1969-8

Analytical Study of Heat Conduction and Thermal Stress in Solids with Pressure Dependent Contact Resistance

Dhruvkumar S. Patel
Brigham Young University - Provo

ANALYTICAL STUDY OF HEAT CONDUCTION AND THERMAL STRESS IN SOLIDS WITH PRESSURE DEPENDENT CONTACT RESISTANCE

A Thesis

Presented to the

Department of Mechanical Engineering Science

Brigham Young University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Dhruvkumar S. Patel

August 1969

This thesis, by Dhruvkumar S. Patel, is accepted in its present form by the Department of Mechanical Engineering Science of Brigham Young University as satisfying the thesis requirements for the degree of Master of Science.

August 4, 1969
Date

Typed by Katherine Shepherd

ACKNOWLEDGMENTS

An expression of gratitude is extended to Dr. Howard S. Heaton, my thesis advisor, for his valuable direction and constant encouragement throughout the thesis work.

My indebtedness is also expressed to Param Pujya Shree Mota, for impregnating in me his spiritual ideals, enlightening my path of prosperity. He has been, is, and will be my real teacher for my whole life.

Appreciation is also conveyed to my parents for their continuous inspiration to me to gain higher education and for their personal care in laying down the foundation of my academic career.

TABLE OF CONTENTS

LIST OF FIGURES

NOMENCLATURE

Symbol	Definition	Units
T_f	Suddenly applied temperature	^oR
T_∞	Initial temperature	^oR
Θ_f	Suddenly applied temperature	Dimensionless
Θ_∞	Initial temperature	Dimensionless
L	Length of the plate	ft.
Θ_1	Temperature distribution in outer cylinder up to the end of the first domain in the inner cylinder	Dimensionless
Θ_s	Temperature distribution in the inner solid cylinder during the first domain	Dimensionless
Θ'_1	Temperature distribution in the outer cylinder from the beginning to the end of the second domain in the inner solid cylinder	Dimensionless
Θ'_s	Temperature distribution in the inner solid cylinder during the second domain	Dimensionless
τ_0	The heat flux propagation in the outer cylinder as well as in the flux plate	ft.
$\bar{\tau}_0$	The heat flux propagation in the outer cylinder	

Symbol	Definition	Units
	as well as in the flat plate	Dimensionless
τ_o'	The heat flux propagation in the inner cylinder	ft.
T_1	Temperature at the interface of the composite cylinders as well as at thc insulated face of the flat plate	^oR
T_2	Temperature at the center of the inner solid cylinder	^oR
F_o	Fourier number $\frac{t}{L^2}$	Dimensionless
α_1	Diffusivity of inner solid cylinder	ft^2/hr
α_2	Diffusivity of outer cylinder	ft^2/hr
h_c	Heat transfer coefficient	$B.T.U./hr.ft^{2\,o}F$
h_{co}	Heat transfer coefficient in stress-free state	$B.T.U./hr.ft^{2\,o}F$

CHAPTER I

INTRODUCTION

In the usual thermal stress analysis of a body, the stresses are found for a predetermined temperature distribution. In problems involving composite materials, thermal contact resistance between materials can significantly affect the temperature distribution. The contact resistance is normally a function of the contact pressure and thermally induced stresses can affect the termperature distribution. Thus in composite bodies the thermal stress problem and the heat transfer problem may be coupled and require a simultaneous solution of both problems.

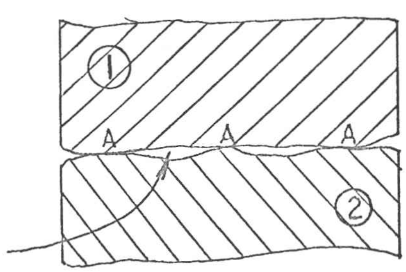
The purpose of this thesis is to investigate this condition and determine the effect of pressure dependent contact resistance on the transient heat transfer in composite bodies with time varying thermal stresses.

This thesis will be concerned primarily with the mathematical analysis of the problem which finds temperature distribution in two contacting bodies with pressure dependent contact resistance. Two cases will be considered. The first case considers the transient heat conduction in rectangular co-ordinates. The second case will deal with the transient heat conduction in circular co-ordinates.

The contact resistance is significantly higher in materials having a higher coefficient of conductivity so the following discussion will primarily consider metal joints.

The influence of the contact resistance is important in many heat transfer problems. It is particularly important in the design of high speed vehicles like supersonic jets and rockets where heat transfer or thermal consideration becomes an important criteria in the design. In modern development, in the area of heat transfer, it is not a prudent assumption to neglect temperature drop along the joints, where the temperature gradient is high.

In normal heat transfer problems, where two contacting surfaces conduct the heat flux, it is usually assumed that there is no temperature drop at the interface. Apparently it seems that no contact resistance would be present at a perfect joint, but a careful microscopic observation of almost all the surfaces reveals many surface asperities which when in contact create many voids or gaps in the joint.



1 and 2 -- Two mating surfaces

voids or air pockets

A - Contact between asperities on both the surfaces

As shown in the sketch, when heat flux passes from one metal to another through the joint, the heat flux concentrates on the area where asperities of the two surfaces are in contact. Since metals have high conductivity, the voids containing air have a comparatively high resistance or low heat transfer coefficient, and a very small amount of heat flux passes through them convectively. This converges the heat flow lines to very small contacting areas. These voids increase the overall contact resistance of the joint and an appreciable temperature drop occurs in the joint.

The effect of the thermally induced stresses on the heat transfer at the joint can be very significant. For example, two metal plates which are in contact with each other and whose free faces are held against the rigid supports, will be considered.

In the stress free state, contact resistance of the joint will be quite high, due to air pockets in the joint. If the plates are heated, the thermal expansion takes place. If this thermal expansion is obstructed by rigid supports, thermal stress is set up at the contact. This thermal stress will vary with the temperature of the two plates. The varying thermal stress at the joint varies the contact resistance which affects the temperature distribution of the plates. As the thermal stress at the joint increases with temperature, the area of metal-to-metal contact increases and the volume of air pockets gets smaller and smaller. Since more and more heat flux passes through metal than passes through air pockets convectively, the contact resistance decreases.

Many empirical relations have been established experimentally between contact pressure and contact resistance, but a linear profile will be selected since this profile serves the purpose of investigating the problem mathematically. Even though the profile taken is very simple, it includes the physically established fact that contact resistance decreases with increasing thermal compressive stress at the joint.

In the first case of rectangular co-ordinates for the mathematical investigation, the flat plate extending infinitely in both the directions with both the faces held between two rigid walls will be considered. One of the faces is insulated and the other face is exposed suddenly to temperature θ_o, higher than initial temperature θ_∞ of the plate, and heat transfer occurs from the source at θ_o to the wall across the joint. As seen before, thermal compressive stress will be set up at the interfaces.

In the second case, to investigate the problem in circular co-ordinates, two concentric cylinders with the inner one being solid will be considered. The axial and the angular temperature variation will be neglected in this problem.

This leads the problem to a one-dimensional analysis in the radial direction. Relative displacement between two cylinders is not allowed and both the cylinders are assumed to expand uniformly in the axial direction, thus giving rise to the uniform axial strain as well as axial stress in both the cylinders.

The composite cylinders are subjected at the periphery of the outer cylinder, to a suddenly applied temperature θ_o, being different from the initial temperature θ_∞. The propagation of heat flux in the radial direction will set up thermal stress at the interface, thereby affecting the contact resistance at the interface.

In both the problems the surfaces are subjected to a step change in temperature, and the heat flux will gradually propagate in one direction. The time taken to reach the steady state will be divided in two parts. The time taken by heat flux to reach from one surface to the other surface will be denoted as first domain. The more incoming heat flux will increase the temperature of the other surface from the initial temperature to the final steady state temperature and this constitutes the second domain of the problem.

In the above two problems the temperature distribution will be found by the integral method. In using the integral method it is required first to find the governing integral equation. For multi-dimensional problems with constant properties the integral equation, as given by Arpaci (1)*, is

$$\int_v \left(\nabla^2\theta - \frac{1}{\alpha}\cdot\frac{\partial\theta}{\partial t} \right) dv = 0. \quad \ldots\ldots\ldots\ldots(1.1)$$

The next step is to select the temperature profile, satisfying the boundary conditions of the problem. The substitution of this profile into

*Numbers in parentheses refer to the List of References.

the governing integral equation will lead to the differential equation with time as the independent variable. This differential equation may be linear or non-linear depending upon the thermal properties. The solution of this governing differential equation gives the required temperature profile.

The exact solution for a pressure-independent contact resistance in the first case is available in chart or tabular form. In the case of pressure dependent contact resistance, the mathematical complexity and difficulty in arriving at its exact solution do not tempt the present investigator in this area to tackle the problem by exact methods using differential calculus.

So far as the author knows, no exact solution technique has been developed. The methods like variational calculus, finite difference approximation, and integral methods which are relatively less rigorous and mathematically complex, but still give relatively good accuracy acceptable for all engineering purposes, seem more promising.

The reason why the integral method is specifically used in the above problem, its relative merits and demerits, its rapid development, and its application in the area of heat transfer, will be justified in the following discussion.

If the temperature of the solid varies over a wide range, the thermal properties become temperature dependent and heat is transferred by conduction as well as radiation, which will give a non-linear governing equation and non-linear boundary conditions. This difficulty necessitated early investigators to develop a new, very simple, and comparatively accurate mathematical technique for solution, known as the integral method, by which approximate solutions to non-linear transient heat conduction problems could be obtained.

The integral method of solution for heat conduction problems does not require the problem to be linearized because it is flexible enough to encompass all the non-linearities.

This method is so versatile that almost any complicated heat transfer problem can be solved by approximation giving accuracy acceptable for engineering purposes. This method can also be used to obtain the approximate solutions to linear problems with complicated spacially dependent thermal properties.

In the thesis problem the governing equation itself is linear but non-linearity is introduced in the boundary condition, due to pressure dependent contact resistance.

Historically, the integral method was first used by Landahl to solve the diffusion equation

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \qquad \begin{array}{l} x > 0 \\ t > 0 \end{array}$$

in the field of biophysics. He selected a simple linear profile to be used in the integral equation for his problem. He also used an exponential profile but the end result was not considered to give significant improvement over the result obtained using a linear profile.

A more sophisticated approach was adopted by the Russian author Veinik in applying the integral method to a great number of heat conduction problems by taking polynomial profile. None of these early investigators showed the accuracy of the method.

A very rigorous mathematical treatise is given by L. V. Kantorovich and V. I. Krylov (2). They took the polynomial profile or circular profile to satisfy the boundary conditions. They also tried to show the technique for higher order approximation, in the integral method, to approximate the integral solution to the exact solution more accurately. They suggested two approaches to reduce the errors in the approximate solution at the points where maximum discrepancies are suspected.

The first approach was to consider "n" relations to be obtained, for the n^{th} order approximation, by multiplying the integrand of the integral formulation of the heat conduction problem by continuous but otherwise largely arbitrary function. Mathematically it can be represented as

$$\int_V \left(\nabla^2 T - \frac{1}{\alpha} \cdot \frac{\partial T}{\partial t} \right) F_i (V) dV = 0$$

$$i = 1, 2, 3, 4, ----- N$$

The above approach has its roots in the calculus of variation. The second approach was based on satisfying the differential equation.

The treatise of Kantorovich and Krylov gives a very good mathematical justification to the usefulness of the integral method but it does not justify properly its use in the area of heat transfer.

Goodman (3) has justified very clearly the importance of the integral method in the area of heat transfer. He has given the integral solutions to many heat transfer problems and has shown that a relatively good accuracy is given by the integral method.

So far a very good mathematical background has been established for integral methods but they are not used as extensively in the field of engineering as the other approximate methods.

In reference (1), the integral method is shown to give accuracy as good as finite difference methods with less mathematical rigor and skill. Moreover, results are in more compact and generalized form, applicable to a variety of heat transfer problems of the same category.

CHAPTER II

FLAT PLATE

This chapter considers the case of a flat plate extending infinitely in two directions with the faces held between two rigid walls. One face is insulated and the other face is suddenly brought into the contact of a rigid wall having infinite conductance, capacitance, and a uniform temperature θ_0. The contact between the plate and the wall will give the contact resistance at interface. As the plate is heated, the thermal expansion will take place. This expansion will be obstructed by the two rigid walls, setting up the thermal stress in the plate and thereby changing the contact resistance. The above problem will be divided into two domains.

I. FIRST DOMAIN

The propagation of the heat flux from the rigid wall through the contact resistance at interface, and up to the insulated face, constitutes the first domain of the problem. In the one-dimensional analysis of the problem co-ordinate system will be such that the distance in x-direction will be measured from the moving boundary of a variable control volume as shown in Figure 1. The control volume, which has a width $\tau_0(t)$, includes the heated region of the plate.

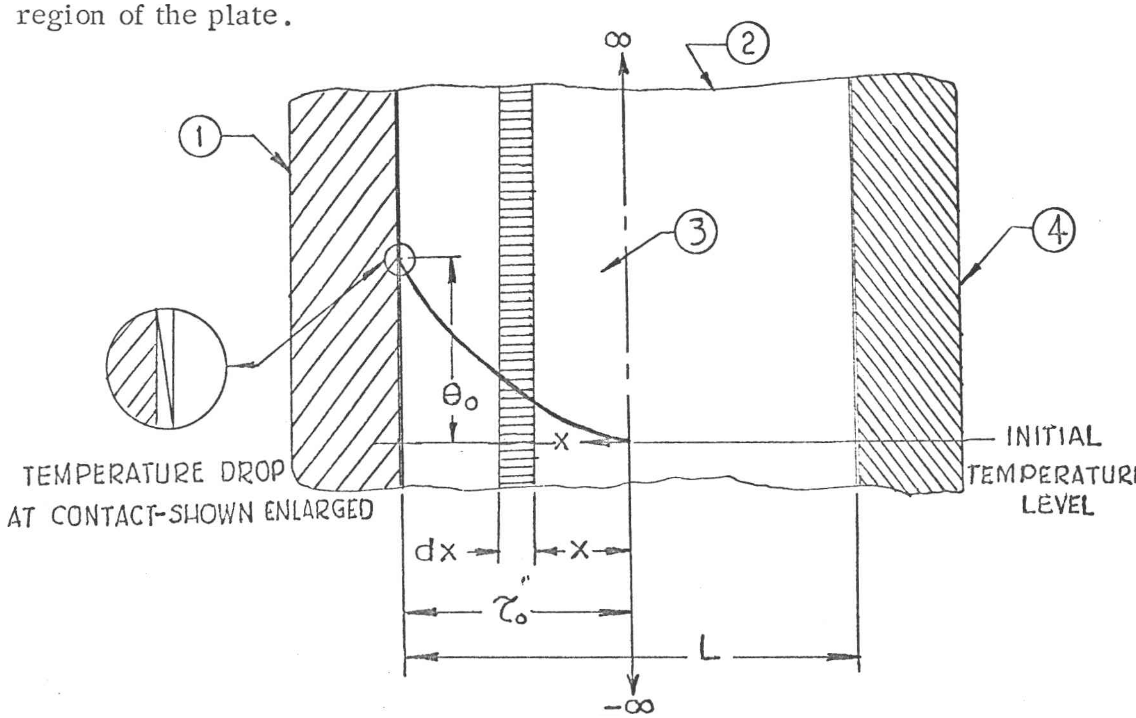


Fig. 1.--The first domain of rectangular case

(1) A rigid wall with infinite conductance; (2) a flat plate; (3) variable control volume; (4) rigid insulated wall.

The assumed linear relation between the contact conductance h_c, and the normal stress σ_x is

$$h_c = h_{co} - C\,\sigma_x$$

The thermal expansion in the differential element is

$$du = \alpha\,(T - T_\infty)\,dx \qquad \text{..........(2.1)}$$

Therefore, the total linear strain in the x-direction in the variable control volume will be

$$\varepsilon_x = \frac{u}{L} = \frac{1}{L}\int_0^{\tau_o(t)} \alpha\,(T - T_\infty)\,dx \qquad \text{..........(2.2)}$$

The stress is given by

$$\sigma_x = -\frac{E}{L}\int_0^{\tau_o(t)} \alpha\,(T - T_\infty)\,dx \qquad \text{..........(2.3)}$$

Therefore,

$$h_c = h_{co} + \frac{C \cdot E \cdot \alpha}{L}\int_0^{\tau_o(t)} (T - T_\infty)\,dx \qquad \text{..........(2.4)}$$

2.1 Integral formulation

The generalized governing equation for integral formulation for a given control volume is

$$\int_V \left(\nabla^2 T - \frac{1}{\alpha}\cdot\frac{\partial T}{\partial t}\right) dv = 0 \qquad \text{..........(1.1)}$$

and it will be used to derive the governing integral equation for first domain. In the case of flat plate the area remains constant but the depth varies. Therefore Equation (1.1) becomes

$$\int_0^{\tau_o(t)} \left(\nabla^2 T - \frac{1}{\alpha}\cdot\frac{\partial T}{\partial t}\right) dx = 0$$

or

$$\frac{d}{dt}\int_0^{\tau_o(t)} (T - T_\infty)\,dx = \nabla T \Big|_{x=0}^{x=\tau_o(t)}$$

But the heat transfer at the moving boundary of the variable control volume is zero. Hence,

$$\nabla T \Big|_{x=0} = 0$$

Therefore, the final form of the governing integral equation is

$$\frac{d}{dt}\int_0^{\tau_0(t)} (T - T_\infty)\, dx = \alpha \frac{\partial T}{\partial x}\Big|_{x=\tau_0(t)} \qquad \text{..........(2.5)}$$

The boundary conditions of the problem are:

1. The heat transfer at interface gives

$$k \frac{\partial T}{\partial x}\Big|_{x=\tau_0(t)} = h_c\left[T_f - T\,(\tau_0(t), t)\right] \qquad \text{..........(2.6)}$$

2. The moving boundary will have zero temperature, i.e.,

$$\left[T(x, t) - T_\infty\right]\Big|_{x=0} = 0 \qquad \text{..........(2.7)}$$

3. No heat transfer at moving boundary gives

$$\frac{\partial T}{\partial x}\Big|_{x=0} = 0 \qquad \text{..........(2.8)}$$

In this problem the temperature variation in the first domain approaches not a known steady state temperature, but an unknown initial temperature of the second domain of the problem. In such a case, unsteady temperature profile cannot be constructed as in the case of the steady state problem. A somewhat different approach will be pursued in constituting the unsteady Kantorovich profile, which requires the selection of functions (polynomial, circular, etc.) satisfying the given boundary conditions.

Although the selection of the order of polynomial is to a large extent arbitrary, a parabolic temperature profile will be preferred to get first order approximation to the problem. Therefore,

$$T(x,t) - T_\infty = Ax^2 + Bx + C \qquad \text{..........(2.9)}$$

For the sake of mathematical convenience, the analysis of the problem will be treated in dimensionless form. Following dimensionless parameters will be used.

1. Dimensionless temperature

$$\theta = \frac{T(x,t) - T_\infty}{T_f - T_\infty} \qquad \text{..........(2.10)}$$

2.

$$\eta = \frac{x}{L} \qquad \text{..........(2.11)}$$

3. Dimensionless penetration depth

$$\bar{\tau}_o(t) = \frac{\tau_o(t)}{L} \qquad \text{..........(2.12)}$$

4. Biot number

$$\frac{h_{co} \cdot L}{k} = Bi \qquad \text{..........(2.13)}$$

5. Fourier number

$$Fo = \frac{\alpha \cdot t}{L^2} \qquad \text{..........(2.14)}$$

To get the governing equation (Equation 2.5) into dimensionless form, it is rearranged as

$$\frac{d}{dt}\int_0^{\bar{\tau}_o(t)}\left(\frac{T-T_\infty}{T_f-T_\infty}\right)\cdot L\cdot d\eta = \frac{\alpha}{L}\cdot\frac{\partial}{\partial\eta}\left(\frac{T-T_\infty}{T_f-T_\infty}\right)\Bigg|_{\eta=\bar{\tau}_o(t)}$$

Substitution of the previously defined parameters gives

$$\frac{d}{dt}\int_0^{\bar{\tau}_o(t)}\theta\cdot d\eta = \frac{\alpha}{L^2}\cdot\frac{\partial\theta}{\partial\eta}\Bigg|_{\eta=\tau_o(t)} \qquad \text{..........(2.15)}$$

The boundary conditions in dimensionless form are given as

$$\theta\,(\eta,t)\,\big|_{\eta=0} = 0 \qquad \text{..........(2.16)}$$

$$\frac{\partial\theta}{\partial\eta}\Bigg|_{\eta=0} = 0 \qquad \text{..........(2.17)}$$

The derivations of the above two conditions are obvious; derivation of the third is shown below.

Rearranging Equation 2.6 with Equation 2.4 in dimensionless parameters

$$k\frac{\partial}{\partial x}\left(\frac{T-T_\infty}{T_f-T_\infty}\right)\Bigg|_{x=\tau_o(t)} = \left[h_{co}+\frac{C\cdot E\cdot\alpha}{L}\left(T_f-T_\infty\right)\int_0^{\tau_o(t)}\left(\frac{T-T_\infty}{T_f-T_\infty}\right)dx\right]$$

$$\times\left[\frac{T_f-T_\infty}{T_f-T_\infty}-\frac{T(\tau_o,t)-T_\infty}{T_f-T_\infty}\right]$$

gives

$$\frac{k}{L}\cdot\frac{d\theta}{d\eta}\Bigg|_{\eta=\bar{\tau}_o(t)} = \left[h_{co}+\frac{C\cdot E\cdot\alpha\,(T_f-T_\infty)\cdot L}{k}\int_0^{\tau_o(t)}\theta\,d\eta\right](1-\theta_s)$$

or

$$\frac{d\theta}{d\eta}\bigg|_{\eta=\bar{\tau}_o(t)} = \left[\frac{h_{co}\cdot L}{k} + \frac{C\cdot E\cdot\alpha\,(T_f - T_\infty)\cdot L}{k}\int_0^{\bar{\tau}_o(t)} \theta\cdot d\eta\right](1-\theta_s)$$

where $\theta_s = \dfrac{T\,(\tau_o(t),\, t) - T_\infty}{T_f - T_\infty}$

The temperature profile in dimensionless form will be

$$\theta\,(\eta\,.\,t) = A\eta^2 + B\eta + C$$

where A, B, and C are functions of dimensionless time (and differ by a constant factor from those in Equation 2.9).

Unit of slope c:

$$\text{Unit of c} \times \frac{lb_f}{ft^2} \times \frac{1}{{}^oF} \times \frac{{}^oF}{1} \times \frac{ft}{1} \times \frac{hf.\ ft.\ {}^oF}{BTU} = \text{dimensionless}$$

which will give

$$\text{Unit of c} = \frac{BTU}{hr.\ lb_f{}^oF}$$

Let $\dfrac{C\cdot E\cdot\alpha\,(T_f - T_\infty)\,L}{k}$ = dimensionless parameter DP

The third boundary condition will be written as

$$\frac{\partial\theta}{\partial\eta}\bigg|_{\eta=\bar{\tau}_o(t)} = \left[Bi + DP\int_0^{\bar{\tau}_o(t)} \theta\cdot d\eta\right](1-\theta_s) \quad \ldots\ldots\ldots\ldots (2.18)$$

The coefficients A, B, and C in the equation for temperature profile in dimensionless form will be evaluated by applying the above three boundary conditions. Application of the second boundary condition (Equation 2.17),

$$\left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=0} = \left.(2A\eta + B)\right|_{\eta=0} = 0$$

will yield $B = 0$

The application of the first boundary condition (Equation 2.16),

$$\left.\theta(\eta, t)\right|_{\eta=0} = \left.A\eta^2 + C\right|_{\eta=0} = 0$$

will yield $C = 0$

Therefore, the temperature profile will be given as

$$\theta(\eta, t) = A\eta^2 \qquad \text{..........(2.19)}$$

Applying the interface boundary condition (Equation 2.18), gives

$$\left.2A\eta\right|_{\eta=\tau_o(t)} = \left[Bi + DP\int_0^{\bar{\tau}_o(t)} A\cdot\eta^2\, d\eta\right]\left(1 - A\bar{\tau}_o^2(t)\right)$$

which upon simplification yields the quadratic equation of the form

$$\frac{A^2\cdot DP\cdot \bar{\tau}^5(t)}{3} - A\left[DP.\frac{\bar{\tau}_o^3(t)}{3} - Bi\cdot\bar{\tau}_o^2(t) - 2\bar{\tau}_o(t)\right] - Bi = 0$$

To reduce the complexity of algebra, let

$$a1 = \frac{DP\cdot \bar{\tau}_o^5(t)}{3}$$

$$b1 = \left[\frac{DP\cdot \tau_o^3(t)}{3} - Bi\,\bar{\tau}^2(t) - 2\bar{\tau}_o(t)\right]$$

$$c1 = -Bi$$

The solution of the above quadratic equation in A will give

$$A = \frac{1}{2a_1}\left[b_1 \pm \sqrt{b_1^2 - 4a_1 \cdot c_1}\right] \qquad \ldots\ldots\ldots\ldots(2.20)$$

The substitution of Equation 2.20 into Equation 2.19

$$\theta(\eta . Fo) = \frac{1}{2a_1}\left[b_1 \pm \sqrt{b_1^2 - 4a_1 \cdot c_1}\right]\eta^2 \qquad \ldots\ldots\ldots\ldots(2.21)$$

For a very small value of c, (DP $\approx$ 0), no pressure effect exists; hence, the convective boundary condition will be given as

$$\left.\frac{\partial\theta}{\partial\eta}\right|_{\eta = \bar{\tau}_o(t)} = Bi\ (1 - \theta s)$$

Therefore,

$$2A\bar{\tau}_o(t) = Bi\ \left(1 - A\bar{\tau}_o^{\,2}(t)\right)$$

or

$$A\left[2\bar{\tau}_o(t) + Bi \cdot \bar{\tau}_o^{\,2}(t)\right] = Bi$$

which will give

$$A = \frac{Bi}{2\bar{\tau}_o(t) + Bi \cdot \bar{\tau}_o^{\,2}(t)} = \frac{c_1}{b_1}$$

In the case of pressure effect

$$A = \frac{1}{2a_1}\left[b_1 \pm \sqrt{b_1^2 - 4a_1 \cdot c_1}\right]$$

Expanding the discriminant in binomial series

$$A = \frac{b_1}{2a_1}\left[1 \pm \left(1 - \frac{1}{2}\cdot\frac{4a_1\cdot c_1}{b_1^2} + \cdots\right.\right]$$

For very small value of slope c, $a_1 \simeq 0$, so terms of power higher than one in binomial expansion are neglected. Only negative sign is feasible for the solution and hence

$$A = \frac{b_1}{2a_1}\left[\frac{1}{2}\cdot\frac{4a_1\cdot c_1}{b_1^2}\right]$$

or

$$A = \frac{c_1}{b_1}$$

Henceforth in the following discussion only the negative sign will be considered (since the solution is feasible only for compressive stresses).

Substituting the temperature profile from Equation 2.19 into the governing integral equation

$$\frac{d}{dt}\int_0^{\tau_0(t)} \theta\cdot d\eta = \frac{\alpha}{L^2}\cdot\frac{\partial\theta}{\partial\eta}\bigg|_{\eta=\tau_0(t)} \qquad \text{..........(2.15)}$$

gives

$$\frac{1}{6}\int_{x(\tau_0=0)}^{x(\tau_0)} y\cdot dx = \frac{\alpha}{L^2}\int_0^t dt \qquad \text{..........(2.22)}$$

where

$$x = A\,\bar{\tau}_0^{\,3}(t)$$

$$y = \frac{1}{A\,\bar{\tau}_0(t)}$$

The integral (2.22) can be solved by numerical integration.

II. SECOND DOMAIN

When the heat flux propagates up to the insulated end, the first domain is completed. More incoming flux will now build up the temperature at the insulated face. The time from where the heat flux reaches the insulated face to when the steady state is finally achieved constitutes the second domain of the problem. The geometry is shown in Figure 2.

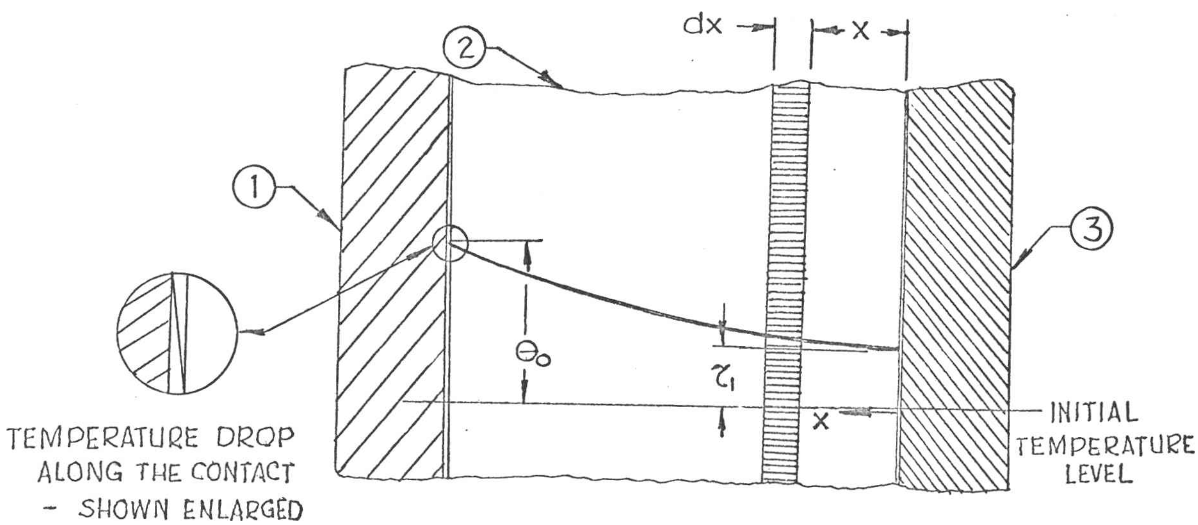


Fig. 2.--The second domain of rectangular case

(1) A rigid wall with infinite conductance; (2) flat plate; (3) rigid insulated wall.

For this domain the governing differential equation will be obtained in the same way as for the first domain; hence,

$$\frac{d}{dt}\int_0^1 \theta \cdot d\eta = \frac{\alpha}{L^2} \cdot \frac{\partial \theta}{\partial \eta}\bigg|_{\eta=1} \qquad \text{..........(2.23)}$$

The boundary conditions of this domain in the dimensionless form are

1. The heat transfer from the insulated wall is zero.

$$\frac{\partial \theta}{\partial \eta}\bigg|_{\eta=0} = 0 \qquad \text{..........(2.24)}$$

2. The temperature at the insulated face will be given by

$$\theta(\eta, t)\big|_{\eta=0} = \tau_1(t) \qquad \text{..........(2.25)}$$

where $\tau_1(t)$ is the unknown parameter in the temperature profile.

3. The convective boundary condition in the dimensionless form will be

$$\frac{d\theta}{d\eta}\bigg|_{\eta=1} = \left[Bi + DP\int_0^1 \theta \, d\eta\right](1 - \theta s) \qquad \text{..........(2.26)}$$

The temperature profile will be assumed to be parabolic

$$\theta(\eta, t) = A\eta^2 + B\eta + C \qquad \text{..........(2.27)}$$

where A, B, and C are the functions of the dimensionless time.

The application of the first boundary condition, Equation 2.24, will yield B = 0.

The application of the second boundary condition (Equation 2.25)

$$\theta(\eta, t)\Big|_{\eta=0} = (A\eta^2 + C)\Big|_{\eta=0} = \tau_1$$

will yield $C = \tau_1$

The temperature profile will be

$$\theta(\eta . t) = A\eta^2 + \tau_1$$

Applying the convective boundary condition

$$\frac{\partial \theta}{\partial \eta}\Big|_{\eta=1} = \left[Bi + DP\int_0^1 \theta \cdot d\eta\right](1 - \theta s)$$

Substituting the temperature profile

$$2A = \left[Bi + DP\int_0^1 (A\eta^2 + \tau_1)\, d\eta\right]\left[1 - (A\eta^2 + \tau_1)\Big|_{\eta=1}\right]$$

Therefore,

$$2A = \left[Bi + \frac{DP \cdot A}{3} + DP \cdot \tau_1\right](1 - \tau_1 - A)$$

and hence,

$$2A = -DP. \frac{A^2}{3} + A\left[(1-\tau_1)\frac{DP}{3} - (Bi + DP \cdot \tau_1)\right] + (1 - \tau_1)(Bi + DP. \tau_1)$$

The arrangement of the above equation in the quadratic form of A is

$$A^2 . \frac{DP}{3} - A\left[(1-\tau_1)\frac{DP}{3} - (Bi + DP. \tau_1) - 2\right]$$

$$- (1 - \tau_1)(Bi + DP. \tau_1) = 0 \qquad \text{.......... (2.28)}$$

Let $a_1 = \frac{DP}{3}$

$b_1 = (1 - \tau_1)\frac{DP}{3} - (Bi + DP.\tau_1) - 2$

$c_1 = -(1 - \tau_1)(Bi + DP.\tau_1)$

The solution of the quadratic equation will yield

$$A = \frac{1}{2a_1}\left[b_1 - \sqrt{b_1^2 - 4a_1c_1}\right] \quad \text{..........(2.29)}$$

As seen in the case of first domain, the negative sign only will give feasible solution. The temperature profile will then be given by

$$\theta(\eta, t) = \frac{1}{2a_1}\left(b_1 - \sqrt{b_1^2 - 4a_1c_1}\right)\eta^2 + \tau_1 \quad \text{..........(2.30)}$$

The substitution of the temperature profile in the governing integral equation will give the final integral form

$$\int_0^1 \frac{d(A + 3\tau_1)}{6A} = \frac{\alpha}{L^2}\int_0 dt \quad \text{..........(2.31)}$$

The integral can be solved by the same numerical technique employed for the first domain.

CHAPTER III

CONCENTRIC CYLINDERS

This chapter considers two concentric cylinders, the inner one being a solid cylinder.. The axial and the angular temperature variation is neglected, hence leading to a one-dimensional problem. In this problem, a step change in the temperature at the outer periphery of the outer cylinder will be applied. The heat transfer problem will be treated in the three parts.

1. The first domain of the outer cylinder will be discussed in the first part. This has little importance in this discussion, so far as the actual problem is concerned, because no contact resistance is involved. To preserve the continuity of discussion it is stated very briefly.

2. The second part discusses the second domain of the outer cylinder and the first domain of the inner cylinder together. This part gives the temperature distributions Θ_1 and Θ_s in the outer and the inner solid cylinders respectively, to the time heat flux propagates to the center of the solid cylinder. The reason why both the domains are considered together will be clear in the actual analysis of the problem.

3. The third part discusses the second domain of the outer cylinder and the second domain of the inner cylinder together. The temperature distributions Θ_s' and Θ_1' in the solid inner cylinder and the outer cylinder respec-

tively are determined, from the point incoming heat flux reaches the center to the point the steady state is achieved.

Part I

The propagation of the heat flux from the outer periphery of the outer cylinder to the inner face of the same cylinder constitutes the first domain of the outer cylinder. The geometry is shown in Figure 3.

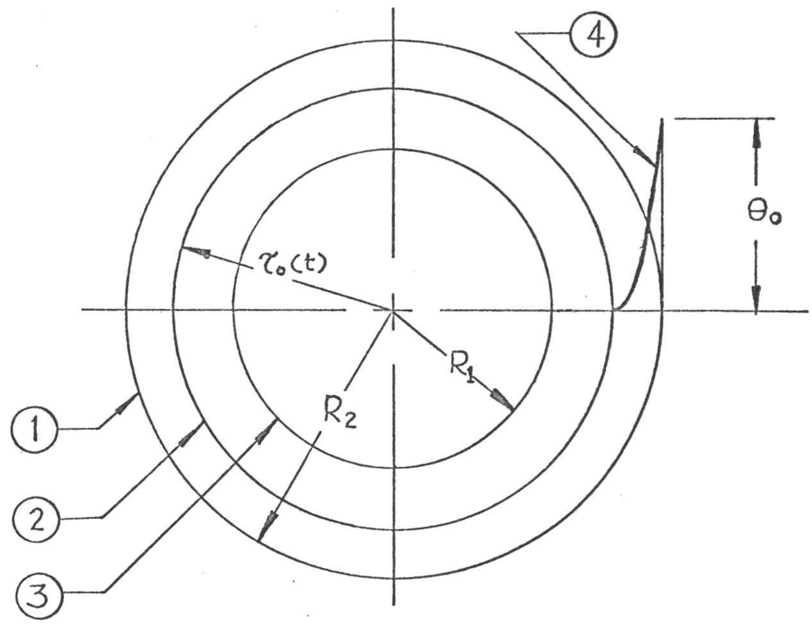


Fig. 3.--The first domain of outer cylinder

(1) Outer hollow cylinder; (2) moving boundary of heat propagation; (3) inside solid cylinder; (4) Temperature distribution in the first domain of outer cylinder.

The generalized governing integral equation for any control volume v

$$\int_v \left(\nabla^2\theta - \frac{1}{\alpha}\cdot\frac{\partial\theta}{\partial t} \right) dv = 0 \qquad \text{..........(1.1)}$$

will be used to determine the governing integral equation in this case. For the one-dimensional analysis of the problem in the radial direction, the area through which the heat flux passes will depend upon the propagation distance r measured from the center of the inner solid cylinder.

Therefore,

$$\text{Volume } dV\ (r) = 2\pi r dr$$

Substituting dv in Equation 1.1, obtain

$$\frac{\partial}{\partial t} \int_{R_2}^{\tau_o(t)} \theta \cdot r . dr = \alpha \left[r \frac{\partial \theta}{\partial r} \right]_{R_2}^{\tau_o(t)} \qquad \text{..........(3.1)}$$

The heat transfer at the moving boundary of the variable control volume is zero, i.e.

$$\left. \frac{\partial \theta}{\partial r} \right|_{\tau_o(t)} = 0$$

Therefore Equation 3.1 becomes

$$\frac{\partial}{\partial t} \int_{\tau_o(t)}^{R_2} \theta \cdot r . dr = \alpha R_2 \left. \frac{\partial \theta}{\partial r} \right|_{R_2} \qquad \text{..........(3.2)}$$

The above governing integral equation of the first domain will be transformed into the dimensionless form for the mathematical convenience. Using the following dimensionless quantities

$$\theta_1 = \frac{T_1 - T_\infty}{T_f - T_\infty} \qquad \text{...............(3.3)}$$

$$\eta = \frac{r}{R_2} \qquad \text{...............(3.4)}$$

$$\bar{\tau}_o(t) = \frac{\tau_o(t)}{R_2} \qquad \text{...............(3.5)}$$

$$F_o = \frac{\alpha . t}{R_2^2} \qquad \text{...............(3.6)}$$

the final form of Equation 3.2 will be

$$\frac{d}{dt}\int_{\bar{\tau}_0(t)}^{1} \theta_1 \cdot \eta \cdot d\eta = \frac{\alpha}{R_2^2} \cdot \frac{\partial \theta_1}{\partial \eta}\Big|_{\eta=1} \qquad \text{. . . . . (3.7)}$$

The boundary conditions of the problem are

1. $\theta_1(\eta, t)\big|_{\eta=1} = 1$

The heat flux at the moving boundary of the variable control volume, i.e.,

2. $\dfrac{\partial \theta_1}{\partial \eta}\Big|_{\eta=\bar{\tau}_0(t)} = 0$ . . . . (3.8)

The temperature at the moving boundary of variable control volume will be zero, i.e.,

3. $\theta_1(\eta, t)\big|_{\eta=\bar{\tau}_0(t)} = 0$ . . . . (3.9)

4. Initial condition: $\theta_1(r, 0) = 0$ . . . . (3.10)

A temperature profile in terms of η will be assumed in this domain. For $\frac{R_1}{R_2} < \eta < 1$ the parabolic temperature profile is

$$\theta_1(\eta, t) = A\eta^2 + B\eta + C$$

where A, B, and C are the functions of time.

A, B, and C can be evaluated by applying the given boundary conditions. The application of the second boundary condition

$$\frac{\partial \theta_1}{\partial \eta}\bigg|_{\eta = \bar{\tau}_o(t)} = 0$$

will yield

$$B = -2A\bar{\tau}_o(t) \qquad \text{..........(3.11)}$$

The application of the first boundary condition

$$\theta_1(\eta, t)\big|_{\eta = 1} = 1$$

will yield

$$A + B + C = 1 \qquad \text{..........(3.12)}$$

Applying the third boundary condition

$$\theta_1(\eta, t)\big|_{\eta = \tau_o(t)} = 0$$

will yield

$$C = A\bar{\tau}_o^2(t) \qquad \text{..........(3.13)}$$

Introducing Equations 3.11 and 3.13 into Equation 3.12, the value of A will be evaluated.

$$A = \frac{1}{(\bar{\tau}_o(t) - 1)^2} \qquad \text{..........(3.14)}$$

Substituting Equation 3.14 in the Equations 3.11 and 3.13,

$$B = \frac{-2\,\bar{\tau}_o(t)}{(\bar{\tau}_o(t) - 1)^2}$$

and

$$C = \frac{\bar{\tau}_o^2(t)}{(\bar{\tau}_o(t) - 1)^2}$$

When the values of A, B, and C are introduced into the actual profile, the result will be given as

$$\Theta_1(\eta, t) = \left(\frac{\eta - \bar{\tau}_o(t)}{1 - \tau_o(t)}\right)^2 \qquad \text{. . . . . (3.15)}$$

Part II

When the increasing heat flux reaches the inner surface of the outer cylinder, it will propagate further, increasing the temperature of that surface. From the point when the incoming heat flux starts building up temperature at this surface to the point the steady state is achieved constitutes the second domain of the outer cylinder. The heat flux propagates further from the contact interface into the inner solid cylinder. Propagation of this heat flux up to the center of the solid cylinder constitutes the first domain of the inner cylinder. The geometry is shown in Figure 4.

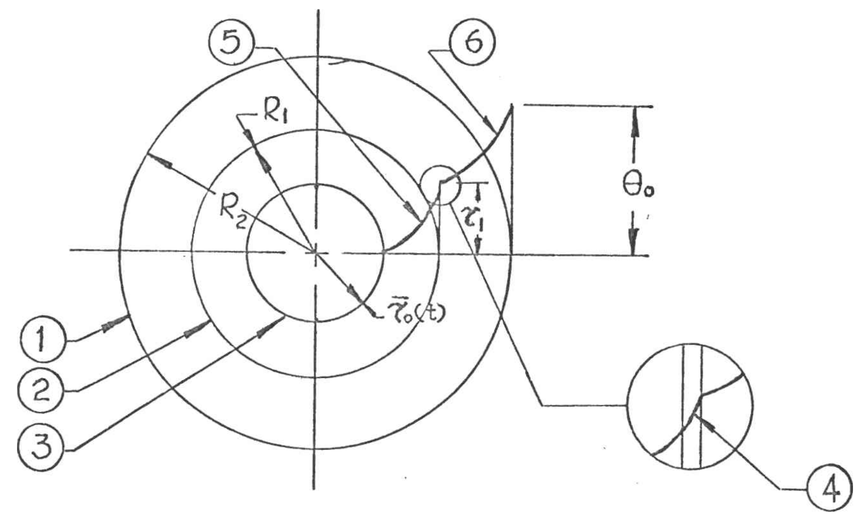


Fig. 4.--The second domain of outside cylinder and the first domain of inside solid cylinder.

(1) Outer hollow cylinder; (2) inside solid cylinder; (3) moving boundary of heat propagation in second domain of solid cylinder; (4) Enlarged view of temperature drop at interface; (5) temperature distribution in first domain of solid cylinder; (6) temperature distribution in second domain of outer cylinder.

The problem for the composite cylinder will be separated in two sections according to the temperature distributions. The temperature distribution for the inner solid cylinder will be denoted by θ_s and for the outer hollow cylinder by θ_1 .

In this part the mathematical difficulty arises because of the complexity of the convective boundary condition at the interface of the two cylinders, i.e.,

$$k_1 \frac{\partial \theta_1}{\partial \eta}\bigg|_{\eta = \frac{R_1}{R_2}} = h_c \left(\theta_{1s} - \theta_{ss}\right) \qquad \ldots\ldots(3.16)$$

(subscript "s" denotes the temperature at the surface)

where

$$hc = hco - C \cdot \sigma_{rs}$$

hco = the heat transfer coefficient in stress-free state and σ_{rs} = the interface radial stress which is a function of θ_1 and θ_s .

Some equations necessary for the derivation of interface pressure have been taken from reference (1). The derivation of all necessary formulae is outlined in the Appendix.

The second domain in the outer cylinder

The governing integral equation is

$$\frac{d}{dt}\int_{R_1}^{R_2} T_1 \cdot r \cdot dr = R_2 \cdot \alpha_1 \cdot \frac{\partial T}{\partial r}\bigg|_{r = R_2} \qquad \ldots\ldots(3.18)$$

The dimensionless quantities following will be used in the dimensionless analysis of the problem.

1. $\eta = \dfrac{r}{R_2}$

2. $\theta_1 = \dfrac{T_1\,(r,t) - T_\infty}{T_F - T_\infty}$

3. $\dfrac{R_1}{R_2} = \beta$

4. $Fo = \dfrac{t \cdot \alpha_1}{R_2^2}$

5. $\bar{\tau}_o' = \dfrac{\tau_o'}{R_2}$

The governing integral equation (3.18) in dimensionless form is

$$\frac{d}{dt}\int_{\beta}^{1} \theta_1 \cdot \eta \cdot d\eta = \frac{\alpha_1}{R_2^2} \cdot \left.\frac{\partial \theta_1}{\partial \eta}\right|_{\eta=1} \qquad \text{.....(3.19)}$$

The boundary conditions of the problem are:

1. $\left.\theta_1\,(\eta.\,t)\right|_{\eta=1} = 1$ ....(3.20)

2. $\left.\theta_1\,(\eta.\,t)\right|_{\eta=\beta} = \tau_1$ ....(3.21)

where τ_1 is an unknown parameter in the temperature profile.

3. The convective boundary condition at the interface is given by

$$k_1 \left.\frac{\partial T_1}{\partial r}\right|_{r=R_1} = h_c\,(T_{1s} - T_{ss}) \qquad \text{..........(3.22)}$$

where T_{1s} = the temperature of the outer cylinder at interface,

T_s = the temperature distribution in the solid core, and

T_{ss} = the temperature of the inner cylinder at the interface.

The above condition in dimensionless form is given as

$$\left.\frac{\partial \theta_1}{\partial \eta}\right|_{\eta=\beta} = \frac{h_c \cdot R_2}{k_1}\left(\theta_{1s} - \theta_{ss}\right) \qquad \ldots\ldots\ldots\ldots (3.23)$$

Taking the value of h_c from Appendix Equation 60 and rearranging Equation 3.23,

$$\frac{h_c R_2}{k_1} = \left[Bi + C_3 \int_{\tau_o(t)}^{\beta} \theta_s \cdot \eta \cdot d\eta + C_4 \int_{\beta}^{1} \theta_1 \cdot \eta \cdot d\eta\right] \qquad \ldots (3.24)$$

where new constants are

$$C_3 = \frac{C_1 R_2^2}{k_1}\left(T_F - T_\infty\right) \quad \text{and}$$

$$C_4 = \frac{C_2 R_2^2}{k_1}\left(T_F - T_\infty\right)$$

Introducing Equation 3.24 into Equation 3.23 gives

$$\left.\frac{\partial \theta_1}{\partial \eta}\right|_{\eta=\beta} = \left[Bi + C_3 \int_{\overline{\tau}_o(t)}^{\beta} \theta_s \cdot \eta \cdot d\eta + C_4 \int_{\beta}^{1} \theta_1 \cdot \eta \cdot d\eta\right]$$

$$\times\left(\theta_{1s} - \theta_{ss}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3.25)$$

The temperature profile is approximated as

$$\theta_1(\eta . t) = A\eta^2 + B\eta + C$$

where A, B, and C are functions of the dimensionless time.

Applying the first boundary condition (Equation 3.21) gives

$$A + B + C = 1 \qquad \ldots\ldots\ldots\ldots(3.26)$$

and applying the second boundary condition (Equation 3.22) gives

$$A\beta^2 + B\beta + C = \tau_1 \qquad \ldots\ldots\ldots\ldots(3.27)$$

Solving 3.26 and 3.27 simulateously, the values of B and C are obtained in terms of A. The results are

$$B = \frac{1 - \tau_1}{1 - \beta} - A(1 + \beta) \qquad \ldots\ldots\ldots\ldots(3.28)$$

and

$$C = \frac{\tau_1 - \beta}{1 - \beta} + A\beta \qquad \ldots\ldots\ldots\ldots(3.29)$$

The coefficient A of the temperature profile will be found by using the convective boundary condition dependent upon the temperature distribution.

<u>The first domain of the inner cylinder</u>

The integral formulation of the first domain of the inner solid cylinder is obtained as before:

$$\frac{d}{dt}\int_{\beta}^{\tau_0'(t)} \theta_s \cdot \eta \cdot d\eta = \frac{k_2}{k_1} \cdot \frac{\alpha_s}{R_2^2} \cdot \frac{\partial \theta_s}{\partial \eta}\bigg|_{\eta = \beta} \qquad \ldots\ldots(3.30)$$

The boundary conditions for this domain are:

$$1. \quad \theta_s(\eta, t)\Big|_{\eta = \tau_0'(t)} = 0 \qquad \ldots\ldots(3.31)$$

2. $$\frac{\partial \theta_s(\eta, t)}{\partial \eta}\bigg|_{\eta = \bar{\tau}_o'(t)} = 0 \quad \ldots\ldots (3.32)$$

3. $$\frac{\partial \theta_s(\eta, t)}{\partial \eta}\bigg|_{\eta = \beta} = h_c \frac{R_2}{k_2}(\theta_{1s} - \theta_{ss})$$

or

$$\frac{\partial \theta_s}{\partial \eta}(\eta, t)\bigg|_{\eta = \beta} = \frac{k_1}{k_2}\left[Bi + C_3 \int_{\bar{\tau}_o'(t)}^{\beta} \theta_s \cdot \eta \cdot d\eta + C_4 \int_{\beta}^{1} \theta_1 \cdot \eta \cdot d\eta\right](\theta_{1s} - \theta_{ss}) \quad \ldots\ldots (3.33)$$

A parabolic temperature profile will be taken for this domain in the form

$$\theta_s(\eta, t) = D\eta^2 + E\eta + F \quad \ldots\ldots\ldots\ldots (3.34)$$

The constants E and F will be found in terms of D using the first two boundary conditions. They are

$$E = -2D\bar{\tau}_o'(t) \quad \ldots\ldots\ldots\ldots (3.35)$$

and

$$F = D\bar{\tau}_o'^2(t) \quad \ldots\ldots\ldots\ldots (3.36)$$

Introducing Equations 3.35 and 3.36 into Equation 3.34,

$$\theta_s(\eta, t) = D\eta^2 + E\eta + F$$

$$= D\eta^2 + (-2D\bar{\tau}_o'(t))\cdot\eta + D\bar{\tau}_o'^2(t)$$

$$= D\left[\eta - \bar{\tau}_o'(t)\right]^2 \qquad \text{..........(3.37)}$$

In the following discussion, the relationship between A and D using interface boundary conditions of both the domains will be established.

A study of Equations 3.23 and 3.33 reveals that

$$\left.\frac{\partial\theta_s(\eta, t)}{\partial\eta}\right|_{\eta=\beta} = \frac{k_1}{k_2}\cdot\left.\frac{\partial\theta_1(\eta, t)}{\partial\eta}\right|_{\eta=\beta}$$

Therefore,

$$\frac{k_1}{k_2}\left(2A\cdot\eta + B\Big|_{\eta=\beta}\right) = 2D(\beta - \bar{\tau}_o'(t))$$

further simplification will yield A in terms of D

$$A = \frac{k_2}{k_1}\left[\frac{2D(\beta - \bar{\tau}_o'(t))}{\beta - 1}\right] + \frac{1 - \tau_1}{(1-\beta)^2} \qquad \text{..........(3.38)}$$

Let us use the interface boundary condition, Equation 3.33, to evaluate constant D

$$\theta_s(\eta, t)\Big|_{\eta=\beta} = \frac{k_1}{k_2}\left[Bi + C_3\int_{\bar{\tau}_o'(t)}^{\beta}\theta_s\,\eta\,d\eta + C_4\int_{\beta}^{1}\theta_1\cdot\eta\cdot d\eta\right](\theta_{1s} - \theta_{ss})$$

Therefore,

$$2D\left(\beta - \bar{\tau}_o'(t)\right) = \frac{k_1}{k_2}\left[Bi + C_3 \int_{\bar{\tau}_o'(t)}^{\beta} D\left(\eta - \bar{\tau}_o'(t)\right)^2 \eta \cdot d\eta\right.$$

$$+ C_4 \int_{\beta}^{1}\left[A\eta^2 + \left\{\frac{(1-\tau_1)}{(1-\beta)} - A(1+\beta)\right\}\eta\right.$$

$$\left.\left. + \left\{\frac{\tau_1 - \beta}{1-\beta} + A\beta\right\}\right]\eta\right] d\eta \left(\tau_1 - D\left(\beta - \bar{\tau}_o'(t)\right)\right) \quad \ldots\ldots\ldots\ldots (3.39)$$

Let us evaluate the integrals

$$I_1 = \int_{\tau_o'(t)}^{\beta} \left(\eta - \bar{\tau}_o'(t)\right)^2 \eta \cdot d\eta$$

let $\eta = \eta_1 + \bar{\tau}_o'(t)$ Therefore,

$$I_1 = \int_0^{\beta - \bar{\tau}_o'(t)} \left[\eta_1^2 \times \left(\eta_1 + \bar{\tau}_o'(t)\right)\right] d\eta$$

or

$$I_1 = \left[\frac{\eta_1^4}{4} + \frac{\eta_1^3}{3} \cdot \bar{\tau}_o'(t)\right]_0^{\beta - \bar{\tau}_o'(t)}$$

which in simplified form will be

$$I_1 = \left(\beta - \bar{\tau}_o'(t)\right)^3 \times \left[\frac{\beta}{4} + \frac{\bar{\tau}_o'(t)}{12}\right] \quad \ldots\ldots\ldots\ldots (3.40)$$

and

$$I_2 = \int_{\beta}^{1} A\eta^3 + \left\{\frac{1-\tau_1}{1-\beta} - A(1+B)\right\}\eta^2 + \left\{\frac{\tau_1 - \beta}{1-\beta} + A\beta\right\}\eta\left.\right] d\eta$$

or

$$I_2 = \frac{A}{4}(1-\beta^4) + \frac{1-\tau_1}{3(1-\beta)}\cdot(1-\beta)^3 - \frac{A}{3}(1+\beta)(1-\beta^3) + \frac{(\tau_1-\beta)(1-\beta)^2}{2(1-\beta)} + \frac{A}{2}\cdot\beta\cdot(1-\beta^2)$$

or

$$I_2 = \left[\frac{1-\beta^4}{4} - \frac{(1+\beta)(1-\beta^3)}{3} + \frac{\beta}{2}(1-\beta^2)\right] + \left[\frac{(1-\tau_1)(1-\beta^3)}{3(1-\beta)} + \frac{(\tau_1-\beta)(1+\beta)}{2}\right] \quad \ldots\ldots(3.41)$$

substituting the value of A in Eq. 3.41 from Eq. 3.38

$$I_2 = D\left[\left\{2\cdot\frac{k_2}{k_1}\cdot\frac{(\beta-\bar{\tau}_o'(t))}{(\beta-1)}\right\}\left\{\frac{1-\beta^4}{4} - \frac{(1+\beta)(1-\beta^3)}{3} + \frac{\beta}{2}(1-\beta^2)\right\}\right] + \left[\left\{\frac{(1-\tau_1)}{(1-\beta)^2}\right\} \times \left\{\frac{1-\beta^4}{4} - \frac{(1+\beta)(1-\beta^3)}{3} + \frac{\beta}{2}(1-\beta^2)\right\} + \frac{(1-\tau_1)(1-\beta^3)}{3(1-\beta)} + \frac{(1+\beta)(\tau_1-\beta)}{2}\right] \quad \ldots\ldots\ldots\ldots(3.42)$$

Let

$$m_1 = \left\{2\cdot\frac{k_2}{k_1}\cdot\frac{(\beta-\bar{\tau}_o'(t))}{(\beta-1)}\right\}\left\{\frac{1-\beta^4}{4} - \frac{(1+\beta)(1-\beta^3)}{3} + \frac{\beta(1-\beta^2)}{2}\right\} \quad \ldots\ldots(3.43)$$

$$n_1 = \left[\left\{\frac{1-\tau_1}{1-\beta}\right\}\left\{\frac{1-\beta^4}{4} - \frac{(1+\beta)(1-\beta^3)}{3} + \frac{\beta(1-\beta^2)}{2}\right\} + \left\{\frac{(1-\tau_1)(1-\beta^3)}{3(1-\beta)} + \frac{(\tau_1-\beta)(1+\beta)}{2}\right\}\right] \quad \ldots\ldots(3.44)$$

$$m_2 = (\beta - \bar{\tau}_o'(t))^3\left[\frac{\beta}{4} + \frac{\bar{\tau}_o'(t)}{12}\right] \quad \ldots\ldots(3.45)$$

Introducing these integrals into Equation 3.39 gives

$$2D(\beta - \bar{\tau}_o'(t)) = \frac{k_1}{k_2}\left[(Bi + n_1 C_4) + D(m_2 C_3 + m_1 C_4)\right.$$

$$\left. \times \left[\tau_1 - D(\beta - \bar{\tau}_o'(t))^2\right]\right]$$

Therefore,

$$D^2 (m_2 C_3 + m_1 C_4)(\beta - \bar{\tau}_o'(t))^2 - D\left[(m_2 C_3 + m_1 C_4)\tau_1\right.$$

$$\left. - 2\frac{k_2}{k_1}(\beta - \bar{\tau}_o'(t)) - (Bi + n_1 C_4)(\beta - \bar{\tau}_o'(t))^2\right]$$

$$-(Bi + n_1 C_4)\tau_1 = 0 \qquad \cdots\cdots (3.46)$$

Let

$$a_1 = (m_2 C_3 + m_1 C_4)(\beta - \tau_o'(t))^2$$

$$b_1 = (m_2 C_3 + m_1 C_4)\tau_1 - 2\frac{k_2}{k_1}(\beta - \bar{\tau}_o'(t))$$

$$-(Bi + n_1 C_4)(\beta - \bar{\tau}_o'(t))^2$$

and $c_1 = (Bi + n_1 C_4)\tau_1$

Therefore,

$$a_1 D^2 - b_1 D + c_1 = 0 \qquad \cdots\cdots\cdots\cdots (3.47)$$

The solution of Equation 3.47 is $D = \frac{1}{2a_1}\left[b_1 - \sqrt{b_1^2 - 4a_1 c_1}\right]$

The negative sign is taken because the solution is feasible for compressive stresses only. Proper justification of this point is given in the last chapter.

Introducing the value of D, the temperature profile for the inner cylinder will be,

$$\Theta_s(\eta, t) = \frac{1}{2a_1}\left[b_1 - \sqrt{b_1^2 - 4a_1c_1}\right](\eta - \bar{\tau}_0'(t))^2$$

For the second domain of the hollow cylinder, the coefficients of A, B, and C are

$$A = \frac{1}{a_1}\left[b_1 - \sqrt{b_1^2 - 4a_1c_1}\right]\frac{(\beta - \bar{\tau}_0'(t))}{(\beta - 1)} - \frac{1 - \tau_1}{1 - \beta^2}$$

$$B = \frac{1 - \tau_1}{1 - \beta} - A(1 + \beta)$$

and

$$C = \frac{\tau_1 - \beta}{1 - \beta} + A\beta$$

It is noticed that the coefficients A, B, C, and D are the functions of $\bar{\tau}_0'(t)$ and $\tau_1(t)$

Substituting the temperature profile of the second domain of the outer cylinder in the governing integral equation (Equation 3.19) gives

$$\frac{d}{dt}(m_1 D + n_1) = \frac{\alpha_1}{R_2^2}(2A + B)$$

Therefore,

$$\int \frac{d(m_1 D + n_1)}{(2A + B)} = \frac{\alpha_1}{R_2^2}\int dt \qquad \ldots\ldots\ldots\ldots (3.48)$$

Next, substitute the temperature profile of the first domain of the inner cylinder in the governing integral equation (Equation 3.30). This gives

$$\frac{d}{dt}\left[\frac{1}{2a_1}\left(b_1 - \sqrt{b_1^2 - 4a_1c_1}\right) \times \left(\beta - \bar{\tau}_o'(t)\right)^3 \times \left(\frac{\beta}{4} + \frac{\bar{\tau}_o'(t)}{12}\right)\right]$$

$$\times \left(\beta - \bar{\tau}_o'(t)\right)$$

$$= 2\,\frac{k_2}{k_1}\cdot\frac{1}{2a_1}\left[b_1 - \sqrt{b_1^2 - 4a_1c_1}\right] \times \frac{\alpha_1}{R_2^2}\left(\beta - \bar{\tau}_o'(t)\right)$$

Therefore the final result is

$$\int \frac{d\left[\frac{1}{2a_1}\left(b_1 - \sqrt{b_1^2 - 4a_1c_1}\right)\left(\beta - \bar{\tau}_o'(t)\right)^3\left(\frac{\beta}{4} + \frac{\tau_o'(t)}{12}\right)\right]}{2\left(\frac{k_2}{k_1}\right)\cdot\frac{1}{2a_1}\left(b_1 - b_1^2 - 4a_1c_1\right)\left(\beta - \bar{\tau}_o'(t)\right)}$$

$$= \frac{\alpha_s}{R_2^2}\int dt \qquad \text{.......... (3.49)}$$

The two integral Equations 3.48 and 3.49 must be solved simultaneously since the integrands of both the integrals are functions of $\tau_o(t)$ and $\tau_1(t)$. The solution can be obtained using numerical integration and a trial and error procedure.

Part III

This part discusses the second domain of the outer cylinder and the inner cylinder respectively. From the point when the heat flux reaches the center of the inner cylinder to the point when the steady state is achieved, constitutes the second domain of the inner cylinder. The geometry is shown in Figure 5.

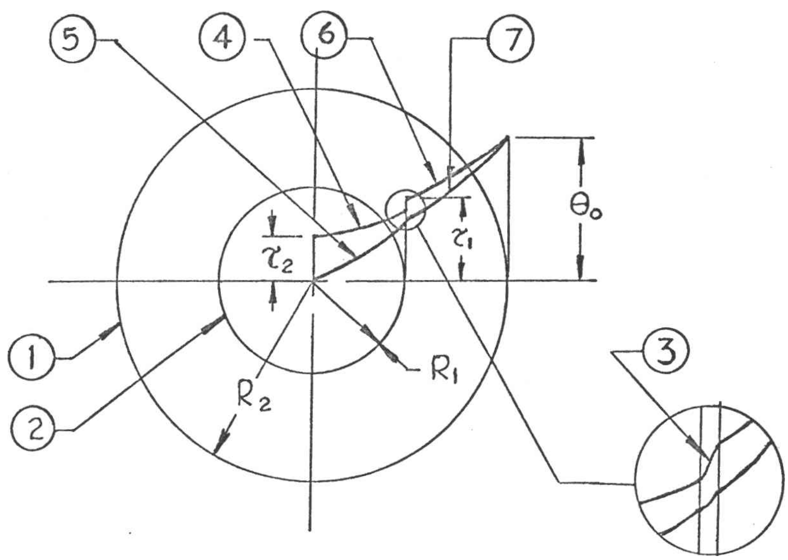


Fig. 5.--The second domains of outside and inside cylinders

(1) Outer cylinder; (2) inside solid cylinder; (3) temperature drop along interface; (4) temperature distribution in second domain of solid cylinder; (5) temperature distribution at the end of first domain of solid cylinder; (6) temperature distribution in the second domain of hollow cylinder after the first domain of solid cylinder is completed; (7) temperature distribution in the second domain of outer cylinder at the completion of second domain of solid cylinder.

The problem will be separated in two sections according to the temperature distributions. The temperature distribution for the inner cylinder will be denoted by θ_5' and for the outer hollow cylinder by θ_1' . This part gives the temperature distributions θ_s' and θ_l' from the point when the second domain of the solid cylinder starts to the point the steady state is achieved.

The second domain of the inner solid cylinder

The integral formulation for the second domain is,

$$\frac{d}{dt}\int_0^{\beta} \theta_s' \cdot \eta \cdot d\eta = \frac{k_2}{k_1} \cdot \frac{\alpha_s}{R_2^2} \left.\frac{\partial \theta_s'}{\partial \eta}\right|_{\eta=\beta} \quad \text{for } 0 < \eta < \beta \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3.50)$$

The boundary conditions of the problem are:

1. $$\left.\theta_s'(\eta, t)\right|_{\eta=0} = \tau_2 \qquad \ldots\ldots (3.51)$$

where $\tau_2(t)$ is the unknown parameter in the temperature profile.

2. $$\left.\frac{\partial \theta_s'(\eta, t)}{\partial \eta}\right|_{\eta=\beta} = \frac{h_c R_2}{k_1}\left[\theta_{ls}'(\beta, t) - \theta_{ss}'(\beta, t)\right]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (3.52)$$

3. $$\left.\frac{\partial \theta_s}{\partial \eta}\right|_{\eta=0} = 0 \qquad \ldots\ldots\ldots\ldots (3.53)$$

A parabolic temperature profile is assumed in the form

$$\theta_s'(\eta, t) = G\eta^2 + H\eta + I \qquad \ldots\ldots\ldots\ldots (3.54)$$

The application of third boundary condition (Equation 3.53) will

yield $$H = 0 \qquad \ldots\ldots\ldots\ldots (3.55)$$

The application of the first boundary condition, Equation 3.51,

$$\theta_s'(\eta,t)\Big|_{\eta=0} \quad G\eta^2 + I\Big|_{\eta=0} = \tau_2$$

will give $$I_2 = \tau_2 \qquad \text{..........(3.56)}$$

Substituting the values of H and I obtained in temperature profile 3.54 gives

$$\theta_s(\eta,t) = G\eta^2 + \tau_2 \qquad \text{..........(3.57)}$$

The second domain of the outer cylinder

The following profile will be assumed for the second domain of the outer cylinder:

$$\theta_1'(\eta,t) = A'\eta^2 + B'\eta + C' \qquad \text{..........(3.58)}$$

The boundary conditions are

$$1. \quad \theta_1'(\eta,t)\Big|_{\eta=1} = 1 \qquad \text{..........(3.59)}$$

$$2. \quad \frac{\partial \theta_1'}{\partial \eta}(\beta,t) = \Big[Bi + C_3\int_0^\beta \theta_s'\cdot\eta\cdot d\eta$$

$$+ C_4\int_\beta^1 \theta_1'\cdot\eta\cdot d\eta\Big](\theta_{1s}' - \theta_{ss}') \text{..........(3.60)}$$

and $$3. \quad \theta_1'(\beta,t) = \tau_1 \qquad \text{..........(3.61)}$$

When the first boundary condition (Equation 3.59) is applied,

$$A' + B' + C' = 1 \qquad \ldots\ldots\ldots\ldots (3.62)$$

The third boundary condition (Equation 3.61) will yield

$$A'\beta^2 + B'\beta + C' = \tau_1 \qquad \ldots\ldots\ldots\ldots (3.63)$$

Solving Equation 3.62 and Equation 3.63 simultaneously, B' and C' in terms of A' are given by

$$B' = -A'(1+\beta) + \frac{(1-\tau_1)}{(1-\beta)} \qquad \ldots\ldots\ldots\ldots (3.64)$$

and

$$C' = \frac{\tau_1 - \beta}{1-\beta} + A'\beta \qquad \ldots\ldots\ldots\ldots (3.65)$$

To relate A' and G, the interface heat transfer condition is used, which is found by combining Equation 3.52 and Equation 3.60.

$$\frac{\partial \theta_s'}{\partial \eta}(\beta.t) = \frac{k_1}{k_2} \cdot \frac{\partial \theta_1(\beta.t)}{\partial \eta}$$

substituting the respective values:

$$2G\eta \Big|_{\eta=\beta} = \frac{k_1}{k_2}(2A'\eta + B')\Big|_{\eta=\beta}$$

Introducing the value of B' (Equation 3.64) gives

$$2G\beta = \frac{k_1}{k_2}\left[A'(\beta-1) + \frac{1-\tau_1}{1-\beta}\right] \qquad \ldots\ldots (3.66)$$

Consider the interface boundary condition,

$$\frac{\partial \theta_s'}{\partial \eta}(\eta, t)\Big|_{\eta=\beta} = \frac{k_1}{k_2}\left[Bi + C_3 \int_0^\beta \theta_s' \cdot \eta \cdot d\eta \right.$$

$$\left. + C_4 \int_\beta^1 \theta_1 \cdot \eta \cdot d\eta \right](\theta_{1s}' - \theta_{ss}')$$

.....(3.67)

Let us evaluate the integral

$$\int_0^\beta \theta_s' \cdot \eta \cdot d\eta$$

substituting the value of θ_s'

$$\int_0^\beta (G\eta^2 + \tau_2) \cdot \eta \cdot d\eta = \left[\frac{G\eta^4}{4} + \frac{\tau_2 \eta^2}{2}\right]_0^\beta$$

therefore

$$\int_0^\beta \theta_s' \cdot \eta \cdot d\eta = \beta^2 \left[\frac{G\beta^2}{4} + \frac{\tau_2}{2}\right]$$

In the above expression introducing the value of G in terms of A' (Equation 3.66)

$$\int_0^\beta \theta_s' \cdot \eta \cdot d\eta = \beta^2 \left[A' \cdot \frac{k_1}{k_2} \cdot \left(\frac{\beta - 1}{2\beta}\right) \cdot \frac{\beta^2}{4} + \frac{1}{2\beta} \cdot \frac{k_1}{k_2} \cdot \frac{(1-\tau_1)\beta^2}{4(1-\beta)} + \frac{\tau_2}{2}\right]$$

or

$$\int_0^\beta \theta_s' \cdot \eta \cdot d\eta = A' \left[\frac{k_1}{k_2} \cdot \frac{\beta^2(\beta+1)}{8}\right] + \left[\frac{\beta^3}{8} \cdot \frac{k_1}{k_2}\left(\frac{1-\tau_1}{1-\beta}\right) + \frac{\beta^2 \tau_2}{2}\right] \qquad ....(3.68)$$

to reduce the complexity of the algebra, let

$$m_2 = \frac{k_1}{k_2} \cdot \frac{\beta^3}{8}(\beta - 1) \qquad ....(3.69)$$

and

$$n_2 = \frac{\beta^3}{8} \times \frac{k_1}{k_2} \times \frac{1-\tau_1}{(1-\beta)} \times \frac{\beta^2 \tau_2}{2} \qquad \ldots\ldots\ldots\ldots(3.70)$$

Therefore,

$$\int_0^{\beta} \theta_s' \cdot \eta \cdot d\eta = m_2 A' + n_2 \qquad \ldots\ldots\ldots\ldots(3.71)$$

Let us evaluate the integral

$$\int_{\beta}^{1} \theta_1' \cdot \eta \cdot d\eta$$

substituting the value of θ_1'

$$\int_{\beta}^{1} (A'\eta^2 + B'\eta + C')\,\eta \cdot d\eta$$

therefore

$$\int_{\beta}^{1} \theta_1' \cdot \eta \cdot d\eta = \left[\frac{A'\eta^4}{4} + \left\{ \left(\frac{1-\tau_1}{1-\beta}\right) - A'(1+\beta) \right\} \frac{\eta^3}{3} + \left\{ \frac{(\tau_1-\beta)}{(1-\beta)} + A'\beta \right\} \frac{\eta^2}{2} \right]_{\beta}^{1}$$

Further simplification yields

$$\int_{\beta}^{1} \theta_1' \cdot \eta \cdot d\eta = A' \left[\frac{1-\beta^4}{4} - \frac{(1+\beta)(1-\beta^3)}{3} + \frac{\beta(1-\beta^2)}{2} \right] + \left[\frac{(1-\tau_1)(1-\beta^3)}{3(1-\beta)} + \frac{(\tau_1-\beta)(1+\beta)}{2} \right] \qquad \ldots\ldots(3.72)$$

Let

$$m_1 = \frac{1-\beta^4}{4} - \frac{(1+\beta)(1-\beta^3)}{3} + \frac{\beta(1-\beta^2)}{2} \qquad \ldots\ldots(3.73)$$

and

$$n_1 = \frac{(1-\tau_1)(1-\beta^3)}{3(1-\beta)} + \frac{(\tau_1-\beta)(1+\beta)}{2} \qquad \ldots\ldots(3.74)$$

Therefore,

$$\int_0^1 \theta_1' \cdot \eta \cdot d\eta = m_1 A' + n_1 \qquad \ldots\ldots\ldots\ldots(3.75)$$

Subtracting the Equation 3.57 from the Equation 3.58 gives

$$\theta_{1S}' - \theta_{SS}' = \tau_1 - G\beta^2 - \tau_2 \qquad \ldots\ldots(3.76)$$

Substituting the values of B', C', and G into Equation 3.76 and simplifying,

$$\theta_{1S}' - \theta_{SS}' = A' \left[\frac{k_1}{k_2} \left(\frac{1-\beta}{2} \right) \beta \right]$$

$$+ \left[(\tau_1 - \tau_2) - \frac{\beta}{2} \cdot \frac{k_1}{k_2} \left(\frac{1-\tau_1}{1-\beta} \right) \right. \qquad \ldots\ldots(3.77)$$

Let

$$m_3 = \frac{k_1}{k_2} \left(\frac{1-\beta}{2} \right) \beta \qquad \ldots\ldots\ldots(3.78)$$

and

$$n_3 = \left[(\tau_1 - \tau_2) - \frac{\beta}{2} \cdot \frac{k_1}{k_2} \left(\frac{1-\tau_1}{1-\beta} \right) \right] \qquad \ldots\ldots\ldots\ldots(3.79)$$

Therefore,

$$\theta_{1S}' - \theta_{SS} = m_3 A' + n_3 \qquad \ldots\ldots\ldots\ldots(3.80)$$

Consider Equation 3.66.

Let $$m_4 = \frac{k_1}{k_2}(\beta - 1) \qquad \text{..........(3.81)}$$

and $$n_4 = \frac{k_1}{k_2}\left(\frac{1-\tau_1}{1-\beta}\right) \qquad \text{..........(3.82)}$$

Therefore,

$$2G\beta = m_4 A' + n_4 \qquad \text{..........(3.83)}$$

Substituting Equations 3.71, 3.75, 3.80, and 3.83 in Equation 3.67 gives

$$m_4 A' + n_4 = \frac{k_1}{k_2}\left[Bi + C_3\left\{m_2 A' + n_2\right\} + C_4\left\{m_1 A' + n_1\right\} + m_3 A' + n_3\right]$$

On simplification it will give a quadratic equation in A':

$$m_3 (m_2 C_3 + C_4 m_1) A'^2 - \left\{-m_3 (Bi + n_2 C_3 + n_1 C_4) - n_3 (m_2 C_3 + m_1 C_4) + \frac{k_2}{k_1}.m_4\right\} A' + \left\{-\frac{k_2}{k_1} n_4 + n_3 (Bi + n_2 C_3 + n_1 C_4)\right\} = 0 \qquad \text{.....(3.84)}$$

To simplify the algebra, let

$$a_1 = m_3 (m_2 C_3 + m_1 C_4) \qquad \text{..........(3.85)}$$

$$b_1 = -m_3 (Bi + n_2 C_3 + n_1 C_4) - n_3 (m_2 C_3 + m_2 C_4) + \frac{k_2}{k_1} m_4 \qquad \text{.....(3.86)}$$

$$C_1 = -\frac{k_2}{k_1} n_4 + n_3 (Bi + n_2 C_3 + n_1 C_4) \qquad \text{.....(3.87)}$$

Therefore,

$$a_1 A'^2 - b_1 A' + c_1 = 0 \qquad \text{...........(3.88)}$$

Solving the above quadratic equation in A':

$$A' = \frac{1}{2a_1}\left[b_1 - \sqrt{b_1^2 - 4a_1 c_1} \right] \qquad \text{...........(3.89)}$$

Substituting the values of θ_s' and θ_1' given by Equations 3.57 and 3.58 in the integral formulations

$$\frac{d}{dt}\int_0^{\beta} \theta_s' \cdot \eta \cdot d\eta = \frac{k_2}{k_1} \cdot \frac{\alpha_s}{R_2^2} \cdot \left.\frac{\partial \theta_s'}{\partial \eta}\right|_{\eta=\beta} \qquad 0 < \eta < \beta \qquad \text{................(3.90)}$$

and

$$\frac{d}{dt}\int_{\beta}^{1} \theta_1' \cdot \eta \cdot d\eta = \frac{\alpha_1}{R_2^2} \cdot \left.\frac{d\theta_1'}{d\eta}\right|_{\eta=1} \qquad \beta < \eta < 1$$

$$\text{................(3.91)}$$

The final results for Equations 3.90 and 3.91 will be

$$\int \frac{d\left[m_2 A' + n_2\right]}{\frac{k_2}{k_1}\left(m_4 A' + n_4\right)} = \frac{\alpha}{R_2^2}\int dt \qquad \text{...........(3.92)}$$

and

$$\int \frac{d\,(m_1 A' + n_1)}{\left[A'(1-\beta) + \frac{(1-\tau_1)}{(1-\beta)}\right]} = \frac{\alpha}{R_2^2}\int dt \qquad \text{...........(3.93)}$$

Both the integrals given by Equations 3.92 and 3.93 are the functions of τ_1 and τ_2 , so they can be solved simultaneously by a numerical technique similar to that described for Part II.

CHAPTER IV

COMPUTER SOLUTION

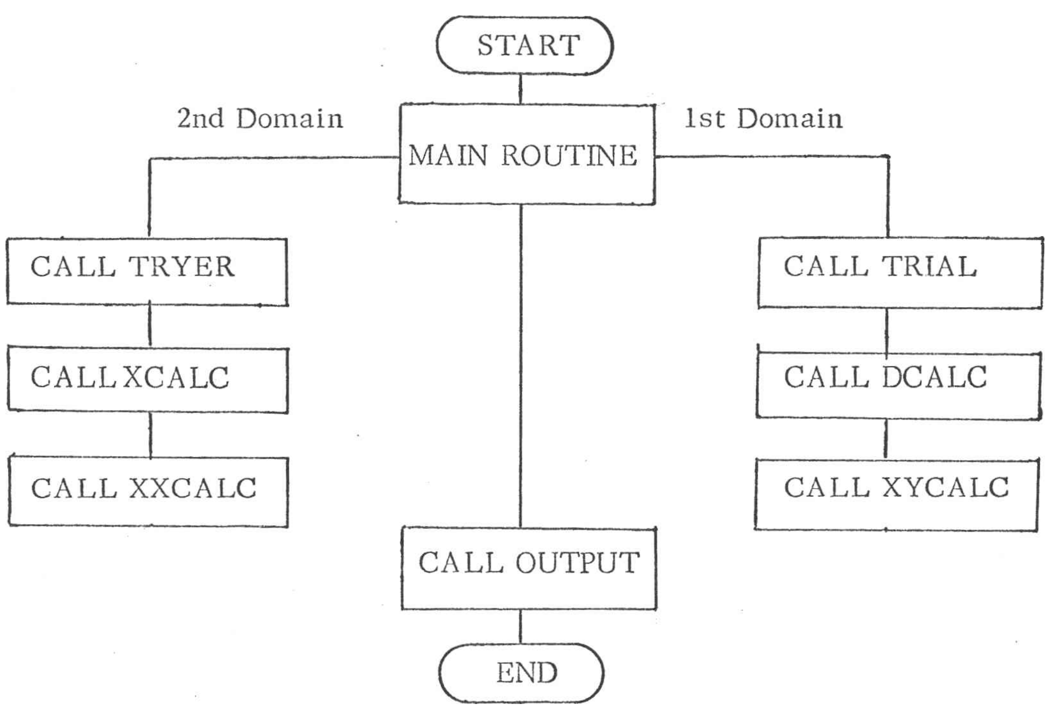
Part I

For the case of the flat plate, a numerical integration technique is used to evaluate the final time integrals. This technique, instead of using the linear approximation assuming trapezoidal integration formulae, assumes parabolic approximation, to get better accuracy of the results. A curve fitting procedure will be used for this purpose.

The set of three points on the actual curve will be selected and a parabola passing through these three points will be found. The area under this parabola is determined by simple integration. The second set of three points which has the first two points common to the last two points of the first set of three points will be considered. Another parabola passing through these three points will be found, and the area under it will be determined in the same way. The average of the two areas under the parabolas passing through the second and third points will be taken, which gives better accuracy than the normal Simpson's rule, to find out the area under the curve. This procedure will be extended to all such consecutive sets of three points, giving the area between any two points twice except the first two and the last two points. This is usually known as the curve fitting procedure. A computer program is prepared to find the area under any curve using the above technique. The equations numbered 2.22 and 2.30 will be solved using this computer program.

Part II

In the case of the composite cylinders, both the final time integrals given by equations 3.48 and 3.49 for Part II and equations 3.92 and 3.93 for Part III are solved simultaneously by using numerical technique known as the trial and error. A computer program is prepared to solve the above equations simultaneously for both the parts. The following flow-chart of the computer program will describe thoroughly the technique used.



Main routine

Main routine controls the values of t, τ_1, and τ_2. It assigns the initial guess and the second guess of τ_1 for each corresponding value of t_o for the first domain, and for corresponding values of τ_2 for the second domain. The TRIAL and TRYER routines will give the true value of τ_1 for corresponding values of τ_o in the case of the first domain and for corresponding values of τ_2 for the second domain, respectively with required accuracy, using the above assigned guesses. For successive initial guesses of τ_1, the preceding true value of τ_1 for the corresponding value of τ_o or τ_2, as the case may be, becomes the initial guess for the following value of τ_1. For the second guess of τ_1 linear interpolation between two preceding true values of τ_1 will be done in the case of the first domain and the exponential interpolation will be done between two preceding true values of τ_1 in the case of the second domain.

XYCALC and XXCALC routines

XYCALC routine calculates the values of the numerator and the denominator of the equations 3.48 and 3.49 for given values of τ and τ_1. XXCALC routine calculates the value of the numerator and the denominator of the equations 3.92 and 3.93 for given values of τ_2 and τ_1.

D CALC and X CALC routines

Referring to equations 3.48 and 3.49

$$\int y_1 \cdot dx1 = \int dt \qquad \dots\dots\dots\dots (4.1)$$

$$\int y_2 \cdot dx_2 = \int dt \qquad \ldots\ldots\ldots\ldots (4.2)$$

where

$$x_1 = \frac{A}{h}\left(1 - \beta h^4\right) + \frac{\beta}{3}\left(1 - \beta^3\right) + \frac{C}{2}\left(1 - \beta^2\right) \qquad \ldots\ldots (4.3)$$

$$y_1 = \frac{1}{\frac{\alpha_1}{R_2^2} - (2A + B)} \qquad \ldots\ldots (4.4)$$

$$x_2 = D \cdot (\beta - \tau)^3 \cdot \left(\frac{\beta}{4} + \frac{\tau}{2}\right) \qquad \ldots\ldots (4.5)$$

$$y_2 = \frac{1}{\frac{K_2}{K_1} \times \frac{\alpha_1}{R_2^2} (\beta - \tau) \times D} \qquad \ldots\ldots (4.6)$$

where $\alpha_1 = K_1 / \varrho_1 C_1$.

Again referring to the equations 3.92 and 3.93

$$\int y_1 \, dx_1 = \int dt \qquad \ldots\ldots\ldots\ldots (4.7)$$

$$\int y_2 \, dx_2 = \int dt \qquad \ldots\ldots\ldots\ldots (4.8)$$

where

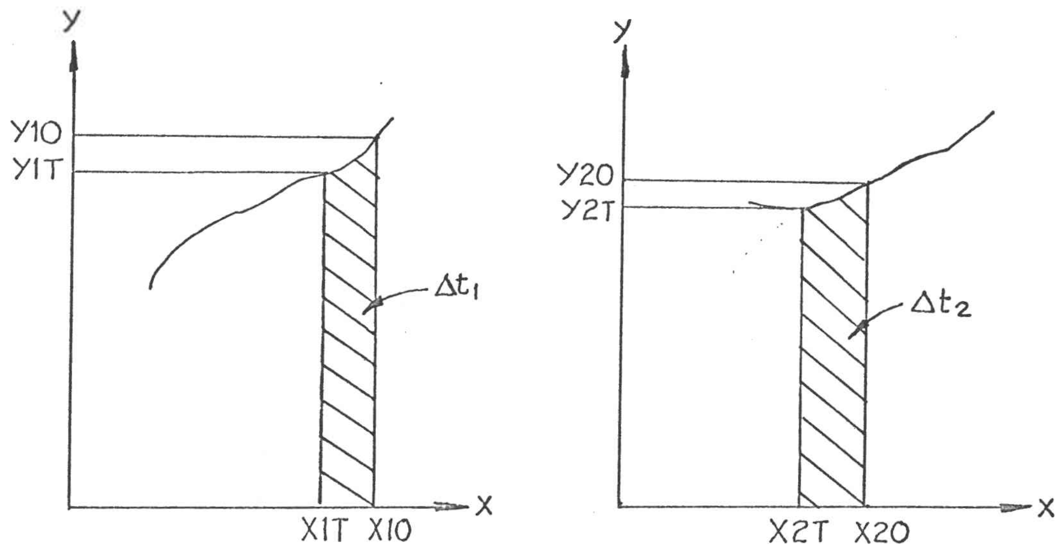
$$x_1 = m_1 A + n_1 \qquad \ldots\ldots (4.9)$$

$$y_1 = \frac{1}{\frac{\alpha_2}{R_2^2}\left\{(1 - \beta) A + \frac{(1 - \tau_1)}{(1 - \beta)}\right\}} \qquad \ldots\ldots (4.10)$$

$$x_2 = m_2 A + n_1 \qquad \ldots\ldots (4.11)$$

$$y_2 = \frac{1}{\frac{K_2}{K_1} \cdot \frac{\alpha_2}{R_2^2} (m_4 \cdot A + n_4)} \qquad \ldots\ldots (4.12)$$

$$\alpha_2 = K_1 / \varrho_2 C_2 \qquad \ldots\ldots (4.13)$$



The above illustration makes the problem more clear. X_1T, Y_1T, X_2T, and Y_2T are obtained by XYCALC routine for preceding true values of T_1 and corresponding values of T in the case of the first domain. Summary X10, Y10, X20, and Y20 will be calculated for another value of T_1 and the corresponding value of T_1. The trapezoidal integration formula is used to evaluate the integral given by equation 4.1 between points X_1T, Y_1T and X10, Y10, and the integral given by equation 4.2 between points X_2T, Y_2T and X10, Y20 to get Δt_1 and Δt_2 respectively. DCALC routine naturally calculates the difference between equations 4.1 and 4.2 in case of the first domain. In the same way XCALC routine calculates the difference between equations 4.7 and 4.8 in the case of the second domain.

TRIAL and TRYER routines

TRIAL routine calculates the true value of τ_1 for respective values of τ in the case of the first domain, and the TRYER routine calculates the true value of τ_1 for corresponding values of τ_2 in case of the second domain. The linear interpolation is done between two guesses, i.e., the initial guess and the second guess, to get the intermediate guess. Again the interpolation between the initial guess and the first intermediate guess is done to get the second intermediate guess. This interpolation procedure is repeated until the intermediate guess obtained satisfies the required accuracy. In essence, these routines try to make

$$\Delta t_1 = \Delta t_2 \simeq 0$$

Δt_1 and Δt_2 are calculated either by DCALC routine or XCALC routine as the case may be. The difference between Δt_1 and Δt_2 should not necessarily be zero; otherwise, these routines fail to work.

CHAPTER V

CONCLUSION

The problem of this thesis was discussed in two parts: (1) one-dimensional heat flow in rectangular coordinates, and (2) one-dimensional radial heat flow in circular coordinates.

The first part treats the problem in rectangular coordinates. It gives the final form for the first domain of the problem as having an equation with a definite integral which has time-dependent propagation of heat flux as limits of integration and integrand as a function of this time-dependent displacement. An evaluation of the integral equation gives the total dimensionless time (Fourier's number) for completion of the first domain.

Analysis of the second domain of the problem leads the problem to final form as an equation having a definite integral with limits as time-dependent temperature parameter at the insulated face and integrand as a function of this time-dependent temperature parameter. Evaluation of this integral gives the total dimensionless time (Fourier's number) for completion of the second domain.

Mathematical difficulties involved in getting an analytical solution forces one to discretize the problem to get the numerical solution for the final integral. A computer program giving the numerical solution for this problem was made and was found to give the required engineering accuracy for results.

The second part, which treats the problem in circular coordinates with one-dimensional radial heat flow only, gives the final forms for two different cases. It is considered to obtain temperature distribution until a steady state is achieved, as two simultaneous integral equations in each case.

Integrands of both simultaneous equations in the case, considering the second domain of the outer cylinder and the first domain of the inner solid cylinder, are the functions of time-dependent propagation of heat flux in the inner cylinder and temperature parameter at the interface of the cylinders.

The integrands of both simultaneous equations in the case, extending the problem to a steady state, are functions of time-dependent temperature parameter at the interface of the cylinders and at the center of the solid cylinder.

In both cases, an attempt has been made to solve the simultaneous integral equations by numeric methods of trial and error as well as by numeric integration. Since there is no definite source of checking the results obtained by the numeric method to establish the perfect validity of the results, the author could not avail himself of the opportunity of presenting the results in the thesis. Establishment of good and perfectly reliable results might prove to be a prospective step in the extension of this thesis and a profitable contribution in this area.

LIST OF REFERENCES

1. Arpaci, V. S. Conduction Heat Transfer. Boston: Addison Wesley Publishing Co., 1966.

2. Kantorvich, L. V., and Krylov, V. I. Approximate Methods of Higher Analysis. New York: Interscience-Noordhoff, 1958.

3. Goodman, Theodore R. "Application of Integral Methods to Transient Non-linear Heat Transfer," Advances in Heat Transfer, Volume I. New York, London: Academic Press, Inc., 1964.

4. Au Norman, N. Stresses and Strains in Multi-layer Anisotropic Cylinders. Elsegundo, California: Aerospace Corporation, 1965.

APPENDIX

DERIVATION OF THE INTERFACE PRESSURE

The determination of the interface pressure for the case of two composite cylinders, the inner one being a solid cylinder, is given by Au (4). To give a more descriptive picture of the stress part of the problem, the author has preferred to give the detailed derivation, and proper organization of the majority of the formulae leading to the final form of the expression for the interface pressure appropriate to this problem.

Let us first find the radial pressure at the interface in terms of Θ_1 and Θ_s . The equation for equilibrium of stresses in the case of circular geometry with symmetric temperature distribution is

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad \ldots\ldots(A)$$

where

r = the radial stress, and

θ = the circumferential stress.

The relations between the stress and strain in the thermal stress problems are:

$$\epsilon_r - \alpha T = \frac{1}{E}\left[\sigma_r - \nu(\sigma_\theta + \sigma_z)\right] \qquad \ldots\ldots(A.1)$$

$$\epsilon_\theta - \alpha T = \frac{1}{E}\left[\sigma_\theta - \gamma\,(\sigma_r + \sigma_z\right] \qquad \ldots\ldots(A.2)$$

$$\epsilon_z - \alpha T = \frac{1}{E}\left[\sigma_z - \gamma\,(\sigma_r + \sigma_\theta\right] \qquad \ldots\ldots(A.3)$$

where

ϵ_r = the radial strain

ϵ_θ - the circumferential strain

σ_z = the stress in axial direction

γ = Poission's ratio

α = the coefficient of thermal expansion.

Assuming the axial displacement to be zero (plane strain), Equation A.3 will become

$$\sigma_z = \gamma\,(\sigma_\theta + \sigma_z) - \alpha E T \qquad \ldots\ldots(A.4)$$

The substitution of Equation A.4 in Equations A.1 and A.2 will give

$$\epsilon_r - (1+\gamma)\alpha T = \frac{1-\gamma^2}{E}\left(\sigma_r - \frac{\gamma}{1-\gamma}\sigma_\theta\right) \qquad \ldots\ldots(A.5)$$

and

$$\epsilon_\theta - (1+\gamma)\alpha T = \frac{1-\gamma^2}{E}\left(\sigma_\theta - \frac{\gamma}{1-\gamma}\sigma_r\right) \qquad \ldots\ldots(A.6)$$

Multiplying Equation A.6 by $\frac{\gamma}{1-\gamma}$ and adding the result to Equation A.5, the value of in terms of σ_r and σ_θ will be obtained.

....(A.7)

To get σ_θ in terms of ϵ_r and ϵ , the same technique is applied.

The final form for σ_θ will be

$$\sigma_\theta = \frac{E\,(1-\gamma)}{(1+\gamma)(1-2\gamma)} \left[\epsilon_\theta + \frac{\gamma}{1-\gamma}\,\epsilon_r - \alpha T \left(\frac{1+\gamma}{1-\gamma} \right) \right] \quad \ldots\ldots (A.8)$$

Introducing the Equations A.7 and A.8 into Equation A, the result is

$$r \frac{d}{dr} \left[\epsilon_r + \frac{\gamma}{1-\gamma}\,\epsilon_\theta \right] + \frac{1-2\gamma}{1-\gamma} \left(\epsilon_r - \epsilon_\theta \right)$$

or

$$r\alpha \left(\frac{1+\gamma}{1-\gamma} \right) \frac{dT}{dr} \quad \ldots\ldots\ldots\ldots (A.9)$$

The radial strain ϵ_r is given by

$$\epsilon_r = \frac{du}{dr} \quad \ldots\ldots\ldots\ldots (A.10)$$

where u is the radial deflection.

The circumferential strain is given by

$$\epsilon_\theta = \frac{u}{r} \quad \ldots\ldots\ldots\ldots (A.11)$$

Substituting Equations A.10 and A.11, the simplified form will be

$$r \frac{d^2u}{dr^2} + \frac{du}{dr} - \frac{u}{r} = r\alpha \left(\frac{1+\gamma}{1-\gamma} \right) \frac{dT}{dr} \quad \ldots\ldots (A.12)$$

Equation A.12 will be arranged as

$$\frac{d^2u}{dr^2} + \frac{1}{r} \cdot \frac{du}{dr} - \frac{u}{r^2} = \alpha \left(\frac{1+\gamma}{1-\gamma} \right) \frac{dT}{dr} \quad \ldots\ldots (A.13)$$

and further simplification will yield

$$\frac{d^2u}{dr^2} + \frac{d}{dr}\left(\frac{u}{r}\right) = \alpha\left(\frac{1+\gamma}{1-\gamma}\right)\frac{dT}{dr} \qquad \ldots.(A.14)$$

Integrating both the sides with respect to r

$$\frac{du}{dr} + \frac{u}{r} = \alpha\left(\frac{1+\gamma}{1-\gamma}\right) T + C_1'$$

and arranging the equation in proper form,

$$r\frac{du}{dr} + u = \alpha\left(\frac{1+\gamma}{1-\gamma}\right) T \cdot r + C_1' r$$

integrating again and dividing the result by r gives

$$u(r) = \alpha\left(\frac{1+\gamma}{1-\gamma}\right) \cdot \frac{1}{r}\int T \cdot r \cdot dr + C_1 r + \frac{C_2}{r} \qquad \ldots.(A.15)$$

Introducing Equation A.15 into Equations A.10 and A.11, the expressions for ϵ_r and ϵ_θ will be

$$\epsilon_r = \frac{du}{dr} = \alpha\left(\frac{1+\gamma}{1-\gamma}\right)\left\{-\frac{1}{r^2}\int T \cdot r . dr\right\} + C_1 - \frac{C_2}{r^2} \qquad \ldots.(A.16)$$

and

$$\epsilon_\theta = \frac{u}{r} = \alpha\left(\frac{1+\gamma}{1-\gamma}\right) \cdot \frac{1}{r^2}\int T . r . dr + C_1 - \frac{C_2}{r^2} \qquad \ldots.(A.17)$$

Substituting the values of ϵ_r and ϵ_θ in Equations A.7 and A.8, the expressions for σ_r and σ_θ will be modified as

$$\sigma_r = \frac{-\alpha E}{(1-\gamma)} \cdot \frac{1}{r^2}\int T \cdot r \cdot dr + \frac{E}{1+\gamma}\left(\frac{C_1}{1-2\gamma} - \frac{C_2}{r^2}\right) \qquad \ldots.(A.18)$$

and

$$\sigma_\theta = \frac{\alpha E}{(1-\gamma)} \cdot \frac{1}{r^2}\int T \cdot r \cdot dr - \frac{\alpha E T}{(1-\gamma)} + \frac{E}{1-\gamma}\left[\frac{c_1}{1-2\gamma} - \frac{c_2}{r^2}\right] \quad \text{.........(A.19)}$$

The case of solid cylinders

In order to evaluate c_1 and c_2 in the general expressions for σ_r and σ_θ the boundary conditions for solid cylinders are as follows:

$$U(r)\Big|_{r=0} = 0$$

$$\sigma(r)\Big|_{r=a} = 0$$

where "a" is the radius of the solid core.

$$U(r)\Big|_{r=0} = \left\{\frac{\alpha(1+\gamma)}{(1-\gamma)} \cdot \frac{1}{r}\int_0^r T \cdot r \cdot dr + C_1 r + \frac{C_2}{r^2}\right\}\Bigg|_{r=0} = 0$$

The first term in the expression for $U(r)\Big|_{r=0}$ has an indeterminate form.

The application of L'Hospital's rule will give its value as zero at $r = 0$ and hence to satisfy the first boundary condition,

$$c_2 = 0$$

To evaluate the second constant, substitute $r = a$ in Equation A.18 and the result will be

$$\frac{C_1}{(1-2\gamma)} = \frac{1+\gamma}{1-\gamma} \cdot \alpha \cdot \frac{1}{a^2}\int_0^a T \cdot r \cdot dr$$

Introducing these two constants in the Equations A.18 and A.19 gives

$$\sigma_r = \frac{\alpha E}{(1-\gamma)}\left[\frac{1}{a^2}\int_0^a T.r.dr - \frac{1}{r^2}\int_0^r T.r.dr\right] \quad \ldots.(A.20)$$

and

$$\sigma_\theta = \frac{\alpha E}{(1-\gamma)}\left[\frac{1}{a^2}\int_0^a T.r.dr + \frac{1}{r^2}\int_0^r T.r.dr - T\right] \ldots.(A.21)$$

The axial stress σ_z in terms of radial stress and circumferential stress can be given as

$$\sigma_z = \gamma(\sigma_r + \sigma_\theta) - \alpha E T$$

Substituting the Equations A.20 and A.21 in the expression for σ_z

$$\sigma_z = \gamma\left[\frac{2E}{(1-\gamma)} \cdot \frac{1}{a^2}\int_0^a T.r.dr - \frac{\alpha E T}{(1-\gamma)}\right] - \alpha E T$$

or

$$\sigma_z = \frac{2\alpha E\gamma}{(1-\gamma)a^2}\int_0^a T.r.dr - \frac{\alpha E T}{(1-\gamma)}\left(\frac{\gamma}{(1-\gamma)} + 1\right)$$

or

$$\sigma_z = \frac{\alpha E}{(1-\gamma)}\left[\frac{2\gamma}{a^2}\int_0^a T.r.dr - T\right] \quad \ldots\ldots\ldots.(A.22)$$

Introducing the values of constants C_1 and C_2 in the expression for U(r), Equation A.15, gives

$$U(r) = \frac{1+\gamma}{1-\gamma} \cdot \alpha\left[\frac{(1-2\gamma)r}{a^2}\int_0^a T.r.dr + \frac{1}{r}\int_0^r T.r.dr\right] \ldots(A.23)$$

The case of a hollow cylinder

The constants C_1 and C_2 in the general expressions for σ_r and σ_θ can be evaluated by using the following boundary conditions for the hollow cylinder.

$$\sigma(r)\Big|_{r=a} = 0$$

$$\sigma_r(r)\Big|_{r=b} = 0$$

where a = the inner radius of the single hollow cylinder, and

b = the outer radius of the single hollow cylinder.

Hence,

$$\sigma_r(r)\Big|_{r=a} = \frac{-\alpha E}{(1-\gamma)} \cdot \frac{1}{a^2}\int_a^a T.r.dr + \frac{E}{(1-\gamma)}\left[\frac{C_1}{(1-2\gamma)} - \frac{C_2}{a^2}\right] = 0$$

gives

$$\frac{C_1}{(1-2\gamma)} - \frac{C_2}{a^2} = 0 \qquad \ldots\ldots\ldots(A.24)$$

At r = b the radial stress is given by

$$\sigma_r(r)\Big|_{r=b} = \frac{-\alpha E}{(1-\gamma)} \cdot \frac{1}{b^2}\int_a^b T.r.dr + \frac{E}{(1-\gamma)}\left(\frac{C_1}{1-2\gamma} - \frac{C_2}{b^2}\right) \qquad \ldots(A.25)$$

Using Equation (A.24) in Equation A.25

$$\sigma_r(r)\Big|_{r=b} = \frac{-\alpha E}{1-\gamma} \cdot \frac{1}{b^2}\int_a^b T.r.dr + \frac{EC_2}{(1-\gamma)}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = 0$$

This will give

$$\frac{EC_2}{(1+\gamma)} = \frac{\alpha E}{(1-\gamma)} \cdot \frac{a^2}{b^2-a^2}\int_a^b T.r.dr \qquad \ldots(A.26)$$

or

$$\frac{EC_1}{(1+\gamma)(1-2\gamma)} = \frac{\alpha E}{(1-\gamma)(b^2-a^2)} \int_a^b T \cdot r \cdot dr \qquad \ldots .(A.27)$$

Substituting Equations A.26 and A.27 in Equations A.18 and A.19, the expressions for σ_r and σ_θ will be obtained.

$$\sigma_r(r) = \frac{\alpha E}{1-\gamma} \cdot \frac{1}{r^2}\left[\frac{(r^2-a^2)}{(b^2-b^2)} \int_a^b T \cdot r \cdot dr - \int_a^r T \cdot r \cdot dr\right] \ldots\ldots\ldots\ldots .(A.28)$$

$$\sigma_\theta(r) = \frac{\alpha E}{(1-\gamma)} \cdot \frac{1}{r^2}\left[\frac{r^2+a^2}{b^2-a^2} \int_a^b T \cdot r\, dr + \int_a^r T \cdot r \cdot dr - Tr^2\right] \ldots\ldots\ldots\ldots .(A.29)$$

Introducing Equations A.28 and A.29 in the expression for the axial stress

$$\sigma_z = \gamma(\sigma_r + \sigma_\theta) - \alpha E T$$

gives the final result

$$\sigma_z = \frac{\alpha E}{(1-\gamma)}\left[\frac{2\gamma}{b^2-a^2} \int_a^b T \cdot r \cdot dr - T\right] \qquad \ldots .(A.30)$$

Substituting the values of C_1 and C_2 in Equation A.15, the radial deflection can be given as

$$U(r) = \frac{1+\gamma}{1-\gamma} \cdot \frac{\alpha}{r}\left[\int_a^r T \cdot r \cdot dr + \frac{(1-2\gamma)r^2+a^2}{b^2-a^2} \int_a^b T \cdot r \cdot dr\right] \ldots .(A.31)$$

An attempt will now be made to find out the radial stress at the interface of two cylinders, the inner one being a solid core.

At this stage discussion will be generalized for N hollow cylinders with solid core at the center. The equations for the stresses will also be derived for N cylinders.

The radial strain in an n^{th} cylinder will be

$$\epsilon_{n\theta} = \frac{U_{n\theta}}{r} \qquad \text{..........(A.32)}$$

The following relation is given by Au (4):

$$\epsilon_{n\theta} = \alpha_n \cdot T_n + \frac{1}{E_n}\left[\sigma_{n\theta} - \gamma_n\left(\sigma_{rn} + \sigma_{zn}\right)\right] \qquad \text{....(A.33)}$$

n = 1, 2, 3, - - - N hollow cylinders

In this discussion the total stress at the interface will be the sum of the stress due to the temperature and the stress due is initial interface pressure.

$$\sigma_{rn} = (\sigma_{rn})_p + (\sigma_{rn})_T$$

and

$$\sigma_{\theta n} = (\sigma_{\theta n})_p + (\sigma_{\theta n})_T$$

where subscripts

p-denotes the stress due to initial interface pressure and

T-denotes the stress due to temperature only.

The following formula is given by Au (4):

$$(\sigma_{rn})_p = -\frac{R_n^2 \cdot R_{n+1}^2 (P_{n+1} - P_n)}{R_{n+1}^2 - R_n^2} \cdot \frac{1}{r^2} + \frac{P_n R_n^2 - P_{n+1} R_{n+1}^2}{R_{n+1}^2 - R_n^2} \quad \text{..(A.34)}$$

and

$$(\sigma_{\theta n})_p = \frac{R_n^2 \cdot R_{n+1}^2 (P_{n+1} - P_n)}{R_{n+1}^2 - R_n^2} \cdot \frac{1}{r^2} + \frac{P_n R_n^2 - P_{n+1} R_{n+1}^2}{R_{n+1}^2 - R_n^2}$$

. . . .(A.35)

The following formulae for hollow cylinders can be generalized from the preceding discussion.

$$(\sigma_{rn})_T = \frac{\alpha_n E_n}{1-\gamma_n} \cdot \frac{1}{r^2} \left[\frac{r^2 - R_n^2}{R_{n+1}^2 - R_n^2} \int_{R_n}^{R_{n+1}} T.\, r.\, dr - \int_{R_n}^{r} T.\, r.\, dr \right]$$

. . . .(A.36)

and

$$(\sigma_{\theta n})_T = \frac{\alpha_n E_n}{1-\gamma_n} \cdot \frac{1}{r^2} \left[\frac{r^2 + R_n^2}{R_{n+1}^2 - R_n^2} \int_{R_n}^{R_{n+1}} T.\, r.\, dr - \int_{R_n}^{r} T.\, r.\, dr - Tr^2 \right]$$

. . . .(A.37)

The total stresses become

$$\sigma_{rn} = \frac{R_n^2 \cdot R_{n+1}^2 (P_{n+1} - P_n)}{R_{n+1}^2 - R_n^2} \cdot \frac{1}{r^2} + \frac{P_n \cdot R_n^2 - P_{n+1} \cdot R_{n+1}^2}{R_{n+1}^2 - R_n^2}$$

$$+ \frac{\alpha_n \cdot E_n}{1-\gamma_n} \cdot \frac{1}{r^2} \left[\frac{r^2 - R_n^2}{R_{n+1}^2 - R_n^2} \int_{R_n}^{R_{n+1}} T.\, r.\, dr - \int_{R_n}^{r} T.\, r.\, dr \right]$$

. . . .(A.38)

and

$$\sigma_{\theta n} = \frac{R_n^2 \cdot R_{n+1}^2 \, (P_{n+1} - P_n)}{R_{n+1}^2 - R_n^2} \cdot \frac{1}{r^2} + \frac{P_n . R_n^2 - P_{n+1} \cdot R_{n+1}^2}{R_{n+1}^2 - R_n^2}$$

$$+ \frac{\alpha_n E_n}{1 - \gamma_n} \cdot \frac{1}{r^2} \left[\frac{r^2 + R_n^2}{R_{n+1}^2 - R_n^2} \int_{R_n}^{R_{n+1}} T \cdot r . dr - \int_{R_n}^{r} T . r . dr - Tr^2 \right] \qquad \text{..............(A.39)}$$

The solid core

The radial strain in the solid core is given by

$$\epsilon_{\theta s} = \frac{U_s}{r}$$

The following relations will be obtained from Au (4):

1. $$\epsilon_{\theta s} = \alpha_s T_s + \frac{1}{E_s} \left[\sigma_{\theta s} - \gamma_s (\sigma_{rs} + \sigma_{zs}) \right] \qquad \text{...(A.40)}$$

where

$$\sigma_{rs} = -p_1 + (\sigma_{rs})_T \qquad \text{..........(A.41)}$$

and

$$\sigma_{\theta s} = -p_1 + (\sigma_{\theta s})_T \qquad \text{..........(A.42)}$$

p_1 is the contact pressure or the initial tightening pressure and from the preceding discussion

$$(\sigma_{rs})_T = \frac{\alpha_s E_s}{1 - \gamma_s} \left[\frac{1}{R_1^2} \int_0^{R_1} T . r . dr - \frac{1}{r^2} \int_0^{r} T . r . dr \right] \qquad \text{....(A.43)}$$

and

$$(\sigma_{\theta s})_T = \frac{\alpha_s E_s}{1-\gamma_s}\left[\frac{1}{R_1^2}\int_0^{R_1} T.r.dr + \frac{1}{r^2}\int_0^{r} T.r.dr - T_s\right] \quad \ldots\ldots\ldots\ldots (A.44)$$

Introducing Equations A.43 and A.44 into Equations A.41 and A.42 respectively, and then substituting the results in the known relation,

$$\sigma_{zs} = \gamma_s(\sigma_{rs} + \sigma_{\theta s}) - \alpha_s E_s T_s$$

gives the result

$$\sigma_{zs} = -2\gamma_s P_1 + \frac{\alpha_s E_s}{1-\gamma_s}\left[\int_0^{R_1} T.r.dr - T_s\right] \quad \ldots (A.45)$$

If all the N cylinders expand uniformly in z-direction and there is no relative displacement in all the cylinders, the uniform strain in z-direction will induce uniform axial stress $E_s \cdot \epsilon_{zs}$ in solid cylinder in that direction. The modified value of σ_{zs} will be

$$\sigma_{zs} = -2\gamma_s P_1 + \frac{\alpha_s E_s}{1-\gamma_s}\left[\frac{2\gamma_s}{R_1^2}\int_0^{R_1} T.r.dr - T\right] + E_s \cdot \epsilon_{zs} \cdots (A.46)$$

where ϵ_{zs} is independent of radius r for all cylinders. To determine the value of ϵ_z for free ends with the above condition of no end stress:

$$\int_0^{R_{n+1}} 2\pi r \cdot \sigma_z \cdot dr = 0$$

or

$$\int_0^{R_1} r\,\sigma_{zs}\,dr + \sum_{n=1}^{N}\int_{R_n}^{R_{n+1}} r\,\sigma_{zn}\,dr = 0 \quad \ldots (A.47)$$

The value of the first integral in Equation A.47 will be found by substituting the value of σ_{zs} as given in Equation A.46 and then performing the actual integration of every term gives

$$\int_0^{R_1} \sigma_{zs}\, r\, dr = -\gamma_s P_1 R_1^2 + \frac{1}{2} E_s \epsilon_{zs} \cdot R_1^2 + \alpha_s E_s \int_0^{R_1} T.\, r.\, dr \quad \ldots\ldots\ldots\ldots (A.48)$$

The value of the second term in Equation A.47 can be obtained as follows:

Introducing the values given by the expressions for σ_{rn} and $\sigma_{\theta n}$ (Equations A.38 and A.39) into the generalized relation

$$\sigma_{zn} = \gamma_n (\sigma_{rn} + \sigma_{\theta n}) - \alpha_n \cdot E_n \cdot T_n$$

gives

$$\sigma_{zn} = 2\gamma_n \left[\frac{P_n.\, R_n^2 - P_{n+1} \cdot R_{n+1}^2}{R_{n+1}^2 - R_n^2} + \frac{\alpha_n E_n}{1-\gamma_n} \left[\frac{2\gamma_n}{R_{n+1}^2 - R_n^2} \times \int_{R_n}^{R_{n+1}} T.\, r.\, dr - T_n \right] \right] + \epsilon_{zn} E_n \quad \ldots\ldots\ldots\ldots (A.49)$$

Substituting the value of σ_{zn} in the given integral

$$\int_{R_n}^{R_{n+1}} \sigma_{zn} \cdot r.\, dr = 2\gamma_n \left[\frac{P_n \cdot R_n^2 - R_{n+1} - R_{n+1}^2}{R_{n+1}^2 - R_n^2} \right] \int_{R_n}^{R_{n+1}} r dr + \int_{R_n}^{R_{n+1}} \epsilon_{zn} E_n - r dr + \frac{\alpha_n E_n}{1-\gamma_n} \left[\frac{2\gamma_n}{R_{n+1}^2 - R_n^2} \int_{R_n}^{R_{n+1}} \left[\int_{R_n}^{R_{n+1}} T.\, r.\, dr \right] \times r.dr - \int_{R_n}^{R_{n+1}} T.\, r.\, dr \right] \quad \ldots\ldots\ldots\ldots (A.50)$$

equals

$$\gamma_n \left[P_n R_n^2 - P_{n+1} \cdot R_{n+1}^2 \right] - \frac{E_n \epsilon_{zn}}{2} \left[R_{n+1}^2 - R_n^2 \right]$$

$$+ \alpha_n E_n - \int_{R_n}^{R_{n+1}} T.\, r.\, dr \qquad \ldots\ldots\ldots\ldots (A.51)$$

Substituting Equations A.48 and A.51 into Equation A.47, and then transposing the terms containing ϵ_z on the left-hand side, gives

$$\epsilon_z = \frac{2}{E_s R_1^2 + \sum_{n=1}^{N} E_n (R_{n+1}^2 - R_n^2)} \left[\alpha_s E_s \int_0^{R_1} T.\, r.\, dr + \right.$$

$$\sum_{n=1}^{N} \alpha_n E_n \int_{R_n}^{R_{n+1}} T.\, r.\, dr + \gamma_s\, p_i\, R_1^2 -$$

$$\left. \sum_{n=1}^{N} \gamma_n \left(P_n \cdot R_n^2 - P_{n+1} \cdot R_{n+1}^2 \right) \right] \ldots\ldots\ldots\ldots\ldots (A.52)$$

Introducing Equations A.41 and A.42 into Equation A.40 gives

$$\epsilon_{\theta s} = \alpha_s T_s - \frac{P_1 (1 - \gamma_s)}{E_s} + \frac{1}{E_s} \left[(\sigma_\theta)_T - \gamma_s (\sigma_r)_T \right]$$

$$- \frac{\gamma_s \sigma_{zs}}{E_s}$$

Substituting the values of $(\sigma_\theta)_T$, $(\sigma_r)_T$, and σ_{zs} into the above equation:

$$\epsilon_{\theta s} = \alpha_s T_s - \frac{P_1 (1 - \gamma_s)}{E_s} \times \frac{1}{E_s} \times \frac{\alpha_s E_s}{(1 - \gamma_s)} \left\{ \left(\frac{1}{R_1^2} \times \right. \right.$$

$$\left. \int_0^{R_1} T.\, r.\, dr + \frac{1}{r^2} \int_0^{r} T.\, r.\, dr - T_s \right) - \gamma_s \left(\frac{1}{R_1^2} \times \right.$$

$$\left. \left. \int_0^{R_1} T.\, r.\, dr - \frac{1}{r^2} \int_0^{r} T.\, r.\, dr \right) \right\} - \frac{\gamma_s}{E_s} \left[-2 P_1 \gamma_s + \right.$$

$$\left. \frac{\alpha_s E_s}{1 - \gamma_s} + E_s \epsilon_z + \left\{ \frac{2 \gamma_s}{R_1^2} \int_0^{R_1} T.\, r.\, dr - T_s \right\} \right]$$

In simplified form,

$$\epsilon_{\theta s} = \frac{P_1}{E_s}\left[1 - \gamma_s - 2\gamma_s^2\right] + \left[1 - \frac{2\gamma_s^2}{1-\gamma_s}\right]\frac{\alpha_s}{R_1^2} \times \int_0^{R_1} T.\,r.\,dr + \frac{(1+\gamma_s)}{(1-\gamma_s)} \times \frac{\alpha_s}{r^2}\int_0^r T.\,r.\,dr - \gamma_s \epsilon_z$$

Substituting this value in the expression for the radial deflection U_s, and then rearranging:

$$U_s(r) = \frac{1+\gamma_s}{1-\gamma_s} \times \alpha_s \left[(1 - 2\gamma_s)\frac{r}{R_1^2}\int_0^{R_1} T.\,r.\,dr + \frac{1}{r}\int_0^r T.\,r.\,dr\right] - \frac{rP_1}{E_s}(1+\gamma_s)(1-2\gamma_s) - r\gamma_s\epsilon_z \quad \ldots\ldots (53)$$

where P_1 is positive in the negative r direction.

From reference 1,

$$U_n = \epsilon_{\theta n} \cdot r$$

$$\epsilon_{\theta n} = \alpha_n T_n + \frac{1}{E_n}\left[\sigma_{\theta n} - \gamma_n(\sigma_{rn} + \sigma_{zn})\right]$$

$$= \alpha_n T_n + \frac{1}{E_n}\left[(\sigma_{\theta n})_P + (\sigma_{\theta n})_T - \gamma_n\left[\left\{(\sigma_{rn})_P + (\sigma_{rn})_T\right\} + \sigma_{zn}\right]\right]$$

$$= \alpha_n T_n + \frac{1}{E_n}\left[\left\{(\sigma_{\theta n})_P - \gamma_n(\sigma_{rn})_P + (\sigma_{rn})_T - \gamma_n(\sigma_{rn})_T\right\}\right] - \frac{\gamma_n\sigma_{zn}}{E_n}$$

Substituting the values of $(\sigma_{\theta n})_p$ $(\sigma_{\theta n})_T$ $(\sigma_{rn})_p$, $(\sigma_{rn})_T$ and σ_{zn} (equations 34, 35, 36, 38, and 49) in the above expression, circumferential strain $(\epsilon_{n\theta})$ in the n^{th} cynlinder is obtained as:

$$
\begin{aligned}
\epsilon_{\theta n} = {} & \alpha_n T_n + \frac{1}{E_n}\Bigg[\Bigg\{ \frac{-R_n^2 \cdot R_{n+1}^2}{R_{n+1}^2 - R_n^2}(P_{n+1} - P_n)\frac{1}{r^2} \\
& + \frac{P_n \cdot R_n^2 - P_{n+1} \cdot R_{n+1}^2}{R_{n+1}^2 - R_n^2}\Bigg\} - \gamma_n \Bigg\{ \frac{R_n^2 \cdot R_{n+1}^2}{R_{n+1}^2 - R_n^2} \times \\
& (P_{n+1} - P_n)\frac{1}{r^2} + \frac{P_n \cdot R_n^2 - P_{n+1} \cdot R_{n+1}^2}{R_{n+1}^2 - R_n^2}\Bigg\}\Bigg] + \\
& \frac{1}{E_n}\Bigg[\Bigg\{ \frac{\alpha_n E_n}{(1-\gamma_n)}\left(\frac{1 + R_n^2}{r^2 (R_{n+1}^2 - R_n^2)} \int_{R_n}^{R_{n+1}} T.r.dr + \frac{1}{r^2} \right. \\
& \left. \int_{R_n}^{r} T.r.dr - T \right)\Bigg\} - \gamma_n \Bigg\{ \frac{\alpha_n \cdot E_n}{(1-\gamma_n)} \left(\frac{1 - R_n^2}{r^2 (R_{n+1}^2 - R_n^2)} \times \right. \\
& \left. \int_{R_n}^{R_{n+1}} T.r.dr - \int_{R_n}^{r} T.r.dr \right)\Bigg\}\Bigg] - \frac{\gamma_n}{E_n}\Bigg[2 \times \gamma_n \Bigg\{ \frac{P_n R_n^2 - P_{n+1} R_{n+1}^2}{R_{n+1}^2 - R_n^2} \Bigg\} \\
& \times \Bigg\{ \frac{2\alpha_n E_n \cdot \gamma_n}{(1-\gamma_n)(R_{n+1}^2 - R_n^2)} + \int_{R_n}^{R_{n+1}} T.r.dr - \frac{\gamma_n E_n}{(1-\gamma_n)} \cdot T_n - \alpha_n E_n T_n + \epsilon_{zn} E_n \Bigg\}\Bigg]
\end{aligned}
$$

By simplifying,

$$
\begin{aligned}
\epsilon_{\theta n} = {} & - \frac{R_n^2 - R_{n+1}^2}{R_{n+1}^2 - R_n^2}(P_{n+1} - P_n) \times \frac{1}{r^2} \times \frac{1 + \gamma_n}{E_n} + \\
& \frac{P_n \cdot R_n^2 - P_{n+1} \cdot R_{n+1}^2}{R_{n+1}^2 - R_n^2}\left(1 - \gamma_n - 2\gamma_n^2\right) + \frac{1 + \gamma_n}{1 - \gamma_n} \\
& \times \frac{\alpha_n}{r^2} \int_{R_n}^{r} T.r.dr + \frac{1 + \gamma_n}{1 - \gamma_n}\left\{ (1 - 2\gamma_n) + \frac{R_n^2}{r^2} \right\} + \\
& \frac{\alpha_n}{R_{n+1}^2 - R_n^2} \int_{R_n}^{R_{n+1}} T.r.dr - \gamma_n \epsilon_{zn}
\end{aligned}
$$

. . . . . (54)

The radial deflection in the n^{th} cylinders will be

$$U_n = r \cdot \epsilon_{\theta n}$$

The radial deflection at the interface will be the same, i.e.,

$$U_s(r)\Big|_{r=R_1} = U_1(r)\Big|_{r=R_1}$$

or

$$\frac{1+\gamma_s}{1-\gamma_s} \cdot \alpha_s \cdot \left[(1-2\gamma_s) \cdot \frac{1}{R_n} \int_0^{R_1} T.r.dr + \frac{1}{R_1} \int_0^{R_1} T.r.dr \right]$$

$$- R_1 (1+\gamma_s)(1-2\gamma_s) \cdot \frac{P_1}{E_s} - \gamma_s R_1 \epsilon_r = -\frac{1}{R_1} \cdot \frac{(1+\gamma_1)}{E_1} \times$$

$$\frac{R_1^2 \cdot R_2^2}{R_2^2 - R_1^2} (P_2 - P_1) + \frac{R_1 (1+\gamma_1)(1-2\gamma_1)}{E_1} \times \frac{(P_1 R_1^2 - P_2 R_2^2)}{R_2^2 - R_1^2} + \frac{\alpha_1 R_1}{R_2^2 - R_1^2}$$

$$\frac{(1+\gamma_1)}{(1-\gamma_1)} \left[(1-2\gamma_1) + \frac{R_1^2}{R_2^2} \right] \int_{R_1}^{R_2} T.r.dr - R_1 r_1 \epsilon_z +$$

$$\frac{(1+\gamma_1)}{(1-\gamma_1)} \cdot \frac{\alpha_1}{R_1^2} \int_{R_1}^{R_1} T \cdot r.dr$$

On simplification, this will give

$$- \left[\frac{(1+\gamma_s)(1-2\gamma_s)}{E_s} R_1^2 (R_2^2 - R_1^2) + \frac{1+\gamma_1}{E_1} R_2^2 \cdot R_1^2 + \right.$$

$$\left. \frac{(1+\gamma_1)(1-2\gamma_1)}{E_1} \cdot R_1^4 \right] P_1 + \left[\frac{2(1+\gamma_1)(1-\gamma_1) R_2^2 R_1^2}{E_1} \right] P_2$$

$$= -2(1+\gamma_s)(R_2^2 - R_1^2) \alpha_s \int_0^{R_1} T.r.dr +$$

$$2(1+\gamma_1)\cdot\alpha_1\cdot R_1^2\int_{R_1}^{R_2} T.\,r.\,dr + \epsilon_z R_1^2(R_2^2-R_1^2)(\gamma_s-\gamma_1) \qquad \ldots\ldots(55)$$

which could be further simplified as

$$a_oP_1 + b_oP_2 = D_o$$

where

$$a_o = -\frac{(1+\gamma_s)(1-2\gamma_s)}{E_s}R_1^2(R_2^2-R_1^2) + \frac{1+\gamma_1}{E_1}R_2^2\cdot R_1^2$$

$$+\frac{(1+\gamma_1)(1-2\gamma_1)}{E_1}\cdot R_1^4$$

$$b_o = \frac{2(1+\gamma_1)(1-\gamma_1)\cdot R_2^2\cdot R_1^2}{E_1}$$

and

$$D_o = -2(1+\gamma_s)(R_2^2-R_1^2)\alpha_s\int_0^{R_1} T.\,r.\,dr + 2(1+\gamma_1)$$

$$+\alpha_1 R_1^2\int_{R_1}^{R_2} T.\,r.\,dr + \epsilon_z R_1^2(R_2^2-R_1^2)(\gamma_s-\gamma_1)$$

But in the present discussion n = 1 and outside pressure $P_2 = 0$.

Hence,

$$a_o P_1 = D_o$$

When the value of ϵz will be substituted in the expression for D_o, it will contain the terms of P_1 and P_2. Substituting $P_2 = 0$ and transposing P_1 on the left hand side, the modified D_o, i.e., D_o', will be

$$D_o' = 2 \left\{ \frac{\alpha_s E_s (R_2^2 - R_1^2) R_1^2 (\gamma_s - \gamma_1)}{R_1^2 (E_s - E_1) + E_1 R_2^2} - (1 + \gamma_s) \times (R_2^2 - R_1^2) \alpha_s \right\} \times \int_0^{R_1} T.\, r.\, dr + 2 \left\{ \alpha_1 E_1 \times \frac{(R_2^2 - R_1^2) R_1^2 (\gamma_s - \gamma_1)}{R_1^2 (E_s - E_1) + E_1 R_2^2} + 2 (1 + \gamma_1) \alpha_1 R_1^2 \right\} \times \int_{R_1}^{R_2} T.\, r.\, dr \quad \ldots\ldots (56)$$

Now let

$$E_o' = 2 \left\{ G_o' \cdot \alpha_s E_s - (1 + \gamma_s)(R_2^2 - R_1^2) \alpha_s \right\}$$

and

$$F_o' = 2 \left\{ G_o' \alpha_1 E_1 + 2 (1 + \gamma_1)(\alpha_1)(R_1^2) \right\}$$

where

$$G_o' = \frac{(R_2^2 - R_1^2)(R_1^2)(\gamma_s - \gamma_1)}{R_1^2 (E_s - E_1) + E_1 R_2^2}$$

Introducing these expressions in equation 56, the modified D_o' will be given as

$$D_o' = E_o' \cdot \int_0^{R_1} T.\, r.\, dr + F_o' \int_{R_1}^{R_2} T.\, r.\, dr \quad \ldots\ldots (57)$$

The modified a_o and b_o are obtained as

$$a_o' = \frac{(1 + \gamma_s)(1 - 2\gamma_s)}{E_s} R_1^2 (R_2^2 - R_1^2) + \frac{1 + \gamma_1}{E_1} (R_2^2 - R_1^2)$$

$$+ \frac{(1+\gamma_1)(1-2\gamma_1)}{E_1} . R_1^4 - \frac{2 R_1^4 (R_2^2 - R_1^2)(\gamma_s - \gamma_1)^2}{R_1^2 (E_s - E_1) + E_1 R_2^2} \quad \ldots\ldots (58)$$

and

$$b_o' = \frac{2(1+\gamma_1)(1-\gamma_1) R_2^2 . R_1^2}{E_1} - \frac{R_2^2 . R_1^2 . \gamma_1 (R_2^2 - R_1^2)(\gamma_s - \gamma_1)}{R_1^2 (E_s . E_1) + E_1 R_2^2}$$

a_o' has units of $\frac{ft^4}{lb/ft^2} = \frac{ft^6}{lb}$

and

D_o' has units of $\frac{1}{{}^oF} \cdot ft^2 \cdot {}^oF \cdot ft^2 = ft^4$

Therefore,

$P_1 = \frac{D_o'}{a_o'}$ has units of $\frac{lb}{ft}$, which agrees with the actual units of P_1.

The heat transfer coefficient at the interface will be taken as

$$h_c = h_{co} - c . \sigma_r \Big|_{r = R_1} \quad \ldots\ldots (59)$$

where h_{co} is the initial heat transfer coefficient before the external temperature is applied at the outer periphery of the outer hollow cylinder. The stress in equation 59 is given by

$$\sigma_r (r) \Big|_{r = R_1} = - P_1 + (\sigma_r)_T \Big|_{r = R_1}$$

but $(\sigma r) T \Big|_{r = R_1} = 0$

Therefore

$$h_e = h_{co} + C \, . \, P_1 = h_{co} + C \frac{D_o'}{a_o'}$$

Substituting the values of d_o' and a_o' and letting $\frac{C \ E_o'}{a_o'} = C_1$ and

$\frac{C \ F_o'}{a_o'} = C_2$ in the resulting expression gives

$$h_c = h_{co} + C_1 \int_0^{R_1} T.r.dr + C_2 \int_{R_1}^{R_2} T.r.dr \quad \ldots . (59)$$

The above expression clearly shows that h_c depends upon the temperature distribution of both cylinders.

The initial temperature of the composite cylinder is taken as T_∞ and it will be treated as reference temperature. The stresses will be calculated from this reference temperature and hence at temperature T_∞. The assembly will be assumed to be in stress free state.

Therefore, equation 3.71 can be modified as

$$h_c = h_{co} + C_1 \int_0^{R_1} (T_s - T_\infty) \, r . dr + C_2 \int_{R_1}^{R_2} (T_1 - T_\infty) \, r . dr \quad \ldots\ldots\ldots . (60)$$

where

T_s = the temperature distribution in the solid core, and

T_∞ = the temperature distribution in the hollow cylinder.

ANALYTICAL STUDY OF HEAT CONDUCTION AND THERMAL STRESS IN SOLIDS WITH PRESSURE DEPENDENT CONTACT RESISTANCE

An Abstract of

A Thesis

Presented to the

Department of Mechanical Engineering Science

Brigham Young University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Dhruvkumar S. Patel

August 1969

\

ABSTRACT

The object of this thesis was to find the influence of pressure dependent contact resistance on the temperature distribution in solid metal bodies having rectangular and circular geometries.

A rectangular plate extending infinitely in two directions with one face being held against the rigid insulated wall and the other being held against the rigid wall with infinite conductance and a constant temperature difference from the plate was considered. The contact resistance at the non-insulated face was considered to be dependent on thermal stress produced at the face due to the temperature gradient.

For investigating the problem in circular coordinates, two concentric cylinders, the inner one being a solid cylinder, were considered. The outer periphery of the outer cylinder was subjected to a step change in temperature. The contact resistance at the interfact of two cylinders was considered to be dependent on thermal stresses produced by temperature gradient.

The integral method was used as an approximate method to determine the temperature distribution in both the cases. For obtaining numerical results, computer programs were prepared for both the cases.

APPROVED: