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Congruences for Fourier Coefficients of Modular Functions of Levels 2 and 4

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A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

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ABSTRACT

Congruences for Fourier Coefficients of Modular Functions of Levels 2 and 4

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We give congruences modulo powers of 2 for the Fourier coefficients of certain level 2 modular functions with poles only at 0, answering a question posed by Andersen and Jenkins. The congruences involve a modulus that depends on the binary expansion of the modular form's order of vanishing at ∞. We also demonstrate congruences for Fourier coefficients of some level 4 modular functions.

Keywords: Weakly holomorphic modular forms, congruences, Fourier coefficients

Contents

Chapter 1. Introduction and statement of results

A modular form $f(z)$ of level N and weight k is a function which is holomorphic on the complex upper half plane, satisfies the equation

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

and is holomorphic at the cusps of $\Gamma_0(N)$. Letting $q = e^{2\pi i z}$, these functions have Fourier series representations of the form $f(z) = \sum_{n=n_0}^{\infty} a(n)q^n$. A weakly holomorphic modular form is a modular form that is allowed to be meromorphic at the cusps. We define $M_k^!(N)$ to be the space of all weight k level N weakly holomorphic modular forms and $M_k^\sharp(N)$ to be the subspace of forms which are holomorphic away from the cusp at ∞.

The Fourier coefficients of many modular forms have interesting arithmetic properties. For instance, let $\Delta(z)$ be the unique normalized cusp form of weight 12 for the group $\mathrm{SL}_2(\mathbb{Z})$. We write

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n.$$

Ramanujan [17] proved the congruences for $\tau(n)$ given by

$$\tau(pn) \equiv 0 \pmod{p} \text{ where } p \in \{2,3,5,7\}.$$

Such congruences also exist for weakly holomorphic modular forms. Lehner, in [12, 13], proved that the classical j-function $j(z) = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n)q^n$ has the beautiful congruence

$$c(2^a3^b5^c7^d) \equiv 0 \left(\mathrm{mod}\ 2^{3a+8}3^{2b+3}5^{c+1}7^d\right) \text{ for } a,b,c,d \geq 1. \tag{1.1}$$

Such congruences have been extended from a single form to every element of a canonical basis for a space of forms. Kolberg [10, 11], Aas [1], and Allatt and Slater [2] strengthened

Lehner's congruence for the j-function, and Griffin, in [6], extended Kolberg's and Aas's results to all elements of a canonical basis for $M_0^\sharp(1)$. Congruences and other results have been proven for the spaces $M_k^\sharp(N)$ for many $N > 1$. For instance, Andersen, Jenkins, and Thornton [3, 8, 9] proved congruences for every element of a canonical basis for $M_0^\sharp(N)$ for many N, including the the genus 0 primes $N = 2$, 3, 5, and 7, and some prime powers, including $N = 4$.

Another way to generalize these results is to work with forms in $M_k^\flat(N)$, which is similar to $M_k^\sharp(N)$ with elements that are holomorphic away from the cusp at 0. Taking

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

to be the Dedekind eta function, a Hauptmodul for $\Gamma_0(p)$ where $p = 2$, 3, 5, 7, or 13 is

$$\phi^{(p)}(z) = \left(\frac{\eta(pz)}{\eta(z)} \right)^{\frac{24}{p-1}} = q + O(q^2).$$

These functions vanish at ∞ and have a pole at 0. Also, the functions $\left(\phi^{(p)}\right)^m(z)$ for $m \geq 0$ form a basis for $M_0^\flat(p)$. Andersen and Jenkins in [3] used powers of $\phi^{(p)}(z)$ to prove congruences involving

$$\psi^{(p)}(z) = \frac{1}{\phi^{(p)}(z)} = q^{-1} + \cdots \in M_0^\sharp(p),$$

and made the following remark: "Additionally, it appears that powers of the function $\phi^{(p)}(z)$ have Fourier coefficients with slightly weaker divisibility properties... It would be interesting to more fully understand these congruences." In response, the author, Jenkins, and Keck proved congruences for the forms $\phi^m(z)$ where $\phi = \phi^{(2)}$.

Theorem 1.1. [7, Theorem 1] *Write $\phi^m(z) = \sum_{n=m}^{\infty} a(m,n)q^n$. Let $n = 2^\alpha n'$ where $2 \nmid n'$. Consider the first α digits of the binary expansion of m, $a_\alpha \ldots a_2 a_1$, padding the left with* *zeroes if necessary. Let i' be the index of the rightmost 1, if it exists. Let*

$$\gamma(m,\alpha) = \begin{cases} \#\{i \mid a_i = 0, i > i'\} + 1 & \text{if } i' \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$a(m, 2^\alpha n') \equiv 0 \pmod{2^{3\gamma(m,\alpha)}}.$$

We note that this congruence is not sharp. For $m = 1$, Allatt and Slater in [2] proved a stronger result that provides an exact congruence for many n. The function $\gamma(m, \alpha)$ depends on α and the structure of the binary expansion of m, in contrast to (1.1) and most of the results previously mentioned, where the power of a prime in a congruence's modulus is an affine function of α.

A natural next step is to investigate congruences for forms in composite levels where we require a pole at 0 or at another cusp. We will demonstrate congruences for some level 4 modular functions. The congruence subgroup $\Gamma_0(4)$ has 3 cusps, which we take to be ∞, 0, and $\frac{1}{2}$, so we consider several forms which have different orders of vanishing at these cusps. We write $\phi^{(4)}_{c,c'}$ to be the normalized form in $M^!_0(4)$ which has a simple pole at the cusp c and a simple zero at the cusp c'. We also introduce the notation

$$\left(\phi^{(4)}_{c,c'}\right)^m (z) = \sum_{n=n_0} a^{(4)}_{c,c'}(m,n) q^n.$$

The additional results of this thesis not contained in [7] are as follows.

Theorem 1.2. *Let $(c, c') = (0, \infty)$, $(0, 1/2)$, $(1/2, \infty)$, or $(1/2, 0)$. Let $n = 2^\alpha n'$ where $2 \nmid n'$. Let $\alpha' = \lfloor \log_2(m) \rfloor + 1$, which is the number of digits in the binary expansion of m. Then, if $\alpha \geq \alpha' + 1$,*

$$a^{(4)}_{c,c'}(m, 2^\alpha n') \equiv 0 \pmod{2^{3(\alpha-\alpha')}}.$$

This congruence is not sharp. In particular, we have the following conjectures.

Conjecture 1.3. *Let α, n', and $\gamma(m,\alpha)$ be as in Theorem 1.1. If $(c,c')=(0,\infty)$ or $(1/2,\infty)$, then*

$$a^{(4)}_{c,c'}(m,2^{\alpha}n') \equiv 0 \pmod{2^{3\gamma(\alpha,m)}}.$$

Conjecture 1.4. *If $(c,c')=(0,1/2)$ or $(1/2,0)$, then*

$$a^{(4)}_{c,c'}(m,2^{\alpha}n') \equiv \begin{cases} 0 \pmod{2^{3(\alpha-1)+3}} & \text{if } m \text{ is even and } \alpha \geq 2, \\ 0 \pmod{2^{3\alpha+3}} & \text{if } m \text{ is odd, or } m \text{ is even and } \alpha = 0,1. \end{cases}$$

Chapters 2 and 3 are joint work with Jenkins and Keck, and are essentially the contents of [7]. Chapter 2 contains results needed for proving Theorem 1.1, and this theorem is proved in Chapter 3. We construct the functions $\phi^{(4)}_{c,c'}$ in Chapter 4. Results for $(c,c')=(\infty,0)$ and $(\infty,1/2)$ follow from [8] which is explained in Chapter 4. In Chapter 5, we prove Theorem 1.2, and we discuss Conjectures 1.3 and 1.4.

Chapter 2. Lemmas for Theorem 1.1

The operator U_p on a function $f(z)$ is given by

$$U_p f(z) = \frac{1}{p}\sum_{j=0}^{p-1} f\left(\frac{z+j}{p}\right).$$

Let $M^!_k(N)$ be the space of weakly holomorphic modular forms of weight k and level N. We have $U_p : M^!_k(N) \to M^!_k(N)$ if p divides N, and if $p^2|N$, then $U_p : M^!_k(N) \to M^!_k(N/p)$. If $f(z)$ has the Fourier expansion $\sum_{n=n_0}^{\infty} a(n)q^n$, then the effect of U_p on $f(z)$ is given by $U_p f(z) = \sum_{n=n_0}^{\infty} a(pn)q^n$.

The following result describes how U_p applied to a modular function behaves under the Fricke involution. This will help us in Lemma 2.4 to write $U_2\phi^m$ as a polynomial in ϕ.

Lemma 2.1. [4, Theorem 4.6] *Let p be prime and let $f(z)$ be a level p modular function.*

Then

$$p(U_p f)\left(\frac{-1}{pz}\right) = p(U_p f)(pz) + f\left(\frac{-1}{p^2 z}\right) - f(z).$$

The Fricke involution $\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$ swaps the cusps of $\Gamma_0(2)$, which are 0 and ∞. We will use this fact in the proof of Lemma 2.4, and the following relations between ϕ and $\psi = \frac{1}{\phi}$ will help us compute this involution.

Lemma 2.2. [3, Lemma 3] *The functions ϕ and ψ satisfy the relations*

$$\phi\left(\frac{-1}{2z}\right) = 2^{-12}\psi(z),$$
$$\psi\left(\frac{-1}{2z}\right) = 2^{12}\phi(z).$$

The following lemma is a special case of a result by Lehner [13]. It provides a polynomial used in the proof of Theorem 3.1 whose roots are the modular forms that appear in $U_2\phi$.

Lemma 2.3. [13, Theorem 2] *There exist integers b_j such that*

$$U_2\phi(z) = 2(b_1\phi(z) + b_2\phi^2(z)).$$

Furthermore, let $h(z) = 2^{12}\phi(z/2)$, $g_1(z) = 2^{14}\left(b_1\phi(z) + b_2\phi^2(z)\right)$, and $g_2(z) = -2^{14}b_2\phi(z)$. Then

$$h^2(z) - g_1(z)h(z) + g_2(z) = 0.$$

In the following lemma, we extend the result from the first part of Lemma 2.3 by writing $U_2\phi^m$ as an integer polynomial in ϕ. In particular, we give the least and greatest powers of the polynomial's nonzero terms.

Lemma 2.4. *For all $m \geq 1$, $U_2\phi^m \in \mathbb{Z}[\phi]$. In particular,*

$$U_2\phi^m = \sum_{j=\lceil m/2 \rceil}^{2m} d(m,j)\phi^j$$

where $d(m,j) \in \mathbb{Z}$, and $d(m, \lceil m/2 \rceil)$ and $d(m, 2m)$ are not 0.

Proof. Using Lemmas 2.1 and 2.2, we have that

$$\begin{aligned} U_2\phi^m(-1/2z) &= U_2\phi^m(2z) + 2^{-1}\phi^m(-1/4z) - 2^{-1}\phi^m(z) \\ &= U_2\phi^m(2z) + 2^{-1-12m}\psi^m(2z) - 2^{-1}\phi^m(z) \\ &= 2^{-1-12m}q^{-2m} + O(q^{-2m+2}) \\ 2^{1+12m}U_2\phi^m(-1/2z) &= q^{-2m} + O(q^{-2m+2}). \end{aligned}$$

Because ϕ^m is holomorphic at ∞, $U_2\phi^m$ is holomorphic at ∞. So $U_2\phi^m(-1/2z)$ is holomorphic at 0 and, since it starts with q^{-2m}, must be a polynomial of degree $2m$ in ψ. Let $b(m,j) \in \mathbb{Z}$ such that

$$2^{1+12m}U_2\phi^m(-1/2z) = \sum_{j=0}^{2m} b(m,j)\psi^j(z),$$

and we note that $b(m, 2m)$ is not 0. Now replace z with $-1/2z$ and use Lemma 2.2 to get

$$2^{1+12m}U_2\phi^m(z) = \sum_{j=0}^{2m} b(m,j)2^{12j}\phi^j(z),$$

which gives

$$U_2\phi^m(z) = \sum_{j=0}^{2m} b(m,j)2^{12(j-m)-1}\phi^j(z).$$

If m is even, the leading term of the above sum is $q^{m/2}$, and if m is odd, the leading term is $q^{(m+1)/2}$, so the sum starts with $j = \lceil m/2 \rceil$ as desired. Notice that $b(m,j)2^{12(j-m)-1}$ is an integer because the coefficients of ϕ^m are integers. □

We may repeatedly use Lemma 2.4 to write $U_2^\alpha \phi^m$ as a polynomial in ϕ. Let

$$f(\ell) = \lceil \ell/2 \rceil\,, \ f^0(\ell) = \ell, \ \text{and} \ f^k(\ell) = f(f^{k-1}(\ell)). \tag{2.1}$$

Using Lemma 2.4, the smallest power of q appearing in $U_2^\alpha \phi^m$ is $f^\alpha(m)$. Lemma 2.5 provides a connection between $\gamma(m,\alpha)$ and the integers $f^\alpha(m)$.

Lemma 2.5. *The function $\gamma(m,\alpha)$ as defined in Theorem 1.1 is equal to the number of odd integers in the list*

$$m, f(m), f^2(m), \ldots, f^{\alpha-1}(m).$$

Proof. Write the binary expansion of m as $a_r \ldots a_2 a_1$, and consider its first α digits, $a_\alpha \ldots a_2 a_1$, where $a_i = 0$ for $i > r$ if $\alpha > r$. If all $a_i = 0$, then all of the integers in the list are even. Otherwise, suppose that $a_i = 0$ for $1 \le i < i'$ and $a_{i'} = 1$. Apply f repeatedly to m, which deletes the beginning 0s from the expansion, until $a_{i'}$ is the rightmost remaining digit; that is, $f^{i'-1}(m) = a_\alpha \ldots a_{i'-1} a_{i'}$. In particular, this integer is odd. Having reduced to the odd case, we now treat only the case where m is odd.

If m in the list is odd, then $a_1 = 1$, which corresponds to the $+1$ in the definition of $\gamma(m,\alpha)$. Also, $f(m) = \lceil m/2 \rceil = (m+1)/2$. Applied to the binary expansion of m, this deletes a_1 and propagates a 1 leftward through the binary expansion, flipping 1s to 0s, and then terminating upon encountering the first 0 (if it exists), which changes to a 1. As in the even case, we apply f repeatedly to delete the new leading 0s, producing one more odd output in the list once all the 0s have been deleted. Thus, each 0 to the left of $a_{i'}$ corresponds to one odd number in the list. □

Chapter 3. Proof of Theorem 1.1

Theorem 1.1 will follow from Theorem 3.1. Let $v_p(n)$ be the p-adic valuation of n.

Theorem 3.1. *Let $f(\ell)$ be as in (2.1). Let $\gamma(m,\alpha)$ be as in Theorem 1.1, and let $\alpha \ge 1$. Define*

$$c(m,j,\alpha) = \begin{cases} -1 & \text{if } f^{\alpha-1}(m) \text{ is even and is not } 2j, \\ 0 & \text{otherwise.} \end{cases}$$

Write $U_2^\alpha \phi^m = \sum_{j=f^\alpha(m)}^{2^\alpha m} d(m,j,\alpha)\phi^j$. *Then*

$$v_2(d(m,j,\alpha)) \geq 8(j - f^\alpha(m)) + 3\gamma(m,\alpha) + c(m,j,\alpha). \tag{3.1}$$

Theorem 3.1 is an improvement on the following result by Lehner [13].

Theorem 3.2. [13, Equation 3.4] *Write* $U_2^\alpha \phi^m$ *as* $\sum d(m,j,\alpha)\phi^j \in \mathbb{Z}[\phi]$. *Then*

$$v_2(d(m,j,\alpha)) \geq 8(j-1) + 3(\alpha - m + 1) + (1-m).$$

In particular, Lehner's bound sometimes only gives the trivial result that the 2-adic valuation of $d(m,j,\alpha)$ is greater than some negative integer.

We prove Theorem 3.1 by induction on α. The base case is similar to Lemma 6 from [3], which gives a subring of $\mathbb{Z}[\phi]$ which is closed under the U_2 operator. The polynomials in this subring are useful because their coefficients are highly divisible by 2. Here, we employ a similar technique to prove divisibility properties of the polynomial coefficients in Lemma 2.4. This method goes back to Watson [18, Section 3]. Another approach to proving the base case can be found in [5, Lemma 4.1.1]. We then induct to extend the divisibility results to the polynomials that arise from repeated application of U_2.

Proof of Theorem 3.1. For the base case, we let $\alpha = 1$, and seek to prove the statement

$$U_2 \phi^m = \sum_{j=\lceil m/2 \rceil}^{2m} d(m,j,1)\phi^j$$

with

$$v_2(d(m,j,1)) \geq 8(j - \lceil m/2 \rceil) + c(m,j) \tag{3.2}$$

where

$$c(m,j) = \begin{cases} 3 & m \text{ is odd,} \\ 0 & m = 2j, \\ -1 & \text{otherwise.} \end{cases}$$

The term $c(m,j)$ combines $c(m,j,\alpha)$ and $3\gamma(m,\alpha)$ for notational convenience. We prove (3.2) by induction on m.

We follow the proof techniques used in Lemmas 5 and 6 of [3]. From the definition of U_2, we have

$$U_2\phi^m(z) = 2^{-1}\left(\phi^m\left(\frac{z}{2}\right) + \phi^m\left(\frac{z+1}{2}\right)\right) = 2^{-1-12m}\left(h_0^m(z) + h_1^m(z)\right)$$

where $h_\ell(z) = 2^{12}\phi\left(\frac{z+\ell}{2}\right)$. To understand this form, we construct a polynomial whose roots are $h_0(z)$ and $h_1(z)$. Let $g_1(z) = 2^{16}\cdot 3\phi(z) + 2^{24}\phi^2(z)$ and $g_2(z) = -2^{24}\phi(z)$. Then by Lemma 2.3, the polynomial $F(x) = x^2 - g_1(z)x + g_2(z)$ has $h_0(z)$ as a root. It also has $h_1(z)$ as a root because under $z \mapsto z+1$, $h_0(z) \mapsto h_1(z)$ and the g_ℓ are fixed.

Recall Newton's identities for the sum of powers of roots of a polynomial. For a polynomial $\prod_{i=1}^n (x - x_i)$, let $S_\ell = x_1^\ell + \cdots + x_n^\ell$ and let g_ℓ be the ℓth symmetric polynomial in the $x_1, \ldots, x_n$. Then

$$S_\ell = g_1 S_{\ell-1} - g_2 S_{\ell-2} + \cdots + (-1)^{\ell+1}\ell g_\ell.$$

We apply this to the polynomial $F(x)$, which has only two roots, to find that

$$h_0^m(z) + h_1^m(z) = S_m = g_1 S_{m-1} - g_2 S_{m-2}.$$

Furthermore,

$$U_2\phi^m = 2^{-1-12m} S_m. \tag{3.3}$$

Lastly, let R be the set of polynomials of the form $d(1)\phi + \sum_{n=2}^N d(n)\phi^n$ where for $n \geq 2$, $v_2(d(n)) \geq 8(n-1)$. Now we rephrase the theorem statement in terms of S_m and elements of R. When m is odd, we wish to show that for some $r \in R$, $U_2\phi^m = 2^{-8(\lceil m/2 \rceil - 1)+3} r$. Performing straightforward manipulations using (3.3), this is equivalent to $S_m = 2^{8(m+1)}r$ for some $r \in R$. Similarly, when m is even and is not $2j$, we wish to show that $U_2\phi^m = 2^{-8(\lceil m/2\rceil-1)-1}r$ for some $r \in R$. This again reduces to showing that $S_m = 2^{8(m+1)}r$ for some $r \in R$. If $m = 2j$, then (3.2) gives $8(j - \lceil 2j/2\rceil) + 0 = 0$, which means the polynomial has integer coefficients, which is true by Lemma 2.4.

When $m = 1$ or 2, we have that $S_m = 2^{8(m+1)}r$ for some $r \in R$, as

$$S_1 = g_1 = 2^{8(2)}(3\phi + 2^8\phi^2),$$
$$S_2 = g_1S_1 - 2g_2 = 2^{8(3)}(2\phi + 2^8 3^2\phi^2 + 2^{17}\phi^3 + 2^{24}\phi^4).$$

Now assume the equality is true for positive integers less than m with m at least 3. Then for some $r_1, r_2 \in R$,

$$\begin{aligned} S_m &= g_1S_{m-1} - g_2S_{m-2} \\ &= (2^{16}(3\phi + 2^8\phi^2))(2^{8m}r_1) + (2^{24}\phi)(2^{8(m-1)}r_2) \\ &= 2^{8(m+1)}[(3\cdot 2^8\phi + 2^{16}\phi^2)r_1 + 2^8\phi r_2], \end{aligned}$$

completing the proof where $\alpha = 1$.

Assume the theorem is true for $U_2^\alpha\phi^m = \sum_{j=s}^{2^\alpha m} d(j)\phi^j$, meaning

$$v_2(d(j)) \geq 8(j - f^\alpha(m)) + 3\gamma(m,\alpha) + c(m,j,\alpha). \tag{3.4}$$

Note that $s = f^\alpha(m)$. Letting $s' = f(s)$ and $U_2\phi^j = \sum_{i=\lceil j/2\rceil}^{2j} b(j,i)\phi^i$, we define $d'(j)$ as the integers satisfying the following equation:

$$\begin{aligned}
U_2^{\alpha+1}\phi^m &= U_2\left(\sum_{j=s}^{2^\alpha m} d(j)\phi^j\right)\\
&= \sum_{j=s}^{2^\alpha m} d(j)U_2\phi^j\\
&= \sum_{j=s}^{2^\alpha m}\ \sum_{i=\lceil j/2\rceil}^{2j} d(j)b(j,i)\phi^i\\
&= \sum_{j=s'}^{2^{\alpha+1} m} d'(j)\phi^j. \qquad (3.5)
\end{aligned}$$

We wish to prove that

$$v_2(d'(j)) \geq 8(j - f^{\alpha+1}(m)) + 3\gamma(m,\alpha+1) + c(m,j,\alpha+1). \qquad (3.6)$$

We will prove inequalities that imply (3.6). Observe that

$$c(m,j,\alpha+1) = \begin{cases} -1 & \text{if } s \text{ is even and not } 2j,\\ 0 & \text{if } s \text{ is odd or } s = 2j,\end{cases}$$

and

$$\gamma(m,\alpha+1) = \begin{cases} \gamma(m,\alpha) & \text{if } s \text{ is even},\\ \gamma(m,\alpha)+1 & \text{if } s \text{ is odd}.\end{cases}$$

Also, $c(m,s,\alpha) = 0$ because if $f^{\alpha-1}(m)$ is even, then $s = f^{\alpha-1}(m)/2$ so $f^{\alpha-1}(m) = 2s$. Therefore, $v_2(d(s)) \geq 3\gamma(m,\alpha)$ by (3.4).

If s is even, we will show that

$$v_2(d'(j)) \geq \max\left\{8\left(j - s'\right) - 1 + v_2(d(s)), v_2(d(s))\right\}, \qquad (3.7)$$

because then if $j = s'$, we have

$$\begin{aligned} v_2(d'(s')) &\geq v_2(d(s)) \\ &\geq 8(s' - s') + 3\gamma(m, \alpha) + c(m, s', \alpha + 1), \end{aligned}$$

and for all j,

$$\begin{aligned} v_2(d'(j)) &\geq 8(j - s') + 3\gamma(m, \alpha) + c(m, j, \alpha + 1) \\ &= 8(j - f^{\alpha+1}(m)) + 3\gamma(m, \alpha + 1) + c(m, j, \alpha + 1), \end{aligned}$$

so that (3.7) implies (3.6). If s is odd we will show that

$$v_2(d'(j)) \geq 8\,(j - s') + 3 + v_2(d(s)), \tag{3.8}$$

because then

$$\begin{aligned} v_2(d'(j)) &\geq 8(j - s') + 3\gamma(m, \alpha) + 3 \\ &= 8(j - s') + 3(\gamma(m, \alpha) + 1) \\ &= 8(j - f^{\alpha+1}(m)) + 3\gamma(m, \alpha + 1) + c(m, j, \alpha + 1), \end{aligned}$$

which is (3.6).

For the sake of brevity, we treat here only the case where s is odd. The case where s is even has a similar proof. This case breaks into subcases. We will only show the proof where $j \leq 2s$, but the other cases are $2s < j \leq 2^{\alpha-1}m$ and $2^{\alpha-1}m < j \leq 2^{\alpha+1}m$, using the same subcases for when s is even. These subcases are natural to consider because in the first range of j-values, the $d(s)$ term is included for computing $d'(j)$, in the second range, there are no $d(s)$ or $d(2^\alpha m)$ terms, and in the third range, there is a $d(2^\alpha m)$ term.

Let $j \leq 2s$. Using (3.5), we know that $d'(j) = \sum_{i=s}^{2j} d(i)b(i, j)$ by collecting the coefficients of ϕ^j. Let $\delta(i)$ be given by

$$\delta(i) = v_2(d(i)) + v_2(b(i,j)).$$

Let $D = \{\delta(i) \mid s \le i \le 2j\}$. Therefore we have

$$\begin{aligned} v_2(d'(j)) &\ge \min\{v_2(d(i)) + v_2(b(i,j)) \mid s \le i \le 2j\} \\ &= \min D. \end{aligned}$$

We claim that $\delta(i)$ achieves its minimum with $\delta(s)$, which proves (3.8). For that element of D, we know by inequality (3.2) that

$$\delta(s) \ge v_2(d(s)) + 8(j - s') + 3.$$

Now suppose $i > s$. Then every element of D satisfies the following inequality:

$$\begin{aligned} \delta(i) &= v_2(d(i)) + 8\left(j - \lceil i/2 \rceil\right) + c(i,j) \\ &\ge 8\,(i - s) - 1 + v_2(d(s)) + 8\left(j - \lceil i/2 \rceil\right) + c(i,j) \\ &\ge 8\left(s + 1 - s + j - \lceil (s+1)/2 \rceil\right) - 2 + v_2(d(s)) \\ &= 8\,(j - s') + 6 + v_2(d(s)), \end{aligned}$$

but this is clearly greater than $\delta(s)$. Therefore, if $j \le 2s$ and s is odd, then $v_2(d'(j)) \ge 8\,(j - s') + 3 + v_2(d(s))$. The other cases are similar. $\square$

Now Theorem 1.1 follows easily from Theorem 3.1.

Theorem 1.1. [7, Theorem 1] *Write* $\phi^m(z) = \sum_{n=m}^{\infty} a(m,n) q^n$. *Let* $n = 2^\alpha n'$ *where* $2 \nmid n'$. *Consider the first* α *digits of the binary expansion of* m, $a_\alpha \ldots a_2 a_1$, *padding the left with* *zeroes if necessary. Let i' be the index of the rightmost 1, if it exists. Let*

$$\gamma(m,\alpha) = \begin{cases} \#\{i \mid a_i = 0, i > i'\} + 1 & \text{if } i' \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$a(m, 2^\alpha n') \equiv 0 \pmod{2^{3\gamma(m,\alpha)}}.$$

Proof. Letting $j = f^\alpha(m)$ in (3.1), the right hand side reduces to

$$3\gamma(m,\alpha) + c(m, f^\alpha(m), \alpha).$$

Notice that $c(m, f^\alpha(m), \alpha) = 0$, because if $f^{\alpha-1}(m)$ is even, then $f^\alpha(m) = f^{\alpha-1}(m)/2$ so $f^{\alpha-1}(m) = 2f^\alpha(m)$. The right hand side of (3.1) is minimized when $j = f^\alpha(m)$, so we conclude that $v_2(a(m, 2^\alpha n')) \geq 3\gamma(m,\alpha)$. □

Chapter 4. Constructing the level 4 Hauptmoduln

The forms $\phi^{(4)}_{c,c'}$ can be constructed using the theory of η-quotients. We need the following theorem to compute η-quotients of the desired weight, level, and character.

Theorem 4.1. [15, 16] *Let N be a positive integer, and suppose that $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an η-quotient which satisfies the following congruences:*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24} \quad \text{and} \quad \sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}.$$

Then $f(z)$ is weakly modular of weight $k = \frac{1}{2}\sum_{\delta|N} r_\delta$ for the group $\Gamma_0(N)$ with character

$$\chi(d) = \left(\frac{(-1)^k s}{d}\right) \quad \text{where} \quad s = \prod_{\delta|N} \delta^{r_\delta}.$$

We will use the following theorem to compute vanishing of η-quotients.

Theorem 4.2. [14] *Let c, d, and N be positive integers with $d|N$ and $\gcd(c,d)=1$. If $f(z)$ is an η-quotient of level N, then the order of vanishing of $f(z)$ at the cusp c/d is given by*

$$\frac{N}{24}\sum_{\delta|N}\frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d,N/d)d\delta}.$$

To construct the forms $\phi_{c,c'}^{(4)}$, we follow Theorem 4.1 to see that the following will guarantee a form in $M_0^!(4)$:

$$r_1+2r_2+4r_4\equiv 0 \pmod{24}, \tag{4.1}$$

$$4r_1+2r_2+r_4\equiv 0 \pmod{24}, \tag{4.2}$$

$$2^{r_2}4^{r_4} = \text{square of a rational number}, \tag{4.3}$$

$$r_1+r_2+r_4=0=k. \tag{4.4}$$

If we want $\phi_{0,\infty}^{(4)}$, for example, we impose the additional condition that the form have a simple pole at 0 and a simple zero at ∞. We accomplish this by using Theorem 4.2. To this end, we compute

$$\begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}\infty=\frac{1}{4}, \quad \text{and} \quad \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}0=\frac{1}{1}.$$

Therefore, the vanishing of a level 4 η-quotient at the cusp c/d with $\gcd(c,d)=1$ and $d|4$ is equal to

$$\frac{1}{6\gcd(d,4/d)d}\left(r_1+\frac{\gcd(d,2)^2r_2}{2}+\frac{\gcd(d,4)^2r_4}{4}\right).$$

So, the η-quotient for $\phi_{0,\infty}^{(4)}$ must satisfy

$$-1=\frac{1}{6}\left(r_1+\frac{r_2}{2}+\frac{r_4}{4}\right),$$

and

$$1 = \frac{1}{24}\left(r_1 + 2r_2 + 4r_4\right).$$

These are equivalent to, respectively,

$$4r_1 + 2r_2 + r_4 = -24$$

$$r_1 + 2r_2 + 4r_4 = 24$$

which are strengthenings of (4.2) and (4.1) respectively. We now have the linear system formed from these two equations and (4.4),

$$\begin{pmatrix} 4 & 2 & 1 \\ 1 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_4 \end{pmatrix} = \begin{pmatrix} -24 \\ 24 \\ 0 \end{pmatrix}$$

which has the unique solution $(r_1, r_2, r_4) = (-8, 0, 8)$. A quick verification shows that this solution satisfies (4.3), and the order of vanishing of the corresponding form at 1/2 is 0. Therefore,

$$\phi_{0,\infty}^{(4)}(z) = \frac{\eta(4z)^8}{\eta(z)^8} = q + 8q^2 + 44q^3 + 192q^4 + O(q^5).$$

A similar computation provides the shapes of the remaining η-quotients which are found in Table 4.1. From the table, it is easy to see several symmetries, and we will prove these in Chapter 5.

The forms $\phi_{\infty,0}^{(4)}$ and $\phi_{\infty,1/2}^{(4)}$ are subsumed in the work of Jenkins and Thornton in [8]. In [8], the form $f_{0,m}^{(4)}(z)$ is the element of $M_0^\sharp(4)$ that starts with q^{-m} and has the largest possible gap in the Fourier expansion thereafter. This is written as

$$f_{0,m}^{(4)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_0^{(4)}(m,n)q^n.$$

Modular function	η-quotient	q-expansion
$\phi^{(4)}_{\infty,0}(z)$	$\dfrac{\eta(z)^8}{\eta(4z)^8}$	$q^{-1} - 8 + 20q - 62q^3 + 216q^5 + O(q^7)$
$\phi^{(4)}_{\infty,1/2}(z)$	$\dfrac{\eta(2z)^{24}}{\eta(z)^8\eta(4z)^{16}}$	$q^{-1} + 8 + 20q - 62q^3 + 216q^5 + O(q^7)$
$\phi^{(4)}_{1/2,0}(z)$	$\dfrac{\eta(z)^{16}\eta(4z)^8}{\eta(2z)^{24}}$	$1 - 16q + 128q^2 - 704q^3 + O(q^4)$
$\phi^{(4)}_{0,1/2}(z)$	$\dfrac{\eta(2z)^{24}}{\eta(z)^{16}\eta(4z)^8}$	$1 + 16q + 128q^2 + 704q^3 + O(q^4)$
$\phi^{(4)}_{0,\infty}(z)$	$\dfrac{\eta(4z)^8}{\eta(z)^8}$	$q + 8q^2 + 44q^3 + 192q^4 + O(q^5)$
$\phi^{(4)}_{1/2,\infty}(z)$	$\dfrac{\eta(z)^8\eta(4z)^{16}}{\eta(2z)^{24}}$	$q - 8q^2 + 44q^3 - 192q^4 + O(q^5)$

Table 4.1: The η-quotients and q-expansions of the modular functions for whose powers we will prove congruences.

These forms make up a canonical basis for the space $M_0^\sharp(4)$, and satisfy the congruence

$$a_0^{(4)}(2^\alpha m', 2^\beta n') \equiv \begin{cases} 0 \pmod{2^{4(\alpha-\beta)+8}} & \text{if } \alpha > \beta, \\ 0 \pmod{2^{3(\beta-\alpha)+8}} & \text{if } \beta > \alpha, \end{cases}$$

where m' and n' are odd [8, Theorem 2]. The $f^{(4)}_{0,m}(z)$ basis is more convenient than the bases $\left(\phi^{(4)}_{\infty,0}\right)^m$ and $\left(\phi^{(4)}_{\infty,1/2}\right)^m$ because a given form is expressible in terms of the $f^{(4)}_{0,m}$ basis by simply reading off the coefficients of the nonpositive powers of q. For this reason, we will not examine congruences for $\left(\phi^{(4)}_{\infty,0}\right)^m$ and $\left(\phi^{(4)}_{\infty,1/2}\right)^m$.

Chapter 5. Congruences in level 4

5.1 Proof of Theorem 1.2

The main idea for proving Theorem 1.2 is to use the U_2 operator to bring level 4 forms down to the space $M_0^\flat(2)$ and to apply Theorem 1.1. Recall that $\phi = \phi^{(2)} \in M_0^\flat(2)$. The following two lemmas show that U_2 applied to $\left(\phi_{c,c'}^{(4)}\right)^m$ can be expressed as an integer polynomial in the level 2 form ϕ.

Lemma 5.1. *For some integers $d(m,n)$, we have that*

$$U_2\left(\phi_{1/2,0}^{(4)}\right)^m = U_2\left(\phi_{0,1/2}^{(4)}\right)^m = \sum_{n=0}^{m} d(m,n)\phi^n.$$

Proof. Let $f = \phi_{1/2,0}^{(4)}$. Firstly, because $2^2|4$, we have that $U_2 f \in M_0^!(2)$. Because the action of the U_p operator on a q-expansion is $U_p \sum a(n)q^n = \sum a(pn)q^n$, we can see from the q-expansion of f (Table 4.1) that $U_2 f^m$ is holomorphic at ∞.

Now, we will determine the order of vanishing of $U_2 f^m$ at 0. By the definition of U_2, we have that

$$2(U_2 f^m) = f^m \left| \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + f^m \right| \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

Applying the Fricke involution W_2, we have

$$(2U_2 f^m) \left| \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} = f^m \right| \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} + f^m \left| \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \right| \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}. \tag{5.1}$$

The form f^m has a pole of order m at $1/2$ and a zero of order m at 0. The first term of (5.1) is an expansion of f^m at 0, given by the Fricke involution W_4. Therefore, this term contributes no negative powers of q. The second term is the expansion of f^m at $1/2$ with the substitution $z \mapsto (2z+1)/2$ which sends $q \mapsto -q$. Therefore, this term contributes a pole of order m.

Therefore, $U_2 f^m$ is a form in the space $M_0^\flat(2)$ with a pole of order m at 0. We conclude that it is a polynomial in ϕ of degree m. Because $U_2 f^m$ has integer coefficients, the polynomial has integer coefficients.

For the form $\phi^{(4)}_{0,1/2}$, we reduce to the previous case. The matrix $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ swaps the cusps 0 and 1/2 and sends q to $-q$. Therefore, the coefficients of $\phi^{(4)}_{0,1/2}$ and $\phi^{(4)}_{1/2,0}$ are the same up to sign. In particular, the even-indexed coefficients are equal and the odd-indexed coefficients are equal but opposite in sign. The same reasoning applies to the mth powers of these forms. Because U_2 only picks off even-indexed coefficients, the coefficients it gathers are of the same sign. We conclude that

$$U_2 \left(\phi^{(4)}_{0,1/2}\right)^m = U_2 \left(\phi^{(4)}_{1/2,0}\right)^m .$$

□

The following lemma is similar to the previous one, except that the resulting polynomial in ϕ has its smallest power equal to $\lceil m/2 \rceil$.

Lemma 5.2. *For some integers $d(m,n)$, we have that*

$$U_2 \left(\phi^{(4)}_{0,\infty}\right)^m = (-1)^m U_2 \left(\phi^{(4)}_{0,\infty}\right)^m = \sum_{n=\lceil m/2 \rceil}^{m} d(m,n)\phi^n.$$

Proof. Let $f = \phi^{(4)}_{0,\infty}$. Again, $U_2 f^m$ is a level 2 form, and it is holomorphic at ∞ by examining its q-expansion. By a similar argument, equation (5.1) shows that the pole at 0 is of order m. If m is even, the least power of q in $U_2 f^m$ is $m/2$, and if m is odd, the least power is $(m+1)/2$. Thus the least power of ϕ in $U_2 f^m$ is $\lceil m/2 \rceil$.

By a similar argument to that presented in Lemma 5.1, the coefficients of $\phi^{(4)}_{0,\infty}$ and $\phi^{(4)}_{1/2,\infty}$ are equal up to sign. Because we are normalizing the forms to have leading coefficient 1 and these forms begin with an odd power of q, the odd-indexed coefficients are equal and the even-indexed coefficients are equal but opposite in sign. The same will be true of any odd power of the two forms for the same reason. In this case, U_2 of both forms is equal but opposite in sign. The pattern reverses when we take an even power of the functions because we no longer have to apply a normalizing -1 to every coefficient. In this case, the image of U_2 on both forms is equal, concluding the proof. $\square$

We now prove Theorem 1.2. We use Lemmas 5.1 and 5.2 to bring $\phi_{c,c'}^{(4)}$ down to level 2, and then we apply Theorem 1.1.

Theorem 1.2. *Let* $(c, c') = (0, \infty)$, $(0, 1/2)$, $(1/2, \infty)$, *or* $(1/2, 0)$. *Let* $n = 2^\alpha n'$ *where* $2 \nmid n'$. *Let* $\alpha' = \lfloor \log_2(m) \rfloor + 1$, *which is the number of digits in the binary expansion of* m. *Then, if* $\alpha \geq \alpha' + 1$,

$$a_{c,c'}^{(4)}(m, 2^\alpha n') \equiv 0 \pmod{2^{3(\alpha-\alpha')}}.$$

Proof. Let $\phi^m(z) = \sum_{n=m}^{\infty} a(m,n)q^n$. Using Lemmas 5.1 and 5.2, we have that

$$\begin{aligned} U_2\left(\phi_{c,c'}^{(4)}\right)^m(z) &= \sum_{n=0}^{\infty} a_{c,c'}^{(4)}(m, 2n)q^n \\ &= \sum_{n=0}^{m} d(m,n)\phi^n(z) \\ &= \sum_{n=0}^{m} d(m,n) \sum_{j=n}^{\infty} a(n,j)q^j \\ &= 1 + \sum_{n=1}^{\infty} q^n \sum_{j=1}^{m} d(m,j)a(j,n). \end{aligned}$$

By comparing coefficients, for $n \geq 1$, we have the equation

$$a_{c,c'}^{(4)}(m, 2n) = \sum_{j=1}^{m} d(m,j)a(j,n). \tag{5.2}$$

Letting $n = 2^\beta n'$, we compute the inequality

$$\begin{aligned} v_2\left(a_{c,c'}^{(4)}(m,2n)\right) &= v_2\left(a_{c,c'}^{(4)}(m,2\cdot 2^\beta n')\right) \\ &\geq \min_{j=1}^{m}\left\{v_2(d(m,j)a(j,2^\beta n'))\right\} \\ &\geq \min_{j=1}^{m}\left\{v_2(a(j,2^\beta n'))\right\} \\ &\geq \min_{j=1}^{m}\left\{3\gamma(j,\beta)\right\}, \end{aligned} \tag{5.3}$$

by (5.2) and Theorem 1.1. Therefore, we see that

$$v_2\left(a_{c,c'}^{(4)}(m,2n)\right) = v_2\left(a_{c,c'}^{(4)}(m,2\cdot 2^\beta n')\right) \geq \min_{j=1}^{m}\left\{3\gamma(j,\beta)\right\}. \tag{5.4}$$

The value of (5.4) may be 0. To illustrate an example, recall the definition of $\gamma(j,\beta)$: Consider the first β digits of the binary expansion of j, padding the left with zeroes if necessary, written $a_\beta \cdots a_2 a_1$. Let i' be the least index i such that $a_i = 1$, if it exists. Then

$$\gamma(j,\beta) = \begin{cases} \#\left\{i \mid a_i = 0 \text{ and } \beta \geq i > i'\right\} + 1 & \text{if } i' \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if β is small, $\gamma(j,\beta)$ is 0 until β reaches the position of the rightmost 1 in the binary expansion of j. For example, $\gamma(16,\beta) = 0$ for $1 \leq \beta \leq 4$ because $16_{10} = 10000_2$. But, if we take β large enough, the function $\gamma(j,\beta)$ counts the leftmost 1 and the leading 0s of the binary expansion of j.

Now, let j vary between 1 and m. The integer $\alpha' = \lfloor \log_2(m) \rfloor + 1$ is the leftmost position of a 1 in any of the binary expansions of the j. If $\beta \geq \alpha'$, then each of $\gamma(j,\beta)$ will be at least 1, and incrementing β will increment every one of the $\gamma(j,\beta)$. We conclude that if $\beta \geq \alpha'$, then

$$v_2\left(a_{c,c'}^{(4)}(m,2n)\right) = v_2\left(a_{c,c'}^{(4)}(m,2\cdot 2^\beta n')\right) \geq 3(\beta - \alpha' + 1).$$

For a meaningful result, we also need the assumption that $\beta \geq 1$ because $\alpha' \geq 1$.

We translate the result to be in terms of the notation used in the theorem statement. Let $\ell = 2^{\alpha}\ell' = 2 \cdot 2^{\beta} n'$, so that

$$a^{(4)}_{c,c'}(m, \ell) = a^{(4)}_{c,c'}(m, 2 \cdot 2^{\beta} n').$$

In particular, this implies that $\alpha = \beta + 1$. Further, we have that

$$\beta \geq \alpha' \iff \alpha \geq \alpha' + 1$$
$$\beta \geq 1 \iff \alpha \geq 2.$$

We conclude with the result

$$\alpha \geq 2 \text{ and } \alpha \geq \alpha' + 1 \Rightarrow v_2\left(a^{(4)}_{c,c'}(m, 2^{\alpha}\ell')\right) \geq 3(\alpha - \alpha').$$

The theorem is proved once we observe that ℓ here is n in the theorem statement. □

5.2 Conjectures for forms in level 4

Conjecture 1.3 should be true for the same reason that Theorem 1.1 is true. Recall that $U_2\phi^m = d_{\lceil m/2 \rceil}\phi^{\lceil m/2 \rceil} + \cdots + d_{2m}\phi^{2m}$ for some integers d_i. The two key ideas in the proof of Theorem 1.1 are:

(1) The valuation $v_2\left(d_{\lceil m/2 \rceil}\right)$ is at least 3 when m is odd, and is at least 0 otherwise.

(2) If $2^a || d_{\lceil m/2 \rceil}$, then $2^a | d_i$ for $i > \lceil m/2 \rceil$.

These are proved in the base case of Theorem 3.1, and here we summarize the process. It is easy to prove that the Fourier expansion of ϕ^m begins with

$$\phi^m(z) = q^m + 24mq^{m+1} + \cdots .$$

If m is odd, the leading term of $U_2\phi^m$ is $24mq^{(m+1)/2}$. So $d_{\lceil m/2\rceil} = d_{(m+1)/2} = 24m$ and the 2-adic valuation of this coefficient is $v_2(24m) = 3$. If m is even, then the leading term of $U_2\phi^m$ is $q^{m/2}$. The second condition above guarantees that the 2-adic valuations of the remaining coefficients is at least 3. Proving this is more difficult, and for this we employed Watson's method [18] which used the modular equation for ϕ.

This same pattern occurs for $\phi^{(4)}_{0,\infty}$ and $\phi^{(4)}_{1/2,\infty}$. From their Fourier expansions, it is again easy to see that

$$\left(\phi^{(4)}_{0,\infty}\right)^m(z) = q^m + 8mq^{m+1} + \cdots$$

$$\left(\phi^{(4)}_{1/2,\infty}\right)^m(z) = q^m - 8mq^{m+1} + \cdots$$

We take, for example, $\left(\phi^{(4)}_{0,\infty}\right)^m$. From Lemma 5.2, we have that

$$U_2\left(\phi^{(4)}_{0,\infty}\right)^m = c_{\lceil m/2\rceil}\phi^{\lceil m/2\rceil} + \cdots + c_m\phi^m$$

for some $c_i \in \mathbb{Z}$. The first terms in these Fourier expansions are

$$\begin{cases} 8mq^{(m+1)/2} & \text{if } m \text{ is odd,} \\ q^{m/2} & \text{if } m \text{ is even.} \end{cases}$$

The first case contributes 2^3 to $c_{\lceil m/2\rceil}$, and the second case gives no information. This is the same pattern we saw for ϕ. The obstacle is obtaining condition 2 for the polynomials presented in Lemma 5.2. To use Watson's method again, we need a modular equation for $\phi^{(4)}_{0,\infty}$ and $\phi^{(4)}_{1/2,\infty}$. Lehner computes these for $\phi^{(p)}$ using Lemma 2.1, and such a result for level 4 forms has thus far eluded the author.

Conjecture 1.4 essentially states that coefficients for $\left(\phi^{(4)}_{1/2,0}\right)^m$ and $\left(\phi^{(4)}_{0,1/2}\right)^m$ follow a congruence similar to (1.1) and similar to congruences for the canonical bases for $M_0^\sharp(N)$. A slight strengthening of Conjecture 1.4 is that, for the same $\phi^{(4)}_{c,c'}$, we have for $n > 1$,

$$v_2\left(a^{(4)}_{c,c'}(m, 2^\alpha n')\right) \begin{cases} = 3\alpha + v_2\left(a^{(4)}_{c,c'}(m, n')\right) & \text{if } m \text{ is odd, or } m \text{ is even and } \alpha = 0, \\ = 3(\alpha - 1) + v_2\left(a^{(4)}_{c,c'}(m, 2n')\right) & \text{if } m \text{ is even and } \alpha \geq 2, \\ \geq 3 & \text{if } \alpha = 0. \end{cases}$$

This was formulated by observing odd-indexed coefficients and following their 2-adic valuation as the index is multiplied by 2 repeatedly. Certainly, we have Lemma 5.1, which allows us to potentially apply Theorem 1.1 after applying U_2, but we still lack the same result from the second bullet point above. Further, Conjecture 1.4 does not depend on the binary expansion of m in any major way, so using Theorem 1.1 would not provide the desired result either way.

Bibliography

[1] Hans-Fredrik Aas. Congruences for the coefficients of the modular invariant $j(\tau)$. *Math. Scand.*, 15:64–68, 1964.

[2] P. Allatt and J. B. Slater. Congruences on some special modular forms. *J. London Math. Soc. (2)*, 17(3):380–392, 1978.

[3] Nickolas Andersen and Paul Jenkins. Divisibility properties of coefficients of level p modular functions for genus zero primes. *Proc. Amer. Math. Soc.*, 141(1):41–53, 2013.

[4] Tom M. Apostol. *Modular functions and Dirichlet series in number theory*, volume 41 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.

[5] Frank Calegari. Congruences between modular forms. http://swc.math.arizona.edu/aws/2013/2013CalegariLectureNotes.pdf, 42, 2013. Accessed January 2018.

[6] Michael Griffin. Divisibility properties of coefficients of weight 0 weakly holomorphic modular forms. *Int. J. Number Theory*, 7(4):933–941, 2011.

[7] Paul Jenkins, Ryan Keck, and Eric Moss. Congruences for coefficients of level 2 modular functions with poles at 0. To appear in *Archiv der Mathematik*. https://rdcu.be/005b. doi:10.1007/s00013-018-1207-8.

[8] Paul Jenkins and D. J. Thornton. Congruences for coefficients of modular functions. *Ramanujan J.*, 38(3):619–628, 2015.

[9] Paul Jenkins and D. J. Thornton. Weakly holomorphic modular forms in prime power levels of genus zero. *Integers*, 18:Paper No. A18, 12, 2018.

[10] O. Kolberg. The coefficients of $j(\tau)$ modulo powers of 3. *Arbok Univ. Bergen Mat.-Natur. Ser.*, 1962(16):7, 1962.

[11] O. Kolberg. Congruences for the coefficients of the modular invariant $j(\tau)$. *Math. Scand.*, 10:173–181, 1962.

[12] Joseph Lehner. Divisibility properties of the Fourier coefficients of the modular invariant $j(\tau)$. *Amer. J. Math.*, 71:136–148, 1949.

[13] Joseph Lehner. Further congruence properties of the Fourier coefficients of the modular invariant $j(\tau)$. *Amer. J. Math.*, 71:373–386, 1949.

[14] Gérard Ligozat. *Courbes modulaires de genre* 1. Société Mathématique de France, Paris, 1975. Bull. Soc. Math. France, Mém. 43, Supplément au Bull. Soc. Math. France Tome 103, no. 3.

[15] Morris Newman. Construction and application of a class of modular functions. *Proc. London. Math. Soc. (3)*, 7:334–350, 1957.

[16] Morris Newman. Construction and application of a class of modular functions. II. *Proc. London Math. Soc. (3)*, 9:373–387, 1959.

[17] S. Ramanujan. Congruence properties of partitions [Proc. London Math. Soc. (2) **18** (1920), Records for 13 March 1919]. In *Collected papers of Srinivasa Ramanujan*, page 230. AMS Chelsea Publ., Providence, RI, 2000.

[18] G. N. Watson. Ramanujans Vermutung über Zerfällungszahlen. *J. Reine Angew. Math.*, 179:97–128, 1938.