(t)\widehat{D}(t,u+\ln(\widehat{\zeta}(t)))+U(t,e^u\widehat{c}_1(t)).\end{aligned}$$

The next section will look at the discretization of $\widehat{V}(t,u)$ and $\widehat{D}(t,u)$. As in Taylor's problem, only a finite number of u values are needed so we will disregard the absorbing boundary conditions.

7.7.3 Post-Default Discretization.

Consider where coefficients of $\widehat{D}_u(t,u)$ are negative or positive. Define

$$\widehat{b}_2(t,u)=\widehat{b}_2^+(t,u)-\widehat{b}_2^-(t,u),$$

with

$$\begin{aligned}\widehat{b}_2^+(t,u) =& r(t)+\widehat{\pi}_2(t)(\mu(t)-r(t)), \\ \widehat{b}_2^-(t,u) =& \widehat{c}_2(t)+\tfrac{1}{2}\widehat{\pi}_2^2(t)\sigma^2(t),\end{aligned}$$

both necessarily positive for all $t \in [0, T]$.

The finite difference scheme is defined in (5.21). Substituting these and rearranging gives a result similar to (5.31). We will use the form

$$\begin{aligned}\widehat{D}(t,u) =& \tfrac{\delta}{1+\delta\lambda_M(t)}\Big[\tfrac{\widehat{D}(t+\delta,u)}{\delta} + \lambda_M(t)B(t,e^u) + r(t)\widehat{D}_u^+(t,u) \\ &+ \sup_{\widehat{c}_2}\big\{-\widehat{c}_2(t)\widehat{D}_u^-(t,u) + U(t,\widehat{c}_2(t)e^u)\big\} \\ &+ \sup_{\widehat{\pi}_2}\big\{\widehat{\pi}_2(t)(\mu(t)-r(t))\widehat{D}_u^+(t,u) - \tfrac{1}{2}\widehat{\pi}_2^2(t)\sigma^2(t)\big(\widehat{D}_u^-(t,u) - \widehat{D}_{uu}(t,u)\big)\big\}\Big].\end{aligned} \tag{7.28}$$

This will be solved backwards in time as described in Section 5.3.3.

7.7.4 Pre-Default Discretization.

As above, we must consider where coefficients of $\widehat{V}_u(t,u)$ are negative or positive. Define

$$\widehat{b}_1(t,u) = \widehat{b}_1^+(t,u) - \widehat{b}_1^-(t,u),$$

with

$$\begin{aligned}\widehat{b}_1^+(t,u) &= r(t) + \eta(t) + \widehat{\pi}_1(t)(\mu(t) - r(t)),\\ \widehat{b}_1^-(t,u) &= \widehat{c}_1(t) + \eta(t)\widehat{\zeta}(t) + \tfrac{1}{2}\widehat{\pi}_1^2(t)\sigma^2(t),\end{aligned}$$

both necessarily positive for all $t \in [0, T]$.

The finite difference scheme is given in (5.20). Substituting these into (7.27) gives a result similar to (5.35). We write it in the form

$$\begin{aligned}
\widehat{V}(t,u) =& \frac{\delta}{1+\delta(\lambda_M(t)+\lambda_D(t))}\Bigg[\frac{\widehat{V}(t+\delta,u)}{\delta} + (r(t)+\eta(t))\widehat{V}_u^+(t,u) \\
&+ \sup_{\widehat{c}_1}\big\{-\widehat{c}_1(t)\widehat{V}_u^-(t,u) + U(t,e^u\widehat{c}_1(t))\big\} \\
&+ \sup_{\widehat{\pi}_1}\big\{\widehat{\pi}_1(t)(\mu(t)-r(t))\widehat{V}_u^+(t,u) - \tfrac{1}{2}\widehat{\pi}_1^2(t)\sigma^2(t)\big(\widehat{V}_u^-(t,u) - \widehat{V}_{uu}(t,u)\big)\big\} \\
&+ \sup_{\widehat{\zeta}\geq 1-e^{-u}I(t)}\big\{-\eta(t)\widehat{\zeta}(t)\widehat{V}_u^-(t,u) + \lambda_M(t)B(t,e^u\widehat{\zeta}(t)) + \lambda_D(t)\widehat{D}(t,u+\ln(\widehat{\zeta}(t)))\big\}\Bigg].
\end{aligned} \tag{7.29}$$

This is solved backwards in time as described in Section 5.3.4.

7.8 Model Parameters

Below is listed all the parameters to be used in this model.

r	μ	σ	γ	ρ	T	$\lambda_M(t), \eta_M(t)$	$\lambda_D(t), \eta_D(t)$
.01	.06	.20	-3	.03	40	$\frac{1}{d}\exp\left(\frac{R+t-c}{d}\right)$	$[0, 0.03]$

Table 7.1: Parameters used in the Matched Payout Model.

R, c, and d are defined as in Section 2.7.3.

7.9 Annuity Pricing

As before the price of the annuity is given by (3.24), where the annuities company does not take its own default into account. The retiree wishes to solve (3.26).

7.10 Computations and Results

We have already seen that no exact solution exists to this problem. Instead, we will jump to the numerical results to the discretized HJB equations. Using Python 3.6 on a 64 core supercomputer with 1GB of memory per core, The run-time to calculate over all choices of $\lambda_D(t) \in [0, 0.01, 0.02, 0.03]$ and an α-step size of 3% of $\alpha_{\max}$ is about 90 minutes. We chose $\delta = 0.01$ and $h = 0.02$ as in previous models.

We are also using the terminal conditions

$$\widehat{D}(T, u) = B(T, e^u),$$
$$\widehat{V}(T, u) = B(T, e^u),$$

provided by the HJB equations (7.26) and (7.27), respectively.

7.10.1 Numerical Solution, Unconstrained.

Using (7.28) and (7.29), without the constraint on $\widehat{\zeta}$, we are interested in solving (3.26). As in previous models, let

$$\theta = \frac{\alpha^*}{w}.$$

Then we get the following table.

λ_D	0	.01	.02	.03
θ	—	0.0	0.0	0.0

Table 7.2: Optimal annuitization as a ratio of wealth for the unconstrained Matched Payout Model. This shows no annuitization is optimal, except in the case $\lambda_D = 0$ which has no optimal level of annuitization due to short-selling.

No surprises here, since short-selling insurance is an optimal choice for the retiree in a complete market with risky annuity, and so he will take advantage by holding onto all his wealth and just short-sell as much as possible to create a risk-free instantaneous term annuity. We graph the value function for various default risks, λ_D, and annuity payment rates, α.

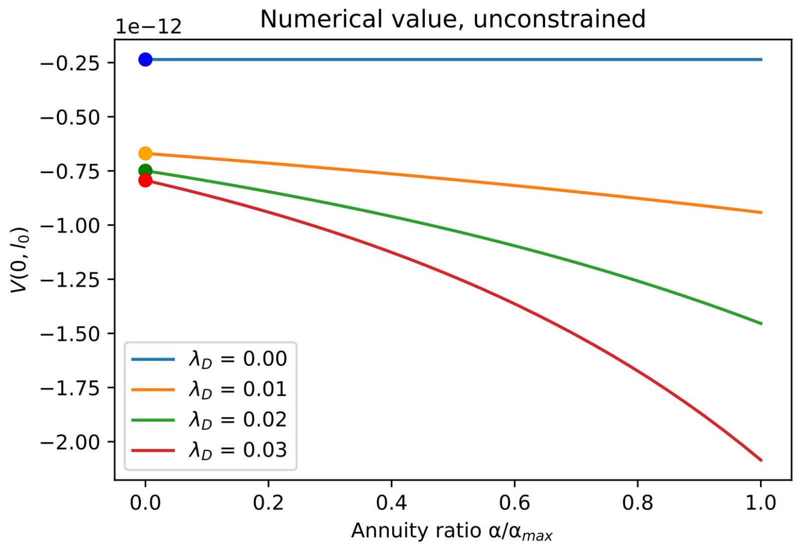

Figure 7.1: Value function for several annuity payment rates α over different default rates $\lambda_D(t)$ in the unconstrained Matched Payout Model.

Next take a look at the constrained version of the problem, which will better reflect reality.

7.10.2 Numerical Solution, Constrained.

Proceeding as before, we consider numerical solutions to (7.28) and (7.29), this time using the constraint

$$\widehat{\zeta} \geq 1 - e^{-u} I(t).$$

This time, calculations give the following.

	$\lambda_D = 0$	.01	.02	.03
θ	1.0	0.21	0.12	0.06

Table 7.3: Optimal annuitization as a ratio of wealth for the constrained Matched Payout Model. θ is much lower than in Table 5.3, an effect of the constraint on insurance payouts.

This is a significant decrease from Table 5.3. However, it makes sense that the retiree would decrease the amount to annuitize since there was the restriction on payouts to begin with, as represented by the restriction (7.1). In the Taylor Model, optimization occurs independently over $\widehat{\zeta}_M(t)$ and $\widehat{\zeta}_D(t)$. In this model, there is no separation that can occur, which impacts the annuitization amount.

Finally, the graph of $V(0, l_0)$ for various λ_D and α ratios is given in Figure 7.2. This graph shows that very little annuitization is optimal in the face of risk. This provides a quantitative solution to the annuity puzzle: despite earlier suggestions that full or near-full annuitization is optimal, it appears very little annuitization is optimal in the face of default risk. This could explain why people do not annuitize as much as some might suggest.

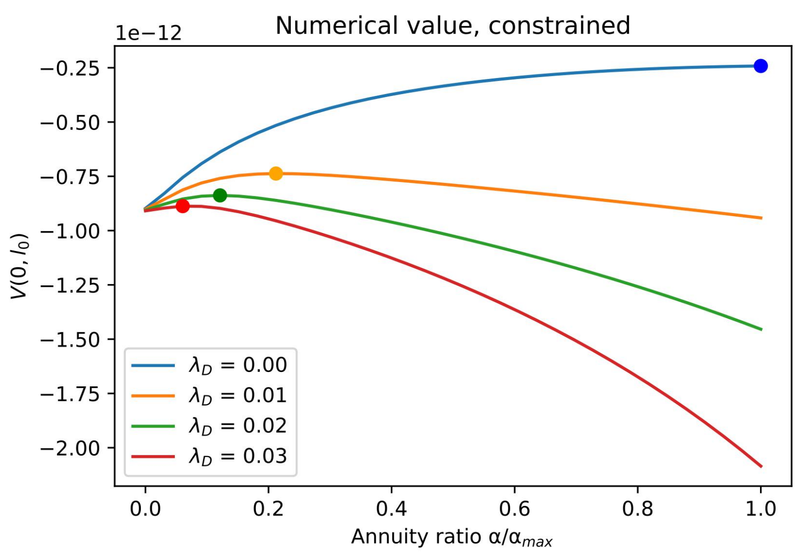

Figure 7.2: Value function for several annuity payment rates α over different default rates $\lambda_D(t)$ in the constrained Matched Payout Model. For $\lambda_D \neq 0$, very little annuitization is optimal, providing a numerical solution to the annuity puzzle.

7.10.3 Numerical Solution, Actuarially Unfair.

We conclude the chapter by considering the actuarially unfair case, with

$$\eta(t) > \lambda(t).$$

The post-default HJB is independent of $\eta(t)$, so it will remain the same in the unfair situation. The pre-default HJB is solved numerically in (7.29), and is dependent on $\eta(t)$. Consider below the last term of (7.29) with its dependency on $\eta(t)$:

$$\sup_{\widehat{\zeta}\geq 1-e^{-u}I(t)} \big\{ -\eta(t)\widehat{\zeta}(t)\widehat{V}_u^-(t,u) + \lambda_M(t)B(t,e^u\widehat{\zeta}(t)) + \lambda_D(t)\widehat{D}(t,u+\ln(\widehat{\zeta}(t)))\big\}.$$

Solve this by taking the derivative with respect to $\widehat{\zeta}$ and setting it equal to 0. This gives

$$\lambda_M(t)e^u B_w(t,\widehat{\zeta}(t)e^u) + \tfrac{\lambda_D(t)}{\widehat{\zeta}}\widehat{D}_u(t,u+\ln(\widehat{\zeta})) = \eta(t)\widehat{V}_u^-(t,u).$$

The right side goes to ∞ as $\eta(t)$ does. Similar to Section 5.3.4, if

$$\widehat{\zeta}(t) \to z > 0,$$

then the left side cannot go to ∞. Therefore we again conclude $\widehat{\zeta} \to 0$.

However, $\widehat{\zeta}(t)$ is constrained, so for large enough $\eta(t)$, then

$$\widehat{\zeta}(t) = 1 - e^{-u}I(t). \tag{7.30}$$

This also forces

$$\widehat{\zeta}(t) \to 1,$$

since

$$I(t) \to 0.$$

Plug (7.30) into (7.27) and take the limit $\eta \to \infty$. The relevant term is

$$\lim_{\eta\to\infty} \eta(t) - \eta(t)\widehat{\zeta}(t) = \lim_{\eta\to\infty} e^{-u}\eta(t)I(t).$$

We solve this limit as in Section 5.3.4 and find

$$\lim_{\eta \to \infty} e^{-u} \eta(t) I(t) = e^{-u} \alpha(t).$$

Equation (7.27) becomes

$$0 = \widehat{V}_t(t,u) - (\lambda_M(t) + \lambda_D(t))\widehat{V}(t,u) + \sup_{(\widehat{c}_1, \widehat{\pi}_1)} \widehat{\Psi}_V(t,u; \widehat{c}_1, \widehat{\pi}_1), \tag{7.31}$$

where

$$\begin{aligned} \widehat{\Psi}_V(t,u;\widehat{c}_1,\widehat{\pi}_1) =& \big(r(t) - \widehat{c}_1(t) + e^{-u}\alpha(t) + \widehat{\pi}_1(t)(\mu(t) - r(t)) \\ & - \tfrac{1}{2}\widehat{\pi}_1^2(t)\sigma^2(t)\big)\widehat{V}_u(t,u) + \frac{1}{2}\widehat{\pi}_1^2(t)\sigma^2(t)\widehat{V}_{uu}(t,u) \\ & + \lambda_M(t)B(t,e^u) + \lambda_D(t)\widehat{D}(t,u) + U(t, e^u\widehat{c}_1(t)). \end{aligned}$$

This is the actuarially unfair HJB given in (5.42). This is expected, since the Matched Payout Model is a restricted case of the Taylor Model.

Now consider the solution to (7.29) with actuarially unfair $\eta(t)$. In particular, let

$$\eta(t) = (1+\epsilon)\lambda(t),$$

where $\epsilon = .25$ and

$$\lambda(t) = \lambda_M(t) + \lambda_D(t).$$

Figure 7.3 shows the graph of the actuarially unfair model, and Figure 7.4 shows a comparison of the constrained model with $\epsilon = 0$ and $\epsilon = .25$.

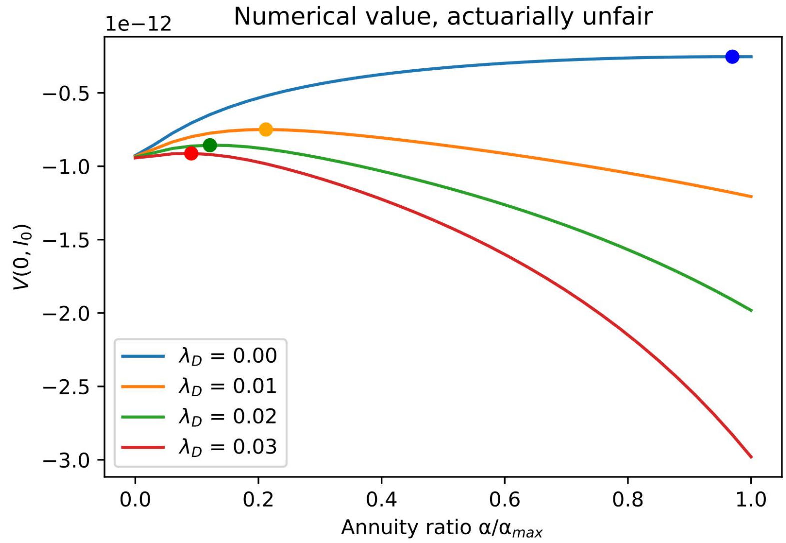

Figure 7.3: Value function for several annuity payment rates α over different default rates $\lambda_D(t)$ in the actuarially unfair Matched Payout Model. For $\lambda_D \neq 0$, very little annuitization is optimal, providing a numerical solution to the annuity puzzle.

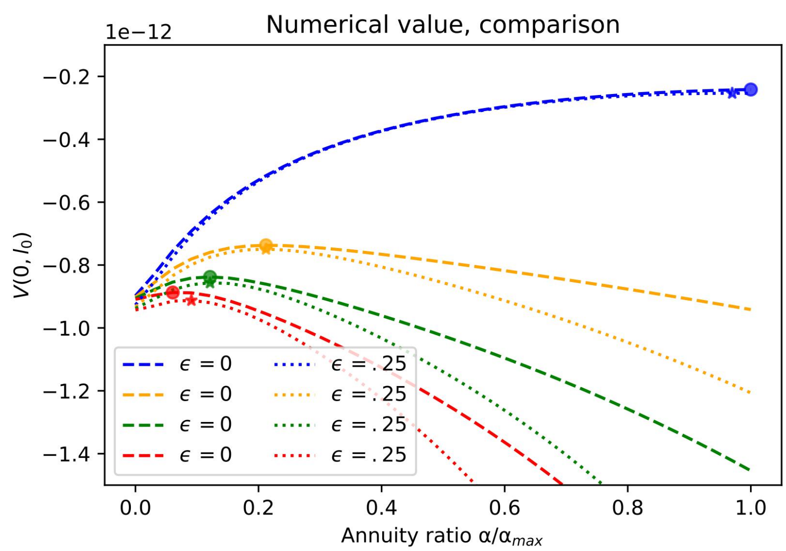

Figure 7.4: A comparison of the value function in the Matched Payout Model for $\epsilon = 0$ and $\epsilon = .25$. Notice the decrease in $V(0, l_0)$). The optimal annuity ratio increases for $\lambda_D = .3$ (red line).

We list below the updated maximized value function $V(0, l_0)$ and optimal annuity/wealth ratio, $\theta = \frac{A(\alpha^*)}{w} = \frac{\alpha^*}{\alpha_{max}}$, as compared to the constrained and unconstrained calculations. We break this into four comparisons, one table representing one value of $\lambda_D(t) \in [0, .03]$. From these tables, we do see the value functions drop as $\eta(t)$ increases.

$\lambda_D(t) = 0.0$	θ	$V(0, l_0)$
Unconstrained	0.0	-2.3573×10^{-13}
Constrained	1.0	-2.4178×10^{-13}
Unfair	.97	-2.5361×10^{-13}

Table 7.4: $V(0, l_0)$ and θ for the unconstrained, constrained, and actuarially unfair Matched Payout Model with $\lambda_D(t) = 0.0$.

$\lambda_D(t) = .01$	θ	$V(0, l_0)$
Unconstrained	0.0	-6.6921×10^{-13}
Constrained	.21	-7.3727×10^{-13}
Unfair	.21	-7.4947×10^{-13}

Table 7.5: $V(0, l_0)$ and θ for the unconstrained, constrained, and actuarially unfair Matched Payout Model with $\lambda_D(t) = .01$.

$\lambda_D(t) = .02$	θ	$V(0, l_0)$
Unconstrained	0.0	-7.4929×10^{-13}
Constrained	.12	-8.3785×10^{-13}
Unfair	.12	-8.5658×10^{-13}

Table 7.6: $V(0, l_0)$ and θ for the unconstrained, constrained, and actuarially unfair Matched Payout Model with $\lambda_D(t) = .02$.

$\lambda_D(t) = .03$	θ	$V(0, l_0)$
Unconstrained	0.0	-7.9346×10^{-13}
Constrained	.06	-8.8729×10^{-13}
Unfair	.09	-9.1251×10^{-13}

Table 7.7: $V(0, l_0)$ and θ for the unconstrained, constrained, and actuarially unfair Matched Payout Model with $\lambda_D(t) = .03$.

7.10.4 Conclusion. Unlike the results in the Taylor Model, Tables 7.4-7.7 show that little annuitization is optimal when faced with the possibility of annuity default, even when default risk is low. This provides a possible solution to the annuity puzzle, showing numerically that people should not fully or near-fully annuitize their wealth at retirement. By combining life and default insurance into a single instrument, we see a decrease in $V(0, l_0)$ compared to the Taylor Model, as expected. However, the value function for the Matched Payout Model is still an improvement over the situation where the retiree cannot purchase any insurance, as seen in the actuarially unfair model. Thus, we have created a potential insurance instrument to cover both mortality and default risk, and improve the retiree's utility of wealth by allowing for borrowing.

Chapter 8. Conclusion

8.1 The Annuity Puzzle

This paper sets out to give a possible solution to the annuity puzzle. The annuity puzzle refers to the following conundrum: if the literature says that fully annuitizing one's savings at retirement is optimal, then why are people not doing it? Many have sought to answer this through social justifications, but until now there has been little by way of numerical justification.

We have just considered four different models. The Basic Model considered the case of an uncertain lifetime, but certain income stream and no life insurance. The Richard Model added to this by allowing for the purchase of life insurance, giving nice exact and numerical solutions to the problem. The Taylor and Matched Payout Models took this a step further by allowing for annuity insolvency, and allowing for default insurance in the form of a CDS under certain conditions.

In the first three models, we saw that full or near-full annuitization was optimal. However, in the Matched Payout Model, we were able to find, numerically, that full annuitization is not optimal. In fact, in face of annuity default, very little annuitization is optimal. The problem only gets worse when the retiree is unable to purchase insurance (life or default) to hedge against these risks,. Thus, we find numerically that individuals should not fully or near-fully annuitize their savings at retirement.

8.2 A Matched Payout Model

The greatest contribution of this paper is the creation of an insurance vehicle that insurance agencies can utilize to increase retiree annuitization. The problem with credit default swaps is, they don't exactly exist. There is not a good way to hedge against annuity insolvency. The Matched Payout Model creates a new form of payout that can account for life insurance and default insurance, and protect the retiree in more ways. It turned the instantaneous life insurance purchase problem into a marketable loan repayment process, giving financial institutions a marketable loan instrument.

We see the negative impact on the value function in Tables 7.4-7.7 in an actuarially unfair situation. As retirees become risky (close to death), the cost of a loan goes up so retirees are less inclined to borrow, so their value from utility decreases.

8.3 Future Research

An individual faces many more options than just consumption, investment in two market vehicles, and insurance purchases. There are 401K's, Social Security Benefits, and other retirement packages. There are also steeper health concerns, such as a sudden hospital visit with huge expenses. We have only considered the lightest of models for a retiree's lifetime. It can be very beneficial to consider a fuller model, which takes these other financial instruments and shocks into account.

In all the models we looked at, it is hard to make a direct comparison of the retiree's value, since the models are designed in different ways. It would be interesting to find numerical solutions to models where the retiree has a bequest motive, makes investment and consumption choices, and does not purchase life insurance, with both a defaultable and non-defaultable annuity. That result is something that is hinted at by the actuarially unfair models, with $\eta(t) \to \infty$, but is too unstable to calculate. One option is to consider applying machine learning techniques to the basic models discussed (no insurance premiums), such as reinforcement learning By possibly looking at Martingales, similar to Ye in [11], we may be able to work out those solutions for true comparisons of the retiree's value function. In this way, we would truly see the advantage that life and default insurance has on the retiree's wealth function.

Finally, Ye and Pliska [12] discussed the possibility of an optimal loading factor for the actuarially unfair Richard Model, which would allow life insurance companies to maximize profits from loaded life insurance premiums. This concept could be extended to the Matched Payout Model to help financial institutions maximize profit from the new insurance instrument used in that model. This would make the model even more marketable and enticing for insurance companies to implement.

Appendix A. Proofs and Derivations

A.1 Proof of Lemma 2.1

Note that

$$\int_t^{T\wedge\tau} U(s,c(s))ds = \mathbb{1}_{\{\tau\geq T\}}\int_t^T U(s,c(s))ds + \mathbb{1}_{\{\tau<T\}}\int_t^\tau U(s,c(s))ds.$$

Start by rewriting (2.15) as

$$\begin{aligned} J(t,w;c,\pi) =& E\Bigg[\mathbb{1}_{\{\tau\geq T\}}\int_t^T U(s,c(s))ds + \mathbb{1}_{\{\tau<T\}}\int_t^\tau U(s,c(s))ds \\ & + B(\tau,W(\tau))\mathbb{1}_{\{\tau<T\}} + L(T,W(T))\mathbb{1}_{\{\tau\geq T\}}\Bigg|\tau\geq t,\mathcal{F}_t\Bigg]. \end{aligned}$$

Assuming $\tau \geq T$ implies that the time domain for $U(s,c(s))$ is $s\in[t,T]$. Then

$$E\big[\mathbb{1}_{\{\tau\geq T\}}\big|\tau\geq t\big] = E\left[\frac{\overline{F}(T)}{\overline{F}(t)}\right] = E\big[\overline{F}(T|t)\big],$$

and so

$$E\big[L(T,W(T))\mathbb{1}_{\{\tau\geq T\}}\big|\tau\geq t\big] = E\big[L(T,W(T))\overline{F}(T|t)\big].$$

For $\tau < T$, we necessarily have

$$\begin{aligned} E\left[\mathbb{1}_{\{\tau<T\}}\int_t^\tau U(s,c(s))ds\middle|\tau\geq t\right] =& E\left[\frac{\int_t^T\left(f(u)\int_t^u U(s,c(s))ds\right)du}{\overline{F}(t)}\right] \\ =& E\left[\int_t^T\left(f(u|t)\int_t^u U(s,c(s))ds\right)du\right]. \end{aligned}$$

Thus

$$\begin{aligned} J(t,w;c,\pi) =& E\Bigg[\overline{F}(T|t)\int_t^T U(s,c(s))ds + \int_t^T\left(f(u|t)\int_t^u U(s,c(s))ds\right)du \\ & + \int_t^T f(u|t)B(u,W(u))du + \overline{F}(T|t)L(T,W(T))\Bigg|\mathcal{F}_t\Bigg]. \end{aligned} \tag{A.1}$$

Taking the second term in (A.1), use Fubini's theorem to interchange the order of integration as follows:

$$\begin{aligned}\int_t^T \int_t^u f(u|t)U(s,c(s))dsdu &= \int_t^T \int_s^T f(u|t)U(s,c(s))duds \\ &= \int_t^T \left(\int_s^T f(u|t)du\right) U(s,c(s))ds \\ &= \int_t^T \left(\overline{F}(s|t) - \overline{F}(T|t)\right) U(s,c(s))ds, \qquad \text{(A.2)}\end{aligned}$$

where the last step was obtained by

$$\begin{aligned}\int_s^T f(u|t)du =& \frac{1}{\overline{F}(t)} \int_s^T f(u)du \\ =& \frac{1}{\overline{F}(t)}(F(T) - F(s)) \\ =& \frac{1}{\overline{F}(t)}(\overline{F}(s) - \overline{F}(T)) \\ =& \overline{F}(s|t) - \overline{F}(T|t).\end{aligned}$$

The final step of the proof is to combine (A.1) with (A.2) to get (2.16).

A.2 Proof of Dynamic Programming Principle, Lemma 2.2

The proof outlined here is a simplification of the proof given by Ye in [11].

Recall $\mathcal{A}(t,w)$ is the set of all admissible tuples (c,π) for given liquid wealth w and time t. Then for any $(c,\pi) \in \mathcal{A}(t,w)$ with corresponding wealth $W_{t,w}^{c,\pi}(\cdot)$, Lemma 2.1 gives

$$
\begin{aligned}
J(t,w;c,\pi) =& E\Big[\int_t^T \big(f(u|t)B(u,W_{t,w}^{c,\pi}(u)) + \overline{F}(u|t)U(u,c(u))\big)du \\
&+ \overline{F}(T|t)L(T,W_{t,w}^{c,\pi}(T))\Big|\mathcal{F}_t\Big] \\
=& E\Big[\int_s^T \big(f(u|t)B(u,W_{t,w}^{c,\pi}(u)) + \overline{F}(u|t)U(u,c(u))\big)du + \overline{F}(T|t)L(T,W_{t,w}^{c,\pi}(T)) \\
&+ \int_t^s \big(f(u|t)B(u,W_{t,w}^{c,\pi}(u)) + \overline{F}(u|t)U(u,c(u))\big)du\Big|\mathcal{F}_t\Big],
\end{aligned} \tag{A.3}
$$

where we simply split the integral into two parts at time $s \in [t,T]$. From (2.9) we get

$$
\begin{aligned}
f(u|t) &= \lambda(u)\exp\Big(-\int_t^u \lambda(v)dv\Big) \\
&= \exp\Big(-\int_t^s \lambda(v)dv\Big)\lambda(u)\exp\Big(-\int_s^u \lambda(v)dv\Big) \\
&= \exp\Big(-\int_t^s \lambda(v)dv\Big)f(u|s),
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\overline{F}(u|t) &= \exp\Big(-\int_t^u \lambda(v)dv\Big) \\
&= \exp\Big(-\int_t^s \lambda(v)dv\Big)\exp\Big(-\int_s^u \lambda(v)dv\Big) \\
&= \exp\Big(-\int_t^s \lambda(v)dv\Big)\overline{F}(u|s).
\end{aligned}
$$

Plugging these into (A.3) and factoring out the exponential gives

$$
\begin{aligned}
&J(t,w;c,\pi) = \\
&\quad E\Big[\exp\Big(-\int_t^s \lambda(v)dv\Big)\Big\{\int_s^T \big(f(u|s))B(u,W_{t,w}^{c,\pi}(u)) + \overline{F}(u|t)U(u,c(u))\big)du \\
&\quad + \overline{F}(T|s)L(T,W_{t,w}^{c,\pi}(T))\Big\} + \int_t^s \big(f(u|t)B(u,W_{t,w}^{c,\pi}(u)) + \overline{F}(u|t)U(u,c(u)\big)\,du\Big|\mathcal{F}_t\Big].
\end{aligned} \tag{A.4}
$$

Next we use the Markov property on $W^{c,\pi}_{t,w}(\cdot)$ which gives

$$\begin{aligned}
E\Big[L(T,W^{c,\pi}_{t,w}(T))\Big|\mathcal{F}_t\Big] &= E\Big[L(T,W^{c,\pi}_{t,w}(T))\Big|\mathcal{F}_s\Big|\mathcal{F}_t\Big] \\
&= E\Big[L(T,W^{c,\pi}_{s,W_{t,w}(s)}(T))\Big|\mathcal{F}_t\Big].
\end{aligned}$$

Also, (c,π) restricted to $[s,T]$ is inherently in $\mathcal{A}(s,W^{c,\pi}_{t,w}(s))$. Then from (A.4) we rewrite the terms in curly brackets as $J(s,W^{c,\pi}_{t,w}(s);c,\pi)$ and get

$$\begin{aligned}
J(t,w;c,\pi) =& E\Big[\exp\Big(-\int_t^s \lambda(v)dv\Big)J(s,W^{c,\pi}_{t,w}(s);c,\pi) \\
&+\int_t^s \big(f(u|t)B(u,W^{c,\pi}_{t,w}(u))+\overline{F}(u|t)U(u,c(u))\big)du\Big|\mathcal{F}_t\Big] \\
\leq& E\Big[\exp\Big(-\int_t^s \lambda(v)dv\Big)V(s,W^{c,\pi}_{t,w}(s)) \\
&+\int_t^s \big(f(u|t)B(u,W^{c,\pi}_{t,w}(u))+\overline{F}(u|t)U(u,c(u))\big)du\Big|\mathcal{F}_t\Big],
\end{aligned}$$

where

$$J(t,w;c,\pi) \leq \sup_{(c,\pi)\in\mathcal{A}(t,w)} J(t,w;c,\pi) = V(t,w).$$

Since (c,π) is arbitrary, then similar to the last argument above,

$$\begin{aligned}
V(t,w) \leq \sup_{(c,\pi)\in\mathcal{A}(t,w)} & E\Big[\exp\Big(-\int_t^s \lambda(v)dv\Big)V(s,W^{c,\pi}_{t,w}(s)) \\
&+\int_t^s \big(f(u|t)B(u,W^{c,\pi}_{t,w}(u))+\overline{F}(u|t)U(u,c(u))\big)du\Big|\mathcal{F}_t\Big].
\end{aligned} \tag{A.5}$$

Next we will prove the reverse inequality. Given $(c,\pi)\in\mathcal{A}(t,w)$, for $\epsilon>0$ and $\omega\in\Omega$, the property of supremums says there is

$$l_{\omega,\epsilon} = (c_{\omega,\epsilon},\pi_{\omega,\epsilon}) \in \mathcal{A}(s,W^{c,\pi}_{t,w}(s))$$

such that

$$J(s,W^{c,\pi}_{t,w}(s);l_{\omega,\epsilon}) \geq V(s,W^{c,\pi}_{t,w}(s)) - \epsilon.$$

Next let

$$\bar{l}(u) = \begin{cases} (c(u), \pi(u)) & \text{if } u \in [t, s], \\ l_{\omega,\epsilon}(u) & \text{if } u \in [s, T]. \end{cases}$$

For ϵ small, the ending wealth at T should be equal whether we start at t or s. Hence,

$$W^{l_{\omega,\epsilon}}_{s,W^{c,\pi}_{t,w}(s)}(T) = W^{\bar{l}}_{t,W}(T).$$

From (A.4) we can say

$$\begin{aligned}
V(t,w) \geq & J(t,w;\bar{l}) \\
= & E\bigg[\exp\bigg(-\int_t^s \lambda(v)dv\bigg)\bigg\{\int_s^T \big(f(u|s)B(u, W^{l_{\omega,\epsilon}}_{t,w}(u)) + \overline{F}(u|s)U(u, c_{\omega,\epsilon}(u))\big)du \\
& + \overline{F}(t|s)L\big(T, W^{l_{\omega,\epsilon}}_{s,W^{c,\pi}_{t,w}(s)}(T)\big)\bigg\} \\
& + \int_t^s \big(f(u|t)B(u, W^{c,\pi}_{t,w}(u)) + \overline{F}(u|t)U(u, c(u))\big)du\bigg|\mathcal{F}_t\bigg] \\
\geq & E\bigg[\exp\bigg(-\int_t^s \lambda(v)dv\bigg)\big(V(t, W^{c,\pi}_{t,w}(t)) - \epsilon\big) \\
& + \int_t^s \big(f(u|t)B(u, W^{c,\pi}_{t,w}(u)) + \overline{F}(u|t)U(u, c(u))\big)du\bigg|\mathcal{F}_t\bigg].
\end{aligned}$$

This inequality holds for any $(c, \pi) \in \mathcal{A}(t, w)$ and $\epsilon > 0$, so

$$\begin{aligned}
V(t,w) \geq \sup_{(c,\pi)\in\mathcal{A}(t,w)} & E\bigg[\exp\bigg(-\int_t^s \lambda(v)dv\bigg)V(s, W^{c,\pi}_{t,w}(s)) \\
& + \int_t^s \big(f(u|t)B(u, W^{c,\pi}_{t,w}(u)) + \overline{F}(u|t)U(u, c(u))\big)du\bigg|\mathcal{F}_t\bigg].
\end{aligned} \tag{A.6}$$

Combining (A.5) and (A.6) gives the result.

A.3 Proof of Dynamic Programming Equation, Theorem 2.3

Let $s = t + h$ in the Dynamic Programming Principle, Lemma 2.1. Recall the liquid wealth $W(t)$ satisfies (2.12). Let

$$\begin{aligned}\beta(t) =& r(t)W(t) - c(t) + \alpha(t) + \pi(t)(\mu(t) - r(t)),\\ \phi(t) =& \sigma(t)\pi(t),\end{aligned}$$

and rewrite (2.12) as

$$dW(t) = \beta(t)dt + \phi(t)d\mathcal{W}(t).$$

The Taylor expansion of $V(t, W(t))$ about t and $W(t)$ gives Itô's Lemma:

$$\begin{aligned}dV =& V_t dt + V_W dW + \tfrac{1}{2}V_{WW}(dW)^2 + error\\ =& V_t dt + V_W(\beta(t)dt + \phi(t)d\mathcal{W}(t)) + \tfrac{1}{2}V_{WW}(\beta(t)dt + \phi(t)d\mathcal{W}(t))^2 + error\\ =& V_t dt + \beta(t)V_W dt + \phi(t)V_W d\mathcal{W}(t) + \tfrac{1}{2}V_{WW}\big(\beta^2(t)(dt)^2 + 2\beta(t)\phi(t)dtd\mathcal{W}(t)\\ &+ \phi^2(t)(d\mathcal{W}(t))^2\big) + error\\ =& V_t dt + \beta(t)V_W dt + \phi(t)V_W d\mathcal{W}(t) + \tfrac{1}{2}\phi^2(t)V_{WW}(d\mathcal{W}(t))^2\\ =& V_t dt + \beta(t)V_W dt + \phi(t)V_W d\mathcal{W}(t) + \tfrac{1}{2}\phi^2(t)V_{WW}dt\\ =& \big(V_t + \beta(t)V_W + \tfrac{1}{2}\phi^2(t)V_{WW}\big)dt + \phi(t)V_W d\mathcal{W}(t),\end{aligned}$$

where $(d\mathcal{W}(t))^2 = dt$ and we discard any terms with power of dt greater than 1. Integrating from t to $t + h$ gives

$$\begin{aligned}V(t+h,&W(t+h)) - V(t, W(t)) =\\ &\int_t^{t+h}\Big\{V_t(s, W(s)) + \beta(s)V_W(s, W(s)) + \tfrac{1}{2}\phi^2(s)V_{WW}(s, W(s))\Big\}ds\\ &+ \int_t^{t+h}\phi(s)V_W(s, W(s))d\mathcal{W}(s).\end{aligned}$$

Rearranging gives

$$
\begin{aligned}
V(t+h, W(t+h)) = & \\
& V(t, W(t)) + \int_t^{t+h} \Big\{ V_t(s, W(s)) + \beta(s) V_w(s, W(s)) + \tfrac{1}{2}\phi^2(s) V_{ww}(s, W(s)) \Big\} ds \\
& + \int_t^{t+h} \phi(s) V_w(s, W(s)) d\mathcal{W}(s).
\end{aligned}
\tag{A.7}
$$

Next, note for small h that the Taylor expansion of

$$
\exp\Big(- \int_t^{t+h} \lambda(v) dv \Big)
$$

gives

$$
\exp\Big(- \int_t^{t+h} \lambda(v) dv \Big) = 1 - \lambda(t) h + \mathcal{O}(h^2).
$$

Then according to Lemma 2.2, after subtracting $V(t, w)$ from both sides, we get

$$
\begin{aligned}
0 = \sup_{(c,\pi) \in \mathcal{A}(t, W(t))} & E\Big[(1 - \lambda(t) h + \mathcal{O}(h^2)) V(t+h, W(t+h)) - V(t, w) \\
& + \int_t^{t+h} \big(f(u|t) B(u, W(u)) + \overline{F}(u|t) U(u, c(u)) \big) du \Big| \mathcal{F}_t \Big].
\end{aligned}
$$

Plug (A.7) into this equation to get

$$
\begin{aligned}
0 = \sup_{(c,\pi) \in \mathcal{A}(t, W(t))} & E\Big[(1 - \lambda(t) h + \mathcal{O}(h^2)) \Big(V(t, W(t)) + \int_t^{t+h} \Big\{ V_t(s, W(s)) \\
& + \beta(s) V_w(s, W(s)) + \tfrac{1}{2}\phi^2(s) V_{ww}(s, W(s)) \Big\} ds + \int_t^{t+h} \phi(s) V_w(s, W(s)) d\mathcal{W}(s) \Big) \\
& - V(t, w) + \int_t^{t+h} \big(f(u|t) B(u, W(u)) + \overline{F}(u|t) U(u, c(u)) \big) du \Big| \mathcal{F}_t \Big].
\end{aligned}
$$

Distributing gives

$$\begin{aligned}0 = \sup_{(c,\pi)\in\mathcal{A}(t,W(t))} E\Bigg[& V(t,W(t)) + \int_t^{t+h} \Big\{ V_t(s,W(s)) + \beta(s)V_w(s,W(s)) \\ & + \tfrac{1}{2}\phi^2(s)V_{ww}(s,W(s))\Big\} ds + \int_t^{t+h} \phi(s)V_w(s,W(s))d\mathcal{W}(s) - \lambda(t)hV(t,W(t)) \\ & - \lambda(t)h\int_t^{t+h} \Big\{ V_t(s,W(s)) + \beta(s)V_w(s,W(s)) + \tfrac{1}{2}\phi^2(s)V_{ww}(s,W(s))\Big\} ds \\ & - \lambda(t)h\int_t^{t+h} \phi(s)V_w(s,W(s))d\mathcal{W}(s) - V(t,w) \\ & + \int_t^{t+h} \big(f(u|t)B(u,W(u)) + \overline{F}(u|t)U(u,c(u))\big)du + \mathcal{O}(h^2)\Bigg|\mathcal{F}_t\Bigg].\end{aligned}$$

Note that

$$E\big[d\mathcal{W}(t)\big] = 0,$$

so

$$E\bigg[\int_t^{t+h} V_w(s,W(s))\phi(s)d\mathcal{W}(s)\bigg] = 0.$$

Next, multiply by $1/h$ and take the limit as $h \to 0$ to get

$$\begin{aligned}0 = \sup_{(c,\pi)\in\mathcal{A}(t,W(t))} E\Bigg[& V_t(t,W(t)) + \beta(t)V_w(t,W(t)) + \tfrac{1}{2}\phi^2(t)V_{ww}(t,W(t)) \\ & - \lambda(t)V(t,W(t)) + f(t|t)B(t,W(t)) + \overline{F}(t|t)U(t,c(t))\Bigg|\mathcal{F}_t\Bigg].\end{aligned}$$

By (2.9) and (2.10),

$$\begin{aligned}f(t|t) &= \lambda(t), \\ \overline{F}(t|t) &= 1.\end{aligned}$$

Thus,

$$
\begin{aligned}
0 = & \sup_{(c,\pi)\in\mathcal{A}(t,W(t))} E\Big[V_t(t,W(t)) + V_w(t,W(t))\beta(t) + \tfrac{1}{2}V_{ww}(t,W(t))\phi^2(t) \\
& - \lambda(t)V(t,W(t)) + \lambda(t)B(t,W(t)) + U(t,c(t))\Big|\mathcal{F}_t\Big] \\
= & V_t(t,W(t)) - \lambda(t)V(t,W(t)) + \sup_{(c,\pi)\in\mathcal{A}(t,W(t))} \Psi(t,W(t);c,\pi) \\
= & V_t(t,w) - \lambda(t)V(t,w) + \sup_{(c,\pi)\in\mathcal{A}(t,w)} \Psi(t,w;c,\pi)
\end{aligned}
$$

as desired.

A.4 Proof of Verification Theorem 2.4

Let $(c,\pi)\in\mathcal{A}(t,w)$ be arbitrary with corresponding wealth process $W(t)$. As in the proof for Theorem 2.3, let

$$
\begin{aligned}
\beta(t) = & r(t)W(t) - c(t) + \alpha(t) + \pi(t)(\mu(t) - r(t)), \\
\phi(t) = & \sigma(t)\pi(t),
\end{aligned}
$$

and apply Itô's Lemma to

$$
\mathcal{V}(s,W(s)) = \exp\Big(-\int_t^s \lambda(v)dv\Big)V(s,W(s)).
$$

Note in this instance that

$$
\begin{aligned}
\mathcal{V}_w(s,W(s)) = & \exp\Big(-\int_t^s \lambda(v)dv\Big)V_w(s,W(s)), \\
\mathcal{V}_{ww}(s,W(s)) = & \exp\Big(-\int_t^s \lambda(v)dv\Big)V_{ww}(s,W(s)), \\
\mathcal{V}_t(s,W(s)) = & \exp\Big(-\int_t^s \lambda(v)dv\Big)V_t(s,W(s)) + V(s,W(s))\exp\Big(-\int_t^s \lambda(v)dv\Big)(-\lambda(s)).
\end{aligned}
$$

Itô's Lemma gives

$$d\mathcal{V} = \big(\mathcal{V}_t + \beta(t)\mathcal{V}_w + \tfrac{1}{2}\phi^2(t)\mathcal{V}_{ww}\big)dt + \phi(t)\mathcal{V}_w d\mathcal{W}.$$

Integrating $d\mathcal{V}$ from t to T gives

$$\begin{aligned}\int_t^T d\mathcal{V}(s, W(s))ds =& \mathcal{V}(T, W(T)) - \mathcal{V}(t, W(t))\\ =& \exp\bigg(-\int_t^T \lambda(v)dv\bigg)V(T, W(T)) - V(t, W(t))\\ =& \exp\bigg(-\int_t^T \lambda(v)dv\bigg)L(T, W(T)) - V(t, W(t)). \qquad \text{(from (2.17))}\end{aligned}$$

Some algebra reveals

$$\begin{aligned}V(t, w) =& \exp\bigg(-\int_t^T \lambda(v)dv\bigg)L(T, W(T)) - \int_t^T \exp\bigg(-\int_t^u \lambda(v)dv\bigg)\bigg\{V_t(u, W(u))\\ &+ \beta(t)V_w(u, W(u)) - \lambda(u)V(u, W(u)) + \tfrac{1}{2}\phi^2(t)V_{ww}(u, W(u))\bigg\}du\\ &- \int_t^T \exp\bigg(-\int_t^u \lambda(v)dv\bigg)\phi(t)V_w(u, W(u))d\mathcal{W}(u).\end{aligned}$$

Take the expectation to get

$$\begin{aligned}V(t, w) =& E\bigg[\exp\bigg(-\int_t^T \lambda(v)dv\bigg)L(T, W(T)) - \int_t^T \exp\bigg(-\int_t^u \lambda(v)dv\bigg)\bigg\{V_t(u, W(u))\\ &+ \beta(t)V_w(u, W(u)) - \lambda(u)V(u, W(u)) + \tfrac{1}{2}\phi^2(t)V_{ww}(u, W(u))\bigg\}du\bigg]\\ =& E\bigg[\exp\bigg(-\int_t^T \lambda(v)dv\bigg)L(T, W(T))\\ &+ \int_t^T \big(f(u|t)B(u, W(u)) + \overline{F}(u|t)U(u, c(u))\big)du\\ &- \int_t^T \big(f(u|t)B(u, W(u)) + \overline{F}(u|t)U(u, c(u))\big)du\\ &- \int_t^T \exp\bigg(-\int_t^u \lambda(v)dv\bigg)\bigg\{V_t(u, W(u))\\ &+ \beta(t)V_w(u, W(u)) - \lambda(u)V(u, W(u)) + \tfrac{1}{2}\phi^2(t)V_{ww}(u, W(u))\bigg\}du\bigg]\end{aligned}$$

$$
\begin{aligned}
=&J(t,w;c,\pi) - E\Big[\int_t^T \exp\Big(-\int_t^u \lambda(v)dv\Big)\Big\{\lambda(u)B(u,W(u)) + U(u,c(u)) \\
&+ \beta(t)V_w(u,w(u)) + \tfrac{1}{2}\phi^2(t)V_{ww}(u,W(u)) \\
&+ V_t(u,W(u)) - \lambda(u)V(u,W(u))\Big\}du\Big] \\
=&J(t,w;c,\pi) - E\Big[\int_t^T \exp\Big(-\int_t^u \lambda(v)dv\Big)\Big\{\Psi(u,W(u);c,\pi) \\
&+ V_t(u,W(u)) - \lambda(u)V(u,W(u))\Big\}du\Big] \\
\geq&J(t,w;c,\pi) - E\Big[\int_t^T \exp\Big(-\int_t^u \lambda(v)dv\Big)\Big\{\sup_{(c,\pi)\in\mathcal{A}(t,w)} \Psi(u,W(u);c,\pi) \\
&+ V_t(u,W(u)) - \lambda(u)V(u,W(u))\Big\}du\Big] \\
=&J(t,w;c,\pi) - E\Big[\int_t^T \exp\Big(-\int_t^u \lambda(v)dv\Big)\Big\{0\Big\}du\Big] \\
=&J(t,w;c,\pi).
\end{aligned}
$$

Thus we get (2.18). Next we choose $(c^*, \pi^*) \in \mathcal{A}(t,w)$ and use the result above to get

$$
\begin{aligned}
V(t,w) =&J(t,w;c^*,\pi^*) - E\Big[\int_t^T \exp\Big(-\int_t^u \lambda(v)dv\Big)\Big\{V_t(u,W^*(u)) \\
&- \lambda(u)V(u,W^*(u)) + \Psi(u,W^*(u);c^*,\pi^*)\Big\}du\Big].
\end{aligned}
$$

This, together with the fact that $V(t,w) \geq J(t,w;c^*,\pi^*)$ implies

$$
V_t(u,W^*(u)) - \lambda(u)V(u,W^*(u)) + \Psi(u,W^*(u);c^*,\pi^*) \leq 0.
$$

This gives equality if and only if (c^*, π^*) is optimal, and the second part of the theorem is proved.

A.5 Derivation of (2.27)

$$
\begin{aligned}
0 =& V_t(t,w) + \tfrac{1}{2}(\pi^*)^2\sigma^2(t)V_{ww}(t,w) + e^{-\rho t}\frac{(c^*)^\gamma}{\gamma} + \lambda(t)B(t,w) - \lambda(t)V(t,w) \\
&+ \big(\alpha(t) - c^* + r(t)w + \pi^*(\mu(t)-r(t))\big)V_w(t,w) \\
=& V_t(t,w) + \frac{e^{-\rho t}}{\gamma}\big(V_w(t,w)e^{\rho t}\big)^{\gamma/(\gamma-1)} + \lambda(t)B(t,w) - \lambda(t)V(t,w) \\
&+ \frac{1}{2}\bigg(\frac{(\mu(t)-r(t))V_w(t,w)}{\sigma^2(t)V_{ww}(t,w)}\bigg)^2\sigma^2(t)V_{ww}(t,w) + \bigg[\alpha(t) - \big(V_W(t,w)e^{\rho t}\big)^{1/(\gamma-1)} + r(t)w \\
&- \frac{(\mu(t)-r(t))V_w(t,w)}{\sigma^2(t)V_{ww}(t,w)}(\mu(t)-r(t))\bigg]V_w(t,w) \\
=& V_t(t,w) + \frac{1}{\gamma}V_w(t,w)^{\gamma/(\gamma-1)}(e^{\rho t})^{1/(\gamma-1)} + \lambda(t)B(t,w) - \lambda(t)V(t,w) + \alpha(t)V_w(t,w) \\
&+ r(t)wV_w(t,w) - (V_w(t,w))^{\gamma/(\gamma-1)}(e^{\rho t})^{1/(\gamma-1)} - \frac{1}{2}\bigg(\frac{\mu(t)-r(t)}{\sigma(t)}\bigg)^2\frac{V_w^2(t,w)}{V_{ww}(t,w)} \\
=& V_t(t,w) + \frac{1}{\gamma}V_w(t,w)^{\gamma/(\gamma-1)}(e^{\rho t/\gamma})^{\gamma/(\gamma-1)} + \lambda(t)B(t,w) - \lambda(t)V(t,w) + \alpha(t)V_w(t,w) \\
&+ r(t)wV_w(t,w) - (V_w(t,w))^{\gamma/(\gamma-1)}(e^{\rho t/\gamma})^{\gamma/(\gamma-1)} - \frac{1}{2}\bigg(\frac{\mu(t)-r(t)}{\sigma(t)}\bigg)^2\frac{V_w^2(t,w)}{V_{ww}(t,w)} \\
=& \frac{1-\gamma}{\gamma}\big(e^{\rho t/\gamma}V_w(t,w)\big)^{\gamma/(\gamma-1)} + \lambda(t)B(t,w) + V_t(t,w) - \lambda(t)V(t,w) \\
&+ (\alpha(t) + r(t)w)V_w(t,w) - \frac{1}{2}\bigg(\frac{\mu(t)-r(t)}{\sigma(t)}\bigg)^2\frac{V_w^2(t,w)}{V_{ww}(t,w)}.
\end{aligned}
$$

A.6 Derivation of (3.17)

$$
\begin{aligned}
0 =& V_t(t,w) - \lambda(t)V(t,w) + \tfrac{1}{2}(\pi^*)^2\sigma^2(t)V_{ww}(t,w) + U(t,c^*) + \lambda(t)B(t,\zeta^*) \\
&+ \big(\alpha(t) - c^* - \eta(t)(\zeta^* - w) + r(t)w + \pi^*(\mu(t)-r(t))\big)V_w(t,w) \\
=& V_t(t,w) - \lambda(t)V(t,w) + \tfrac{1}{2}(\pi^*)^2\sigma^2(t)V_{ww}(t,w) + e^{-\rho t}\frac{(c^*)^\gamma}{\gamma} + \lambda(t)e^{-\rho t}\frac{(\zeta^*)^\gamma}{\gamma} \\
&+ \big(\alpha(t) - c^* - \eta(t)(\zeta^* - w) + r(t)w + \pi^*(\mu(t)-r(t))\big)V_w(t,w) \\
=& V_t(t,w) - \lambda(t)V(t,w) + \frac{1}{2}\bigg(\frac{(\mu(t)-r(t))V_w(t,w)}{\sigma^2(t)V_{ww}(t,w)}\bigg)^2\sigma^2(t)V_{ww}(t,w) \\
&+ \frac{e^{-\rho t}}{\gamma}\big(V_w(t,w)e^{\rho t}\big)^{\gamma/(\gamma-1)} + \frac{\lambda(t)}{\gamma}e^{-\rho t}\bigg(\frac{\eta(t)}{\lambda(t)}e^{\rho t}V_w(t,w)\bigg)^{\gamma/(\gamma-1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\alpha(t)-\big(V_w(t,w)e^{\rho t}\big)^{1/(\gamma-1)}-\eta(t)\bigg(\frac{\eta(t)}{\lambda(t)}e^{\rho t}V_w(t,w)\bigg)^{1/(\gamma-1)}\right.\\
&\left.+\eta(t)w+r(t)w-\frac{(\mu(t)-r(t))V_w(t,w)}{\sigma^2(t)V_{ww}(t,w)}(\mu(t)-r(t))\right]V_w(t,w)\\
=&V_t(t,w)-\lambda(t)V(t,w)+\big(\alpha(t)+(\eta(t)+r(t))w\big)V_w(t,w)\\
&-\frac{1}{2}\bigg(\frac{\mu(t)-r(t)}{\sigma(t)}\bigg)^2\frac{V_w^2(t,w)}{V_{ww}(t,w)}+\left[\frac{1}{\gamma}(e^{\rho t})^{1/(\gamma-1)}-(e^{\rho t})^{1/(\gamma-1)}\right.\\
&\left.+\bigg(\frac{\lambda(t)^{1/(1-\gamma)}}{\eta(t)^{\gamma/(1-\gamma)}}\bigg)\bigg(\frac{1}{\gamma}(e^{\rho t})^{1/(\gamma-1)}-(e^{\rho t})^{1/(\gamma-1)}\bigg)\right]V_w(t,w)^{\gamma/(\gamma-1)}\\
=&V_t(t,w)-\lambda(t)V(t,w)+\big(\alpha(t)+(\eta(t)+r(t))w\big)V_w(t,w)\\
&-\frac{1}{2}\bigg(\frac{\mu(t)-r(t)}{\sigma(t)}\bigg)^2\frac{V_w^2(t,w)}{V_{ww}(t,w)}+\frac{1-\gamma}{\gamma}e^{\rho t/(\gamma-1)}\left[\frac{\lambda(t)^{1/(1-\gamma)}}{\eta(t)^{\gamma/(1-\gamma)}}+1\right]V_w(t,w)^{\gamma/(\gamma-1)}.
\end{aligned}
$$

A.7 Proof of Lemma 4.5

Apply the expectation operator to (4.21) to get

$$
\begin{aligned}
V(t,w)=&\sup_{(c,\pi,\zeta_M,\zeta_D)}E\bigg[\int_t^{\tau_M\wedge T\wedge\tau_D}U(s,c(s))ds+B(\tau_M,\zeta_M(\tau_M))\mathbb{1}_{\tau_M<T\wedge\tau_D}\\
&+B(T,W(T))\mathbb{1}_{T\le\tau_M\wedge\tau_D}+D(\tau_D,\zeta_D(\tau_D))\mathbb{1}_{\tau_D<T\wedge\tau_M}\bigg]\\
=&\sup_{(c,\pi,\zeta_M,\zeta_D)}E\bigg[\int_t^T U(s,c(s))\mathbb{1}_{s\le\tau_D}\mathbb{1}_{s\le\tau_M}ds\\
&+\int_t^T f_M(s|t)B(s,\zeta_M(s))\mathbb{1}_{s\le\tau_D}ds+B(T,W(T))\mathbb{1}_{T\le\tau_M}\mathbb{1}_{T\le\tau_D}\\
&+\int_t^T f_D(s|t)D(s,\zeta_D(s))\mathbb{1}_{s\le\tau_M}ds\bigg]\\
=&\sup_{(c,\pi,\zeta_M,\zeta_D)}E\bigg[\int_t^T U(s,c(s))\overline{F}_D(s|t)\overline{F}_M(s|t)ds\\
&+\int_t^T f_M(s|t)B(s,\zeta_M(s))\overline{F}_D(s|t)ds+B(T,W(T))\overline{F}_M(T|t)\overline{F}_D(T|t)\\
&+\int_t^T f_D(s|t)D(s,\zeta_D(s))\overline{F}_M(s|t)ds\bigg]\\
=&\sup_{(c,\pi,\zeta_M,\zeta_D)}E\bigg[\int_t^T U(s,c(s))\overline{F}(s|t)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_t^T \lambda_M(s)\overline{F}_M(s|t)B(s,\zeta_M(s))\overline{F}_D(s|t)ds + B(T,W(T))\overline{F}(T|t)\\
&+\int_t^T \lambda_D(s)\overline{F}_D(s|t)D(s,\zeta_D(s))\overline{F}_M(s|t)ds\Bigg]\\
=&\sup_{(c,\pi,\zeta_M,\zeta_D)} E\Bigg[\int_t^T U(s,c(s))\overline{F}(s|t)ds + \int_t^T \lambda_M(s)B(s,\zeta_M(s))\overline{F}(s|t)ds\\
&+B(T,W(T))\overline{F}(T|t) + \int_t^T \lambda_D(s)D(s,W(s))\overline{F}(s|t)ds\Bigg]\\
=&\sup_{(c,\pi,\zeta_M,\zeta_D)} E\Bigg[\int_t^T \big(U(s,c(s)) + \lambda_M(s)B(s,\zeta_M(s))\\
&+\lambda_D(s)D(s,\zeta_D(s))\big)\overline{F}(s|t)ds + B(T,W(T))\overline{F}(T|t)\Bigg].
\end{aligned}
$$

A.8 Proof of Pre-Default Verification Theorem 4.8

Let $(c_1, \pi_1, \zeta_{M_1}, \zeta_{D_1})$ be an arbitrary control policy with corresponding wealth process $W_1(t)$. Let

$$
\begin{aligned}
\beta(t) =&\alpha_D(t) + \alpha_{ND}(t) - c_1(t) - \eta_M(t)\zeta_{M_1}(t) - \eta_D(t)\zeta_{D_1}(t)\\
&+ (r(t) + \eta_M(t) + \eta_D(t))W_1(t) + (\mu(t) - r(t))\pi_1(t),\\
\phi(t) =&\sigma(t)\pi_1(t),\\
\lambda(t) =&\lambda_M(t) + \lambda_D(t),
\end{aligned}
$$

and apply Itô's Lemma to

$$
\mathcal{V}(s, W_1(s)) = \exp\left(-\int_t^s \lambda(v)dv\right)V(s, W_1(s)).
$$

Note in this instance that

$$
\begin{aligned}
\mathcal{V}_w(s, W_1(s)) &= \exp\left(-\int_t^s \lambda(v)dv\right)V_w(s, W_1(s)),\\
\mathcal{V}_{ww}(s, W_1(s)) &= \exp\left(-\int_t^s \lambda(v)dv\right)V_{ww}(s, W_1(s)),
\end{aligned}
$$

$$\mathcal{V}_t(s, W_1(s)) = \exp\Big(- \int_t^s \lambda(v)dv\Big) V_t(s, W_1(s)) + V(s, W_1(s)) \exp\Big(- \int_t^s \lambda(v)dv\Big)(-\lambda(s)),$$

where $\mathcal{V}_t(s, W_1(s))$ was obtained by the product rule for differentiating with respect to t. Now Itô's Lemma gives

$$d\mathcal{V} = \big(\mathcal{V}_t + \mathcal{V}_w \beta(t) + \tfrac{1}{2}\mathcal{V}_{ww}\phi^2(t)\big)dt + \mathcal{V}_w \phi(t) d\mathcal{W}.$$

Integrating $d\mathcal{V}$ from t to T gives

$$\begin{aligned}
\int_t^T d\mathcal{V}(s, W_1(s))ds =& \mathcal{V}(T, W_1(T)) - \mathcal{V}(t, W_1(t)) \\
=& \exp\Big(- \int_t^T \lambda(v)dv\Big) V(T, W_1(T)) - V(t, W_1(t)) \\
=& \exp\Big(- \int_t^T \lambda(v)dv\Big) B(T, W_1(T)) - V(t, W_1(t)). \qquad \text{(from (4.25))}
\end{aligned}$$

Some algebra reveals

$$\begin{aligned}
V(t, W_1(t)) =& \exp\Big(- \int_t^T \lambda(v)dv\Big) B(T, W_1(T)) - \int_t^T \exp\Big(- \int_t^u \lambda(v)dv\Big) \big(V_t(u, W_1(u)) \\
&+ V_w(u, W_1(u))\beta(u) - \lambda(u)V(u, W_1(u)) + \tfrac{1}{2}V_{ww}(u, W_1(u))\phi^2(t)\big)du \\
&- \int_t^T \exp\Big(- \int_t^u \lambda(v)dv\Big) V_w(u, W_1(u))\phi(u)d\mathcal{W}(u).
\end{aligned}$$

Take the expectation to get

$$\begin{aligned}
V(t, W_1(t)) =& E\Big[\exp\Big(- \int_t^T \lambda(v)dv\Big) B(T, W_1(T)) \\
&- \int_t^T \exp\Big(- \int_t^u \lambda(v)dv\Big) \big(V_t(u, W_1(u)) + V_w(u, W_1(u))\beta(t) \\
&- \lambda(u)V(u, W_1(u)) + \tfrac{1}{2}V_{ww}(u, W_1(u))\phi^2(u)\big)du\Big] \\
=& E\Big[\overline{F}(T|t)B(T, W_1(T)) + \int_t^T \big(\lambda_M(u)B(u, \zeta_{M_1}(u)) + \lambda_D(u)D(u, \zeta_{D_1}(u)) \\
&+ U(u, c(u))\big)\overline{F}(u|t)du - \int_t^T \big(\lambda_M(u)B(u, \zeta_{M_1}(u)) + \lambda_D(u)D(u, \zeta_{D_1}(u))
\end{aligned}$$

$$
\begin{aligned}
&+ U(u, c(u))\big)\overline{F}(u|t)du - \int_t^T \overline{F}(u|t)\big(V_t(u, W_1(u)) + V_w(u, W_1(u))\beta(u) \\
&- \lambda(u)V(u, W_1(u)) + \tfrac{1}{2}V_{ww}(u, W_1(u))\phi^2(u)\big)du\Bigg] \\
=& J(t, W_1; c_1, \pi_1, \zeta_{M_1}, \zeta_{D_1}) - E\Bigg[\int_t^T \overline{F}(u|t)\big(\lambda_M(u)B(u, \zeta_{M_1}(u)) \\
&+ \lambda_D(u, \zeta_{D_1}(u)) + U(u, c_1(u)) + V_t(u, W_1(u)) + V_w(u, W_1(u))\beta(u) \\
&- \lambda(u)V(u, W_1(u)) + \tfrac{1}{2}V_{ww}(u, W_1(u))\phi^2(u)\big)du\Bigg] \\
=& J(t, W_1; c_1, \pi_1, \zeta_{M_1}, \zeta_{D_1}) - E\Bigg[\int_t^T \overline{F}(u|t)\big(V_t(u, W_1(u)) \\
&- \lambda(u)V(u, W_1(u)) + \Psi(u, W_1(u); c_1, \pi_1, \zeta_{M_1}, \zeta_{D_1})\big)du\Bigg] \qquad \text{(A.8)} \\
\geq& J(t, W_1; c_1, \pi_1, \zeta_{M_1}, \zeta_{D_1}) \\
&- E\Bigg[\int_t^T \overline{F}(u|t)\big(V_t(u, W_1(u)) - \lambda(u)V(u, W_1(u)) \\
&+ \sup_{(c_1, \pi_1, \zeta_{M_1}, \zeta_{D_1})} \Psi(u, W_1(u); c_1, \pi_1, \zeta_{M_1}, \zeta_{D_1})\big)du\Bigg] \\
=& J(t, W_1; c_1, \pi_1, \zeta_{M_1}, \zeta_{D_1}) - E\Bigg[\int_t^T \overline{F}(u|t)(0)du\Bigg] \\
=& J(t, W_1; c_1, \pi_1, \zeta_{M_1}, \zeta_{D_1}),
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi(u, w; c_1, \pi_1, \zeta_{M_1}, \zeta_{D_1}) =& \lambda_M(u)B(u, \zeta_{M_1}(u)) + \lambda_D(u)D(u, \zeta_{D_1}(u)) + U(u, c_1(u)) \\
&+ V_w(u, w)\beta(u) + \tfrac{1}{2}V_{ww}(u, w)\phi^2(u).
\end{aligned}
$$

Thus we get (4.26). Next choose $(c_1^*, \pi_1^*, \zeta_{M_1}^*, \zeta_{D_1}^*)$ and use (A.8) to get

$$
\begin{aligned}
V(t, w) =& J(t, w; c_1^*, \pi_1^*, \zeta_{M_1}^*, \zeta_{D_1}^*) - E\Bigg[\int_t^T \overline{F}(u|t)\big(V_t(u, W_1^*(u)) \\
&- \lambda(u)V(u, W_1^*(u)) + \Psi(u, W_1^*(u); c_1^*, \pi_1^*, \zeta_{M_1}^*, \zeta_{D_1}^*)\big)du\Bigg].
\end{aligned}
$$

This, together with the fact that $V(t,w) \geq J(t,w;c_1^*,\pi_1^*,\zeta_{M_1}^*,\zeta_{D_1}^*)$ implies

$$V_t(u,W_1^*(u)) - \lambda(u)V(u,W_1^*(u)) + \Psi(u,W_1^*(u);c_1^*,\pi_1^*) \leq 0.$$

The HJB equation (4.25) gives equality if and only if $(c_1^*,\pi_1^*,\zeta_{M_1}^*,\zeta_{D_1}^*)$ is optimal, and the second part of the theorem is proved.

A.9 Derivation of (4.46)

$$\begin{aligned}
0 =& V_t(t,W_1(t)) - (\lambda_M(t)+\lambda_D(t))V(t,W_1(t)) \\
&+ \sup_{(c_1,\pi_1,\zeta_{M_1},\zeta_{D_1})} \Big\{ b_1(t,W_1(t))V_w(t,W_1(t)) + \tfrac{1}{2}\pi_1^2(t)\sigma^2(t)V_{ww}(t,W_1(t)) \\
&+ U(t,c_1(t)) + \lambda_M(t)B(t,\zeta_{M_1}(t)) + \lambda_D(t)D(t,\zeta_{D_1}(t)) \Big\} \\
=& V_t(t,W_1(t)) - (\lambda_M(t)+\lambda_D(t))V(t,W_1(t)) + \frac{1}{\gamma}(e^{\rho t})^{1/(\gamma-1)}V_w(t,W_1(t))^{\gamma/(\gamma-1)} \\
&+ \lambda_M(t)\frac{1}{\gamma}(e^{\rho t})^{1/(\gamma-1)}\left(\frac{\eta_M(t)}{\lambda_M(t)}\right)^{\gamma/(\gamma-1)} V_w(t,W_1(t))^{\gamma/(\gamma-1)} \\
&+ \lambda_D(t)\frac{1}{\gamma}(e^{\rho t})^{1/(\gamma-1)}e(t)\left(\frac{\eta_D(t)}{\lambda_D(t)}\right)^{\gamma/(\gamma-1)} V_w(t,W_1(t))^{\gamma/(\gamma-1)} \\
&+ \Bigg[\alpha_{ND}(t) + \alpha_D(t) - \big(V_w(t,W_1(t))e^{\rho t}\big)^{1/(\gamma-1)} - \eta_M(t)\left(\frac{\eta_M(t)}{\lambda_M(t)}e^{\rho t}V_w(t,W_1(t))\right)^{1/(\gamma-1)} \\
&- \eta_D(t)\left(\frac{\eta_D(t)}{\lambda_D(t)}e^{\rho t}V_w(t,W_1(t))\right)^{1/(\gamma-1)} e(t) + \big(r(t)+\eta_M(t)+\eta_D(t)\big)W_1(t) \\
&- \left(\frac{\mu(t)-r(t)}{\sigma(t)}\right)^2 \frac{V_w(t,W_1(t))}{V_{ww}(t,W_1(t))}\Bigg] V_w(t,W_1(t)) + \frac{1}{2}\left(\frac{\mu(t)-r(t)}{\sigma(t)}\right)^2 \frac{V_w^2(t,W_1(t))}{V_{ww}(t,W_1(t))} \\
=& V_t(t,W_1(t)) - (\lambda_M(t)+\lambda_D(t))V(t,W_1(t)) + \Bigg[\frac{1}{\gamma}(e^{\rho t})^{1/(\gamma-1)} \\
&+ \lambda_M(t)\frac{1}{\gamma}(e^{\rho t})^{1/(\gamma-1)}\left(\frac{\eta_M(t)}{\lambda_M(t)}\right)^{\gamma/(\gamma-1)} + \lambda_D(t)\frac{1}{\gamma}(e^{\rho t})^{1/(\gamma-1)}e(t)\left(\frac{\eta_D(t)}{\lambda_D(t)}\right)^{\gamma/(\gamma-1)} \\
&- (e^{\rho t})^{1/(\gamma-1)} - \eta_M(t)\left(\frac{\eta_M(t)}{\lambda_M(t)}e^{\rho t}\right)^{1/(\gamma-1)} \\
&- \eta_D(t)\left(\frac{\eta_D(t)}{\lambda_D(t)}e^{\rho t}\right)^{1/(\gamma-1)} e(t)\Bigg] \big(V_w(t,W_1(t))\big)^{\gamma/(\gamma-1)} + \Big[\alpha_{ND}(t) + \alpha_D(t)
\end{aligned}$$

$$
\begin{aligned}
&+ \eta_D(t) I_{ND}(t) + \big(r(t) + \eta_M(t) + \eta_D(t)\big) W_1(t)\Bigg] V_w(t, W_1(t)) \\
&- \frac{1}{2}\left(\frac{\mu(t) - r(t)}{\sigma(t)}\right)^2 \frac{V_w^2(t, W_1(t))}{V_{ww}(t, W_1(t))} \\
=& V_t(t, W_1(t)) - (\lambda_M(t) + \lambda_D(t)) V(t, W_1(t)) + \left(\frac{1-\gamma}{\gamma}\right)(e^{\rho t})^{1/(\gamma-1)}\Bigg[1 \\
&+ \frac{\eta_M(t)^{\gamma/(\gamma-1)}}{\lambda_M(t)^{1/(\gamma-1)}} + e(t)\frac{\eta_D(t)^{\gamma/(\gamma-1)}}{\lambda_D(t)^{1/(\gamma-1)}}\Bigg]\big(V_w(t, W_1(t))\big)^{\gamma/(\gamma-1)} \\
&+ \Bigg[\alpha_{ND}(t) + \alpha_D(t) + \eta_D(t) I_{ND}(t) + \big(r(t) + \eta_M(t) + \eta_D(t)\big) W_1(t)\Bigg] V_w(t, W_1(t)) \\
&- \frac{1}{2}\left(\frac{\mu(t) - r(t)}{\sigma(t)}\right)^2 \frac{V_w^2(t, W_1(t))}{V_{ww}(t, W_1(t))}.
\end{aligned}
$$

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