1963-5

A Study of the Vibration Characteristics of a Thin Shell Hyperbolic Paraboloid

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A STUDY OF THE VIBRATION CHARACTERISTICS OF A THIN SHELL HYPERBOLIC PARABOLOID

A Thesis Presented to
the Faculty of
the Department of Mechanical Engineering
Brigham Young University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science

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by
Farrin W. West
May 1963

This thesis, by Farrin W. West, is accepted in its present form by the Department of Mechanical Engineering of Brigham Young University as satisfying the thesis requirements for the degree of Master of Science.

May 7, 63
Date

TO THE MEMORY OF
MY STEP-FATHER

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CHAPTER I

INTRODUCTION

Recent progress in structural engineering has been marked by the increased use of thin shells as load carrying members. In particular, shells of revolution have been widely utilized in the missile industry whereas shells of arbitrary shape have been incorporated in the architectural design of today's modern buildings.

Missile applications have been extensively studied both statically and dynamically. However, architechural applications have generally been limited to the statics problem with little consideration being given to the dynamic behavior of these unique structures.

Statement of the problem

It was the purpose of this study to investigate the dynamic behavior of thin shells with a particular emphasis being given to the hyperbolic paraboloid, which is a common shell structure used in architectural design. The major objective was to determine the lowest natural frequencies of vibration as a function of the physical characteristics of the shell.

Purpose of the study

An accepted principle in engineering analysis is to develop a mathematical model prior to the construction of a physical prototype. The purpose of this study was to lay the analytical ground-work which is required before an experimental analysis can begin.

A hyperbolic paraboloid was selected because this particular surface readily lends itself to analysis and may easily be constructed in the laboratory.

Method of approach

In order to gain an understanding of the problem the first step in the analysis consisted of the derivation of the equations of motion of a thin shell. These equations were first developed for a shell of arbitrary shape defined by a general curvilinear coordinate system.

The next step was the application of the general theory to the special case of a thin shallow shell defined in rectangular coordinates. The equations were then simplified by neglecting the longitudinal inertia terms and considering only the transverse vibrations. The conditions under which this simplification is valid were established through a fractional analysis of the governing equations.

The frequency determinate was evaluated for the case of a hyperbolic paraboloid, of general profile, simply supported at its edges. The lowest natural frequencies were then calculated and presented as a function of the physical dimensions of the shell.

Finally an experimental study was proposed to help answer the questions that arose in developing the mathematical model.

Review of the literature

An extensive literature search was conducted to determine the present status of the problem. In this search many excellent articles were found that dealt with the statics problem in shell design but relatively few articles discussed the dynamics problem.[2,5]

Foremost among those authors who have published papers in the area of thin shells was Eric Reissner.[2] Many of his studies were supported by the Office of Naval Research.

Since most of the periodical literature omitted the derivation of the basic equations it was necessary to turn to the various texts on shell theory.

The earliest text found was "A Treatise On The Mathematical Theory of Elasticity" by A.E.H. Love[6] written in 1892. His work is considered by many present day authorities as being the classic reference in the area of elasticity. However, it is more difficult to read than are some of the newer books.

Among the more recent texts available are "The Theory of Thin Shells" by V.V. Novozhilov,[1] "Stresses in Shells" by Wilhelm Flugge,[4] and "The Theory of Plates and Shells" by S. Timoshenko.[7]

It should be noted here that the majority of this thesis is based on the general theory as presented by V.V. Novozhilov and the applied work of Eric Reissner. (Both of which have been previously referenced.)

CHAPTER II

THE GENERAL THEORY OF THIN SHELLS*

Introduction

A shell is a body bounded by two curved surfaces, called the faces. The middle surface is defined as the locus of points which lie at equal distances from the faces.

The thickness of the shell is the distance between the two faces measured perpendicular to the middle surface. In practice the thickness of a shell does not vary with respect to the other dimensions. The geometry of a shell is completely defined when the middle surface, thickness, and the edges are specified.

I. BASIC ASSUMPTIONS

The present discussion of the theory of thin shells is restricted to the study of the deformation of elastic bodies under the influence of given loads. Thus, it will be assumed that the material of the shell is isotropic and obeys Hook's law and that the displacements at a point are small in comparison with the thickness of the shell.

This latter assumption leads to linear differential equations which are solved with much less difficulty than their non-linear counterparts.

In the following analysis a shell will be called thin, if the maximum value of the ratio δ/R (where δ is the shell thickness and R is the radius of curvature of the middle surface), can be neglected in comparison with unity. This restriction simplifies the basic equations considerably.

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*This entire chapter consists of an outline of the derivation of the equations of thin shells as presented in the book "The Theory of Thin Shells" by V.V. Novozhilov, (Ref. 1). The material herein will be presented without further reference to the original text.

Stated more precisely the practical range of thin shells is,

$$\frac{1}{1000} \leqq \frac{\delta}{R} \leqq \frac{1}{50} .$$

In addition to the foregoing assumptions and definitions the theory of thin shells is based on the two fundamental hypotheses proposed by G. Kirchhoff.

a) The straight fibers of a plate which are perpendicular to the middle surface before deformation remain so after deformation and do not change length.

b) The normal stresses acting on planes parallel to the middle surface may be neglected in comparison with the other stresses.

Thus it is apparent that the analysis of a thin shell is reduced to a study of the deformation of the middle surface in much the same way that the analysis of a beam is reduced to the study of the deformation of the neutral axis.

II. GENERAL CURVILINEAR COORDINATE SYSTEM

In the classical approach to the theory of thin shells considerable use is made of a general curvilinear coordinate system. This system is related to the Cartesian coordinate system through the use of the parameters α_1 and α_2 such that,

$$\begin{aligned} x &= f_1(\alpha_1, \alpha_2), \\ y &= f_2(\alpha_1, \alpha_2), \\ z &= f_3(\alpha_1, \alpha_2), \end{aligned} \tag{1}$$

where f_1, f_2, f_3 are continuous, single-valued functions of the parameters α_1, α_2.

If α_2 is held constant and α_1 is varied then a family of curves called coordinate lines α_1 will be generated. In a similar manner coordinate lines α_2 are found. The result is the development of a double curvature surface.

Furthermore, every point on the surface is specified by the intersection of two coordinate lines (see Fig. 1).

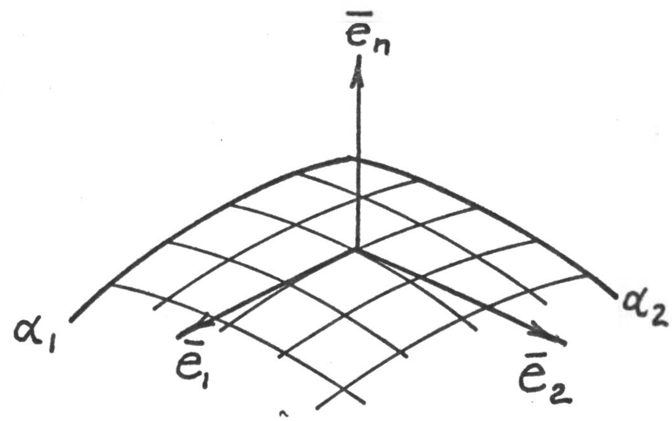

Fig. 1

Since the choice of system of parameters is arbitrary it is convenient to choose α_1 and α_2 as the lines of principal curvature of the surface. This particular system will be orthogonal, due to the fact that the directions of principal curvature on a surface are mutually perpendicular. In the derivation of the general equations of thin shells this system of orthogonal curvilinear coordinates is adopted.

The three scalar equations (1) correspond to a single vector equation,

$$\bar{r} = \bar{r}(\alpha_1, \alpha_2) \tag{2}$$

whose projections on the x, y, z axes are given by equation (1).

It may be shown that an increase in one of the curvilinear coordinates corresponds to a shift of the vector $\bar{r}$ on the surface in the direction corresponding to the increased coordinate. Hence the vector

$$d\bar{s} = \frac{\partial \bar{r}}{\partial \alpha_1} d\alpha_1 + \frac{\partial \bar{r}}{\partial \alpha_2} d\alpha_2 \tag{3}$$

is equal in magnitude and direction to the segment joining the points (α_1, α_2) and $(\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2)$ of the surface. The square of the length of this segment is given by

$$|d\bar{s}|^2 = \left|\frac{\partial \bar{r}}{\partial \alpha_1}\right|^2 d\alpha_1^2 + 2\left(\frac{\partial \bar{r}}{\partial \alpha_1} \cdot \frac{\partial \bar{r}}{\partial \alpha_2}\right) d\alpha_1 d\alpha_2 + \left|\frac{\partial \bar{r}}{\partial \alpha_2}\right|^2 d\alpha_2^2. \tag{3a}$$

Introducing now the notation

$$\left|\frac{\partial \bar{r}}{\partial \alpha_1}\right|^2 = A_1^2 = \left(\frac{\partial x}{\partial \alpha_1}\right)^2 + \left(\frac{\partial y}{\partial \alpha_1}\right)^2 + \left(\frac{\partial z}{\partial \alpha_1}\right)^2, \tag{4}$$

$$\left|\frac{\partial \bar{r}}{\partial \alpha_2}\right|^2 = A_2^2 = \left(\frac{\partial x}{\partial \alpha_2}\right)^2 + \left(\frac{\partial y}{\partial \alpha_2}\right)^2 + \left(\frac{\partial z}{\partial \alpha_2}\right)^2, \tag{5}$$

where the quantities A_1, A_2 are called the Lame' parameters and the principal radii of curvature R_1, R_2 are established by the conditions of Codazzi and the condition of Gauss respectively:

$$\begin{aligned} \frac{\partial}{\partial \alpha_1}\left(\frac{A_2}{R_2}\right) &= \frac{1}{R_1} \frac{\partial A_2}{\partial \alpha_1}, \\ \frac{\partial}{\partial \alpha_2}\left(\frac{A_1}{R_1}\right) &= \frac{1}{R_2} \frac{\partial A_1}{\partial \alpha_2}, \end{aligned} \tag{6}$$

$$\frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_1} \frac{\partial A_2}{\partial \alpha_1}\right) + \frac{\partial}{\partial \alpha_2}\left(\frac{1}{A_2} \frac{\partial A_1}{\partial \alpha_2}\right) = -\frac{A_1 A_2}{R_1 R_2}. \tag{7}$$

These relations were derived directly from the partial derivatives of $\bar{e}_1, \bar{e}_2, \bar{e}_n$ with respect to α_1, α_2.

III. THE LAW OF VARIATION OF DISPLACEMENTS ACROSS THE THICKNESS OF THE SHELL

Let a point on the middle surface before deformation be designated as O (see Fig. 2) and let the vector $\bar{r}$ describe this point. Now, as a result of deformation let this point undergo a small displacement $\bar{\Delta}$ and designate the new position as O'.

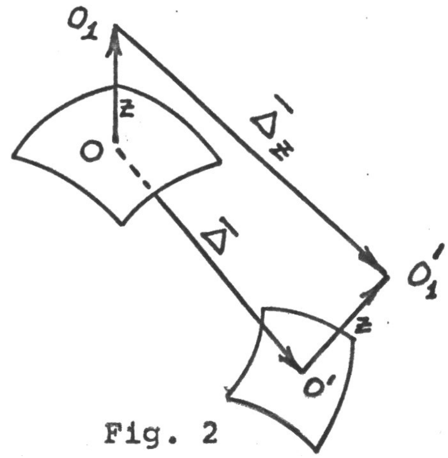

Fig. 2

The deformed middle surface can be described by the vector

$$\bar{R} = \bar{r} + \bar{\Delta} = \bar{r} + u\,\bar{e}_1 + v\,\bar{e}_2 + w\,\bar{e}_n \quad , \tag{8}$$

where u, v, w are the projections of the displacements of the middle surface in the direction of $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_n$ respectively.

Using the methods of vector analysis the displacement of an arbitrary point in the shell may be expressed in terms of the displacement of its corresponding point on the middle surface. This relation is given by

$$\begin{aligned} u_{(z)} &= u + z\,\vartheta \,, \\ v_{(z)} &= v + z\,\psi \,, \\ w_{(z)} &= w, \end{aligned} \tag{9}$$

where $u_{(z)}, v_{(z)}, w_{(z)}$ are the displacements of the arbitrary point, and

$$\vartheta = -\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} + \frac{u}{R_1},$$

$$\psi = -\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} + \frac{v}{R_2}. \tag{10}$$

Equation (9) is referred to as the "law of variation of displacements across the thickness of the shell." This law is consistant with the basic assumptions in that the variation of displacements is linear and the normal displacement $w_{(z)}$ is independent of z.

IV. THE DEFORMATION OF A SHELL AND THE DEFORMATION OF ITS MIDDLE SURFACE

If A_1' is defined as one of the Lame' parameters of the deformed middle surface, then from equation (8)

$$\left(A_1'\right)^2 = \left|\frac{\partial \bar{R}}{\partial \alpha_1}\right|^2 \approx A_1(1+\varepsilon_1) \tag{11}$$

where
$$\varepsilon_1 = \frac{1}{A_1}\frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} v + \frac{w}{R_1}.$$

Now consider a segment of the line α_1 in the middle surface whose length before deformation is given by

$$ds_1 = A_1\, d\alpha_1$$

and after deformation,

$$ds_1' = A_1' d\alpha_1 .$$

The strain experienced by this element is

$$\frac{ds_1' - ds_1}{ds_1} = \varepsilon_1 . \tag{12a}$$

In like manner the strain in the direction α_2 is given by

$$\varepsilon_2 = \frac{1}{A_2}\frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} u + \frac{w}{R_2}. \tag{12b}$$

The shear strain, ω , is equal to the cosine of the angle between the lines α_1 and α_2 after deformation. Hence,

$$\omega = (\bar{e}_1' \cdot \bar{e}_2'),$$

which upon substitution of the values for $\bar{e}_1'$, $\bar{e}_2'$ becomes

$$\omega = \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{v}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{u}{A_1}\right). \tag{12c}$$

Considering a surface parallel to the middle surface at a distance z (see Fig. 3),

$$ds_1 = A_1 d\alpha_1 = m_1 m_2 ,$$

$$m_1 m_2 = R_1 \Delta\theta ,$$

$$n_1 n_2 = R_1^{(z)} \Delta\theta,$$

$$R_1^{(z)} = R_1 + z ,$$

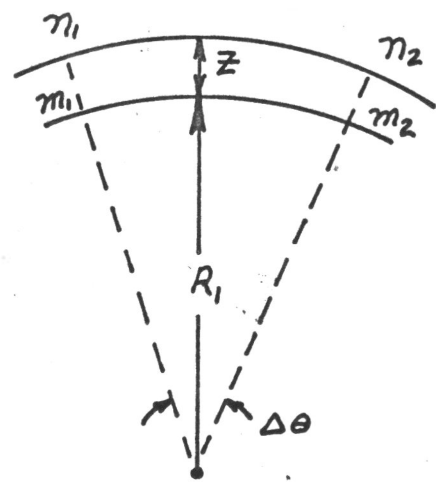

Fig. 3

Hence,

$$ds_1^{(z)} = n_1 n_2 = A_1\left(1+\frac{z}{R_1}\right)d\alpha_1 . \tag{13}$$

From equations (12) and (13) the deformation of any surface parallel to the middle surface can be expressed in terms of the deformation of the middle surface as follows:

$$\varepsilon_1^{(z)} = \frac{1}{1+z/R_1}\left(\varepsilon_1 + zK_1\right),$$

$$\varepsilon_2^{(z)} = \frac{1}{1+z/R_2}\left(\varepsilon_2 + zK_2\right), \tag{14}$$

$$\omega^{(z)} = \frac{1}{1+z/R_1 \quad 1+z/R_2}\Big\{\left(1-\frac{z^2}{R_1R_2}\right)\omega$$

$$+2\left[1+\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{z}{2}\right]z\tau\Big\},$$

where,

$$K_1 = -\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1} - \frac{u}{R_1}\right) - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2} - \frac{v}{R_2}\right),$$

$$K_2 = -\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2} - \frac{v}{R_2}\right) - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_1} - \frac{u}{R_1}\right),$$

$$\tau = -\frac{1}{A_1 A_2}\left(\frac{\partial^2 w}{\partial\alpha_1\partial\alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial w}{\partial\alpha_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial w}{\partial\alpha_2}\right) +$$

$$+\frac{1}{R_1}\left(\frac{1}{A_2}\frac{\partial u}{\partial\alpha_2} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial\alpha_2}u\right) + \frac{1}{R_2}\left(\frac{1}{A_1}\frac{\partial v}{\partial\alpha_1} - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial\alpha_1}v\right).$$

The parameters K_1, K_2, τ characterize the change of curvature and the torsion of the middle surface.

Equations (14) may be simplified by expanding the z/R terms in a power series of z and neglecting terms z^2 and higher. This step is justified by the thin shell assumptions. Thus, it is found that

$$\varepsilon_1^{(z)} \approx \varepsilon_1 + z(K_1 - \varepsilon_1/R_1),$$

$$\varepsilon_2^{(z)} \approx \varepsilon_2 + z(K_2 - \varepsilon_2/R_2),$$

$$\omega^{(z)} \approx \omega + 2z\left[\tau - \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{\omega}{2}\right].$$

In these equations the first terms represent the strain distributed throughout the plane of the shell and the second terms represent the bending strain. Also the coefficients of z in the first two of the formulae above are equal to the changes of curvature of the middle surface in the directions α_1, α_2.

V. COMPATIBILITY CONDITIONS

To insure that the middle surface remains continuous after deformation the six deformation parameters ε_1, ε_2, ω, K_1, K_2, τ must satisfy the following compatibility conditions for shells of arbitrary form (first deduced by Gol'denweizer using the conditions of Gauss-Codazzi).

$$A_2 \frac{\partial K_2}{\partial \alpha_1} + \frac{\partial A_2}{\partial \alpha_1}(K_2 - K_1) - A_1 \frac{\partial \tau}{\partial \alpha_2} - 2\frac{\partial A_1}{\partial \alpha_2}\tau + \frac{1}{R_2}\frac{\partial A_1}{\partial \alpha_2}\omega +$$

$$+ \frac{1}{R_1}\left[A_1 \frac{\partial \omega}{\partial \alpha_2} + \frac{\partial A_1}{\partial \alpha_2}\omega - A_2 \frac{\partial \varepsilon_2}{\partial \alpha_1} - \frac{\partial A_2}{\partial \alpha_1}(\varepsilon_2 - \varepsilon_1)\right] = 0,$$

$$A_1 \frac{\partial K_1}{\partial \alpha_2} + \frac{\partial A_2}{\partial \alpha_1}(K_1 - K_2) - A_2 \frac{\partial \tau}{\partial \alpha_1} - 2\frac{\partial A_2}{\partial \alpha_1}\tau + \frac{1}{R_1}\frac{\partial A_2}{\partial \alpha_1}\omega +$$

$$+ \frac{1}{R_2}\left[A_2 \frac{\partial \omega}{\partial \alpha_1} + \frac{\partial A_2}{\partial \alpha_1}\omega - A_2 \frac{\partial \varepsilon_1}{\partial \alpha_2} - \frac{\partial A_1}{\partial \alpha_2}(\varepsilon_1 - \varepsilon_2)\right] = 0, \qquad (16)$$

$$\frac{K_1}{R_2} + \frac{K_2}{R_1} + \frac{1}{A_1 A_2}\left\{\frac{\partial}{\partial \alpha_1}\frac{1}{A_1}\left[A_2 \frac{\partial \varepsilon_2}{\partial \alpha_1} + \frac{\partial A_2}{\partial \alpha_1}(\varepsilon_2 - \varepsilon_1) - \frac{A_1}{2}\frac{\partial \omega}{\partial \alpha_2} - \frac{\partial A_1}{\partial \alpha_2}\omega\right]\right.$$

$$\left. + \frac{\partial}{\partial \alpha_2}\frac{1}{A_2}\left[A_1 \frac{\partial \varepsilon_1}{\partial \alpha_2} + \frac{\partial A_1}{\partial \alpha_2}(\varepsilon_1 - \varepsilon_2) - \frac{A_2}{2}\frac{\partial \omega}{\partial \alpha_1} - \frac{\partial A_2}{\partial \alpha_1}\omega\right]\right\} = 0.$$

VI. FORCES AND MOMENTS

Consider an element cut from the shell as shown in Fig. 4.

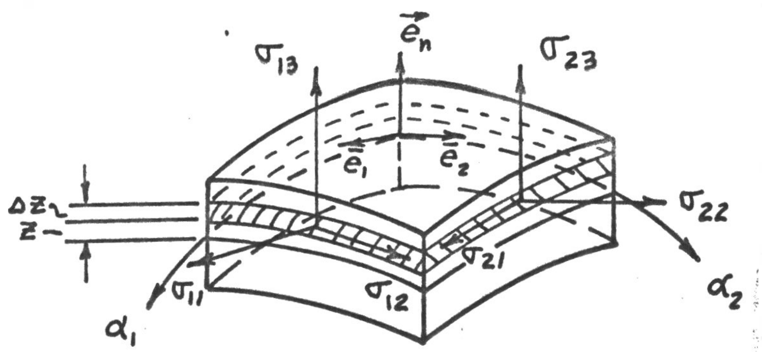

Fig. 4

Let

σ_{11}, σ_{22} be the normal stresses, acting on the faces of this element,
σ_{12}, σ_{21} be the shear stresses on these faces, acting parallel to the middle surface,
σ_{13}, σ_{23} be the shear stresses on these faces, acting normal to the middle surface.

The resultants of the stresses σ_{11}, σ_{12}, σ_{13} acting on the individual faces may be written as,

$$S_{11} = A_2 \, d\alpha_2 \int_{-\delta/2}^{+\delta/2} \sigma_{11} (1 + z/R_2) \, dz,$$

$$S_{12} = A_2 \, d\alpha_2 \int_{-\delta/2}^{+\delta/2} \sigma_{12} (1 + z/R_2) \, dz,$$

$$S_{13} = A_2 \, d\alpha_2 \int_{-\delta/2}^{+\delta/2} \sigma_{13} (1 + z/R_2) \, dz.$$

And the forces per unit length of a line α_2 are expressed by the following,

$$T_1 = \frac{S_{11}}{A_2 d\alpha_2} = \int_{-\delta/2}^{+\delta/2} \sigma_{11}(1+z/R_2)\,dz,$$

$$T_{12} = \frac{S_{12}}{A_2 d\alpha_2} \int_{-\delta/2}^{+\delta/2} \sigma_{12}(1+z/R_2)\,dz, \tag{17}$$

$$N_1 = \frac{S_{13}}{A_2 d\alpha_2} = \int_{-\delta/2}^{+\delta/2} \sigma_{13}(1+z/R_2)\,dz$$

In a like manner the forces per unit length of a line α_1 become,

$$T_2 = \int_{-\delta/2}^{+\delta/2} \sigma_{22}(1+z/R_1)\,dz,$$

$$T_{21} = \int_{-\delta/2}^{+\delta/2} \sigma_{21}(1+z/R_1)\,dz,$$

$$N_2 = \int_{-\delta/2}^{+\delta/2} \sigma_{23}(1+z/R_1)\,dz.$$

Consider next the moments of the stresses σ_{11}, σ_{12} acting on one of the faces of the element about the directions $\bar{e}_2, \bar{e}_1$ respectively. These are the bending and twisting moments per unit length of the line α_2 and are expressed by,

$$M_1 = \int_{-\delta/2}^{+\delta/2} z(1+z/R_2)\,\sigma_{11}\,dz,$$

$$M_{12} = \int_{-\delta/2}^{+\delta/2} z(1+z/R_2)\,\sigma_{12}\,dz. \tag{19}$$

Also the bending and twisting moments per unit length of a line α_1 are

$$M_2 = \int_{-\delta/2}^{+\delta/2} z(1+z/R_1)\,\sigma_{22}\,dz$$

$$M_{21} = \int_{-\delta/2}^{+\delta/2} z(1+z/R_1)\,\sigma_{21}\,dz \tag{20}$$

Thus the six forces T_1, T_{12}, N_1, T_2, T_{21}, N_2 and the four moments M_1, M_{12}, M_2, M_{21} completely characterize the state of stress of the shell.

VII. EQUILIBRIUM OF A SHELL ELEMENT

Introducing the vector notation,

$$\bar{T}^{(1)} = (T_1\bar{e}_1 + T_{12}\bar{e}_2 + N_1\bar{e}_n)\,A_2\,d\alpha_2 \quad , \quad \bar{M}^{(1)} = (M_1\bar{e}_2 - M_{12}\,e_1)\,A_2\,d\alpha_2$$

$$\bar{T}^{(2)} = (T_2\bar{e}_1 + T_{21}\bar{e}_2 + N_2\bar{e}_n)\,A_1\,d\alpha_1 \quad , \quad \bar{M}^{(2)} = (M_{21}\bar{e}_2 - M_2\,\bar{e}_1)\,A_1\,d\alpha_1$$

the forces on a free body diagram may be represented as shown in Fig. 5.

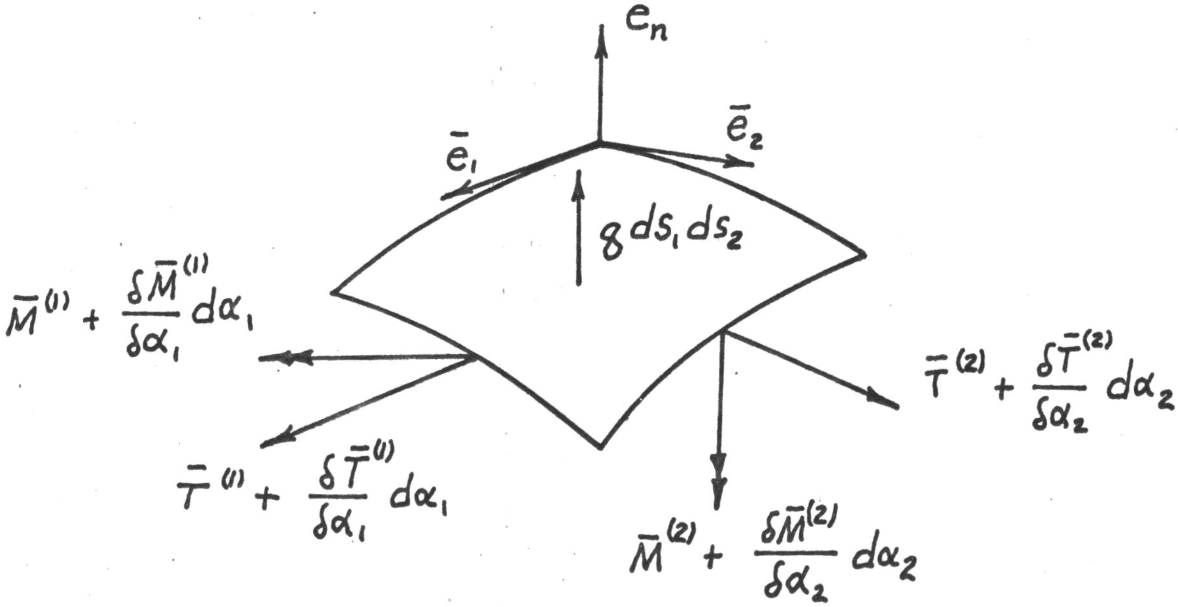

Fig. 5

The condition of equilibrium requires that,

$$\sum \bar{F} = 0,$$

$$\sum \bar{M} = 0,$$

which leads to the following differential equations.

$$\frac{1}{A_1 A_2}\left[\frac{\partial A_2 T_1}{\partial \alpha_1} + \frac{\partial A_1 T_{21}}{\partial \alpha_2} + \frac{\partial A_1}{\partial \alpha_2} T_{12} - \frac{\partial A_2}{\partial \alpha_1} T_2\right] \frac{N_1}{R_1} + q_1 = 0,$$

$$\frac{1}{A_1 A_2}\left[\frac{\partial A_2 T_{12}}{\partial \alpha_1} + \frac{\partial A_1 T_2}{\partial \alpha_2} + \frac{\partial A_2}{\partial \alpha_1} T_{21} - \frac{\partial A_1}{\partial \alpha_2} T_1\right] \frac{N_2}{R_2} + q_2 = 0,$$

$$\frac{1}{A_1 A_2}\left[\frac{\partial A_2 N_1}{\partial \alpha_1} + \frac{\partial A_1 N_2}{\partial \alpha_2}\right] - \frac{T_1}{R_1} - \frac{T_2}{R_2} + q_n = 0. \tag{21}$$

$$\frac{1}{A_1 A_2}\left[\frac{\partial A_2 M_1}{\partial \alpha_1} + \frac{\partial A_1 M_{21}}{\partial \alpha_2} + \frac{\partial A_1}{\partial \alpha_2} M_{12} - \frac{\partial A_2}{\partial \alpha_1} M_2\right] - N_1 = 0,$$

$$\frac{1}{A_1 A_2}\left[\frac{\partial A_2 M_{12}}{\partial \alpha_1} + \frac{\partial A_1 M_2}{\partial \alpha_2} + \frac{\partial A_2}{\partial \alpha_1} M_{21} - \frac{\partial A_1}{\partial \alpha_2} M_1\right] - N_2 = 0,$$

$$T_{12} - T_{21} + \frac{M_{12}}{R_1} - \frac{M_{21}}{R_2} = 0.$$

These six equations of equilibrium involve ten unknowns and hence the problem is statically indeterminate. However, the solution is made possible by introducing the relations between stress and strain of an elastic body. This will be done in the next section.

VIII. THE STRAIN ENERGY FOR A SHELL

From the basic theory of elasticity it is known that the equation for the potential energy of a material which is homogeneous, isotropic, and obeys Hooke's law is given by,

$$V = \frac{1}{2}\int(\sigma_{11}e_{11} + \sigma_{22}e_{22} + \sigma_{33}e_{33} + \sigma_{12}e_{12} + \sigma_{13}e_{13} + \sigma_{23}e_{23})\,do \quad (22)$$

where

$\sigma_{11}, \sigma_{22}, \sigma_{33}$ are the normal stresses on three mutually perpendicular faces of an element, cut from the elastic body,
e_{11}, e_{22}, e_{33} are the corresponding strains,
$\sigma_{12}, \sigma_{13}, \sigma_{23}$ are the shear stresses on these faces,
e_{12}, e_{13}, e_{23} are the corresponding shear deformations,
do is the volume of the element.

Also, by Hooke's law,

$$\begin{aligned}
\sigma_{11} &= \frac{E}{1+\mu}\left(e_{11} + \frac{\mu}{1-2\mu}e\right),\\
\sigma_{22} &= \frac{E}{1+\mu}\left(e_{22} + \frac{\mu}{1-2\mu}e\right),\\
\sigma_{33} &= \frac{E}{1+\mu}\left(e_{33} + \frac{\mu}{1-2\mu}e\right),\\
\sigma_{12} &= \frac{E}{2(1+\mu)}e_{12},
\end{aligned} \quad (23)$$

E is Young's modulus,

μ is Poisson's ratio,

$e = e_{11} + e_{22} + e_{33}$.

Utilization of these concepts leads to the potential energy of deformation for a thin shell given by,

$$V = \frac{E\delta}{2(1-\mu^2)} \iint \left[(\varepsilon_1 + \varepsilon_2)^2 - 2(1-\mu)\left(\varepsilon_1\varepsilon_2 - \frac{\omega^2}{4}\right)\right] A_1 A_2 \, d\alpha_1 d\alpha_2 + \tag{24}$$
$$+ \frac{E\delta^3}{24(1-\mu^2)} \iint \left[(K_1 + K_2)^2 - 2(1-\mu)(K_1 K_2 - \tau^2)\right] A_1 A_2 \, d\alpha_1 d\alpha_2$$

In this equation the first term represents the potential energy of extension and shear, the second term represents the potential energy of bending and torsion.

IX. RELATIONS BETWEEN THE FORCES, MOMENTS, AND THE STRAINS OF THE MIDDLE SURFACE

Combining the results of the last three sections and neglecting quantities of order δ/R compared with unity the following approximate relations between the forces, moments, and the strains of the middle surface are obtained.

$$T_1 = \frac{E\delta}{1-\mu^2}(\varepsilon_1 + \mu\varepsilon_2), \qquad T_2 = \frac{E\delta}{1-\mu^2}(\varepsilon_2 + \mu\varepsilon_1),$$
$$T_{12} = T_{21} = \frac{E\delta}{(1-\mu)}\,\frac{\omega}{2} \tag{25}$$
$$M_1 = \frac{E\delta^3}{12(1-\mu^2)}(K_1 + \mu K_2), \qquad M_2 = \frac{E\delta^3}{12(1-\mu^2)}(K_2 + \mu K_1),$$
$$M_{12} = M_{21} = \frac{E\delta^3}{12(1+\mu)}\tau$$

Thus equations (21) and (25) are sufficient to solve the problem of the theory of thin shells.

However, use of the equations (25) results in some contradictions in the shell theory. Specifically, not all the deformations of the elastic body can be expressed as potential energy of deformation. Also the six equilibrium equations (21) are not identically satisfied.

Even though the use of the approximate equations leads to the above contradictions the author of the original derivation (V.V. Novozhilov) contends that the mathematical inaccuracies do not exceed the inaccuracies due to the initial assumptions.

CHAPTER III

TRANSVERSE VIBRATIONS OF SHALLOW SHELLS

Introduction

The previous chapter was concerned with the general theory of thin shells and is applicable to any coordinate system. This general approach is useful in deriving the basic equations but before a particular problem can be solved the coordinate system must be specified.

Thus in this chapter the equations of a thin shell will be presented using the Cartesian coordinate system. Furthermore, the theory will be restricted to the case of shallow shells, i.e., those whose height to span ratio is small compared to unity. This restriction leads to simplified equations in which the longitudinal inertia terms are negligible.

I. BASIC DIFFERENTIAL EQUATIONS

Let the middle surface be defined in Cartesian coordinates by an equation of the form

$$z = f(x, y),$$

where z is a second-degree equation representing a surface which has a rectangular projection on the base plane (see Fig. 6).

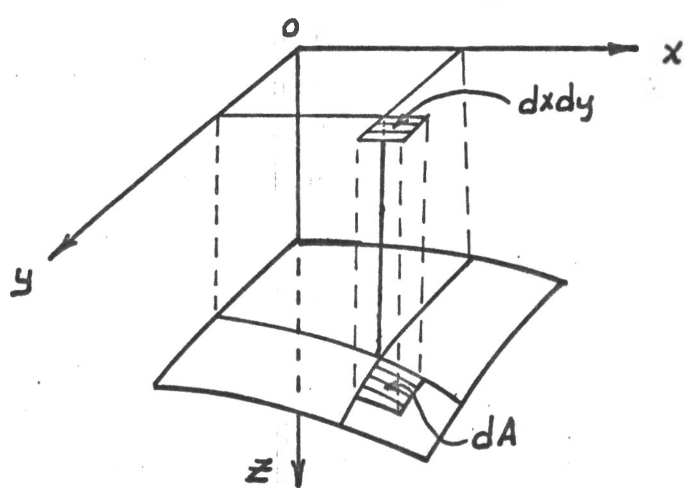

Fig. 6

Following a procedure similar to that in Chapter 2, the German scientist K. Marguerre formulated the equations for the linear theory of shallow shells. These are presented in the following form.[2]

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \rho\delta \frac{\partial^2 u}{\partial t^2}, \tag{1a}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = \rho\delta \frac{\partial^2 v}{\partial t^2}, \tag{1b}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x} N_x + \frac{\partial z}{\partial y} N_{xy}\right] + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x} N_{xy} + \frac{\partial z}{\partial y} N_y\right] = \rho\delta \frac{\partial^2 w}{\partial t^2}, \tag{2}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0, \tag{3a}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0, \tag{3b}$$

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right), \tag{4a}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right), \tag{4b}$$

$$M_{xy} = -(1-\nu) D \frac{\partial^2 w}{\partial x \, \partial y}, \tag{5a}$$

$$\frac{\partial u}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial w}{\partial x} = \frac{N_x - \nu N_y}{E\delta}, \tag{5b}$$

$$\frac{\partial v}{\partial y} + \frac{\partial z}{\partial y}\frac{\partial w}{\partial y} = \frac{N_y - \nu N_x}{E\delta}, \tag{5c}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial z}{\partial y}\frac{\partial w}{\partial x} = \frac{N_{xy}}{G\delta}. \tag{5d}$$

In the foregoing relations the middle surface is given by the equation $z = z(x,y)$ and the components of displacement in the x, y, z directions are u, v, w respectively. In addition,

t	time
ρ	density of the shell material
δ	shell thickness
N	tangential stress resultant
Q	transverse stress resultant
M	stress couples
D	bending stiffness, $E\delta^3/12(1-\nu^2)$
E	Young's modulus
ν	Poisson's ratio

Introduction of equations (3),(4),(5) into (1) and (2) leads to three simultaneous equations in u, v, w. For E and δ constant these equations reduce to the following form.

$$\frac{1-\nu}{2}\nabla^2 u + \frac{1+\nu}{2}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + \frac{1-\nu}{2}\left(\frac{\partial z}{\partial x}\nabla^2 w + \frac{\partial w}{\partial x}\nabla^2 z\right)$$

$$+\frac{1+\nu}{2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial z}{\partial y}\frac{\partial w}{\partial y}\right) = \frac{\rho}{E}\frac{\partial^2 u}{\partial t^2}, \tag{6a}$$

$$\frac{1-\nu}{2}\nabla^2 v + \frac{1+\nu}{2}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + \frac{1-\nu}{2}\left(\frac{\partial z}{\partial y}\nabla^2 w + \frac{\partial w}{\partial y}\nabla^2 z\right) \tag{6b}$$

$$+\frac{1+\nu}{2}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial z}{\partial y}\frac{\partial w}{\partial y}\right) = \frac{\rho}{E}\frac{\partial^2 v}{\partial t^2},$$

$$-D\nabla^4 w + \frac{E\delta}{1-\nu^2}\left\{\frac{\partial^2 z}{\partial x^2}\left[\frac{\partial u}{\partial x}+\frac{\partial z}{\partial x}\frac{\partial w}{\partial x}+\nu\left(\frac{\partial v}{\partial y}+\frac{\partial z}{\partial y}\frac{\partial w}{\partial y}\right)\right]+\frac{\partial^2 z}{\partial y^2}\left[\frac{\partial v}{\partial y}+\right.\right.$$

$$\left.+\frac{\partial z}{\partial y}\frac{\partial w}{\partial y}+\nu\left(\frac{\partial u}{\partial x}+\frac{\partial z}{\partial x}\frac{\partial w}{\partial x}\right)\right]+(1-\nu)\frac{\partial^2 z}{\partial x\partial y}\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\right.$$

$$\left.\left.+\frac{\partial z}{\partial x}\frac{\partial w}{\partial y}+\frac{\partial z}{\partial y}\frac{\partial w}{\partial x}\right]\right\} = \rho\delta\left[\frac{\partial^2 w}{\partial t^2}-\frac{\partial z}{\partial x}\frac{\partial^2 u}{\partial t^2}-\frac{\partial z}{\partial y}\frac{\partial^2 v}{\partial t^2}\right] \tag{6c}$$

It may be appropriate at this point to quote a statement made by Eric Reissner concerning the complexity of these equations.[2]

> "...an indication of the difficulties associated with this system of equations (is) that no successful attempt is known of an evaluation of the frequency determinant obtained from this theory for what is probably the simplest of these spherical-shell problems, the problem of the complete shell segment with clamped edge."

II. SIMPLIFIED EQUATIONS*

A common method of solving simultaneous equations in problems of statics is to introduce a new function, called the stress function.

In problems of dynamics this approach is prevented by the presence of the longitudinal inertia terms in equations (1). However, for the class of dynamic problems in which vibrations take place principally transversely the equations (6) may be reduced to two simultaneous equations involving the vertical displacements, w, and the Airy stress function, F.

The stress function, F, is defined such that

$$N_x = \frac{\partial^2 F}{\partial y^2}, \quad N_y = \frac{\partial^2 F}{\partial x^2}, \quad N_{xy} = (-)\frac{\partial^2 F}{\partial x \partial y}, \tag{7}$$

satisfy equations (1) when the longitudinal inertia terms are neglected. Utilization is also made of strain compatibility which gives an equation of the form,

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial w}{\partial x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial y} + \frac{\partial z}{\partial y}\frac{\partial w}{\partial y}\right) - \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial z}{\partial y}\frac{\partial w}{\partial x}\right) = 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial^2 w}{\partial y \partial x} - \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 z}{\partial y^2}\frac{\partial^2 w}{\partial x^2}. \tag{8}$$

*The next two sections consist of a complete review of a paper written by Eric Reissner on the vibrations of thin shells (see Ref. 2).

Substituting equations (5) and (7) into (8) gives the first of two simultaneous equations in F and w,

$$\nabla^4 F = E\delta\left[2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 z}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right]. \quad \text{(I)}$$

Similarily substitution of equations (3),(4), and (7) into (2) yields the second simultaneous equation,

$$D\nabla^4 w + \rho\delta\frac{\partial^2 w}{\partial t^2} = -2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 z}{\partial y^2}\frac{\partial^2 F}{\partial x^2}. \quad \text{(II)}$$

III. JUSTIFICATION OF THE SIMPLIFIED DIFFERENTIAL EQUATIONS

The conditions under which the longitudinal inertia terms may be neglected involves an extensive order-of-magnitude analysis of the displacement differential equations. A common method of approach is outlined as follows.*

1. Normalize the main variables.
2. Normalize the coefficients of the derivative terms by dividing by any one of the coefficients.
3. Arrange the resulting parameters to get more suitable ones as desired from physical considerations.
4. Through the use of fractional analysis the insignificant terms are eliminated thus simplifying the original problem.

Refer now to equation (6). The variables are normalized by introducing the following dimensionless parameters,

$$\xi = \frac{x}{l}, \qquad \eta = \frac{y}{l}, \qquad \tau = \omega t, \qquad \text{(9a,b,c)}$$

$$w = Wf, \qquad u = Ug, \qquad v = Vk, \qquad \text{(10a,b,c)}$$

---

*See Appendix I.

such that the functions f, g, K and their derivatives with respect to ξ, η, τ are at most of order of magnitude unity. The constants W, U, V are arbitrary but ℓ and t are characteristics of the system. In addition, let

$$Z_x = \frac{H}{L}\psi_x \quad , \qquad Z_y = \frac{H}{L}\psi_y , \qquad \text{(11a,b)}$$

$$Z_{xx} = \frac{H}{L^2}\psi_{xx} \quad , \quad Z_{yy} = \frac{H}{L^2}\psi_{yy} \quad , \quad Z_{xy} = \frac{H}{L^2}\psi_{xy} , \qquad \text{(12)}$$

where once again ψ_x, ψ_y, ψ_{xx}, ψ_{xy}, ψ_{yy} are at most of order of magnitude unity and where $(H/L)^2$ is negligibly small compared with unity. This last statement specifies a shallow shell i.e., one for which the maximum height to span ratio is less than, say 1/8.

Substitution of equations (9) to (12) into equation (6) yields the following: From (6a),

$$\frac{1-\nu}{2}\left[\frac{U}{\ell^2}g_{\xi\xi} + \frac{U}{\ell^2}g_{\eta\eta}\right] + \frac{1+\nu}{2}\left[\frac{U}{\ell^2}g_{\xi\xi} + \frac{V}{\ell^2}K_{\xi\eta}\right]$$

$$+ \frac{1-\nu}{2}\left[\frac{H}{L}\psi_x\left(\frac{W}{\ell^2}f_{\xi\xi} + \frac{W}{\ell^2}f_{\eta\eta}\right) + \frac{W}{\ell}f_\xi\left(\frac{H}{L^2}\psi_{xx} \ \frac{H}{L^2}\psi_{yy}\right)\right]$$

$$+ \frac{1+\nu}{2}\left[\left(\frac{H}{L}\psi_x\right)\left(\frac{W}{\ell^2}f_{\xi\xi}\right) + \left(\frac{W}{\ell}f_\xi\right)\left(\frac{H}{L^2}\psi_{xx}\right)\right] +$$

$$+ \left[\left(\frac{H}{L}\psi_y\right)\left(\frac{W}{\ell^2}f_{\eta\eta}\right) + \left(\frac{W}{\ell}f_\eta\right)\left(\frac{H}{L^2}\psi_{yy}\right)\right] = \frac{\rho\omega^2}{E}Ug_{\tau\tau} .$$

Considering only the representative types of terms, the above equation becomes

$$\frac{U}{\ell^2}g_{\xi\xi} + \frac{H}{L}\psi_x\frac{W}{\ell^2}f_{\xi\xi} + \frac{H}{L^2}\psi_{xx}\frac{W}{\ell}f_\xi + \cdots = \frac{\rho\omega^2}{E}Ug_{\tau\tau} . \qquad \text{(13a)}$$

Since equation (6b) is analogous in structure to (6a) it will contain the same representative types of terms and need not be considered separately. Now, from (6c),

$$\frac{E\delta^3}{12(1-\nu^2)} \frac{W}{\ell^4}\left[-f_{\xi\xi\xi\xi} + \cdots\right] + \frac{E\delta}{1-\nu^2} \frac{H}{L^2} \psi_{xx}\left\{\left[\frac{U}{\ell} g_\xi + \frac{H}{L}\psi_x \frac{W}{\ell} f_\xi\right]\right.$$

$$\left. + \cdots\right\} + \cdots = \rho\delta\omega^2\left[W f_{\tau\tau} - \frac{H}{L}\psi_x U g_{\tau\tau} - \cdots\right]. \tag{13b}$$

It is expected that longitudinal straining will exert an influence on transverse vibrations, therefore it is necessary that in (13a) terms involving both f and g remain significant, i.e., of order unity. This is accomplished by setting

$$U = \frac{H}{L} W. \tag{14}$$

Substituting this value for U into (13a) yields,

$$\frac{HW}{L\ell^2} g_{\xi\xi} + \frac{HW}{L\ell^2}\psi_x f_{\xi\xi} + \frac{HW}{L^2\ell}\psi_{xx} f_\xi + \cdots = \frac{\rho\omega^2}{E}\frac{HW}{L} g_{\tau\tau},$$

which, upon dividing through by $HW/L\ell^2$, reduces to the following equation:

$$g_{\xi\xi} + \psi_x f_{\xi\xi} + \frac{\ell}{L}\psi_{xx} f_\xi + \cdots = \frac{\rho\omega^2\ell^2}{E} g_{\tau\tau}. \tag{15}$$

Since terms in (15) can be at most order unity* it is a necessary restriction on the simplified equations that the relation between ℓ and L is established as,

$$\frac{\ell}{L} = O(1), \text{ (including } \ll 1). \tag{16}$$

*See Appendix I

This restriction means that representative vibration wave lengths ℓ are at most of the order of magnitude of the representative shell shape wave lengths L. This condition is particularly important in problems involving shallow shells.

If it happens that the coefficient on the right of (15), $\rho\omega^2\ell^2/E$, is small compared to unity then longitudinal inertia will be negligible. To obtain information on this point substitute $U=\frac{H}{L}W$ into equation (13b).

$$\frac{E\delta^3}{12(1-\nu^2)}\frac{W}{\ell^4}\left[-f_{\xi\xi\xi\xi}+\cdots\right]+\frac{E\delta}{1-\nu^2}\frac{H}{L^2}\psi_{xx}\left\{\left[\frac{HW}{L\ell}g_\xi+\frac{HW}{L\ell}\psi_x f_\xi\right]\right.$$

$$\left.+\cdots\right\}+\cdots=\rho\delta\omega^2\left[Wf_{\tau\tau}-\frac{H^2}{L^2}W\psi_x g_{\tau\tau}+\cdots\right].$$

Dividing through by EW and collecting terms,

$$\frac{1}{12(1-\nu^2)}\frac{\delta^3}{\ell^4}\left[-f_{\xi\xi\xi\xi}+\cdots\right]+\frac{1}{1-\nu^2}\frac{H^2\delta}{L^3\ell}\psi_{xx}\left\{\left[g_\xi+\psi_x f_\xi\quad+\cdots\right]\right.$$

$$\left.+\cdots\right\}+\cdots=\frac{\rho\delta\omega^2}{E}\left[f_{\tau\tau}-\frac{H^2}{L^2}\psi_x g_{\tau\tau}+\cdots\right]. \tag{17}$$

Since transverse inertia is to be important, it is necessary to match the coefficient of $f_{\tau\tau}$ with at least one of the coefficients on the left side of (17).

Thus, let

$$\frac{1}{12(1-\nu^2)}\frac{\delta^2}{\ell^4}=\frac{\rho\omega^2}{E}, \tag{18a}$$

or

$$\frac{1}{1-\nu^2}\frac{H^2}{L^3\ell}=\frac{\rho\omega^2}{E}. \tag{18b}$$

With this, and taking into account the fact that for shallow shells $(H/L)^2 \ll 1$,

$$\left[-f_{\xi\xi\xi\xi}+\cdots\right]+12\,\frac{H^2\ell^3}{\delta^2 L^3}\,\psi_{xx}\left\{\left[g_\xi+\psi_x f_\xi\right]+\cdots\right\}+\cdots=f_{\tau\tau}, \quad (19a)$$

or

$$\frac{1}{12}\,\frac{L^3\delta^2}{\ell^3 H^2}\left[-f_{\xi\xi\xi\xi}+\cdots\right]+\psi_{xx}\left\{\left[g_\xi+\psi_x f_\xi\right]+\cdots\right\}+\cdots=f_{\tau\tau}. \quad (19b)$$

Since the coefficients of all significant terms must be at most order unity then equation (19a) is applicable as long as,

$$12\,\frac{H^2\ell^3}{\delta^2 L^3}=O(1)\,, \quad (\text{OR} \ll 1) \qquad (20a)$$

while equation (19b) is applicable as long as,

$$\frac{1}{12}\,\frac{L^3\delta^2}{\ell^3 H^2}=O(1)\,, \quad (\text{OR} \ll 1) \qquad (20b)$$

And the two regions of parameter values complement each other.

By introducing (18a) and (18b) into (15a) the coefficient on the right of (15a) becomes,

$$\frac{1}{12(1-\nu^2)}\,\frac{\delta^2}{\ell^2}\,,$$

and

$$\frac{1}{(1-\nu^2)}\,\frac{H^2}{L^2}\,\frac{\ell}{L}\,,$$

respectively.

From these two relations it is assumed that not only is $H^2/L^2 \ll 1$ but also $\delta^2/\ell^2 \ll 1$. In fact, this assumption is necessary if the effects of transverse shear deformation are to be neglected. Now, since $\ell/L = O(1)$ then it is observed that both the above longitudinal inertia coefficients are small compared with unity. Accordingly, as long as the representative wave length ℓ and frequency ω are related as in (18a) of (18b) and as long as $\ell/L = O(1)$ then the effect of longitudinal inertia in the theory of shallow shells is negligible compared with the effect of transverse inertia.

Summary

From the foregoing discussion the conditions where by the simplified differential equations I and II are justified are as follows:

$$\left(\frac{H}{L}\right)^2 << 1 ,$$

$$\frac{\ell}{L} = O(1) ,$$

and the representative wave length ℓ and the frequency ω are related by

$$\frac{1}{12(1-\nu^2)} \frac{\delta^2}{\ell^4} = \frac{\rho\omega^2}{E} ,$$

or

$$\frac{1}{(1-\nu^2)} \frac{H^2}{L^3 \ell} = \frac{\rho\omega^2}{E} .$$

CHAPTER IV

APPLICATION OF THE SIMPLIFIED SHELL EQUATIONS TO A PARTICULAR HYPERBOLIC PARABOLOID STRUCTURE

Consider the hyperbolic paraboloid shown in Fig. 7, whose middle surface is defined by the equation,

$$z = \frac{2H}{L^2}\left(x^2 - y^2\right). \qquad (1)$$

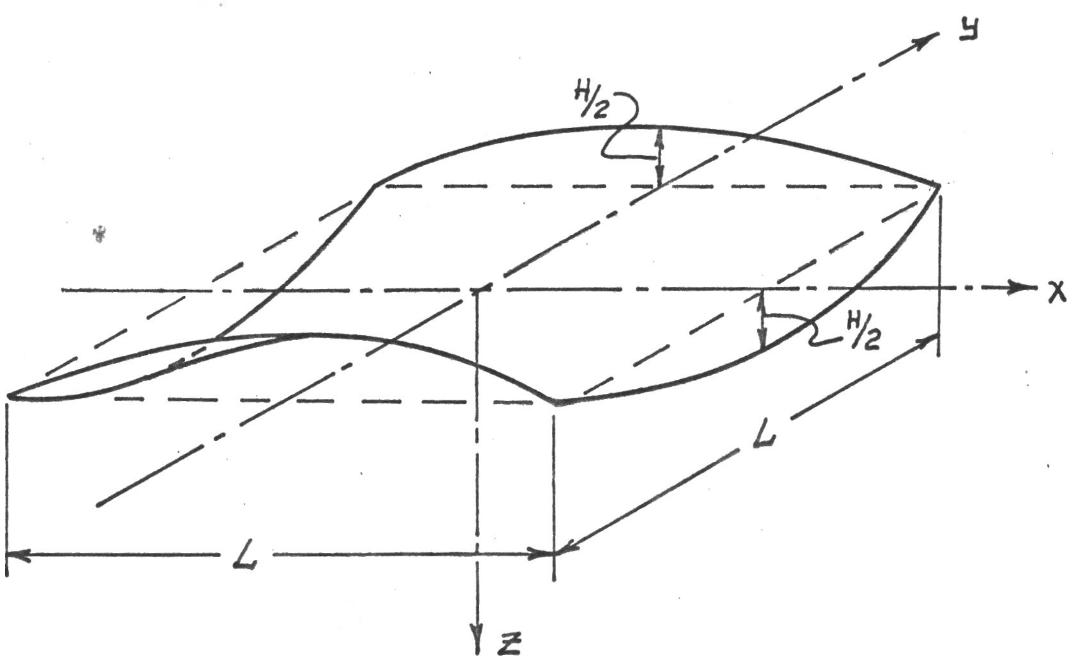

Fig. 7

If the edges are supported by a structure which is rigid in its own plane but non-resistant perpendicular to its own plane then it may be taken that the displacements at the edges are negligible compared to the displacements of the shell itself. Also the stresses normal to the edge must go to zero. Such a supporting structure is called a "diaphragm." [4] These conditions may be expressed as,

$$x = y = \pm \frac{L}{2}, \quad \begin{cases} w = 0 & (2) \\ N_x = N_y = 0 & (3) \end{cases}$$

In addition to the displacements at the edges being negligible it is assumed that for a thin shell the effect of the moments introduced by the supporting structure are also negligible. This assumption is justified by the fact that the effect of the moments only penetrates the shell for a distance usually not exceeding the thickness of the shell.[7] This momentless edge condition may be expressed as,

$$x = y = \pm \frac{L}{2}, \quad \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} = 0 \tag{4}$$

Statement of the problem

To find the lowest natural frequencies and modes of vibration of a shallow shell described by the equation (1) and subject to the boundary conditions (2), (3), and (4). The modes will be determined by the solution of the set of partial differential equations (I) and (II) of Chapter III. The frequencies of vibration will be determined by the characteristic equation.

Solution

Substituting the given equation (1) for the middle surface into the simplified differential equations (I) and (II) of the previous chapter leads to the following set of equations.

$$\nabla^4 F = \frac{4HE\delta}{L^2}\left[\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}\right] \tag{5}$$

$$\nabla^4 w = -\frac{\rho\delta}{D}\frac{\partial^2 w}{\partial t^2} + \frac{4H}{DL^2}\left[\frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial x^2}\right] \tag{6}$$

The boundary conditions for w are derived from equation (2). and equation (4). Hence,

$$x = \pm L/2 \quad ; \quad w = 0 \, , \; w_{xx} = 0 \, ,$$
$$y = \pm L/2 \quad ; \quad w = 0 \, , \; w_{yy} = 0 \, . \tag{7a,b}$$

The boundary conditions for F are derived from equation (3) and the relations between the stress function and the normal forces (see Chapter III, Eq. 7). Hence,

$$x = \pm L/2 \quad ; \quad F_{yy} = 0 \, , \quad F_{xx} = 0 \, ,$$
$$y = \pm L/2 \quad ; \quad F_{xx} = 0 \, , \quad F_{yy} = 0 \, . \tag{8a,b}$$

The form of the equations (5) and (6) and the boundary conditions allows the following modes of vibration*

$$w(x,y,t) = Ae^{i\omega t} \begin{cases} \cos \alpha_{2n+1} x \cos \beta_{2n+1} y \, , \\ \sin \alpha_{2n} x \sin \beta_{2n} y \, , \end{cases} \tag{9a}$$

$$F(x,y,t) = Be^{i\omega t} \begin{cases} \cos \alpha_{2n+1} x \cos \beta_{2n+1} y \, , \\ \sin \alpha_{2n} x \sin \beta_{2n} y \, , \end{cases} \tag{9b}$$

where, from the boundary conditions,

$$\alpha_m = \frac{m\pi}{L} \, , \quad \beta_m = \frac{m\pi}{L} \, . \tag{10}$$

---

*See Appendix II for complete discussion.

Consider the particular mode,

$$w_{pq}(x,y,t) = A_{pq}\, e^{i\omega_{pq}t} \sin\alpha_p x \sin\beta_q y$$

$$F_{pq}(x,y,t) = B_{pq}\, e^{i\omega_{pq}t} \sin\alpha_p x \sin\beta_q y$$

Substitution into equations (1) and (2) yields,

$$\left[\frac{4HE\delta}{L^2}(\alpha_p^2 - \beta_q^2)\right] A_{pq} + (\alpha_p^2 + \beta_q^2)^2 B_{pq} = 0$$

$$\left[(\alpha_p^2 + \beta_q^2)^2 - \frac{\rho\delta}{D}\omega_{pq}^2\right] A_{pq} + \left[\frac{4H}{DL^2}(\beta_q^2 - \alpha_p^2)\right] B_{pq}$$

The vanishing of the determinant of the above equation leads to the frequency equation (also referred to as the characteristic equation). Thus, solving for ω_{pq},

$$\omega_{pq}^2 = \frac{D}{\rho\delta}(\alpha_p^2 + \beta_q^2)^2 + \frac{16H^2E}{\rho L^4}\left[\frac{(\beta_q^2 - \alpha_p^2)}{(\alpha_p^2 + \beta_q^2)}\right]^2$$

The first term on the right corresponds to the frequency of a flat plate. The second term is called the shell-curvature correction factor.[2] Notice that when p equals q the shell-curvature correction term vanishes. This is clearly a trivial solution.*

$$(\beta_q^2 - \alpha_p^2) = \frac{\pi^2}{L^2}(q^2 - p^2) = 0 \quad \text{FOR } p = q$$

---

*See Appendix II for a complete discussion of this particular problem.

CHAPTER V

PROPOSED EXPERIMENTAL INVESTIGATION

Objective

In order to further investigate the vibration characteristics of the hyperbolic paraboloid it is proposed that an experimental study be undertaken. The object of this study would be as follows:

1. To establish the boundary conditions imposed upon the mathematical model.
2. To verify the analytical predictions. In particular, to determine the physical significance of the trivial solutions of the frequency equation.
3. To extend the results of the theory to such non-homogeneous, anisotropic materials as steel reinforced concrete and glass reinforced plastic.

To aid the experimenter in his study the frequency equation has been programmed on the Bendix G-15 digital computer and the lowest non-trivial frequencies have been calculated for several possible prototypes (see Fig. 8 and Table I).

Prototype construction

The prototype should be made of a homogeneous, isotropic material that is inexpensive and easy to work with. A possible choice is Zytel, a nylon manufactured by Dupont. This material has the following physical characteristics:

$$E = 0.2 \times 10^6 \text{ lb/in}^2,$$
$$\gamma = 71.0 \text{ lb/ft}^3,$$
$$\nu = 0.4.$$

The hyperbolic paraboloid surface may be generated by running wire diagonally between two paralled sheets of plywood, upon which a pattern of holes have been drilled representing the desired parabola. This surface is then covered with fiberglass and epoxy. After curing, the resultant shell is used to make a plaster of Paris mold. The prototype is then formed by heating a sheet of plastic, placing it on the mold, and allowing it to cool.

Method of support

The structure should be rigidly mounted only at the corners. The edges are to be supported by a hinged structure that prevents vertical displacement but allows transverse movement with respect to the plane of the shell. Further, the edge supports must be capable of transmitting shear loads to the corner mounts. The complete assembly may then be mounted on a suitable vibration platform.

Instrumentation

The frequencies of vibration and the corresponding displacements should be measured at several locations simultaneously to determine the mode shapes. A suitable transducer for this purpose would be the linear transformer velocity pickup.[8] The transformer core, which can be made very light weight, would be mounted directly on the surface of the shell and the coil would be fixed directly above the core. The output velocity would then be integrated to obtain displacement as a function of time. The fundamental frequency could be determined with the use of an oscilloscope in series with a low-pass filter.

Anticipated results

The following graph and table indicate the anticipated results for the case of a hyperbolic paraboloid made of Zytel plastic. The calculations are based on a model with the following physical dimensions.

Base: 3 ft. by 3 ft.

Height: 1 in. to 6 in.

Thickness: 1/16 in. and 1/8 in.

The value of frequency presented is the lowest numerical value calculated that was consistant with the mathematical restrictions imposed upon the solution (see Chapter IV).

The corresponding mode shape is indicated by the subscripts p and q defined by equations (9) and (10) of the previous chapter.

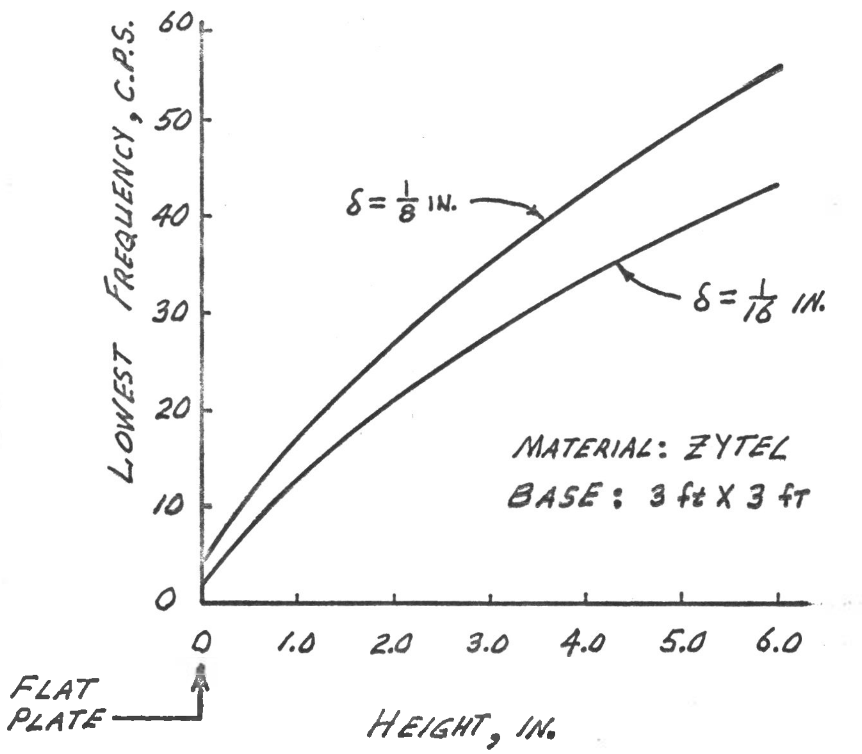

Fig. 8

LOWEST FREQUENCY v.s. HEIGHT AND THICKNESS OF A THIN SHELL HYPERBOLIC PARABOLOID

Height	1/16 in. Thick		1/8 in. Thick	
	f_{pq}	Mode	f_{pq}	Mode
0.0	1.9	1,1	3.9	1,1
1.0	13.7	1,2	16.2	1,2
2.0	20.9	2,3	27.4	1,2
3.0	27.8	2,3	35.6	2,3
4.0	34.4	3,4	41.7	2,3
5.0	38.8	3,4	48.4	2,3
6.0	43.5	3,4	55.5	2,3

TABLE I

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BIBLIOGRAPHY

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APPENDIX

APPENDIX A

FRACTIONAL ANALYSIS[3]

Definition of terms

Fractional analysis: Any procedure for obtaining some information about the answer to a problem in the absence of methods or time for finding a complete solution.

Variables: Those quantities which can vary inside a given system.

Parameters: Those quantities which are constant in any one system, but which can vary from one system to another.

I. PROCEDURE FOR NORMALIZATION

The term "normalize" will be used to mean, to make non-dimensional according to the specified procedure which now follows:

(1) Make all the variables in the equation non-dimensional in a prescribed way to be specified below.

(2) Divide by the coefficient in front of one of the terms to make the entire equation non-dimensional.

Method of Step (1):

(a) Define all dependent non-dimensional variables so that they are approximately unity over a finite distance but nowhere exceed unity in the domain of concern.

(b) Define all independent variables so that their increment is approximately unity over the same domain. This means that the boundary conditions will run from zero to one (or one to two, etc.) in the new variable.

It will be noted that the dimensionless parameters of a normalized equation are the dimensionless coefficients of the equation.

It is sufficient to normalize the governing equations and boundary conditions and extract the dimensionless parameters by inspection.

II. THEOREMS ON HOMOGENITY

Homogenity is usually defined as follows. If upon insertion of the quantity λX for a variable X in a given equation λ identically cancels, then the equation is said to be homogeneous in X.

Theorem I. If the governing equations are homogeneous in a given variable, and if this variable can be expressed in the same homogeneous form in the boundary conditions, then that variable can be eliminated as a parameter in the governing dimensionless groups by proper normalization.

Theorem II. If a given parameter is homogeneous in each term of the governing equations, and if this parameter does not appear in the boundary conditions, then the solution for the dependent variable is independent of the given parameter.

III. APPROXIMATION THEORY

In normalizing an equation each term in the non-dimensional variables is made approximately unity in magnitude. Under these circumstances it is possible to compare the magnitude of the pi groups. If any of these are small compared to the others, we can attempt to drop them from our model correlation. We also drop the associated terms from the equation to see if we can find a simpler approximate equation that is more readily soluble but still gives a satisfactory solution. This procedure with its limitations and conditions is called "approximation theory." Two theorems related to this theory will now be stated.

Theorem III. Given a dependent variable T, a function of K independent variables X_i; sufficient but not necessary conditions for (1) Provision for uniform estimates in the range $0 \leq X_i \leq L_i$, and (2) Accurate use of approximation theory in this same range for a very broad class of equations, to be defined are that in the range $0 \leq X_i \leq 1$;

$$\bar{T} = \frac{T}{T_a} = O(1)$$

and

$$\frac{\partial^r \bar{T}}{\partial \bar{X}_i{}^r} = O(1)$$

where $\bar{X}_i = \frac{X_i}{L_i}$, n is the highest order derivative in each X_i, and r is any integer such that $r \leq n$.

Note: A problem is said to have uniform behavior if it is possible to supply a single extimate for each variable in such a way that each term containing variables in the governing equation is made approximately unity over a finite range in the domain of interest and equal to or less than unity throughout the remainder of the domain. It is useful to normalize is such a way that the dependent variable is made of order unity and the independent variable runs from zero to unity in terms of the boundary conditions.

Theorem IV. If the governing equations can be reformulated by transformation to non-dimensional variables so that each term in the equation containing variables (and functions of the variables) can be made unity order, and if the limits of integration of all orders required can all be made of finite size in terms of the same non-dimensional variables, then approximation theory can be applied.

APPENDIX B

DETAILED DERIVATION OF THE EQUATIONS PRESENTED IN CHAPTER IV

For $z = \frac{2H}{L^2}(x^2 - y^2)$, Equations (I) and (II) of Chapter III become

$$\nabla^4 F = K_1 \left[\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \right] \tag{1}$$

$$\nabla^4 w = K_2 \frac{\partial^2 w}{\partial t^2} + K_3 \left[\frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial x^2} \right] \tag{2}$$

where, $K_1 = \frac{4HE\delta}{L^2}$, $K_2 = -\frac{\rho\delta}{D}$, $K_3 = \frac{4H}{DL^2}$

Assume product solutions of the form,

$$w(x, y, t) = G(x, y)\, T(t) \tag{3}$$

$$F(x, y, t) = H(x, y)\, S(t) \tag{4}$$

Substituting equations (3) and (4) into equation (1), yields

$$\frac{\nabla^4 H(x,y)}{K_1 \left[G_{xx}(x,y) - G_{yy}(x,y) \right]} = \frac{T(t)}{S(t)} \tag{5}$$

In equation (5) the right-hand side is independent of (x, y), while the left-hand side is independent of (t); hence their common value must be a constant, say λ. Thus,

$$T(t) = \lambda \left(S(t) \right) .$$

Equations (3) and (4) may now be rewritten, using equation (6), as

$$w(x, y, t) = G(x, y)\ T(t) . \tag{3a}$$

$$F(x, y, t) = H(x, y)\left[\frac{1}{\lambda} T(t)\right] . \tag{4a}$$

Substituting Eqs. (3a) and (4a) into Eq. (2),

$$\left[\nabla^4 G(x,y)\right] T(t) = K_2\, G(x,y)\, T''(t)$$
$$+ \frac{K_3}{\lambda}\left[H_{yy}(x,y) - H_{xx}(x,y)\right] T(t) . \tag{7}$$

Dividing through by $T(t)$, transposing terms, and then dividing through by $K_2 G(x,y)$; the variables are separated and equation (7) may be written as

$$\frac{\nabla^4 G(x,y) - \frac{K_3}{\lambda}\left[H_{yy}(x,y) - H_{xx}(x,y)\right]}{K_2\, G(x,y)} = \frac{T''(t)}{T(t)} . \tag{8}$$

In Eq. (8) the right-hand side is independent of (x , y), while the left-hand side is independent of (t); hence their common value must be a constant, say μ . Thus,

$$T''(t) - \mu T(t) = 0 . \tag{9}$$

Since the vibration problem is harmonic in nature, μ must be negative, say $\mu = -\omega^2$. Thus,

$$T''(t) + \omega^2 T(t) = 0 \tag{10}$$

The solution of equation (10) is readily found to be

$$T(t) = A e^{i\omega t} . \tag{11}$$

The form of Eqs. (1) and (2), i.e. even order derivatives, suggest the following relations for $G(x,y)$ and $H(x,y)$.

$$G(x,y) = D_1 \sin\alpha_1 x \sin\beta_1 y , \tag{12}$$

$$H(x,y) = D_2 \sin\alpha_2 x \sin\beta_2 y . \tag{13}$$

Substituting Eqs. (12) and (13) into Eq. (5),

$$\frac{D_2(\alpha_2^2 + \beta_2^2)^2 \sin\alpha_2 x \sin\beta_2 y}{K_1 D_1 (\beta_1^2 - \alpha_1^2) \sin\alpha_1 x \sin\beta_1 y} = \lambda . \tag{14}$$

In order for Eq.(14) to be equal to a constant for all values of X and y then it is necessary that

$$\alpha_1 = \alpha_2 ,$$

$$\beta_1 = \beta_2 .$$

Thus, Eqs. (12) and (13) assume the form

$$G(x,y) = D_1 \sin\alpha x \sin\beta y , \tag{15}$$

$$H(x,y) = D_2 \sin\alpha x \sin\beta y . \tag{16}$$

Another set of solutions may be shown to be

$$G(x,y) = \bar{D}_1 \cos\bar{\alpha} x \cos\bar{\beta} y , \tag{17}$$

$$H(x,y) = \bar{D}_2 \cos\bar{\alpha} x \cos\bar{\beta} y . \tag{18}$$

The complete form of the solution is found by applying the following boundary conditions.

On w, $\quad x = \pm \frac{L}{2}; \quad w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0,$ (19)

$\quad y = \pm \frac{L}{2}; \quad w = 0, \quad \frac{\partial^2 w}{\partial y^2} = 0.$ (20)

On F, $\quad x = \pm \frac{L}{2}; \quad F_{yy} = 0, \quad F_{xx} = 0,$ (21)

$\quad y = \pm \frac{L}{2}; \quad F_{xx} = 0, \quad F_{yy} = 0.$ (22)

For $x = \pm \frac{L}{2}$ and all y and t,

$$\left.\begin{aligned} w &= 0 \\ w_{xx} &= 0 \\ F_{yy} &= 0 \\ F_{xx} &= 0 \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} &\sin \alpha \tfrac{L}{2} \sin \beta y = 0, \\ &\cos \bar{\alpha} \tfrac{L}{2} \cos \bar{\beta} y = 0. \end{aligned}\right.$$

$$\therefore \quad \alpha = 2n \frac{\pi}{L} \qquad n = 1, 2, 3, \ldots$$

$$\bar{\alpha} = (2n+1) \frac{\pi}{L} \qquad n = 1, 2, 3, \ldots \tag{23}$$

For $y = \pm \frac{L}{2}$ and all x and t,

$$\left.\begin{aligned} w &= 0 \\ w_{yy} &= 0 \\ F_{xx} &= 0 \\ F_{yy} &= 0 \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} &\sin \alpha x \sin \beta \tfrac{L}{2} = 0, \\ &\cos \bar{\alpha} x \cos \bar{\beta} \tfrac{L}{2} = 0. \end{aligned}\right.$$

$$\therefore \quad \beta = 2n \frac{\pi}{L} \qquad n = 1, 2, 3, \ldots$$

$$\bar{\beta} = (2n+1) \frac{\pi}{L} \qquad n = 1, 2, 3, \ldots \tag{24}$$

From Eqs. (3a), (4a), (15), (16), (17), and (18) it is apparent that the solution allows the following modes of vibration.

$$w(x,y,t) = Ae^{iwt}\begin{cases}\cos\alpha_{2n+1}x\cos\beta_{2n+1}y\\ \sin\alpha_{2n}x\sin\beta_{2n}y\end{cases} \tag{25}$$

$$F(x,y,t) = Be^{iwt}\begin{cases}\cos\alpha_{2n+1}x\cos\beta_{2n+1}y\\ \sin\alpha_{2n}x\sin\beta_{2n}y\end{cases} \tag{26}$$

Where, from Eqs. (23) and (24),

$$\alpha_m = \frac{m\pi}{L},$$

$$\beta_m = \frac{m\pi}{L}.$$

Eqs. (23) and (24) imply that for p and q both odd integers allows the following mode of vibration.

$$w = Ae^{iwt}\cos\alpha_p x\cos\beta_q y,$$

$$F = Be^{iwt}\cos\alpha_p x\cos\beta_q y.$$

And when p and q are both even integers then

$$w = Ae^{iwt}\sin\alpha_p x\sin\beta_q y,$$

$$F = Be^{iwt}\sin\alpha_p x\sin\beta_q y.$$

However, when ρ and g are equal the shell-curvature correction term of the frequency equation vanishes and the shell apparently behaves like a flat plate. Since this behavior is not logical from physical reasoning, then it is concluded that the frequency equation does not apply to the shell problem when ρ and g are equal.

NOMENCLATURE

(Chapters III & IV)

$x, y, z.$	Cartesian Coordinates
$u, v, w.$	Displacements in the x-,y-,z- Directions respectively
t	Time
ρ	Density of shell material
δ	Shell thickness
N	Tangential stress resultant
Q	Transverse stress resultant
M	Stress couples
D	Bending stiffness, $E\delta^3/12(1-\nu^2)$.
E	Young's modulus
ν	Poisson's ratio
F	Airy stress function
$\alpha_m \cdot \beta_m$.	Defined by Equation (10), page 32
ω_{pq}	Characteristic Equation (Frequency)
$m \cdot n \cdot p \cdot q$.	Any integer greater than zero
ℓ	Characteristic wave length
H	Shell height (Page 30)
L	Shell length (Page 30)

A STUDY OF THE VIBRATION CHARACTERISTICS OF A
THIN SHELL HYPERBOLIC PARABOLOID

An Abstract of a Thesis
Presented to the Faculty of
the Department of Mechanical Engineering
Brigham Young University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science

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by
Farrin W. West
May 1963

ABSTRACT

The vibration characteristics of a thin shell structure were analyzed. In particular, the frequency determinant was derived for a thin shell hyperbolic paraboloid simply supported at the edges.

The method of solution consisted of a fractional analysis of the governing differential equations, wherein the insignificant terms were neglected and the relationship between the remaining terms was established. The frequency equation was then derived for the special case of shallow shells considering only the transverse inertia terms.

It was found that under certain conditions the frequency equation yielded an almost trivial solution. An experimental investigation was proposed to determine the physical significance of such behavior.

An important result of the analysis was that as the curvature of the hyperbolic paraboloid was reduced to zero the solution of the frequency equation approached that of a flat plate, a condition that is a necessary physical limit.

May 7, 63
Date