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Clean Indices of Common Rings

Benjamin L. Schoonmaker
Brigham Young University

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Pace P. Nielsen, Chair
David Cardon
Mark Hughes
Stephen Humphries
Paul Jenkins

Department of Mathematics

Brigham Young University

ABSTRACT

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Benjamin L. Schoonmaker
Department of Mathematics, BYU
Doctor of Philosophy

Lee and Zhou introduced the clean index of rings in 2004. Motivated by this work, Basnet and Bhattacharyya introduced both the weak clean index of rings and the nil clean index of rings and Cimpean and Danchev introduced the weakly nil clean index of rings. In this work, we calculate each of these indices for the rings $\mathbb{Z}/n\mathbb{Z}$ and matrix rings with entries in $\mathbb{Z}/n\mathbb{Z}$. A generalized index is also introduced.

Keywords: clean rings, clean index, weak clean rings, nil-clean rings, nil clean index

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Contents

Chapter 1. Introduction

For this paper, a ring R will be assumed to be an associative ring with identity. We use the notation idem(R) to denote the set of idempotents of R and $U(R)$ to denote the set of units of R. We also use $\mathbb{M}_n(R)$ to denote the $n \times n$ matrices with entries in R.

Definition 1.0.1. A ring is called a *clean ring* if every element can be written as a sum of a unit and an idempotent.

Clean rings were introduced by Nicholson in [11] as a subclass of the rings with the exchange property. The latter class are exactly the rings with the property that idempotents lift modulo every one-sided ideal. These properties are among the reasons researchers have stayed interested in clean rings for nearly 40 years. Special sub-classes of clean rings include strongly clean rings and uniquely clean rings. In strongly clean rings, the idempotent and unit commute; in uniquely clean rings there is only one way to write each element as a sum of a unit and an idempotent. Additionally, cleanness is defined at an element level.

Definition 1.0.2. An element $r \in R$ is called a *clean element* if $r = e + u$ for some $e \in \text{Idem}(R)$ and $u \in U(R)$. The expression $e + u$ is called a *clean expression* for r. In this case, we say r is *cleaned* by e.

In [9], Lee and Zhou introduced the clean index of rings as a way to study clean elements in rings.

Definition 1.0.3. Let $\mathcal{E}(a) = \{e \in \text{Idem}(R) : a - e \in U(R)\}$. The *clean index* of R, denoted in(R), is

$$\text{in}(R) = \sup\{|\mathcal{E}(a)| : a \in R\}.$$

If the supremum is not finite, we say the clean index is infinite and write in$(R) = \infty$.

In [2, Proposition 15], it was shown that clean rings also have the property that every element r can be written as $u - e$ where $u \in U(R)$ and $e \in \text{Idem}(R)$. This fact led to the study, beginning with Ahn and Anderson [1] of *weakly clean rings*, rings in which every element can be written as $u + e$ or $u - e$.

Combining the ideas of the clean index of rings and weakly clean rings, Basnet and Bhattacharyya introduced the weak clean index of rings in [3].

Definition 1.0.4. [3, Definition 1.1] For a ring R and $a \in R$, define the set $\chi(a) = \{e \in \text{Idem}(R) : a - e \in U(R) \text{ or } a + e \in U(R)\}$. The *weak clean index* of R, denoted $\text{Win}(R)$, is defined as $\text{Win}(R) = \sup_{a \in R}\{|\chi(a)|\}$, where $|\chi(a)|$ denotes the cardinality of $\chi(a)$ and the supremum is taken in $\mathbb{N} \cup \{\infty\}$.

In a study of some of the properties of clean rings, Diesl introduced nil clean rings in [6]. A ring is *nil clean* if every element can be written as the sum of an idempotent and a nilpotent element. Nil cleanness can likewise be defined at the elemental level.

Definition 1.0.5. An element $r \in R$ is called a *nil clean element* if $r = e + n$ for some $e \in \text{Idem}(R)$ and n a nilpotent element. The expression $e + n$ is called a *nil clean expression* for r. In this case, we say r is *nil cleaned* by e.

Inspired by the clean index, Basnet and Bhattacharyya created the nil clean index in [4].

Definition 1.0.6. For an element a in a ring R, we form the set $\eta(a) = \{e^2 = e \in R : a - e \text{ is nilpotent}\}$. The *nil clean index*, denoted $\text{Nin}(R)$, is $\sup_{a \in R} |\eta(a)|$ where the supremum is taken over $\mathbb{N} \cup \{\infty\}$.

Similar to the weak clean index, Cimpean and Danchev introduced the weakly nil-clean index in [5] as part of their work in characterizing uniquely weak nil-clean rings. In a fashion similar to previous work, for a ring R and $a \in R$, they define the set

$$\alpha(a) = \{e \in R : e^2 = e \text{ and } a - e \text{ or } a + e \text{ is nilpotent in } R\}.$$

The weakly nil-clean index is then defined as follows:

Definition 1.0.7. [5, Definitions 2.2, 2.3] The *weakly nil-clean index* of an element a in ring R is

$$\mathrm{wnc}(a) = |\alpha(a)|$$

and the *weakly nil-clean index* of R is

$$\mathrm{wnc}(R) = \sup\{\mathrm{wnc}(a) : a \in R\}.$$

This paper considers each of these indices separately. We first generalize some of the properties proved by Lee and Zhou in [9]. We then calculate the clean index of the rings $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{M}_2(\mathbb{Z}/n\mathbb{Z})$. The clean index of matrix rings of larger dimensions with entries in $\mathbb{Z}/n\mathbb{Z}$ is also discussed. We then calculate the weak clean index, the nil clean index, and the weakly nil-clean index of $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{M}_2(\mathbb{Z}/n\mathbb{Z})$. We also create a generalized version of the clean and nil clean index and consider some properties of this generalized index. Lastly, we identify some open questions that follow from this research.

Chapter 2. Clean Index of Rings

We begin with the clean index as introduced by Lee and Zhou. The following are some properties of the clean index given in Lemmas 1-3 in [9].

Proposition 2.0.1. *(i) For any ring R,* $\mathrm{in}(R) \geq 1$.

(ii) If a ring has at most n idempotents or at most n units, then $\mathrm{in}(R) \leq n$.

(iii) If S is a subring of R, possibly with different identity, then $\mathrm{in}(S) \leq \mathrm{in}(R)$.

(iv) Let $R = S \times T$ be a direct product of rings, then $\mathrm{in}(R) = \mathrm{in}(S)\,\mathrm{in}(T)$.

Proposition 2.0.1 part (iv) can be generalized to arbitrary products. In order to do so, we first prove the following lemma.

Lemma 2.0.2. *Let I be an indexing set and set $R := \prod_{i\in I} R_i$ with* $\mathrm{in}(R_i) = 1$ *for each $i \in I$. Then* $\mathrm{in}(R) = 1$.

Proof. By Proposition 2.0.1 (i), we know $\mathrm{in}(R) \geq 1$. Let $r \in R$ have clean expressions $r = e + u = f + v$ where $e, f \in \mathrm{Idem}(R)$ and $u, v \in U(R)$. Projecting these expressions to the i^{th} coordinate for any $i \in I$, we see $e_i + u_i$ and $f_i + v_i$ are clean expressions for r_i. Because $\mathrm{in}(R_i) = 1$, we must have $e_i = f_i$ and $u_i = v_i$. Since i was an arbitrary coordinate, we conclude that $e = f$ and $u = v$. Thus $|\mathcal{E}(r)| \leq 1$ for all $r \in R$, completing the proof. □

We now state the generalization of Proposition 2.0.1 (iv) for arbitrary products. This proof is similar to the proof of Proposition 2.0.1 (iv) that appears as [9, Lemma 3].

Theorem 2.0.3. *Let I be an indexing set. If $R = \prod_{i\in I} R_i$, then* $\mathrm{in}(R) = \prod_{i\in I} \mathrm{in}(R_i)$.

Proof. For any $i \in I$, we know R_i is isomorphic to a subring of R, so $\mathrm{in}(R_i) \leq \mathrm{in}(R)$ by Proposition 2.0.1 (iii). Thus, if $\mathrm{in}(R_i) = \infty$ for any $i \in I$, we have $\mathrm{in}(R) = \infty$ and the theorem holds.

We may now assume that $\mathrm{in}(R_i) = n_i < \infty$ for all $i \in I$. For each $i \in I$, there exists an $s_i \in R_i$ such that $|\mathcal{E}(s_i)| = n_i$. Let $s_i = e_{ij_i} + u_{ij_i}$ $(j_i = 1, \ldots, n_i)$ be clean expressions for s_i. Then $(s_i)_{i \in I} \in R_i$ has clean expressions

$$(s_i)_{i \in I} = (e_{ij_i})_{i \in I} + (u_{ij_i})_{i \in I}.$$

Thus $|\mathcal{E}((s_i)_{i \in I})| \geq \prod_{i \in I} n_i$. If $n_i > 1$ for more than finitely many $i \in I$, we have $|\mathcal{E}((s_i)_{i \in I})| \geq \infty$, proving the desired result.

Assume, then, that for only finitely many $i \in I$, we have $n_i > 1$. Then by possibly re-indexing, we can consider R as $S \times T$ where $S = \prod_{j=1}^{k} R_j$ for some finite k and where $T = \prod_{a \in A} R_a$ with $A = I - \{1, \ldots, k\}$ is an appropriate indexing set and $\mathrm{in}(R_a) = 1$ for each $a \in A$. By Proposition 2.0.1 (iv), $\mathrm{in}(S) = \prod_{j=1}^{k} n_j$ and by Lemma 2.0.2 $\mathrm{in}(T) = 1$. Applying Proposition 2.0.1 (iv) once more, we get $\mathrm{in}(R) = \prod_{j=1}^{k} n_j = \prod_{i \in I} n_i$ as desired. □

In [9] and [10], Lee and Zhou completely characterize rings of clean index 1, 2, 3, and 4. They also characterize clean rings of clean index 5 and show there are no clean rings with clean index 6 or 7.

In characterizing these rings, Lee and Zhou considered upper triangular rings. In their results, the following lemma is implied.

Lemma 2.0.4. *Let A and B be rings and ${}_AM_B$ be a bimodule. Define $R := \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. If $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in R$ has a clean expression in R,*

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} e & n \\ 0 & f \end{pmatrix} + \begin{pmatrix} u & p \\ 0 & v \end{pmatrix},$$

then $a = e + u$ and $b = f + v$ are clean expressions for a and b.

Proof. Let $\begin{pmatrix} e & n \\ 0 & f \end{pmatrix}$ be an idempotent in R. Then

$$\begin{pmatrix} e & n \\ 0 & f \end{pmatrix} = \begin{pmatrix} e & n \\ 0 & f \end{pmatrix}^2 = \begin{pmatrix} e^2 & en + nf \\ 0 & f^2 \end{pmatrix}.$$

Thus, $e^2 = e$ and $f^2 = f$, so e and f are idempotent. Now let $\left(\begin{smallmatrix} u & p \\ 0 & v \end{smallmatrix}\right)$ be a unit. Let $\left(\begin{smallmatrix} g & h \\ 0 & k \end{smallmatrix}\right)$ be its inverse. Then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u & p \\ 0 & v \end{pmatrix} \begin{pmatrix} g & h \\ 0 & k \end{pmatrix} = \begin{pmatrix} ug & uh + pk \\ 0 & vk \end{pmatrix}.$$

Therefore, $ug = gu = 1_A$ and $vk = kv = 1_B$, and u and v are units. The result follows. □

In [9, Theorem 12(3)], Lee and Zhou proved that if a ring R can be written as $R := \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ with $\text{in}(A) = \text{in}(B) = 1$ and M a bimodule of order 2, then $\text{in}(R) = 2$. With the help of the previous lemma, their result can be generalized as follows.

Theorem 2.0.5. *Let A and B be rings and ${}_AM_B$ be a bimodule. Define $R := \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$. If $|M| = n$ and $\text{in}(A) = \text{in}(B) = 1$, then $\text{in}(R) = n$.*

Proof. Lee and Zhou prove in [10, Lemma 4] that $\text{in}(R) \geq n$. We now show that $\text{in}(R) \leq n$. Let $X = \left(\begin{smallmatrix} a & m \\ 0 & b \end{smallmatrix}\right)$. Then we can write any clean expression for X as

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} e & p \\ 0 & f \end{pmatrix} + \begin{pmatrix} u & m-p \\ 0 & v \end{pmatrix}$$

where, by Lemma 2.0.4, $e + u = a$ and $f + v = b$ are clean expressions of a and b and $ep + pf = p$. This gives

$$\mathcal{E}(X) = \left\{ \begin{pmatrix} e & p \\ 0 & f \end{pmatrix} : e \in \mathcal{E}(a), b \in \mathcal{E}(b), ep + pf = p \right\}.$$

Since $|\mathcal{E}(a)| \leq 1$ and $|\mathcal{E}(b)| \leq 1$ by hypothesis, there can be a maximum of one possibility for each of e and f. So the size of $\mathcal{E}(X)$ is determined by the number of elements $p \in M$ satisfying $ep + pf = p$. Thus, $|\mathcal{E}(X)| \leq |M| = n$. □

2.1 Clean Index of Integers mod n

In this section, we calculate the clean index of the ring $\mathbb{Z}/n\mathbb{Z}$ for any integer n. To do this, we first note that if the prime factorization of n is $p_1^{m_1} \cdots p_k^{m_k}$, then by the Chinese Remainder theorem, we have

$$\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/p_1^{m_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{m_k}\mathbb{Z}.$$

By 2.0.1 part (iv), to calculate the clean index of $\mathbb{Z}/n\mathbb{Z}$, we need only find the clean index of $\mathbb{Z}/p^m\mathbb{Z}$ for prime numbers p.

As in [9], a ring R is an *elemental ring* if the only idempotents are trivial and $1 = u + v$ for some $u, v \in U(R)$. A class of elemental rings is the set of rings $\mathbb{Z}/p\mathbb{Z}$ for p an odd prime.

Lemma 2.1.1. *Let p be an odd prime. The rings $\mathbb{Z}/p^m\mathbb{Z}$ are elemental rings.*

Proof. Let $a \in \mathbb{Z}$. Then $\overline{a}$ is a unit of $\mathbb{Z}/p^m\mathbb{Z}$ if and only if $p \nmid a$. If $p \mid a$, then $\overline{a}^m = 0$, so $\overline{a}$ is nilpotent. Since a non-zero nilpotent element is not idempotent, the only idempotents are 0 and 1. As p is odd, $\overline{2}$ and $\overline{-1}$ are units which add to $\overline{1}$. Therefore $\mathbb{Z}/p^m\mathbb{Z}$ is an elemental ring. □

Corollary 2.1.2. *Let p be an odd prime. Then for $m \geq 1$, we have* $\text{in}(\mathbb{Z}/p^m\mathbb{Z}) = 2$.

Proof. This follows from the previous result and from [9, Theorem 12], where Lee and Zhou show that an elemental ring has clean index 2. □

We must also find the clean index of $\mathbb{Z}/p^m\mathbb{Z}$ when $p = 2$.

Lemma 2.1.3. *If $R = \mathbb{Z}/2^m\mathbb{Z}$ and $m \geq 1$, then* $\text{in}(R) = 1$.

Proof. Note that if $u \in U(R)$, then $u \equiv 1 \mod 2$. If $a \in R$ and $a \notin U(R)$, then $a^m \equiv 0 \mod 2^m$, so a is nilpotent. Thus the only idempotents are 0 and 1. If $u \in U(R)$, then $u = 0 + u$ is a clean expression, but $u \neq 1 + v$ for any other unit v because $1 + v \equiv 0 \mod 2$. If $a \notin U(R)$, then $a \equiv 0 \mod 2$. Thus $a = 1 + (-1 + a)$ is a clean expression, but $a \neq 0 + v$ for any $v \in U(R)$. Therefore, $\text{in}(R) = 1$. □

Theorem 2.1.4. *If $R = \mathbb{Z}/n\mathbb{Z}$, then* $\text{in}(R) = |\text{Idem}(R)| = 2^k$ *where k is the number of distinct odd primes that divide n.*

Proof. Let $n = 2^{m_0}p_1^{m_1}\cdots p_k^{m_k}$ be the prime decomposition of n, where $m_0 \geq 0$, $m_i \geq 1$ and p_i are distinct odd primes. Then by the Chinese Remainder Theorem,

$$R = \mathbb{Z}/2^{m_0}\mathbb{Z} \times \mathbb{Z}/p_1^{m_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{m_k}\mathbb{Z}.$$

As mentioned above, Proposition 2.0.1 part (iv) states that if $R = S \times T$, then $\text{in}(R) = \text{in}(S) \times \text{in}(T)$. Thus, by Corollary 2.1.2 and Lemma 2.1.3, $\text{in}(R) = 1 \cdot 2^k = 2^k$. □

Since we have calculated the clean index for the rings $\mathbb{Z}/p^m\mathbb{Z}$, one may wonder whether the clean index of the p-adic integers can be calculated. Indeed it can, though the calculation does not require the above results.

Theorem 2.1.5. *Let p be a prime and $\hat{\mathbb{Z}}_p$ denote the p-adic integers. Then*

$$\text{in}(\hat{\mathbb{Z}}_p) = \begin{cases} 1 & \textit{if } p = 2 \\ 2 & \textit{otherwise.} \end{cases}$$

Proof. It is well known that $\hat{\mathbb{Z}}_p$ is a local ring with maximal ideal $p\hat{\mathbb{Z}}_p$. Then [9, Lemma 1 (5)] shows that for a local ring R, we have $\text{in}(R) \leq 2$. Part (6) of the same lemma says that for a local ring R with Jacobson radical $J(R)$, we have $\text{in}(R) = 2$ if and only if $R/J(R) \ncong \mathbb{Z}/2\mathbb{Z}$. This theorem follows directly from these results. □

2.2 Clean Index of 2×2 Matrix Rings over Integers mod n

First, we note that for a 2×2 idempotent matrix $X := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ over a commutative ring to be an idempotent, we must have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \qquad (2.2.1)$$

From the main diagonal entries, we obtain $a - a^2 = d - d^2 = bc$. From the entries off the diagonal we need $b(a+d) = b$ and $c(a+d) = c$. It is well known that if the commutative ring is an integral domain, we have either $b = c = 0$ with a and d idempotents, or $a + d = 1$ and $bc = ad$ (i.e., $\mathrm{tr}(X) = 1$ and $\det(X) = 0$). This same description can be extended to commutative rings with only trivial idempotents, as in the following lemma.

Lemma 2.2.2. *Let R be a commutative ring with only trivial idempotents, and let $S := \mathbb{M}_2(R)$. Then $A \in S$ is an idempotent if and only if one of the following conditions hold:*

(i) A is a diagonal matrix with idempotent entries on the diagonal, or

(ii) A has trace 1 and determinant 0.

Proof. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. If A satifies condition (i), then clearly $A^2 = A$. Let A satisfy condition (ii). Since $\mathrm{tr}(A) = a + d = 1$, the equations from the off-diagonal entries in equation (2.2.1) are satisfied. We also get that $a = 1 - d$ and $d = 1 - a$. Then equations from the diagonal entries of equation (2.2.1), namely $a(1-a) = bc$ and $d(1-d) = bc$, are satisfied by substituting $bc = ad$, which is a result of $\det(A) = 0$. Therefore, $A^2 = A$.

To prove the forward direction of the lemma, we let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an idempotent matrix in S. If a is an idempotent in R, then $a - a^2 = 0$. Therefore $bc = 0$ and $d - d^2 = 0$. Hence d is an idempotent. Since R has only trivial idempotents, we have three cases. If both a and d are 0, then from the off-diagonal entry equations in equation (2.2.1), we have $b \cdot 0 = 0$ and $c \cdot 0 = 0$. So A satisfies condition (i). If $a = d = 1$, then from the off-diagonal entries in equation (2.2.1), $2b = b$ and $2c = c$. Again, we have $b = c = 0$, so A satisfies condition (i). If one of a and d is 0 and the other is 1, we have $\mathrm{tr}(A) = 1$. Since ad and bc are 0, we also have $\det(A) = 0$ and A satisfies condition (ii). Now we assume that $a - a^2 \neq 0$. We have $A^2 = A$, so we must have $\det(A)^2 = \det(A)$. Since $\det(A)$ is an idempotent and R has only trivial idempotents, we get $ad - bc = 0$ or $ad - bc = 1$. If $ad - bc = 1$, from the first entry in equation (2.2.1) we have $a(a + d - 1) = 1$, implying $a + d - 1$ is a unit. However, the second entry in equation (2.2.1) gives $b(a + d - 1) = 0$, implying $a + d - 1$ is 0 or a zero-divisor, a contradiction. We conclude that $\det(A) = 0$. This means that $ad = bc$. Using that fact and the diagonal entries from (2.2.1), we get

$$\begin{aligned} a + d &= a^2 + bc + d^2 + bc \\ &= a^2 + ad + d^2 + ad \\ &= (a + d)^2. \end{aligned}$$

So we get $a + d$ is an idempotent hence $a + d = 0$ or $a + d = 1$. If $a + d = 0$, we get $bc = 0$, which contradicts our assumption that $bc = a - a^2 \neq 0$. Therefore, $\operatorname{tr}(A) = 1$. □

We now use these properties of 2×2 idempotent matrices to calculate the clean index of 2×2 matrices over a field.

Theorem 2.2.3. *If F is a field, $|F| > 2$, and $R := \mathbb{M}_2(F)$, then $\operatorname{in}(R) = |\operatorname{idem}(R)|$. Furthermore, the elements that realize this index are precisely the elements $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$ for any $x \in F \setminus \{0, 1\}$.*

Proof. We need to show that there is some element $A \in R$ with a clean expression for each idempotent of R, that is, $\mathcal{E}(A) = \operatorname{idem}(R)$. Let $A = \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$ where $x \notin 0, 1$. Since $x \in U(F)$, we see $A \in U(R)$ and hence $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{E}(A)$. Similarly,

$$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} x - 1 & 0 \\ 0 & x - 1 \end{pmatrix}$$

is a clean expression since $x - 1 \in U(F)$. Every other idempotent in R takes the form $\left(\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right)$ where $bc = a(1 - a)$ for some $a, b, c \in F$. Each of these idempotents is in $\mathcal{E}(A)$ as long as $U = \left(\begin{smallmatrix} x-a & -b \\ -c & x-(1-a) \end{smallmatrix}\right)$ is a unit. So we calculate $\det(U)$:

$$\begin{aligned}
\det(U) &= (x-a)(x-(1-a)) - bc \\
&= x^2 - x(1-a) - xa + a(1-a) - a(1-a) \\
&= x^2 - x + xa - xa \\
&= x^2 - x.
\end{aligned}$$

Since $x \notin \{0,1\}$, $x^2 - x \neq 0$, and hence U is a unit. Therefore, $\mathcal{E}(A) = \text{idem}(R)$.

Now we want to show that if B is not one of these matrices, then $|\mathcal{E}(B)| < |\text{idem}(R)|$. To do this, we need only demonstrate that for some idempotent in R, there is not a clean expression of B. We have four cases, $B = 0$, $B = I_2$, B is diagonal but not a multiple of I, and B is not diagonal.

Case 1: If $B = 0$, then

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}.$$

Note that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is an idempotent, but $\left(\begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix}\right)$ is not a unit. So $|\mathcal{E}(0)| < |\text{idem}(R)|$.

Case 2: If $B = I_2$, then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Both of these matrices are non-unit idempotents, so $|\mathcal{E}(I_2)| < |\text{idem}(R)|$.

Case 3: Let $B = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ with $a \neq d$. Let $E = \left(\begin{smallmatrix} e & f \\ g & 1-e \end{smallmatrix}\right)$ be a non-trivial idempotent in $\mathbb{M}_2(F)$. Then the determinant of $B - E$ is

$$(a-e)(d-(1-e)) - fg = ad - a + ae - ed.$$

Setting the determinant equal to 0 and solving for e, we get $e = \frac{ad-a}{a-d}$, which is well-defined since $a \neq d$. So the matrix B is not cleaned by any notrivial idempotent with $e = \frac{ad-a}{a-d}$. Since at least one such idempotent exists, namely $\left(\begin{smallmatrix} e & e \\ 1-e & 1-e \end{smallmatrix}\right)$, $|\mathcal{E}(B)| < |\mathrm{idem}(R)|$.

Case 4: Let $B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a matrix that is not a diagonal matrix. Without loss of generality, assume $b \neq 0$. There is an idempotent with its first row having values a and b, namely $E = \left(\begin{smallmatrix} a & b \\ a(1-a)b^{-1} & 1-a \end{smallmatrix}\right)$. We see

$$B - E = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ a(1-a)b^{-1} & 1-a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c - a(1-a)b^{-1} & d-(1-a) \end{pmatrix} \notin U(R).$$

We have once again $|\mathcal{E}(B)| < |\mathrm{idem}(R)|$. □

Corollary 2.2.4. *Let p be an odd prime. For the ring $R := \mathbb{M}_2(\mathbb{Z}/p\mathbb{Z})$ we have* $\mathrm{in}(R) = p^2 + p + 2$.

Proof. From Theorem 2.2.3, we only need to show that $|\mathrm{idem}(R)| = p^2 + p + 2$. For a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, note that the only idempotents with $b = c = 0$ and $a = d$ are the trivial idempotents. So for the rest of the idempotents, we have $d = 1 - a$ and $ad = bc$. First let $a \notin \{0, 1\}$. Then $ad \neq 0$ since $\mathbb{Z}/p\mathbb{Z}$ is a field, and so there are $p - 1$ choices for b and then c is fixed by the choice of b. So there are $(p-2)(p-1)$ idempotents of this type. If $a = 0$ or $a = 1$, then $ad = 0$, so $b = 0$ or $c = 0$. If $b = 0$, there are p choices for c and, similarly, if $c = 0$, there are p choices for b. In this counting, the matrix with $b = c = 0$ is double counted, so there are $2p - 1$ idempotents for each of $a = 1$ and $a = 0$. In total we have

$$|\mathrm{idem}(R)| = 2 + (p-2)(p-1) + 2(2p-1) = p^2 + p + 2.$$

We conclude that $\mathrm{in}(R) = p^2 + p + 2$. □

Lee and Zhou in [10, Lemma 20(2)] found that $\mathrm{in}(\mathbb{M}_2(\mathbb{Z}/2\mathbb{Z})) = 5$. To fully characterize the clean index of $\mathbb{M}_2(\mathbb{Z}/n\mathbb{Z})$, we also examine the idempotents and clean expressions of $\mathbb{M}_2(\mathbb{Z}/p^m\mathbb{Z})$.

Lemma 2.2.5. *Each nontrivial idempotent of* $\mathbb{M}_2(\mathbb{Z}/p^m\mathbb{Z})$ *lifts to exactly* p^2 *idempotents of* $\mathbb{M}_2(\mathbb{Z}/p^{m+1}\mathbb{Z})$.

Proof. As a result of Lemma 2.2.2 any nontrivial idempotent in $\mathbb{M}_2(\mathbb{Z}/p^m\mathbb{Z})$ has the form $e := \left(\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right)$, so an idempotent lift in $\mathbb{M}_2(\mathbb{Z}/p^m\mathbb{Z})$ is uniquely of the form

$$\tilde{e} = \begin{pmatrix} a + up^m & b + vp^m \\ c + wp^m & (1-a) + tp^m \end{pmatrix}$$

where $0 \leq t, u, v, w \leq p - 1$. For $\tilde{e}$ to be an idempotent, the trace must be 1, so $t = -u$ and we can write

$$\tilde{e} = \begin{pmatrix} a + up^m & b + vp^m \\ c + wp^m & 1 - (a + up^m) \end{pmatrix}.$$

We also need the determinant to be zero. That is, we need

$$(a + up^m)(1 - (a + up^m)) - (b + vp^m)(c + wp^m) = 0.$$

Now

$$(a + up^m)(1 - (a + up^m)) - (b + vp^m)(c + wp^m) \equiv (1 - 2a)up^m - bwp^m - cvp^m \mod p^{m+1}.$$

If p does not divide $(1 - 2a)$, then for any v, w with $0 \leq v, w \leq p - 1$, there is a unique value for u that solves the equation $(1 - 2a)up^m - bwp^m - cvp^m \equiv 0 \mod p^{m+1}$, so there are p^2 idempotent lifts in that case. On the other hand, if p does divide $(1 - 2a)$, then p divides neither a nor $1 - a$. Since p does not divide $a(1 - a)$, it does not divide bc. So for any u, v, with $0 \leq u, v \leq p - 1$, there is a unique value for w that solves the equation $(1 - 2a)up^m - bwp^m - cvp^m \equiv 0 \mod p^{m+1}$. So, again, there are p^2 idempotent lifts. □

Remark 2.2.6. We note here that the only idempotent lifts of the zero and identity matrices are the zero and identity matrices. For instance, the matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in $\mathbb{M}_2(\mathbb{Z}/p^m\mathbb{Z})$ lifts to $\left(\begin{smallmatrix} ap^m & bp^m \\ cp^m & dp^m \end{smallmatrix}\right)$ in $\mathbb{M}_2(\mathbb{Z}/p^{m+1}\mathbb{Z})$. Since none of these lifts has trace equal to 1, the only idempotent lifts are those that have idempotents on the diagonal and $b = c = 0$. However, ap^m (resp. dp^m) is idempotent in $\mathbb{Z}/p^{m+1}\mathbb{Z}$ if and only if $a = 0$ (resp. $d = 0$), so the only idempotent lift of the zero matrix is the zero matrix. A similar argument holds for the identity matrix.

Lemma 2.2.7. *Let p be a prime. Each of the lifts of a unit in $\mathbb{M}_2(\mathbb{Z}/p^m\mathbb{Z})$, is a unit in $\mathbb{M}_2(\mathbb{Z}/p^{m+1}\mathbb{Z})$.*

Proof. If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a unit in $\mathbb{M}_2(\mathbb{Z}/p^m\mathbb{Z})$, then p does not divide $ad - bc$. The determinant of a lift is

$$(a + up^m)(d + xp^m) - (b + vp^m)(c + wp^m) = (ad - bc) + ap^m x + dp^m u - bp^m w - cp^m v.$$

Since p divides all the terms except $ad - bc$, and p does not divide $ad - bc$, we see p does not divide the determinant of the lift. So the determinant is a unit of $\mathbb{Z}/p^{m+1}\mathbb{Z}$ and the lift is a unit of $\mathbb{M}_2(\mathbb{Z}/p^{m+1}\mathbb{Z})$. □

Lemma 2.2.8. *Let p be a prime. If an element A of $\mathbb{M}_2(\mathbb{Z}/p^m\mathbb{Z})$ is cleaned by an idempotent E, then any lift of A is cleaned by each of the idempotent lifts of E.*

Proof. If $A - E$ is a unit, and A' is a lift of A and E' is an idempotent lift of E, then $A' - E'$ is a lift of $A - E$. By Lemma 2.2.7 $A' - E'$ is a unit. □

The converse is true because the projection map from $\mathbb{M}_2(\mathbb{Z}/p^{m+1}\mathbb{Z})$ to $\mathbb{M}_2(\mathbb{Z}/p^m\mathbb{Z})$ is a ring homomorphism, so idempotents project to idempotents and units project to units.

Theorem 2.2.9. *If p is an odd prime, then* $\operatorname{in}(\mathbb{M}_2(\mathbb{Z}/p^m\mathbb{Z})) = p^{2m} + p^{2m-1} + 2$. *Furthermore, the elements that realize this index are the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $a \equiv d \not\equiv 0, 1 \mod p$ and $b, c \equiv 0 \mod p$.*

Proof. We work by induction on $m \geq 1$. Corollary 2.2.4 provides the base case. For the inductive step, we let $A := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element with clean expressions for each idempotent in $\mathbb{M}_2(\mathbb{Z}/p^m\mathbb{Z})$. Two of the expressions are from the trivial idempotents, so there are $p^{2m} + p^{2m-1}$

expressions with nontrivial idempotents. By Lemma 2.2.8, there is a clean expression of the lift of A for each idempotent lift of an idempotent. From Lemma 2.2.5, there are p^2 idempotent lifts of each non-trivial idempotent. This gives us a clean expression of the lift of A with all $p^{2m+2}+p^{2m+1}+2$ idempotents of $\mathbb{M}_2(\mathbb{Z}_{p^{m+1}})$.

Finally, let $B := \left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right)$ be any matrix in $\mathbb{M}_2(\mathbb{Z}/p^{m+1}\mathbb{Z})$ and say $\mathcal{E}(B) = p^{m+2}+p^{2m+1}+2$. Let $\overline{B} := \left(\begin{smallmatrix} \overline{w} & \overline{x} \\ \overline{y} & \overline{z} \end{smallmatrix}\right)$ represent the image of B in $\mathbb{M}_2(\mathbb{Z}/p^m\mathbb{Z})$. Since B is cleaned by every idempotent in $M_2(\mathbb{Z}/p^{m+1}\mathbb{Z})$, we must have $\overline{B}$ cleaned by every idempotent in $M_2(\mathbb{Z}/p^m\mathbb{Z})$. By the inductive step, $\overline{w} \equiv \overline{x} \not\equiv 0, 1 \bmod p$ and $\overline{x}, \overline{y} \equiv 0 \bmod p$. Clearly, this gives us $w \equiv x \not\equiv 0, 1 \bmod p$ and $x, y \equiv 0 \bmod p$. □

Theorem 2.2.10. *The clean index of* $\mathbb{M}_2(\mathbb{Z}/2^m\mathbb{Z})$ *is* $2^{2m-1}+2^{2m-2}+2$*, and the elements that realize this are of the form* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *where either* $a \equiv b \equiv c \equiv 1 \bmod 2$ *and* $d \equiv 0 \bmod 2$ *or* $b \equiv c \equiv d \equiv 1 \bmod 2$ *and* $a \equiv 0 \bmod 2$.

Proof. Again, we proceed by induction on m. Lee and Zhou in [10, Lemma 20(2)] give that $\operatorname{in}(\mathbb{M}_2(\mathbb{Z}/2\mathbb{Z})) = 5$. The two elements that realize the clean index are $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Both have clean expressions for the two trivial idempotents. This provides the base case since every other element of $\mathbb{M}_2(\mathbb{Z}/2\mathbb{Z})$ is cleaned by at most three idempotents. The lifts of these elements can be cleaned by at most $4^{m-1}\cdot 3$ idempotents in $\mathbb{M}_2(\mathbb{Z}/2^m\mathbb{Z})$ while the lifts of $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ are cleaned by $4^{m-1}\cdot 3+2$ idempotents. Proceeding with the induction, assume that for some m, the theorem holds for $R = \mathbb{M}_2(\mathbb{Z}/2^m\mathbb{Z})$. Let A be a matrix that realizes the clean index in R. By Lemma 2.2.8 each lift A' of A is cleaned by the trivial idempotents and $2^2(2^{2m-1}+2^{2m-2})$ non-trivial idempotents. This gives $2^2(2^{2m-1}+2^{2m-2})+2$ clean expressions for A'.

Finally, assume that $B \in \mathbb{M}_2(\mathbb{Z}/2^{m+1}\mathbb{Z})$ has $|\mathcal{E}(B)| = 2^{2m+1}+2^{2m}+2$ clean expressions. If we project B to $\mathbb{M}_2(\mathbb{Z}/2\mathbb{Z})$, its image is cleaned by at most 3 nontrivial idempotents, so B is cleaned by at most $4^{2m}\cdot 3 = 2^{2m+1}+2^{2m}$ nontrivial idempotents. Thus B is cleaned by both trivial idempotents. Furthermore, the image of B in $\mathbb{M}_2(\mathbb{Z}/2^m\mathbb{Z})$ is cleaned by $2^{2m-1}+2^{m-1}+2$ idempotents. The induction hypothesis then completes the proof. □

Corollary 2.2.11. *The clean index for $R := \mathbb{M}_2(\mathbb{Z}/n\mathbb{Z})$, where the prime decomposition for n is $2^a p_1^{e_1} \cdots p_m^{e_m}$, is*

$$\mathrm{in}(R) = (p_1^{2e_1} + p_1^{2e_1-1} + 2) \cdots (p_m^{2e_m} + p_m^{2e_m-1} + 2)$$

if $a = 0$ and

$$\mathrm{in}(R) = (2^{2a-1} + 2^{2a-2} + 2)(p_1^{2e_1} + p_1^{2e_1-1} + 2) \cdots (p_m^{2e_m} + p_m^{2e_m-1} + 2)$$

otherwise.

Proof. This follows from Proposition 2.0.1 (4), Theorems 2.2.9 and 2.2.10 and the fact that matrix rings respect direct products. $\square$

2.3 Clean Index of $n \times n$ Matrix Rings over Integers mod p^k

The number of idempotents in the ring of $n \times n$ matrices over $\mathbb{Z}/p^k\mathbb{Z}$ is found in [12] to be

$$\sum_{r=0}^{n} \frac{p^{2r(n-r)(k-1)}\,|\mathrm{GL}(n,p)|}{|\mathrm{GL}(r,p)|\,|\mathrm{GL(n\text{-}r, p)}|}.$$

The order of the general linear group is determined by counting the number of linearly independent vectors that can fill the columns of each matrix as is given by

$$\mathrm{GL}(n,p) = \prod_{m=0}^{n-1} (p^n - p^m).$$

Combining the two formulas, we get that the number of idempotents in the $n \times n$ matrix ring over $\mathbb{Z}/p^k\mathbb{Z}$ is

$$\sum_{r=0}^{n} \frac{p^{2r(n-r)(k-1)} \prod_{m=0}^{n-1}(p^n - p^m)}{\prod_{m=0}^{r-1}(p^r - p^m) \prod_{m=0}^{n-r-1}(p^{n-r} - p^m)}.$$

The following result is a consequence of Lemma 3 in [8], although a different proof is given here.

Lemma 2.3.1. *Given a ring R, with an element $u \in U(R)$ such that $1 - u \in U(R)$, then u is cleaned by all the idempotents with which it commutes.*

Proof. Note that both u and $u - 1$ are units, so $0, 1 \in \mathcal{E}(u)$. If e is a nontrivial idempotent that commutes with u in R, then e decomposes R as $eR \oplus (1 - e)R$. Since each piece is u-invariant, we examine what $u - e$ does on each piece. If $es \in eR$, then $(u - e)(es) = ue(s) - e^2(s) = u(es) - es = (u - 1)(es)$. For $(1 - e)t \in (1 - e)R$, $(u - e)((1 - e)t) = u(1 - e)t - e(1 - e)t = u(1 - e)t$. Since $u - e$ acts like $u - 1$ and u on eR and $(1 - e)R$, respectively, and both u and $u - 1$ are units, $u - e$ is a unit. Thus, $e \in \mathcal{E}(u)$. □

From this result, we can derive the clean index for two types of rings.

Corollary 2.3.2. *Let $R = \mathbb{M}_n(\mathbb{Z}/p\mathbb{Z})$ with p an odd prime. Then* $\text{in}(R) = |\text{Idem}(R)|$.

Proof. Let $X := aI$ where $a \not\equiv 0, 1 \bmod p$. In $\mathbb{M}_n(\mathbb{Z}/p\mathbb{Z})$, the element X is a unit and $X - I$ is a unit. So by Lemma 2.3.1, X is cleaned by all idempotents with which it commutes. Since X is a multiple of the identity, it is in the center of R. Thus $\mathcal{E}(X) = \text{Idem}(R)$. □

Corollary 2.3.3. *Let $R := \mathbb{M}_n(\hat{\mathbb{Z}}_p)$ for p an odd prime. Then* $\text{in}(R) = \infty$.

Proof. Let $a \in \hat{\mathbb{Z}}$ with $a \not\equiv 0, 1 \bmod p$, and let $X := aI \in R$. Then X is central in R, and X is a unit. Additionally, $X - I$ is a unit. So by Lemma 2.3.1, we see that X is cleaned by all the idempotents of R. Since there are infinitely many idempotents in R, we have the desired result. □

As with the 2×2 matrices, we use lifting to extend the results of Corollary 2.3.2 to the rings $\mathbb{M}_n(\mathbb{Z}/p^k\mathbb{Z})$.

Lemma 2.3.4. *If $X \in \mathbb{M}_n(\mathbb{Z}/p^k\mathbb{Z})$ is cleaned by an idempotent E, then each lift $X' \in \mathbb{M}_n(\mathbb{Z}/p^{k+1}\mathbb{Z})$ is cleaned by each of the idempotent lifts of E.*

Proof. Define an ideal of $R = \mathbb{M}_n(\mathbb{Z}/p^{k+1}\mathbb{Z})$ as $J := p^kR$. Note that J is contained in the radical of R, and that we can consider $\mathbb{M}_n(\mathbb{Z}/p^k\mathbb{Z})$ as R/J. Since idempotents lift modulo a nil ideal [7, Theorem 21.28], we let E' be some idempotent lift of E in R. We need to show that $X' - E'$ is a unit in R. Clearly, $X' - E'$ is a lift of $X - E$. By [7, Proposition 4.8], $X' - E'$ is a unit. □

Theorem 2.3.5. *If $X \in \mathbb{M}_n(\mathbb{Z}/p^k\mathbb{Z})$ for p an odd prime, and if X is a lift of $aI \in \mathbb{M}_n(\mathbb{Z}/p\mathbb{Z})$ for $a \not\equiv 0, 1 \bmod p$, then $\mathcal{E}(X) = \text{Idem}(\mathbb{M}_n(\mathbb{Z}/p^k\mathbb{Z}))$.*

Proof. We prove the theorem by induction on k. Corollary 2.3.2 provides the base case. Using the same notation as Lemma 2.3.4, let $X \in R$ be a lift of aI. Then X is a lift of some $\bar{X} \in R/J(R)$, and $\bar{X}$ is also a lift of aI. By the induction hypothesis, $\bar{X}$ has a clean expression for all idempotents. By Lemma 2.3.4, X has a clean expression with each nontrivial idempotent of R. □

While the proof of Theorem 2.3.5 would work for $\mathbb{M}_n(\mathbb{Z}/2^k\mathbb{Z})$ with the appropriate element, the conditions for the clean index of $\mathbb{M}_n(\mathbb{Z}/2\mathbb{Z})$, which is not equal to the number of idempotents in $\mathbb{M}_n(\mathbb{Z}/2\mathbb{Z})$, have proved elusive. By direct calculation, the first from [10, Lemma 20], we have $\text{in}(\mathbb{M}_2(\mathbb{Z}/2\mathbb{Z})) = 5$, $\text{in}(\mathbb{M}_3(\mathbb{Z}/2\mathbb{Z})) = 30$, $\text{in}(\mathbb{M}_4(\mathbb{Z}/2\mathbb{Z})) = 382$, and $\text{in}(\mathbb{M}_5(\mathbb{Z}/2\mathbb{Z})) = 8466$. In the cases of $\mathbb{M}_3(\mathbb{Z}/2\mathbb{Z})$ and $\mathbb{M}_4(\mathbb{Z}/2\mathbb{Z})$ the calculations were done by creating a multi-set of all sums $e + u$ with $e \in \text{Idem}(R)$ and $u \in U(R)$, and calculating the greatest multiplicity of an element in the set. For $\mathbb{M}_5(\mathbb{Z}/2\mathbb{Z})$, the set of units was so large that the multi-set required too much memory. A different tactic was employed using the following application of Lemma 1 part (iv) in [9].

Lemma 2.3.6. *For $A, B \in \mathbb{M}_n(R)$, if A and B are similar, $|\mathcal{E}(A)| = |\mathcal{E}(B)|$.*

Because the invariant factors of a matrix are also invariant under similarity, the calculation for clean index involved finding all possible invariant factors and writing a matrix A for each factor. The computer then calculated the set $\mathcal{E}(A)$. We also made an investigation in the smaller matrix rings over $\mathbb{Z}/2\mathbb{Z}$ with regards to the invariant factors of matrices and how they related to the number of clean expressions. In presenting the results, we use the following notation. For $m \in \mathbb{Z}^+$, let E_m represent the set of matrices A with $|\mathcal{E}(A)| = m$.

For $\mathbb{M}_2(\mathbb{Z}/2\mathbb{Z})$ we have:

E_m	invariant factors	number of matrices
E_1	x, x or $x+1, x+1$	2
E_3	x^2 or x^2+1 or x^2+x	12
E_5	x^2+x+1	2

Furthermore, the elements in E_5 are cleaned by both trivial idempotents and half the idempotents of rank 1 (i.e., those that are similar to $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$).

For $\mathbb{M}_3(\mathbb{Z}/2\mathbb{Z})$ we have:

E_m	invariant factors	number of matrices
E_1	x, x, x or $x+1, x+1, x+1$	2
E_9	$x+1, x^2+1$ or x, x^2	12
E_{10}	x, x^2+x or $x+1, x^2+x$	56
E_{18}	x^3+x or x^3+x^2	168
E_{21}	x^3 or x^3+x^2+x+1	84
E_{23}	x^3+x^2+x or x^3+1	112
E_{30}	x^3+x^2+1 or x^3+x+1	48

Here the matrices that realize the clean index are cleaned by both trivial idempotents, half of the idempotents of rank 1, and half of the idempotents of rank 2.

For $\mathbb{M}_4(\mathbb{Z}/2\mathbb{Z})$ we have:

Results for $\mathbb{M}_4(\mathbb{Z}/2\mathbb{Z})$		
E_m	Invariant Factors	Number of Matrices
E_1	x, x, x, x or $x+1, x+1, x+1, x+1$	2
E_{33}	x, x, x^2 or $x+1, x+1, x^2+1$	210
E_{36}	x, x, x^2+x or $x+1, x+1, x^2+x$	240
E_{124}	x^2+x, x^2+x	560
E_{145}	x^2+1, x^2+1 or x^2, x^2	420
E_{148}	$x+1, x^3+1$ or x, x^3+x^2	5040
E_{156}	x, x^3+x or $x+1, x^3+x^2$	3360
E_{161}	x, x^3 or $x+1, x^3+x^2+x+1$	2520
E_{173}	x, x^3+x^2+x or $x+1, x^3+1$	2240
E_{236}	x^4+x^3 or x^4+x^2 or $x^4+x^3+x^2+x$	15120
E_{281}	x^4 or x^4+1	5040
E_{285}	x^4+x^3+x+1 or $x^4+x^3+x^2$	6720
E_{289}	x^4+x^2+x+1 or $x^4+x^3+x^2+1$	
	or x^4+x^3+x or x^4+x^2+x	11520
E_{362}	x^4+x+1 or x^4+x^3+1 or $x^4+x^3+x^2+x+1$	4032
E_{366}	x^4+x^2+1	1680
E_{382}	x^2+x+1, x^2+x+1	112

In $\mathbb{M}_4(\mathbb{Z}/2\mathbb{Z})$, the elements that realize the clean index are cleaned by both trivial idempotents, 60 of the idempotents of rank 1, 260 of rank 2, and 60 of rank 3. In these clean expressions, none of the idempotents of rank 1 or rank 3 have complements that clean, but 140 of the idempotents of rank 2 have cleaning complements.

For $\mathbb{M}_5(\mathbb{Z}/2\mathbb{Z})$ we get the following:

Results for $\mathbb{M}_5(\mathbb{Z}/2\mathbb{Z})$	
E_m	Invariant Factors
E_1	x, x, x, x, x or $x+1, x+1, x+1, x+1, x+1$
E_{129}	x, x, x, x^2 or $x+1, x+1, x+1, x^2+1$
E_{136}	x, x, x, x^2+x or $x+1, x+1, x+1, x^2+x$
E_{1729}	x, x^2, x^2 or $x+1, x^2+1, x^2+1$
E_{1744}	x, x^2+x, x^2+x or $x+1, x^2+x, x^2+x$
E_{1800}	x, x, x^3+x^2 or $x+1, x+1, x^3+x$
E_{1857}	x, x, x^3 or $x+1, x+1, x^3+x^2+x+1$
E_{1872}	x, x, x^3+x or $x+1, x+1, x^3+x^2$
E_{1937}	x, x, x^3+x^2+x or $x+1, x+1, x^3+1$
E_{3408}	x^2+x, x^3+x or x^2+x, x^3+x^2
E_{4168}	x^2, x^3+x^2 or x^2+1, x^3+x
E_{4296}	x, x^4+x^3 or $x+1, x^4+x^3+x^2+x$
E_{4304}	x, x^4+x^2 or x, x^4+x or $x+1, x^4+x^2$
E_{4368}	$x, x^4+x^3+x^2+x$ or $x+1, x^4+x^3$
E_{4577}	x^2, x^3 or x^2+1, x^3+x^2+x+1
E_{4769}	x, x^4 or $x+1, x^4+1$
E_{4785}	$x, x^4+x^3+x^2$ or $x+1, x^4+x^3+x+1$
E_{4881}	x, x^4+x^3+x or x, x^4+x^2+x or $x+1, x^4+x^3+x^2+1$
E_{6032}	$x^5+x^4+x^3+x^2$ or x^5+x^3
E_{6056}	x^5+x^4 or x^5+x
E_{6072}	$x^5+x^4+x^2+x$ or x^5+x^2

Continuation of Results for $M_5(\mathbb{Z}/2\mathbb{Z})$	
E_m	Invariant Factors
E_{6096}	$x^5 + x^3 + x^2 + x$ or $x^5 + x^4 + x^3 + x$
E_{7025}	x^5 or $x^5 + x^4 + x + 1$
E_{7041}	$x^5+x^4+x^3+1$ or $x^5+x^4+x^2$ or $x^5+x^4+x^3$ or $x^5+x^4+x^2$ or $x^5 + x^3 + x^2 + 1$ or $x^5 + x^2 + x + 1$
E_{7097}	x^5+x^2+x or x^5+x^3+x+1 or x^5+x^4+x or $x^5+x^4+x^2+1$ or $x^5 + x^4 + x^3 + x^3 + x^2 + x$ or $x^5 + 1$
E_{7113}	$x^5 + x^3 + x$ or $x^5 + x^4 + x^3 + x^2 + x + 1$
E_{7177}	$x^2 + x + 1, x^3 + x^2 + x$ or $x^2 + x + 1, x^3 + 1$
E_{8434}	$x^5 + x^2 + 1$ or $x^5 + x^3 + 1$ or $x^5 + x^3 + x^2 + x + 1$ or $x^5 + x^4 + x^2 + x + 1$ or $x^5 + x^4 + x^3 + x + 1$ or $x^5 + x^4 + x^3 + x^2 + 1$
E_{8466}	$x^5 + x + 1$ or $x^5 + x^4 + 1$

In $\mathbb{M}_5(\mathbb{Z}/2\mathbb{Z})$, if $A \in E_{8466}$, then A is cleaned by both trivial idempotents, 248 idempotents of rank 1, 3984 idempotents of rank 2, 3984 idempotents of rank 3, and 248 idempotents of rank 4. Of the idempotents of rank 1 and of rank 4, 112 of each have complements that clean A. Of the idempotents of rank 2 and rank 3, all 3984 have complements that clean A.

Chapter 3. Weak Clean Index

We move next to the weak clean index. Recall that the set $\chi(a)$ from Definition 1.0.4 is $\chi(a) = \{e \in \text{Idem}(R) : a - e \in U(R) \text{ or } a + e \in U(R)\}$. It is clear that for all $a \in R$, the statement $\mathcal{E}(a) \subseteq \chi(a)$ holds, so $\text{in}(R) \leq \text{Win}(R)$.

Basnet and Bhattacharyya prove in [3] many properties of the weak clean index similar to the properties of the clean index (see, for instance, Lemmas 2.1 and 2.2). Notably, however, the property regarding the weak clean index of a product of rings is weaker than the corresponding property of the clean index. It appears as Lemma 3.2 in [3], and is included here for future reference.

Lemma 3.0.1. *Let S and T be rings with $\text{Win}(S) = 1$ and let $R := S \times T$. Then $\text{Win}(R) = \text{Win}(T)$.*

3.1 Weak Clean Index of Integers mod n

As before, we compute the weak clean index for the rings $\mathbb{Z}/n\mathbb{Z}$. We begin by giving a result that appeared as Theorem 3.1 in [3]. Recall that a ring is abelian if all the idempotents are central.

Theorem 3.1.1. *For a ring R, $\text{Win}(R) = 1$ if and only if R is abelian and for any non-zero idempotent e, $e \neq u + v$ for any $u, v \in U(R)$.*

We use this result to prove our first result about the weak clean index of some of the rings $\mathbb{Z}/n\mathbb{Z}$.

Lemma 3.1.2. *For any $m \in \mathbb{N}$, we have $\text{Win}(\mathbb{Z}/2^m\mathbb{Z}) = 1$.*

Proof. As we showed in Lemma 2.1.3, the only nonzero idempotent of $\mathbb{Z}/2^m\mathbb{Z}$ is 1. Note that because units are precisely the elements in $\mathbb{Z}/2^m\mathbb{Z}$ that are equivalent to 1 mod 2, we cannot write 1 as the sum of two units of $\mathbb{Z}/2^m\mathbb{Z}$. Since $\mathbb{Z}/2^m\mathbb{Z}$ is also abelian, Theorem 3.1.1 applies and gives the result. $\square$

A combination of Lemma 3.1.2 and results from [3] are sufficient to calculate the weak clean index of all the rings $\mathbb{Z}/n\mathbb{Z}$.

Theorem 3.1.3. *For any $n \in \mathbb{Z}$, we have* $\mathrm{Win}(\mathbb{Z}/n\mathbb{Z}) = 2^k$ *where k is the number of distinct odd prime factors of n.*

Proof. Let $n = 2^{m_0} p_1^{m_1} \cdots p_k^{m_k}$ be the prime factorization of n where p_i are distinct odd primes. If $k = 0$, then Lemma 3.1.2 applies. Otherwise, we recognize that $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/2^{m_0}\mathbb{Z} \times (\mathbb{Z}/p_1^{m_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{m_k}\mathbb{Z})$. If $m_0 = 0$, then the first factor is trivial and can be ignored. If $m_0 \neq 0$, then by Lemma 3.1.2, we have $\mathrm{Win}(\mathbb{Z}/2^{m_0}\mathbb{Z}) = 1$. Thus, in either case, by Lemma 3.0.1, we have $\mathrm{Win}(\mathbb{Z}/n\mathbb{Z}) = \mathrm{Win}(\mathbb{Z}/p_1^{m_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{m_k}\mathbb{Z})$. Since the rings $\mathbb{Z}/p_1^{m_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{m_k}\mathbb{Z}$ are clean rings, and in each of these rings 2 is a unit, Lemma 2.1(vii) in [3] gives us $\mathrm{Win}(\mathbb{Z}/p_1^{m_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{m_k}\mathbb{Z}) = |\mathrm{Idem}(\mathbb{Z}/p_1^{m_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{m_k}\mathbb{Z})|$. Each ring $\mathbb{Z}/p_i^{m_i}\mathbb{Z}$ is elemental by Lemma 2.1.1, so each has 2 idempotents. We conclude that $\mathrm{Win}(\mathbb{Z}/p_1^{m_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{m_k}\mathbb{Z}) = 2^k$. □

We also here provide the weak clean index of the p-adic integers. This follows directly from results of Basnet and Bhattacharyya.

Corollary 3.1.4. *Let $R := \hat{\mathbb{Z}}_p$. Then* $\mathrm{Win}(R) = \mathrm{in}(R)$.

Proof. Lemma 2.1 parts (v) and (vi) in [3] in combination with Theorem 2.1.5 yield this result. □

3.2 Weak Clean Index of Matrix Rings over Integers mod n

We now turn our attention to the matrix rings over $\mathbb{Z}/n\mathbb{Z}$.

Theorem 3.2.1. *Let R be a ring with the property that $U(R) = U(R) + 2R$. Then* $\mathrm{Win}(R) = \mathrm{in}(R)$.

Proof. For any element $a \in R$ and any idempotent $e \in R$, the condition $U(R) = U(R) + 2R$ means that $a - e \in U(R)$ if and only if $a + e \in U(R)$. In this case, $\mathcal{E}(R) = \chi(R)$. □

Corollary 3.2.2. *Let $d \in \mathbb{N}$ and $R := \mathbb{M}_d(\mathbb{Z}/2^k\mathbb{Z})$. Then* $\mathrm{Win}(R) = \mathrm{in}(R)$.

Proof. We first note that the case $d = 1$ is clear from the results of Lemma 2.1.3 and Lemma 3.1.2.

To show that the corollary holds for $d > 1$, we show that any such R matches the conditions of Theorem 3.2.1. Recall that in $M_d(\mathbb{Z}/2^k\mathbb{Z})$, an element B is a unit if and only if $\det(B) \equiv 1 \mod 2$. We note that $\det(A)$ is a polynomial in the entries of A. We also note that $2E \equiv 0 \mod 2$. Therefore, we have $\det(A) \equiv \det(A + 2E) \mod 2$, and if A is a unit, so is $A + 2E$. We see then that $\mathbb{M}_d(\mathbb{Z}/2^k\mathbb{Z})$ meets the condition of Theorem 3.2.1, so we have $\mathrm{Win}(\mathbb{M}_d(\mathbb{Z}/2^k\mathbb{Z})) = \mathrm{in}(\mathbb{M}_d(\mathbb{Z}/2^k\mathbb{Z}))$.

□

Remark 3.2.3. In terms of clean expressions, it should be noted that Corollary 3.2.2 demonstrates that for a matrix $A \in \mathbb{M}_d(\mathbb{Z}/2^k\mathbb{Z})$ and idempotent $E \in M_d(\mathbb{Z}/2^k\mathbb{Z})$, every time $A+E$ is a unit, $A - E$ is also a unit. In other words, if there exists a unit $U \in M_d(\mathbb{Z}/2^k\mathbb{Z})$ with $A = U + E$, then there exists a unit $V \in M_d(\mathbb{Z}/2^k\mathbb{Z})$ with $A = V - E$.

Corollary 3.2.2 also allows us to calculate the weak clean index for all matrix rings over the rings $\mathbb{Z}/n\mathbb{Z}$.

Theorem 3.2.4. *Let $R := \mathbb{M}_d(\mathbb{Z}/n\mathbb{Z})$. Then* $\mathrm{Win}(R) = \mathrm{in}(R)$.

Proof. Let $n = p_1^{m_1} \cdots p_k^{m_k}$ be the prime factorization of n. Then

$$\mathbb{M}_d(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{M}_d(\mathbb{Z}/p_1^{m_1}\mathbb{Z}) \times \cdots \times \mathbb{M}_d(\mathbb{Z}/p_k^{m_k}\mathbb{Z}).$$

We prove this by cases on the primes p_i and on k.

Case 1: When $k = 1$ and $p_1 = 2$, we have the case proved in Corollary 3.2.2.

Case 2: When $k = 1$ and p_1 is an odd prime, we have shown in Theorem 2.3.5 that $\mathrm{in}(R) = |\mathrm{Idem}(R)|$. Since $\mathrm{in}(R) \leq \mathrm{Win}(R) \leq |\mathrm{Idem}(R)|$, we have the result.

Case 3: When $k > 1$ and p_i are all odd primes, by Theorem 2.3.5 together with Proposition 2.0.1 (iv), we again have $\mathrm{in}(R) = |\mathrm{Idem}(R)|$. Thus $\mathrm{Win}(R) = \mathrm{in}(R)$.

Case 4: Now let $k > 1$ and $p_1 = 2$. By definition, $\mathrm{in}(R) \leq \mathrm{Win}(R)$, so we need only show $\mathrm{Win}(R) \leq \mathrm{in}(R)$. Let $A \in R$. For any $E \in \chi(A)$, we can represent E as $(E_{\bmod 2^{m_1}}, E_{\bmod p_2^{m_2}}, \ldots, E_{\bmod p_k^{m_k}})$ where $E_{\bmod p_i^{m_i}}$ is the projection of E to $\mathbb{M}_d(\mathbb{Z}/p_i^{m_i}\mathbb{Z})$. As discussed in Remark 3.2.3, for each $E \in \chi(A)$, it must be the case that $E_{\bmod 2^{m_1}}$ cleans $A_{\bmod 2^{m_1}}$. Thus, there are no more than $\mathrm{in}(\mathbb{M}_n(\mathbb{Z}/2^{m_1}\mathbb{Z}))$ choices for $E_{\bmod 2^{m_1}}$. This gives

$$|\chi(A)| \leq \mathrm{in}(\mathbb{M}_n(\mathbb{Z}/2^{m_1}\mathbb{Z}))\, |\mathrm{Idem}(\mathbb{M}_n(\mathbb{Z}/p_2^{m_2}\mathbb{Z})| \cdots |\mathrm{Idem}(\mathbb{M}_n(\mathbb{Z}/p_k^{m_k}\mathbb{Z}))|\,.$$

Using Theorem 2.3.5 and Proposition 2.0.1 (iv) together, we see the right side of the inequality is $\mathrm{in}(R)$. So we have $|\chi(A)| \leq \mathrm{in}(R)$ for all $A \in R$. Therefore, $\mathrm{Win}(R) \leq \mathrm{in}(R)$ and we have proved the result in this case.

□

We now consider the nil clean index as defined in Definition 1.0.6. Basnet and Bhattacharyya proved in Lemmas 2.1, 2.2, and 2.3 of [4] that the properties in Proposition 2.0.1 hold if we replace the clean index with the nil clean index. They also showed in Lemma 2.8 of [4] that $\mathrm{Nin}(R) \leq \mathrm{in}(R)$ for all rings R with unity.

4.1 Nil Clean Index of Integers mod n

Computing the nil clean index of the rings $\mathbb{Z}/n\mathbb{Z}$ is straightforward. In [4, Theorem 3.2], Basnet and Bhattacharyya prove the following:

Theorem 4.1.1. $\mathrm{Nin}(R) = 1$ *if and only if* R *is abelian.*

As a direct consequence, we have the following:

Corollary 4.1.2. *For* $R := \mathbb{Z}/n\mathbb{Z}$*, we have* $\mathrm{Nin}(R) = 1$

4.2 Nil Clean Index of Matrix Rings over Integers mod n

To find the nil clean index of the 2×2 matrix rings over the rings $\mathbb{Z}/n\mathbb{Z}$, we prove the following theorem about 2×2 matrix rings over fields.

Theorem 4.2.1. *Let* F *be a finite field of order* n*. Let* $R := \mathbb{M}_2(F)$*. Then we have* $\mathrm{Nin}(R) = 2n - 1$*. Furthermore, if* $n \neq 2$*, the elements that realize this are precisely the nontrivial idempotents.*

Proof. Note that nil cleanness is preserved by similarity. Let $A := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then A is a nontrivial idempotent and all other nontrivial idempotents are similar to A. So we begin by demonstrating that $|\eta(A)| = 2p - 1$. Since neither $A - 0$ nor $A - 1$ are nilpotent, the elements of $\eta(A)$ can only be nontrivial idempotents. Nontrivial idempotents have the form $\left(\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right)$ with $bc = a - a^2$. So we need to determine when $\left(\begin{smallmatrix} 1-a & -b \\ -c & a-1 \end{smallmatrix}\right)$ is nilpotent.

If a 2×2 matrix with entries in a field is nilpotent, then its square is 0. So we compute the square of the matrix above.

$$\begin{pmatrix} 1-a & -b \\ -c & a-1 \end{pmatrix}^2 = \begin{pmatrix} bc+(1-a)^2 & 0 \\ 0 & bc+(a-1)^2 \end{pmatrix}$$

Thus, we have $bc = -(a-1)^2$. Since $bc = a - a^2$, we get $a - a^2 = -(a-1)$. Solving this equation, we get $a = 1$. This gives us $bc = 0$, so $b = 0$ or $c = 0$. If $b = 0$, there are n choices for c; likewise, there are n choices for b if $c = 0$. Since the choice of $b = 0$ and $c = 0$, is double counted, we have $2n-1$ distinct matrices. So $\eta(A) = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ c & 0 \end{smallmatrix}\right), b, c \in F\}$ and $|\eta(A)| = 2n-1$. Therefore $\mathrm{Nin}(R) \geq 2n-1$.

Now let $B \in R$ be a matrix that is not similar to A. We want to show that $|\eta(B)| \leq 2n-1$. Because every matrix over a field is similar to its rational cannonical form, we can reduce to dividing cases by the rational canonical form.

Case 1: Let $B = \left(\begin{smallmatrix} 0 & x \\ 1 & y \end{smallmatrix}\right)$. Then $0 \in \eta(B)$ when B is nilpotent. Since

$$\begin{pmatrix} 0 & x \\ 1 & y \end{pmatrix}^2 = \begin{pmatrix} x & xy \\ y & x+y^2 \end{pmatrix},$$

we see that $0 \in \eta(B)$ only when $x = 0$ and $y = 0$.

If $I_2 \in \eta(B)$, we have $\left(\begin{smallmatrix} -1 & x \\ 1 & y-1 \end{smallmatrix}\right)$ is nilpotent. Here

$$\begin{pmatrix} -1 & x \\ 1 & y-1 \end{pmatrix}^2 = \begin{pmatrix} 1+x & -x+x(y-1) \\ y-2 & x+y-1 \end{pmatrix}.$$

Thus we get $x = -1$ and $y = 2$.

For a nontrivial idempotent $\left(\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right)$ to be in $\eta(B)$ we must have that $\left(\begin{smallmatrix} -a & x-b \\ 1-c & y+a-1 \end{smallmatrix}\right)$ is nilpotent. Taking the square, we get

$$\begin{pmatrix} -a & x-b \\ 1-c & y+a-1 \end{pmatrix}^2 = \begin{pmatrix} a^2+(1-c)(-b+x) & (-b+x)(-1+y) \\ (1-c)(-1+y) & (1-c)(-b+x)+(-1+a+y)^2 \end{pmatrix}. \tag{4.2.2}$$

From the off-diagonal entries, we get either $y = 1$, or $b = x$ and $c = 1$.

Subcase A: If $b = x$ and $c = 1$, then from the equation $a^2 + (1 - c)(-b + x) = 0$ corresponding to the upper left entry we get $a = 0$. From the lower left entry, the equation $(1 - c)(-b + x) + (-1 + a + y)^2 = 0$, we get $y = 1$. So the idempotent that cleans B has the form

$$\begin{pmatrix} a & b \\ c & 1-a \end{pmatrix} = \begin{pmatrix} 0 & x \\ 1 & 1 \end{pmatrix}.$$

From the relationship $a - a^2 = bc$ we get that $x = 0$. Thus

$$\begin{pmatrix} 0 & x \\ 1 & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

an idempotent.

Subcase B: Suppose instead that $y = 1$. Then both the diagonal entries of equation 4.2.2 give

$$0 = a^2 + (1 - c)(-b + x) = a^2 - b + x + bc - cx.$$

Since $a = bc + a^2$ for an idempotent, this gives $a - b + x - cx = 0$. If $c = 0$, we have $x = b - a$. Because $bc = a - a^2$, we get $a = 0$ or $a = 1$. If $a = 0$, then $b = x$, so $\left(\begin{smallmatrix} 0 & x \\ 0 & 1 \end{smallmatrix}\right) \in \eta(B)$. If $a = 1$, then $b = x + 1$, so $\left(\begin{smallmatrix} 1 & x+1 \\ 0 & 0 \end{smallmatrix}\right) \in \eta(B)$. If instead $c \neq 0$, then $b = c^{-1}(a - a^2)$. Fixing c then fixes a and b. There are $n - 1$ choices for c. Thus, if $B = \left(\begin{smallmatrix} 0 & x \\ 1 & 1 \end{smallmatrix}\right)$, then $|\eta(B)| \leq n + 1$.

In summary, if $B = \left(\begin{smallmatrix} 0 & x \\ 1 & y \end{smallmatrix}\right)$ and is not a nontrivial idempotent, then $|\eta(B)| = 1$ if $\left(\begin{smallmatrix} 0 & x \\ 1 & y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 2 \end{smallmatrix}\right)$, $|\eta(B)| \leq n + 1$ if $\left(\begin{smallmatrix} 0 & x \\ 1 & y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & x \\ 1 & 1 \end{smallmatrix}\right)$ and $x \neq 0$, and $|\eta(B)| = 0$ otherwise.

Case 2: Let $B = \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$. Then if $0 \in \eta(B)$, we have $\left(\begin{smallmatrix} x^2 & 0 \\ 0 & y^2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ which happens if and only if $x = 0$ and $y = 0$. If $I_2 \in \eta(B)$, we see $\left(\begin{smallmatrix} (x-1)^2 & 0 \\ 0 & (y-1)^2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ which happens if and only if $x = 1$ and $y = 1$.

If $\left(\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right)$ is a nontrivial idempotent and in $\eta(B)$, then we have

$$\begin{pmatrix} x - a & -b \\ -c & y - 1 + a \end{pmatrix}^2 = \begin{pmatrix} bc + (x-a)^2 & -b(x - 1 + y) \\ -c(x - 1 + y) & bc + (-1 + a + y)^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.2.3}$$

From the off-diagonal entries, we get $x + y = 1$, or $b = 0$ and $c = 0$.

Subcase A: If $b = 0$ and $c = 0$, then $a - a^2 = 0$, so $a = 1$ or $a = 0$. If $a = 1$, then from the top left entry, we get $bc + (x - 1)^2 = (x - 1)^2 = 0$ so $x = 1$. From the bottom right entry, we get $bc + (a - 1 + y)^2 = y^2 = 0$ so $y = 0$. In this case, B is an idempotent. If $a = 0$, from the same equations we get $x = 0$ and $y = 1$, so B is an idempotent.

Subcase B: If instead of $b = 0$ and $c = 0$, we have $x + y = 1$, then we notice that if $x = 1$ or $x = 0$, then B is an idempotent. Assuming B is not idempotent, we see that the two main diagonal entries of equation 4.2.3 give the equation $bc + (x - a)^2 = 0$. Substituting $a - a^2$ for bc, we see that a is fixed by x. Notice that $a = 0$ or $a = 1$ only when $x = 0$ or $x = 1$. Then, since $bc = a - a^2 \neq 0$, there are $n - 1$ choices for b, which fixes c. So in this case $|\eta(B)| \leq n - 1$.

In summary, if $B = \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$ and B is not an idempotent, then $|\eta(B)| \leq n - 1$ if $x + y = 1$ and $|\eta(B)| = 0$ otherwise.

Since the two cases describe all possible rational canonical forms, we see that $\text{Nin}(R) = 2n - 1$. Notice also that when $n \geq 3$, we also have $2n - 1 > n + 1$, so the elements that realize the nil clean index are precisely the trivial idempotents. □

Corollary 4.2.4. *As a direct consequence, for a prime p, the nil clean index of $\mathbb{M}_2(\mathbb{Z}/p\mathbb{Z})$ is $2p - 1$.*

It should be noted that for $\mathbb{M}_2(\mathbb{F}_2)$, the elements that realize the nil clean index are the trivial idempotents and the elements $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. These latter two elements are also the elements that realize the clean index for $\mathbb{M}_2(\mathbb{F}_2)$. They are each either cleaned or nil cleaned by every idempotent.

The method of proof of theorem 4.2.1 also yields the following result.

Corollary 4.2.5. *If R has infinite cardinality, then* $\text{Nin}(\mathbb{M}_2(R)) = \infty$.

Proof. The element $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ has infinitely many nil-clean expressions of the form

$$\begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -r \\ 0 & 0 \end{pmatrix},$$

one for every element $r \in R$. □

To continue finding the nil clean index for the 2×2 matrices over $\mathbb{Z}/n\mathbb{Z}$, we again consider lifting from $\mathbb{M}_2(\mathbb{Z}/p^k\mathbb{Z})$ to $\mathbb{M}_2(\mathbb{Z}/p^{k+1}\mathbb{Z})$.

Lemma 4.2.6. *Let $N \in \mathbb{M}_2(\mathbb{Z}/p^k\mathbb{Z})$ be nilpotent, and N' a lift of N in $\mathbb{M}_2(\mathbb{Z}/p^{k+1}\mathbb{Z})$. Then N' is nilpotent.*

Proof. If N is nilpotent, then $N^m = 0$ for some positive integer m. So $(N')^m = \begin{pmatrix} ap^k & bp^k \\ cp^k & dp^k \end{pmatrix}$. Then

$$((N')^m)^2 = \begin{pmatrix} a^2p^{2k} + bcp^{2k} & abp^{2k} + bdp^{2k} \\ acp^{2k} + bcp^{2k} & cbp^{2k} + d^2p^{2k} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So N' is nilpotent. □

Lemma 4.2.7. *Let R be a ring with a nil ideal I, let $a + I \in R/I$, and let $e + I \in \eta(a + I)$. Then every idempotent lift $e' \in R$ of $e + I$ has $e' \in \eta(A)$.*

Proof. Because $e + I \in \eta(a + I)$, we have $(a + I) - (e + I)$ is nilpotent in R/I. Let n be an integer with $(a - e)^n + I = 0$. Then $(a - e')^n \in I$ for every idempotent lift e' of $e + I$. Since I is a nil-ideal, $(a - e')^n$ is nilpotent. Thus, $e' \in \eta(a)$. □

As a direct consequence we get the following lemma.

Lemma 4.2.8. *Let p be a prime number. If $A \in \mathbb{M}_2(\mathbb{Z}/p^k\mathbb{Z})$ is nil cleaned by an idempotent E, then a lift A' of A in $\mathbb{M}_2(\mathbb{Z}/p^{k+1}\mathbb{Z})$ is nil cleaned by each of the idempotent lifts of E.*

Using these lifting properties, we can determine which elements realize the nil clean index in $\mathbb{M}_2(\mathbb{Z}/p^k\mathbb{Z})$.

Theorem 4.2.9. *Let p be a prime number, $k \in \mathbb{N}$, and $R = \mathbb{M}_2(\mathbb{Z}/p^k\mathbb{Z})$. Then* $\mathrm{Nin}(R) = p^{2(k-1)}(2p-1)$ *and the elements that realize this are nil-cleaned only by nontrivial idempotents.*

Proof. We will prove by induction. Corollary 4.2.4 provides the base case. Note that as established in Theorem 4.2.1, the elements that realize the nil clean index are nil-cleaned only by nontrivial idempotents.

For the inductive step, assume that for $j \in \mathbb{N}$ and $S := \mathbb{M}_2(\mathbb{Z}/p^{j-1}\mathbb{Z})$, the theorem holds. Let $A \in S$ have $|\eta(A)| = \operatorname{Nin}(S)$. Then A is nil-cleaned only by nontrivial idempotents. Let A' be a lift of S in $\mathbb{M}_2(\mathbb{Z}/p^j\mathbb{Z})$. Then by Lemma 4.2.7 $\eta(A')$ contains the idempotents lifts of $\eta(A)$. By Lemma 2.2.5 there are p^2 such lifts. Thus we have $|\eta(A')| = p^2 \operatorname{Nin}(S)$ and $\operatorname{Nin}(\mathbb{M}_2(\mathbb{Z}/p^j\mathbb{Z})) \geq p^2 \operatorname{Nin}(S)$. Since nontrivial idempotents only lift to nontrivial idempotents, we have that A' is only nil-cleaned by nontrivial idempotents.

Let $B \in \mathbb{M}_2(\mathbb{Z}/p^j\mathbb{Z})$ and let $\overline{B}$ be its projection in S. Assume that $\left|\eta(\overline{B})\right| < \operatorname{Nin}(S)$. Then $\eta(B)$ contains the idempotent lifts of the elements of $\eta(\overline{B})$. Since $\overline{B}$ may be cleaned by either trivial or nontrivial idempotents, there are at most p^2 idempotent lifts of each element of $\eta(\overline{B})$. Thus $|\eta(B)| \leq p^2 \left|\eta(\overline{B})\right| < p^2 \operatorname{Nin}(S)$. Therfore, $\operatorname{Nin}(\mathbb{Z}/p^j\mathbb{Z}) = p^2 \operatorname{Nin}(S) = p^{2(j-1)}(2p-1)$. □

Corollary 4.2.10. *Let* $n \in \mathbb{N}$ *have prime factorization* $n = p_1^{m_1} \cdots p_k^{m_k}$*, and let* $R := \mathbb{M}_2(\mathbb{Z}/n\mathbb{Z})$*. Then* $\operatorname{Nin}(R) = (p_1^{2(m_1-1)}(2p_1 - 1)) \cdots (p_k^{2(m_k-1)}(2p_m - 1))$.

Proof. This follows immediately from Theorem 4.2.9 and the result for nil clean index analogous to Proposition 2.0.1 part (iv) (see [4, Lemma 2.2]). □

We also consider the weakly nil clean index. Cimpean and Danchev also demonstrate some of the properties identified for the other indices. Notably [5, Lemma 2.9] establishes the relationship between the four indices discussed thus far. For all rings with unity, R, the following inequality holds:

$$\mathrm{Nin}(R) \leq \mathrm{wnc}(R) \leq \mathrm{in}(R) \leq \mathrm{Win}(R).$$

5.1 Weakly Nil Clean Index of Integers mod n

As with the nil clean index of the rings $\mathbb{Z}/n\mathbb{Z}$, this follows directly from previous results. Proposition 2.11 of [5] asserts that $\mathrm{wnc}(R) = 1$ if and only if R is abelian. The following corollary to this proposition follows directly.

Corollary 5.1.1. *Let $n \in \mathbb{N}$. Then* $\mathrm{wnc}(\mathbb{Z}/n\mathbb{Z}) = 1$.

5.2 Weakly Nil Clean Index of Matrix Rings over Integers mod n

In [5], Cimpean and Danchev calculate $\mathrm{wnc}(\mathbb{M}_2(\mathbb{Z}/2\mathbb{Z})) = 3$ in Proposition 2.29 and also $\mathrm{wnc}(\mathbb{M}_2(\mathbb{Z}/3\mathbb{Z})) = 5$ in Example 2.26. It should be noted that for $R = \mathbb{M}_2(\mathbb{Z}/2\mathbb{Z})$ or $R = \mathbb{M}_2(\mathbb{Z}/3\mathbb{Z})$, we have $\mathrm{wnc}(R) = \mathrm{Nin}(R)$. Here we generalize this result.

Theorem 5.2.1. *Let F be a finite field of order n. Set $R := \mathbb{M}_2(F)$. Then* $\mathrm{wnc}(R) = \mathrm{Nin}(R)$.

Proof. For a ring, S, Let $\beta(a) := \{e \in S : e^2 = e \text{ and } a + e \text{ is nilpotent in } S\}$. Notice that $\alpha(a) = \beta(a) \cup \eta(a)$ where $\eta(a)$ is as defined above. To guide our proof, we observe that if $e \in S$ is an idempotent and $e \in \eta(a)$ for some $a \in S$, then $e \in \beta(-a)$.

Let $n = 2$. Then $a = -a$, so $\beta(a) = \eta(a)$ for all $a \in R$. Thus $\alpha(a) = \eta(a)$. The result immediately follows.

When F has order $n \geq 3$, we have from Theorem 4.2.1 the following sizes of η sets:

$$\begin{aligned}
\left|\eta\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right)\right| &= 2n-1, \\
\left|\eta\left(\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\right)\right| &= 1, \\
\left|\eta\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 2 \end{smallmatrix}\right)\right)\right| &= 1, \\
\left|\eta\left(\left(\begin{smallmatrix} 0 & x \\ 1 & 1 \end{smallmatrix}\right)\right)\right| &\leq n+1, \text{for } x \neq 0 \in F, \\
\left|\eta\left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right)\right| &= 1, \\
\left|\eta\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)\right| &= 1, \\
\left|\eta\left(\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)\right)\right| &\leq n-1, \text{for } x \neq 0 \in F, y \neq 0 \in F, x+y=1,
\end{aligned}$$

with $|\eta(A)| = 0$ for any A not similar to one of these matrices.

From our observation above, we conclude that

$$\begin{aligned}
\left|\beta\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right)\right| &= 2n-1, \\
\left|\beta\left(\left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}\right)\right)\right| &= 1, \\
\left|\beta\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & -2 \end{smallmatrix}\right)\right)\right| &= 1, \\
\left|\beta\left(\left(\begin{smallmatrix} 0 & -x \\ -1 & -1 \end{smallmatrix}\right)\right)\right| &\leq n+1, \text{for } x \neq 0 \in F, \\
\left|\beta\left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right)\right| &= 1, \\
\left|\beta\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right)\right| &= 1, \\
\left|\beta\left(\left(\begin{smallmatrix} -x & 0 \\ 0 & -y \end{smallmatrix}\right)\right)\right| &\leq n-1, \text{for } x \neq 0 \in F, y \neq 0 \in F, x+y=1,
\end{aligned}$$

with $|\beta(A)| = 0$ if A is not similar to one of these matrices.

Since the calculations in Theorem 4.2.1 were done up to similarity, we may consider the β sets in the same way. To count the elements in the α sets and determine the weakly nil clean index, we need only to determine the rational canonical form of the matrices which have a non-empty β set.

The rational canonical form of $\left(\begin{smallmatrix}-1 & 0\\ 0 & 0\end{smallmatrix}\right)$ is $\left(\begin{smallmatrix}0 & 0\\ 1 & -1\end{smallmatrix}\right)$. Its η set is empty.

The rational canonical form of $\left(\begin{smallmatrix}0 & 0\\ -1 & 0\end{smallmatrix}\right)$ is $\left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)$, so for a matrix A similar to $\left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)$, we have $|\eta(A)| = 1$. It should be noted that $\beta(A) = \eta(A) = \left\{\left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right)\right\}$. So in this case, $|\alpha(A)| = 1$.

The rational canonical form of $\left(\begin{smallmatrix}0 & 1\\ -1 & -2\end{smallmatrix}\right)$ is $\left(\begin{smallmatrix}0 & -1\\ 1 & -2\end{smallmatrix}\right)$. Except in the case that $F = \mathbb{Z}/3\mathbb{Z}$, these matrices have empty η sets. When $F = \mathbb{Z}/3\mathbb{Z}$, since $-2 = 1$, we have $\left(\begin{smallmatrix}0 & -1\\ 1 & -2\end{smallmatrix}\right) = \left(\begin{smallmatrix}0 & 2\\ 1 & 1\end{smallmatrix}\right)$. If A is similar to these matrices, then $|\eta(A)| \leq n+1$ and $|\alpha(A)| \leq n+2$. Because $n+2 = 2n-1$ when $n = 3$, we still have $|\alpha(A)| \leq \mathrm{Nin}(R)$ in this case.

The rational canonical form of $\left(\begin{smallmatrix}0 & -x\\ -1 & -1\end{smallmatrix}\right)$ is $\left(\begin{smallmatrix}0 & x\\ 1 & -1\end{smallmatrix}\right)$. Except in the case that $F = \mathbb{Z}/3\mathbb{Z}$, these matrices have empty η sets, so the α set has size less then or equal to $n+1$. When $F = \mathbb{Z}/3\mathbb{Z}$ and in the case that $x = -1$, we have $\left(\begin{smallmatrix}0 & x\\ 1 & -1\end{smallmatrix}\right) = \left(\begin{smallmatrix}0 & -1\\ 1 & 2\end{smallmatrix}\right)$. If A is similar to these matrices, then $|\eta(A)| = 1$ and $|\alpha(A)| \leq n+2$. However, since $n = 3$, we still have $|\alpha(A)| \leq 5 = \mathrm{Nin}(\mathbb{M}_2(\mathbb{Z}/3\mathbb{Z}))$.

The matrix $\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$ is its own rational canonical form. In this case, the η set is empty.

Finally, if $\left(\begin{smallmatrix}x & 0\\ 0 & y\end{smallmatrix}\right)$ with $x+y = 1$ is a rational canonical form, then so is $\left(\begin{smallmatrix}-x & 0\\ 0 & -y\end{smallmatrix}\right)$. However, if $x+y = 1$, then $-x-y = -1$, so $\eta\left(\left(\begin{smallmatrix}-x & 0\\ 0 & -y\end{smallmatrix}\right)\right)$ is empty.

So in every case, we have $|\alpha(A)| \leq 2n-1$, thus $\mathrm{wnc}(R) \leq \mathrm{Nin}(R)$, completing the proof. □

We now try to extend this result to all rings of 2×2 matrices with entries from $\mathbb{Z}/n\mathbb{Z}$. In order to do so, we need the following result.

Lemma 5.2.2. *Let p be a prime number and k a positive integer. Let $A \in \mathbb{M}_2(\mathbb{Z}/p^k\mathbb{Z})$ and E an idempotent in $\mathbb{M}_2(\mathbb{Z}/p^k\mathbb{Z})$. Let A' be a lift of A in $\mathbb{M}_2(\mathbb{Z}/p^{k+1}\mathbb{Z})$. If $A+E$ is nilpotent in $\mathbb{M}_2(\mathbb{Z}/p^k\mathbb{Z})$, then $A'+E'$ is nilpotent in $\mathbb{M}_2(\mathbb{Z}/p^{k+1}\mathbb{Z})$ for each idempotent lift E' of E.*

Proof. This follows as a special case of the proof of Lemma 4.2.7 with the appropriate sign changes. □

From the lemma, we can examine the weak clean index of the 2×2 matrices over $\mathbb{Z}/p^k\mathbb{Z}$ for prime numbers p.

Theorem 5.2.3. *Let p be a prime number and $k \geq 1$ and integer. Set $R := \mathbb{M}_2(\mathbb{Z}/p^k\mathbb{Z})$. Then* $\operatorname{wnc}(R) = \operatorname{Nin}(R)$.

Proof. We prove the theorem by induction on k. Theorem 5.2.1 provides the base case. For the inductive step, we need only show that $\operatorname{wnc}(R) \leq \operatorname{Nin}(R)$ since the reverse is true for all rings. Let $A \in R$ and $\overline{A}$ be the projection of A in $\mathbb{M}_2(\mathbb{Z}/p^{k-1}\mathbb{Z})$. Then, by the inductive hypothesis, $\alpha(\overline{A}) \leq \operatorname{Nin}(\mathbb{M}_2(\mathbb{Z}/p^{k-1}\mathbb{Z}))$. By Lemmas 4.2.8 and 5.2.2, for each $E \in \alpha(\overline{A})$, every idempotent lift of E is an element of $\alpha(A)$. Since each idempotent has at most p^2 idempotent lifts, $\alpha(A) \leq p^2 \operatorname{Nin}(\mathbb{M}_2(\mathbb{Z}/p^{k-1}\mathbb{Z}))$. From Theorem 4.2.9, we have $\operatorname{Nin}(\mathbb{M}_2(\mathbb{Z}/p^{k-1}\mathbb{Z})) = p^{2(k-2)}(2p-1)$. Thus,

$$\alpha(A) \leq p^{2(k-1)}(2p-1) = \operatorname{Nin}(R).$$

We have chosen A to be an arbitrary element of R, so $\operatorname{wnc}(R) \leq \operatorname{Nin}(R)$, proving the result. □

We can now use the following result about direct products to discover the weak nil-clean index for $\mathbb{M}_2(\mathbb{Z}/n\mathbb{Z})$.

Theorem 5.2.4. *Let R and S be rings with* $\operatorname{wnc}(R) = \operatorname{Nin}(R)$ *and* $\operatorname{wnc}(S) = \operatorname{Nin}(S)$. *Then* $\operatorname{wnc}(R \times S) = \operatorname{Nin}(R \times S)$.

Proof. By [5, Lemma 2.9] we have $\operatorname{Nin}(R \times S) \leq \operatorname{wnc}(R \times S)$, so here we only need to prove that $\operatorname{wnc}(R \times S) \leq \operatorname{Nin}(R \times S)$.

For $A \in R \times S$, let $A = (A_R, A_S)$. Let $A \in R \times S$ and $E \in \alpha(A)$. Then $A - E$ or $A + E$ is nilpotent in $R \times S$. Suppose by way of contradiction that $E_R \notin \alpha(A_R)$. Then neither $A_R - E_R$ nor $A_R + E_R$ is nilpotent in R. Then neither $A - E$ nor $A + E$ is nilpotent in $R \times S$, contradicting $E \in \alpha(A)$. Thus, $E_R \in \alpha(A_R)$ and, similarly, $E_S \in \alpha(A_S)$. Therefore, we have the following inequalities:

$$\begin{aligned}|\alpha(A)| &\leq |\alpha(A_R)|\,|\alpha(A_S)| \\ &\leq \mathrm{wnc}(R)\,\mathrm{wnc}(S) \\ &= \mathrm{Nin}(R)\,\mathrm{Nin}(S) \\ &= \mathrm{Nin}(R \times S)\end{aligned}$$

where the last line follows from [4, Lemma 2.3]. Since A is an arbitrary element, we have $\mathrm{wnc}(R \times S) \leq \mathrm{Nin}(R \times S)$, as desired. □

Corollary 5.2.5. *Let* $R := \mathbb{M}_2(\mathbb{Z}/n\mathbb{Z})$. *Then* $\mathrm{wnc}(R) = \mathrm{Nin}(R)$.

Proof. This follows directly from Theorems 5.2.3 and 5.2.4. □

By doing calculations on a computer, the nil clean index and weakly nil clean indices of the rings $\mathbb{M}_3(\mathbb{Z}/2\mathbb{Z})$ and $\mathbb{M}_3(\mathbb{Z}/3\mathbb{Z})$ were investigated. The calculation consisted of forming sets of all idempotent and nilpotent elements of the ring, and finding the sets $\eta(A)$ and $\alpha(A)$ for a matrix A representing each set of invariant factors for the matrices in each ring. It was calculated that $\mathrm{Nin}(\mathbb{M}_2(\mathbb{Z}/2\mathbb{Z})) = \mathrm{wnc}(\mathbb{M}_2(\mathbb{Z}/2\mathbb{Z})) = 10$ and, more notably, that $\mathrm{Nin}(\mathbb{M}_2(\mathbb{Z}/3\mathbb{Z})) = 33$ while $\mathrm{wnc}(\mathbb{M}_3(\mathbb{Z}/3\mathbb{Z})) = 36$. In [5], Cimpean and Danchev state that their Problem 4 relates to the existence of rings R for which $\mathrm{wnc}(R) > \mathrm{Nin}(R)$. While the example of $\mathbb{M}_3(\mathbb{Z}/3\mathbb{Z})$ doesn't relate to Problem 4, the weakly nil clean index of a direct product of rings, it does prove the existence of rings for which the weakly nil clean index is not equal to the nil clean index.

Chapter 6. A Generalized Index

We now try to generalize the idea of the clean and nil clean index and prove a property of the generalized index. We begin with a definition.

Definition 6.0.1. A property P is a *product property* if $a \in R$ has P in R and $b \in S$ has P in S if and only if $(a, b) \in R \times S$ has P in $R \times S$.

Note that an element being an idempotent, a unit, nilpotent, and regular are all examples of product properties. It follows then that being clean or nil-clean are also product properties. However, weakly clean and weakly nil-clean are not product properties.

We now create an index on product properties. We will follow a similar construction as the creation of the clean index.

Definition 6.0.2. Let R be a ring and P and Q be product properties on R. Let $a \in R$. Define the pair of a in R with respect to P and Q to be a pair (p, q) where $a = p + q$, p has P in R, and q has Q in R. Let $S_{R,P,Q}(a)$ be the set of such pairs. We define *the index of a in R with respect to P and Q* as

$$\operatorname{ind}_{R,P,Q}(a) = |S_{R,P,Q}(a)| .$$

Here we note that if we let P be the property "is an idempotent" and Q be the property "is a unit," then for a ring R with $a \in R$, we have $S_{R,P,Q}(a) = \mathcal{E}(a)$ as defined in Definition 1.0.3. If instead, we let Q be "is nilpotent," we have $S_{R,P,Q}(a) = \eta(a)$ as defined in 1.0.5.

Definition 6.0.3. Let R be a ring and P and Q be product properties. We define the index-set of R with respect to P and Q as

$$I_{P,Q}(R) := \{|S_{R,P,Q}(a)| : a \in R\}.$$

We see that the index-set can distinguish rings more finely than can the above indices.

Definition 6.0.4. Let R be a ring and P and Q be product properties. The *finite index of R with respect to P and Q* we denote by $\text{ind}_{P,Q}(R)$ and define it as $\sup I_{P,Q}(R)$ if the supremum exists and ∞ otherwise.

The index is called the finite index because it only distinguishes rings for which the index is finite. For rings with infinite index, we do not distinguish the cardinality of the index-set. Furthermore, in the case that the index-set has cardinality $\aleph_0$, we do not distinguish between those rings for which each element has finite index and those which contain elements of infinite index. However, the use of this index leads to a generalization of Proposition 2.0.1 part (iv) the analogous result for the nil clean index.

Theorem 6.0.5. *Let P and Q be product properties. Let J be an indexing set and $R := \prod_{j\in J} R_j$. Then $I_{P,Q}(R) = \{\prod_{j\in J} \left|S_{R_j,P,Q}(a_j)\right| : (a_j)$ is every possible tuple$\}$.*

Proof. We show that for each $(a_j) \in R$, we have $\text{ind}_{R,P,Q}((a_j)) = \prod_{j\in J} \text{ind}_{R_j,P,Q}(a_j)$. Let $(a_j) \in R$. Then, by definition, $\text{ind}_{R,P,Q}((a_j)) = |S_{R,P,Q}((a_j))|$. Now $S_{R,P,Q}((a_j))$ contains ordered pairs $((p_j),(q_j))$ where $(a_j) = (p_j) + (q_j)$, (p_j) has P in R, and (q_j) has Q in R. By the definition of addition in a product, we have $a_j = p_j + q_j$ for each $j \in J$. Since P and Q are product properties, for each $j \in J$, we see that p_j has P in R_j and q_j has Q in R_j. Thus the ordered pair $(p_j, q_j) \in S_{R_j,P,Q}(a_j)$ for each $j \in J$. Therefore, there is a bijection between $S_{R,P,Q}((a_j))$ and $\prod_{j\in J} S_{R_j,P,Q}(a_j)$. The cardinalities of these two sets are therefore equal. Since $I_{P,Q}(R)$ is the set of $|S_{R,P,Q}((a_j))|$ as (a_j) varies over R, we have the result. $\square$

Corollary 6.0.6. *Let P and Q be product properties. Let J be an indexing set and $R = \prod_{j\in J} R_j$. Then $\text{ind}_{P,Q}(R) = \prod_{j\in J} \text{ind}_{P,Q}(R_j)$.*

Proof. This follows directly from Theorem 6.0.5. $\square$

Since it is possible to choose product properties P and Q and a ring R such that $\text{ind}_{P,Q}(R) = 0$ and choose a second ring S such that $\text{ind}_{P,Q}(S) \neq 0$, Corollary 6.0.6 shows that the generalization of Proposition 2.0.1 part (iii) does not hold.

Chapter 7. Open Questions

This chapter deals with open questions suggested by this research. The first was mentioned already in chapter 1.

Question 7.0.1. What is $\text{in}(\mathbb{M}_n(\mathbb{Z}/2\mathbb{Z}))$?

Answering this question will allow a complete calculation of $\text{in}(\mathbb{M}_n(\mathbb{Z}/n\mathbb{Z}))$.

Two similar question have to do with other indexes of the same rings:

Question 7.0.2. What is $\text{Nin}(\mathbb{M}_n(\mathbb{Z}/n\mathbb{Z}))$?

Question 7.0.3. What is $\text{wnc}(\mathbb{M}_n(\mathbb{Z}/n\mathbb{Z}))$?

Knowing the answer to questions 7.0.2 and 7.0.3 may help in the characterization of rings for which $\text{Nin}(R) \neq \text{wnc}(R)$ as we have seen that $\mathbb{M}_3(\mathbb{Z}/3\mathbb{Z})$ provides one such example.

A question relating to a partial characterization of rings R for which $\text{Nin}(R) = \text{wnc}(R)$ is the following.

Question 7.0.4. If I is a nil-ideal and $\text{Nin}(R/I) = \text{wnc}(R/I)$, does $\text{Nin}(R) = \text{wnc}(R)$?

This would be a generalization of Theorem 4.2.9. The method of proof may be similar, but in establishing the equality in Theorem 4.2.9, we use that fact that each nontrivial idempotent lifts the same number of times (in this case p^2) for each lift from $\mathbb{M}_2(\mathbb{Z}/p^k\mathbb{Z})$ to $\mathbb{M}_2(\mathbb{Z}/p^{k+1}\mathbb{Z})$, which is not the case in general.

The multiplication property for the weak nil clean index as stated in Lemma 3.0.1 is fairly weak. Theorem 3.2.4 suggests a stronger property may be possible. The following questions provide two such strengthened versions.

Question 7.0.5. If $R = S \times T$, and $\text{Win}(S) = \text{in}(S)$, does $\text{Win}(R) = \text{Win}(S)\,\text{Win}(T)$?

Question 7.0.6. If $R = S \times T$, and $\text{Win}(S) = |\text{Idem}(S)|$, does $\text{Win}(R) = \text{Win}(S)\,\text{Win}(T)$?

The proof of Theorem 3.2.4 cannot be generalized directly as it uses the property particular to $\mathbb{M}_d(\mathbb{Z}/2\mathbb{Z})$ discussed in remark 3.2.3.

A question about characterizing the elements that achieve the nil-clean index in certain rings follows.

Question 7.0.7. If F is a finite field of order $n > 2$, and $R = \mathbb{M}_d(F)$, are the nontrivial idempotents precisely the elements that realize $\mathrm{Nin}(R)$?

This is suggested by Theorem 4.2.1 and holds true for $\mathbb{M}_3(\mathbb{Z}/3\mathbb{Z})$.

Bibliography

[1] Myung-Sook Ahn and D.D. Anderson. Weakly clean rings and almost clean rings. *Rocky Mountain J. Math.*, 36(3):783–798, June 2006.

[2] D. D. Anderson and V. P. Camillo. Commutative rings whose elements are a sum of a unit and idempotent. *Communications in Algebra*, 30(7):3327–3336, 2002.

[3] D. K. Basnet and J. Bhattacharyya. Weak clean index of a ring, December 2016. arXiv:1612.08357.

[4] Dhiren Kumar Basnet and Jayanta Bhattacharyya. Nil clean index of rings. *International Electronic Journal of Algebra*, 15:145–156, 2014.

[5] Peter Danchev and Andrada Cimpean. Weakly nil-clean index and uniquely weakly nil-clean rings. *International Electronic Journal of Algebra*, 21:180–197, 2017.

[6] Alexander J. Diesl. Nil clean rings. *Journal of Algebra*, 383:197–211, 2013.

[7] Tsit-Yuen Lam. *A first course in noncommutative rings*. Graduate texts in mathematics. Springer, New York, 2001.

[8] Tsiu-Kwen Lee and Yiqiang Zhou. A class of exchange rings. *Glasgow Mathematical Journal*, 50(3):509–522, September 2008.

[9] Tsiu-Kwen Lee and Yiqiang Zhou. Clean index of rings. *Communications in Algebra*, 40(3):807–822, 2012.

[10] Tsiu-Kwen Lee and Yiqiang Zhou. Rings of clean index 4 and applications. *Communications in Algebra*, 41(1):238–259, 2013.

[11] W. K. Nicholson. Lifting idempotents and exchange rings. *Transactions of the American Mathematical Society*, 229:269–278, 1977.

[12] Benjamin Steinberg. Number of idempotent $n \times n$ matrices over $\mathbb{Z}_m$? (answer). MathOverflow. URL:http://mathoverflow.net/q/121743 (version: 2013-02-15).