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ABSTRACT

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We present a family of fibrations over the Hawaiian earring that are inverse limits of regular covering spaces over the Hawaiian earring. These fibrations satisfy unique path lifting, and as such serve as a good extension of covering space theory in the case of non-semi-locally simply connected spaces. We give a condition for when these fibrations are path-connected.

Keywords: Hawaiian earring, inverse limit, fibration, path-connected

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Contents

Chapter 1. Introduction

The fundamental group is an important algebraic invariants in topology. Classically, covering space theory is the primary approach to studying and computing the fundamental group. There is a correspondence between automorphisms of covering spaces over a base space X and subgroups of the fundamental group of X. However, this correspondence is incomplete if X is not a path-connected, locally path-connected, semilocally simply connected space.

One of the most simple nonsemilocally simply connected spaces is the Hawaiian earring. Every neighborhood of the basepoint contains an essential loop. Consequently, the Hawaiian earring does not admit a universal covering space. We probe deeper into the structure of the Hawaiian earring by generalizing covering spaces to inverse limits of towers of covering spaces.

Chapter 2. Preliminaries

We first recall the definitions of covering spaces and fibrations, as well as their connection to the fundamental group of a space.

2.1 Covering Spaces

Definition 2.1. [1] A *covering space* is a triple (E, p, X) where $p : E \to X$ is a continuous surjective map which evenly covers X. That is, for each $x \in X$ there is an open neighborhood U_x of x such that $p^{-1}(U_x)$ is homeomorphic to a disjoint union of copies of U_x, called *slices*, and that p restricted to one of these slices is a homeomorphism to U_x. The map p is called a *covering map*.

Definition 2.2. A *deck transformation* of a covering space $p : E \to X$ is a homeomorphism $f : E \to E$ such that $f = p \circ f$.

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

The set of deck transformations of a covering space form a group under composition called the *deck transformation group*, denoted $\mathrm{Aut}(E \to X)$ or $\mathrm{Aut}(E)$ when X is understood.

Definition 2.3. [2] A triple (E, p, X) where $p : E \to X$ is a continuous map is said to have the *homotopy lifting property* with respect to a space Y if for all homotopies $H : I \times Y \to X$ and maps $f_0 : 0 \times Y \to E$ such that $H_0 = f_0 \circ p$, there exists a lift $\widetilde{H} : I \times Y \to E$ of H with $\widetilde{H}_0 = f_0$.

$$\begin{tikzcd} 0 \times Y \arrow[r, "f_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ I \times Y \arrow[r, "H"] \arrow[ur, dashed, "\exists \widetilde{H}"] & X \end{tikzcd}$$

Definition 2.4. A *(Hurewicz) fibration* is a triple (E, p, X) where $p : E \to X$ has the homotopy lifting with respect to all spaces. A *unique path lifting (UPL)* fibration is a fibration for which lifts of paths are unique.

We now show that covering spaces are fibrations with unique path lifting. It is this property that is preserved when we pass to the inverse limit of a tower. Throughout the following proofs we denote the backwards parametrization of a path by overlining.

Proposition 2.5. *[2] Let $p : (E, e_0) \to (X, x_0)$ be a covering map, and $f : (Y, y_0) \to (X, x_0)$ where Y is path-connected and locally path-connected. A lift $\widetilde{f} : Y \to E$ of f exists if and only if $f_*(\pi_1(Y, y_0))$ is in a conjugacy class of $p_*(\pi_1(E, e_0))$.*

Proof. Suppose we have a lift $\widetilde{f}$, and $\alpha : (I, 0, 1) \to (E, e_0, \widetilde{f_0}(y_0))$. So we have a change of basepoint homomorphism $h_\alpha : \pi_1(E, \widetilde{f}(y_0)) \to \pi_1(E, e_0)$. Then since $p \circ \widetilde{f} = f$, $h_{p \circ \overline{\alpha}} \circ p_* \circ h_\alpha \circ \widetilde{f}_* = f_*$, so we must have $\mathrm{im}\, f_* \subseteq [p \circ \overline{\alpha}](\mathrm{im}\, p_*)[p \circ \alpha]$.

Now suppose $\operatorname{im} f_* \subseteq [p\circ\alpha](\operatorname{im} p_*)[p\circ\overline{\alpha}]$. Let $\gamma, \gamma' : (I, 0, 1) \to (Y, y_0, y)$ and define $\widetilde{f}(y) = p \circ \widetilde{\alpha * f} \circ \gamma_{e_0}$. Since $\gamma * \overline{\gamma'}$ is a loop in Y, and $\operatorname{im} f_* \subseteq [p\circ\alpha](\operatorname{im} p_*)[p\circ\alpha]$, $p \circ \widetilde{\alpha * f} \circ \gamma_{e_0} * \overline{p \circ \widetilde{\alpha * f} \circ \gamma'_{e_0}}$ is a loop in E. So $p \circ \widetilde{\alpha * f} \circ \gamma_{e_0}(1) = p \circ \widetilde{\alpha * f} \circ \gamma'_{e_0}(1)$. Thus $\widetilde{f}$ is well defined. Also $p \circ \widetilde{f} = p(\widetilde{f}(y)) = p(p \circ \widetilde{\alpha * f} \circ \gamma_{e_0}(1)) = f \circ (p \circ \alpha * \gamma(1)) = f(y)$.

Now let $e \in \widetilde{f}(y)$ and U be an evenly covered neighborhood about e. Then take $V \subseteq f^{-1}(p(U))$ a path connected neighborhood of y. Then let δ be a path based at y contained in V. Then $\widetilde{f}(\delta(1)) = f \circ \widetilde{(\gamma * \delta)}_{e_0}(1)$. By construction $f \circ (\gamma * \delta)(1) \in p(U)$, so $f \circ \widetilde{(\gamma * \delta)}_{e_0}(1)$ is in U. Thus $\widetilde{f}(V) \subseteq U$. So $\widetilde{f}$ is continuous. ☐

If we have two such lifts that agree on a point, then they are the same lift.

Proof. Suppose $\widetilde{f}_1(y) = \widetilde{f}_2(y)$. Then given $y' \in Y$, let $\gamma : (I, 0, 1) \to (Y, y, y')$. Then $\widetilde{f}_1(y') = \widetilde{f}(\gamma(1)) = \widetilde{f \circ \gamma}_{\widetilde{f}_1(y)}(1) = \widetilde{f \circ \gamma}_{\widetilde{f}_2(y)}(1) = \widetilde{f}_2(\gamma(1)) = \widetilde{f}_2(y')$. ☐

Corollary 2.6. *Covering spaces are fibrations.*

Proof. Let $p : E \to X$ be a covering space. Given a homotopy $H : Y \times I \to X$ and a lift of $\widetilde{H}_0$ of $H_0 : Y \to X$, we have $H_*(\pi_1(Y \times I) = H_{0*}(\pi_1(Y))$ because H, and H_0 are homotopic maps. Since H_0 admits a lift, by Proposition 2.5, $H_{0*}(\pi_1(Y))$ is contained in a conjugacy class of $p_*(\pi_1(E))$. Hence H admits a lift as well. ☐

We now show unique path lifting.

Proposition 2.7. *Given a covering space $p : E \to X$, a path $\alpha : [0, 1] \to X$ and $e \in p^{-1}\alpha(0)$, there exists a unique lift $\widetilde{\alpha}$ of α with $p \circ \widetilde{\alpha} = \alpha$ and $\widetilde{\alpha}(0) = e$.*

$$\begin{tikzcd} & E \arrow[d, "p"] \\ I \arrow[ur, dashed, "\exists!\widetilde{\alpha}"] \arrow[r, "\alpha"'] & X \end{tikzcd}$$

Proof. Since I is compact, its image in X is compact, so we can cover the image of I with finitely many evenly covered neighborhoods. Call this cover $\mathcal{U}$. Take one such neighborhood $U_1 \in \mathcal{U}$ containing $\alpha(0)$, and define $\widetilde{\alpha} = p|_{U_{1,e}}^{-1} \circ \alpha|_{I_1}$ where $U_{1,e}$ is the slice of U_1 containing e and I_1 is the open interval of $\alpha^{-1}(U_1)$ that contains 0. We then choose $U_{i+1} \in \mathcal{U}$ such that $\alpha^{-1}(U_{i+1})$ has as a component an open interval I_{i+1} which intersects and extends I_i, and extend $\widetilde{\alpha} = p|_{U_{i+1,e}}^{-1} \circ \alpha|_{I_{i+1}}$ where $U_{i+1,e}$ is the slice of U_{i+1} that intersects the slice of $U_{i,e}$ containing $\widetilde{\alpha}(I_i \cap I_{i+1})$. This process terminate because $\mathcal{U}$ is finite, and completes the definition of $\widetilde{\alpha}$ since $\mathcal{U}$ covers the image of α. Given β any other lift beginning at e, we can find that $\beta = \alpha$ on U_1 by homeomorphically pushing β down and back up along p, and then continuing to U_i as before. $\square$

Corollary 2.8. *Covering spaces are UPL fibrations with unique lifts of homotopies.*

Proof. Suppose we have a map $f : (I \times Y, (0, y_0)) \to (X, x_0)$. Then there exists a map $\widetilde{f}_0 : (\{0\} \times Y, (0, y_0)) \to (X, x_0)$ if and only if $f|_{\{0\} \times Y_*}(\pi_1(\{0\} \times Y, (0, y_0)) \subseteq p_*(\pi_1(E, e_0))$. If this is the case then $f_*(\pi_1(I \times Y, (0, y_0))) \subseteq p_*(\pi_1(E, e_0))$ since $I \times Y$ is homotopic to Y. So the homotopy f lifts to $\widetilde{f}$ with $\widetilde{f}(0, y) = \widetilde{f}_0(y)$. Furthermore, this lift is uniquely determined by $\widetilde{f}_0(y_0)$. $\square$

We now discuss the relationship between covering spaces and the fundamental group of the base space. The correspondence is more general than we present here, but we restrict our attention to what is relevant for this thesis.

Proposition 2.9. *Given a covering space $p : E \to X$, the induced homomorphism on the fundamental group is injective.*

Proof. Let $[\alpha] \in \ker p_*$. Then $p \circ \alpha$ is nullhomotopic in X. By the homotopy lifting property, this nullhomotopy lifts to a nullhomotopy in E which begins at α by unique path lifting. Thus $[\alpha]$ is the identity of $\pi_1(E, e_0)$. $\square$

Definition 2.10. A path-connected, locally path-connected covering space is *normal* if the deck transformation group of the covering space acts transitively on point fibers.

Proposition 2.11. *[2] Given a normal covering space $p : (E, e_0) \to (X, x_0)$, there is a surjective homomorphism $\pi_1(X, x_0) \to \mathrm{Aut}(E)$.*

Proof. Let $\widetilde{\delta}_b$ denote the unique lift of a path $\delta : (I, 0) \to (X, p(b))$ based at $b \in E$.

Suppose σ is a deck transformation. Then for γ a path from e_0 to e. Then σ must map e to $\widetilde{p \circ \gamma}_{\sigma(e_0)}(1)$. Thus a deck transformation is completely determined by how it acts on point fibers because of unique path lifting.

Furthermore, we note that for any path $\gamma : (I, 0, 1) \to (E, e_0, \sigma(e_0))$ and $[\beta] \in \pi_1(E, \sigma(e_0))$ we have a change of basepoint homomorphism $h_\gamma : \pi_1(E, \sigma(e_0)) \to \pi_1(E, e_0) : [\beta] \mapsto [\gamma * \beta * \overline{\gamma}]$. Then $p_*(h_\gamma([\beta])) = [p \circ \gamma][p \circ \beta][p \circ \overline{\gamma}]$. So $p_*(\pi_1(E, \sigma(e_0))) = [p \circ \gamma] p_*(\pi_1(E, e_0))[p \circ \overline{\gamma}]$. Since σ is a deck transformation, we also have $p_*(\pi_1(E, \sigma(e_0))) = p_*(\sigma_*(\pi_1(E, e_0))) = p_*(\pi_1(E, e_0))$. So $p_*(\pi_1(E, e_0)) = [p \circ \gamma] p_*(\pi_1(E, e_0))[p \circ \overline{\gamma}]$. Given any $[\alpha] \in \pi_1(X, x_0)$, α lifts to some path between fibers, and since there is a deck transformation taking e_0 to $\widetilde{\alpha}_{e_0}(1)$. Letting $\gamma = \widetilde{\alpha}_{e_0}$ in the above argument shows $p_*(\pi_1(E, e_0)) = [\alpha] p_*(\pi_1(E, e_0))[\overline{\alpha}]$. So $\operatorname{im} p_*$ is normal in $\pi_1(X, x_0)$.

We now define a homomorphism $\varphi : \pi_1(X, x_0) \to \mathrm{Aut}(E)$. Let $[\alpha] \in \pi_1(X, x_0)$. Then the lift of α based at e_0 ends at some e_1 in the fiber of x_0. Since the covering space is normal, there is a deck transformation σ_α taking e_0 to e_1, and this transformation is uniquely determined by this action as we have mentioned before. So we set $\varphi([\alpha]) = \sigma_\alpha$. Note that since path homotopies lift, any other representative of $[\alpha]$ will lift with the same endpoint via the lifted homotopy. So φ is well defined.

Now suppose we also have $[\beta] \in \pi_1(X, x_0)$. Then $\sigma_{\alpha*\beta}(e_0) = \widetilde{\alpha * \beta}_{e_0}(1) = \widetilde{\alpha}_{\widetilde{\beta}_{e_0}(1)}(1) = \widetilde{\alpha}_{\sigma_\beta}(e_0)(1) = \sigma_\alpha(\sigma_\beta(e_0))$. So σ homomorphism. The kernel is exactly those paths which trivially act on the fiber (i.e. lift to loops), and thus is exactly $\pi_1(X, x_0)$. $\square$

Proposition 2.12. *[2] A path-connected, locally path-connected covering space is normal if and only if $p_*(\pi_1(E))$ is normal in $\pi_1(X)$.*

Proof. We know the forward direction by Proposition 2.11. If we suppose that $p_*(\pi_1(E, e_0))$ was normal in $\pi_1(X, x_0)$, then let γ be a path connecting e_0 to any other fiber point of x_0. Then the change of base point homomorphism induced by γ yields $p_*(\pi_1(E, \gamma(1))) = [p \circ \gamma] p_*(\pi_1(E, e_0))[p \circ \overline{\gamma}] = p_*(\pi_1(E, e_0))$. Thus there exists a unique lift of p along p which takes e_0 to e_1 by Proposition 2.5. Call this lift σ. Since $p \circ \sigma = p$ we need only check that σ is injective on fibers. If it were not injective on fibers then there is some nonloop path γ that is mapped to a loop under σ, but $p \circ \gamma$ lifts to a path by unique path lifting, but this lift is also $\sigma \circ \gamma$, which would be a loop. Hence the map is injective on fibers. $\square$

Definition 2.13. A subgroup H of $\pi_1(X, x_0)$ is called a *covering subgroup* if there exists a covering space $p : E \to X$ such that $p_*(\pi_1(E)) = H$.

Definition 2.14. Set $C_n = \{(x, y) \in \mathbb{R}^2 | (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$ for $n \geq 1$. Then the *Hawaiian earring* is the space $X = \bigcup_{n=1}^{\infty} C_n$ with the subspace topology.

2.2 Fibrations from towers of covering Spaces

Definition 2.15. Given a tower $(X_i, f_i : X_i \to X_{i-1})$ of topological spaces and continuous maps

$$\cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

the *inverse limit* of the tower is defined to be the subspace of the product

$$\varprojlim X_i = \left\{ (x_0, x_1, \dots) \in \prod_{i=0}^{\infty} X_i \,\middle|\, f_i(x_i) = x_{i-1} \forall i \geq 1 \right\}$$

endowed with the the subspace topology from the product topology.

Lemma 2.16 (universal property)**.** *Let (X, f) be a tower of topological spaces and $g_i : Y \to X_i$ be a collection of continuous maps satisfying $f_i \circ g_i = g_{i-1}$. Then there exists a unique continuous map $g : Y \to \varprojlim X_i$ with $p_i \circ g = g_i$ where $p_j : \varprojlim X_i \to X_j$ is projection.*

Proof. Define $g : Y \to \prod_{i=0}^{\infty} X_i$ by $g(y) = (g_i(y))$. By construction it is continuous and has image $\varprojlim X_i$. Suppose $g' : Y \to E$ also satisfied these hypotheses. Then $p_i \circ g'(y) = g_i(y)$. So $g' = g$. $\square$

We now define the family of fibrations consideration.

Definition 2.17. A *regular fibration* over X is an inverse limit of a tower of regular covering spaces over X.

Proposition 2.18. *Regular fibrations have unique path lifting.*

Proof. Let (X_*, f_*) be a tower of normal covering spaces with $X_0 = X$, $E = \varprojlim X_n$, $p_i : E \to X_i$, and $p : E \to X$. Let $\alpha : I \to X$ and $(e_i) \in p^{-1}(\alpha(0))$. Then we have lifts $\widetilde{\alpha}_i : (I, 0) \to (X_i, e_i)$. Then $\widetilde{\alpha} = (\widetilde{\alpha}_i)$ is a lift of α to E since it is continuous as the product of continuous functions and $\alpha = p^i \circ \widetilde{\alpha}_i = p^i \circ p_i \circ \widetilde{\alpha}$, so it is in fact a lift.

Now suppose we have β another lift of α. So $\alpha = p \circ \beta = p^i \circ p_i \circ \beta$. Thus $p_i \circ \beta$ is a lift of α to X_i. Since X_i is a covering space this lift is unique, so $p_i \circ \beta = \widetilde{\alpha}_i$. Thus $\beta = \widetilde{\alpha}_i$. So $\beta = \widetilde{\alpha}$. □

Chapter 3. Path-connectedness and $\varprojlim^1$

When we combine the connection between covering spaces and the fundamental group, we obtain towers of groups associated to a tower of regular covering spaces. Passing to the inverse limit reveals a necessary and sufficient criterion for when a fibration arising from a tower is path-connected. We develop that criterion here, and investigate it further in the following chapters.

Proposition 3.1. *There is the following short exact sequence of towers of groups:*

$$\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
* \arrow[r] & \pi_1(\widetilde{X}_i) \arrow[r] \arrow[d, hook] & \pi_1(X) \arrow[r] \arrow[d, "id"] & \pi_1(X)/\pi_1(\widetilde{X}_i) \arrow[r] \arrow[d, two heads] & * \\
* \arrow[r] & \pi_1(\widetilde{X}_{i-1}) \arrow[r] \arrow[d] & \pi_1(X) \arrow[r] \arrow[d] & \pi_1(X)/\pi_1(\widetilde{X}_{i-1}) \arrow[r] \arrow[d] & * \\
& \vdots & \vdots & \vdots &
\end{tikzcd}$$

Proof. We describe the homomorphisms and exactness of each row:

Given $[\alpha] \in \pi_1(\widetilde{X})$, the first map is $\rho^i_*([\alpha]) = [\rho^i \circ \alpha]$. Suppose $[\rho^i \circ \alpha] = 1$ the identity of $\pi_1(X)$. Since ρ_i is a covering map, we have unique lifts of paths, so there is a null homotopic based loop in X which lifts to α. Let H be a homotopy from x_0 to $\rho^\circ \alpha$. Then, again by the covering property, which guarantees the homotopy lifting property, H lifts to a homotopy from e_0 to α. So α is null homotopic. So ρ^i is injective. The second map, being a quotient being the quotient by the kernel of ρ^i, is surjective, and immediately yields exactness at $\pi_1(X)$.

Now we check commutativity of the diagram. Clearly the left and center towers commute because all the maps are inclusions. Since $\pi_1(\widetilde{X}_i) \subset \pi_1(\widetilde{X}_{i-1})$, by the universal property of quotients the map $\pi_1(X) \to \pi_1(X)/\pi_1(\widetilde{X}_{i-1})$ descends to a map $\pi_1(X)/\pi_1(\widetilde{X}_i) \to \pi_1(X)/\pi_1(\widetilde{X}_{i-1})$. $\square$

We now take a detour to develop some basic theory for such a sequence of towers of groups.

3.1 Towers of Groups and $\varprojlim^1$

Definition 3.2. Given a tower of groups and group homomorphisms $(G_i, f_i : G_i \to G_{i-1})$

$$\cdots \xrightarrow{f_3} G_2 \xrightarrow{f_2} G_1 \xrightarrow{f_1} G_0$$

the *inverse limit* of the tower is defined to be the subgroup of the product

$$\varprojlim G_i = \left\{ (g_0, g_1, \dots) \in \prod_{i=0}^{\infty} G_i \,\middle|\, f_i(g_i) = g_{i-1} \forall i \geq 1 \right\}.$$

Note that $\varprojlim G_i$ is a group and satisfies the same universal property described in Lemma 2.16.

Definition 3.3. [3] A tower of (possibly non-abelian) groups and homomorphisms

$$\cdots \longrightarrow G_n \xrightarrow{j} G_{n-1} \longrightarrow \cdots \longrightarrow G_{-1} = *$$

give rise to a left action of the product group $\prod G_n$ on the product set $\prod G_n$ given by

$$(g_0, \dots, g_i, \dots) \cdot (x_0, \dots, x_i, \dots) = (g_0 x_0 j(g_1)^{-1}, \dots, g_i x_i j(g_{i+1})^{-1}, \dots).$$

Clearly

$$\varprojlim G_n = \left\{ g \in \prod G_n \middle| g \cdot * = * \right\}$$

and we define $\varprojlim^1 G_n$ as the orbit set

$$\varprojlim{}^1 G_n = \prod G_n / \text{action}$$

i.e. $\varprojlim^1 G_n$ is the set of equivalence classes of $\prod G_n$ under the equivalence relation given by

$$x \sim y \Leftrightarrow y = g \cdot x \text{ for some } g \in \prod G_n.$$

Proposition 3.4. *[3] A short exact sequence of towers of groups*

$$* \longrightarrow G'_* \longrightarrow G_* \longrightarrow G''_* \longrightarrow *$$

yields an exact sequence of groups and pointed sets

$$\begin{tikzcd}
* \arrow[r] & \varprojlim G'_n \arrow[r] & \varprojlim G_n \arrow[r] & \varprojlim G''_n \arrow[dll] & \\
& \varprojlim^1 G'_n \arrow[r] & \varprojlim^1 G_n \arrow[r] & \varprojlim^1 G''_n \arrow[r] & *
\end{tikzcd}$$

Proof. First we show $\varprojlim \alpha : \varprojlim G'_i \to \varprojlim G_i$ is an injective homomorphism.

Suppose $(\alpha_i(g'_i)) = (\alpha_i(h'_i))$. Then for all $i \geq 0$ $\alpha_i(g'_i) = \alpha_i(h'_i)$. Since α_i is injective, $g'_i = h'_i$ for all $i \geq 0$. So $(g'_i) = (h'_i)$. Furthermore, $(\varprojlim \alpha)(g'_i h'_i) = (\alpha_i(g'_i h'_i)) = (\alpha_i(g'_i)\alpha_i(h'_i)) = (\alpha_i(g'_i))(\alpha_i(h'_i)) = (\varprojlim \alpha)(g_i)(\varprojlim \alpha)(h_i)$. So $\varprojlim \alpha$ is a homomorphism and we have exactness at $\varprojlim G'_i$.

We now show that $\varprojlim \beta$ and exactness at $\varprojlim G_i$ (note $\varprojlim \beta$ is a homomorphism by the same argument as used for $\varprojlim \alpha$).

Let $(g_i) \in \ker \varprojlim \beta$. then $(\varprojlim \beta)(g_i) = 0$. So $\beta_i(g_i) = 0$ for $i \geq 0$. Since each row is exact, there is g_i' such that $\alpha_i(g_i') = g_i$. Since α_i is injective, g_i' is unique. We claim $(\varprojlim \alpha)(g_i') = (g_i)$. First we must verify (g_i') is a coherent sequence. Since (g_i) is coherent, $f_i(g_i) = g_{i-1}$. Since α is a morphism of towers we have $f_i(g_i) = f_i(\alpha_i(g_i')) = \alpha_{i-1}(f_i'(g_i'))$. So $\alpha_{i-1}(f_i'(g_i')) = g_{i-1}$. Since g_{i-1}' is the unique element such that $\alpha_{i-1}(g_{i-1}') = g_{i-1}$, we must have $f_i'(g_i') = g_{i-1}'$. So $(g_i') \in \varprojlim G_i'$. Now $(\varprojlim \alpha)(g_i') = (\alpha_i(g_i')) = (g_i)$.

Given $(g_i) \in \operatorname{im} \varprojlim \alpha$, we have $(g_i') \varprojlim G_i'$ such that $\alpha_i(g_i') = g_i$. So $(\varprojlim \beta)(g_i) = (\beta_i(\alpha_i(g_i'))) = (e_i'')$. So $(g_i) = \ker \varprojlim \beta$.

Thus we have exactness at $\varprojlim G_i$.

Following [4], we define the connecting homomorphism $\delta : \varprojlim G_i'' \to \varprojlim^1 G_i'$.

Take $(g_i'') \in \varprojlim G_i''$. Since each β_i is surjective, there exists $g_i \in G_i$ so that $\beta_i(g_i) = g_i''$. Noe that (g_i) may not be a coherent sequence, but $\beta_i(g_i f_{i+1}(g_{i+1}^{-1})) = \beta_i(g_i) f_{i+1}''(\beta_{i+1}(g_{i+1}^{-1})) = g_i'' g_i''^{-1} = e_i''$. Thus by exactness there is a unique g_i' so that $\alpha_i(g_i') = g_i f_{i+1}(g_{i+1}^{-1})$. This yields a (not necessarily coherent sequence) $(g_i') \in \prod G_i'$. We define $\delta((g_i'')) = [(g_i')]$.

We now show that δ is well defined.

Suppose that $(g_i), (h_i) \in \prod G_i$ with $\beta_i(g_i) = \beta_i(h_i)$. Then we have unique sequences $(g_i'), (h_i') \in \prod G_i'$ satisfying $\alpha_i(g_i') = g_i f_{i+1}(g_{i+1}^{-1})$ and $\alpha_i(h_i') = h_i f_{i+1}(h_{i+1}^{-1})$. We now seek a sequence $(r_i') \in \prod G_i'$ such that $r_i' g_i' f_{i+1}'({r_{i+1}'}^{-1}) = h_i'$.

Since $\beta_i(g_i) = \beta_i(h_i)$, $\beta_i(h_i g_i^{-1}) = e_i''$. So there is a unique element of G_i', which assign r_i', so that $\alpha_i(r_i') = h_i g_i^{-1}$. Then $\alpha_i(r_i' g_i' f_{i+1}'({r_{i+1}'}^{-1})) = \alpha_i(r_i')\alpha_i(g_i') f_{i+1}(\alpha_{i+1}({r_{i+1}'}^{-1})) =$
$h_i g_i^{-1} g_i f_{i+1}(g_{i+1}^{-1}) f_{i+1}((h_{i+1} g_{i+1}^{-1})^{-1}) =$
$h_i g_i^{-1} g_i f_{i+1}(g_{i+1}^{-1}) f_{i+1}(g_i) f_{i+1}(h_i^{-1}) = h_i f_{i+1}(f_{i+1}(h_{i+1}^{-1}))$. Since h_i' is the unique elements with $\alpha_i(h_i') = h_i f_{i+1}(h_{i+1}^{-1})$, we must have $r_i' g_i' f_{i+}'({r_{i+1}'}^{-1}) = h_i'$. So $(g_i') \sim (h_i')$, this δ is well-defined.

We now show $\ker \delta = \operatorname{im} \varprojlim \beta$.

Let $(g_i'') \in \ker \delta$. Then given $(g_i) \in \prod G_i$ such that $|beta_i(g_i) = g_i''$ and unique (g_i') such that $\alpha_i(g_i') = g_i f_{i+1}(g_i^{-1})$, we have that $[(g_i')] = [(e_i')]$. So there is a sequence $(r_i') \in \prod G_i'$ so that $(r_i') \cdot (e_i') = (r_i' f_{i+1}'({r_{i+1}'}^{-1})) = (g_i')$. So $\alpha_i(r_i' f_{i+1}'({r_{i+1}'}^{-1})) = \alpha_i(r_i') f_{i+1}(\alpha_{i+1}({r_{i+1}'}^{-1})) = g_i f_{i+1}(g_{i+1}^{-1}$, thus $f_{i+1}(\alpha_i({r_{i+1}'}^{-1})) = \alpha_i({r_i'}^{-1}) g_i$. So $(\alpha_i({r_i'}^{-1}))$ is a coherent sequence and $(\varprojlim \beta)(\alpha({r_i'}^{-1}) g_i) = (\beta_i(\alpha_i({r_i'}^{-1} \beta(g_i)))) = (\beta_i(g_i)) = (g_i'')$. So $(g_i'') \in \operatorname{im} \varprojlim \beta$.

Let $(g_i'') \in \varprojlim \beta$. Then there exists $(g_i) \in \varprojlim G_i$ so that $\beta_i(g_i) = g_i''$. So $\delta((g_i'')) = [(r_i)]$ where r_i' is the unique sequence of elements of G_i' so that $\alpha_i(r_i') = g_i f_{i+1}(g_{i+1}^{-1})$. Since (g_i) is coherent, $g_i f_{i+1}(g_{i+1}^{-1}) = g_i g_i^{-1} = e_i$. So $r_i' = e_i'$, and $\delta((g_i'')) = [(e_i')]$.

So we have exactness at $\varprojlim G_i''$.

Next is exactness at $\varprojlim^1 G_i'$.

First we show $\varprojlim^1 \alpha : \varprojlim^1 G_i' \to \varprojlim^1 G_i$ is well-defined.

We define $(\varprojlim^1 \alpha)[(g_i')] = [(\alpha_i(g_i'))]$. Suppose $(g_i') \sim (h_i')$. So there exists $(r_i') \in \prod G_i'$ so that $r_i' g_i' f_{i+1}'({r_{i+1}}^{-1}) = h_i'$. So $\alpha_i(r_i') \alpha_i(g_i') f_{i+1}(\alpha_{i+1}({r_{i+1}'}^{-1})) = \alpha_i(h_i')$.So $(\alpha_i(g_i')) \sim (\alpha_i(h_i'))$ via the sequence $\alpha_i(r_i') \in \prod G_i$. So $\varprojlim^1 \alpha$ is well-defined.

Let $[(g_i')] \in \ker \varprojlim^1 \alpha$. So there is $(r_i) \in \prod G_i$ so that $r_i e_i f_{i+1}(r_{i+1}^{-1})$ for all $i \geq 0$. Applying β_i yields $e_i'' = \beta_i(r_i) f_{i+1}''(\beta_{i+1}(r_i^{-1}))$. Then $\beta_i(r_i) = f_{i+1}''(\beta_i(r_i))$, so $(\beta_i(r_i)) \in \varprojlim G_i''$. We then compute $\delta((\beta_i(r_i)))$. We see (r_i) is a sequence so that $(\beta_i(r_i)) = (\beta_i(r_i))$, so we take the unique element of G_i' whose image under α is $r_i f_{i+1}(r_{i+1}^{-1})$, but we already know this is g_i'. So $\delta((\beta_i(r_i))) = [(g_i')]$. So $\ker \varprojlim^1 \alpha \subseteq \operatorname{im} \delta$.

Now suppose $[(g_i')] \in \operatorname{im} \delta$. So there is $(g_i'') \in \varprojlim G_i''$ so that there is $(g_i) \in \prod G_i$ with $\beta_i(g_i) = g_i''$, $h_i' = \alpha^{-1}(g_i f_{i+1}(g_{i+1}^{-1}))$, and $(r_i') \in \prod G_i''$ with $r_i' h_i' f_{i+1}'({r_{i+1}'}^{-1}) = g_i'$. So $\alpha(g_i) = \alpha_i(r_i' h_i' f_{i+1}'({r_{i+1}'}^{-1})) = \alpha_i(r_i') \alpha(h_i') f_{i+1}(\alpha_{i+1}({r_{i+1}'}^{-1})) = \alpha_i(r_i') g_i f_{i+1}(g_{i+1}^{-1}) f_{i+1}(\alpha_{i+1}({r_{i+1}'}^{-1})) = (\alpha_i(r_i') g_i) e_i'(f_{i+1}((\alpha_{i+1}(r_{i+1}' g_{i+1})^{-1})))$. So $(\alpha_i(g_i'))\tilde{(}e_i)$ via the sequence $(\alpha_i(r_i') g_i)$. So we have equality $(\varprojlim^1 \alpha)([g_i']) = ([e_i])$.

So we have exactness at $\varprojlim G_i$.

Now we show exactness at $\varprojlim^1 G_i$ (note that $\varprojlim^1 \beta$ is well-defined similarly to $\varprojlim^1 \alpha$).

Let $[(g_i)] \in \ker \varprojlim^1 \beta$. So there is $(r''_i) \in \prod G''_i$ so that $r''_i \beta(g_i) f''_{i+1}({r''_{i+1}}^{-1}) = e''_i$. Since each β_i is surjective, here is r_i with $\beta_i(r_i) = r''_i$. So $e''_i = \beta_i(r_i)\beta_i(g_i) f''_{i+1}(\beta_{i+1}({r_{i+1}}^{-1})) = \beta_i(r_i g_i f_{i+1}({r_{i+1}}^{-1}))$. Since each row is exact, there is a unique g'_i with $\alpha_i(g'_i) = r_i g_i f_{i+1}(r_{i+1}^{-1})$. Then $(\varprojlim^1 \alpha)[(g'_i)] = [(r_i g_i f_{i+1}(r_{i+1}^{-1}))] = [(g_i)]$. So $\ker \varprojlim^1 \beta \subseteq \operatorname{im} \varprojlim^1 \alpha$.

Now suppose $[(g_i)] \in \operatorname{im} \varprojlim^1 \alpha$. So here is $[(g'_i)] \in \varprojlim^1 G'_i$ and $(r_i) \in \prod G_i$ with $r_i \alpha_i(g'_i) f'_{i+1}({r'_{i+1}}^{-1}) = g_i$. Then we compute $\beta_i(g_i) = \beta_i(r_i)\beta_i(\alpha_i(g'_i)) f''_{i+1}(\beta_{i+1}(r_{i+1}^{-1})) = \beta_i(r_i) e''_i f''_{i+1}(\beta_{i+1}(r_{i+1}^{-1}))$. Thus $(\beta_i(g_i) \sim (e''_i)$. So $[(g_i)] \in \ker \varprojlim^1 \beta$.

So we have exactness at $\varprojlim^1 G_i$.

Finally surjectivity of $\varprojlim^1 \beta$ yields exactness at $\varprojlim^1 G''_i$ and concludes the theorem.

Let $[(g''_i)] \in \varprojlim^1 G''_i$. Then since each β_i is surjective there is $g'_i \in G_i$ with $\beta_i(g_i) = g''_i$. Then $(\varprojlim^1 \beta)[(g_i)] = [(\beta_i(g_i)] = [(g''_i)]$. So $\varprojlim^1 \beta$ is surjective. □

Proposition 3.5. *[3] If G_n is a tower of surjections then $\varprojlim^1 G_n = *$.*

3.2 Path-connectedness

Proposition 3.6. *[5, Theorem 4.10] For a tower of normal covering spaces with inverse limit E we have $\varprojlim \pi_1(X, x_0)/\pi_1(\widetilde{X}_i, \widetilde{x}_0) \cong \varprojlim \operatorname{Aut}(\widetilde{X}_i) \hookrightarrow \operatorname{Aut}(E)$. Furthermore, the action of $\varprojlim \pi_1(X, x_0)/\pi_1(\widetilde{X}_i, \widetilde{x}_0)$ on fibers is free and transitive.*

Proof. The isomorphism $\varprojlim \pi_1(X, x_0)/\pi_1(\widetilde{X}_i, \widetilde{x}_0) \cong \varprojlim \operatorname{Aut}(\widetilde{X}_i)$ is a consequence of Proposition 2.11.

Suppose $(\sigma_i) \in \varprojlim \operatorname{Aut}(\widetilde{X}_i)$. That means $\sigma_i : \widetilde{X}_i \to \widetilde{X}_i$ such that $p^i \circ \sigma_i = p^i$ and $p^i_{i-1} \circ \sigma_i = \sigma_{i-1} \circ p^i_{i-1}$. Then define $\sigma : E \to E$ be $\sigma((y_i)) = (\sigma_i(y_i))$. We first verify that $(\sigma_i(y_i)) \in E$. Since $p^i_{i-1}(\sigma_i(y_i)) = \sigma_{i-1}(p^i_{i-1}(y_i)) = \sigma_{i-1}(y_{i-1})$, the codomain of σ is in fact E. Since each σ_i is a deck transformation, we have $p((\sigma_i(y_i))) = p^j(\sigma_j(y_j)) = p^j(y_j) = p((y_i))$. The product of continuous maps $\sigma_1 \times \sigma_2 \times \cdots$ is continuous and σ is a restriction of this product map, so σ continuous. Finally σ has inverse $(y_i) \mapsto (\sigma_i^{-1}(y_i))$. So σ is an automorphism of E over X.

Given $(\tau_i) \in \varprojlim \mathrm{Aut}(\widetilde{X}_i)$ with induced automorphism τ on E, we see $\sigma(\tau((y_i))) = \sigma((\tau_i(y_i))) = (\sigma_i(\tau_i(y_i)) = ((\sigma_i \circ \tau_i)(y_i))$ so this is a homomorphism. The kernel is trivial because if σ is the identity, then $p_i(\sigma) = \sigma_i$ is the identity.

Let $(e_i), (e_i') \in p^{-1}(x_0)$. Since p^i is a normal covering map and $e_i, e_i' \in {p^i}^{-1}(x_0)$, there is a unique deck transformation of $\sigma_i \in \mathrm{Aut}(\widetilde{X}_i)$ that maps e_i to e_i'. These deck transformations form a coherent sequence: $p^i_{i-1}(\sigma_i(e_i)) = p^i_{i-1}(e_i') = e_{i-1}' = \sigma_{i-1}(e_{i-1}) = \sigma_{i-1}(p^i_{i-1}(e_i))$. Since deck transformations are determined by their action on a point, we don't need to check $p^i_{i-1} \circ \sigma_i = \sigma_{i-1} \circ p^i_{i-1}$ anywhere else (doing so amounts to checking that $p\sigma$ commutes over paths, which it is the case by unique path lifting). So the image $\varprojlim \mathrm{Aut}(X_i)$ acts transitively on point fibers. Since all of the terms in the sequence (σ_i) are unique, the sequence is too, thus the image is free as well. $\square$

Proposition 3.7. *[5, Theorem 4.5] The map $\pi_1(X, x_0) \to \varprojlim \mathrm{Aut}(\widetilde{X}_i, \widetilde{x}_i^0)$ is surjective if and only if E is path connected.*

Proof. Suppose E is path connected. Then let α be a path connecting e_0 to any other fiber point e_1 of $\widetilde{x}_0$. Every deck transformation in the image is determined by its action on the basepoint. The $p \circ \alpha$ maps to the sequence of deck transformations that exactly yields the unique deck transformation σ of E in the image of $\varprojlim \mathrm{Aut}(\widetilde{X}_i)$ that takes e_0 to e_1. So the map is surjective because we can obtain a loop which lifts to a path connecting the basepoint to any other fiber point.

Now suppose the map is surjective. Then there is a unique deck transformation in the image of $\pi_1(X, x_0)$ that takes the basepoint to any other fiber point. A loop in the preimage of this deck transformation must then be a loop with lifts to a path connecting the basepoint to that fiber point. Given any point $y \in E$, we can take a path from $p(y)$ to x_0. Then the lift of this path based at y is a path from y to a fiber point of x_0 since it is a lift. Thus it is sufficient to see we have paths between fiber points to obtain general path connectedness. Hence E is path connected. $\square$

Chapter 4. Product structures on towers of regular covering spaces

We make strong use of the following lemma.

Lemma 4.1. *Let $H_1, H_2 \trianglelefteq G$ with $H_2 \subset H_1$. Then we have the exact sequence*

$$1 \longrightarrow H_1/H_2 \longrightarrow G/H_2 \longrightarrow G/H_1 \longrightarrow 1$$

For this sequence, the following are equivalent:

(1) There exists a retract $r : G/H_2 \to H_1/H_2$.

(2) This sequence splits as a direct product:

$$\begin{tikzcd}
 & & G/H_2 \arrow[dr] \arrow[dd, "\cong"] & & \\
1 \arrow[r] & H_1/H_2 \arrow[ur] \arrow[dr] & & G/H_1 \arrow[r] & 1 \\
 & & H_1/H_2 \times G/H_1 \arrow[ur] & &
\end{tikzcd}$$

(3) There is a normal subgroup N_2 of G satisfying

(i) $H_1 \cap N_2 = H_2$

(ii) $G = H_1 N_2$.

Proof. Since $H_2 \subset H_1$, by the universal property of quotients we get a unique group homomorphism $G/H_2 \to G/H_1$ given by $gH_2 \mapsto gH_1$. The kernel of this map is $\{gH_2 \mid g \in H_1\}$. This is exactly the image of H_1 under the quotient by H_2.

Now we show the three equivalent conditions for splitting as a product.

(1⇒2) Let $\iota : H_1/H_2 \to G/H_2$ and $q : G/H_2 \to G/H_1$. Define $r \times q : G/H_2 \to H_1/H_2 \times G/H_1$ by $(r \times q)(gH_2) = (r(gH_2), q(gH_2))$. Let $gH_2 \in \ker(r \times q)$. Then we have that $q(gH_2) = 1$, so $g \in H_1$. Thus $1 = r(gH_2) = r(\iota(gH_2)) = gH_2$. Therefore $g \in H_2$ and $r \times q$ is injective.

Now let $gH_2 \in H_1/H_2$ and $fH_1 \in G/H_1$. Then let $f' \in H_1$ representative of $r(fH_2)$. Since r is a retract $r(gH_2) = gH_1$ and $r(f'H_2) = f'H_2$. Then

$$\begin{aligned}(r \times q)(gff'^{-1}H_2) &= (r(gH_2)r(fH_2)r(f'^{-1}H_2), q(gH_2)q(fH_2)q(f'^{-1}H_2)) \\ &= (gH_2f'H_2f'^{-1}H_2, gH_1fH_1f'^{-1}H_1) \\ &= (gH_2, fH_1)\end{aligned}$$

so $(r \times q)$ is surjective.

(2⇒1) Projection onto the first coordinate is a retract.

(1⇒3) Let $\varphi : G \to G/H_2$ be the quotient map. Set $N_2 = \ker(r \circ \varphi)$. Then by the first isomorphism theorem $G/N_2 \cong H_1/H_2$. Suppose $h \in H_2$. Then since $h \in \ker \varphi$, clearly $h \in N_2$. Since $H_2 \subset H_1$, we have $H_2 \subset H_1 \cap N_2$.

Now suppose $h \in H_1 \cap N_2$. Then $\iota(hH_2) = \varphi(h)$ since $h \in H_1$. Applying r yields $hH_2 = r(\varphi(h))$, which is trivial since $h \in N_2$. So $h \in H_2$.

Let $g \in G$. Since $H_1/(H_1 \cap N_2)$ and $N_2/(H_1 \cap N_2)$ are normal subgroups of $G/(H_1 \cap H_2)$ with trivial intersection, we have $G/(H_1 \cap N_2) = H_1/(H_1 \cap N_2)N_2/(H_1 \cap N_2)$. So $g(H_1 \cap N_2) = hn(H_1 \cap N_2)$ for some $h \in H_1$ and $n \in N_2$. So $g = hnk$ for some $k \in H_1 \cap N_2$. Since $nk \in N_2$, we have $g \in H_1N_2$.

(3⇒1) Since $G = H_1N_2$, we can write $g = hn$ where $h \in H_1$ and $n \in N_2$. Define $r'(g) = h(H_1 \cap N_2)$. First, we check that this is well-defined. Suppose $g = hn = h'n'$ where $h' \in H_1$ and $n' \in N_2$. Then $h'^{-1}h = n'^{-1}n$. So $h'^{-1}h \in H_1 \cap N_2$, and $h(H_1 \cap N_2) = h'(H_1 \cap N_2)$.

We now check that it is a homomorphism. Suppose $g = hn, g' = h'n'$. Since N_2 is normal, there is n'' such that $h'n' = n''h'$. Then $r(gg') = r(hnh'n') = r(hh'n''n') = hh'(H_1 \cap N_2) = h(H_1 \cap N_2)h'(H_1 \cap N_2)) = r(g)r(g')$.

If $r'(g) = 1$, then $g \in N_2$. So $H_1 \cap N_2 \subset \ker r'$, so r' descends to a homomorphism $r : G/(H_1 \cap N_2) \to H_1/(H_1 \cap N_2)$. Lastly, we check that r is a retract. Let $h(H_1 \cap N_2) \in H_1$. Then $r(h(H_1 \cap N_2)) = r'(h) = h(H_1 \cap N_2)$. So r is in fact a retract for the inclusion $H_1/(H_1 \cap N_2) \hookrightarrow G/(H_1 \cap N_2)$. $\square$

We now apply this to the case of a tower of regular covering spaces.

Definition 4.2. We say a tower of regular covers $\widetilde{X}_*$ over a base space X has a *product structure* if the exact sequence

$$1 \longrightarrow \mathrm{Aut}(\widetilde{X}_i \to \widetilde{X}_{i-1}) \longrightarrow \mathrm{Aut}(\widetilde{X}_i \to X) \longrightarrow \mathrm{Aut}(\widetilde{X}_{i-1} \to \widetilde{X}) \longrightarrow 1$$

splits as a product for all i.

Proposition 4.3. *A tower $\widetilde{X}_*$ has a product structure if and only if there is a collection $\{N_i\}_{i=1}^{\infty}$ of normal covering subgroups of $\pi_1(X)$ satisfying*

(i) $p^i_*(\pi_1(\widetilde{X}_n)) = \bigcap_{i=1}^n N_i$

(ii) $\pi_1(X) = (\bigcap_{j=1}^{i-1} N_j)N_i$.

Furthermore, there are isomorphisms $\pi_1(X)/p^i_*(\pi_1(\widetilde{X}_i)) \cong \prod_{j=1}^i \pi_1(\widetilde{X}_{j-1})/p^j_{j-1_*}(\pi_1(\widetilde{X}_i))$ *commute with the connecting maps, where the connecting homomorphism*

$$\prod_{j=1}^{i} \pi_1(\widetilde{X}_{j-1})/p^j_{j-1_*}(\pi_1(\widetilde{X}_i)) \to \prod_{j=1}^{i-1} \pi_1(\widetilde{X}_{j-1})/p^j_{j-1_*}(\pi_1(\widetilde{X}_j))$$

is given by projection onto the first $i-1$ factors. In particular

$$\varprojlim \pi_1(X)/p^i_*(\pi_1(\widetilde{X}_i)) \cong \prod_{i=1}^{\infty} \pi_1(\widetilde{X}_{i-1})/p^i_{i-1_*}(\pi_1(\widetilde{X}_i))$$

Proof. Set $H_i = p^i_*(\pi_1(\widetilde{X}_i))$. Then by application of Lemma 4.1, there exists $N_i \trianglelefteq G$ such that $H_{i-1} \cap N_i = H_i$ and $H_{i-1}N_i$. With the convention that $H_0 = G$, we have the base case $H_1 = N_1$. Now assume, by way of induction, that $H_{n-1} = \bigcap_{i=1}^{n-1} N_i$. Then $H_n = H_{n-1} \cap N_n = (\bigcap_{i=1}^{n-1}) \cap N_n = \bigcap_{i=1}^n N_i$.

Alternatively, if we have covering subgroups $\{N_i\}$ satisfying (i) and (ii), the exact sequence $1 \to \pi_1(\widetilde{X}_{i-1})/\pi_1(\widetilde{X}_i) \to \pi_1(X)/\pi_1(\widetilde{X}_i) \to \pi_1(X)/\pi_1(\widetilde{X}_{i-1}) \to 1$ splits by (3) $\Rightarrow$ (2) in Lemma 4.1. So the tower corresponding to $\{\bigcap_{i=1}^{n}(N_i)\}$ has a product structure.

So we now have by assumption that $\mathrm{Aut}(\widetilde{X}_i \to X) \cong \mathrm{Aut}(\widetilde{X}_i \to \widetilde{X}_{i-1}) \times \mathrm{Aut}(\widetilde{X}_{i-1} \to X)$, where projection to the second coordinate corresponds exactly to the bonding map $\mathrm{Aut}(\widetilde{X}_i \to X) \to \mathrm{Aut}(\widetilde{X}_{i-1} \to X)$:

$$\begin{tikzcd}
\mathrm{Aut}(\widetilde{X}_i \to X) \arrow[rr, "\cong"] \arrow[dr, "q^i_{i-1}"] & & \mathrm{Aut}(\widetilde{X}_i \to \widetilde{X}_{i-1}) \times \mathrm{Aut}(\widetilde{X}_{i-1} \to X) \arrow[dl] \\
& \mathrm{Aut}(\widetilde{X}_{i-1} \to X) &
\end{tikzcd}$$

In fact this gives the following isomorphism of towers (and thus of inverse limits):

$$\begin{tikzcd}
\varprojlim \mathrm{Aut}(\widetilde{X}_i \to X) \arrow[r, "\cong"] \arrow[d] & \varprojlim \prod_{j=1}^{i} \mathrm{Aut}(\widetilde{X}_j \to \widetilde{X}_{j-1}) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathrm{Aut}(\widetilde{X}_i \to X) \arrow[r, "\cong"] \arrow[d, "q^i_{i-1}"] & \prod_{j=1}^{i} \mathrm{Aut}(\widetilde{X}_j \to \widetilde{X}_{j-1}) \arrow[d] \\
\mathrm{Aut}(\widetilde{X}_{i-1} \to X) \arrow[r, "\cong"] \arrow[d] & \prod_{j=1}^{i-1} \mathrm{Aut}(\widetilde{X}_j \to \widetilde{X}_{j-1}) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathrm{Aut}(\widetilde{X}_1 \to X) \arrow[r, "\cong"] & \mathrm{Aut}(\widetilde{X}_1 \to X)
\end{tikzcd}$$

It is easy to see that $\varprojlim \prod_{j=1}^{i} \mathrm{Aut}(\widetilde{X}_j \to \widetilde{X}_{j-1}) \cong \prod_{j=1}^{\infty} \mathrm{Aut}(\widetilde{X}_j \to \widetilde{X}_{j-1})$. $\square$

Chapter 5. Product towers that yield to path-connected regular fibrations

We now present our main result.

Theorem 5.1. *Let X be the Hawaiian earring and let E be a regular fibration over X arising from a tower with a product structure. If for all $k \in \mathbb{N}$, there is $\ell \in \mathbb{N}$ such that for $i > \ell$, the group $\pi_1(\widetilde{X}_{i-1})/\pi_1(\widetilde{X}_i)$ is generated by loops of diameter less than $1/k$, then E is path-connected,*

Proof. Set $H_i = p^i_*(\pi_1(\widetilde{X}_i)$. First, assume that for all $k \in \mathbb{N}$, there is $\ell_k \in \mathbb{N}$ such that for $i > \ell$, the group H_{i-1}/H_{i-1} is generated by loops of diameter less than $1/k$. Let N_i be the subgroups obtained by applying Proposition 4.3. In the proof of Proposition 4.3, N_i is the kernel of the composition $G \to G/H_i \to H_{i-1}/H_i$ where the second map is the retract from the product structure. So $G/N_i \cong H_{i-1}/H_i$, and in particular if $\{g_j H_i\}$ is a generating set for H_{i-1}/H_i, then $\{g_j N_i\}$ is a generating set for G/N_i. Thus, since H_{i-1}/H_i is generated by loops of diameter less than $1/k$ for $i > \ell_k$, so is G/N_i. Then we can take $(\sigma_i) \in \varprojlim \operatorname{Aut}(\widetilde{X}_i \to X) \cong \prod_{i=1}^{\infty} \operatorname{Aut}(\widetilde{X}_i \to \widetilde{X}_{i-1}) \cong \prod_{i=1}^{\infty} G/N_i$. So we have an element $(g_i N_i) \in \prod G/N_i$ corresponding to (σ_i). For $i > \ell_k$ we may find w_i such that $g_i N_i = w_i N_i$ where w_i is a reduced word in $\{a_j \mid j > k\}$. Furthermore, by the product structure we may assume $w_i \in N_j$ for $j \neq i$. We define $g = w_1 w_2 \cdots$. Note that g is a well-defined element of the G since every big free generator a_k appears only in the factors w_i where $i \leq \ell_k$, and w_i is a legal word in G. Since we have $gN_i = (w_1 \cdots w_{i-1} N_i)(w_i N_i)(w_{i+1} \cdots N_i) = w_i N_i$. Hence, $g \mapsto (w_i N_i)$ under the product of projection maps, and by construction $(w_i N_i) = (g_i N_i)$. So $\pi_1(X) \to \prod_{i=1}^{\infty} G/N_i \cong \varprojlim \operatorname{Aut}(\widetilde{X}_i \to X)$ is onto, and thus $E = \varprojlim \widetilde{X}_i$ is path-connected.

□

Conjecture 5.2. *If a regular fibration E does not arise as the inverse limit of a product tower then it is not path-connected.*

The following examples illustrate the various cases covered in the above theorem.

Example 5.3 ($\mathbb{Z}$-tower)**.** Define

$$H_i = \langle\langle a_k, [a_m, a_n] \mid k > i, 1 \leq m < n \leq i \rangle\rangle$$

Then the subgroups obtained from the product structure are $N_i = \langle\langle a_k \mid k \neq i \rangle\rangle$, so $H_{i-1}/H_i \cong G/N_i \cong \mathbb{Z}$ and $\varprojlim \mathrm{Aut}(\widetilde{X}_i \to X) = \mathbb{Z}^\infty$. The regular fibration obtained from this descending sequence of subgroups is the integer lattice $\mathbb{Z}^\infty \subset \mathbb{R}^\infty$, with the subspace topology induced by the product topology on $\mathbb{R}^\infty$.

Example 5.4 (F_2-tower with product structure)**.** Define

$$H_i = \langle\langle a_k, [a_m, a_n] \mid k > 2i, 1 \leq m < n \leq 2i, \text{ where } n \neq m+1 \text{ if } m = 1 \pmod 2 \rangle\rangle$$

Then $G/H_i \cong F(a_1, a_2) \times F(a_3, a_4) \times \cdots \times F(a_{2(i-1)-1}, a_{2(i-1)}) \times F(a_{2i-1}, a_{2i})$. Then the subgroups obtained from the product structure are $N_i = \langle\langle a_k \mid k \neq 2i, k \neq 2i+1 \rangle\rangle$, so $H_{i-1}/H_i \cong G/N_i \cong F(a_{2i}, a_{2i+1})$. The resulting inverse limit group is $\varprojlim \mathrm{Aut}(\widetilde{X}_i \to X) \cong F(a_1, a_2) \times F(a_3, a_4) \times \cdots$. The resulting regular fibration here is an infinite lattice of Cayley graphs of F_2.

Example 5.5 (F_2-tower without product structure)**.** Define

$$H_i = \langle\langle a_k, [a_m, a_n] \mid k > 2i-1, 1 \leq m < n \leq 2i, \text{ where } n \neq m+1 \text{ if } m = 1 \pmod 2 \rangle\rangle$$

So $G/H_i \cong F(a_1, a_2) \times F(a_3, a_4) \times \cdots \times F(a_{2(i-1)-1}, a_{2(i-1)}) \times F(a_{2i-1})$. For this tower the exact sequence

$$1 \to H_{i-1}/H_i \to G/H_i \to G/H_{i-1} \to 1$$

does not split for any $i > 1$. For simplicity we only argue the case of $i = 2$. Then note that $H_1/H_2 = \langle a_1^{-m} a_2 a_1^m H_1 \mid m \geq 0 \rangle \langle a_3 H_1 \rangle$. This group is not finitely generated. Since

H_2 is a covering subgroup of G, G/H_2 is finitely generated, so there can be no surjection $G/H_1 \to H_1/H_2$. Hence, a retract cannot exist.

Despite this, this tower clearly results in the same fibration as the previous example.

Example 5.6 (solenoid on a_1)**.** Define

$$H_i = \langle\langle a_1^{2i}, a_k \mid k > 1 \rangle\rangle$$

Then $G/H_i \cong \mathbb{Z}/2^i\mathbb{Z}$. Even though G/H_i and H_{i-1}/H_i are abelian for all i, this tower does not yield a product structure since the short exact sequence does not split for any $i > 1$: $\mathbb{Z}/2^i\mathbb{Z} \ncong \mathbb{Z}/2^{i-1}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. One can see that regular fibration obtained from this tower is the 2-adic solenoid with Hawaiian earrings attached at each point preimage of the basepoint of the base space. It is not path-connected.

Example 5.7 (not a product)**.** Define

$$H_i = \langle\langle a_k, [a_m, a_n] \mid k > i+1, 1 \leq m < n \leq i+1, \text{ and } |m-n| > 1 \rangle\rangle$$

Then G/H_i is generated by $i+1$ generators $\{a_1, \ldots, a_{i+1}\}$ where every generator commutes unless their indices are adjacent. Using this we can construct a coherent sequence of deck transformations that are not in the image of $\pi_1(X) \to \varprojlim \operatorname{Aut}(\widetilde{X}_i \to X)$:

$$\begin{aligned}
\sigma_1 &= [a_1, a_2]H_1 \\
\sigma_2 &= [a_1, a_2][a_1, a_2^{-1}a_3a_2]H_3 \\
\sigma_3 &= [a_1, a_2][a_1, a_2^{-1}a_3a_2][a_1, a_2^{-1}a_3^{-1}a_4a_3a_2]H_4 \\
&\vdots
\end{aligned}$$

However, $[a_1, a_2][a_1, a_2^{-1}a_3a_2][a_1, a_2^{-1}a_3^{-1}a_4a_3a_2]\ldots$ is not a valid word in the Hawaiian earring group because a_1 appears infinitely often.

Chapter 6. Conclusion

We now have a characterization of when a regular fibration arising from a product tower over the Hawaiian earring is path-connected. This leads to the following questions:

- Is there an appropriate generalization of the arguments presented here that extend this characterization to all one-dimensional Peano continua? The notion of a product structure on a descending sequence of normal subgroups is a purely algebraic, and perhaps $\varprojlim^1$ can be computed directly from this information.
- What UPL fibrations exists that do not arise as the inverse limit of towers of regular covering spaces? Are any of these path-connected?

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