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An Optimization Technique Applied to the Determination of Modal Damping Coefficients

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AN OPTIMIZATION TECHNIQUE APPLIED TO THE DETERMINATION OF MODAL DAMPING COEFFICIENTS

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CHAPTER I

INTRODUCTION

Although analytical techniques are generally available for the simulation of dynamic systems, often these techniques require a great deal of computer time and storage. This difficulty was vividly pointed out by a previous thesis undertaken in this Department by R. L. Webster (1). Mr. Webster undertook the determination of orthotropic material properties using dynamic response test data from plates. A major difficulty which was encountered was the enormous amount of computer time required for the simulation. This thesis is an investigation of a method which when fully developed, may provide the ability to simulate such systems much more efficiently.

The Finite Element Method

Perhaps the most popular method for simulation of distributed dynamic systems is the finite element technique. In this method, the distributed mass, elasticity, and damping properties are lumped, forming a system of masses, springs, and dashpots. The resulting equations which model the system are often a large system of matrix differential equations which describe the motion of the lumped parameters. It is an approximate model in the sense that a distributed system cannot be exactly represented by a lumped

system. The dimensionality or the number of degrees of freedom represented by this system of equations depends upon the number of degrees of freedom necessary to model the system within the desired accuracy and is a function of the ability of the individual who performs the modeling. There are techniques of reducing the dimensionality of the system by reducing the degrees of freedom of the system to include only "master" displacements (2). These techniques are also somewhat dependent upon one's modeling ability, and are also limited in their application. It is not the purpose of this thesis to examine these techniques in detail, but one should realize that they do exist.

Matrix Exponential

One method of solving the aforementioned system of differential equations is the matrix exponential method (3). A great advantage associated with this method is the fact that for linear systems, the exponential matrix need only be computed once. Computing the exponential matrix may use appreciable computer time, however, if the dimensionality is large. This method can be readily applied to non-linear systems as well, but since this requires calculation of the exponential matrix at each time step, computer time is very significantly increased.

Modal Analysis

Modal analysis is also a useful tool for solving such systems of differential equations. The principal difficulty with regard to computer

time associated with this method is the time required to solve the eigenvalue problem. The principal advantages are two:

1. It provides a method of reducing the dimensionality of the problem once the eigenvalue problem is solved by writing the solution in terms of the principal eigenvalues or modes only.

2. It provides a means of uncoupling the system of equations representing conservative (undamped) systems because of the orthogonality of the modes (called normal modes because of the orthogonality).

The second advantage, it should be noted, applies to conservative systems only. Since non-conservative systems do not in general have normal modes, the system of equations cannot be readily uncoupled. This greatly hampers the utility of the modal analysis approach to non-conservative systems.

The Ritz Method

Another form of modal analysis is the Ritz method (4). This is a variational approach in which a functional is defined which has a minimum value when the proper solution is substituted into it. It is commonly used to model distributed systems directly, rather than in conjunction with the finite element technique.

After a functional is defined, a solution form is assumed which contains certain parameters which are then optimized by finding the values of the parameters for which the functional has a minimum value. This method works well once a functional has been defined. Hamilton's Principle provides

a means of easily defining a functional for conservative systems. For non-conservative systems, the task is again more difficult as a functional is not easily defined.

It should be pointed out that the Ritz method is a method of approximating rather than finding exactly the solution. This method is subject to error due to the fact that some of the modes of vibration are neglected, as well as the fact that the assumed solution form may not be the best to describe the modes considered. For engineering applications, however, the increased efficiency of calculation is usually much more valuable than the accuracy lost. For conservative systems, this method has been used with excellent results (4).

Application to Non-conservative Systems

Since both the modal analysis technique applied to matrix systems as well as the Ritz method have the discussed difficulties when applied to damped systems, the matrix exponential method is presently the most useful method for simulating them. Modal analysis, expecially in the Ritz form, needs only a method of efficiently handling damping to greatly increase its utility. However, since the form of modal analysis as applied to matrix systems is well developed and more intuitive than the Ritz method as applied to distributed systems, this type of system was used for beginning this investigation. The approach is variational in nature in that a functional is defined such that it is a minimum at the proper solution. A solution form is assumed in which the modes are uncoupled and each mode has a coefficient of damping associated with it. These damping coefficients

will then be selected such that the functional is minimized. In this manner optimum modal damping coefficients are chosen such that the system can be modeled in terms of an uncoupled system of equations which are optimized using information from the original uncoupled system of equations. Although beyond the scope of this thesis, it is hoped that the form of the functional defined and investigated can be extended to allow using the Ritz method for modeling damped systems.

Because of the complexity of the functional defined relative to functionals describing conservative systems, numerical methods will be used to optimize the parameters such that the function is minimized. Two such optimization methods will be used in this investigation, the Davidon method (6) and the Powell method (5). The former requires the ability to calculate not only the functional but also the gradient of the functional, while the latter requires calculation of only the functional.

CHAPTER II

DERIVATION OF EQUATIONS

The Functional (7)

The law of conservation of energy will be used to derive the functional that will be used. At any instant, an energy balance for a dynamic system, given an initial displacement and/or velocity and allowed to vibrate freely, can be written in the form

$$E_o = E_p + E_k + E_d \qquad (2.1\text{-}1)$$

where E_o = the initial energy in the system at $t = 0$

E_p = the instantaneous potential energy in the system

E_k = the instantaneous kinetic energy in the system

E_d = the total energy dissipated by the system from the time $t = 0$ until the instant in question

Rearranging equation (2.1-1),

$$E_o - E_p - E_k - E_d = 0 \qquad (2.1\text{-}2)$$

Note that each of the terms above can be written in terms of the displacements and velocities of the system. Exact expressions for these displacements must satisfy this equation. Since the assumed solutions will be

approximate solutions, equation (2.1-2) will not be exactly true, but the expression on the left will be equal to some non-zero value $\mathcal{E}$, which will be called the energy deficit. Equation (2.1-2) for this case then becomes

$$E_o - E_p - E_k - E_d = \mathcal{E}(t) \qquad (2.1\text{-}3)$$

where $\mathcal{E}$ is clearly a function of time as well as a function of the damping coefficients. An optimum solution would be one in which $\mathcal{E}$ is as nearly zero as possible over some interval of time. In other words, an optimum solution would be one which minimized the integral over some length of time of a positive definite function of $\mathcal{E}$ such as $|\mathcal{E}|$ or $\mathcal{E}^2$. Preliminary work indicated that the best function to use was $\mathcal{E}^2$. Therefore the functional becomes

$$I = \int_o^{t_f} \mathcal{E}^2 dt = I(\mathcal{f}_1, \mathcal{f}_2, \ldots \mathcal{f}_m) \qquad (2.1\text{-}4)$$

and an optimum solution of the problem is defined as the solution which causes I to be a minimum.

Assumed Solution Form

The assumed solution vector $\underline{X}$ to a system of differential equations has the form

$$\underline{X} = \sum_{i=1}^{m} \underline{\phi}_i e^{-\mathcal{f}_i w_i t} \left(a_i \sin\left\{w_i\sqrt{1-\mathcal{f}_i^2}\,t\right\} + b_i \cos\left\{w_i\sqrt{1-\mathcal{f}_i^2}\,t\right\}\right) \qquad (2.2\text{-}1)$$

where $\underline{X}$ is the vector of nodal displacements

$\underline{\phi}_i$ is the vector mode shape of the i^{th} mode

ζ_i is the damping coefficient of the i^{th} mode

w_i is the circular frequency of the i^{th} mode

t is time

a_i and b_i are constants evaluated from initial conditions

m is the number of modes

The above equation can be written as

$$\underline{X} = \sum_{i=1}^{m} \underline{\phi}_i q_i = [\phi] \underline{q} \qquad (2.2\text{-}2)$$

where $[\phi]$ is the modal matrix consisting of the mode shapes column listed (see "Modal Analysis" in Appendix) and $\underline{q}$ is the vector of modal displacements. The elements of $\underline{q}$ are then expressed by the equation

$$q_i = e^{-\zeta_i w_i t} \left(a_i \sin\left\{w_i \sqrt{1-\zeta_i^2} t\right\} + b_i \cos\left\{w_i \sqrt{1-\zeta_i^2} t\right\}\right) \qquad (2.2\text{-}3)$$

where each of the terms has the same definition as in equation (2.1-1).

Each of the energy terms in the minimization integral can now be expressed in terms of this assumed solution. The instantaneous potential energy,

$$E_p = \frac{1}{2} \underline{X}^T [K] \underline{X}$$

the instantaneous kinetic energy,

$$E_k = \frac{1}{2} \dot{\underline{X}}^T [M] \dot{\underline{X}}$$

and the energy dissipated by the system from t=0 to the time t,

$$E_d = \int_o^t (\dot{\underline{X}}^T [C] \dot{\underline{X}}) \, dt$$

where $[K]$, $[M]$, and $[C]$ are the stiffness, mass, and damping matrices respectively, and $\dot{\underline{X}}$ represents the derivative of the vector $\underline{X}$ with respect to time, i.e.,

$$\dot{X} = \begin{bmatrix} \frac{\partial X_1}{\partial t} \\ \\ \frac{\partial X_2}{\partial t} \\ \vdots \\ \frac{\partial X_n}{\partial t} \end{bmatrix}$$

Substituting $\underline{X} = [\phi] \underline{q}$ and $\underline{X}^T = \underline{q}^T [\phi]^T$, these expressions can be written

$$E_p = \frac{1}{2} \underline{q}^T [\phi]^T [K] [\phi] \underline{q}$$

$$E_k = \frac{1}{2} \dot{\underline{q}}^T [\phi]^T [M] [\phi] \dot{\underline{q}}$$

$$E_d = \int_o^t (\dot{\underline{q}}^T [\phi]^T [C] [\phi] \dot{\underline{q}}) \, dt$$

Recalling that

$$[\phi]^T [M] [\phi] = [M_n]$$

and

$$[\phi]^T [K] [\phi] = [M_n] [\omega^2]$$

are diagonal matrices which, if the normal modes are normalized properly,

can be reduced to

$$[\phi]^T[M][\phi] = [I]$$

$$[\phi]^T[K][\phi] = [\omega^2]$$

where $[I]$ is the unity matrix and $[\omega^2]$ is a diagonal matrix with the square of the m undamped natural frequencies listed on the diagonal. Also, defining

$$[D] = [\phi]^T[C][\phi]$$

as the transformed damping matrix which is not diagonal. (In the actual case where it happens to be diagonal, the terms listed on the diagonal contain the modal damping coefficients and this procedure is unnecessary.)

Substituting these expressions into the energy equations yields

$$E_p = \frac{1}{2} q^T[\omega^2] q = \frac{1}{2} \sum_{i=1}^{m} (w_i q_i)^2 \qquad (2.2\text{-}4)$$

$$E_k = \frac{1}{2} \dot{q}^T [I] \dot{q} = \frac{1}{2} \sum_{i=1}^{m} \dot{q}_i^2 \qquad (2.2\text{-}5)$$

$$E_d = \int_o^t (\dot{q}^T [D] \dot{q}) \, dt \qquad (2.2\text{-}6)$$

and the energy terms are expressed as functions of the modal displacements.

The initial conditions can be somewhat arbitrarily chosen. For convenience the initial conditions

$$q_{o_i} = 0, \quad \dot{q}_i = \dot{q}_{o_i}, \qquad i = 1, 2, \ldots m$$

were chosen. Substituting these into equation (2.2-3) to determine a_i and b_i yields

$$b_i = 0 \qquad a_i = \frac{\dot{q}_{oi}}{w_i \sqrt{1-\zeta_i^2}}$$

and equation (2.2-3) becomes

$$q_i = \frac{\dot{q}_{oi}}{w_i \sqrt{1-\zeta_i^2}} \; e^{-\zeta_i w_i t} \; \sin\left\{w_i \sqrt{1-\zeta_i}\, t\right\} \tag{2.2-7}$$

Differentiating to find $\dot{q}_i$,

$$\dot{q}_i = \frac{\dot{q}_{oi}}{w_i \sqrt{1-\zeta_i^2}} \; e^{-\zeta_i w_i t} \; (-\zeta_i w_i \; \sin\left\{w_i \sqrt{1-\zeta_i^2}\, t\right\} +$$

$$w_i \sqrt{1-\zeta_i^2} \; \cos\left\{w_i \sqrt{1-\zeta_i^2}\, t\right\}) \tag{2.2-8}$$

Also from the initial conditions, the initial energy in the system can be expressed:

$$E_o = \frac{1}{2} \sum_{i=1}^{m} \dot{q}_{o_i}^{\,2} \tag{2.2-9}$$

Substituting (2.2-7) into (2.2-4) and (2.2-8) into (2.2-5) and (2.2-6) enables calculation of

$$\mathcal{E}(t) = E_o - E_p - E_k - E_d$$

for any t. Numerical integration is used to evaluate the integral E_d. The functional

$$I = \int_o^{t_f} \mathcal{E}^2 dt$$

is then found also using a numerical integration routine.

The integral I is thus expressed as a function of the parameters $\mathcal{f}_1, \mathcal{f}_2, \ldots \mathcal{f}_m$. Note that all other parameters are known a priori. The w_i's have previously been determined by modal analysis of the undamped system. The time interval t_f throughout which the function is integrated should be long enough to contain sufficient information about the system's response and will be somewhat arbitrarily picked after examining $\mathcal{E}(t)$ for a few systems.

The Gradients

To use a Davidon search method to minimize I, the gradient of I, defined as the vector of partial derivatives of I with respect to each of the $\mathcal{f}_i$,

$$\underline{g} = \begin{bmatrix} \frac{\partial I}{\partial \mathcal{f}_1} \\ \frac{\partial I}{\partial \mathcal{f}_2} \\ \vdots \\ \frac{\partial I}{\partial \mathcal{f}_m} \end{bmatrix}$$

must be determined. To do this we note that

$$I = \int_o^t \mathcal{E}^2 dt$$

and

$$\frac{\partial I}{\partial \mathcal{f}_i} = \frac{\partial}{\partial \mathcal{f}_i} \int_o^t \mathcal{E}^2 dt = \int_o^t \frac{\partial \mathcal{E}^2}{\partial \mathcal{f}_i} \, dt$$

$$= \int_0^t 2\varepsilon(t) \frac{\partial \varepsilon(t)}{\partial f_i} \, dt \qquad (2.3\text{-}1)$$

$\varepsilon(t)$ has already been calculated to determine I. Therefore, each component of the gradient requires only the evaluation of $\partial \varepsilon(t) / \partial f_i$.

Since

$$\varepsilon = E_o - E_p - E_k - E_d$$

$$\frac{\partial \varepsilon}{\partial f_i} = - \frac{\partial E_p}{\partial f_i} - \frac{\partial E_k}{\partial f_i} - \frac{\partial E_d}{\partial f_i} \qquad (2.3\text{-}2)$$

as $$\frac{\partial E_o}{\partial f_i} = 0 \qquad i = 1, 2, \ldots m$$

Examining each of the terms in equation (2.3-2),

$$E_p = \frac{1}{2} \sum_{i=1}^{m} (w_i \, q_i)^2$$

$$\frac{\partial E_p}{\partial f_i} = w_i^2 \, q_i \, \frac{\partial q_i}{\partial f_i} \qquad (2.3\text{-}3)$$

$$E_k = \frac{1}{2} \sum_{i=1}^{m} \dot{q}_i^2$$

$$\frac{\partial E_k}{\partial f_i} = \dot{q}_i \, \frac{\partial \dot{q}_i}{\partial f_i} \qquad (2.3\text{-}4)$$

Note that $\frac{\partial \dot{q}_i}{\partial f_j} = 0$ for $i \neq j$, and therefore only one term of each of the summations above remains after differentiation.

Also,

$$E_d = \int_o^t (\dot{q}^T [D] \dot{q})\, dt$$

$$\frac{\partial E_d}{\partial \mathcal{f}_i} = \frac{\partial}{\partial \mathcal{f}_i} \int_o^t (\dot{q}^T [D] \dot{q})\, dt = \int_o^t \frac{\partial}{\partial \mathcal{f}_i} (\dot{\underline{q}}^T [D] \dot{\underline{q}})\, dt$$

$$= \int_o^t (\dot{q}^T [D] \frac{\partial \dot{\underline{q}}}{\partial \mathcal{f}_i} + \frac{\partial \dot{\underline{q}}^T}{\partial \mathcal{f}_i} [D] \dot{\underline{q}})\, dt \qquad (2.3\text{-}5)$$

where the vectors

$$\frac{\partial \dot{\underline{q}}}{\partial \mathcal{f}_i} = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial q_i}{\partial \mathcal{f}_i} \\ 0 \\ \cdot \\ \cdot \\ o \end{bmatrix} \qquad \text{and} \quad \frac{\partial \underline{q}^T}{\partial \mathcal{f}_i} = (\frac{\partial \dot{\underline{q}}}{\partial \mathcal{f}_i})^T$$

To determine $\frac{\partial q_i}{\partial \mathcal{f}_i}$ equation (2.2-7) is differentiated with respect to $\mathcal{f}_i$:

$$q_i = \dot{q}_{o_i} \frac{e^{-\mathcal{f}_i w_i t}}{w_i \sqrt{1-\mathcal{f}_i^2}} \sin \{ w_i \sqrt{1-\mathcal{f}_i^2}\, t \}$$

$$\frac{\partial \dot{q}_i}{\partial \mathcal{f}_i} = \dot{q}_{o_i} e^{-\mathcal{f}_i w_i t} \left[\left(\frac{\mathcal{f}_i}{w_i (1-\mathcal{f}_i^2)^{3/2}} - \frac{t}{\sqrt{1-\mathcal{f}_i^2}} \right) \sin \{ w_i \sqrt{1-\mathcal{f}_i^2}\, t \right.$$

$$\left. - \frac{t\, w_i\, \mathcal{f}_i}{\sqrt{1-\mathcal{f}_i^2}} \cos \{ w_i \sqrt{1-\mathcal{f}_i t} \} \right] \qquad (2.3\text{-}6)$$

and to determine $\frac{\partial \dot{q}_i}{\partial \mathcal{f}_i}$ equation (2.2-8) is differentiated with respect to $\mathcal{f}_i$.

$$\dot{q}_i = \frac{\dot{q}_{o_i} e^{-\zeta_i w_i t}}{w_i \sqrt{1-\zeta_i^2}} \left(-\zeta_i w_i \sin\left\{w_i \sqrt{1-\zeta_i^2}\, t\right\} + w_i \sqrt{1-\zeta_i^2} \right.$$

$$\left. \cos\left\{w_i \sqrt{1-\zeta_i^2}\, t\right\}\right)$$

$$\frac{\partial \dot{q}_i}{\partial \zeta_i} = \dot{q}_{o_i} e^{-\zeta_i w_i t} \left[\left(-\frac{\zeta_i^2}{(1-\zeta_i^2)^{3/2}} + \frac{\zeta_i w_i t - 1}{\sqrt{1-\zeta_i^2}} + \frac{(1-\zeta_i^2)\, t}{\zeta_i} \right) \right.$$

$$\left. \sin\left\{w_i \sqrt{1-\zeta_i^2}\, t\right\} - w_i t \,(1+\zeta_i) \cos\left\{w_i \sqrt{1-\zeta_i^2}\, t\right\} \right] \qquad (2.3\text{-}7)$$

The components of the gradient can now be determined by back-substitution. These equations are listed in summary form in Appendix A.

Damping Coefficients Greater than One

When the form of the modal displacements were assumed (equation 2.2-3), it was assumed that the absolute value of ζ_i would always be less than one. This will almost always be the case. If ζ_i is greater than one, the i^{th} mode is overdamped (i.e., has greater damping than critical damping), which should seldom be the case for the class of problems which are being considered. Since a search routine will be used to determine the optimum damping coefficients, however, it is possible that one or more of these ζ_i may at some time during the optimization process be greater than one. This situation is handled by substituting hyperbolic functions for the trigonometric functions in equations (2.2-1) and (2.2-3). For any mode for which ζ_i is greater than one (i.e., $\sqrt{1-\zeta_i^2}$ is a complex number), equation

(2.2-3) becomes

$$q_i = e^{-\zeta_i w_i t} \left(a_i \sinh \left\{ w_i \sqrt{\zeta_i^2 - 1}\, t \right\} + b_i \cosh \left\{ w_i \sqrt{\zeta_i^2 - 1}\, t \right\}\right) \tag{2.4-1}$$

The other equations are then derived in a manner exactly analogous to the case where the trigonometric functions are used. The resulting equations are included in the summary in Appendix A. When the q_i are calculated, the computer first examines the value of ζ_i and goes to the proper branch of the program to calculate q_i, thereby avoiding complex arithmetic.

Excitation of Modes

As was pointed out during the derivation of the functional, the initial displacement given each of the modes was chosen as zero while the initial velocities were each given a value $\dot{q}_{oi}$. The relative values of these initial "modal velocities" is a measure of the relative levels at which we excite each of the modes. The original feeling of the author was that it would be preferrable to excite each of the modes equally and all of the initial modal velocities were chosen such that

$$\dot{q}_{oi} = 1.0 \qquad i = 1, 2, \ldots m$$

Later in the development it was felt that it would be preferrable to excite the modes in the same way that a step impulse force applied to each mass (node of the system) would excite them. This is equivalent to giving each mass an equivalent initial velocity. Therefore,

$$\dot{X}_{o_i} = 1.0 \qquad i = 1, 2, \ldots n$$

and

$$\underline{\dot{q}}_o = [\phi]^{-1} \underline{X}_o = [\phi]^T [M] \underline{X}_o$$

The effect of this change in the method of excitation and the motivation for it will be further discussed in Chapter V.

CHAPTER III

NUMERICAL METHODS AND COMPUTER PROGRAM

The various FORTRAN IV subroutines in the program used in this study and the functions which they performed were as follows:

OPDAMP.--This is the main program which reads in the data, calls the other subroutines, and does the final system simulation using the optimum damping coefficients. This subroutine also calculates the time step and the total interval of time used for evaluation of the integrals involved. The time step was chosen rather arbitrarily as 1/20 of the natural period of the highest frequency mode. In a final development of a system such as this, a numerical analysis in order to determine the size of the time step necessary for the accuracy required would provide for maximum program efficiency. This was not justified in this preliminary derivation and investigation of the functional, however, and thus the arbitrary nature of the time step. Similarly, the total time interval was set as three natural periods of the lowest frequency mode, with the stipulation that a maximum of two hundred time steps be used. This limit was set to facilitate programming and dimensioning of arrays used in the program.

MODES*.--A subroutine used to solve the eigenvalue problem. The eigenvalues or natural frequencies of the system are calculated along with the eigenvectors or mode shapes associated with each mode. The mode shapes are normalized such that when they are arranged into the modal matrix, $[\phi]$, that $[\phi]^T [M] [\phi] = [I]$. This subroutine provides the modal matrix $[\phi]$.

DELTA.--A subroutine which uses the modal matrix as well as the damping matrix to calculate the transformed damping matrix $[D]$.

FIND**.--A subroutine which uses the Powell minimization technique to minimize a function of several variables by means of conjugate directions. This method is outlined in Reference 5.

SEARCH**.--A subroutine which uses the Davidon minimization routine to minimize a function of several variables. This method is outlined in Reference 6.

FECAL.--A subroutine which calculates the value of the functional for a given set of coordinates (modal damping coefficients).

GEE.--A subroutine which calculates the gradient of the functional for a given set of coordinates.

EXACT.--A subroutine which simulates the response of a given system using the matrix exponential method. The exponential matrix itself was formed in the following subroutine whereas this subroutine does the actual

simulation.

MTXEXP**.--A subroutine which forms the exponential matrix. This subroutine, as well as a detailed description of the matrix exponential method, is found in Reference 3.

SINT.--A subroutine which accomplishes the numerical integration, using a modified form of Simpson's rule. A parabolic fit to the first three values of the function integrated allows "starting" of Simpson's rule in such a way that the subroutine does not require an even number of intervals in order to perform the integration.

Other subroutines included various subroutines to perform the matrix operations required to do the calculations. Part of these were provided on-line by the computer while some were provided by this program. Because of their straightforwardness and generality, it is not felt that they need be discussed here.

The total program was developed piecemeal, each subroutine being checked individually where possible. The function and gradient subroutines were used to examine the "surface" of the functional for some trial cases. In its completed form, the total program solves the undamped eigenvalue problem, determines the optimum damping coefficients, and simulates the system using the resulting uncoupled equations. It also simulates the system using the matrix exponential method, and the two solutions are compared in the form of a plotted output.

A complete program listing will be placed on file with the office copy of this thesis at the Department of Mechanical Engineering Science, Brigham Young University.

The subroutines marked (*) were obtained from Dr. H. N. Christiansen. The subroutines marked (**) were obtained from Dr. J. C. Free.

CHAPTER IV

EXAMINATION OF THE FUNCTIONAL SURFACE

Examination of the profile of the functional not only allows verification that the functional does in fact have a minimum in the correct place, but also yields insight into the convergence of the minimization routines used to find the minimum.

One Dimension

The simplest form of the functional is the one-dimensional form, i.e., the form which is a function of only one variable or damping coefficient. A one dimensional functional represents a single second order differential equation.

As mentioned previously, the form of the functional $I = \int_0^{t_f} \varepsilon^2 dt$ was chosen rather than the form $I = \int_0^{t_f} |\varepsilon| \, dt$ because of preliminary examination of their profiles. Figure 1 is an example of the latter form while Figure 2 represents the former. Both are functionals representing the same differential equation. As can be seen, the form $\int_0^{t_f} |\varepsilon| \, dt$ has a discontinuous derivative at the minimum. This would create a difficulty for the optimization routines.

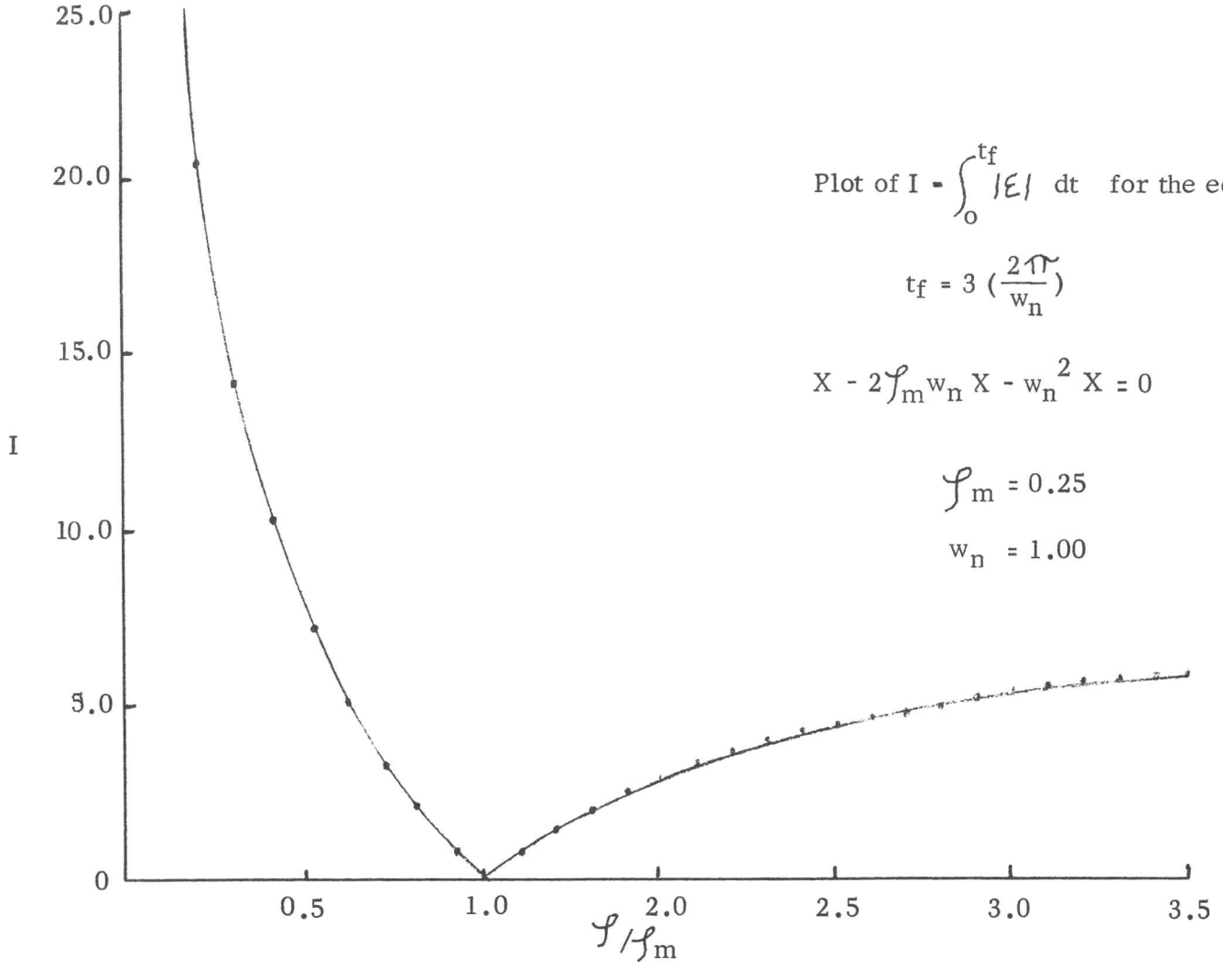


Figure 1.--Plot of integral of absolute value of energy deficit form of functional

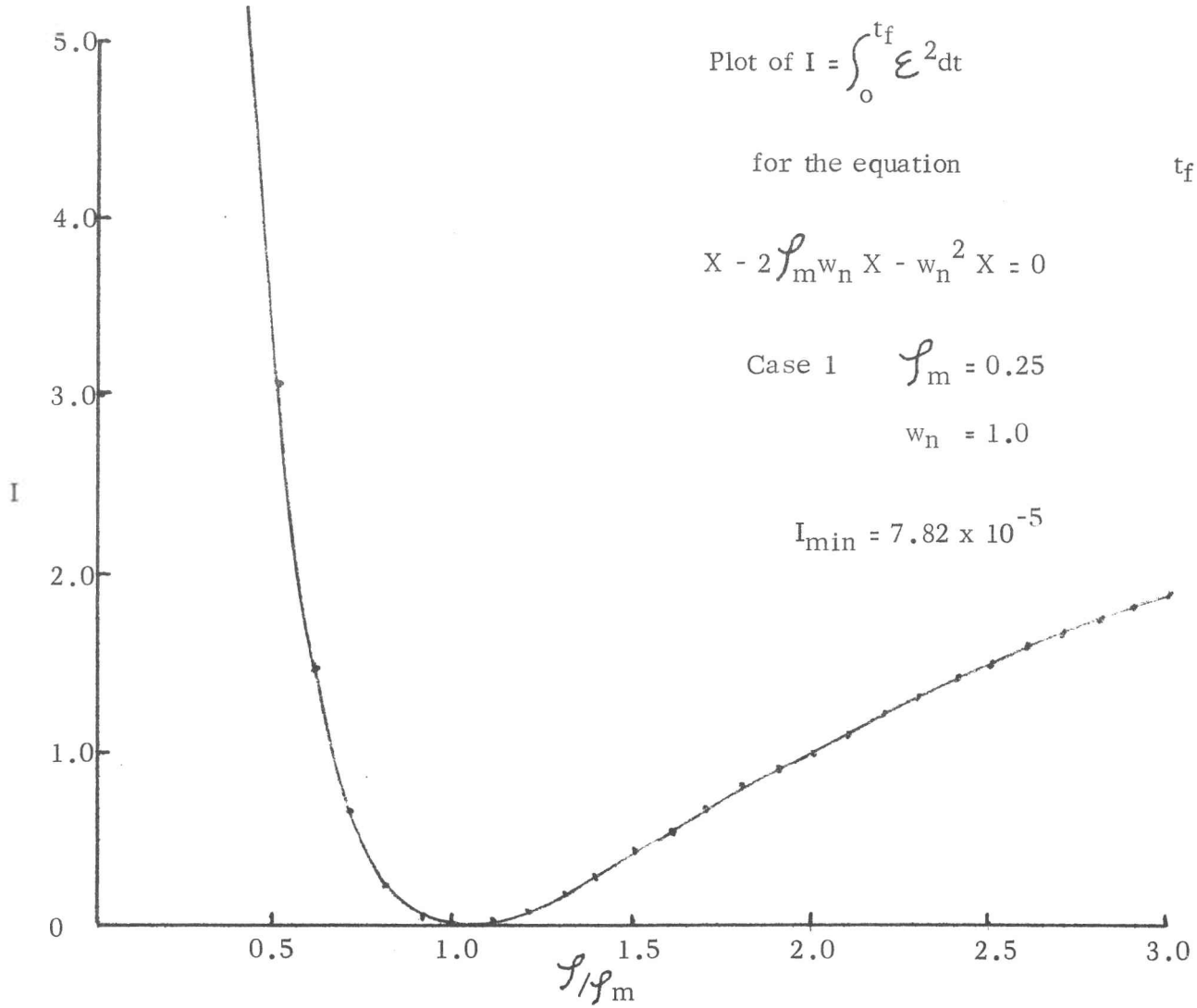


Figure 2.--Plot of one dimensional functional--Case 1

Figures 3 and 4 are plots of the integral of the energy deficit squared functionals for two other differential equations. The equation represented in Figure 3 differs by that represented in Figure 2 in that it has a higher frequency, whereas the equation of Figure 4 has a lower value of damping tha that of Figure 2. Since Figures 2, 3, and 4 represent different cases of the chosen form of the functional, they are listed as Case 1, Case 2, and Case 3, respectively.

Two Dimensions

The two dimensional functional allows us to observe the interaction of two parameters (or two modes) while allowing us to visualize the interaction as a plot in three dimensional space. The two dimensional function requires three space coordinates to represent it, the functional itself requiring the third. Represented in this way, the functional is a surface, its value defined by its height above a "zero" plane. Two dimensional functionals are represented in Figures 5 and 6 by a grid of numbers, each number representing the value of the functional at that coordinate location. Figure 5 plots a functional representing a system of two uncoupled differential equations, whereas Figure 6 represents a system of two coupled equations (i.e., $[D]$ not diagonal).

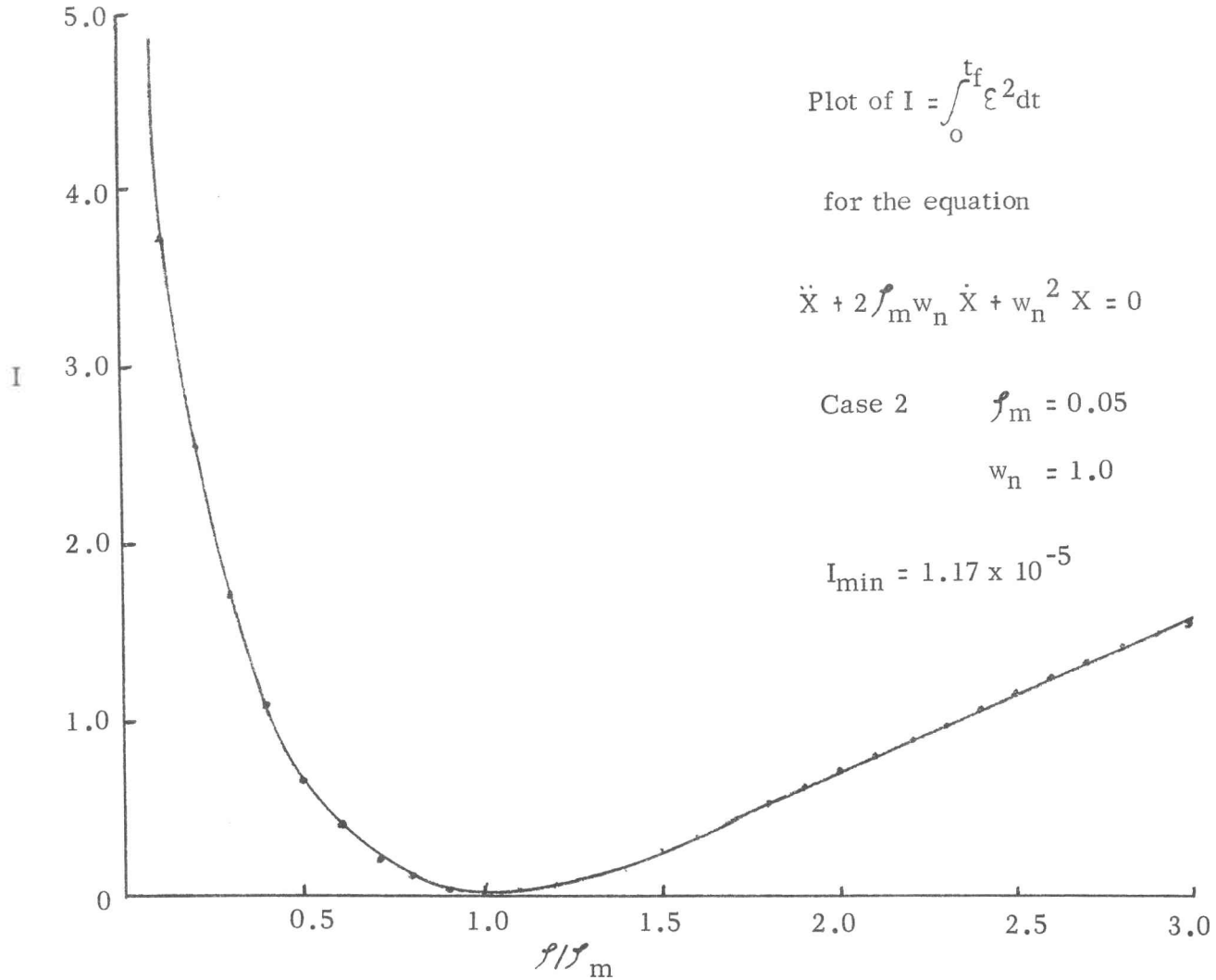
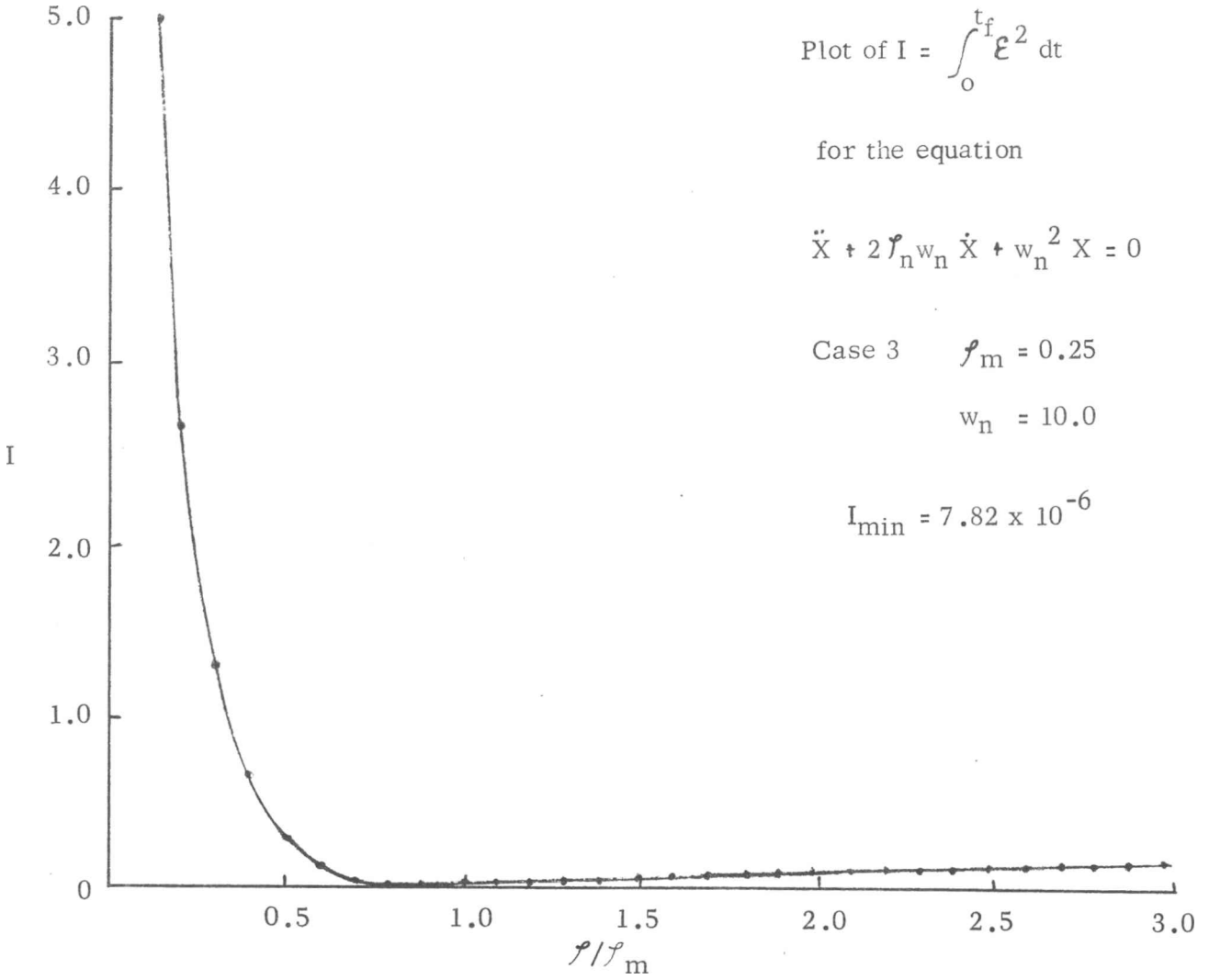


Figure 3.--Plot of one dimensional functional--Case 2



Plot of $I = \int_0^{t_f} \varepsilon^2 \, dt$

for the equation $t_f = 3\left(\frac{2\pi}{w_n}\right)$

$\ddot{X} + 2\zeta_n w_n \dot{X} + w_n^2 X = 0$

Case 3 $\zeta_m = 0.25$

$w_n = 10.0$

$I_{min} = 7.82 \times 10^{-6}$

Figure 4.--Plot of one dimensional functional--Case 3

.6	13.2	5.69	2.30	.809	.225	.0765	.136	.292	.488	.695	.900	1.10	1.28	1.45	1.61
	13.5	5.79	2.36	.840	.236	.0725	.120	.267	.454	.655	.855	1.04	1.23	1.39	1.54
.5	13.7	6.04	2.51	.927	.272	.0681	.0845	.206	.373	.556	.741	.920	1.09	1.25	1.39
	14.0	6.21	2.62	.990	.300	.0700	.0653	.169	.323	.995	.670	.841	1.00	1.15	1.29
.4	14.3	6.42	2.76	1.07	.341	.0775	.0465	.129	.266	.423	.586	.747	.900	1.04	1.18
	14.7	6.71	2.95	1.19	.402	.0951	.0308	.0865	.201	.340	.488	.635	.777	.911	1.04
.3	15.3	7.11	3.22	1.36	.498	.134	.0265	.0440	.128	.243	.371	.500	.627	.748	.863
	16.14	7.71	3.62	1.63	.660	.216	.0413	.0120	.0551	.136	.234	.339	.446	.549	.648
.2	17.4	8.64	4.28	2.08	.961	.397	129	.0226	.00405		.0894		.233	.309	.384
	19.8	10.3	5.53	2.99	1.61	.845	.421	.190	.0693	.0338 .0149		.158 .00911		.0643	.101
.1	25.0	14.2	8.50	5.28	3.40	2.24	1.52	1.04	.720	.501	.0000572 .349	.242	.0325 .165	.117	.0733
	43.0	28.6	20.3	15.1	11.8	9.57	7.99	6.38	5.97	5.30	4.78	4.35	4.01	3.72	3.48
0	323	280	252	232	218	208	200	194	190	186	183	180	178	176	175
	0		.1		.2		.3		.4		.5		.6		.7

$f_2 \uparrow$

$f_1 \rightarrow$

System of Equations in Which $w_1 = 1.0$ $w_2 = 10.0$

$$[D] = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 3.0 \end{bmatrix}$$

Minimum at $_1 = 0.50$, $_2 = 0.15$

Figure 5.--Plot of two dimensional functional with uncoupled damping

.6	13.3	5.77	2.34	.829	.234	.0787	.135	.292	.489	.699	.907	1.11	1.30	1.47	1.63
	13.5	5.88	2.41	.867	.248	.0745	.117	.261	.449	.651	.853	1.05	1.23	1.40	1.56
.5	13.7	6.02	2.50	.915	.267	.0719	.0972	.227	.403	.597	.790	.978	1.16	1.32	1.47
	13.9	6.19	2.60	.978	.294	.0721	.0762	.189	.351	.533	.717	.897	1.07	1.23	1.39
.4	14.3	6.41	2.74	1.06	.334	.0777	.0551	.147	.292	.459	.630	.800	.962	1.11	1.26
	14.7	6.70	2.93	1.17	.393	.0934	.0366	.1008	.223	.372	.528	.683	.834	.977	1.11
.3	15.3	7.10	3.20	1.34	.486	.192	.0267	.0542	.147	.270	.405	.543	.679	.809	.932
	16.1	7.68	3.60	1.61	.645	.207	.0404	.0168	.1675	.156	.262	.376	.490	.601	.708
.2	17.5	8.62	4.26	2.05	.940	.381	.120	.0196	.00782	.0445	.107	.183	.265	.349	.431
	19.8	10.3	5.50	2.95	1.58	.820	.401	.174	.0594	.0106	.00150	.0164	.0457	.0834	.1259
.1	24.9	14.1	8.46	5.24	3.35	2.20	1.47	1.00	.684	.467	.320	.216	.143	.0918	.0566
	42.9	28.5	20.2	15.1	11.7	9.49	7.90	6.74	5.86	5.19	4.66	4.23	3.88	3.60	3.35
0	323	280	251	232	218	207	200	194	189	185	183	180	178	176	174
	0		.1		.2		.3		.4		.5		.6		.7

$f_2 \uparrow$

$f_1 \rightarrow$

System of Equations in Which $w_1 = 1.0$ $w_2 = 10.0$

$$[D] = \begin{bmatrix} 1.0 & -0.5 \\ -0.5 & 3.0 \end{bmatrix}$$

Figure 6.--Plot of two dimensional function with coupled damping

CHAPTER V

DISCUSSION OF RESULTS

The results of this work can be divided into two parts:

1. The nature of the surface described by the functional examined.

2. The comparison of the solution resulting from the uncoupled equations formed with the matrix exponential solution.

The Nature of the Functional Surface

The nature of the one-dimensional functional was very encouraging. A well defined minimum exists at the expected location, and the function is well behaved. The gradient is much steeper as the correct value of damping is approached from a value too small than it is when approaching from a value too large. This can be observed from Figures 2, 3, and 4.

The interaction between modes that occurs in the multiple dimension case, however, created some difficulty. The first trials in examination of the multiple dimensioned functional were conducted with systems of uncoupled equations, as a check on the accuracy of the method, since the proper modal damping coefficients were known.

Examination of the one dimensional function appeared to indicate that it would be wise to underestimate the damping coefficients and then let one of

the optimization routines proceed to the minimum of the functional. When this was tried, it was found that both of the optimization routines progressed very slowly. One trial, three-dimensional case was carried to completion using the Davidon method in approximately two hours of computer time on the Librascope model L-3055. In some other cases, progress appeared even worse. It was evident that the surface of the functional was very difficult for the optimization routines to negotiate. Of the two optimization routines, Powell's method made better progress.

Plotting of two dimensional functionals (Figures 5 and 6) revealed the reason for the difficulty. A very narrow, curved valley appears in the surface as shown in the plots. The gradient of the functional in a direction tangent to the valley is much less than the gradient encountered at the sides of the valley. If the search routines happen to enter the valley at a location far from the minimum, negotiation of the valley toward the minimum becomes very difficult.

Physical Explanation of the Valley

The appearance of the valley encountered in the functional surface can be accounted for on a physical basis. The energy deficit $\mathcal{E}$ from which the functional was formed requires an energy balance on the system in order for it to be zero. If one mode has too large a damping coefficient, the energy in that mode will appear to damp out too quickly. This can be compensated for by a damping coefficient in another mode which is too small, as the energy in the latter mode appears to be damped out too slowly. This can even be

carried to the situation where a negatively damped mode is "introducing energy into the system" to compensate for other modes that have too large a damping coefficient. In one trial case, it actually happened that the search routine was making very slow progress on a functional at a location where one mode erroneously appeared negatively damped whereas the other two modes were given high damping coefficients. The valley therefore represents a locus of damping coefficients that will dissipate the total system energy at approximately the same rate as the correct values.

Choosing Initial Values

Three approaches to overcome the problem presented by the valley in the functional surface are:

1. The choice of a new form of functional which does not have a valley.

2. The design of a minimization method that will use known facts about the valley to more efficiently find its minimum value.

3. The choice of the initial values given the search subroutine at values near enough to the proper values that it does not have to negotiate the valley for long distances.

Of these three, the third appears to be the most feasible.

Comparison of Figures 5 and 6 indicates that the minimum was shifted very little by the addition of the off-diagonal terms in the $[D]$ matrix. Although the shift will not always be this small, a logical place to begin the search is at the location suggested by the diagonal terms in the $[D]$ matrix.

This was found to be fairly successful in the cases where it was used.

Change in Excitation of Modes

As mentioned in Chapter II, each of the modes were originally excited equally, i.e., each initial modal velocity was equal. The presence of the valley in the functional surface provides the possibility that the search routine may find only a relative minimum somewhere in the valley, being unable to negotiate the valley to the absolute minimum. In this case, it would be better to have the smallest error in the damping coefficient which contributes most to the final solution. Giving each mode the same amount of excitation when calculating the functional was unduly biasing the final result, as all modes contribute differently to it. For this reason, each mass (or node) should be given an equal initial velocity and the resulting initial modal velocities used for energy determination. The result of this was that the functional became less sensitive to unexcited modes. This created a difficulty in one case where Powell's minimization method could not find a minimum in one of the coordinate directions. The functional happened to be perfectly level in that direction and the search routine was unable to find a change in the function over the interval it attempted to examine. Davidon's method does not have that difficulty, and was therefore used to do the optimization in this case. Whether or not the change in the method of excitation caused much improvement could not be determined.

Comparison with Matrix Exponential

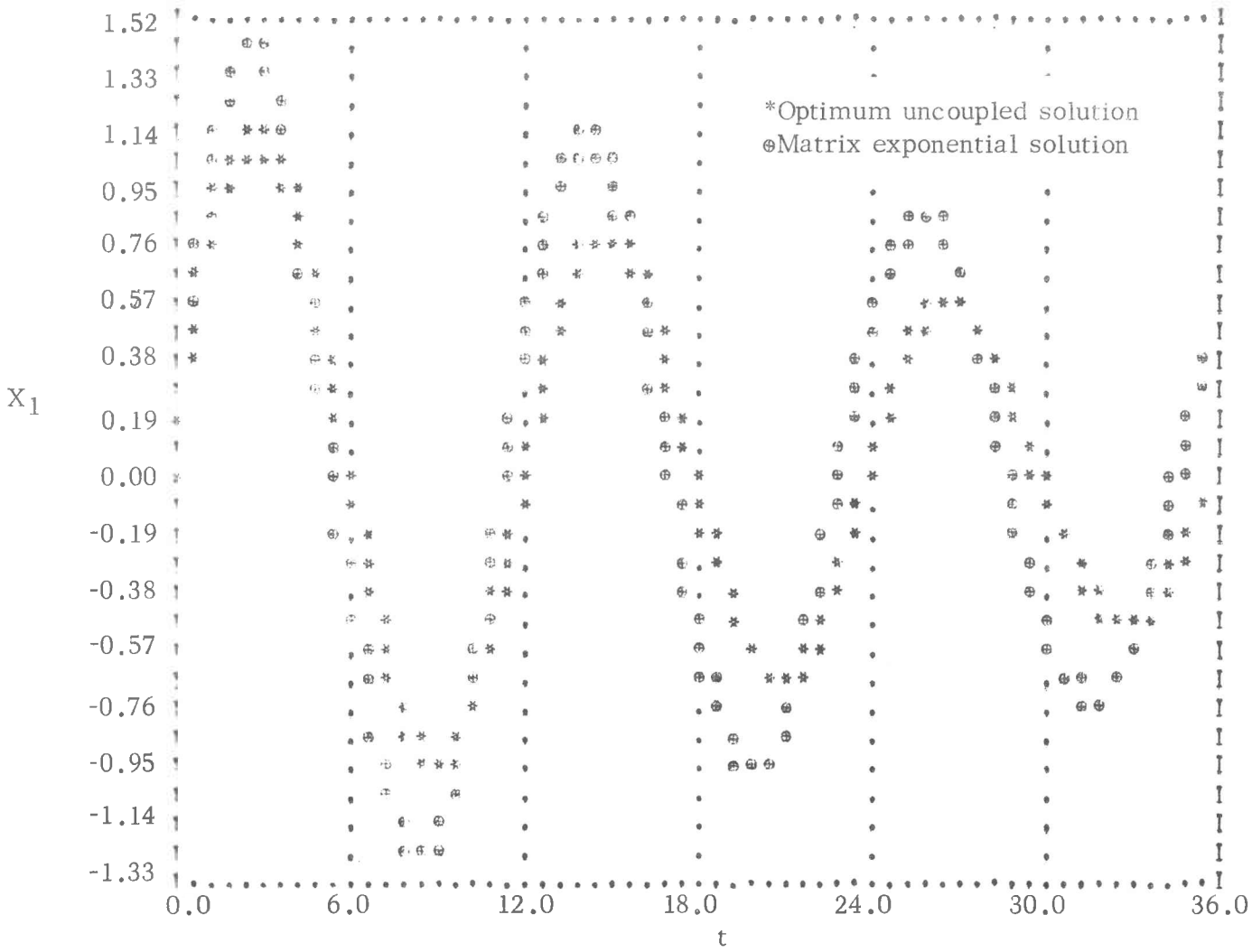
The solution of the system of uncoupled equations resulting from the determination of "optimum" modal damping coefficients was compared with the matrix exponential solution for three, three-dimensional cases graphically in Figures 7 through 15.

In Case 1 (Figures 7-9) the damping coefficient for at least one of the modes which contributes quite significantly to the response of the system appears to be too large. In each of the displacements, the response is more highly damped than the matrix exponential solutions. The search routine evidently concluded in the valley at a location where a dominant mode had a high damping coefficient.

Case 2 represents a negatively damped system. Again the trend is that each of the displacements deviates from the matrix exponential solution in the same manner. The correlation in this case is quite good.

In Case 3, the solution follows very closely the matrix exponential solution, providing highly acceptable results. In each of these cases, the $[D]$ matrix contains fairly significant off-diagonal terms. Table 1 lists the damping coefficients at the beginning of the search (estimated using diagonal terms only) and at the conclusion of the search. The fact that they changed as little as they did indicates that the initial values are well chosen. Whether or not the improvement in the values justifies the search may be questioned for individual cases.

The total error represented in the three cases illustrated is a result



Case 1

$[M]$ Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$[C]$ Matrix

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$[K]$ Matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Figure 7.--Comparison with matrix exponential solution for three-dimensional system $[M]\ddot{X} + [C]\dot{X} + [K]X = 0$, Case 1, Node 1

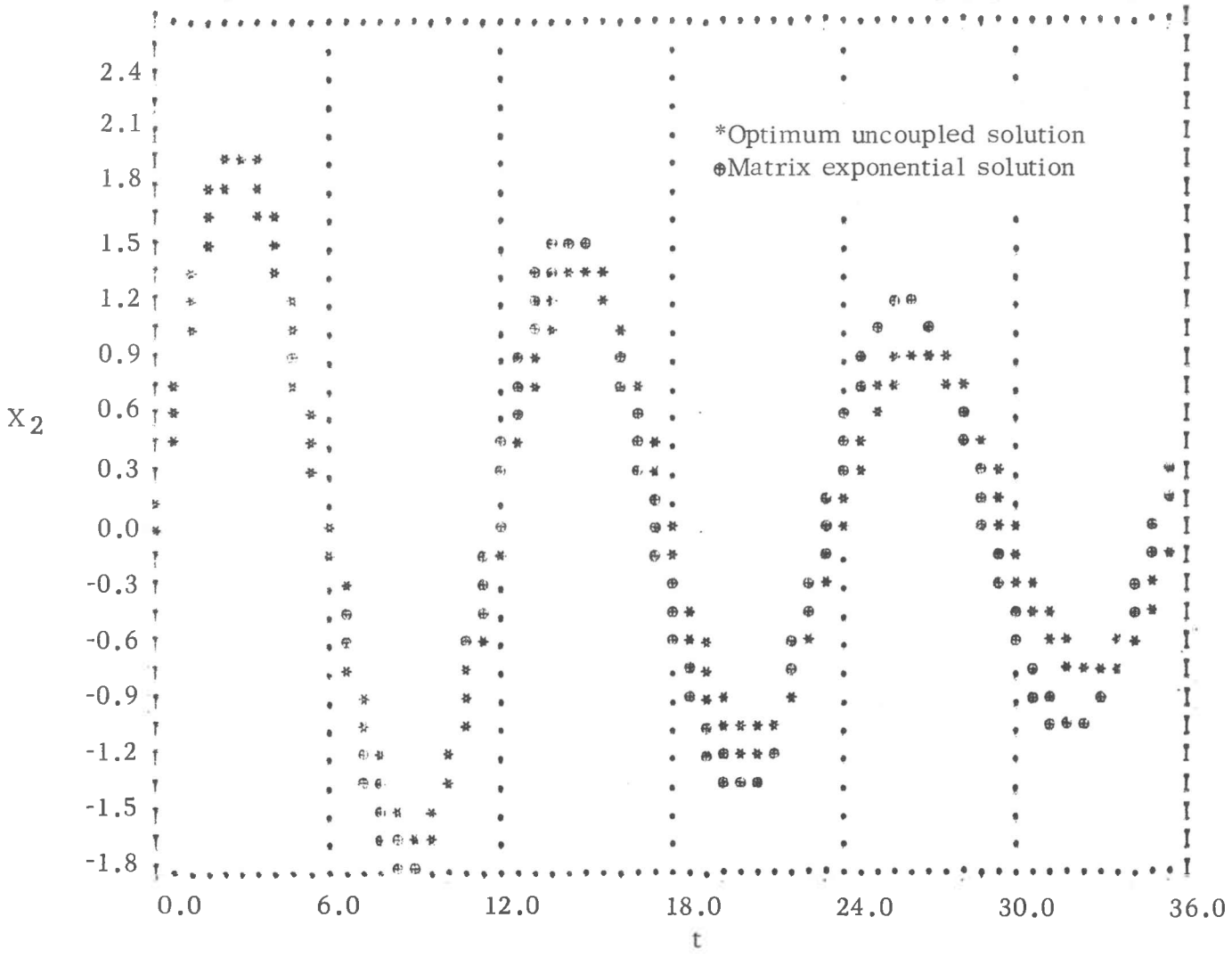


Figure 8.--Comparison with matrix exponential solution for three-dimensional system $[M]\ddot{\underline{X}} + [C]\dot{\underline{X}} + [K]\underline{X} = 0$, Case 1, Node 2

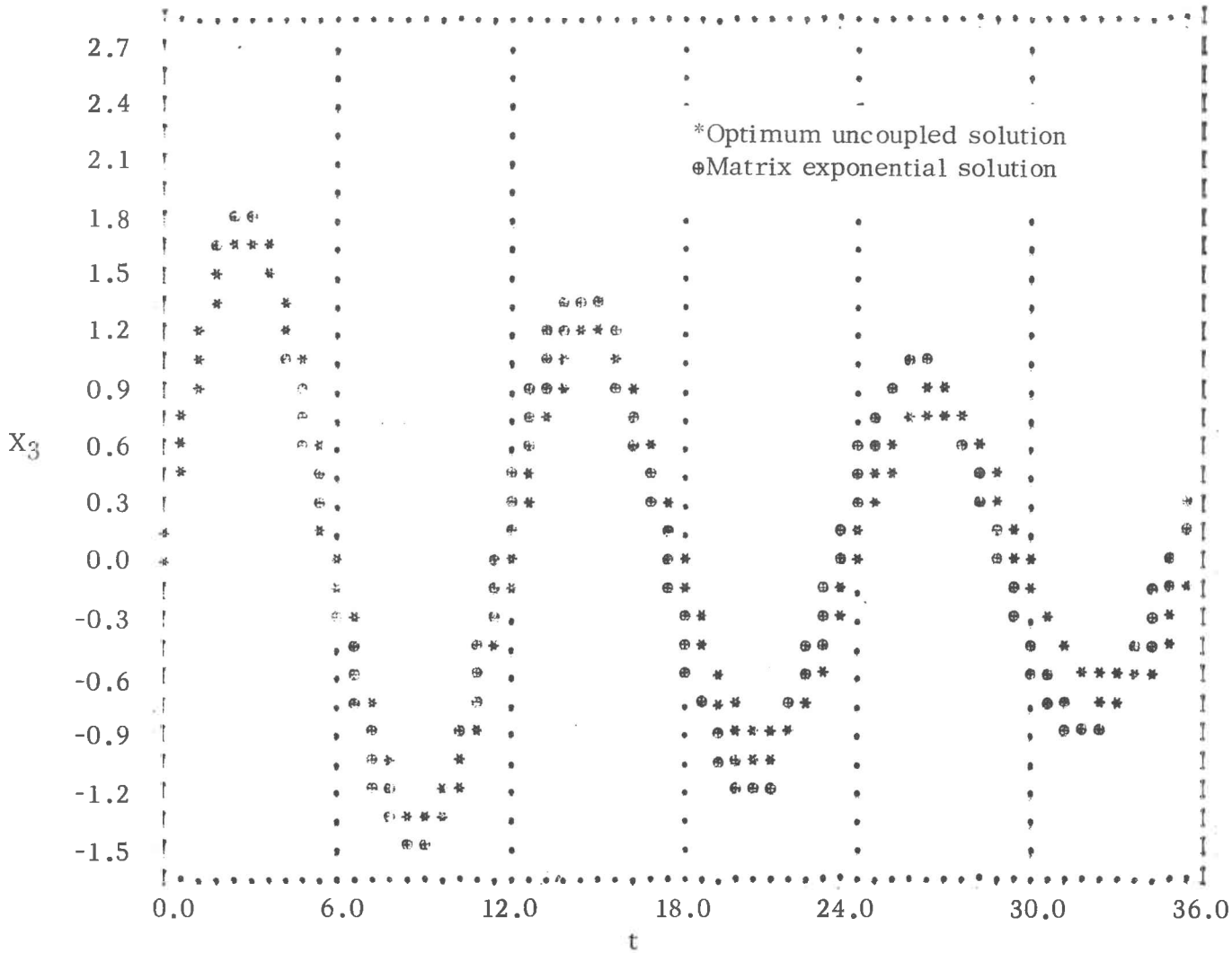


Figure 9.--Comparison with matrix exponential solution for three-dimensional system $[M]\ddot{X} + [C]\dot{X} + [K]X = 0$, Case 1, Node 3

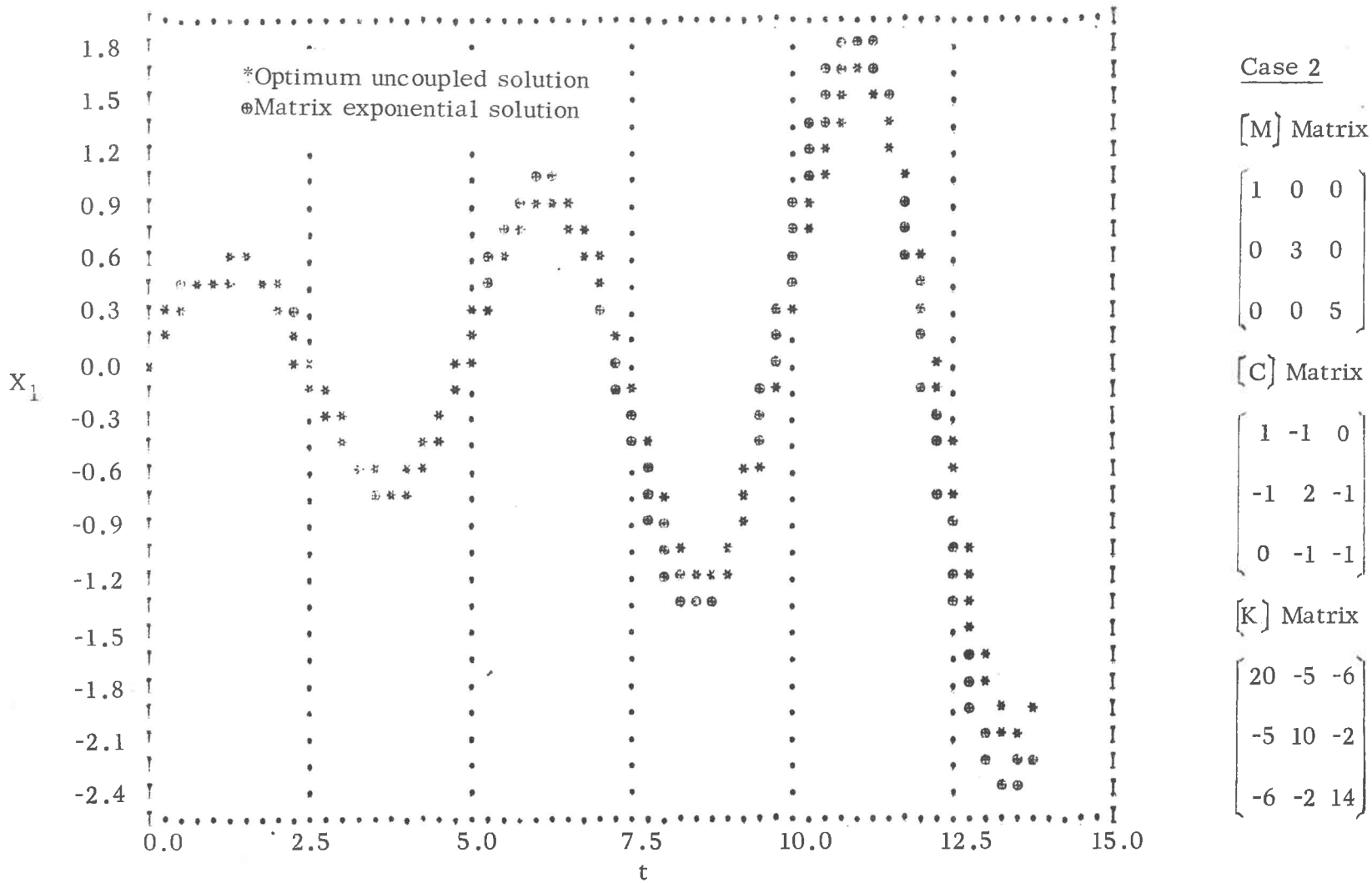


Figure 10.--Comparison with matrix exponential solution for three-dimensional system $[M]\ddot{\underline{X}} + [C]\dot{\underline{X}} + [K]\underline{X} = 0$, Case 2, Node 1

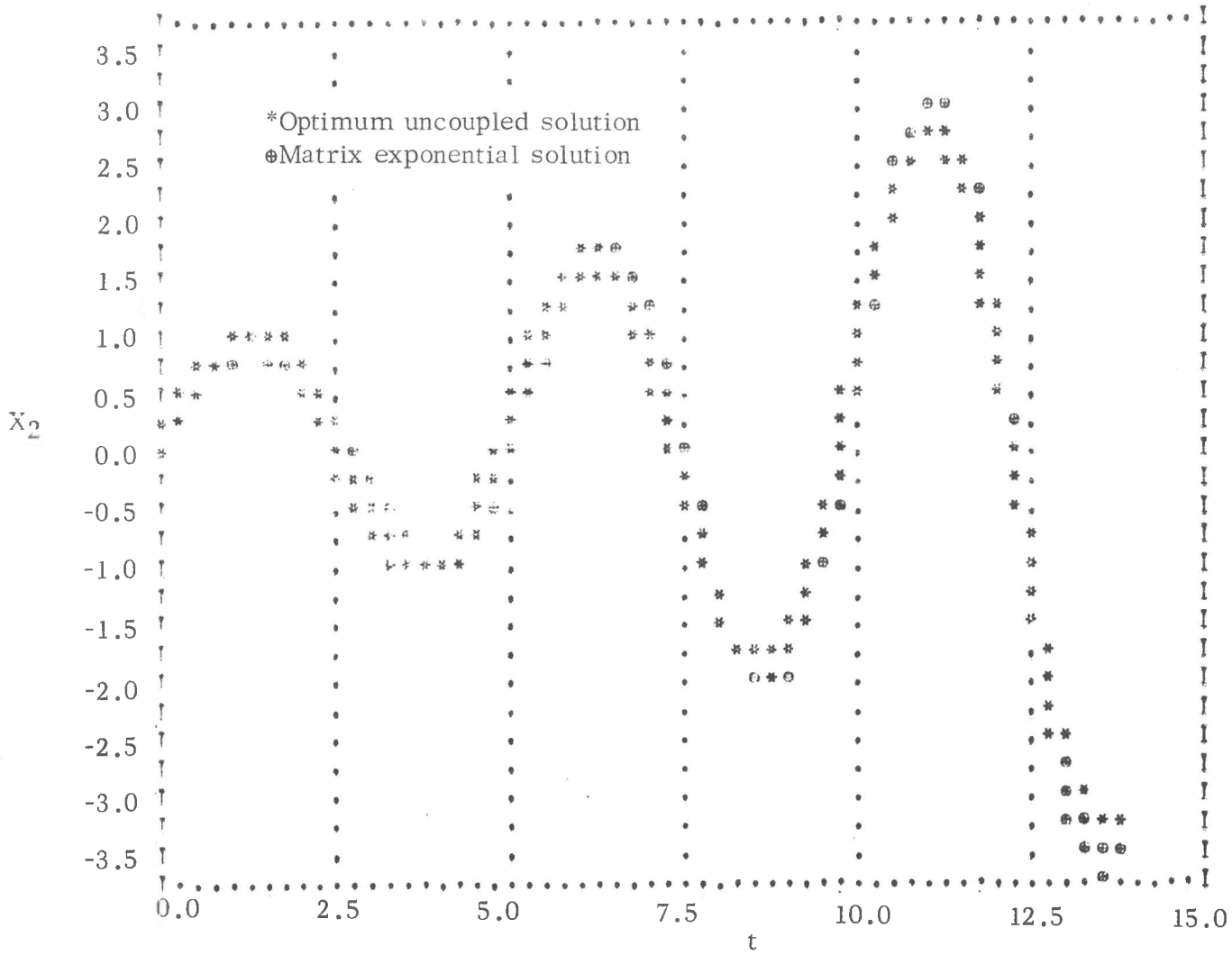


Figure 11.--Comparison with matrix exponential solution for three-dimensional system $[M]\ddot{\underline{X}} + [C]\dot{\underline{X}} + [K]\underline{X} = 0$, Case 2, Node 2

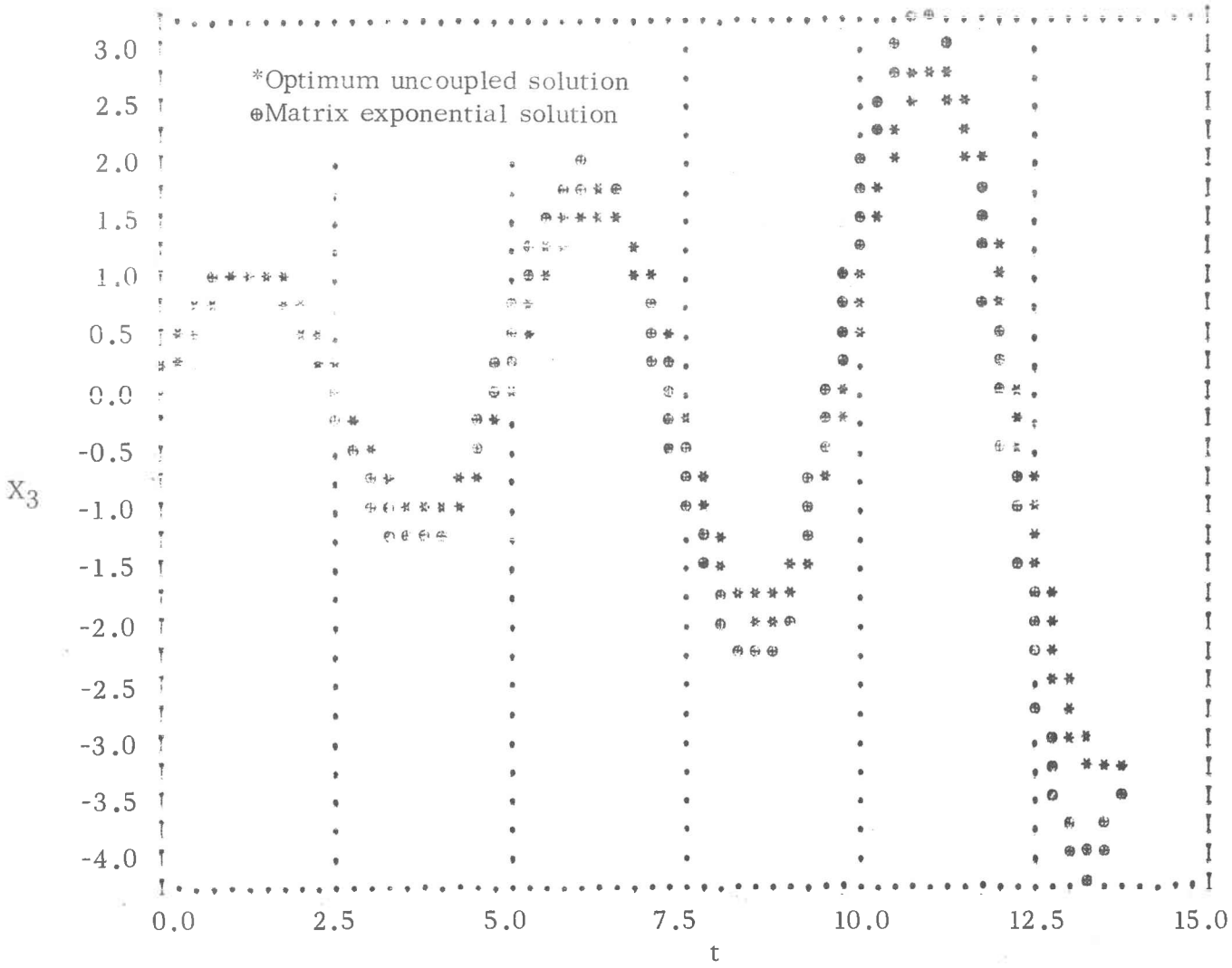
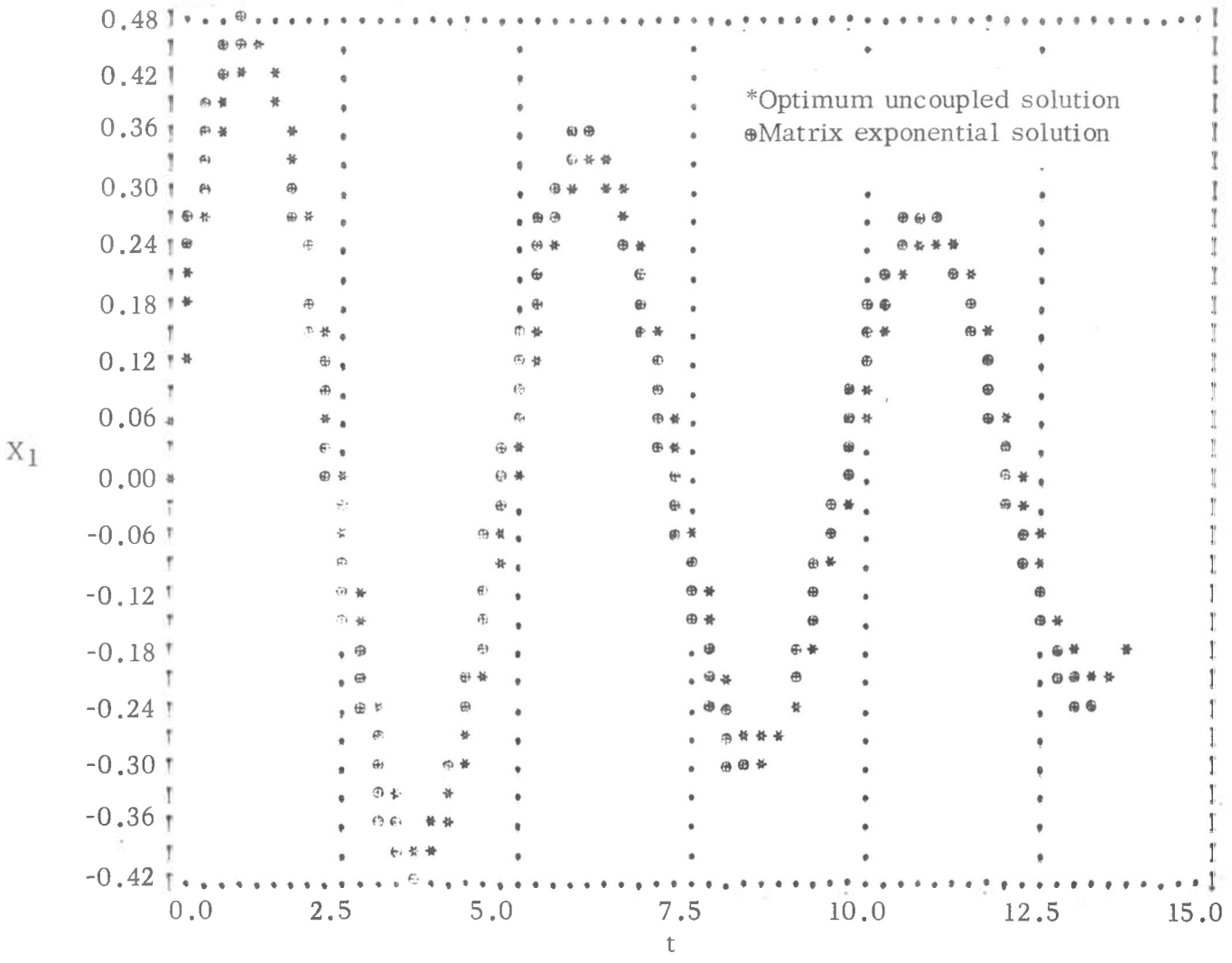


Figure 12.--Comparison with matrix exponential solution for three-dimensional system $[M]\ddot{X} + [C]\dot{X} + [K]X = 0$, Case 2, Node 3



Case 3

$[M]$ Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$[C]$ Matrix

$$\begin{bmatrix} 5 & -2 & -2 \\ -2 & 5 & -3 \\ -2 & -3 & 5 \end{bmatrix}$$

$[K]$ Matrix

$$\begin{bmatrix} 20 & -5 & -6 \\ -5 & 10 & -2 \\ -6 & -2 & 14 \end{bmatrix}$$

Figure 13.--Comparison with matrix exponential solution for three-dimensional system $[M]\ddot{\underline{X}} + [C]\dot{\underline{X}} + [K]\underline{X} = 0$, Case 3, Node 1

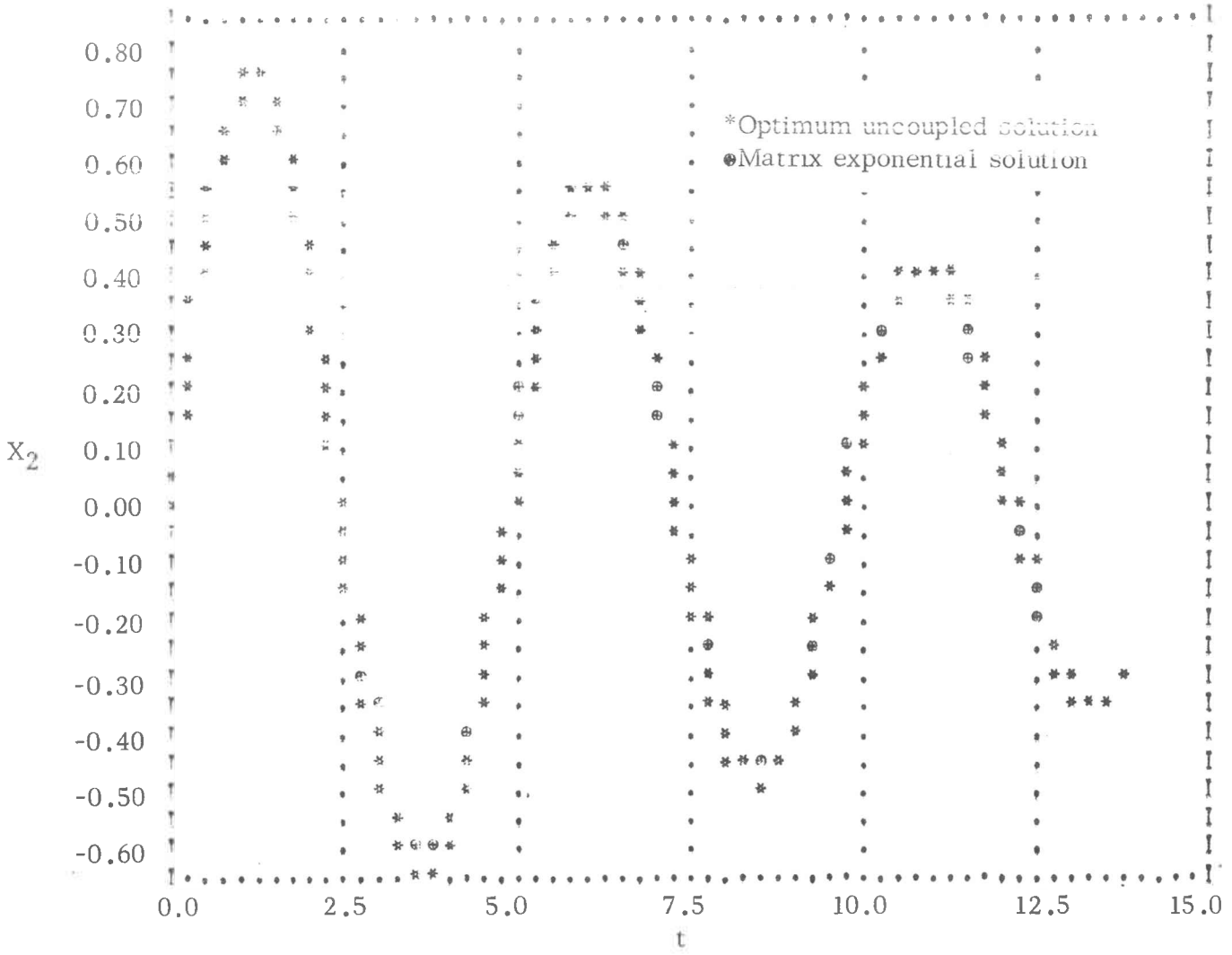


Figure 14.--Comparison with matrix exponential solution for three-dimensional system $[M]\ddot{\underline{X}} + [C]\dot{\underline{X}} + [K]\underline{X} = 0$, Case 3, Node 2

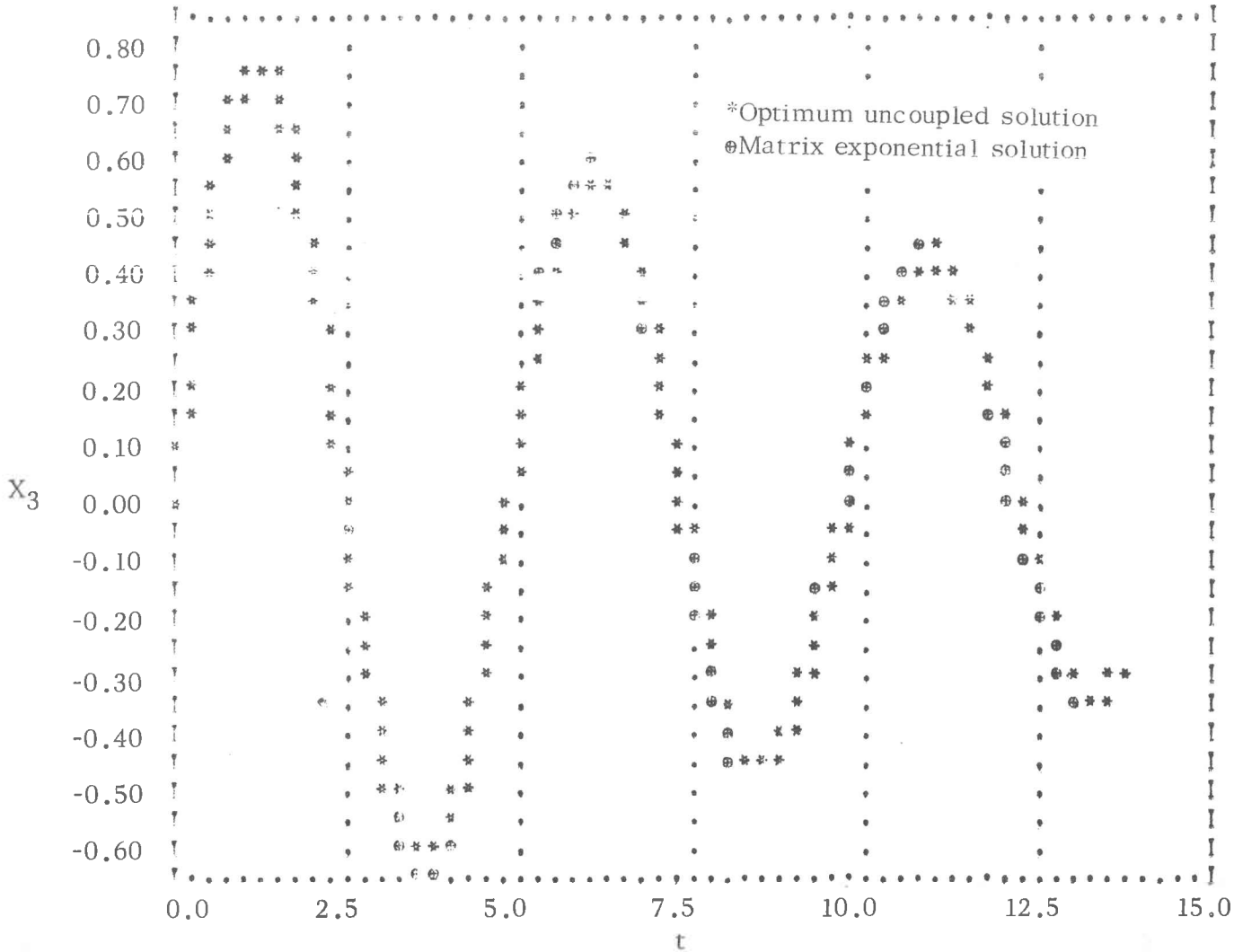


Figure 15.--Comparison with matrix exponential solution for three-dimensional system $[M]\ddot{\underline{X}} + [C]\dot{\underline{X}} + [K]\underline{X} = 0$, Case 3, Node 3

TABLE 1

CHANGE IN DAMPING COEFFICIENTS DURING OPTIMIZATION SEARCH

Case	Mode	Modal Damping Coefficient	
		Beginning	Final
1	1	0.05664	0.05672
	2	0.6666	0.6666
	3	0.7356	0.3909
2	1	-0.08440	-0.08441
	2	0.1451	0.2779
	3	0.1241	0.1259
3	1	0.04629	0.04598
	2	0.5637	0.5637
	3	0.5918	0.5805

of more than just the selection of the damping coefficients. The modal analysis routine relies upon a power method convergent routine, the convergence of which is based upon the frequency or eigenvalue, at which time it is assumed that the eigenvector has also converged. The eigenvector is then used, because of its orthogonality to other eigenvectors, to reduce the system so it will converge to the next eigenvalue and eigenvector. The final simulation is a result of errors propagated throughout this and the whole procedure, as well as errors introduced by choice of the damping coefficients. The matrix exponential solution is also dependent upon a convergent matrix series evaluation to form the exponential matrix. The total deviation that we are examining is a result of all of these effects combined.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

It became obvious almost from the onset that the developed method would not compete with the matrix exponential in computational efficiency. Since it cannot represent the original system exactly, it cannot hope to compete with the matrix exponential in accuracy. It is recommended, therefore, that this method not be used in place of the matrix exponential method. If a system is to be simulated a number of times and with a variety of forcing functions, it may be desirable to reduce the equations to include only the principal modes. This can be accomplished by the conventional modal analysis technique, however, and the matrix exponential can be used to solve the resulting coupled equations. Whether or not the modal analysis is justified depends upon the number of times the system will be simulated and the relative cost of solving the eigenvalue problem and evaluating the exponential matrix. In any case, uncoupling the damping in a set of matrix equations is not justified.

The method investigated is a workable method. The functional developed does have a minimum when the correct damping coefficients are selected. If this method can be extended, applying the same basic approach to finding

modal damping coefficients to associate with the Ritz form of modal analysis, applying directly to a distributed system, it may provide an efficient workable method of simulating distributed dynamic systems.

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APPENDIXES

APPENDIX A

SUMMARY OF EQUATIONS

Functional

$$I = \int_0^{t_f} \varepsilon^2 dt$$

$$\varepsilon = E_0 - E_p - E_K - E_d$$

$$E_0 = \frac{1}{2} \sum_{i=1}^{m} q_{i_0}^2$$

$$E_p = \frac{1}{2} \sum_{i=1}^{m} (\omega_i q_i)^2$$

$$E_K = \frac{1}{2} \sum_{i=1}^{m} q_i^2$$

$$E_d = \int_0^t (\dot{\underline{q}}^T [D] \dot{\underline{q}}) dt$$

Gradient of Functional

$$q^T = \left[\frac{\partial I}{\partial \rho_1} \quad \frac{\partial I}{\partial \rho_2} \cdots \frac{\partial I}{\partial \rho_m} \right]$$

$$\frac{\partial I}{\partial \rho_i} = 2 \int_0^{t_f} \varepsilon \frac{\partial \varepsilon}{\partial \rho_i} dt$$

$$\frac{\partial \varepsilon}{\partial \rho_i} = -\frac{\partial E_p}{\partial \rho_i} - \frac{\partial E_K}{\partial \rho_i} - \frac{\partial E_d}{\partial \rho_i}$$

$$\frac{\partial E_p}{\partial \rho_i} = \omega_i^2 q_i \frac{\partial q_i}{\partial \rho_i}$$

$$\frac{\partial E_K}{\partial \rho_i} = \dot{q}_i \frac{\partial \dot{q}_i}{\partial \rho_i}$$

$$\frac{\partial E_d}{\partial \rho_i} = \int_0^t \left(\frac{\partial \dot{\underline{q}}^T}{\partial \rho_i} [D] \dot{\underline{q}} + \dot{\underline{q}}^T [D] \frac{\partial \dot{\underline{q}}}{\partial \rho_i} \right) dt$$

$$\frac{\partial \dot{q}^T}{\partial \zeta_i} = \left[0 \; 0 \ldots \frac{\partial \dot{q}_i}{\partial \zeta_i} \ldots 0 \right]$$

Modal Displacements

$\zeta < 1.0$

$$q_i = \frac{\dot{q}_{oi}\, e^{-\zeta_i \omega_i t}}{\omega_i \sqrt{1-\zeta_i^2}} \sin\left(\omega_i \sqrt{1-\zeta_i^2}\, t\right)$$

$$\dot{q}_i = \frac{\dot{q}_{oi}\, e^{-\zeta_i \omega_i t}}{\omega_i \sqrt{1-\zeta_i^2}} \left[-\zeta_i \omega_i \sin\left(\omega_i \sqrt{1-\zeta_i^2}\, t\right) + \omega_i \sqrt{1-\zeta_i^2} \cos\left(\omega_i \sqrt{1-\zeta_i^2}\, t\right) \right]$$

$$\frac{\partial q_i}{\partial \zeta_i} = \frac{\dot{q}_{oi}\, e^{-\zeta_i \omega_i t}}{\omega_i \sqrt{1-\zeta_i^2}} \left[\left(\frac{\zeta_i}{\omega_i (1-\zeta_i^2)^{3/2}} - \frac{t}{\sqrt{1-\zeta_i^2}} \right) \sin\left(\omega_i \sqrt{1-\zeta_i^2}\, t\right) - \frac{\zeta_i \omega_i t}{\sqrt{1-\zeta_i^2}} \cos\left(\omega_i \sqrt{1-\zeta_i^2}\, t\right) \right]$$

$$\frac{\partial \dot{q}_i}{\partial \zeta_i} = \dot{q}_{oi}\, e^{-\zeta_i \omega_i t} \left[\left(\frac{-\zeta_i^2}{(1-\zeta_i^2)^{3/2}} + \frac{\zeta_i \omega_i t - 1}{\sqrt{1-\zeta_i^2}} + \zeta_i \omega_i^2 t \right) \sin\left(\omega_i \sqrt{1-\zeta_i^2}\, t\right) + \left(-\omega_i t + \frac{\zeta_i^2 \omega_i^2 t}{\sqrt{1-\zeta_i^2}} \right) \cos\left(\omega_i \sqrt{1-\zeta_i^2}\, t\right) \right]$$

$\zeta = 1.0$

$$q_i = \dot{q}_{oi}\, t\, e^{-\zeta_i \omega_i t}$$

$$\dot{q}_i = \dot{q}_{oi}\, e^{-\zeta_i \omega_i t} \left(1 - t \zeta_i \omega_i\right)$$

$$\frac{\partial q_i}{\partial \zeta_i} = -\dot{q}_{oi}\, t^2 \omega_i\, e^{-\zeta_i \omega_i t}$$

$$\frac{\partial \dot{q}_i}{\partial \zeta_i} = -\dot{q}_{oi} \omega_i t \left(\omega_i \zeta_i t - 2\right) e^{-\zeta_i \omega_i t}$$

$\zeta > 1.0$

$$q_i = \frac{\dot{q}_{0i}\, e^{-\zeta_i \omega_i t}}{\omega_i \sqrt{\zeta_i^2 - 1}} \sinh\left(\omega_i \sqrt{\zeta_i^2 - 1}\, t\right.$$

$$\dot{q}_i = \frac{\dot{q}_{0i}\, e^{-\zeta_i \omega_i t}}{\omega_i \sqrt{\zeta_i^2 - 1}} \left[-\zeta_i \omega_i \sinh\left(\omega_i \sqrt{\zeta_i^2 - 1}\, t\right) + \omega_i \sqrt{\zeta_i^2 - 1}\right.$$
$$\left.\cosh\left(\omega_i \sqrt{\zeta_i^2 - 1}\, t\right)\right]$$

$$\frac{\partial q_i}{\partial \zeta_i} = \dot{q}_{0i}\, e^{-\zeta_i \omega_i t} \left[\left(-\frac{\zeta_i}{\omega_i (\zeta_i^2 - 1)^{3/2}} - \frac{t}{\sqrt{\zeta_i^2 - 1}}\right) \sinh\left(\omega_i \sqrt{\zeta_i^2 - 1}\, t\right)\right.$$
$$\left. + \frac{\zeta_i \omega_i t}{\sqrt{\zeta_i^2 - 1}} \cosh\left(\omega_i \sqrt{\zeta_i^2 - 1}\, t\right)\right]$$

$$\frac{\partial \dot{q}_i}{\partial \zeta_i} = \dot{q}_{0i}\, e^{-\zeta_i \omega_i t} \left[\left(\frac{\zeta_i^2}{(\zeta_i^2 - 1)^{3/2}} + \frac{\zeta_i \omega_i t - 1}{\sqrt{\zeta_i^2 - 1}} + \zeta_i \omega_i t\right)\right.$$
$$\sinh\left(\omega_i \sqrt{\zeta_i^2 - 1}\, t\right) + \left(-\omega_i t - \frac{\zeta_i^2 \omega_i^2 t}{\sqrt{\zeta_i^2 - 1}}\right)$$
$$\left.\cosh\left(\omega_i \sqrt{\zeta_i^2 - 1}\, t\right)\right]$$

APPENDIX B

MODAL ANALYSIS

Modal Analysis

Modal Analysis is a useful tool in reducing the dimensionality of a problem, and has been well explained in textbooks. Included here is an overview of the procedure.

The general form of the dynamics equation is

$$[M]\ddot{\underline{x}} + [C]\dot{\underline{x}} + [K]\underline{x} = \underline{f} \tag{1}$$

where $\underline{x}$ is a vector representing the motion of the n degrees of freedom employed in the dynamic model. The nxn matrices M, C, and K are derived from the physics of the problem and the vector $\underline{f}$ represents the n forces applied at the n degrees of freedom.

For undamped, free vibrations, equation (1) reduces to

$$[M]\ddot{\underline{x}} + [K]\underline{x} = 0 \tag{2}$$

which for harmonic motion of circular frequency w becomes

$$-\omega^2[M]\underline{x} + [K]\underline{x} = 0$$

or

$$\left([K] - \omega^2[M]\right)\underline{x} = 0 \tag{3}$$

In general there are n discrete values of w, or eigenvalues, which will permit a non-trivial solution of (3). These eigenvalues represent the natural frequencies of the system. Associated with each of these natural frequencies is an eigenvector or mode shape $\underline{x} = \underline{\phi}_n$. Finding these natural frequencies and their associated mode shapes is an eigenvalue problem and is discussed in testbooks.

The mode shapes are orthogonal or normal and are therefore called normal modes. If these n mode shapes are column listed in an nxn matrix $[\phi]$, then

$$[\phi]^T [M] [\phi] = [\tilde{m}_n]$$

is a diagonal matrix called the transformed mass matrix and

$$[\phi]^T [K] [\phi] = [\tilde{\omega^2}] [\tilde{m}_n]$$

where $[\tilde{\omega^2}]$ is also a diagonal matrix consisting of the squares of the n natural frequencies. Further, if the normal modes $\underline{\phi}_n$ are properly normalized so that each of the diagonal components of the transformed mass matrix is equal to 1.0

$$[\tilde{m}_n] = [I]$$

then

$$[\phi]^T [M] [\phi] = [I] \tag{4}$$

and

$$[\phi]^T [K] [\phi] = [\tilde{\omega^2}] [I] = [\tilde{\omega^2}] \tag{5}$$

where $[I]$ is the unity matrix.

Since the normal modes provide a basis for the displacement vector $\underline{x}$, we can write the displacement vector as a linear combination of the normal modes.

$$\underline{x}(t) = q_1(t)\,\underline{\phi}_1 + q_2(t)\,\underline{\phi}_2 + \cdots q_n(t)\,\underline{\phi}_n$$

$$= [\underline{\phi}_1\ \underline{\phi}_2 \cdots \underline{\phi}_n] \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix} = [\phi]\,\underline{q}(t) \tag{6}$$

Substituting (6) into 1 yields

$$[M][\phi]\ddot{\underline{q}} + [C][\phi]\dot{\underline{q}} + [K][\phi]\underline{q} = \underline{f}$$

Post multiplying by $[\phi]^T$

$$[\phi]^T[M][\phi]\ddot{\underline{q}} + [\phi]^T[C][\phi]\dot{\underline{q}} + [\phi]^T[K][\phi]\underline{q} = [\phi]^T\underline{f}$$

Substituting (4) and (5)

$$\ddot{\underline{q}} + [\phi]^T[C][\phi]\dot{\underline{q}} + [\tilde{\omega}^2]\underline{q} = [\phi]^T\underline{f} \qquad (7)$$

For convenience let us define

$$[\Delta] = [\phi]^T[C][\phi] \qquad (8)$$

and the equation (7) becomes

$$\ddot{\underline{q}} + [\Delta]\dot{\underline{q}} + [\tilde{\omega}^2]\underline{q} = [\phi]^T\underline{f} \qquad (9)$$

At this point some discussion of the matrix $[\Delta]$ is in order. $[\Delta]$ in general is not a diagonal matrix. If it happens to be diagonal, or if the diagonal elements are much larger than the off diagonal elements so that it can be assumed to be diagonal with reasonable accuracy, then the system represented by (9) consists of n uncoupled scalar equations, each of which can be solved independently. Herein lies the utility as well as the difficulty of the modal analysis technique. For undamped, or very lightly damped systems, it provides a method of uncoupling the dynamic equations. By examining $[\phi]^T\underline{f}$ one can determine the relative participation of each of the modes in the total solution. By elimination of modes with relatively low participation, the dimensionality of the problem can be reduced such that only the dominant modes are retained. Modal analysis therefore has the

ability to reduce the dimensionality of an n degree of freedom system to m, where m is the number of dominant modes. The difficulty involved, however, is that of determining a method of diagonalizing the $[\Delta]$ matrix or, in physical terms, determining the damping associated with each mode.

Reduction in Dimensionality

While performing the analysis just described, the dimensionality is reduced if the analysis is begun using only the dominant modes. In this case the matrix $[\phi]$ which consists of a column listing of the m dominant normal modes is an nxm matrix, where n is the number of degrees of freedom of the system and m is the number of dominant modes used. $\underline{q}$ is now a vector of m components.

Therefore

$[\phi]^T [M] [\phi] = [I]$ is an m x m matrix,

$[\phi]^T [C] [\phi] = [\Delta]$ is an m x m matrix

and $[\phi]^T [K] [\phi] = [\omega^2]$ is an m x m matrix.

$[\phi]^T \underline{f}$ results in a vector of m components.

The resulting equation (9)

$$\ddot{\underline{q}} + [\Delta] \dot{\underline{q}} + [\omega^2] \underline{q} = [\phi]^T \underline{f}$$

is therefore now a system of m equations in m unknowns, rather than n equations in n unknowns. The contributions of the lower participating modes have been neglected.

AN OPTIMIZATION TECHNIQUE APPLIED TO THE DETERMINATION OF MODAL DAMPING COEFFICIENTS

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ABSTRACT

A variational approach to the determination of optimum modal damping coefficients for dynamic systems is presented. A functional is defined, derived, and examined. Numerical optimization techniques are used to determine the modal damping coefficients that minimize the functional. Application of the theory is to a matrix system of differential equations and the results are compared to the matrix exponential solution of the equations. Conclusions are drawn regarding the utility of the method and the direction of future investigations.

COMMITTEE APPROVAL: