Congruences for Fourier Coefficients of Modular Functions of Levels 2 and 4

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Congruences for Fourier Coefficients of Modular Functions of Levels 2 and 4

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We give congruences modulo powers of 2 for the Fourier coefficients of certain level 2 modular functions with poles only at 0, answering a question posed by Andersen and Jenkins. The congruences involve a modulus that depends on the binary expansion of the modular form’s order of vanishing at $\infty$. We also demonstrate congruences for Fourier coefficients of some level 4 modular functions.

Keywords: Weakly holomorphic modular forms, congruences, Fourier coefficients


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Chapter 1. Introduction and Statement of Results

A modular form $f(z)$ of level $N$ and weight $k$ is a function which is holomorphic on the complex upper half plane, satisfies the equation

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

and is holomorphic at the cusps of $\Gamma_0(N)$. Letting $q = e^{2\pi i z}$, these functions have Fourier series representations of the form $f(z) = \sum_{n=n_0}^{\infty} a(n)q^n$. A weakly holomorphic modular form is a modular form that is allowed to be meromorphic at the cusps. We define $M_k^!(N)$ to be the space of all weight $k$ level $N$ weakly holomorphic modular forms and $M_k^*(N)$ to be the subspace of forms which are holomorphic away from the cusp at $\infty$.

The Fourier coefficients of many modular forms have interesting arithmetic properties. For instance, let $\Delta(z)$ be the unique normalized cusp form of weight 12 for the group $SL_2(\mathbb{Z})$. We write

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n.$$

Ramanujan [17] proved the congruences for $\tau(n)$ given by

$$\tau(pn) \equiv 0 \pmod{p} \text{ where } p \in \{2, 3, 5, 7\}.$$

Such congruences also exist for weakly holomorphic modular forms. Lehner, in [12, 13], proved that the classical $j$-function $j(z) = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n)q^n$ has the beautiful congruence

$$c(2a 3^b 5^c 7^d) \equiv 0 \pmod{2^{3a+8}3^{2b+3}5^{c+1}7^d} \text{ for } a, b, c, d \geq 1. \quad (1.1)$$

Such congruences have been extended from a single form to every element of a canonical basis for a space of forms. Kolberg [10, 11], Aas [1], and Allatt and Slater [2] strengthened
Lehner’s congruence for the $j$-function, and Griffin, in [6], extended Kolberg’s and Aas’s results to all elements of a canonical basis for $M_k^\#(1)$. Congruences and other results have been proven for the spaces $M_k^\#(N)$ for many $N > 1$. For instance, Andersen, Jenkins, and Thornton [3, 8, 9] proved congruences for every element of a canonical basis for $M_0^\#(N)$ for many $N$, including the the genus 0 primes $N = 2, 3, 5, 7$, and some prime powers, including $N = 4$.  

Another way to generalize these results is to work with forms in $M_k^\#(N)$, which is similar to $M_k^\#(N)$ with elements that are holomorphic away from the cusp at 0. Taking

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

for $\eta(z)$ to be the Dedekind eta function, a Hauptmodul for $\Gamma_0(p)$ where $p = 2, 3, 5, 7, 13$ is

$$\phi(p)(z) = \left( \frac{\eta(pz)}{\eta(z)} \right)^{\frac{24}{p-1}} = q + O(q^2).$$

These functions vanish at $\infty$ and have a pole at 0. Also, the functions $(\phi(p))^m(z)$ for $m \geq 0$ form a basis for $M_0^\#(p)$. Andersen and Jenkins in [3] used powers of $\phi(p)(z)$ to prove congruences involving

$$\psi(p)(z) = \frac{1}{\phi(p)(z)} = q^{-1} + \cdots \in M_0^\#(p),$$

and made the following remark: “Additionally, it appears that powers of the function $\phi(p)(z)$ have Fourier coefficients with slightly weaker divisibility properties... It would be interesting to more fully understand these congruences.” In response, the author, Jenkins, and Keck proved congruences for the forms $\phi^m(z)$ where $\phi = \phi(2)$.

**Theorem 1.1.** [7, Theorem 1] Write $\phi^m(z) = \sum_{n=m}^{\infty} a(m, n)q^n$. Let $n = 2^n n'$ where $2 \nmid n'$. Consider the first $\alpha$ digits of the binary expansion of $m$, $a_\alpha \ldots a_2 a_1$, padding the left with
zeroes if necessary. Let $i'$ be the index of the rightmost 1, if it exists. Let

$$\gamma(m, \alpha) = \begin{cases} 
\#\{i \mid a_i = 0, i > i'\} + 1 & \text{if } i' \text{ exists}, \\
0 & \text{otherwise}.
\end{cases}$$

Then

$$a(m, 2^\alpha n') \equiv 0 \pmod{2^{\gamma(m, \alpha)}}.$$ 

We note that this congruence is not sharp. For $m = 1$, Allatt and Slater in [2] proved a stronger result that provides an exact congruence for many $n$. The function $\gamma(m, \alpha)$ depends on $\alpha$ and the structure of the binary expansion of $m$, in contrast to (1.1) and most of the results previously mentioned, where the power of a prime in a congruence’s modulus is an affine function of $\alpha$.

A natural next step is to investigate congruences for forms in composite levels where we require a pole at 0 or at another cusp. We will demonstrate congruences for some level 4 modular functions. The congruence subgroup $\Gamma_0(4)$ has 3 cusps, which we take to be $\infty$, 0, and $\frac{1}{2}$, so we consider several forms which have different orders of vanishing at these cusps.

We write $\phi_{c,c'}^{(4)}$ to be the normalized form in $M_0^!(4)$ which has a simple pole at the cusp $c$ and a simple zero at the cusp $c'$. We also introduce the notation

$$\left(\phi_{c,c'}^{(4)}\right)^m(z) = \sum_{n=n_0} a_{c,c'}^{(4)}(m, n)q^n.$$ 

The additional results of this thesis not contained in [7] are as follows.

**Theorem 1.2.** Let $(c, c') = (0, \infty), (0, 1/2), (1/2, \infty)$, or $(1/2, 0)$. Let $n = 2^\alpha n'$ where $2 \nmid n'$. Let $\alpha' = \lfloor \log_2(m) \rfloor + 1$, which is the number of digits in the binary expansion of $m$. Then, if $\alpha \geq \alpha' + 1,$

$$a_{c,c'}^{(4)}(m, 2^\alpha n') \equiv 0 \pmod{2^{3(\alpha - \alpha')}}.$$ 

This congruence is not sharp. In particular, we have the following conjectures.
Conjecture 1.3. Let $\alpha$, $n'$, and $\gamma(m, \alpha)$ be as in Theorem 1.1. If $(c, c') = (0, \infty)$ or $(1/2, \infty)$, then
\[ a_{c,c'}^{(4)}(m, 2^\alpha n') \equiv 0 \pmod{2^{3\gamma(\alpha, m)}}. \]

Conjecture 1.4. If $(c, c') = (0, 1/2)$ or $(1/2, 0)$, then
\[ a_{c,c'}^{(4)}(m, 2^\alpha n') \equiv \begin{cases} 0 \pmod{2^{3(\alpha-1)+3}} & \text{if } m \text{ is even and } \alpha \geq 2, \\ 0 \pmod{2^{3\alpha+3}} & \text{if } m \text{ is odd, or } m \text{ is even and } \alpha = 0, 1. \end{cases} \]

Chapters 2 and 3 are joint work with Jenkins and Keck, and are essentially the contents of [7]. Chapter 2 contains results needed for proving Theorem 1.1, and this theorem is proved in Chapter 3. We construct the functions $\phi_{c,c'}^{(4)}$ in Chapter 4. Results for $(c, c') = (\infty, 0)$ and $(\infty, 1/2)$ follow from [8] which is explained in Chapter 4. In Chapter 5, we prove Theorem 1.2, and we discuss Conjectures 1.3 and 1.4.

Chapter 2. Lemmas for Theorem 1.1

The operator $U_p$ on a function $f(z)$ is given by
\[ U_p f(z) = \frac{1}{p} \sum_{j=0}^{p-1} \left( \frac{z+j}{p} \right). \]

Let $M_k^!(N)$ be the space of weakly holomorphic modular forms of weight $k$ and level $N$. We have $U_p : M_k^!(N) \to M_k^!(N)$ if $p$ divides $N$, and if $p^2 | N$, then $U_p : M_k^!(N) \to M_k^!(N/p)$. If $f(z)$ has the Fourier expansion $\sum_{n=n_0}^{\infty} a(n) q^n$, then the effect of $U_p$ on $f(z)$ is given by
\[ U_p f(z) = \sum_{n=n_0}^{\infty} a(pm) q^n. \]

The following result describes how $U_p$ applied to a modular function behaves under the Fricke involution. This will help us in Lemma 2.4 to write $U_2 \phi^m$ as a polynomial in $\phi$.

Lemma 2.1. [4, Theorem 4.6] Let $p$ be prime and let $f(z)$ be a level $p$ modular function.
Then
\[
p(U_p f) \left( \frac{-1}{pz} \right) = p(U_p f)(pz) + f \left( \frac{-1}{p^2z} \right) - f(z).
\]

The Fricke involution \( \left( \begin{array}{cc} 0 & -1 \\ 2 & 0 \end{array} \right) \) swaps the cusps of \( \Gamma_0(2) \), which are 0 and \( \infty \). We will use this fact in the proof of Lemma 2.4, and the following relations between \( \phi \) and \( \psi = \frac{1}{\phi} \) will help us compute this involution.

**Lemma 2.2.** [3, Lemma 3] The functions \( \phi \) and \( \psi \) satisfy the relations

\[
\begin{align*}
\phi \left( \frac{-1}{2z} \right) &= 2^{-12}\psi(z), \\
\psi \left( \frac{-1}{2z} \right) &= 2^{12}\phi(z).
\end{align*}
\]

The following lemma is a special case of a result by Lehner [13]. It provides a polynomial used in the proof of Theorem 3.1 whose roots are the modular forms that appear in \( U_2 \phi \).

**Lemma 2.3.** [13, Theorem 2] There exist integers \( b_j \) such that

\[
U_2 \phi(z) = 2(b_1 \phi(z) + b_2 \phi^2(z)).
\]

Furthermore, let \( h(z) = 2^{12}\phi(z/2) \), \( g_1(z) = 2^{14} (b_1 \phi(z) + b_2 \phi^2(z)) \), and \( g_2(z) = -2^{14}b_2 \phi(z) \). Then

\[
h^2(z) - g_1(z)h(z) + g_2(z) = 0.
\]

In the following lemma, we extend the result from the first part of Lemma 2.3 by writing \( U_2 \phi^m \) as an integer polynomial in \( \phi \). In particular, we give the least and greatest powers of the polynomial’s nonzero terms.

**Lemma 2.4.** For all \( m \geq 1 \), \( U_2 \phi^m \in \mathbb{Z}[\phi] \). In particular,

\[
U_2 \phi^m = \sum_{j=\lceil m/2 \rceil}^{2m} d(m, j) \phi^j
\]
where $d(m, j) \in \mathbb{Z}$, and $d(m, \lfloor m/2 \rfloor)$ and $d(m, 2m)$ are not 0.

**Proof.** Using Lemmas 2.1 and 2.2, we have that

$$U_2 \phi^m(-1/2z) = U_2 \phi^m(2z) + 2^{-1} \phi^m(-1/4z) - 2^{-1} \phi^m(z)$$

$$= U_2 \phi^m(2z) + 2^{-1-12m} \psi^m(2z) - 2^{-1} \phi^m(z)$$

$$= 2^{-1-12m} q^{-2m} + O(q^{-2m+2})$$

$$2^{1+12m} U_2 \phi^m(-1/2z) = q^{-2m} + O(q^{-2m+2}).$$

Because $\phi^m$ is holomorphic at $\infty$, $U_2 \phi^m$ is holomorphic at $\infty$. So $U_2 \phi^m(-1/2z)$ is holomorphic at 0 and, since it starts with $q^{-2m}$, must be a polynomial of degree $2m$ in $\psi$. Let $b(m, j) \in \mathbb{Z}$ such that

$$2^{1+12m} U_2 \phi^m(-1/2z) = \sum_{j=0}^{2m} b(m, j) \psi^j(z),$$

and we note that $b(m, 2m)$ is not 0. Now replace $z$ with $-1/2z$ and use Lemma 2.2 to get

$$2^{1+12m} U_2 \phi^m(z) = \sum_{j=0}^{2m} b(m, j) 2^{12j} \phi^j(z),$$

which gives

$$U_2 \phi^m(z) = \sum_{j=0}^{2m} b(m, j) 2^{12(j-m)-1} \phi^j(z).$$

If $m$ is even, the leading term of the above sum is $q^{m/2}$, and if $m$ is odd, the leading term is $q^{(m+1)/2}$, so the sum starts with $j = \lfloor m/2 \rfloor$ as desired. Notice that $b(m, j) 2^{12(j-m)-1}$ is an integer because the coefficients of $\phi^m$ are integers. 

We may repeatedly use Lemma 2.4 to write $U_2^\alpha \phi^m$ as a polynomial in $\phi$. Let

$$f(\ell) = \lfloor \ell/2 \rfloor, \ f^0(\ell) = \ell, \text{ and } f^k(\ell) = f(f^{k-1}(\ell)).$$

(2.1)
Using Lemma 2.4, the smallest power of $q$ appearing in $U_2^\alpha \phi^m$ is $f^\alpha(m)$. Lemma 2.5 provides a connection between $\gamma(m, \alpha)$ and the integers $f^\alpha(m)$.

**Lemma 2.5.** The function $\gamma(m, \alpha)$ as defined in Theorem 1.1 is equal to the number of odd integers in the list

\[ m, f(m), f^2(m), \ldots, f^{\alpha-1}(m). \]

**Proof.** Write the binary expansion of $m$ as $a_r \ldots a_2 a_1$, and consider its first $\alpha$ digits, $a_\alpha \ldots a_2 a_1$, where $a_i = 0$ for $i > r$ if $\alpha > r$. If all $a_i = 0$, then all of the integers in the list are even. Otherwise, suppose that $a_i = 0$ for $1 \leq i < i'$ and $a_{i'} = 1$. Apply $f$ repeatedly to $m$, which deletes the beginning 0s from the expansion, until $a_{i'}$ is the rightmost remaining digit; that is, $f^{i'-1}(m) = a_\alpha \ldots a_{i'-1} a_{i'}$. In particular, this integer is odd. Having reduced to the odd case, we now treat only the case where $m$ is odd.

If $m$ in the list is odd, then $a_1 = 1$, which corresponds to the +1 in the definition of $\gamma(m, \alpha)$. Also, $f(m) = \lceil m/2 \rceil = (m + 1)/2$. Applied to the binary expansion of $m$, this deletes $a_1$ and propagates a 1 leftward through the binary expansion, flipping 1s to 0s, and then terminating upon encountering the first 0 (if it exists), which changes to a 1. As in the even case, we apply $f$ repeatedly to delete the new leading 0s, producing one more odd output in the list once all the 0s have been deleted. Thus, each 0 to the left of $a_{i'}$ corresponds to one odd number in the list. 

**Chapter 3. Proof of Theorem 1.1**

Theorem 1.1 will follow from Theorem 3.1. Let $v_p(n)$ be the $p$-adic valuation of $n$.

**Theorem 3.1.** Let $f(\ell)$ be as in (2.1). Let $\gamma(m, \alpha)$ be as in Theorem 1.1, and let $\alpha \geq 1$. Define

\[
c(m, j, \alpha) = \begin{cases} 
-1 & \text{if } f^{\alpha-1}(m) \text{ is even and is not } 2j, \\
0 & \text{otherwise.}
\end{cases}
\]
Write $U_2^\alpha \phi^m = \sum_{j=f^\alpha(m)}^{2^\alpha m} d(m, j, \alpha) \phi^j$. Then

$$v_2(d(m, j, \alpha)) \geq 8(j - f^\alpha(m)) + 3\gamma(m, \alpha) + c(m, j, \alpha).$$  \hspace{1cm} (3.1)

Theorem 3.1 is an improvement on the following result by Lehner [13].

**Theorem 3.2.** [13, Equation 3.4] Write $U_2^\alpha \phi^m$ as $\sum d(m, j, \alpha) \phi^j \in \mathbb{Z}[\phi]$. Then

$$v_2(d(m, j, \alpha)) \geq 8(j - 1) + 3(\alpha - m + 1) + (1 - m).$$

In particular, Lehner’s bound sometimes only gives the trivial result that the 2-adic valuation of $d(m, j, \alpha)$ is greater than some negative integer.

We prove Theorem 3.1 by induction on $\alpha$. The base case is similar to Lemma 6 from [3], which gives a subring of $\mathbb{Z}[\phi]$ which is closed under the $U_2$ operator. The polynomials in this subring are useful because their coefficients are highly divisible by 2. Here, we employ a similar technique to prove divisibility properties of the polynomial coefficients in Lemma 2.4. This method goes back to Watson [18, Section 3]. Another approach to proving the base case can be found in [5, Lemma 4.1.1]. We then induct to extend the divisibility results to the polynomials that arise from repeated application of $U_2$.

**Proof of Theorem 3.1.** For the base case, we let $\alpha = 1$, and seek to prove the statement

$$U_2 \phi^m = \sum_{j=[m/2]}^{2m} d(m, j, 1) \phi^j$$

with

$$v_2(d(m, j, 1)) \geq 8(j - [m/2]) + c(m, j) \hspace{1cm} (3.2)$$
where

\[ c(m, j) = \begin{cases} 
3 & m \text{ is odd}, \\
0 & m = 2j, \\
-1 & \text{otherwise}.
\end{cases} \]

The term \( c(m, j) \) combines \( c(m, j, \alpha) \) and \( 3\gamma(m, \alpha) \) for notational convenience. We prove (3.2) by induction on \( m \).

We follow the proof techniques used in Lemmas 5 and 6 of [3]. From the definition of \( U_2 \), we have

\[
U_2 \phi^m(z) = 2^{-1} \left( \phi^m \left( \frac{z}{2} \right) + \phi^m \left( \frac{z + 1}{2} \right) \right) = 2^{-1-12m} (h_0^m(z) + h_1^m(z))
\]

where \( h_\ell(z) = 2^{12\ell} \left( \frac{z+\ell}{2} \right) \). To understand this form, we construct a polynomial whose roots are \( h_0(z) \) and \( h_1(z) \). Let \( g_1(z) = 2^{16} \cdot 3\phi(z) + 2^{24}\phi^2(z) \) and \( g_2(z) = -2^{24}\phi(z) \). Then by Lemma 2.3, the polynomial \( F(x) = x^2 - g_1(z)x + g_2(z) \) has \( h_0(z) \) as a root. It also has \( h_1(z) \) as a root because under \( z \mapsto z + 1 \), \( h_0(z) \mapsto h_1(z) \) and the \( g_\ell \) are fixed.

Recall Newton’s identities for the sum of powers of roots of a polynomial. For a polynomial \( \prod_{i=1}^n (x - x_i) \), let \( S_\ell = x_1^\ell + \cdots + x_n^\ell \) and let \( g_\ell \) be the \( \ell \)th symmetric polynomial in the \( x_1, \ldots, x_n \). Then

\[
S_\ell = g_1 S_{\ell-1} - g_2 S_{\ell-2} + \cdots + (-1)^{\ell+1} \ell g_\ell.
\]

We apply this to the polynomial \( F(x) \), which has only two roots, to find that

\[
h_0^m(z) + h_1^m(z) = S_m = g_1 S_{m-1} - g_2 S_{m-2}.
\]

Furthermore,

\[
U_2 \phi^m = 2^{-1-12m} S_m. \tag{3.3}
\]

Lastly, let \( R \) be the set of polynomials of the form \( d(1)\phi + \sum_{n=2}^N d(n)\phi^n \) where for \( n \geq 2 \), \( v_2(d(n)) \geq 8(n-1) \). Now we rephrase the theorem statement in terms of \( S_m \) and elements of \( R \). When \( m \) is odd, we wish to show that for some \( r \in R, U_2 \phi^m = 2^{-8([m/2]-1)+3r} \). Performing
straightforward manipulations using (3.3), this is equivalent to \( S_m = 2^{8(m+1)}r \) for some \( r \in R \). Similarly, when \( m \) is even and is not \( 2j \), we wish to show that \( U_2\phi^m = 2^{-8([m/2]−1)}r \) for some \( r \in R \). This again reduces to showing that \( S_m = 2^{8(m+1)}r \) for some \( r \in R \). If \( m = 2j \), then (3.2) gives \( 8(j − [2j/2]) + 0 = 0 \), which means the polynomial has integer coefficients, which is true by Lemma 2.4.

When \( m = 1 \) or 2, we have that \( S_m = 2^{8(m+1)}r \) for some \( r \in R \), as

\[
S_1 = g_1 = 2^{8(2)}(3\phi + 2^8\phi^2),
\]

\[
S_2 = g_1S_1 - 2g_2 = 2^{8(3)}(2\phi + 2^83^2\phi^2 + 2^{17}\phi^3 + 2^{24}\phi^4).
\]

Now assume the equality is true for positive integers less than \( m \) with \( m \) at least 3. Then for some \( r_1, r_2 \in R \),

\[
S_m = g_1S_{m-1} - g_2S_{m-2}
\]

\[
= (2^{16}(3\phi + 2^8\phi^2))(2^{8m}r_1) + (2^{24}\phi)(2^{8(m-1)}r_2)
\]

\[
= 2^{8(m+1)}[(3 \cdot 2^8\phi + 2^{16}\phi^2)r_1 + 2^8\phi r_2],
\]

completing the proof where \( \alpha = 1 \).

Assume the theorem is true for \( U_2\phi^m = \sum_{j=8}^{2^m} d(j)\phi^j \), meaning

\[
v_2(d(j)) \geq 8(j - f^\alpha(m)) + 3\gamma(m, \alpha) + c(m, j, \alpha). \tag{3.4}
\]

Note that \( s = f^\alpha(m) \). Letting \( s' = f(s) \) and \( U_2\phi^j = \sum_{i=[j/2]}^{2j} b(j, i)\phi^i \), we define \( d'(j) \) as the
integers satisfying the following equation:

\[
U_2^{\alpha+1} \phi^m = U_2 \left( \sum_{j=s}^{2^m} d(j) \phi^j \right) \\
= \sum_{j=s}^{2^m} d(j) U_2 \phi^j \\
= \sum_{j=s}^{2^m} \sum_{i=\lfloor j/2 \rfloor}^{2j} d(j) b(j, i) \phi^i \\
= \sum_{j=s'}^{2^{m+1}} d'(j) \phi^j. \tag{3.5}
\]

We wish to prove that

\[
v_2(d'(j)) \geq 8(j - f^{\alpha+1}(m)) + 3\gamma(m, \alpha + 1) + c(m, j, \alpha + 1). \tag{3.6}
\]

We will prove inequalities that imply (3.6). Observe that

\[
c(m, j, \alpha + 1) = \begin{cases} 
-1 & \text{if } s \text{ is even and not } 2j, \\
0 & \text{if } s \text{ is odd or } s = 2j,
\end{cases}
\]

and

\[
\gamma(m, \alpha + 1) = \begin{cases} 
\gamma(m, \alpha) & \text{if } s \text{ is even}, \\
\gamma(m, \alpha) + 1 & \text{if } s \text{ is odd}.
\end{cases}
\]

Also, \(c(m, s, \alpha) = 0\) because if \(f^{\alpha-1}(m)\) is even, then \(s = f^{\alpha-1}(m)/2\) so \(f^{\alpha-1}(m) = 2s\). Therefore, \(v_2(d(s)) \geq 3\gamma(m, \alpha)\) by (3.4).

If \(s\) is even, we will show that

\[
v_2(d'(j)) \geq \max \{8(j - s') - 1 + v_2(d(s)), v_2(d(s))\}, \tag{3.7}
\]
because then if \( j = s' \), we have

\[
v_2(d'(s')) \geq v_2(d(s)) \\
\geq 8(s' - s') + 3\gamma(m, \alpha) + c(m, s', \alpha + 1),
\]

and for all \( j \),

\[
v_2(d'(j)) \geq 8(j - s') + 3\gamma(m, \alpha) + c(m, j, \alpha + 1) \\
= 8(j - f^{\alpha+1}(m)) + 3\gamma(m, \alpha + 1) + c(m, j, \alpha + 1),
\]

so that (3.7) implies (3.6). If \( s \) is odd we will show that

\[
v_2(d'(j)) \geq 8(j - s') + 3 + v_2(d(s)), \tag{3.8}
\]

because then

\[
v_2(d'(j)) \geq 8(j - s') + 3\gamma(m, \alpha) + 3 \\
= 8(j - s') + 3(\gamma(m, \alpha) + 1) \\
= 8(j - f^{\alpha+1}(m)) + 3\gamma(m, \alpha + 1) + c(m, j, \alpha + 1),
\]

which is (3.6).

For the sake of brevity, we treat here only the case where \( s \) is odd. The case where \( s \) is even has a similar proof. This case breaks into subcases. We will only show the proof where \( j \leq 2s \), but the other cases are \( 2s < j \leq 2^{\alpha-1}m \) and \( 2^{\alpha-1}m < j \leq 2^{\alpha+1}m \), using the same subcases for when \( s \) is even. These subcases are natural to consider because in the first range of \( j \)-values, the \( d(s) \) term is included for computing \( d'(j) \), in the second range, there are no \( d(s) \) or \( d(2^{\alpha}m) \) terms, and in the third range, there is a \( d(2^{\alpha}m) \) term.

Let \( j \leq 2s \). Using (3.5), we know that \( d'(j) = \sum_{i=s}^{2j} d(i) b(i, j) \) by collecting the coefficients
of $\phi$. Let $\delta(i)$ be given by

$$\delta(i) = v_2(d(i)) + v_2(b(i, j)).$$

Let $D = \{\delta(i) \mid s \leq i \leq 2j\}$. Therefore we have

$$v_2(d'(j)) \geq \min \{v_2(d(i)) + v_2(b(i, j)) \mid s \leq i \leq 2j\} = \min D.$$

We claim that $\delta(i)$ achieves its minimum with $\delta(s)$, which proves (3.8). For that element of $D$, we know by inequality (3.2) that

$$\delta(s) \geq v_2(d(s)) + 8(j - s') + 3.$$

Now suppose $i > s$. Then every element of $D$ satisfies the following inequality:

$$\delta(i) = v_2(d(i)) + 8(j - [i/2]) + c(i, j)
\geq 8(i - s) - 1 + v_2(d(s)) + 8(j - [i/2]) + c(i, j)
\geq 8(s + 1 - s + j - [(s + 1)/2]) - 2 + v_2(d(s))
= 8(j - s') + 6 + v_2(d(s)),$$

but this is clearly greater than $\delta(s)$. Therefore, if $j \leq 2s$ and $s$ is odd, then $v_2(d'(j)) \geq 8(j - s') + 3 + v_2(d(s))$. The other cases are similar. \hfill \Box

Now Theorem 1.1 follows easily from Theorem 3.1.

**Theorem 1.1.** [7, Theorem 1] Write $\phi^m(z) = \sum_{n=m}^{\infty} a(m, n)q^n$. Let $n = 2^\alpha n'$ where $2 \nmid n'$. Consider the first $\alpha$ digits of the binary expansion of $m$, $a_\alpha \ldots a_2 a_1$, padding the left with
zeroes if necessary. Let \( i' \) be the index of the rightmost 1, if it exists. Let

\[
\gamma(m, \alpha) = \begin{cases} 
\# \{ i \mid a_i = 0, i > i' \} + 1 & \text{if } i' \text{ exists,} \\
0 & \text{otherwise.}
\end{cases}
\]

Then

\[
a(m, 2^s n') \equiv 0 \pmod{2^{3\gamma(m, \alpha)}}.
\]

**Proof.** Letting \( j = f^\alpha(m) \) in (3.1), the right hand side reduces to

\[
3\gamma(m, \alpha) + c(m, f^\alpha(m), \alpha).
\]

Notice that \( c(m, f^\alpha(m), \alpha) = 0 \), because if \( f^{\alpha-1}(m) \) is even, then \( f^\alpha(m) = f^{\alpha-1}(m)/2 \) so \( f^{\alpha-1}(m) = 2f^\alpha(m) \). The right hand side of (3.1) is minimized when \( j = f^\alpha(m) \), so we conclude that \( v_2(a(m, 2^s n')) \geq 3\gamma(m, \alpha) \).

\[
\square
\]

**CHAPTER 4. CONSTRUCTING THE LEVEL 4 HAUPTMODULN**

The forms \( \phi^{(4)}_{c, c'} \) can be constructed using the theory of \( \eta \)-quotients. We need the following theorem to compute \( \eta \)-quotients of the desired weight, level, and character.

**Theorem 4.1.** [15, 16] Let \( N \) be a positive integer, and suppose that \( f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta} \) is an \( \eta \)-quotient which satisfies the following congruences:

\[
\sum_{\delta \mid N} \delta r_\delta \equiv 0 \pmod{24} \quad \text{and} \quad \sum_{\delta \mid N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}.
\]

Then \( f(z) \) is weakly modular of weight \( k = \frac{1}{2} \sum_{\delta \mid N} r_\delta \) for the group \( \Gamma_0(N) \) with character

\[
\chi(d) = \left( \frac{(-1)^k s}{d} \right) \quad \text{where} \quad s = \prod_{\delta \mid N} \delta^{r_\delta}.
\]
We will use the following theorem to compute vanishing of $\eta$-quotients.

**Theorem 4.2.** [14] Let $c$, $d$, and $N$ be positive integers with $d|N$ and $\gcd(c, d) = 1$. If $f(z)$ is an $\eta$-quotient of level $N$, then the order of vanishing of $f(z)$ at the cusp $c/d$ is given by

$$N \sum_{\delta|N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd(d, N/d) d \delta}.$$

To construct the forms $\phi^{(4)}_{c,c'}$, we follow Theorem 4.1 to see that the following will guarantee a form in $M^!_0(4)$:

- $r_1 + 2r_2 + 4r_4 \equiv 0 \pmod{24}$, (4.1)
- $4r_1 + 2r_2 + r_4 \equiv 0 \pmod{24}$, (4.2)
- $2^{r_2} 4^{r_4}$ is square of a rational number, (4.3)
- $r_1 + r_2 + r_4 = 0 = k$. (4.4)

If we want $\phi^{(4)}_{0,\infty}$, for example, we impose the additional condition that the form have a simple pole at 0 and a simple zero at $\infty$. We accomplish this by using Theorem 4.2. To this end, we compute

$$\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \infty = \frac{1}{4}, \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} 0 = \frac{1}{1}.$$

Therefore, the vanishing of a level 4 $\eta$-quotient at the cusp $c/d$ with $\gcd(c, d) = 1$ and $d|4$ is equal to

$$\frac{1}{6 \gcd(d, 4/d)d} \left( r_1 + \frac{\gcd(d, 2)^2 r_2}{2} + \frac{\gcd(d, 4)^2 r_4}{4} \right).$$

So, the $\eta$-quotient for $\phi^{(4)}_{0,\infty}$ must satisfy

$$-1 = \frac{1}{6} \left( r_1 + \frac{r_2}{2} + \frac{r_4}{4} \right),$$

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and

\[ 1 = \frac{1}{24} (r_1 + 2r_2 + 4r_4). \]

These are equivalent to, respectively,

\[ 4r_1 + 2r_2 + r_4 = -24 \]
\[ r_1 + 2r_2 + 4r_4 = 24 \]

which are strengthenings of (4.2) and (4.1) respectively. We now have the linear system formed from these two equations and (4.4),

\[
\begin{pmatrix}
4 & 2 & 1 \\
1 & 2 & 4 \\
1 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
r_1 \\
r_2 \\
r_4 \\
\end{pmatrix}
= \begin{pmatrix}
-24 \\
24 \\
0 \\
\end{pmatrix}
\]

which has the unique solution \((r_1, r_2, r_4) = (-8, 0, 8)\). A quick verification shows that this solution satisfies (4.3), and the order of vanishing of the corresponding form at 1/2 is 0. Therefore,

\[ \phi_{4,0,\infty} (z) = \frac{\eta(4z)^8}{\eta(z)^8} = q + 8q^2 + 44q^3 + 192q^4 + O(q^5). \]

A similar computation provides the shapes of the remaining \(\eta\)-quotients which are found in Table 4.1. From the table, it is easy to see several symmetries, and we will prove these in Chapter 5.

The forms \(\phi_{4,0,\infty}^{(4)}\) and \(\phi_{4,1/2,\infty}^{(4)}\) are subsumed in the work of Jenkins and Thornton in [8]. In [8], the form \(f_{0,m}^{(4)}(z)\) is the element of \(M_0^2(4)\) that starts with \(q^{-m}\) and has the largest possible gap in the Fourier expansion thereafter. This is written as

\[ f_{0,m}^{(4)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_{0}^{(4)}(m,n)q^n. \]
<table>
<thead>
<tr>
<th>Modular function</th>
<th>η-quotient</th>
<th>q-expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_{\infty,0}^{(4)}(z)$</td>
<td>$\frac{\eta(z)^8}{\eta(4z)^8}$</td>
<td>$q^{-1} - 8 + 20q - 62q^3 + 216q^5 + O(q^7)$</td>
</tr>
<tr>
<td>$\phi_{\infty,1/2}^{(4)}(z)$</td>
<td>$\frac{\eta(2z)^{24}}{\eta(z)^8\eta(4z)^{16}}$</td>
<td>$q^{-1} + 8 + 20q - 62q^3 + 216q^5 + O(q^7)$</td>
</tr>
<tr>
<td>$\phi_{1/2,0}^{(4)}(z)$</td>
<td>$\frac{\eta(z)^{16}\eta(4z)^8}{\eta(2z)^{24}}$</td>
<td>$1 - 16q + 128q^2 - 704q^3 + O(q^4)$</td>
</tr>
<tr>
<td>$\phi_{0,1/2}^{(4)}(z)$</td>
<td>$\frac{\eta(2z)^{24}}{\eta(z)^{16}\eta(4z)^8}$</td>
<td>$1 + 16q + 128q^2 + 704q^3 + O(q^4)$</td>
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<td>$q + 8q^2 + 44q^3 + 192q^4 + O(q^5)$</td>
</tr>
<tr>
<td>$\phi_{1/2,\infty}^{(4)}(z)$</td>
<td>$\frac{\eta(z)^8\eta(4z)^{16}}{\eta(2z)^{24}}$</td>
<td>$q - 8q^2 + 44q^3 - 192q^4 + O(q^5)$</td>
</tr>
</tbody>
</table>

Table 4.1: The η-quotients and q-expansions of the modular functions for whose powers we will prove congruences.

These forms make up a canonical basis for the space $M_{0}^{\sharp}(4)$, and satisfy the congruence

$$a_{0}^{(4)}(2^{\alpha}m', 2^{\beta}n') \equiv \begin{cases} 0 \pmod{2^{4(\alpha-\beta)+8}} & \text{if } \alpha > \beta, \\ 0 \pmod{2^{3(\beta-\alpha)+8}} & \text{if } \beta > \alpha, \end{cases}$$

where $m'$ and $n'$ are odd [8, Theorem 2]. The $f_{0,m}^{(4)}(z)$ basis is more convenient than the bases $(\phi_{\infty,0}^{(4)})^m$ and $(\phi_{\infty,1/2}^{(4)})^m$ because a given form is expressible in terms of the $f_{0,m}^{(4)}$ basis by simply reading off the coefficients of the nonpositive powers of $q$. For this reason, we will not examine congruences for $(\phi_{\infty,0}^{(4)})^m$ and $(\phi_{\infty,1/2}^{(4)})^m$. 

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5.1 **Proof of Theorem 1.2**

The main idea for proving Theorem 1.2 is to use the $U_2$ operator to bring level 4 forms down to the space $M^0_0(2)$ and to apply Theorem 1.1. Recall that $\phi = \phi^{(2)} \in M^0_0(2)$. The following two lemmas show that $U_2$ applied to $(\phi^{(4)})^m$ can be expressed as an integer polynomial in the level 2 form $\phi$.

**Lemma 5.1.** For some integers $d(m, n)$, we have that

$$U_2 \left( \phi^{(4)}_{1/2,0} \right)^m = U_2 \left( \phi^{(4)}_{0,1/2} \right)^m = \sum_{n=0}^{m} d(m, n) \phi^n.$$

**Proof.** Let $f = \phi^{(4)}_{1/2,0}$. Firstly, because $2^2|4$, we have that $U_2 f \in M^0_0(2)$. Because the action of the $U_p$ operator on a $q$-expansion is $U_p \sum a(n)q^n = \sum a(pm)q^n$, we can see from the $q$-expansion of $f$ (Table 4.1) that $U_2 f^m$ is holomorphic at $\infty$.

Now, we will determine the order of vanishing of $U_2 f^m$ at 0. By the definition of $U_2$, we have that

$$2(U_2 f^m) = f^m \left| \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right| + f^m \left| \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \right|.$$

Applying the Fricke involution $W_2$, we have

$$(2U_2 f^m) \left| \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \right| = f^m \left| \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} \right| + f^m \left| \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \right| \left| \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \right|. \hspace{1cm} (5.1)$$

The form $f^m$ has a pole of order $m$ at $1/2$ and a zero of order $m$ at 0. The first term of (5.1) is an expansion of $f^m$ at 0, given by the Fricke involution $W_4$. Therefore, this term contributes no negative powers of $q$. The second term is the expansion of $f^m$ at $1/2$ with the substitution $z \mapsto (2z + 1)/2$ which sends $q \mapsto -q$. Therefore, this term contributes a pole of order $m$. 

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Therefore, $U_2 f^m$ is a form in the space $M^0_0(2)$ with a pole of order $m$ at 0. We conclude that it is a polynomial in $\phi$ of degree $m$. Because $U_2 f^m$ has integer coefficients, the polynomial has integer coefficients.

For the form $\phi_{0,1/2}^{(4)}$, we reduce to the previous case. The matrix \( \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \) swaps the cusps 0 and 1/2 and sends $q$ to $-q$. Therefore, the coefficients of $\phi_{0,1/2}^{(4)}$ and $\phi_{1/2,0}^{(4)}$ are the same up to sign. In particular, the even-indexed coefficients are equal and the odd-indexed coefficients are equal but opposite in sign. The same reasoning applies to the $m$th powers of these forms. Because $U_2$ only picks off even-indexed coefficients, the coefficients it gathers are of the same sign. We conclude that

$$U_2 \left( \phi_{0,1/2}^{(4)} \right)^m = U_2 \left( \phi_{1/2,0}^{(4)} \right)^m.$$

\[ \square \]

The following lemma is similar to the previous one, except that the resulting polynomial in $\phi$ has its smallest power equal to $\lceil m/2 \rceil$.

**Lemma 5.2.** For some integers $d(m,n)$, we have that

$$U_2 \left( \phi_{0,\infty}^{(4)} \right)^m = (-1)^m U_2 \left( \phi_{0,\infty}^{(4)} \right)^m = \sum_{n=\lceil m/2 \rceil}^{m} d(m,n) \phi^n.$$

**Proof.** Let $f = \phi_{0,\infty}^{(4)}$. Again, $U_2 f^m$ is a level 2 form, and it is holomorphic at $\infty$ by examining its $q$-expansion. By a similar argument, equation (5.1) shows that the pole at 0 is of order $m$. If $m$ is even, the least power of $q$ in $U_2 f^m$ is $m/2$, and if $m$ is odd, the least power is $(m+1)/2$. Thus the least power of $\phi$ in $U_2 f^m$ is $\lceil m/2 \rceil$.

By a similar argument to that presented in Lemma 5.1, the coefficients of $\phi_{0,\infty}^{(4)}$ and $\phi_{1/2,\infty}^{(4)}$ are equal up to sign. Because we are normalizing the forms to have leading coefficient 1 and these forms begin with an odd power of $q$, the odd-indexed coefficients are equal and the even-indexed coefficients are equal but opposite in sign. The same will be true of any odd power of the two forms for the same reason. In this case, $U_2$ of both forms is equal but opposite in sign. The pattern reverses when we take an even power of the functions because
we no longer have to apply a normalizing $-1$ to every coefficient. In this case, the image of $U_2$ on both forms is equal, concluding the proof.

We now prove Theorem 1.2. We use Lemmas 5.1 and 5.2 to bring $\phi_{c,c'}^{(4)}$ down to level 2, and then we apply Theorem 1.1.

**Theorem 1.2.** Let $(c, c') = (0, \infty), (0, 1/2), (1/2, \infty)$, or $(1/2, 0)$. Let $n = 2^\alpha n'$ where $2 \nmid n'$. Let $\alpha' = [\log_2(m)] + 1$, which is the number of digits in the binary expansion of $m$. Then, if $\alpha \geq \alpha' + 1$,

$$a_{c,c'}^{(4)}(m, 2^\alpha n') \equiv 0 \pmod{2^3(\alpha - \alpha')}.$$  

**Proof.** Let $\phi^m(z) = \sum_{n=m}^{\infty} a(m, n)q^n$. Using Lemmas 5.1 and 5.2, we have that

$$U_2(\phi_{c,c'}^{(4)})^m(z) = \sum_{n=0}^{\infty} a_{c,c'}^{(4)}(m, 2n)q^n = \sum_{n=0}^{m} d(m, n)q^n \sum_{j=1}^{\infty} a(m, j)q^j = 1 + \sum_{n=1}^{\infty} q^n \sum_{j=1}^{m} d(m, j)a(j, n).$$

By comparing coefficients, for $n \geq 1$, we have the equation

$$a_{c,c'}^{(4)}(m, 2n) = \sum_{j=1}^{m} d(m, j)a(j, n).$$  \hfill (5.2)
Letting $n = 2^\beta n'$, we compute the inequality

$$v_2 \left( a_{c,c'}^{(4)}(m, 2n) \right) = v_2 \left( a_{c,c'}^{(4)}(m, 2 \cdot 2^\beta n') \right) \geq \min_{j=1}^m \left\{ v_2(d(m, j) a(j, 2^\beta n')) \right\} \geq \min_{j=1}^m \left\{ v_2(a(j, 2^\beta n')) \right\} \geq \min_{j=1}^m \{3 \gamma(j, \beta)\},$$

(5.3)

by (5.2) and Theorem 1.1. Therefore, we see that

$$v_2 \left( a_{c,c'}^{(4)}(m, 2n) \right) = v_2 \left( a_{c,c'}^{(4)}(m, 2 \cdot 2^\beta n') \right) \geq \min_{j=1}^m \{3 \gamma(j, \beta)\}.$$  

(5.4)

The value of (5.4) may be 0. To illustrate an example, recall the definition of $\gamma(j, \beta)$:

Consider the first $\beta$ digits of the binary expansion of $j$, padding the left with zeroes if necessary, written $a_\beta \cdots a_2 a_1$. Let $i'$ be the least index $i$ such that $a_i = 1$, if it exists. Then

$$\gamma(j, \beta) = \begin{cases} 
# \{q \mid a_q = 0 \text{ and } \beta \geq q > i'\} + 1 & \text{if } i' \text{ exists,} \\
0 & \text{otherwise.} 
\end{cases}$$

Therefore, if $\beta$ is small, $\gamma(j, \beta)$ is 0 until $\beta$ reaches the position of the rightmost 1 in the binary expansion of $j$. For example, $\gamma(16, \beta) = 0$ for $1 \leq \beta \leq 4$ because $16_{10} = 10000_2$. But, if we take $\beta$ large enough, the function $\gamma(j, \beta)$ counts the leftmost 1 and the leading 0s of the binary expansion of $j$.

Now, let $j$ vary between 1 and $m$. The integer $\alpha' = \lfloor \log_2(m) \rfloor + 1$ is the leftmost position of a 1 in any of the binary expansions of the $j$. If $\beta \geq \alpha'$, then each of $\gamma(j, \beta)$ will be at least 1, and incrementing $\beta$ will increment every one of the $\gamma(j, \beta)$. We conclude that if $\beta \geq \alpha'$, then

$$v_2 \left( a_{c,c'}^{(4)}(m, 2n) \right) = v_2 \left( a_{c,c'}^{(4)}(m, 2 \cdot 2^\beta n') \right) \geq 3(\beta - \alpha' + 1).$$

For a meaningful result, we also need the assumption that $\beta \geq 1$ because $\alpha' \geq 1$. 

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We translate the result to be in terms of the notation used in the theorem statement. Let \( \ell = 2^n \ell' = 2 \cdot 2^\beta n' \), so that

\[
a_{c,c'}^{(4)}(m, \ell) = a_{c,c'}^{(4)}(m, 2 \cdot 2^\beta n').
\]

In particular, this implies that \( \alpha = \beta + 1 \). Further, we have that

\[
\beta \geq \alpha' \iff \alpha \geq \alpha' + 1
\]
\[
\beta \geq 1 \iff \alpha \geq 2.
\]

We conclude with the result

\[
\alpha \geq 2 \text{ and } \alpha \geq \alpha' + 1 \Rightarrow v_2 \left( a_{c,c'}^{(4)}(m, 2^n \ell') \right) \geq 3(\alpha - \alpha').
\]

The theorem is proved once we observe that \( \ell \) here is \( n \) in the theorem statement.

5.2 Conjectures for forms in level 4

Conjecture 1.3 should be true for the same reason that Theorem 1.1 is true. Recall that

\[
U_2 \phi^m = d_{[m/2]} \phi^{[m/2]} + \cdots + d_{2m} \phi^{2m}
\]

for some integers \( d_i \). The two key ideas in the proof of Theorem 1.1 are:

1. The valuation \( v_2(d_{[m/2]}) \) is at least 3 when \( m \) is odd, and is at least 0 otherwise.
2. If \( 2^a \| d_{[m/2]} \), then \( 2^a \| d_i \) for \( i > [m/2] \).

These are proved in the base case of Theorem 3.1, and here we summarize the process. It is easy to prove that the Fourier expansion of \( \phi^m \) begins with

\[
\phi^m(z) = q^m + 24mq^{m+1} + \cdots.
\]
If $m$ is odd, the leading term of $U_2 \phi^m$ is $24mq^{(m+1)/2}$. So $d_{[m/2]} = d_{(m+1)/2} = 24m$ and the 2-adic valuation of this coefficient is $v_2(24m) = 3$. If $m$ is even, then the leading term of $U_2 \phi^m$ is $q^{m/2}$. The second condition above guarantees that the 2-adic valuations of the remaining coefficients is at least 3. Proving this is more difficult, and for this we employed Watson’s method [18] which used the modular equation for $\phi$.

This same pattern occurs for $\phi^{(4)}_{0,\infty}$ and $\phi^{(4)}_{1/2,\infty}$. From their Fourier expansions, it is again easy to see that
\[
\left( \phi^{(4)}_{0,\infty} \right)^m(z) = q^m + 8mq^{m+1} + \cdots
\]
\[
\left( \phi^{(4)}_{1/2,\infty} \right)^m(z) = q^m - 8mq^{m+1} + \cdots
\]
We take, for example, $\left( \phi^{(4)}_{0,\infty} \right)^m$. From Lemma 5.2, we have that
\[
U_2 \left( \phi^{(4)}_{0,\infty} \right)^m = c_{[m/2]} \phi^{[m/2]} + \cdots + c_m \phi^m
\]
for some $c_i \in \mathbb{Z}$. The first terms in these Fourier expansions are
\[
\begin{cases}
8mq^{(m+1)/2} & \text{if } m \text{ is odd,} \\
q^{m/2} & \text{if } m \text{ is even.}
\end{cases}
\]
The first case contributes $2^3$ to $c_{[m/2]}$, and the second case gives no information. This is the same pattern we saw for $\phi$. The obstacle is obtaining condition 2 for the polynomials presented in Lemma 5.2. To use Watson’s method again, we need a modular equation for $\phi^{(4)}_{0,\infty}$ and $\phi^{(4)}_{1/2,\infty}$. Lehner computes these for $\phi^{(p)}$ using Lemma 2.1, and such a result for level 4 forms has thus far eluded the author.

Conjecture 1.4 essentially states that coefficients for $\left( \phi^{(4)}_{1/2,0} \right)^m$ and $\left( \phi^{(4)}_{0,1/2} \right)^m$ follow a congruence similar to (1.1) and similar to congruences for the canonical bases for $M^2_0(N)$. A
slight strengthening of Conjecture 1.4 is that, for the same $\phi_{c,c'}^{(4)}$, we have for $n > 1$,

$$v_2 \left( a_{c,c'}^{(4)}(m, 2^n n') \right) \begin{cases} 
 3\alpha + v_2 \left( a_{c,c'}^{(4)}(m, n') \right) & \text{if } m \text{ is odd, or } m \text{ is even and } \alpha = 0, \\
 3(\alpha - 1) + v_2 \left( a_{c,c'}^{(4)}(m, 2n') \right) & \text{if } m \text{ is even and } \alpha \geq 2, \\
 \geq 3 & \text{if } \alpha = 0.
\end{cases}$$

This was formulated by observing odd-indexed coefficients and following their 2-adic valuation as the index is multiplied by 2 repeatedly. Certainly, we have Lemma 5.1, which allows us to potentially apply Theorem 1.1 after applying $U_2$, but we still lack the same result from the second bullet point above. Further, Conjecture 1.4 does not depend on the binary expansion of $m$ in any major way, so using Theorem 1.1 would not provide the desired result either way.
Bibliography


