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The Arithmetic of Modular Grids

Grant Steven Molnar

A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

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ABSTRACT

The Arithmetic of Modular Grids

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Let $M_k^{(\infty)}(\Gamma, \nu)$ denote the space of weight k weakly holomorphic weight modular forms with poles only at the cusp (∞), and let $\widehat{M}_k^{(\infty)}(\Gamma, \nu) \subseteq M_k^{(\infty)}(\Gamma, \nu)$ denote the space of weight k weakly holomorphic modular forms in $M_k^{(\infty)}(\Gamma, \nu)$ which vanish at every cusp other than (∞). We construct canonical bases for these spaces in terms of Maass-Poincaré series, and show that the coefficients of these bases satisfy Zagier duality.

Keywords: weakly holomorphic modular forms, harmonic Maass forms, Zagier duality, Bruinier-Funke pairing

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Contents

Chapter 1. Introduction and Statement of Results

1.1 Introduction

In [36], Zagier defined canonical bases for spaces of weight 1/2 and weight 3/2 weakly holomorphic modular forms for $\Gamma_0(4)$ in the Kohnen plus space. The weight 1/2 basis elements $f_n = q^{-n} + \sum_{j\geq 1} a_{1/2}(n,j)q^j$ and weight 3/2 basis elements $g_m = q^{-m} + \sum_{j\geq 0} a_{3/2}(m,j)q^j$ have Fourier coefficients satisfying the striking duality

$$a_{1/2}(n,m) = -a_{3/2}(m,n),$$

so that the nth Fourier coefficient of the mth basis element in one weight is the negative of the mth Fourier coefficient of the nth basis element in the other weight. This duality is very clear when the Fourier expansions of the first few basis elements are computed, as follows.

$$\begin{array}{llrrrrl}
f_0(z) = & 1 & +\,2q\,+ & 2q^4\,+ & 0q^5\,+ & 0q^8\,+ & \cdots, \\
f_3(z) = & q^{-3} & -\,248q\,+ & 26752q^4\,- & 85995q^5\,+ & 1707264q^8\,+ & \cdots, \\
f_4(z) = & q^{-4} & +\,492q\,+ & 143376q^4\,+ & 565760q^5\,+ & 18473000q^8\,+ & \cdots, \\
f_7(z) = & q^{-7} & -\,4119q\,+ & 8288256q^4\,- & 52756480q^5\,+ & 5734772736q^8\,+ & \cdots, \\
f_8(z) = & q^{-8} & +\,7256q\,+ & 26124256q^4\,+ & 190356480q^5\,+ & 29071392966q^8\,+ & \cdots.
\end{array}$$

$$\begin{array}{llrrrrl}
g_1(z) = & q^{-1}-2 & +\,248q^3\,- & 492q^4\,+ & 4119q^7\,- & 7256q^8\,+ & \cdots, \\
g_4(z) = & q^{-4}-2 & -\,26752q^3\,- & 143376q^4\,- & 8288256q^7\,- & 26124256q^8\,+ & \cdots, \\
g_5(z) = & q^{-5}+0 & +\,85995q^3\,- & 565760q^4\,+ & 52756480q^7\,- & 190356480q^8\,+ & \cdots, \\
g_8(z) = & q^{-8}+0 & -\,1707264q^3\,- & 18473000q^4\,- & 5734772736q^7\,- & 29071392966q^8\,+ & \cdots.
\end{array}$$

Looking at these expressions, we see that the coefficients appearing in the rows of one basis appear, with opposite sign, in the columns of the other basis.

Thus, these coefficients form a "modular grid", where each row contains the coefficients of a basis element of one space, and each column contains the negatives of the coefficients of the basis elements of the other weight. Zagier gave two proofs of this duality. The first proof relies on showing that these basis elements have a common two-variable generating function that encodes the modular grid as the Fourier coefficients in both variables. Similar generating functions for bases of modular forms have been used by Eichler [17, 18], Borcherds [5, 6], Faber [20, 21] and others [4] in studying the Hecke algebra and the zeros of modular forms. The second proof of duality given by Zagier, attributed to Kaneko, is simpler and involves considering the constant term of the product $f_n(z)g_m(z)$, which is $a(n,m)+b(m,n)$. However $f_n(z)g_m(z)$ lives in a space of weight 2 forms which are derivatives, and hence have constant term equal to 0.

Duke and Jenkins [16] used similar generating functions to prove duality between spaces of forms on $\mathrm{SL}_2(\mathbb{Z})$ of any even integer weights k and $2-k$; additionally they used these generating functions to study the zeros of these forms and prove congruences for the Fourier coefficients of the basis elements. Zagier duality has been proven in various settings [17, 18, 5, 6, 4, 1, 2, 7, 8, 11, 12, 13, 14, 19, 23, 25, 26, 27, 28, 29, 30, 34, 35, 37, 38, 22]. In many cases, Zagier's and Kaneko's proofs are difficult to adapt directly, and authors have adapted them in ad hoc ways to obtain duality for the modular forms they study. In general, generating functions become increasingly complicated and difficult to find in the presence of higher dimension spaces of cusp forms, while the existence of weight 2 Eisenstein series in higher levels allow for non-zero constant terms, which interfere with Kaneko's proof in these settings.

Harmonic Maass forms provide an alternate path to duality. The Maass–Poincaré series form a basis for a space of harmonic Maass forms with poles at specified cusps. Explicit formulas for their coefficients [8] demonstrate that duality holds in this setting, creating "mock modular grids" [11, 3]. However, unlike the proofs of Zagier and Kaneko, these formulas do not give any intuitive reason why we should expect duality, only that it exists.

1.2 Statement of Results

In this thesis, we generalize both Zagier's and Kaneko's proofs for duality to a very general setting, without relying on explicit formulas for coefficients of the Maass-Poincaré series.

In particular, suppose $k \in \frac{1}{2}\mathbb{Z}$ with $k \neq \frac{1}{2}, 1, \frac{3}{2}$, suppose $N \in \mathbb{N}$, and let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ be any discrete group with finite index above $\Gamma(N)$. Moreover, let ν be a multiplier for Γ which acts trivially on $\{\mu^{-1} \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \mu \mid \mu \in \mathrm{SL}_2(\mathbb{Z})\}$.

If k is as above, we let $M_k^{(\infty)}(\Gamma, \nu)$ denote the space of weight k weakly holomorphic modular forms for Γ and ν with poles allowed only at the cusp (∞), and let $\widehat{M}_k^{(\infty)}(\Gamma, \nu) \subseteq M_k^{(\infty)}(\Gamma, \nu)$ be the subspace of weakly holomorphic modular forms which vanish at every cusp besides (∞).

Remark. The notation $S_k^\sharp$ is used to denote analogous spaces in [1, 23, 26, 27, 28, 29, 30, 35]. We have altered the notation here to avoid confusion with $S_k^!$, which represents the space of weakly holomorphic cusp forms which have vanishing constant terms at *every* cusp.

The multiplier ν governs which powers of $q \coloneqq e^{2\pi i z}$ can occur in the Fourier expansions of modular forms for Γ and ν. This is independent of the weight, as can be seen by considering the image of ν on the translations in Γ (See Chapter 2). Let $\mathbb{Z}^{(\nu)} \subseteq \frac{1}{N}\mathbb{Z}$ be defined as in (2.2), with numerators satisfying certain congruence conditions defined by ν. We note that $\mathbb{Z}^{(\overline{\nu})}$ will satisfy opposite congruence conditions, so that $\mathbb{Z}^{(\overline{\nu})} = -\mathbb{Z}^{(\nu)}$. In Chapter 4 we define canonical bases for $M_k^{(\infty)}(\Gamma, \nu)$ and $\widehat{M}_k^{(\infty)}(\Gamma, \nu)$ given by

$$\{f_{k,m}^{(\nu)}(z) := q^{-m} + \sum_{\substack{n\in\mathbb{Z}^{(\nu)}\\ n\notin\widetilde{I}_k^{(\nu)}}} a_k^{(\nu)}(m,n)q^n \quad | \quad m \in \widetilde{I}_k^{(\nu)}\},$$

$$\{g_{k,m}^{(\nu)}(z) := q^{-m} + \sum_{\substack{n\in\mathbb{Z}^{(\nu)}\\ n\notin\widetilde{J}_k^{(\nu)}}} b_k^{(\nu)}(m,n)q^n \quad | \quad m \in \widetilde{J}_k^{(\nu)}\}$$

respectively, where the indexing sets $\widetilde{I}_k^{(\nu)}$ and $\widetilde{J}_k^{(\nu)}$ are defined in Chapter 4.

With this notation we have the following.

Theorem 1.1. *If $k \neq \frac{1}{2}, 1, \frac{3}{2}$, then the spaces $M_{2-k}^{(\infty)}(\Gamma,\nu)$ and $\widehat{M}_k^{(\infty)}(\Gamma,\overline{\nu})$ are dual. In particular, the coefficients of the forms $f_{2-k,m}^{(\nu)}$ and $g_{k,n}^{(\overline{\nu})}$ satisfy*

$$a_{2-k}^{(\nu)}(m,n) = -b_k^{(\overline{\nu})}(n,m).$$

This theorem includes the main results of [16, 1, 23, 26, 27, 28, 29, 30, 35] as corollaries. Our proof of this theorem shows that duality can be seen as a consequence of the Bruinier–Funke pairing; this generalizes Kaneko's proof in a way that applies equally well to weakly holomorphic forms of essentially arbitrary level.

Chapter 2. Modularity and Cusps

Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. Given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, let $j(\gamma, z) := (cz + d)$. If $k \in \frac{1}{2}\mathbb{Z}$, then a weight k multiplier for Γ is a map $\nu : \Gamma \to \mathbb{C}^\times$ satisfying the multiplication law

$$\nu(\gamma_1)\nu(\gamma_2)j(\gamma_1, \gamma_2 z)^k j(\gamma_2, z)^k = \nu(\gamma_1\gamma_2)j(\gamma_1\gamma_2, z)^k, \tag{2.1}$$

for any matrices $\gamma_1, \gamma_2 \in \Gamma$.

If k is an integer, this reduces to the condition that ν is a multiplicative homomorphism. If k is a half-integer then (here and throughout) we take the principal branch of the square root, and this multiplication law is closely related to the group law of the metaplectic group [31]. We may extend this notation to $\mathrm{GL}_2^+(\mathbb{R})$ (the 2×2 real matrices with positive determinant) by defining

$$j(\gamma, z) \coloneqq \det(\gamma)^{-1/2}(cz + d),$$

and requiring both that ν is trivial on scalar matrices $\gamma = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ with $a > 0$ and that

$$\nu(-\mathrm{I}) \coloneqq (-1)^{-k} = j(-\mathrm{I}, z)^{-k}.$$

Given a function $f : \mathbb{H} \to \mathbb{C}$ and $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$, the action of the weight k Petersson slash operator is defined by

$$f(z)|_k\gamma \coloneqq j(\gamma, z)^{-k} f(\gamma z).$$

If a group Γ has a weight k multiplier system ν, then for $\gamma \in \Gamma$ we define

$$f(z)|_{k,\nu}\gamma \coloneqq \nu(\gamma)^{-1} j(\gamma, z)^{-k} f(\gamma z).$$

Equation (2.1) is equivalent to

$$f(z)|_{k,\nu}\gamma_1|_{k,\nu}\gamma_2 = f(z)|_{k,\nu}(\gamma_1\gamma_2).$$

We say that f is modular of weight k for Γ with multiplier ν if $f(z)|_{k,\nu}\gamma = f(z)$ for every $\gamma \in \Gamma$.

By construction, $\bullet|_{k,\nu}\gamma$ acts trivially whenever $\gamma = a\mathrm{I} \in \mathbb{R}_{\neq 0}\mathrm{I}$. Because of this, we may identify $\mathrm{SL}_2(\mathbb{R})$ with $\mathrm{GL}_2^+(\mathbb{R})/\mathbb{R}^+ I$. In particular, we may scale matrices in a subgroup $\Gamma \subseteq \mathrm{GL}_2^+(\mathbb{R})$ as convenient.

2.1 Cusps

Given a discrete subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$, we define the modular curve $Y(\Gamma) \coloneqq \Gamma\backslash\mathbb{H}$. The compactified modular curve $X(\Gamma)$ (if it exists) is the unique connected compact Riemann surface containing $Y(\Gamma)$. It turns out that $X(\Gamma)$ may be obtained from $Y(\Gamma)$ by adding a finite collection of points $\Omega(\Gamma)$ called cusps to $Y(\Gamma)$ (see Chapter 2 of [15]).

The extended rationals and the extended upper half plane are the sets

$$\mathbb{Q}^\star \coloneqq \mathbb{Q} \cup \{\infty\} \quad \text{and} \quad \mathbb{H}^\star \coloneqq \mathbb{H} \cup \mathbb{Q}^\star,$$

respectively. Then the curve $X(N) \coloneqq X(\Gamma(N))$ can be described as the quotient with the extended upper half plane $X(N) \simeq \Gamma(N)\backslash\mathbb{H}^\star$, with $\Omega(N) \coloneqq \Omega(\Gamma(N))$ given by the equivalence classes $\Gamma(N)\backslash\mathbb{Q}^\star$.

Proposition 2.1. *Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ containing $\Gamma(N)$ for some N. Then $[\Gamma : \Gamma(N)]$ is finite. Moreover, Γ acts on $\mathbb{Q}^\star$ and $\Omega(\Gamma) \simeq \Gamma\backslash\mathbb{Q}^\star$.*

Proof. Let $\Gamma^\star = \Gamma(N)\backslash\Gamma$. The curve $Y(N) \coloneqq Y(\Gamma(N))$ has finite (but positive) volume with respect to to the hyperbolic measure on $\mathbb{H}$. Since Γ contains $\Gamma(N)$, the volume of $Y(\Gamma) \simeq \Gamma^\star\backslash Y(N)$ is also finite. Since Γ is discrete, the volume is positive. Therefore, the quotient map $Y(N) \to Y(\Gamma)$ has finite degree, equal to $[\Gamma : \Gamma(N)]$. This quotient map lifts to a quotient on $X(N)$. Since $X(N)$ is compact, the image is also. Hence, $X(\Gamma)$ exists and is $X(\Gamma) \simeq \Gamma^\star\backslash X(N)$. The quotient map must send cusps to cusps, and so

$$\Omega(\Gamma) \simeq \Gamma^\star\backslash\Omega(N) \simeq \Gamma\backslash\mathbb{Q}^\star. \qquad \square$$

The fact that Γ acts on $\mathbb{Q}^\star$ can also be proven algebraically. This fact implies that each $\gamma \in \Gamma$ (with Γ viewed as a subgroup of $\mathrm{GL}_2^+(\mathbb{R})/\mathbb{R}^+ I$) may be scaled appropriately so that all its entries are rational. More specifically, each γ has a canonical choice of scaling so that each entry is integral, but so that the entries do not share a common factor greater than 1. For $\rho \in \mathbb{Q}^\star$, let $(\rho) = \Gamma\rho$ denote the cusp containing ρ.

For example, consider $\Gamma_0(2)^+ = \langle \Gamma_0(2), W_2 \rangle$, where W_2 is the Fricke involution which normalizes $\Gamma_0(2)$. The Fricke involution can be represented by the determinant 1 matrix $\left(\begin{smallmatrix} 0 & \frac{-1}{\sqrt{2}} \\ \sqrt{2} & 0 \end{smallmatrix}\right)$, or more simply by the determinant 2 matrix $\left(\begin{smallmatrix} 0 & -1 \\ 2 & 0 \end{smallmatrix}\right)$. A computation shows that $\Omega(\Gamma_0(2)^+) = \{(\infty)\}$, whereas $\Omega(\Gamma_0(2)) = \{(\infty), (0)\}$.

2.2 Expansions at cusps

Let Γ, ν, k, and N be as above. Suppose is $f : \mathbb{H} \to \mathbb{C}$ is smooth and modular for Γ and ν of weight k. Then since $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma(N) \subseteq \Gamma$, f has a Fourier expansion of the form

$$f(z) = \sum_{n \in \frac{1}{N}\mathbb{Z}} a(n, y) e^{2\pi \mathrm{i} n x},$$

where (here and throughout) $z = x + \mathrm{i}y$. Indeed, let $\mu \in \mathrm{SL}_2(\mathbb{Z})$ be arbitrary, and let $f^\mu = f|_k\mu$. We see f^μ is modular on $\Gamma^\mu := \mu^{-1}\Gamma\mu$ with multiplier ν^μ defined so that $\nu^\mu(\mu^{-1}\gamma\mu) = \nu(\gamma)$. We say that f^μ is an expansion of f at the cusp (μ^{-1}). Furthermore, as $\Gamma(N)$ is normal in $\mathrm{SL}_2(\mathbb{Z})$, we have $[\Gamma^\mu : \Gamma(N)] = [\Gamma : \Gamma(N)] < \infty$, and so f^μ has a Fourier expansion of the form

$$f^\mu(z) = \sum_{n \in \frac{1}{N}\mathbb{Z}} a^\mu(n, y) e^{2\pi \mathrm{i} n x}.$$

By the Euclidean algorithm, a modular function f may be expanded at any cusp $(\rho) \in \Omega(\Gamma)$ using a matrix $\mu \in \mathrm{SL}_2(\mathbb{Z})$. For $\mu \in \mathrm{SL}_2(\mathbb{Z})$ (indeed, for $\mu \in \mathrm{GL}_2(\mathbb{Q})$), we write $(\mu) = (\mu\infty)$.

Let $\Gamma^\mu_\infty \subseteq \Gamma^\mu$ denote the group of upper-triangular matrices in Γ^μ. As Γ^μ is of finite index over $\Gamma(N)$, the group Γ^μ_∞ is generated (up to sign) by a single element $T_\mu = \left(\begin{smallmatrix} 1 & \frac{s}{t} \\ 0 & 1 \end{smallmatrix}\right)$, where $\frac{s}{t}$ is positive and rational. Moreover, the numerator s divides N, as otherwise we could replace s with $\gcd(s, N)$. In fact t also divides N, although we will neither prove nor use this. We define $\omega(\Gamma, \mu) := \frac{s}{t}$ to be the cuspidal width of (μ) in Γ with respect to μ.

Let $T = T_{\mathrm{I}}$ be the generator of $\Gamma_\infty = \Gamma^{\mathrm{I}}_\infty$, and let $\omega = \omega(\Gamma, \mathrm{I})$. Further, let $\nu(T) = \zeta$.

Notice that

$$\zeta^{\frac{N}{\omega}} = \nu(T^{\frac{N}{\omega}}) = \nu\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) = 1,$$

and so $\zeta = \mathrm{e}^{2\pi\,\mathrm{i}\,c\frac{\omega}{N}}$, for some integer c. Define

$$\mathbb{Z}^{(\nu)} := \left\{ \frac{m}{N} \mid \; m \in \mathbb{Z},\; m \equiv c \pmod{\tfrac{N}{\omega}} \right\} \tag{2.2}$$

Then $f(Tz) = f(z + \omega) = \zeta f(z)$, and we conclude

$$f(z) = \sum_{n \in \mathbb{Z}^{(\nu)}} a(n, y) e^{2\pi\,\mathrm{i}\,nx}. \tag{2.3}$$

We refer to (2.3) as the Fourier expansion of f at the cusp (∞). Any linear change of variable also gives an expansion about the cusp (∞), but (2.3) is the canonical choice. If we define

$$\mathbb{Z}_\nu^{(\mu)} := \mathbb{Z}^{(\nu^\mu)}, \tag{2.4}$$

we also have

$$f^\mu(z) = \sum_{n \in \mathbb{Z}_\nu^{(\mu)}} a^\mu(n, y) e^{2\pi\,\mathrm{i}\,nx}. \tag{2.5}$$

Chapter 3. Harmonic Maass forms

Here we recall the definition of harmonic Maass forms of weight k for a group Γ and multiplier ν. As above, we set $z = x + \mathrm{i}\,y$ with x and y real, and $q = \mathrm{e}^{2\pi\,\mathrm{i}\,z}$. The weight k hyperbolic Laplacian is defined by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \mathrm{i}\,ky \left(\frac{\partial}{\partial x} + \mathrm{i}\,\frac{\partial}{\partial y} \right).$$

Definition 3.1. Let $\Gamma \supset \Gamma(N)$ and ν be as in the previous section. Then a real analytic function $F : \mathbb{H} \to \mathbb{C}$ is a *harmonic Maass form* of weight k for Γ and multiplier ν if:

(i) The function $F(z)$ is modular for Γ and ν, so that

$$F|_{k,\nu}\gamma = F$$

for every matrix $\gamma \in \Gamma$;

(ii) The function F is harmonic, so that $\Delta_k F = 0$;

(iii) The function F has a meromorphic *principal part* at each cusp.

Using (2.2), we can restate part (3) of the definition as the following: if $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $F^\gamma = F|_k\gamma$, then there is some polynomial $P^\gamma(q^{-\frac{1}{N}}) \in \mathbb{C}[q^{-\frac{1}{N}}]$ so that

$$\lim_{y\to\infty} F^\gamma - P^\gamma(q^{-\frac{1}{N}}) = 0.$$

We denote the space of harmonic Maass forms of weight k for Γ and ν by $H_k(\Gamma, \nu)$. If we strengthen this growth condition to require that

$$F^\gamma - P^\gamma(q^{-\frac{1}{N}}) = O(e^{-\frac{1}{N}y}),$$

then we obtain the subspace $H_k^!(\Gamma, \nu) \subseteq H_k(\Gamma, \nu)$. These spaces only differ in weights 2 and 3/2. In these cases the larger space contains non-holomorphic Eisenstein series, not present in the smaller space. We also define $H_k^{(\infty)}(\Gamma, \nu) \subseteq H_k^!(\Gamma, \nu)$ as the subspace of harmonic Maass forms with holomorphic principal part at every cusp other than (∞).

The differential equation given by $\Delta_k F = 0$ implies that harmonic Maass forms have Fourier expansions which split into two components: one part which is a holomorphic q-series, and one part which is non-holomorphic.

Lemma 3.2 ([10, Proposition 3.2]). *Let $F(z) \in H_k(\Gamma, \nu)$, with Γ and ν as above. Then we have that*

$$F(z) = F^{+}(z) + F^{-}(z)$$

where F^{+} is the holomorphic part of F or mock modular form, *given by*

$$F^{+}(z) := \sum_{\substack{n \in \mathbb{Z}^{\nu} \\ n \gg -\infty}} c_F^{+}(n) q^n,$$

and F^{-} is the non-holomorphic part given by

$$F^{-}(z) := c_F^{-}(0) y^{1-k} + \sum_{n \in \mathbb{Z}^{\nu}_{<0}} c_F^{-}(n) \Gamma(k-1, 4\pi y|n|) q^n.$$

3.1 Differential operators, the Petersson inner product, and the Bruinier–Funke Pairing

Differential operators yield some important relations between spaces of harmonic Maass forms and weakly holomorphic modular forms of dual weight. Define the operators

$$D^{k-1} := \left(\frac{1}{2\pi i} \frac{\partial}{\partial z}\right)^{k-1} \qquad \text{and} \qquad \xi_k := 2iy^k \overline{\frac{\partial}{\partial \overline{z}}}$$

where for the first operator we require that $k \geq 2$ be an integer. These maps yield the exact sequences

$$0 \to M_{2-k}(\Gamma, \nu) \hookrightarrow H_{2-k}(\Gamma, \nu) \xrightarrow{D^{k-1}} S_k^{\perp}(\Gamma, \nu) \to 0, \tag{3.1}$$

$$0 \to M^{!}_{2-k}(\Gamma, \nu) \hookrightarrow H^{!}_{2-k}(\Gamma, \nu) \xrightarrow{\xi_{2-k}} S_k(\Gamma, \overline{\nu}) \to 0. \tag{3.2}$$

Here, we note that in the first sequence, $M_{2-k}(\Gamma, \nu)$ is 0 unless $k = 2$ and ν is trivial, in which case it is $\mathbb{C}$. The space $S_k^{\perp}(N; \mathbb{C})$ is a distinguished subspace of $M_k^{!}(\Gamma, \nu)$ consisting of those forms with vanishing constant term at all cusps and which are orthogonal to the cusp forms $S_k(\Gamma,\nu)$ with respect to the regularized Petersson inner product described below. In the second sequence, note that for $2-k=\frac{3}{2},2$, we have $H_{2-k}(\Gamma,\nu)\neq H^!_{2-k}(\Gamma,\nu)$. In these weights, we also have the exact sequence

$$0\to M^!_{2-k}(\Gamma,\nu)\hookrightarrow H_{2-k}(\Gamma,\nu)\xrightarrow{\xi_{2-k}} M_k(\Gamma,\overline{\nu})\to 0. \tag{3.3}$$

The D^{k-1} operator preserves integrality of coefficients, and so extends to a map

$$D^{k-1}: M^!_{2-k}(\Gamma,\nu)\xrightarrow{D^{k-1}} M^!_k(\Gamma,\nu).$$

The usual Petersson inner product for $M_k(\Gamma,\nu)\times M_k(\Gamma,\nu)$ is given by

$$\langle f,g\rangle_{k,\Gamma}:=\int_{\mathcal{F}_\Gamma} y^k f(z)\overline{g(z)}\mathrm{d}\mu,$$

wherever this expression is defined (it is sufficient for f or g to vanish at each cusp). Here, $\mathrm{d}\mu:=\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$ and $\mathcal{F}_\Gamma\subseteq\mathbb{H}$ is any fundamental domain for Γ. Note that if we replace Γ by any subgroup Γ' with $[\Gamma:\Gamma']<\infty$, the inner product changes by a factor of $[\Gamma:\Gamma']$. The inner product is sometimes normalized to remove this dependence; however, we choose not to normalize it. We will only be interested in whether or not it vanishes, and the non-normalized version simplifies the equation in Theorem 3.3 below.

The integral may be regularized for convergence as in [10], extending the inner product to $M^!_k(\Gamma,\nu)\times M_k(\Gamma,\nu)$ with the same vanishing conditions as above. This regularization is accomplished by truncating the fundamental domain at a fixed height above the real axis and by circular arcs around each cusp. As the height of the truncated domain goes to infinity and the radius of the arcs around each cusp goes to zero, the integral converges and the limit is defined to be the inner product. For $k=0$, the regularized inner product $\langle f,g\rangle_{0,\Gamma}$ is well-defined if $f\in M^!_0(\Gamma,\nu)$ and $g\in\mathbb{C}$. For $k=2$, the regularized inner product $\langle f,g\rangle_{2,\Gamma}$ is well-defined if $f\in H_2(\Gamma,\nu)$ and $g\in S_2(\Gamma,\nu)$.

The Bruinier–Funke pairing $\{\bullet, \bullet\}$ is given by

$$\{f, G\} \coloneqq \langle f, \xi_{2-k} G \rangle_k.$$

with $f \in H_k(\Gamma, \nu)$ and $g \in H_{2-k}(\Gamma, \nu)$, wherever the inner product is defined. Bruinier and Funke showed that although $\{f, G\}$ depends only on $\xi_{2-k} G$, not the choice of pre-image G itself, the pairing $\{f, G\}$ can be calculated in terms of the coefficients of G and f.

Theorem 3.3 ([10, Proposition 3.5])**.** *Suppose $f \in H_k(\Gamma, \nu)$ and $G \in H_{2-k}(\Gamma, \overline{\nu})$ have Fourier expansions at cusps given by*

$$f|_k \gamma = \sum_{n \in \mathbb{Q}} a^{\gamma}(n) q^n$$

and

$$(G|_k \gamma)^+ = \sum_{n \in \mathbb{Q}} b^{\gamma,+}(n) q^n.$$

Then

$$\{f, G\} = \sum_{(\rho) \in \Omega(\Gamma)} \omega(\Gamma, \gamma_\rho) \sum_{n \in \mathbb{Q}} a^{\gamma_\rho}(n) b^{\gamma_\rho,+}(-n).$$

provided the pairing on the left is defined. Here $\gamma_\rho \in \mathrm{SL}_2(\mathbb{Z})$ is any matrix with $(\rho) = (\gamma_\rho \infty)$.

The inner sum in the theorem is the *residue* of $f(z)G(z)$ at the cusp (ρ), which is the constant term of $(f \cdot G)|_2 \gamma_\rho$, weighted by the width of the fundamental domain of Γ^{γ_ρ}.

The original version of this theorem given in [10] was written in terms of vector-valued modular forms for $\mathrm{SL}_2(\mathbb{Z})$. If $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, we may rewrite the expression in Theorem 3.3 in a manner more similar to the original:

$$\{f, G\} = \sum_{\gamma \in \Gamma \backslash \mathrm{SL}_2(\mathbb{Z})} \sum_{n \in \mathbb{Q}} a^{\gamma}(n) b^{\gamma,+}(-n).$$

In this case, the widths $\omega(\Gamma, \gamma)$ which appear in Theorem 3.3 can be interpreted as counting the number of elements in $\Gamma \backslash \mathrm{SL}_2(\mathbb{Z})$ which give expansions at the same cusp.

Sketch of Proof. The theorem is an application of Stokes' Theorem, using the truncated fundamental domains described above for the regularized inner product. With Stoke's theorem, the integral becomes the sum of line-integrals on the arcs $C_{\rho,r}$ of radius r around each cusp ρ. In the limit the becomes a sum of residues at each cusp. To calculate the residue at a given cusp ρ, we make the change of variable $z \mapsto \gamma_\rho z$, which gives

$$\lim_{r\to\infty} \int_{C_{\rho,r}} f(z)G(z)\mathrm{d}z = \lim_{T\to\infty} \int_{x'+iT}^{x'+w+iT} (f \cdot G)|_2 \gamma_\rho(z)\mathrm{d}z$$

for some real x', and where $w = \omega(\Gamma, \gamma_\rho)$. □

3.2 Maass-Poincaré Series

Following Niebur [32], Bringmann-Ono [9], etc., we define the Maass Poincaré series as follows. Define the seed function $\phi_{s,k,m} : \mathbb{H} \to \mathbb{C}$ by

$$\phi_{s,k,m}(z) \coloneqq |4\pi m|^{-s}|y|^{-\frac{k}{2}}\, \mathrm{M}_{\frac{k}{2}\operatorname{sgn}(m), s-\frac{1}{2}}(|4\pi m y|)\, \mathrm{e}^{2\pi\,\mathrm{i}\, m x},$$

where $\mathrm{M}_{\kappa,\mu}(w)$ is the usual M-Whittaker function which satisfies

$$\left(\frac{\partial^2}{\partial w^2} - \frac{1}{4} + \frac{\kappa}{w} + \frac{1/4-\mu^2}{w^2}\right)\mathrm{M}_{\kappa,\mu}(w) = 0.$$

This definition holds for $m \in \mathbb{R}_{\neq 0}$ and $s \in \mathbb{C}$ such that $2s \notin \mathbb{Z}_{\leq 0}$. We extend to $m = 0$ by taking the limit

$$\phi_{s,k,0}(z) \coloneqq \lim_{m\to 0} \phi_{s,k,m}(z) = y^{s-k/2}.$$

Using standard formulas for the derivative of $M_{\kappa,\mu}$ we find

$$R_k\ \phi_{s,k,m}(z) = \left(s + \tfrac{k}{2}\right)\ \phi_{s,2+k,m}(z), \tag{3.4}$$

$$\xi_k\ \phi_{s,k,m}(z) = \left(\overline{s} - \tfrac{k}{2}\right)\ \phi_{\overline{s},2-k,-m}(z). \tag{3.5}$$

where $R_k := \left(2\,\mathrm{i}\,\frac{\partial}{\partial z} + \frac{k}{y}\right)$ is the Maass raising operator which for k positive and even satisfies $-(4\pi)^{k-1} R_{k-2} \circ \ldots R_{4-k} \circ R_{2-k} = D^{k-1}$.

The second identity implies that $\phi_{s,k,m}(z)$ is holomorphic if $s = \frac{k}{2}$ with $k \geq 1/2$. In this case $\phi_{k/2,k,m}(z) = q^m$. Iterating the second identity, we find

$$\Delta_k \ \phi_{s,k,m}(z) = -\xi_k^2 \ \phi_{s,k,m}(z) = \left(s(1-s) - \tfrac{k}{2}(1-\tfrac{k}{2})\right) \ \phi_{s,k,m}(z),$$

which implies that $\phi_{s,k,m}(z)$ is harmonic if $s = \frac{k}{2}$ or $1 - \frac{k}{2}$. If $k \geq \frac{1}{2}$ and $m < 0$ we have

$$\begin{aligned} \phi_{k/2,2-k,m}(z) &= |4\pi m|^{1-k}(k-1)\gamma(k-1, 4\pi|m|y) \cdot q^m & (3.6)\\ &= |4\pi m|^{1-k}\left(\Gamma(k) - (k-1)\Gamma(k-1, 4\pi|m|y)\right) \cdot q^m, & (3.7) \end{aligned}$$

where $\gamma(s,x)$ and $\Gamma(s,x)$ are the lower and upper incomplete Γ-functions respectively.

If $m \in \mathbb{Z}^{(\nu)}$ and $\operatorname{Re} s > 1$, then the Maass–Poincare series

$$\mathcal{P}^{(\nu)}(s,k,m;\ z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \phi_{s,k,m}|_{k,\nu}\gamma$$

converges absolutely. From (3.5) we have that

$$\xi_k \mathcal{P}^{(\nu)}(s,k,m;\ z) = (\overline{s} - \tfrac{k}{2})\mathcal{P}^{(\nu)}(\overline{s}, 2-k, -m;\ z). \tag{3.8}$$

In particular, for $k > 2$ we have that $\mathcal{P}^{(\nu)}(\frac{k}{2}, k, m;\ z)$ is weakly holomorphic, while $\mathcal{P}^{(\nu)}(\frac{k}{2}, 2-k, m;\ z)$ is harmonic.

The series $\mathcal{P}^{(\nu)}(s,k,m;\ z)$, viewed as a function in s, may be meromorphically continued to $s = 1$ and $s = 3/2$. The continuation to $s = 3/2$ is non-trivial and requires estimates for the conditional convergence of the Selberg–Kloosterman zeta function [24, 33]. The continuation may result in a simple pole for $\mathcal{P}^{(\nu)}(s,k,m;\ z)$ at $s = 1$ or $s = 3/2$ if $k = 0$ or $k = 1/2$ respectively. In particular, if $k = 3/2$ or $k = 2$ and $s = k/2$, then (3.8) still holds, but we may need to interpret the right hand side as a residue. In this case $\mathcal{P}^{(\nu)}(s,k,m;\ z)$ is harmonic and $\mathcal{P}^{(\nu)}(s,2-k,-m;\ z)$ is holomorphic.

With these known convergence conditions, we define the harmonic Maass–Poincaré series as follows. For $k>1$ and $m\in\mathbb{Z}^{(\nu)}_{\geq 0}$ let

$$\mathcal{P}^{(\nu)}_{k,m}(z) := \mathcal{P}^{(\nu)}(\tfrac{k}{2},k,-m;\ z), \tag{3.9}$$

for $k<1$ and $m\in\mathbb{Z}^{(\nu)}_{>0}$ let

$$\mathcal{P}^{(\nu)}_{k,m}(z) := \frac{|4\pi m|^{1-k}}{\Gamma(2-k)}\mathcal{P}^{(\nu)}(1-\tfrac{k}{2},k,-m;\ z). \tag{3.10}$$

Exact formulas for the coefficients of $\mathcal{P}^{(\nu)}_{k,m}(z)$ can be obtained using Poisson summation. See for instance [9]. However, elementary bounds can give basic information about the principal parts of the Maass–Poincaré series. We have that $\phi_{s,k,m}(z)=y^{s-k/2}+O(y^{s+1-k/s})$ as $y\to\infty$. If $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2^+(\mathbb{R})$ with $c\neq 0$, then $\phi_{s,k,m}(z)|_k\gamma=O(y^{-s-k/2})$. Using Poisson summation to sum over the non-upper triangular matrices in the group, we find that these contribute a term which is $O(y^{1-s-k/2})$, where the implied constant depends on s and k. In particular, if $k<1$ and $s=1-k/2$, we find that $P^{(\nu)}_{k,m}(z)=q^{-m}+O(1)$ for all $m\in\mathbb{Z}^{(\nu)}_{\geq 0}$, and $P^{(\nu)}_{k,m}(z)|_k\gamma=O(1)$ for any $\gamma\in\mathrm{GL}_2^+(\mathbb{Q})$ not upper triangular. If $k>1$ and $s=k/2$, the same results hold, but can be strengthened from $O(1)$ to $O(y^{1-k})$. Therefore in these cases the constant term at every cusp must vanish.

We write $q^{-m}+\sum_{n\in\mathbb{Z}^{(\nu)}}\mathcal{A}^{(\nu)}_k(m,n)\,q^n$ for the holomorphic part of $\mathcal{P}^{(\nu)}_{k,m}$.

Proposition 3.4. *Let k, Γ, and ν be as above, and let*

$$F=\sum_{m\in\mathbb{Z}^{(\nu)}_{\geq 0}} a_m\cdot\mathcal{P}^{(\nu)}_{k,m}$$

be a sum of Maass-Poincaré series, with $a_0=0$ unless $k>1$.

Then F is weakly holomorphic if and only if the Bruinier–Funke pairing $\{F, h\} = 0$ for every holomorphic modular form $h \in M_{2-k}(\Gamma, \overline{\nu})$. In particular, F is is weakly holomorphic if $k > 2$.

If $k < 1$, then F is weakly holomorphic if and only if the Bruinier–Funke pairing $\{F, h\} = 0$ for every cusp form $h \in S_{2-k}(\Gamma, \overline{\nu})$.

Proof. The statement is trivial for $k > 2$.

We have that F is harmonic, and so has a non-holomorphic part $F^- = c_F^-(0)y^{1-k} + \sum_{n\in\mathbb{Z}^\nu} c_F^-(n)\Gamma(k-1, 4\pi y|n|)q^n$. Since $\mathcal{P}_{k,m}^{(\nu)}(z) = q^{-m} + O(1)$, we have that $c_F^-(n) = 0$ for all $n > 0$, and for $n = 0$ unless $k = 2$ or $3/2$. This holds similarly at every cusp. Therefore $\xi_k F$ is a holomorphic modular form in $M_{2-k}(\Gamma, \overline{\nu})$, and if $k < 1$ we have that $\xi_k F \in S_{2-k}(\Gamma, \overline{\nu})$. Recall that F is weakly holomorphic if and only if $\xi_k F = 0$, which in turn is true if and only if $\{F, h\} = 0$ for every $h \in M_{2-k}(\Gamma, \overline{\nu})$. If $k < 1$, we need only check the pairing against the cusp forms. □

Chapter 4. Weakly Holomorphic Modular Forms

4.1 Construction of Bases

The space of weight k holomorphic modular forms for a group Γ and multiplier ν is denoted by $M_k(\Gamma, \nu)$. This space has finite dimension, and so it has a basis of forms whose Fourier expansions at (∞) are in reduced echelon form. We denote this basis by

$$\{F_{k,m}^{(\nu)}(z) = q^m + \sum_{\substack{n\in\mathbb{Z}_{\geq 0}^{(\nu)} \\ n\notin I_k^{(\nu)}}} A_k^{(\nu)}(m,n)q^n \quad | \quad m \in I_k^{(\nu)}\}.$$

Here, we have defined the finite set $I_k^{(\nu)} \subseteq \mathbb{Z}_{\geq 0}^{(\nu)}$ implicitly to be the set of indices for the reduced basis. Note that if $k < 0$, then $I_k^{(\nu)}$ is empty.

The space $S_k(\Gamma,\nu)$ of weight k holomorphic cusp forms for Γ and ν also has finite dimension, and has a basis of forms whose Fourier expansions at (∞) are in reduced echelon form. We denote this basis by

$$\{G_{k,m}^{(\nu)}(z) = q^m + \sum_{\substack{n\in\mathbb{Z}_{>0}^{(\nu)}\\ n\notin J_k^{(\nu)}}} B_k^{(\nu)}(m,n)q^n \quad | \quad m\in J_k^{(\nu)}\}.$$

Here, as before, we have implicitly defined the finite set $J_k^{(\nu)}\subseteq\mathbb{Z}_{>0}^{(\nu)}$ to be the set of indices for the reduced basis.

We will use the holomorphic basis elements $F_{k,m}^{(\nu)}(z)$ and $G_{k,m}^{(\nu)}(z)$ and the Maass–Poincaré series $\mathcal{P}_{k,m}^{(\nu)}(z)$ to define the weakly holomorphic forms $f_{k,m}^{(\nu)}(z)$ and $g_{k,m}^{(\nu)}(z)$ appearing in Theorem 1.1. In particular, let $f_{k,m}^{(\nu)}(z) := F_{k,-m}^{(\nu)}(z)$ for $-m\in I_k^{(\nu)}$, and $g_{k,m}^{(\nu)}(z) := G_{k,-m}^{(\nu)}(z)$ for $-m\in J_k^{(\nu)}$.

The forms $f_{k,m}^{(\nu)}(z)$ and $g_{k,m}^{(\nu)}(z)$ are indexed by sets $\widetilde{I}_k^{(\nu)}$ and $\widetilde{J}_k^{(\nu)}$ respectively, which are defined so that they satisfy the disjoint unions

$$\begin{aligned}\widetilde{I}_k^{(\nu)}\sqcup J_{2-k}^{(\overline{\nu})} &= -\left(\mathbb{Z}_{<0}^{(\nu)}\sqcup I_k^{(\nu)}\right),\\ \widetilde{J}_k^{(\nu)}\sqcup I_{2-k}^{(\overline{\nu})} &= -\left(\mathbb{Z}_{\le 0}^{(\nu)}\sqcup J_k^{(\nu)}\right).\end{aligned}$$

A short manipulation shows that these imply the relations

$$\big(-\widetilde{J}_{2-k}^{(\overline{\nu})}\big)\sqcup\widetilde{I}_k^{(\nu)} = \mathbb{Z}^{(\nu)}, \tag{4.1}$$

$$\big(-\widetilde{I}_{2-k}^{(\overline{\nu})}\big)\sqcup\widetilde{J}_k^{(\nu)} = \mathbb{Z}^{(\nu)}. \tag{4.2}$$

The Maass–Poincaré series may also be reduced against the forms $F_{k,n}^{(\nu)}(z)$ and $G_{k,n}^{(\nu)}(z)$ at (∞), to obtain forms

$$\widetilde{\mathcal{P}}_{k,m}^{(\nu)}(z) \coloneqq \mathcal{P}_{k,m}^{(\nu)}(z) - \sum_{n \in I_k^{(\nu)}} \mathcal{A}_k^{(\nu)}(m,n)\, F_{k,n}^{(\nu)}(z),$$

$$\widehat{\mathcal{P}}_{k,m}^{(\nu)}(z) \coloneqq \mathcal{P}_{k,m}^{(\nu)}(z) - \sum_{n \in J_k^{(\nu)}} \mathcal{A}_k^{(\nu)}(m,n)\, G_{k,n}^{(\nu)}(z).$$

Let us write $q^{-m} + \sum_{n \in \mathbb{Z}^{(\nu)}} \widetilde{\mathcal{A}}_k^{(\nu)}(m,n)\, q^n$ for the holomorphic part of $\widetilde{\mathcal{P}}_{k,m}^{(\nu)}$, and $q^{-m} + \sum_{n \in \mathbb{Z}^{(\nu)}} \widehat{\mathcal{A}}_k^{(\nu)}(m,n)\, q^n$ for the holomorphic part of $\widehat{\mathcal{P}}_{k,m}^{(\nu)}$. Clearly

$$\widetilde{\mathcal{A}}_k^{(\nu)}(m,n) = \mathcal{A}_k^{(\nu)}(m,n) - \sum_{\ell \in I_k^{(\nu)}} \mathcal{A}_k^{(\nu)}(m,\ell) \left(\delta_{\ell n} + A_k^{(\nu)}(\ell,n)\right),$$

$$\widehat{\mathcal{A}}_k^{(\nu)}(m,n) = \mathcal{A}_k^{(\nu)}(m,n) - \sum_{\ell \in J_k^{(\nu)}} \mathcal{A}_k^{(\nu)}(m,\ell) \left(\delta_{\ell n} + B_k^{(\nu)}(\ell,n)\right),$$

where $\delta_{\ell n} = \begin{cases} 1 & \text{if } \ell = n, \\ 0 & \text{else.} \end{cases}$

The remaining forms $f_{k,m}^{(\nu)}(z)$ and $g_{k,m}^{(\nu)}(z)$ may then be defined in terms of these reduced Maass–Poincaré series as follows. If $k \neq 2$ or ν is nontrivial, for $m \in \widetilde{I}_k^{(\nu)}$, $-m \notin I_k^{(\nu)}$ we define

$$f_{k,m}^{(\nu)}(z) \coloneqq \widetilde{\mathcal{P}}_{k,m}^{(\nu)}(z) - \sum_{n \in J_{2-k}^{(\overline{\nu})}} B_{2-k}^{(\overline{\nu})}(n,m) \widetilde{\mathcal{P}}_{k,n}^{(\nu)}(z). \tag{4.3}$$

If $k = 2$ and ν is trivial, let $A_{2,m}(\gamma)$ denote the constant term of $F_{2,m}^{(\nu)}(z)|_k \gamma$. We define

$$f_{2,m}^{(\nu)}(z) \coloneqq \widetilde{\mathcal{P}}_{2,m}^{(\nu)}(z) + \left(\sum_{n \in I_2^{(\nu)}} \mathcal{A}_2^{(\nu)}(m,n) \sum_{(\mu) \in \Omega(\Gamma)} \omega(\Gamma,\mu) A_{2,n}^{(\nu)}(\mu) \right) \mathcal{P}_{2,0}^{(\nu)}(z). \tag{4.4}$$

If $k \neq 0$ or ν is nontrivial, for $m \in \widetilde{J}_k^{(\nu)}$, $-m \notin J_k^{(\nu)}$ we define

$$g_{k,m}^{(\nu)}(z) := \widehat{\mathcal{P}}_{k,m}^{(\nu)}(z) - \sum_{n \in I_{2-k}^{(\overline{\nu})}} A_{2-k}^{(\overline{\nu})}(n,m) \widehat{\mathcal{P}}_{k,n}^{(\nu)}(z). \tag{4.5}$$

If $k = 0$ and ν is trivial, we define

$$g_{0,m}^{(\nu)}(z) := \widehat{\mathcal{P}}_{0,m}^{(\nu)}(z) + \mathcal{A}_2^{(\overline{\nu})}(0,m) - \sum_{n \in I_2^{(\overline{\nu})}} A_2^{(\overline{\nu})}(n,m) \left(\widehat{\mathcal{P}}_{0,n}^{(\nu)}(z) + \mathcal{A}_2^{(\overline{\nu})}(0,n) \right). \tag{4.6}$$

These are each harmonic Maass forms, being sums of Maass–Poincaré series. We will see that they are also weakly holomorphic, although this is non-trivial for $k < 1$.

We write

$$q^{-m} + \sum_{m \in \mathbb{Z}^{(\nu)}} a_k^{(\nu)}(m,n) q^n$$

for the holomorphic part of $f_{k,m}^{(\nu)}(z)$ and

$$q^{-m} + \sum_{m \in \mathbb{Z}^{(\nu)}} b_k^{(\nu)}(m,n) q^n$$

for the holomorphic part of $g_{k,m}^{(\nu)}(z)$. Then by construction we have

$$a_k^{(\nu)}(m,n) = \begin{cases} -B_{2-k}^{(\overline{\nu})}(-n,m) & \text{if } -n \in J_{2-k}^{(\overline{\nu})}, \\ 0 & \text{if } n \in I_k^{(\nu)}, \end{cases}$$

and

$$b_k^{(\nu)}(m,n) = \begin{cases} -A_{2-k}^{(\overline{\nu})}(-n,m) & \text{if } -n \in I_{2-k}^{(\overline{\nu})}, \\ 0 & \text{if } n \in J_k^{(\nu)} \end{cases}$$

for m in $\widetilde{I}_k^{(\nu)}$ and $m \in \widetilde{J}_k^{(\nu)}$ respectively. Note that unless $0 \leq k \leq 2$, at least one case in each statement is vacuous.

Let us also define $E_{k,\gamma}^{(\nu)}(z) = \mathcal{P}_{k,0}^{(\nu^\gamma)}(z)|_k \gamma^{-1}$ to be the weight k Eisenstein series with a constant at (γ), and write $\sum_{n\in\mathbb{Z}^{(\nu)}} s^{(\nu)}_{k,\gamma}(n)q^n$ for the holomorphic part of $E^{(\nu)}_{k,\gamma}$. For Γ a congruence group, it is well-known that when $k<1$ we have $\langle E^{(\nu)}_{k,\gamma}, g\rangle_{k,\Gamma}=0$ for all cusp forms $g\in S_k(\Gamma,\nu)$ (see pages 206-207 of [15]). This argument can be extended in a straightforward way to any level N Fuchsian group.

With this notation, we have the following.

Lemma 4.1. *For $k<1$, the constant term of $\mathcal{P}^{(\nu)}_{k,m}|_k\gamma$ is $-\frac{s^{(\overline{\nu})}_{2-k,\gamma}(n)}{\omega(\Gamma,\gamma)}$.*

Proof. Let $k<1$, and consider $\left\{E^{(\overline{\nu})}_{2-k,\gamma}, \mathcal{P}^{(\nu)}_{k,m}\right\}$. Note that $E^{(\overline{\nu})}_{2-k,\gamma}$ is holomorphic if $2-k>2$, so this pairing makes sense. By the previous paragraph, $\left\{E^{(\overline{\nu})}_{2-k,\gamma}, \mathcal{P}^{(\nu)}_{k,m}\right\}=0$. Write $\mathcal{A}^{(\nu)}_{k,\gamma}(m)$ for the constant term of $\mathcal{P}^{(\nu)}_{k,m}|_k\gamma$. By Theorem 3.3,

$$\left\{E^{(\overline{\nu})}_{2-k,\gamma}, \mathcal{P}^{(\nu)}_{k,m}\right\} = s^{(\overline{\nu})}_{2-k,\gamma}(m) + \omega(\Gamma,\gamma)\mathcal{A}^{(\nu)}_{k,\gamma}(m),$$

since the product $E^{(\overline{\nu})}_{2-k,\gamma}\mathcal{P}^{(\nu)}_{k,m}$ vanishes at every cusp other than (∞) and (γ), and the result follows. □

Lemma 4.2. *For $k\neq \frac{1}{2},1,\frac{3}{2}$, the forms $f^{(\nu)}_{k,m}(z)$ and $g^{(\nu)}_{k,m}(z)$ defined as above are weakly holomorphic, and form bases for $M^{(\infty)}_k(\Gamma,\nu)$ and $\widehat{M}^{(\infty)}_k(\Gamma,\nu)$ respectively.*

Proof. We begin with the $f^{(\nu)}_{k,m}(z)$ with $m\in\widetilde{I}^{(\nu)}_k$. By (3.2), $\xi_k f^{(\nu)}_{k,m}\in S_{2-k}(\Gamma,\overline{\nu})$. By Proposition 3.4, if $k\neq 2$, $f^{(\nu)}_{k,m}(z)$ is weakly holomorphic if and only if $\left\{G^{(\overline{\nu})}_{2-k,n}, f^{(\nu)}_{k,m}\right\}=0$ for each n in $J^{(\overline{\nu})}_{2-k}$. By Theorem 3.3, and because $f^{(\nu)}_{k,m}G^{(\overline{\nu})}_{2-k,n}$ vanishes at every cusp other than (∞), we calculate

$$\left\{G^{(\overline{\nu})}_{2-k,n}, f^{(\nu)}_{k,m}\right\} = a^{(\nu)}_k(m,-n) + B^{(\overline{\nu})}_{2-k}(n,m) + \sum_{\ell\in\mathbb{Z}^{(\nu)}} a^{(\nu)}_k(m,\ell)B^{(\overline{\nu})}_{2-k}(n,-\ell) = 0.$$

Here, the terms outside the summation cancel, whereas in each of the summands, either $a^{(\nu)}_k(m,\ell)=0$ (if $\ell\in\widetilde{I}^{(\nu)}_k$), or $B^{(\overline{\nu})}_{2-k}(n,-\ell)=0$ (if $-\ell\in\mathbb{Z}^{(\overline{\nu})}_{\leq 0}\cup J^{(\overline{\nu})}_{2-k}=-\left(\widetilde{J}^{(\overline{\nu})}_{2-k}\cup I^{(\nu)}_k\right)$). By (4.1), this is covers every ℓ. Thus the $f^{(\nu)}_{k,m}(z)$ are weakly holomorphic.

Now suppose $k = 2$. If ν is nontrivial, then $S_0(\Gamma, \nu) = M_0(\Gamma, \nu) = \{0\}$ and Proposition 3.4 holds vacuously. If ν is trivial, then by Proposition 3.4, $f_{2,m}^{(\nu)}$ is weakly holomorphic if and only if $\left\{1, f_{2,m}^{(\nu)}\right\} = 0$. But by (4.4), we have

$$\left\{1, f_{2,m}^{(\nu)}\right\} = (1-1) \sum_{n \in I_k^{(\nu)}} \mathcal{A}_k^{(\nu)}(m, n) \sum_{(\mu) \in \Omega(\Gamma)} \omega(\Gamma, \mu) A_{2,n}(\mu) = 0$$

and $f_{2,m}^{(\nu)}$ is weakly holomorphic as desired.

In order to see that $f_{k,m}^{(\nu)}(z)$ with $m \in \widetilde{I}_k^{(\nu)}$ is a basis for $M_k^{(\infty)}(\Gamma, \nu)$, recall that the Maass–Poincaré series $\mathcal{P}_{k,m}^{(\nu)}(z)$ together with the holomorphic modular forms $M_k(\Gamma, \nu)$ span the harmonic Maass forms $H_k^{(\infty)}(\Gamma, \nu)$. The co-dimension of $M_k^{(\infty)}(\Gamma, \nu)$ inside $H_k^{(\infty)}(\Gamma, \nu)$ is the dimension of $S_{2-k}(\Gamma, \overline{\nu})$. By considering the possible orders of poles, we see that the space spanned by the $f_{k,m}^{(\nu)}(z)$ is a subset of $M_k^{(\infty)}(\Gamma, \nu)$ with the same co-dimension, and is therefore the whole space. By construction the forms $\left\{f_{k,m}^{(\nu)}(z)\right\}_{m \in \widetilde{I}_k^{(\nu)}}$ are row-reduced.

In order to verify that $g_{k,m}^{(\nu)}(z)$ is weakly holomorphic, it suffices to show that $\left\{F_{2-k,n}^{(\overline{\nu})}, g_{k,m}^{(\nu)}\right\} = 0$ for each n in $I_k^{(\overline{\nu})}$. In fact, showing that $\left\{G_{2-k,n}^{(\overline{\nu})}, g_{k,m}^{(\nu)}\right\} = 0$ would be sufficient (unless $k = 2$ and ν is trivial), but our construction of $g_{k,m}^{(\nu)}(z)$ makes the former identity easier to verify.

By the above argument, we see $g_{k,m}^{(\nu)} F_{2-k,n}^{(\overline{\nu})}$ vanishes at each cusp other than infinity, and so as before we need only consider the contribution at the cusp (∞) which is

$$\left\{F_{2-k,n}^{(\overline{\nu})}, g_{k,m}^{(\nu)}\right\} = b_k^{(\nu)}(m, -n) + A_{2-k}^{(\overline{\nu})}(n, m) + \sum_{\ell \in \mathbb{Z}^{(\nu)}} b_k^{(\nu)}(m, \ell) A_{2-k}^{(\overline{\nu})}(n, -\ell) = 0. \qquad (4.7)$$

As before, the terms outside the summation cancel directly, whereas each term of the summand itself is 0. Thus the $g_{k,m}^{(\nu)}(z)$ are all in $\widehat{M}_k^{(\infty)}(\Gamma, \nu)$.

We now have an extra step: we must show that the $g_{k,m}^{(\nu)}(z)$ vanish away from infinity. If $k > 1$, the paragraph before Proposition 3.4 tells us that the constant term of $\widehat{\mathcal{P}}_{k,m}^{(\nu)}(z)$ at the cusp (ρ) is 0 unless $(\rho) = (\infty)$; then in this case $g_{k,m}^{(\nu)}(z) \in \widehat{M}_{\nu}^{(\infty)} k\,(\Gamma, \nu)$.

Let $k < 1$. Suppose first that $k \neq 0$. Given a cusp $(\gamma) \neq (\infty)$, Lemma 4.1 tells us that the constant of $\widehat{\mathcal{P}}^{(\nu)}_{k,m}|_k\gamma$ is $-\frac{s^{(\overline{\nu})}_{2-k,\gamma}(n)}{\omega(\Gamma,\gamma)}$ Hence, the constant term of $g^{(\nu)}_{k,m}|_k\gamma$ is

$$-\frac{1}{\omega(\Gamma,\gamma)}\left(s^{(\overline{\nu})}_{2-k,\gamma}(m)-\sum_{n\in I^{(\overline{\nu})}_{2-k}} A^{(\overline{\nu})}_{2-k}(n,m)s^{(\overline{\nu})}_{2-k,\gamma}(n)\right).$$

Since the $F^{(\overline{\nu})}_{2-k,n}(z)$ form a basis for $M_{2-k}(\Gamma,\overline{\nu})$, and $E^{(\overline{\nu})}_{2-k,\gamma}(z)$ is in this space, we have

$$E^{(\overline{\nu})}_{2-k,\gamma}=\sum_{n\in I^{(\overline{\nu})}_{2-k}} s^{(\overline{\nu})}_{2-k,\gamma}(n)F^{(\overline{\nu})}_{2-k,n}(z).$$

Considering just the the mth coefficient, we find

$$s^{(\overline{\nu})}_{2-k,\gamma}(m)=\sum_{n\in I^{(\overline{\nu})}_{2-k}} A^{(\overline{\nu})}_{2-k}(n,m)s^{(\overline{\nu})}_{2-k,\gamma}(n),$$

and so the constant term of $g^{(\nu)}_{k,m}|_k\gamma$ is zero.

Now suppose $k=0$. If ν is nontrivial, then $S_0(\Gamma,\nu)=M_0(\Gamma,\nu)=\{0\}$ and $E^{\overline{\nu}}_{2,\gamma}\in M_2(\Gamma,\nu)$ by Proposition 3.4, so the above argument goes through unchanged. If ν is trivial, then by definition the constant term of $g^{(\nu)}_{0,m}|_0\gamma$ is

$$-\frac{1}{\omega(\Gamma,\gamma)}\left(s^{(\overline{\nu})}_{2,\gamma}(m)-\omega(\Gamma,\gamma)\mathcal{A}^{(\overline{\nu})}_{2}(0,m)-\sum_{n\in I^{(\overline{\nu})}_{2}} A^{(\overline{\nu})}_{2}(n,m)\left(s^{(\overline{\nu})}_{2,\gamma}(n)-\omega(\Gamma,\gamma)\mathcal{A}^{(\overline{\nu})}_{2}(0,n)\right)\right).$$

Now observe $E^{\overline{\nu}}_{2,\gamma}-\omega(\Gamma,\gamma)E^{\overline{\nu}}_{2,I}=E^{\overline{\nu}}_{2,\gamma}-\omega(\Gamma,\gamma)\mathcal{P}^{(\overline{\nu})}_{2,0}$ is holomorphic by Proposition 3.4. Then we have

$$E^{\overline{\nu}}_{2,\gamma}-\omega(\Gamma,\gamma)\mathcal{P}^{(\overline{\nu})}_{2,0}=\sum_{n\in I^{(\overline{\nu})}_{2}}\left(s^{(\overline{\nu})}_{2,\gamma}(n)-\omega(\Gamma,\gamma)\mathcal{A}^{(\overline{\nu})}_{2}(0,n)\right)F^{(\overline{\nu})}_{2-k,n}(z).$$

Considering just the mth term, we find

$$s_{2,\gamma}^{(\overline{\nu})}(m) = \omega(\Gamma,\gamma)\mathcal{A}_2^{(\overline{\nu})}(0,m) + \sum_{n\in I_2^{(\overline{\nu})}} A_2^{(\overline{\nu})}(n,m)\left(s_{2,\gamma}^{(\overline{\nu})}(n) - \omega(\Gamma,\gamma)\mathcal{A}_2^{(\overline{\nu})}(0,n)\right),$$

and so the constant term of $g_{0,m}^{(\nu)}|_0\gamma$ is zero.

As before, a co-dimension argument shows that in fact the $g_{k,m}^{(\nu)}(z)$ span $\widehat{M}_k^{(\infty)}(\Gamma,\nu)$, and by construction the forms $\left\{g_{k,m}^{(\nu)}(z)\right\}_{m\in\widetilde{J}_k^{(\nu)}}$ are row-reduced. □

Corollary 4.3. *If* $f = \sum_{n\in\mathbb{Z}^{(\nu)}} a_n q^n \in M_k^{(\infty)}(\Gamma,\nu)$ *then* $f = \sum_{n\in\widetilde{I}_k^{(\nu)}} a_n f_{k,-n}^{(\nu)}$*, and if* $f \in \widehat{M}_k^{(\infty)}(\Gamma,\nu)$ *then* $f = \sum_{n\in\widetilde{J}_k^{(\nu)}} a_n g_{k,-n}^{(\nu)}$.

4.2 Proof of Theorem 1.1

Suppose that $k > 1$. We apply the Bruinier–Funke pairing to $g_{k,n}^{(\overline{\nu})}$ and $f_{2-k,m}^{(\nu)}$. As these are both weakly holomorphic functions, we see that $\left\{g_{k,n}^{(\overline{\nu})}, f_{2-k,m}^{(\nu)}\right\} = 0$. On the other hand, since $f_{2-k,m}^{(\nu)} g_{k,n}^{(\overline{\nu})}$ vanishes at every cusp other than (∞), Theorem 3.3 gives

$$\left\{g_{k,n}^{(\overline{\nu})}, f_{2-k,m}^{(\nu)}\right\} = a_{2-k}^{(\nu)}(m,n) + b_k^{(\overline{\nu})}(n,m) + \sum_{\ell\in\mathbb{Z}_\nu} a_{2-k}^{(\nu)}(m,\ell) b_k^{(\overline{\nu})}(n,-\ell). \tag{4.8}$$

Note that $a_{2-k}^{(\nu)}(m,\ell)$ is 0 unless $\ell \in \mathbb{Z}^{(\nu)} \setminus \widetilde{I}_{2-k}^{(\nu)}$, or equivalently using (4.1), unless $\ell \in \widetilde{J}_k^{(\overline{\nu})}$. However, $b_k^{(\overline{\nu})}(n,-\ell) = 0$ for $-\ell \in \widetilde{J}_k^{(\nu)}$, so (4.8) reduces to

$$\left\{g_{k,n}^{(\overline{\nu})}, f_{2-k,m}^{(\nu)}\right\} = a_{2-k}^{(\nu)}(m,n) + b_k^{(\overline{\nu})}(n,m) = 0.$$

The case $k < 1$ is similar, but we consider $\left\{f_{2-k,m}^{(\nu)}, g_{k,n}^{(\overline{\nu})}\right\}$ instead of $\left\{g_{k,n}^{(\overline{\nu})}, f_{2-k,m}^{(\nu)}\right\}$. This concludes the proof.

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