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ABSTRACT

Congruences for Fourier Coefficients of Modular Functions of Levels 2 and 4

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We give congruences modulo powers of 2 for the Fourier coefficients of certain level 2 modular functions with poles only at 0, answering a question posed by Andersen and Jenkins. The congruences involve a modulus that depends on the binary expansion of the modular form’s order of vanishing at \( \infty \). We also demonstrate congruences for Fourier coefficients of some level 4 modular functions.

Keywords: Weakly holomorphic modular forms, congruences, Fourier coefficients
A modular form \( f(z) \) of level \( N \) and weight \( k \) is a function which is holomorphic on the complex upper half plane, satisfies the equation

\[
f \left( \frac{az + b}{cz + d} \right) = (cz + d)^k f(z) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),
\]

and is holomorphic at the cusps of \( \Gamma_0(N) \). Letting \( q = e^{2\pi iz} \), these functions have Fourier series representations of the form \( f(z) = \sum_{n=n_0}^{\infty} a(n)q^n \). A weakly holomorphic modular form is a modular form that is allowed to be meromorphic at the cusps. We define \( M_k^!(N) \) to be the space of all weight \( k \) level \( N \) weakly holomorphic modular forms and \( M_k^+(N) \) to be the subspace of forms which are holomorphic away from the cusp at \( \infty \).

The Fourier coefficients of many modular forms have interesting arithmetic properties. For instance, let \( \Delta(z) \) be the unique normalized cusp form of weight 12 for the group \( \text{SL}_2(\mathbb{Z}) \). We write

\[
\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n.
\]

Ramanujan [17] proved the congruences for \( \tau(n) \) given by

\[
\tau(pn) \equiv 0 \pmod{p} \text{ where } p \in \{2, 3, 5, 7\}.
\]

Such congruences also exist for weakly holomorphic modular forms. Lehner, in [12, 13], proved that the classical \( j \)-function \( j(z) = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n)q^n \) has the beautiful congruence

\[
c(2^a3^b5^c7^d) \equiv 0 \pmod{2^{3a+8}3^{2b+3}5^{c+1}7^d} \text{ for } a, b, c, d \geq 1. \tag{1.1}
\]

Such congruences have been extended from a single form to every element of a canonical basis for a space of forms. Kolberg [10, 11], Aas [1], and Allatt and Slater [2] strengthened...
Lehner’s congruence for the \( j \)-function, and Griffin, in [6], extended Kolberg’s and Aas’s results to all elements of a canonical basis for \( M_0^\#(1) \). Congruences and other results have been proven for the spaces \( M_k^\#(N) \) for many \( N > 1 \). For instance, Andersen, Jenkins, and Thornton [3, 8, 9] proved congruences for every element of a canonical basis for \( M_0^\#(N) \) for many \( N \), including the the genus 0 primes \( N = 2, 3, 5, \) and 7, and some prime powers, including \( N = 4 \).

Another way to generalize these results is to work with forms in \( M_k^\#(N) \), which is similar to \( M_k^\#(N) \) with elements that are holomorphic away from the cusp at 0. Taking

\[
\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)
\]
to be the Dedekind eta function, a Hauptmodul for \( \Gamma_0(p) \) where \( p = 2, 3, 5, 7, \) or 13 is

\[
\phi(p)(z) = \left( \frac{\eta(pz)}{\eta(z)} \right)^{\frac{24}{p-1}} = q + O(q^2).
\]

These functions vanish at \( \infty \) and have a pole at 0. Also, the functions \((\phi(p))^m(z)\) for \( m \geq 0 \) form a basis for \( M_0^\#(p) \). Andersen and Jenkins in [3] used powers of \( \phi(p)(z) \) to prove congruences involving

\[
\psi(p)(z) = \frac{1}{\phi(p)(z)} = q^{-1} + \cdots \in M_0^\#(p),
\]

and made the following remark: “Additionally, it appears that powers of the function \( \phi(p)(z) \) have Fourier coefficients with slightly weaker divisibility properties... It would be interesting to more fully understand these congruences.” In response, the author, Jenkins, and Keck proved congruences for the forms \( \phi^m(z) \) where \( \phi = \phi^{(2)} \).

**Theorem 1.1.** [7, Theorem 1] Write \( \phi^m(z) = \sum_{n=m}^{\infty} a(m, n)q^n \). Let \( n = 2^a n' \) where \( 2 \nmid n' \). Consider the first \( \alpha \) digits of the binary expansion of \( m \), \( a_\alpha \cdots a_2 a_1 \), padding the left with
zeroes if necessary. Let $i'$ be the index of the rightmost 1, if it exists. Let

$$\gamma(m, \alpha) = \begin{cases} \# \{ i \mid a_i = 0, i > i' \} + 1 & \text{if } i' \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$a(m, 2^{\alpha}n') \equiv 0 \pmod{2^{3\gamma(m, \alpha)}}.$$  

We note that this congruence is not sharp. For $m = 1$, Allatt and Slater in [2] proved a stronger result that provides an exact congruence for many $n$. The function $\gamma(m, \alpha)$ depends on $\alpha$ and the structure of the binary expansion of $m$, in contrast to (1.1) and most of the results previously mentioned, where the power of a prime in a congruence’s modulus is an affine function of $\alpha$.

A natural next step is to investigate congruences for forms in composite levels where we require a pole at 0 or at another cusp. We will demonstrate congruences for some level 4 modular functions. The congruence subgroup $\Gamma_0(4)$ has 3 cusps, which we take to be $\infty$, 0, and $\frac{1}{2}$, so we consider several forms which have different orders of vanishing at these cusps. We write $\phi_{c,c'}^{(4)}$ to be the normalized form in $M_0^!(4)$ which has a simple pole at the cusp $c$ and a simple zero at the cusp $c'$. We also introduce the notation

$$\left( \phi_{c,c'}^{(4)} \right)^m(z) = \sum_{n=n_0} a_{c,c'}^{(4)}(m, n)q^n.$$  

The additional results of this thesis not contained in [7] are as follows.

**Theorem 1.2.** Let $(c, c') = (0, \infty), (0, 1/2), (1/2, \infty), \text{or } (1/2, 0)$. Let $n = 2^\alpha n'$ where $2 \nmid n'$. Let $\alpha' = \lfloor \log_2(m) \rfloor + 1$, which is the number of digits in the binary expansion of $m$. Then, if $\alpha \geq \alpha' + 1$,

$$a_{c,c'}^{(4)}(m, 2^{\alpha}n') \equiv 0 \pmod{2^{3(\alpha - \alpha')}}.$$  

This congruence is not sharp. In particular, we have the following conjectures.
Conjecture 1.3. Let $\alpha$, $n'$, and $\gamma(m, \alpha)$ be as in Theorem 1.1. If $(c, c') = (0, \infty)$ or $(1/2, \infty)$, then

$$a_{c,c'}(4)(m, 2^\alpha n') \equiv 0 \pmod{2^{3\gamma(\alpha, m)}}.$$ 

Conjecture 1.4. If $(c, c') = (0, 1/2)$ or $(1/2, 0)$, then

$$a_{c,c'}(4)(m, 2^\alpha n') \equiv \begin{cases} 
0 \pmod{2^3(\alpha - 1) + 3} & \text{if } m \text{ is even and } \alpha \geq 2, \\
0 \pmod{2^{3\alpha + 3}} & \text{if } m \text{ is odd, or } m \text{ is even and } \alpha = 0, 1.
\end{cases}$$

Chapters 2 and 3 are joint work with Jenkins and Keck, and are essentially the contents of [7]. Chapter 2 contains results needed for proving Theorem 1.1, and this theorem is proved in Chapter 3. We construct the functions $\phi_{c,c'}^{(4)}$ in Chapter 4. Results for $(c, c') = (\infty, 0)$ and $(\infty, 1/2)$ follow from [8] which is explained in Chapter 4. In Chapter 5, we prove Theorem 1.2, and we discuss Conjectures 1.3 and 1.4.

**Chapter 2. Lemmas for Theorem 1.1**

The operator $U_p$ on a function $f(z)$ is given by

$$U_p f(z) = \frac{1}{p} \sum_{j=0}^{p-1} f \left( \frac{z + j}{p} \right).$$

Let $M_k^!(N)$ be the space of weakly holomorphic modular forms of weight $k$ and level $N$. We have $U_p : M_k^!(N) \to M_k^!(N)$ if $p$ divides $N$, and if $p^2 | N$, then $U_p : M_k^!(N) \to M_k^!(N/p)$. If $f(z)$ has the Fourier expansion $\sum_{n=n_0}^{\infty} a(n)q^n$, then the effect of $U_p$ on $f(z)$ is given by $U_p f(z) = \sum_{n=n_0}^{\infty} a(pm)q^n$.

The following result describes how $U_p$ applied to a modular function behaves under the Fricke involution. This will help us in Lemma 2.4 to write $U_2 \phi^m$ as a polynomial in $\phi$.

**Lemma 2.1.** [4, Theorem 4.6] Let $p$ be prime and let $f(z)$ be a level $p$ modular function.

The following result describes how $U_p$ applied to a modular function behaves under the Fricke involution. This will help us in Lemma 2.4 to write $U_2 \phi^m$ as a polynomial in $\phi$. 

**Lemma 2.1.** [4, Theorem 4.6] Let $p$ be prime and let $f(z)$ be a level $p$ modular function.
Then
\[ p(U_p f) \left( \frac{-1}{p^2 z} \right) = p(U_p f)(p^2 z) + f \left( \frac{-1}{p^2 z} \right) - f(z). \]

The Fricke involution \( \left( \begin{array}{cc} 0 & -1 \\ 2 & 0 \end{array} \right) \) swaps the cusps of \( \Gamma_0(2) \), which are 0 and \( \infty \). We will use this fact in the proof of Lemma 2.4, and the following relations between \( \phi \) and \( \psi = \frac{1}{\phi} \) will help us compute this involution.

**Lemma 2.2.** [3, Lemma 3] The functions \( \phi \) and \( \psi \) satisfy the relations

\[ \phi \left( \frac{-1}{2z} \right) = 2^{-12} \psi(z), \]
\[ \psi \left( \frac{-1}{2z} \right) = 2^{12} \phi(z). \]

The following lemma is a special case of a result by Lehner [13]. It provides a polynomial used in the proof of Theorem 3.1 whose roots are the modular forms that appear in \( U_2 \phi \).

**Lemma 2.3.** [13, Theorem 2] There exist integers \( b_j \) such that

\[ U_2 \phi(z) = 2(b_1 \phi(z) + b_2 \phi^2(z)). \]

Furthermore, let \( h(z) = 2^{12} \phi(z/2) \), \( g_1(z) = 2^{14} (b_1 \phi(z) + b_2 \phi^2(z)) \), and \( g_2(z) = -2^{14} b_2 \phi(z) \).

Then
\[ h^2(z) - g_1(z) h(z) + g_2(z) = 0. \]

In the following lemma, we extend the result from the first part of Lemma 2.3 by writing \( U_2 \phi^m \) as an integer polynomial in \( \phi \). In particular, we give the least and greatest powers of the polynomial’s nonzero terms.

**Lemma 2.4.** For all \( m \geq 1 \), \( U_2 \phi^m \in \mathbb{Z}[\phi] \). In particular,

\[ U_2 \phi^m = \sum_{j=[m/2]}^{2m} d(m, j) \phi^j \]
where \( d(m, j) \in \mathbb{Z} \), and \( d(m, \lceil m/2 \rceil) \) and \( d(m, 2m) \) are not 0.

Proof. Using Lemmas 2.1 and 2.2, we have that

\[
U_2 \phi^m(-1/2z) = U_2 \phi^m(2z) + 2^{-1} \phi^m(-1/4z) - 2^{-1} \phi^m(z)
\]

\[
= U_2 \phi^m(2z) + 2^{-1-12m} \psi^m(2z) - 2^{-1} \phi^m(z)
\]

\[
= 2^{-1-12m} q^{-2m} + O(q^{-2m+2})
\]

\[
2^{1+12m} U_2 \phi^m(-1/2z) = q^{-2m} + O(q^{-2m+2}).
\]

Because \( \phi^m \) is holomorphic at \( \infty \), \( U_2 \phi^m \) is holomorphic at \( \infty \). So \( U_2 \phi^m(-1/2z) \) is holomorphic at 0 and, since it starts with \( q^{-2m} \), must be a polynomial of degree \( 2m \) in \( \psi \). Let \( b(m, j) \in \mathbb{Z} \) such that

\[
2^{1+12m} U_2 \phi^m(-1/2z) = \sum_{j=0}^{2m} b(m, j) \psi^j(z),
\]

and we note that \( b(m, 2m) \) is not 0. Now replace \( z \) with \( -1/2z \) and use Lemma 2.2 to get

\[
2^{1+12m} U_2 \phi^m(z) = \sum_{j=0}^{2m} b(m, j) 2^{12j} \phi^j(z),
\]

which gives

\[
U_2 \phi^m(z) = \sum_{j=0}^{2m} b(m, j) 2^{12(j-m)-1} \phi^j(z).
\]

If \( m \) is even, the leading term of the above sum is \( q^{m/2} \), and if \( m \) is odd, the leading term is \( q^{(m+1)/2} \), so the sum starts with \( j = \lceil m/2 \rceil \) as desired. Notice that \( b(m, j) 2^{12(j-m)-1} \) is an integer because the coefficients of \( \phi^m \) are integers.

We may repeatedly use Lemma 2.4 to write \( U_2^\alpha \phi^m \) as a polynomial in \( \phi \). Let

\[
f(\ell) = \lceil \ell/2 \rceil, \quad f^0(\ell) = \ell, \quad \text{and } f^k(\ell) = f(f^{k-1}(\ell)). \quad (2.1)
\]
Using Lemma 2.4, the smallest power of $q$ appearing in $U_2^\alpha \phi^m$ is $f^\alpha(m)$. Lemma 2.5 provides a connection between $\gamma(m, \alpha)$ and the integers $f^\alpha(m)$.

**Lemma 2.5.** The function $\gamma(m, \alpha)$ as defined in Theorem 1.1 is equal to the number of odd integers in the list

$$m, f(m), f^2(m), \ldots, f^{\alpha-1}(m).$$

**Proof.** Write the binary expansion of $m$ as $a_r \ldots a_2 a_1$, and consider its first $\alpha$ digits, $a_\alpha \ldots a_2 a_1$, where $a_i = 0$ for $i > r$ if $\alpha > r$. If all $a_i = 0$, then all of the integers in the list are even. Otherwise, suppose that $a_i = 0$ for $1 \leq i < i'$ and $a_{i'} = 1$. Apply $f$ repeatedly to $m$, which deletes the beginning $0$s from the expansion, until $a_{i'}$ is the rightmost remaining digit; that is, $f^{i'-1}(m) = a_\alpha \ldots a_{i'-1} a_{i'}$. In particular, this integer is odd. Having reduced to the odd case, we now treat only the case where $m$ is odd.

If $m$ in the list is odd, then $a_1 = 1$, which corresponds to the $+1$ in the definition of $\gamma(m, \alpha)$. Also, $f(m) = \lceil m/2 \rceil = (m + 1)/2$. Applied to the binary expansion of $m$, this deletes $a_1$ and propagates a 1 leftward through the binary expansion, flipping 1s to 0s, and then terminating upon encountering the first 0 (if it exists), which changes to a 1. As in the even case, we apply $f$ repeatedly to delete the new leading 0s, producing one more odd output in the list once all the 0s have been deleted. Thus, each 0 to the left of $a_{i'}$ corresponds to one odd number in the list. \qed

**Chapter 3. Proof of Theorem 1.1**

Theorem 1.1 will follow from Theorem 3.1. Let $v_p(n)$ be the $p$-adic valuation of $n$.

**Theorem 3.1.** Let $f(\ell)$ be as in (2.1). Let $\gamma(m, \alpha)$ be as in Theorem 1.1, and let $\alpha \geq 1$. Define

$$c(m, j, \alpha) = \begin{cases} -1 & \text{if } f^{\alpha-1}(m) \text{ is even and is not } 2j, \\ 0 & \text{otherwise.} \end{cases}$$
Write $U_2^\alpha \phi^m = \sum_{j=f^\alpha(m)}^{2^\alpha m} d(m, j, \alpha) \phi^j$. Then

$$v_2(d(m, j, \alpha)) \geq 8(j - f^\alpha(m)) + 3\gamma(m, \alpha) + c(m, j, \alpha).$$

(3.1)

Theorem 3.1 is an improvement on the following result by Lehner [13].

**Theorem 3.2.** [13, Equation 3.4] Write $U_2^\alpha \phi^m$ as $\sum d(m, j, \alpha) \phi^j \in \mathbb{Z}[\phi]$. Then

$$v_2(d(m, j, \alpha)) \geq 8(j - 1) + 3(\alpha - m + 1) + (1 - m).$$

In particular, Lehner’s bound sometimes only gives the trivial result that the 2-adic valuation of $d(m, j, \alpha)$ is greater than some negative integer.

We prove Theorem 3.1 by induction on $\alpha$. The base case is similar to Lemma 6 from [3], which gives a subring of $\mathbb{Z}[\phi]$ which is closed under the $U_2$ operator. The polynomials in this subring are useful because their coefficients are highly divisible by 2. Here, we employ a similar technique to prove divisibility properties of the polynomial coefficients in Lemma 2.4. This method goes back to Watson [18, Section 3]. Another approach to proving the base case can be found in [5, Lemma 4.1.1]. We then induct to extend the divisibility results to the polynomials that arise from repeated application of $U_2$.

**Proof of Theorem 3.1.** For the base case, we let $\alpha = 1$, and seek to prove the statement

$$U_2 \phi^m = \sum_{j=\lfloor m/2 \rfloor}^{2m} d(m, j, 1) \phi^j$$

with

$$v_2(d(m, j, 1)) \geq 8(j - \lfloor m/2 \rfloor) + c(m, j)$$

(3.2)
where
\[ c(m, j) = \begin{cases} 
3 & m \text{ is odd}, \\
0 & m = 2j, \\
-1 & \text{otherwise}.
\end{cases} \]

The term \( c(m, j) \) combines \( c(m, j, \alpha) \) and \( 3\gamma(m, \alpha) \) for notational convenience. We prove (3.2) by induction on \( m \).

We follow the proof techniques used in Lemmas 5 and 6 of [3]. From the definition of \( U_2 \), we have
\[
U_2 \phi^m(z) = 2^{-1} \left( \phi^m \left( \frac{z}{2} \right) + \phi^m \left( \frac{z + 1}{2} \right) \right) = 2^{-1-12m} (h_0^m(z) + h_1^m(z))
\]
where \( h_{\ell}(z) = 2^{12} \phi \left( \frac{z + \ell}{2} \right) \). To understand this form, we construct a polynomial whose roots are \( h_0(z) \) and \( h_1(z) \). Let \( g_1(z) = 2^{16} \cdot 3\phi(z) + 2^{24}\phi^2(z) \) and \( g_2(z) = -2^{24}\phi(z) \). Then by Lemma 2.3, the polynomial \( F(x) = x^2 - g_1(z)x + g_2(z) \) has \( h_0(z) \) as a root. It also has \( h_1(z) \) as a root because under \( z \mapsto z + 1 \), \( h_0(z) \mapsto h_1(z) \) and the \( g_{\ell} \) are fixed.

Recall Newton’s identities for the sum of powers of roots of a polynomial. For a polynomial \( \prod_{i=1}^{n} (x - x_i) \), let \( S_\ell = x_1^\ell + \cdots + x_n^\ell \) and let \( g_\ell \) be the \( \ell \)th symmetric polynomial in the \( x_1, \ldots, x_n \). Then
\[
S_\ell = g_1 S_{\ell-1} - g_2 S_{\ell-2} + \cdots + (-1)^{\ell+1} \ell g_\ell.
\]

We apply this to the polynomial \( F(x) \), which has only two roots, to find that
\[
h_0^m(z) + h_1^m(z) = S_m = g_1 S_{m-1} - g_2 S_{m-2}.
\]

Furthermore,
\[
U_2 \phi^m = 2^{-1-12m} S_m. \tag{3.3}
\]

Lastly, let \( R \) be the set of polynomials of the form \( d(1)\phi + \sum_{n=2}^{N} d(n)\phi^n \) where for \( n \geq 2, v_2(d(n)) \geq 8(n-1) \). Now we rephrase the theorem statement in terms of \( S_m \) and elements of \( R \). When \( m \) is odd, we wish to show that for some \( r \in R, U_2 \phi^m = 2^{-8([m/2]-1)+3r} \). Performing
straightforward manipulations using (3.3), this is equivalent to $S_m = 2^{8(m+1)}r$ for some $r \in R$.
Similarly, when $m$ is even and is not $2j$, we wish to show that $U_2 \phi^m = 2^{-8 \cdot [m/2]-1}r$ for some $r \in R$. This again reduces to showing that $S_m = 2^{8(m+1)}r$ for some $r \in R$. If $m = 2j$, then (3.2) gives $8(j - [2j/2]) + 0 = 0$, which means the polynomial has integer coefficients, which is true by Lemma 2.4.

When $m = 1$ or $2$, we have that $S_m = 2^{8(m+1)}r$ for some $r \in R$, as

$$S_1 = g_1 = 2^{8(2)}(3\phi + 2^8\phi^2),$$
$$S_2 = g_1S_1 - 2g_2 = 2^{8(3)}(2\phi + 2^8\phi^2 + 2^{15}\phi^3 + 2^{24}\phi^4).$$

Now assume the equality is true for positive integers less than $m$ with $m$ at least 3. Then for some $r_1, r_2 \in R$,

$$S_m = g_1S_{m-1} - g_2S_{m-2} = (2^{16}(3\phi + 2^8\phi^2))(2^{8m}r_1) + (2^{24}\phi)(2^{8(m-1)}r_2)$$
$$= 2^{8(m+1)}[(3 \cdot 2^8\phi + 2^{16}\phi^2)r_1 + 2^8\phi r_2],$$

completing the proof where $\alpha = 1$.

Assume the theorem is true for $U_2^2 \phi^m = \sum_{j=8}^{2^m} d(j)\phi^j$, meaning

$$v_2(d(j)) \geq 8(j - f^\alpha(m)) + 3\gamma(m, \alpha) + c(m, j, \alpha).$$ (3.4)

Note that $s = f^\alpha(m)$. Letting $s' = f(s)$ and $U_2\phi^j = \sum_{i=[j/2]}^{2^j} b(j, i)\phi^i$, we define $d'(j)$ as the
integers satisfying the following equation:

\begin{align*}
U_2^{\alpha+1}\phi^m &= U_2 \left( \sum_{j=s}^{2^m} d(j)\phi^j \right) \\
 &= \sum_{j=s}^{2^m} d(j)U_2\phi^j \\
 &= \sum_{j=s}^{2^m} \sum_{i=\lceil j/2 \rceil}^{2j} d(j)b(j,i)\phi^j \\
 &= \sum_{j=s'}^{2^{\alpha+1}m} d'(j)\phi^j.
\end{align*} 

We wish to prove that

\begin{equation}
v_2(d'(j)) \geq 8(j - f^{\alpha+1}(m)) + 3\gamma(m, \alpha + 1) + c(m, j, \alpha + 1). \tag{3.6}
\end{equation}

We will prove inequalities that imply (3.6). Observe that

\begin{align*}
c(m, j, \alpha + 1) &= \begin{cases} 
-1 & \text{if } s \text{ is even and not } 2j, \\
0 & \text{if } s \text{ is odd or } s = 2j,
\end{cases} \\
\gamma(m, \alpha + 1) &= \begin{cases} 
\gamma(m, \alpha) & \text{if } s \text{ is even,} \\
\gamma(m, \alpha) + 1 & \text{if } s \text{ is odd.}
\end{cases}
\end{align*}

Also, \(c(m, s, \alpha) = 0\) because if \(f^{\alpha-1}(m)\) is even, then \(s = f^{\alpha-1}(m)/2\) so \(f^{\alpha-1}(m) = 2s\). Therefore, \(v_2(d(s)) \geq 3\gamma(m, \alpha)\) by (3.4).

If \(s\) is even, we will show that

\begin{equation}
v_2(d'(j)) \geq \max \{8(j - s') - 1 + v_2(d(s)), v_2(d(s))\}, \tag{3.7}
\end{equation}

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because then if \( j = s' \), we have

\[
v_2(d'(s')) \geq v_2(d(s)) \\
\geq 8(s' - s') + 3\gamma(m, \alpha) + c(m, s', \alpha + 1),
\]

and for all \( j \),

\[
v_2(d'(j)) \geq 8(j - s') + 3\gamma(m, \alpha) + c(m, j, \alpha + 1) \\
= 8(j - f^{\alpha+1}(m)) + 3\gamma(m, \alpha + 1) + c(m, j, \alpha + 1),
\]

so that (3.7) implies (3.6). If \( s \) is odd we will show that

\[
v_2(d'(j)) \geq 8(j - s') + 3 + v_2(d(s)),
\]

because then

\[
v_2(d'(j)) \geq 8(j - s') + 3\gamma(m, \alpha) + 3 \\
= 8(j - s') + 3(\gamma(m, \alpha) + 1) \\
= 8(j - f^{\alpha+1}(m)) + 3\gamma(m, \alpha + 1) + c(m, j, \alpha + 1),
\]

which is (3.6).

For the sake of brevity, we treat here only the case where \( s \) is odd. The case where \( s \) is even has a similar proof. This case breaks into subcases. We will only show the proof where \( j \leq 2s \), but the other cases are \( 2s < j \leq 2^{\alpha-1}m \) and \( 2^{\alpha-1}m < j \leq 2^{\alpha+1}m \), using the same subcases for when \( s \) is even. These subcases are natural to consider because in the first range of \( j \)-values, the \( d(s) \) term is included for computing \( d'(j) \), in the second range, there are no \( d(s) \) or \( d(2^\alpha m) \) terms, and in the third range, there is a \( d(2^\alpha m) \) term.

Let \( j \leq 2s \). Using (3.5), we know that \( d'(j) = \sum_{i=s}^{2j} d(i)b(i, j) \) by collecting the coefficients
of \phi^j. Let \delta(i) be given by
\[ \delta(i) = v_2(d(i)) + v_2(b(i, j)). \]

Let \( D = \{ \delta(i) \mid s \leq i \leq 2j \} \). Therefore we have
\[ v_2(d'(j)) \geq \min \{ v_2(d(i)) + v_2(b(i, j)) \mid s \leq i \leq 2j \} = \min D. \]

We claim that \( \delta(i) \) achieves its minimum with \( \delta(s) \), which proves (3.8). For that element of \( D \), we know by inequality (3.2) that
\[ \delta(s) \geq v_2(d(s)) + 8(j - s') + 3. \]

Now suppose \( i > s \). Then every element of \( D \) satisfies the following inequality:
\[ \delta(i) = v_2(d(i)) + 8(j - \lceil i/2 \rceil) + c(i, j) \geq 8(i - s) - 1 + v_2(d(s)) + 8(j - \lceil i/2 \rceil) + c(i, j) \geq 8(s + 1 - s + j - \lceil (s + 1)/2 \rceil - 2 + v_2(d(s)) = 8(j - s') + 6 + v_2(d(s)), \]

but this is clearly greater than \( \delta(s) \). Therefore, if \( j \leq 2s \) and \( s \) is odd, then \( v_2(d'(j)) \geq 8(j - s') + 3 + v_2(d(s)) \). The other cases are similar. \( \square \)

Now Theorem 1.1 follows easily from Theorem 3.1.

**Theorem 1.1.** [7, Theorem 1] Write \( \phi^m(z) = \sum_{n=m}^{\infty} a(m, n)q^n \). Let \( n = 2^\alpha n' \) where \( 2 \nmid n' \). Consider the first \( \alpha \) digits of the binary expansion of \( m, a_a \ldots a_2a_1 \), padding the left with
zeroes if necessary. Let $i'$ be the index of the rightmost 1, if it exists. Let

$$\gamma(m, \alpha) = \begin{cases} \# \{ i \mid a_i = 0, i > i' \} + 1 & \text{if } i' \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$a(m, 2^\alpha n') \equiv 0 \pmod{2^{3\gamma(m, \alpha)}}.$$

**Proof.** Letting $j = f^\alpha(m)$ in (3.1), the right hand side reduces to

$$3\gamma(m, \alpha) + c(m, f^\alpha(m), \alpha).$$

Notice that $c(m, f^\alpha(m), \alpha) = 0$, because if $f^{\alpha-1}(m)$ is even, then $f^\alpha(m) = f^{\alpha-1}(m)/2$ so $f^{\alpha-1}(m) = 2f^\alpha(m)$. The right hand side of (3.1) is minimized when $j = f^\alpha(m)$, so we conclude that $v_2(a(m, 2^\alpha n')) \geq 3\gamma(m, \alpha)$.

**Chapter 4. Constructing the level 4 Hauptmoduln**

The forms $\phi_{c,c'}^{(4)}$ can be constructed using the theory of $\eta$-quotients. We need the following theorem to compute $\eta$-quotients of the desired weight, level, and character.

**Theorem 4.1.** [15, 16] Let $N$ be a positive integer, and suppose that $f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta}$ is an $\eta$-quotient which satisfies the following congruences:

$$\sum_{\delta \mid N} \delta r_\delta \equiv 0 \pmod{24} \quad \text{and} \quad \sum_{\delta \mid N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}.$$

Then $f(z)$ is weakly modular of weight $k = \frac{1}{2} \sum_{\delta \mid N} r_\delta$ for the group $\Gamma_0(N)$ with character

$$\chi(d) = \left(\frac{-1}{d}\right)^k s$$

where $s = \prod_{\delta \mid N} \delta^{r_\delta}$. 

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We will use the following theorem to compute vanishing of $\eta$-quotients.

**Theorem 4.2.** [14] Let $c$, $d$, and $N$ be positive integers with $d|N$ and $\gcd(c, d) = 1$. If $f(z)$ is an $\eta$-quotient of level $N$, then the order of vanishing of $f(z)$ at the cusp $c/d$ is given by

$$\frac{N}{24} \sum_{\delta|N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd(d, N/d) d \delta}.$$ 

To construct the forms $\phi_{c,c'}^{(4)}$, we follow Theorem 4.1 to see that the following will guarantee a form in $M_0^!(4)$:

$$r_1 + 2r_2 + 4r_4 \equiv 0 \pmod{24}, \quad (4.1)$$

$$4r_1 + 2r_2 + r_4 \equiv 0 \pmod{24}, \quad (4.2)$$

$$2^{r_2} 4^{r_4} = \text{square of a rational number}, \quad (4.3)$$

$$r_1 + r_2 + r_4 = 0 = k. \quad (4.4)$$

If we want $\phi_{0,\infty}^{(4)}$, for example, we impose the additional condition that the form have a simple pole at 0 and a simple zero at $\infty$. We accomplish this by using Theorem 4.2. To this end, we compute

$$\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \infty = \frac{1}{4}, \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} 0 = \frac{1}{1}.$$ 

Therefore, the vanishing of a level 4 $\eta$-quotient at the cusp $c/d$ with $\gcd(c, d) = 1$ and $d|4$ is equal to

$$\frac{1}{6 \gcd(d, 4/d) d} \left( r_1 + \frac{\gcd(d, 2)^2 r_2}{2} + \frac{\gcd(d, 4)^2 r_4}{4} \right).$$ 

So, the $\eta$-quotient for $\phi_{0,\infty}^{(4)}$ must satisfy

$$-1 = \frac{1}{6} \left( r_1 + \frac{r_2}{2} + \frac{r_4}{4} \right).$$
and

\[ 1 = \frac{1}{24} (r_1 + 2r_2 + 4r_4). \]

These are equivalent to, respectively,

\[ 4r_1 + 2r_2 + r_4 = -24 \]
\[ r_1 + 2r_2 + 4r_4 = 24 \]

which are strengthenings of (4.2) and (4.1) respectively. We now have the linear system formed from these two equations and (4.4),

\[
\begin{pmatrix}
4 & 2 & 1 \\
1 & 2 & 4 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
r_1 \\
r_2 \\
r_4
\end{pmatrix}
= \begin{pmatrix}
-24 \\
24 \\
0
\end{pmatrix}
\]

which has the unique solution \((r_1, r_2, r_4) = (-8, 0, 8)\). A quick verification shows that this solution satisfies (4.3), and the order of vanishing of the corresponding form at 1/2 is 0. Therefore,

\[
\phi_{0,\infty}^{(4)}(z) = \frac{\eta(4z)^8}{\eta(z)^8} = q + 8q^2 + 44q^3 + 192q^4 + O(q^5).
\]

A similar computation provides the shapes of the remaining \(\eta\)-quotients which are found in Table 4.1. From the table, it is easy to see several symmetries, and we will prove these in Chapter 5.

The forms \(\phi_{\infty,0}^{(4)}\) and \(\phi_{\infty,1/2}^{(4)}\) are subsumed in the work of Jenkins and Thornton in [8]. In [8], the form \(f_{0,m}^{(4)}(z)\) is the element of \(M_0^2(4)\) that starts with \(q^{-m}\) and has the largest possible gap in the Fourier expansion thereafter. This is written as

\[
f_{0,m}^{(4)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_{0}^{(4)}(m, n)q^n.
\]
Table 4.1: The \( \eta \)-quotients and \( q \)-expansions of the modular functions for whose powers we will prove congruences.

These forms make up a canonical basis for the space \( M_{0}^{\sharp}(4) \), and satisfy the congruence

\[
a_0^{(4)}(2^\alpha m', 2^\beta n') \equiv \begin{cases} 
0 \pmod{2^4(\alpha-\beta)+8} & \text{if } \alpha > \beta, \\
0 \pmod{2^3(\beta-\alpha)+8} & \text{if } \beta > \alpha,
\end{cases}
\]

where \( m' \) and \( n' \) are odd [8, Theorem 2]. The \( f_{0,m}^{(4)}(z) \) basis is more convenient than the bases \( \phi_{\infty,0}^{(4)} \) and \( \phi_{\infty,1/2}^{(4)} \) because a given form is expressible in terms of the \( f_{0,m}^{(4)} \) basis by simply reading off the coefficients of the nonpositive powers of \( q \). For this reason, we will not examine congruences for \( \phi_{\infty,0}^{(4)} \) and \( \phi_{\infty,1/2}^{(4)} \).
Chapter 5. Congruences in level 4

5.1 Proof of Theorem 1.2

The main idea for proving Theorem 1.2 is to use the $U_2$ operator to bring level 4 forms down to the space $M_0^0(2)$ and to apply Theorem 1.1. Recall that $φ = φ^{(2)} \in M_0^0(2)$. The following two lemmas show that $U_2$ applied to $(φ^{(4)}_{c,c'})^m$ can be expressed as an integer polynomial in the level 2 form $φ$.

Lemma 5.1. For some integers $d(m,n)$, we have that

$$U_2 \left( φ^{(4)}_{1/2,0} \right)^m = U_2 \left( φ^{(4)}_{0,1/2} \right)^m = \sum_{n=0}^{m} d(m,n) φ^n.$$

Proof. Let $f = φ^{(4)}_{1/2,0}$. Firstly, because $2^2|4$, we have that $U_2 f \in M_0^0(2)$. Because the action of the $U_p$ operator on a $q$-expansion is $U_p \sum a(n) q^n = \sum a(pm) q^n$, we can see from the $q$-expansion of $f$ (Table 4.1) that $U_2 f^m$ is holomorphic at $∞$.

Now, we will determine the order of vanishing of $U_2 f^m$ at 0. By the definition of $U_2$, we have that

$$2(U_2 f^m) = f^m \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + f^m \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

Applying the Fricke involution $W_2$, we have

$$(2U_2 f^m) \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} = f^m \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} + f^m \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

(5.1)

The form $f^m$ has a pole of order $m$ at 1/2 and a zero of order $m$ at 0. The first term of (5.1) is an expansion of $f^m$ at 0, given by the Fricke involution $W_4$. Therefore, this term contributes no negative powers of $q$. The second term is the expansion of $f^m$ at 1/2 with the substitution $z \mapsto (2z + 1)/2$ which sends $q \mapsto -q$. Therefore, this term contributes a pole of order $m$. 

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Therefore, $U^2 f^m$ is a form in the space $M_0^+(2)$ with a pole of order $m$ at 0. We conclude that it is a polynomial in $\phi$ of degree $m$. Because $U^2 f^m$ has integer coefficients, the polynomial has integer coefficients.

For the form $\phi_{0,1/2}^{(4)}$, we reduce to the previous case. The matrix $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ swaps the cusps 0 and 1/2 and sends $q$ to $-q$. Therefore, the coefficients of $\phi_{0,1/2}^{(4)}$ and $\phi_{1/2,0}^{(4)}$ are the same up to sign. In particular, the even-indexed coefficients are equal and the odd-indexed coefficients are equal but opposite in sign. The same reasoning applies to the $m$th powers of these forms. Because $U_2$ only picks off even-indexed coefficients, the coefficients it gathers are of the same sign. We conclude that

$$U_2 \left( \phi_{0,1/2}^{(4)} \right)^m = U_2 \left( \phi_{1/2,0}^{(4)} \right)^m.$$ 

The following lemma is similar to the previous one, except that the resulting polynomial in $\phi$ has its smallest power equal to $\lceil m/2 \rceil$.

**Lemma 5.2.** For some integers $d(m,n)$, we have that

$$U_2 \left( \phi_{0,\infty}^{(4)} \right)^m = (-1)^m U_2 \left( \phi_{0,\infty}^{(4)} \right)^m = \sum_{n=\lceil m/2 \rceil}^m d(m,n) \phi^n.$$ 

**Proof.** Let $f = \phi_{0,\infty}^{(4)}$. Again, $U_2 f^m$ is a level 2 form, and it is holomorphic at $\infty$ by examining its $q$-expansion. By a similar argument, equation (5.1) shows that the pole at 0 is of order $m$. If $m$ is even, the least power of $q$ in $U_2 f^m$ is $m/2$, and if $m$ is odd, the least power is $(m + 1)/2$. Thus the least power of $\phi$ in $U_2 f^m$ is $\lceil m/2 \rceil$.

By a similar argument to that presented in Lemma 5.1, the coefficients of $\phi_{0,\infty}^{(4)}$ and $\phi_{1/2,\infty}^{(4)}$ are equal up to sign. Because we are normalizing the forms to have leading coefficient 1 and these forms begin with an odd power of $q$, the odd-indexed coefficients are equal and the even-indexed coefficients are equal but opposite in sign. The same will be true of any odd power of the two forms for the same reason. In this case, $U_2$ of both forms is equal but opposite in sign. The pattern reverses when we take an even power of the functions because
we no longer have to apply a normalizing $-1$ to every coefficient. In this case, the image of $U_2$ on both forms is equal, concluding the proof.

We now prove Theorem 1.2. We use Lemmas 5.1 and 5.2 to bring $\phi_{c,c'}^{(4)}$ down to level 2, and then we apply Theorem 1.1.

**Theorem 1.2.** Let $(c, c') = (0, \infty), (0, 1/2), (1/2, \infty), \text{ or } (1/2, 0)$. Let $n = 2^\alpha n'$ where $2 \nmid n'$.

Let $\alpha' = \lfloor \log_2(m) \rfloor + 1$, which is the number of digits in the binary expansion of $m$. Then, if $\alpha \geq \alpha' + 1$,

$$a_{c,c'}^{(4)}(m, 2^\alpha n') \equiv 0 \pmod{2^{3(\alpha - \alpha')}}.$$

**Proof.** Let $\phi^m(z) = \sum_{n=m}^{\infty} a(m, n)q^n$. Using Lemmas 5.1 and 5.2, we have that

$$U_2\left(\phi_{c,c'}^{(4)}\right)^m(z) = \sum_{n=0}^{\infty} a_{c,c'}^{(4)}(m, 2n)q^n$$

$$= \sum_{n=0}^{m} d(m, n)\phi^n(z)$$

$$= \sum_{n=0}^{m} d(m, n) \sum_{j=n}^{\infty} a(n, j)q^j$$

$$= 1 + \sum_{n=1}^{\infty} q^n \sum_{j=1}^{m} d(m, j)a(j, n).$$

By comparing coefficients, for $n \geq 1$, we have the equation

$$a_{c,c'}^{(4)}(m, 2n) = \sum_{j=1}^{m} d(m, j)a(j, n). \quad (5.2)$$
Letting \( n = 2^\beta n' \), we compute the inequality

\[
v_2 \left( a^{(4)}_{c,c'}(m, 2n) \right) = v_2 \left( a^{(4)}_{c,c'}(m, 2 \cdot 2^\beta n') \right)
\geq \min_{j=1}^m \left\{ v_2(d(m, j)a(j, 2^\beta n')) \right\}
\geq \min_{j=1}^m \left\{ v_2(a(j, 2^\beta n')) \right\}
\geq \min_{j=1}^m \{ 3\gamma(j, \beta) \}, \tag{5.3}
\]

by (5.2) and Theorem 1.1. Therefore, we see that

\[
v_2 \left( a^{(4)}_{c,c'}(m, 2n) \right) = v_2 \left( a^{(4)}_{c,c'}(m, 2 \cdot 2^\beta n') \right) \geq \min_{j=1}^m \{ 3\gamma(j, \beta) \} \tag{5.4}
\]

The value of (5.4) may be 0. To illustrate an example, recall the definition of \( \gamma(j, \beta) \):

Consider the first \( \beta \) digits of the binary expansion of \( j \), padding the left with zeroes if necessary, written \( a_\beta \cdots a_2 a_1 \). Let \( i' \) be the least index \( i \) such that \( a_i = 1 \), if it exists. Then

\[
\gamma(j, \beta) = \begin{cases} 
\# \{ i \mid a_i = 0 \text{ and } \beta \geq i > i' \} + 1 & \text{if } i' \text{ exists,} \\
0 & \text{otherwise.}
\end{cases}
\]

Therefore, if \( \beta \) is small, \( \gamma(j, \beta) \) is 0 until \( \beta \) reaches the position of the rightmost 1 in the binary expansion of \( j \). For example, \( \gamma(16, \beta) = 0 \) for \( 1 \leq \beta \leq 4 \) because \( 16_{10} = 10000_2 \). But, if we take \( \beta \) large enough, the function \( \gamma(j, \beta) \) counts the leftmost 1 and the leading 0s of the binary expansion of \( j \).

Now, let \( j \) vary between 1 and \( m \). The integer \( \alpha' = \lceil \log_2(m) \rceil + 1 \) is the leftmost position of a 1 in any of the binary expansions of the \( j \). If \( \beta \geq \alpha' \), then each of \( \gamma(j, \beta) \) will be at least 1, and incrementing \( \beta \) will increment every one of the \( \gamma(j, \beta) \). We conclude that if \( \beta \geq \alpha' \), then

\[
v_2 \left( a^{(4)}_{c,c'}(m, 2n) \right) = v_2 \left( a^{(4)}_{c,c'}(m, 2 \cdot 2^\beta n') \right) \geq 3(\beta - \alpha' + 1).
\]

For a meaningful result, we also need the assumption that \( \beta \geq 1 \) because \( \alpha' \geq 1 \).
We translate the result to be in terms of the notation used in the theorem statement.

Let $\ell = 2^\alpha \ell' = 2 \cdot 2^\beta n'$, so that

$$a_{c,c'}^{(4)}(m, \ell) = a_{c,c'}^{(4)}(m, 2 \cdot 2^\beta n').$$

In particular, this implies that $\alpha = \beta + 1$. Further, we have that

$$\beta \geq \alpha' \iff \alpha \geq \alpha' + 1$$
$$\beta \geq 1 \iff \alpha \geq 2.$$

We conclude with the result

$$\alpha \geq 2 \text{ and } \alpha \geq \alpha' + 1 \Rightarrow v_2 \left( a_{c,c'}^{(4)}(m, 2^\alpha \ell') \right) \geq 3(\alpha - \alpha').$$

The theorem is proved once we observe that $\ell$ here is $n$ in the theorem statement. \qed

5.2 CONJECTURES FOR FORMS IN LEVEL 4

Conjecture 1.3 should be true for the same reason that Theorem 1.1 is true. Recall that $U_2 \phi^m = d_{[m/2]} \phi^{[m/2]} + \cdots + d_{2m} \phi^{2m}$ for some integers $d_i$. The two key ideas in the proof of Theorem 1.1 are:

1. The valuation $v_2 \left( d_{[m/2]} \right)$ is at least 3 when $m$ is odd, and is at least 0 otherwise.

2. If $2^a || d_{[m/2]}$, then $2^a | d_i$ for $i > [m/2]$.

These are proved in the base case of Theorem 3.1, and here we summarize the process. It is easy to prove that the Fourier expansion of $\phi^m$ begins with

$$\phi^m(z) = q^m + 24mq^{m+1} + \cdots.$$
If $m$ is odd, the leading term of $U_2 \phi^m$ is $24mq^{(m+1)/2}$. So $d_{[m/2]} = d_{(m+1)/2} = 24m$ and the 2-adic valuation of this coefficient is $v_2(24m) = 3$. If $m$ is even, then the leading term of $U_2 \phi^m$ is $q^{m/2}$. The second condition above guarantees that the 2-adic valuations of the remaining coefficients is at least 3. Proving this is more difficult, and for this we employed Watson’s method [18] which used the modular equation for $\phi$.

This same pattern occurs for $\phi_{0,\infty}^{(4)}$ and $\phi_{1/2,\infty}^{(4)}$. From their Fourier expansions, it is again easy to see that

$$
\left( \phi_{0,\infty}^{(4)} \right)^m(z) = q^m + 8mq^{m+1} + \cdots
$$

$$
\left( \phi_{1/2,\infty}^{(4)} \right)^m(z) = q^m - 8mq^{m+1} + \cdots
$$

We take, for example, $\left( \phi_{0,\infty}^{(4)} \right)^m$. From Lemma 5.2, we have that

$$
U_2 \left( \phi_{0,\infty}^{(4)} \right)^m = c_{[m/2]} \phi^{[m/2]} + \cdots + c_m \phi^m
$$

for some $c_i \in \mathbb{Z}$. The first terms in these Fourier expansions are

$$
\begin{cases}
8mq^{(m+1)/2} & \text{if } m \text{ is odd}, \\
q^{m/2} & \text{if } m \text{ is even}.
\end{cases}
$$

The first case contributes $2^4$ to $c_{[m/2]}$, and the second case gives no information. This is the same pattern we saw for $\phi$. The obstacle is obtaining condition 2 for the polynomials presented in Lemma 5.2. To use Watson’s method again, we need a modular equation for $\phi_{0,\infty}^{(4)}$ and $\phi_{1/2,\infty}^{(4)}$. Lehner computes these for $\phi^{(p)}$ using Lemma 2.1, and such a result for level 4 forms has thus far eluded the author.

Conjecture 1.4 essentially states that coefficients for $\left( \phi_{1/2,0}^{(4)} \right)^m$ and $\left( \phi_{0,1/2}^{(4)} \right)^m$ follow a congruence similar to (1.1) and similar to congruences for the canonical bases for $M_0^2(N)$.
slight strengthening of Conjecture 1.4 is that, for the same $\phi_{c,c'}^{(4)}$, we have for $n > 1$,

$$v_2 \left( a_{c,c'}^{(4)}(m, 2^n n') \right) \begin{cases} 
3\alpha + v_2 \left( a_{c,c'}^{(4)}(m, n') \right) & \text{if } m \text{ is odd, or } m \text{ is even and } \alpha = 0, \\
3(\alpha - 1) + v_2 \left( a_{c,c'}^{(4)}(m, 2n') \right) & \text{if } m \text{ is even and } \alpha \geq 2, \\
\geq 3 & \text{if } \alpha = 0.
\end{cases}$$

This was formulated by observing odd-indexed coefficients and following their 2-adic valuation as the index is multiplied by 2 repeatedly. Certainly, we have Lemma 5.1, which allows us to potentially apply Theorem 1.1 after applying $U_2$, but we still lack the same result from the second bullet point above. Further, Conjecture 1.4 does not depend on the binary expansion of $m$ in any major way, so using Theorem 1.1 would not provide the desired result either way.
Bibliography


