2018-04-01

A New Family of Topological Invariants

Nicholas Guy Larsen
Brigham Young University

Follow this and additional works at: https://scholarsarchive.byu.edu/etd

Part of the Mathematics Commons

BYU ScholarsArchive Citation
https://scholarsarchive.byu.edu/etd/6757

This Thesis is brought to you for free and open access by BYU ScholarsArchive. It has been accepted for inclusion in All Theses and Dissertations by an authorized administrator of BYU ScholarsArchive. For more information, please contact scholarsarchive@byu.edu, ellen_amatangelo@byu.edu.
A New Family of Topological Invariants

Nicholas Guy Larsen

A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

Gregory Conner, Chair
Curtis Kent
Eric Swenson

Department of Mathematics
Brigham Young University

Copyright © 2018 Nicholas Guy Larsen
All Rights Reserved
ABSTRACT

A New Family of Topological Invariants

Nicholas Guy Larsen
Department of Mathematics, BYU
Master of Science

We define an extension of the $n$th homotopy group $\pi_n$ which can distinguish a larger class of spaces. (E.g., a converging sequence of disjoint circles and the disjoint union of countably many circles, which have isomorphic fundamental groups, regardless of choice of basepoint.) We do this by introducing a generalization of homotopies, called component-homotopies, and defining the $n$th extended homotopy group to be the set of component-homotopy classes of maps from compact subsets of $(0,1)^n$ into a space, with a concatenation operation.

We also introduce a method of tree-adjoinment for “connecting” disconnected metric spaces and show how this method can be used to calculate the extended homotopy groups of an arbitrary metric space.

Keywords: algebraic topology, homotopy, fundamental group
ACKNOWLEDGMENTS

I am indebted to many people for assistance with this thesis, mathematical or otherwise. First I would like to express gratitude to my advisor, Dr. Greg Conner, for his encouragement and his sometimes inordinate confidence in me, as well as for convincing me to pursue a graduate degree in the first place. I would also like to thank Lonette Stoddard, the Mathematics Department Secretary, without whose knowledge and patience I would be totally helpless. I am grateful for the assistance of my committee, Drs. Greg Conner, Curt Kent, and Eric Swenson, in finding solutions to some unexpected difficulties encountered during the writing of this thesis. I would never have been able to get to where I am today without the support of my family, especially my parents and grandparents; I will be trying to repay them for the rest of my life. Finally, I am eternally grateful to my wife for her patience during the whole process of writing this thesis, and for pretending to be interested when I talked to her about homotopies.
## Contents

<table>
<thead>
<tr>
<th>Contents</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contents</td>
<td>iv</td>
</tr>
<tr>
<td>1  The Extended Homotopy Groups</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Component-homotopies</td>
<td>1</td>
</tr>
<tr>
<td>1.3 The extended homotopy groups</td>
<td>6</td>
</tr>
<tr>
<td>2  Properties of the Extended Homotopy Groups</td>
<td>9</td>
</tr>
<tr>
<td>2.1 Relationship to homotopy groups</td>
<td>9</td>
</tr>
<tr>
<td>2.2 Tree-connected spaces</td>
<td>11</td>
</tr>
<tr>
<td>2.3 Examples</td>
<td>16</td>
</tr>
<tr>
<td>A  Lemmas</td>
<td>16</td>
</tr>
<tr>
<td>A.1 Lemmas</td>
<td>17</td>
</tr>
<tr>
<td>Bibliography</td>
<td>18</td>
</tr>
</tbody>
</table>
Chapter 1. The Extended Homotopy Groups

1.1 Introduction

Consider the following subspaces of $\mathbb{R}^2$:

$$X = \bigcup_{i \in \mathbb{N}} \left\{ (x, y) \in \mathbb{R}^2 : d \left( (x, y), \left( \frac{1}{2^{(i-1)}}, \frac{1}{2^{(i+1)}} \right) \right) = \frac{1}{2^{(i+1)}} \right\}$$

and

$$Y = X \cup \{ (x, y) \in \mathbb{R}^2 : d((x, y), (-1, 0)) = 1 \}.$$

So $X$ is a countable collection of circles and $Y$ is the same collection of circles, converging to a point contained in another circle. It is easy to see that for any choice of $x'$, we have $\pi_1(X, x') = \pi_1(Y, x') \cong \mathbb{Z}$.

Imagine that instead of finding the path-homotopy classes of maps $(I, \partial I) \to (X, x')$ or $(I, \partial I) \to (Y, x')$, we were to take a compact subset of the unit interval $I$ with possibly infinitely many components and map each of these as a loop into either $X$ or $Y$. It is clear with some consideration that we could construct such a map onto $Y$ that is surjective, while this is impossible for $X$.

The motivation for this thesis is to formalize this idea in order to define a family of topological invariants which extend the homotopy groups (i.e., contain subgroups that are isomorphic to the homotopy groups) while distinguishing a larger class of spaces.

1.2 Component-homotopies

Let $X$ be a topological space and let $\{X_i\}_{i \in J}$ be the set of path components of $X$, indexed by some set $J$. For each $i \in J$, fix $x_i \in X_i$. For $n \geq 2$, let $I_t^{n-1} = \{ \pi \in I^n : x_{n-t} = t \}$, where $I$ is the unit interval $[0, 1]$.

For $n \in \mathbb{N}$, let $\mathcal{U}_n$ be the collection of nonempty open subsets of $(0, 1)^n$. For each $U \in \mathcal{U}_n$, let $\partial U$ denote the boundary of $U$ in $I^n$. Now define $R_n(X, \{x_i\})$ to be the set of continuous
maps from elements $U$ of $\mathcal{U}_n$ into $X$ mapping $\partial U$ into $\{x_i\}$:

$$R_n(X, \{x_i\}) = \{ \phi : (U, \partial U) \to (X, \{x_i\}) : U \in \mathcal{U}_n \}.$$ 

Notice that an element of $R_n$ can be thought of as many “simultaneous” maps of $I^n$ into $X$, each sending $\partial I^n$ to some element of $\{x_i\}$.

For simplicity, we may refer to $R_n(X, \{x_i\})$ as $R_n$ or $R_n(X)$ when the choices of $X$ and/or $\{x_i\}$ are clear from context.

Let $b$ be the self-homeomorphism of $I^n$ defined by

$$(x_1, \ldots, x_n) \mapsto (1 - x_1, x_2, \ldots, x_n).$$

For each $U \in \mathcal{U}_n$, let $U_-$ denote $b(U)$. By Lemma A.1.1, $\partial(U_-) = (\partial U)_-$, so we will use the notation $\partial U_-$ to refer to both. For each $\phi : (U, \partial U) \to (X, \{x_i\}) \in R_n$, define $\overline{\phi}$ as $\phi \circ b$, which we will call the reverse of $\phi$. We will show that reverses of elements of $R_n$ are also elements of $R_n$.

**Claim 1.2.1.** For each $\phi \in R_n(X, \{x_i\})$, we have $\overline{\phi} \in R_n(X, \{x_i\})$ as well.

**Proof.** Let $\phi : (U, \partial U) \to (X, \{x_i\}) \in R_n$ be arbitrary. We will show that $\overline{\phi} : (U_-, \partial U_-) \to (X, \{x_i\}) \in R_n$.

Since $b$ is a reflection across $\{x \in \mathbb{R}^n : x_1 = 1/2\}$, we have that $U_- = b(U) \in \mathcal{U}_n$. Also since $\overline{\phi} = \phi \circ b$ and $\phi$ is assumed to be continuous, $\overline{\phi}$ must be continuous.

Let $\overline{x} \in \partial(U_-)$. As we have seen, $\partial(U_-) = (\partial U)_-$, so $b(\overline{x}) \in \partial U$. Then

$$\phi(b(\overline{x})) = \overline{\phi}(\overline{x}) \in \{x_i\},$$

which shows that $\overline{\phi} : (U_-, \partial U_-) \to (X, \{x_i\}) \in R_n(X, \{x_i\})$, proving the claim.
Consider the following homeomorphisms of $\mathbb{R}^n$:

$$\ell(x_1, \ldots, x_n) = \left( \frac{x_1}{3}, x_2, \ldots, x_n \right)$$
$$r(x_1, \ldots, x_n) = \left( \frac{x_1 + 2}{3}, x_2, \ldots, x_n \right).$$

Notice that $\ell$ and $r$ have the following properties:

1. $\partial \ell(U) = \ell(\partial U)$ and $\partial r(U) = r(\partial U)$ (by Lemma A.1.1),

2. for $U \in \mathcal{U}_n$, $\ell(U), r(U) \in \mathcal{U}_n$,

3. for any $U, V \in \mathcal{U}_n$, $\ell(U) \cap r(V) = \emptyset$ and $\partial(\ell(U) \sqcup r(V)) = \partial \ell(U) \sqcup \partial r(V)$. Also $\ell(U) \cup \partial \ell(U)$ and $r(V) \cup \partial r(V)$ are disjoint closed sets.

We define a binary operation $\ast$ on $R_n(X, \{x_i\})$. Let $\phi : (U, \partial U) \to X$ and $\psi : (V, \partial V) \to (X, \{x_i\})$ in $R_n$ be arbitrary. Let $W = \ell(U) \cup r(V)$. Define $\phi \ast \psi : (W, \partial W) \to (X, \{x_i\})$ by

$$(\phi \ast \psi)(x_1, \ldots, x_n) = \begin{cases} 
\phi(3x_1, x_2, \ldots, x_n) & (x_1, \ldots, x_n) \in \ell(U) \cup \partial \ell(U) \\
\psi(3x_1 - 2, x_2, \ldots, x_n) & (x_1, \ldots, x_n) \in r(V) \cup \partial r(V).
\end{cases}$$

Equivalently,

$$(\phi \ast \psi)(\pi) = \begin{cases} 
\phi(\ell^{-1}(\pi)) & (x_1, \ldots, x_n) \in \ell(U) \cup \partial \ell(U) \\
\psi(r^{-1}(\pi)) & (x_1, \ldots, x_n) \in r(V) \cup \partial r(V).
\end{cases}$$

Now we will show that $R_n$ is closed under $\ast$.

**Claim 1.2.2.** If $\phi, \psi \in R_n(X, \{x_i\})$, then $\phi \ast \psi \in R_n(X, \{x_i\})$.

**Proof.** Let $\phi : (U, \partial U) \to (X, \{x_i\})$ and $\psi : (V, \partial V) \to (X, \{x_i\})$ in $R_n$ be arbitrary, and set $W = \ell(U) \cup r(V)$.

By property 2 of $\ell$ and $r$, $W = \ell(U) \cup r(V) \in \mathcal{U}_n$.

Recall that $\phi$ is continuous by assumption and that $\ell$ is a homeomorphism. Then since $(\phi \ast \psi) = \phi \circ \ell^{-1}$ on $\ell(U) \cup \partial \ell(U)$, $\phi \ast \psi$ is continuous on $\ell(U) \cup \partial \ell(U)$. Similarly, $\phi \ast \psi$ is
continuous on $r(V) \cup \partial r(V)$. Therefore $\phi \ast \psi$ is continuous by the pasting lemma, noting that $\ell(U) \cup \partial \ell(U)$ and $r(V) \cup \partial r(V)$ are disjoint and closed.

Let $\mathbf{x} \in \partial W$. By property 3 of $\ell$ and $r$, $\mathbf{x}$ is contained in exactly one of $\partial \ell(U)$ and $r(V) \cup \partial r(V)$. Suppose that $\mathbf{x} \in \partial \ell(U) = \ell(\partial U)$. Then $\mathbf{x} = \ell(y)$ for some $y \in \partial U$, and

$$(\phi \ast \psi)(\mathbf{x}) = (\phi \ast \psi)(\ell(y)) = \phi(\ell^{-1}(\ell(y))) = \phi(y) \in \{x_i\}.$$ 

Similarly, if $\mathbf{x} \in \partial r(V)$, $(\phi \ast \psi)(\mathbf{x}) \in \{x_i\}$. Then $(\phi \ast \psi)(\partial W) \subseteq \{x_i\}$, proving the claim. □

With the reverses of elements of $R_n$ and the operation $\ast$, we are close to showing that $R_n$ is a group. However, it is easy to see that $\ast$ fails to be associative. In order for this to be the case, we must define an equivalence relation $\sim$ on $R_n$. For $\phi : (U, \partial U) \to (X, \{x_i\})$, $\psi : (V, \partial V) \to (X, \{x_i\}) \in R_n$, we say that $\phi \sim \psi$ if there exists an open subset $O$ of $I^{n+1}$ and a continuous function $h : (O, \partial O) \to (X, \{x_i\})$ satisfying the following:

1. $O \cap I^n_0 = U \times \{0\}$,
2. $O \cap I^n_1 = V \times \{1\}$,
3. $h(\cdot, 0)|_{U \times \{0\}} = \phi$, and
4. $h(\cdot, 1)|_{V \times \{1\}} = \psi$,

where $I^n_t = \{\mathbf{x} \in I^{n+1} : x_{n+1} = t\} \cong I^n$ for $t \in I$.

Notice that in the special case where $U = V = (0,1)^n$, $\phi \sim \psi$ exactly when $\phi$ and $\psi$ are homotopic, with $h$ as a homotopy. For this reason, when $\phi \sim \psi$ we will say that $\phi$ and $\psi$ are component-homotopic and that $h$ is a component-homotopy between $\phi$ and $\psi$.

**Claim 1.2.3.** The relation $\sim$ is an equivalence relation.

**Proof.** We must show that $\sim$ is reflexive, symmetric, and transitive.
Claim 1.2.4. $\sim$ is reflexive.

*Subproof.* Let $\phi : (U, \partial U) \to (X, \{x_i\})$ be arbitrary. Let $O = U \times I$ and define $h : (O, \partial O)$ by

$$h(\bar{x}, t) = \phi(\bar{x}).$$

Then clearly $\phi \sim \phi$, proving the claim. $\square$

Claim 1.2.5. $\sim$ is symmetric.

*Subproof.* Suppose that $\phi \sim \psi$, where $\phi : (U, \partial U) \to (X, \{x_i\})$ and $\psi : (V, \partial V) \to (X, \{x_i\})$ are elements of $R_n$. Then by definition, there exists an open subset $O$ of $I^{n+1}$ and a component-homotopy $h : (O, \partial O) \to (X, \{x_i\})$ between $\phi$ and $\psi$. Define $O' = b_{n+1}(O)$ and $h' = h \circ b_{n+1}$. Then it is easy to see that $h'$ is a component-homotopy between $\psi$ and $\phi$, proving the claim. $\square$

Claim 1.2.6. $\sim$ is transitive.

*Subproof.* Suppose that $\phi \sim \psi$ and $\psi \sim \theta$, where $\phi : (U, \partial U) \to (X, \{x_i\})$, $\psi : (V, \partial V) \to (X, \{x_i\})$, and $\theta : (W, \partial W) \to (X, \{x_i\})$ are elements of $R_n$. Then there exist $O_1 \subseteq I^{n+1}$ and $h_1 : (O_1, \partial O_1) \to (X, \{x_i\})$ satisfying the conditions of $\sim$ to show that $\phi \sim \psi$ and $O_2 \subseteq I^{n+1}$ and $h_2 : (O_2, \partial O_2) \to (X, \{x_i\})$ showing that $\psi \sim \theta$.

Let $O = \{(x_1, \ldots, x_n, x_{n+1}/2) : \bar{x} \in O_1\} \cup \{(x_1, \ldots, x_n, x_{n+1}/2 + 1/2) : \bar{x} \in O_2\}$ and define $h : (O, \partial O) \to (X, \{x_i\})$ by

$$h(\bar{x}) = \begin{cases} h_1(x_1, \ldots, x_n, 2x_{n+1}) & x_{n+1} \in [0, 1/2] \\ h_2(x_1, \ldots, x_n, 2x_{n+1} - 1) & x_{n+1} \in [1/2, 1]. \end{cases}$$

Then $h$ is a component-homotopy between $\phi$ and $\theta$, proving the claim. $\square$

Since $\sim$ is reflexive, symmetric, and transitive, it is an equivalence relation. $\square$
Now we define
\[ \rho_n(X, \{x_i\}) = R_n(X, \{x_i\})/\sim, \]
so \( \rho_n(X, \{x_i\}) \) is the set of component-homotopy classes of maps \( (U, \partial U) \to (X, \{x_i\}) \) for \( U \in \mathcal{U}_n \). For simplicity we may refer to \( \rho_n(X, \{x_i\}) \) as \( \rho_n \) or \( \rho_n(X) \) when the choice of \( X \) and/or \( \{x_i\} \) is clear from context, as we did with \( R_n \).

1.3 The extended homotopy groups

Recall that the obstacle to showing that \( R_n \) is a group under \(*\) was that \(*\) is not associative on elements of \( R_n \). However, we can show that \(*\) is associative on component-homotopy classes, and therefore that \( \rho_n \) is in fact a group. It is worth noting that this proof is very similar to the proof that \( \pi_n \) is a group, with the only significant differences coming from the fact that the operation \(*\) compresses the first coordinate by a factor of three, while the operation on \( \pi_n \) compresses the first coordinate by a factor of two.

**Claim 1.3.1.** The set \( \rho_n(X, \{x_i\}) \) has a group structure under \(*\), defined by \( [\phi] * [\psi] = [\phi * \psi] \) for \( [\phi], [\psi] \in \rho_n(X, \{x_i\}) \).

**Proof.** First we must show that \(*\) is well-defined with respect to \( \sim \).

**Claim 1.3.2.** The operation \(*\) on \( \rho_n(X, \{x_i\}) \) is well-defined with respect to \( \sim \).

**Subproof.** Let \( [\phi], [\psi] \in \rho_n(X, \{x_i\}) \) be arbitrary. Let \( \phi_1 : (U_1, \partial U_1) \to (X, \{x_i\}) \) and \( \phi_2 : (U_2, \partial U_2) \to (X, \{x_i\}) \) be representatives of \( [\phi] \) and let \( \psi_1 : (V_1, \partial V_1) \to (X, \{x_i\}) \) and \( \psi_2 : (V_2, \partial V_2) \to (X, \{x_i\}) \) be representatives of \( [\psi] \). To show that \(*\) is well-defined, we must show that \( [\phi_1 * \psi_1] = [\phi_2 * \psi_2] \), or equivalently,

\[ (\phi_1 * \psi_1) \sim (\phi_2 * \psi_2). \]

Since \( \phi_1, \phi_2 \in [\phi] \) and \( \psi_1, \psi_2 \in [\psi] \), \( \phi_1 \sim \phi_2 \) and \( \psi_1 \sim \psi_2 \). Then there exists some open \( O_\phi \subseteq I^{n+1} \) and \( h_\phi : (O_\phi, \partial O_\phi) \to (X, \{x_i\}) \), and some open \( O_\psi \subseteq I^{n+1} \) and \( h_\psi : (O_\psi, \partial O_\psi) \to \)
Let $O = \ell(O\phi) \cup r(O\psi)$ and define $h : (O, \partial O) \to (X, \{x_i\})$ by

$$h(\overline{x}) = \begin{cases} h_\phi(\ell^{-1}(\overline{x})) & \overline{x} \in \ell(O\phi) \\ h_\psi(r^{-1}(\overline{x})) & \overline{x} \in r(O\psi). \end{cases}$$

Then $h$ is a component-homotopy, proving the claim. \qed

**Claim 1.3.3.** $\rho_n(X, \{x_i\})$ is closed under $\ast$.

*Subproof.* This follows immediately from Claim 2. \qed

**Claim 1.3.4.** $\rho_n(X, \{x_i\})$ has an identity element under $\ast$.

*Subproof.* Fix $x' \in \{x_i\}$. Let $e_{x'} : (I^n, \partial I^n) \to (X, \{x_i\})$ be the constant map to $x'$. Then clearly $[e_{x'}] \in \rho_n$. We will show that $[e_{x'}]$ is an identity element of $\rho_n$. It suffices to show that for each $\phi \in R_n$, $(\phi \ast e_{x'}) \sim \phi$.

Let $\phi : (U, \partial U) \to (X, \{x_i\})$ be arbitrary. Then since $\sim$ is reflexive, there exists an open set $O_\phi \subseteq I^{n+1}$ and a component-homotopy $h_\phi : (O_\phi, \partial O_\phi) \to (X, \{x_i\})$ between $\phi$ and itself. Let $k$ be the homeomorphism of $I^{n+1}$ defined by

$$k(x_1, \ldots, x_n, t) = \left(\frac{2t + 1}{3}(x_1), x_2, \ldots, x_n, t\right).$$

(Notice that $k(x_1, \ldots, x_n, 0) = \ell(x_1, \ldots, x_n, 0)$ and $k(x_1, \ldots, x_n, 1) = (x_1, \ldots, x_n, 1)$.) Let $O_{x'}$ be an open subset of $I^{n+1}$ such that $O_{x'} \cap I^n_0 = r(I^n)$, and $x_1 > 2/3$ and $x_{n+1} < 1/9$ for all $\overline{x} \in O_{x'}$. Define $O = k(O_\phi) \cup O_{x'}$ and $h : (O, \partial O) \to (X, \{x_i\})$ by

$$h(\overline{x}) = \begin{cases} \phi(k^{-1}(\overline{x})) & \overline{x} \in k(O_\phi) \\ x' & \overline{x} \in O_{x'}. \end{cases}$$

Then $h$ is a component-homotopy between $(\phi \ast e_{x'})$ and $\phi$, which proves the claim. \qed
Claim 1.3.5. Each element of $\rho_n(X, \{x_i\})$ has a two-sided inverse under $\ast$.

Subproof. Let $[\phi] \in \rho_n(X, \{x_i\})$ be arbitrary. Let $\phi : (U, \partial U) \to (X, \{x_i\})$ be a representative of $[\phi]$. By Claim 1, $\overline{\phi} : (U_-, \partial U_-) \to (X, \{x_i\}) \in R_n$. We will show that $[\overline{\phi}]$ is a two-sided inverse for $[\phi]$. It suffices to show that $(\phi \ast \overline{\phi}) \sim e_{x'}$ for some $x' \in \{x_i\}$, since $(\phi) = \phi$.

First notice that since $\phi \sim \phi$, there exists an open set $O_\phi \subseteq I^{n+1}$ and a component-homotopy $h_\phi : (O_\phi, \partial O_\phi) \to (X, \{x_i\})$ between $\phi$ and itself. Let $O_{x'} = \{ \overline{x} \in I^{n+1} : 0 < x_1 < 1, x_{n+1} \geq 2/3 \}$. Let $\sigma$ be a homeomorphism from $I^{n+1}$ to $\{ \overline{x} \in I^{n+1} : x_{n+1} \leq 1/3 \}$ such that $\sigma(\text{int}(I_0^n)) = \ell(\text{int}(I_0^n))$ and $\sigma(\text{int}(I_1^n)) = r(\text{int}(I_1^n))$. Let $O = \sigma(O_\phi) \cup O_{x'}$ and define $h : (O, \partial O) \to (X, \{x_i\})$ by

$$h(\overline{x}) = \begin{cases} h_\phi(\sigma^{-1}(\overline{x})) & \overline{x} \in \sigma(O_\phi) \\ x' & \overline{x} \in O_{x'}. \end{cases}$$

Then $h$ is a component-homotopy between $(\phi \ast \overline{\phi})$ and $e_{x'}$, proving the claim. \hfill \Box

Claim 1.3.6. The operation $\ast$ on $\rho_n(X, \{x_i\})$ is associative.

Subproof. Let $[\phi], [\psi], [\theta] \in \rho_n(X, \{x_i\})$. Let $\phi : (U, \partial U) \to (X, \{x_i\})$, $\psi : (V, \partial V) \to (X, \{x_i\})$, $\theta : (W, \partial W) \to (X, \{x_i\})$ be representatives of these classes respectively. We must show that $([\phi] \ast [\psi]) \ast [\theta] = [\phi] \ast ([\psi] \ast [\theta])$, or equivalently, that $(\phi \ast \psi) \ast \theta \sim \phi \ast (\psi \ast \theta)$. This means that we must show that there exists some open $O \subseteq I^{n+1}$ and a component-homotopy $h : (O, \partial O) \to (X, \{x_i\})$ between $(\phi \ast \psi) \ast \theta$ and $\phi \ast (\psi \ast \theta)$.

Since $\sim$ is reflexive, there exist open sets $O_\phi, O_\psi, O_\theta$ and functions $h_\phi, h_\psi, h_\theta$ satisfying the conditions of $\sim$ to show that $\phi \sim \phi$, $\psi \sim \psi$, and $\theta \sim \theta$, respectively. Define $\sigma_\phi, \sigma_\psi, \sigma_\theta$, and
\(\sigma_\theta\) to be homeomorphisms of \(U \times I\), \(V \times I\), and \(W \times I\), respectively, such that

\[
\begin{align*}
\sigma_\phi(x_1, \ldots, x_n, t) &= \left(\frac{2t + 1}{9}, x_2, \ldots, x_n, t\right), \\
\sigma_\psi(x_1, \ldots, x_n, t) &= \left(\frac{x_1 + 4t + 2}{9}, x_2, \ldots, x_n, t\right), \\
\sigma_\theta(x_1, \ldots, x_n, t) &= \left(\frac{(3 - t)x_1 + 2t}{9}, x_2, \ldots, x_n, t\right).
\end{align*}
\]

Then notice that \(\sigma_\phi(U \times I) \cap I^n_0 = \ell^2(U), \sigma_\phi(U \times I) \cap I^n_1 = \ell(U), \sigma_\psi(V \times I) \cap I^n_0 = \ell(r(V)), \sigma_\psi(V \times I) \cap I^n_1 = r(\ell(V)), \sigma_\theta(W \times I) \cap I^n_0 = r(W), \) and \(\sigma_\theta(W \times I) \cap I^n_1 = r^2(W)\). Let \(O = \sigma_\phi(U \times I) \cup \sigma_\psi(V \times I) \cup \sigma_\theta(W \times I)\) and define \(h : (O, \partial O) \to (X, \{x_1\})\) by

\[
h(\overline{x}) = \begin{cases} 
  h_\phi(\sigma_\phi^{-1}(\overline{x})) & \overline{x} \in \sigma_\phi(U \times I) \\
  h_\psi(\sigma_\psi^{-1}(\overline{x})) & \overline{x} \in \sigma_\psi(V \times I) \\
  h_\theta(\sigma_\theta^{-1}(\overline{x})) & \overline{x} \in \sigma_\theta(W \times I).
\end{cases}
\]

Then \(h\) is a component-homotopy between \((\phi * \psi) * \theta\) and \(\phi * (\psi * \theta)\), proving the claim. \(\Box\)

Since \(*\) is a well-defined, associative operation under which \(\rho_n\) is closed and has two-sided inverses, \(\rho_n\) is a group under \(*\). \(\Box\)

Chapter 2. Properties of the Extended Homotopy Groups

2.1 Relationship to homotopy groups

Now we show that \(\rho_n\) is in fact an extension of \(\pi_n\) by showing that \(\pi_n\) can be isomorphically embedded in \(\rho_n\).
Claim 2.1.1. The homotopy group $\pi_n(X, x')$ is isomorphic to a subgroup of $\rho_n(X, \{x_i\})$.

Proof. Notice that since $x'$ is contained in the same path component as some element of $\{x_i\}$, we can assume without loss of generality that $x' \in \{x_i\}$.

For an arbitrary $\phi : (I^n, \partial I^n) \to (X, x')$, let $[\phi]_{\pi_n}$ denote the homotopy class of $\phi$ (which is an element of $\pi_n$) and $[\phi]_{\rho_n}$ the component-homotopy class of $\phi$ (which is an element of $\rho_n$).

Choose $[\phi] \in \pi_n(X, x')$ and let $\phi \in [\phi]$. Then $\phi$ is a continuous map $(I^n, \partial I^n) \to (X, x')$, which means that $\phi \in R_n(X, \{x_i\})$. Define $f : \pi_n(X, x') \to \rho_n(X, \{x_i\})$ by $f([\phi]_{\pi_n}) = [\phi]_{\rho_n}$.

Suppose that $[\phi]_{\pi_n}, [\psi]_{\pi_n} \in \pi_n(X, x')$. To avoid confusion, we will use $\ast$ to denote the operation in $\pi_n(X, x')$ and $\ast$ to denote the operation that we have defined previously on $\rho_n(X, \{x_i\})$. We will use the standard definition of $\ast$:

$$(\phi \ast \psi)(\bar{x}) = \begin{cases} 
\phi(2x_1, x_2, \ldots, x_n) & x_1 \in [0, 1/2] \\
\psi(2x_1 - 1, x_2, \ldots, x_n) & x_1 \in [1/2, 1].
\end{cases}$$

We claim that $f$ is a homomorphism. We must show that

$$f([\phi \ast \psi]_{\pi_n}) = f([\phi]_{\pi_n}) \ast f([\psi]_{\pi_n})$$

$$[\phi \ast \psi]_{\rho_n} = [\phi]_{\rho_n} \ast [\psi]_{\rho_n}$$

$$(\phi \ast \psi) \sim (\phi \ast \psi).$$

Since $\sim$ is reflexive, there exists an open $O_\phi \subseteq I^{n+1}$ and a component-homotopy $h_\phi : (O_\phi, \partial O_\phi) \to (X, x')$ between $\phi$ and itself. Similarly, there exists an open $O_\psi \subseteq I^{n+1}$ and a component-homotopy $h_\psi : (O_\psi, \partial O_\psi) \to (X, x')$ between $\psi$ and itself. Let $\sigma_\phi$ and $\sigma_\psi$ be
homeomorphisms of $I^{n+1}$ such that

$$
\sigma_\phi(x_1, \ldots, x_n, t) = \left( \frac{(3-t)x_1}{6}, x_2, \ldots, x_n, t \right)
$$

$$
\sigma_\psi(x_1, \ldots, x_n, t) = \left( \frac{(3-t)x_1 + (3+t)}{6}, x_2, \ldots, x_n, t \right).
$$

Then notice that $\sigma_\phi(I^{n+1}) \cap I^n_0 = \{(x_1/2, x_2, \ldots, x_n) : x \in I^n\}, \sigma_\phi(I^{n+1}) \cap I^n_1 = \ell(I^n)$, $\sigma_\psi(I^{n+1}) \cap I^n_0 = \{x_1/2 + 1/2, x_2, \ldots, x_n) : x \in I^n\}$, and $\sigma_\psi(I^{n+1}) \cap I^n_1 = r(I^n)$. Let $O = \sigma_\phi(O_\phi) \cup \sigma_\psi(O_\psi)$ and define $h : (O, \partial O) \to (X, \{x_i\})$ by

$$
h(\bar{x}) = \begin{cases} 
\phi^{-1}(\bar{x}) & \bar{x} \in \sigma_\phi(O_\phi) \\
\psi^{-1}(\bar{x}) & \bar{x} \in \sigma_\psi(O_\psi).
\end{cases}
$$

Then $h$ is a component-homotopy between $\phi \ast \psi$ and $\phi \ast \psi$, proving that $f$ is a homomorphism.

Now suppose that $f([\phi])$ is the identity element of $\rho_n(X, \{x_i\})$ for some $[\phi] \in \pi_n(X, x')$, $\phi : (I^n, \partial I^n) \to (X, x')$. Then $\phi \sim e_{x'}$, where $e_{x'} : (I^n, \partial I^n) \to (X, x')$ is the constant map to $x'$. By definition of $\sim$, there exists an open $O \subseteq I^{n+1}$ and a component-homotopy $h : (O, \partial O) \to (X, x')$ between $\phi$ and $e_{x'}$. Notice that $h$ is an extension of $\phi : (I^n, \partial I^n) \to (X, x')$ to a map $(\text{int}(I^{n+1}), \partial I^{n+1}) \to (X, x')$, which means that $[\phi]$ is trivial in $\pi_n$ and therefore $f$ is injective.

Since $f$ is an injective homomorphism, we have that

$$
\pi_n(X, x') \cong f(\pi_n(X, x')) \leq \rho_n(X, \{x_i\}). \quad \Box
$$

It follows easily from this proof that when $X$ is path-connected, $\rho_n(X) \cong \pi_n(X)$.

### 2.2 Tree-connected spaces

Now we introduce a method for calculating $\rho_n(X, \{x_i\})$ when $X$ is a metric space. Suppose that $X$ is a metric space, with metric $d_X$. Further suppose that the set of basepoints $\{x_i\}_{i \in J}$
is a closed subset of $X$. We define $T(X, \{x_i\})$ to be the quotient space of

\[
\left( \bigcup_{i \in J} [0,1]_i \right) \cup X
\]

formed by

1. $0_j \sim 0_k$ for all $j, k \in J$
2. $1_i \sim x_i$ for all $i \in J$
3. $t_j \sim t_k$ whenever $t \leq 1 - \frac{1}{2}d_X(x_j, x_k)$.

Let $K$ denote the “tree” part of $T(X, \{x_i\})$; i.e., $K = T(X, \{x_i\}) \backslash (X \setminus \{x_i\})$. Then $K$ is a metric space under the shortest-path metric $d_K$. Also notice that we can replace $d_X$ with the topologically equivalent metric $d'_X$ defined by

\[
d'_X(p_i, p_j) = \min\{d_X(p_i, p_j), d_X(p_i, x_i) + 2 + d_X(x_j, p_j)\}
\]

for $p_i \in X_i, p_j \in X_j$. Then it is easy to see that $T(X, \{x_i\})$ is a path-connected metric space, with a metric $d$ which agrees exactly with $d'_X$ on $X$ and $d_K$ on $K$. We claim that we can calculate the $n$th extended homotopy group of $X$ by finding the $n$th homotopy group of $T(X)$.

Claim 2.2.1. $\pi_n(T(X, \{x_i\}), x') \cong \rho_n(X, \{x_i\})$.

Proof. As before, we will let $[\phi]_{\pi_n}$ denote an element of $\pi_n$ and $[\psi]_{\rho_n}$ denote an element of $\rho_n$, where $\phi$ and $\psi$ satisfy the necessary conditions. Also since $T(X, \{x_i\})$ is path-connected, $\pi_n(T(X, \{x_i\}), x')$ is independent (up to isomorphism) of the choice of basepoint, so we may suppose without loss of generality that $x' \in \{x_i\}$.

Let $[\phi]_{\pi_n} \in \pi_n$. Choose $\phi : (I^n, \partial I^n) \rightarrow (X, x') \in [\phi]_{\pi_n}$. Notice that $X$ is closed in $T(X, \{x_i\})$, since $T(X) \setminus X = \bigcup_{i \in J}[0,1]_i$ is open. Also $X \setminus \{x_i\}$ is open. Let $U = \phi^{-1}(X \setminus \{x_i\})$
and define \( \phi' = \phi|_{(U \cup \partial U)} \). Then we claim that \( \phi' : (U, \partial U) \to (X, \{x_i\}) \in R_1 \) (and so \([\phi']_{\rho_n} \in \rho_n\)).

We have that \( \phi'(U) \subseteq X \) by definition. Since \( \phi \) is continuous and \( X - \{x_i\} \) is open, so is \( U \), so \( U \subseteq U_1 \). Also since \( \phi \) is continuous, so is \( \phi' \). To see that \( \phi'(\partial U) \subseteq \{x_i\} \), let \( x \in \partial U \). Then by definition there exists a sequence \( \{s_n\} \) in \( U \) that converges to \( x \). Since \( U \) is open and \( x \in \partial U \), then \( x \not\in U \). Since \( X \) is closed and \( \phi'(\{x_i\}) \subseteq X \), then \( \phi'(x) \in X \), since \( \phi' \) is continuous. Since \( U = \phi^{-1}(X \setminus \{x_i\}) \) and \( x \not\in U \), then \( \phi'(x) \not\in X \setminus \{x_i\} \). Then it must be the case that \( \phi'(x) \in \{x_i\} \) and therefore \( \phi'(\partial U) \subseteq \{x_i\} \).

Now define \( f : \pi_n(T(X, \{x_i\}), x') \to \rho_n(X, \{x_i\}) \) by \( f([\phi]_{\pi_n}) = [\phi']_{\rho_n} \). We claim that \( f \) is an isomorphism.

**Claim 2.2.2.** \( f \) is well-defined.

*Subproof.** Suppose that \([\phi_1]_{\pi_n} = [\phi_2]_{\pi_n} \), where \( \phi_1 \) and \( \phi_2 \) are maps \( (I^n, \partial I^n) \to (T(X, \{x_i\}), x') \). Then there exists a homotopy \( h : (I^{n+1}, \partial I^{n+1}) \to (X, x') \) between \( \phi_1 \) and \( \phi_2 \). Let \( O = h^{-1}(X \setminus \{x_i\}) \), which is an open subset of \( I^{n+1} \). It is easy to see that \( h' = h|_{(O \cup \partial O)} \) is a component-homotopy between \( \phi'_1 \) and \( \phi'_2 \), so \( f([\phi_1]_{\pi_n}) = [\phi'_1]_{\rho_n} = [\phi'_2]_{\rho_n} = f([\phi_2]_{\pi_n}) \) and \( f \) is well-defined.

**Claim 2.2.3.** \( f \) is injective.

*Subproof.** Suppose that \([f(\phi)]_{\rho_n} = [f(\psi)]_{\rho_n} \), where \( \phi \) and \( \psi \) are maps \( (I^n, \partial I^n) \to (T(X, \{x_i\}), x') \). Then there exists a component-homotopy \( h : (O, \partial O) \to (X, \{x_i\}) \) between \( f(\phi) \) and \( f(\psi) \). Since our goal is to show that \([\phi]_{\pi_n} = [\psi]_{\pi_n} \), if we find a homotopy between \( \phi \) and \( \psi \) then we are done.

Let \( W = \text{cl}(O) \cup \partial I^{n+1} \). Notice that we can extend \( h \) continuously to a map \( h_1 \) on \( W \) by defining \( h_1 = \phi(x) \) on \( I^n_0 \), \( h_1 = \psi(x) \) on \( I^n_1 \), and \( h_1 = x' \) otherwise. Now we must show that we can extend \( h_1 \) continuously to the rest of \( I^{n+1} \).
Since it is a finite union of closed sets, \( W \) is closed, so \( \partial W \subseteq W \). Consider \( W^c = I^{n+1} \setminus W \). By definition, \( \partial W = \partial W^c \), so \( h_1(\partial W^c) \) is defined. We wish to show that \( h_1(\partial W^c) = h_1(\partial W) \subseteq K \).

Recall that \( \partial W^c = \partial W \subseteq \partial \text{cl}(O) \cup \partial I^2 \) and let \( x \in W^c \) be arbitrary. If \( x \in \partial \text{cl}(O) \), then by definition \( h_1(x) \in \{ x_i \} \), so \( h_1(x) \in K \). Then suppose that \( x \in \partial I^{n+1} \cap W^c \). If \( x \not\in I_0^\alpha \cup I_1^n \), then \( h_1(x) = x' \in \{ x_i \} \) by construction, so suppose that \( x \in I_0^\alpha \cup I_1^n \), so either \( h_1(x) = \phi(x) \) or \( h_1(x) = \psi(x) \). Also notice that clearly \( \partial W^c \cap O = \emptyset \), since \( O \subseteq W \), so we know that \( x \not\in O \). Let \( U_\phi = \phi^{-1}(X \setminus \{ x_i \}) \), so that we have \( f(\phi) : (U_\phi, \partial U_\phi) \to (T(X, \{ x_i \}), x') \). Define \( U_\psi \) similarly. Recall that by definition, \( O \cap \partial I_0^\alpha = U_\phi \) and \( O \cap \partial I_1^n = U_\psi \), so since \( x \not\in O \), then \( x \not\in U_\phi \) and \( x \not\in U_\psi \). Since by definition, \( U_\phi = \phi^{-1}(X \setminus \{ x_i \}) \) and \( U_\psi = \psi^{-1}(X \setminus \{ x_i \}) \), \( h_1(x) \notin (X \setminus \{ x_i \}) \). But since \( K = T(X, \{ x_i \}) \setminus (X \setminus \{ x_i \}) \), this means that \( h_1(x) \in K \).

Let \( h_2 : \partial W^c \to K \) be defined as \( h_1|_{\partial W^c} \). Notice that \( \partial W^c \) is a closed subset of the normal space \( W^c \) (since \( W^c \subseteq I^{n+1} \)). Since \( K \) is an absolute retract for normal spaces, \( h_2 \) can be extended continuously to a map \( h_3 : \text{cl}(W^c) \to K \). [1] Define \( h' : I^{n+1} \to T(X, \{ x_i \}) \) by

\[
h'(x) = \begin{cases} \ h_1(x) & x \in \text{cl}(W) \\ \ h_3(x) & x \in \text{cl}(W^c). \end{cases}
\]

By construction, \( h_1 \) and \( h_3 \) agree on the intersection of their domains, \( \partial W = \partial W^c \). Then by the pasting lemma, \( h' \) is continuous and therefore \( h' \) is a homotopy between \( \phi \) and \( \psi \), proving the claim.

**Claim 2.2.4.** \( f \) is surjective.

*Subproof.* Let \( [\phi]_{\rho_n} \) be arbitrary. Choose \( \phi : (U, \partial U) \to (X, \{ x_i \}) \in [\phi]_{\rho_n} \). We want to show that there exists \( [\psi]_{\pi_n} \) and \( \psi \in [\psi]_{\pi_n} \) such that \( \psi|_U = \phi \), so that \( f([\psi]_{\pi_n}) = [\phi]_{\rho_n} \).

Let \( V = I^n \setminus \text{cl}(U) \) and let \( V_\alpha \) be a component of \( V \). Notice that \( \phi \) is defined on \( \partial V = \partial U \) and that \( \phi(\partial V) \subseteq \{ x_i \} \).
Define \( \phi' : \partial V \cup \partial I^n \to K \) by \( \phi'(0, a_2, \ldots, a_n) = \phi'(1, b_2, \ldots, b_n) = x' \) and \( \phi'(x) = \phi(x) \) otherwise. Then since \( \partial V \cup \partial I^n \) is a closed subspace of \( V \) and \( K \) is an absolute retract for normal spaces, \( \phi' \) extends continuously to a map \( \phi'' : V \to K \). Define \( \psi : (I^n, \partial I^n) \to (T(X, \{x_i\}), x') \) by
\[
\psi(x) = \begin{cases}
\phi(x) & x \in \text{cl}(U) \\
\phi''(x) & x \in \text{cl}(V).
\end{cases}
\]
Since \( \text{cl}(U) \cap \text{cl}(V) = \partial U = \partial V \) and \( \phi(\partial U) = \phi''(\partial U) \) by construction, \( \psi \) is continuous by the pasting lemma. Then \([\psi] \in \pi_n(T(X, \{x_i\}, x')\) and \( f([\psi]_{\pi_n}) = [\phi]_{\rho_n} \), which proves the claim.

Claim 2.2.5. \( f \) is a homomorphism.

Subproof. Let \([\phi]_{\pi_n} \) and \([\psi]_{\pi_n} \) be arbitrary. Letting \( * \) denote the usual operation in \( \pi_n \), it suffices to show that
\[
\begin{align*}
f([\phi]_{\pi_n} * [\psi]_{\pi_n}) &= f([\phi]_{\pi_n}) * f([\psi]_{\pi_n}) \\
f([\phi \star \psi]_{\pi_n}) &= f([\phi]_{\pi_n}) * f([\psi]_{\pi_n}) \\
([\phi \star \psi]')_{\rho_n} &= [\phi']_{\rho_n} * [\psi']_{\rho_n} \\
(\phi \star \psi)' &\sim \phi' \star \psi'.
\end{align*}
\]

Define \( \theta_0, \theta_1 : (I^n, \partial I^n) \to (T(X, \{x_i\}, x') \) as follows:
\[
\begin{align*}
\theta_0(a_1, \ldots, a_n) &= (\phi \star \psi)(a_1, \ldots, a_n) \\
&= \begin{cases}
\phi(3a_1, a_2, \ldots, a_n) & a_1 \in [0, 1/3] \\
x' & a_1 \in [1/3, 2/3] \\
\psi(3a_1 - 2, a_2, \ldots, a_n) & a_1 \in [2/3, 1].
\end{cases}
\end{align*}
\]
It is easy to see that \([\theta_0]_{\pi_n} = [\theta_1]_{\pi_n} \), so there exists a homotopy \( h : (I^{n+1}, \partial I^{n+1}) \to \)
$(T(X, \{x_i\}), x')$ such that $h|_{I^0_n} = \theta_0$ and $h|_{I^1_n} = \theta_1$. Define $O = h^{-1}(X)$ and $h' = h|_O$. Then $h'$ is a component-homotopy between $(\phi \star \psi)'$ and $\phi' \star \psi'$, since $\theta_1|_O = \phi' \star \psi'$. This shows that $(\phi \star \psi)' \sim \phi' \star \psi'$ and that $f$ is a homomorphism. □

Since $f$ is a bijective homomorphism, we can conclude that

$$\pi_n(T(X, \{x_i\}), x') \cong \rho_n(X, \{x_i\}).$$

\section*{2.3 Examples}

\textbf{Example 2.3.1.} For $i \in \mathbb{N}$, let

$$X_i = \left\{ (x, y) \in \mathbb{R}^2 : d\left((x, y), \left( \frac{1}{2(i-1)}, \frac{1}{2(i+1)} \right) \right) = \frac{1}{2(i+1)} \right\},$$

and let $X_0 = \{(0, 0)\}$. Let $x_i = \left( \frac{1}{2(i-1)}, 0 \right)$ for $i \in \mathbb{N}$ and let $x_0 = (0, 0)$. Let $X = \bigcup_{i=0}^{\infty} X_i$. It is an easy corollary of Claim 6 that $\rho_1(X, \{x_i\})$ is isomorphic to the Hawaiian Earring Group $\mathbb{H}$.

We will see from the next example the importance of the inclusion of the limit point $(0, 0)$ in the previous example.

\textbf{Example 2.3.2.} For $i \in \mathbb{N}$ define $X_i$ and $x_i$ as in the previous example. Let $X' = \bigcup_{i \in \mathbb{N}} X_i$. Then $(X', \{x_i\}) \cong (Y, \{y_i\}) = \bigcup_{i \in \mathbb{N}} (Y_i, y_i)$, where

$$Y_i = \{(x, y) \in \mathbb{R}^2 : d((x, y), (i, 1)) = 1\}$$

and $y_i = (i, 0)$. Letting $Z = (Y, \{y_i\}) \cup \{(x, 0) : x \in \mathbb{R}\}$, it is easy to see that

$$\rho_1(X', \{x_i\}) \cong \rho_1(Y, \{y_i\}) \cong \pi_1(T(Y, \{y_i\}), y_1) \cong \pi_1(Z) \cong \bigoplus_{i \in \mathbb{N}} \mathbb{Z}.$$
Appendix A. Lemmas

A.1 Lemmas

Lemma A.1.1. Suppose \( \phi : X \to Y \) is a homeomorphism and \( A \subseteq X \). Then \( \partial \phi(A) = \phi(\partial A) \).

Proof. If \( \partial \phi(A) = \emptyset \), then by definition, \( \phi(A) \) is both open and closed. Since \( \phi \) is a homeomorphism, \( A \) is both open and closed, and therefore \( \phi(\partial A) = \phi(\emptyset) = \emptyset \) and we are done.

Now suppose that \( \partial \phi(A) \neq \emptyset \). Choose \( p \in \partial \phi(A) \). Let \( U \) be a neighborhood of \( \phi^{-1}(p) \). It suffices to show that \( U \cap (X \setminus A) \neq \emptyset \). Since \( \phi \) is a homeomorphism, \( \phi(U) \) is a neighborhood of \( p \), and by definition, \( \phi(U) \cap (Y \setminus \phi(A)) \neq \emptyset \). Choose \( q \in \phi(U) \cap (Y \setminus \phi(A)) \). Then \( \phi^{-1}(q) \in U \) and \( \phi^{-1}(q) \not\in A \), so \( \phi^{-1}(q) \in U \cap (X \setminus A) \neq \emptyset \) and we are done.

Notice that the reverse inclusion follows from a similar argument by considering the homeomorphism \( \phi^{-1} : Y \to X \).

Lemma A.1.2. For \( U \in \mathcal{U}_n \), \( (\ell(U))_- = r(U_-) \) and \( (r(U))_- = \ell(U_-) \).

Proof. Recall that we used the notation \( U_- \) to signify \( b(U) \), where \( b(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n, -x_{n+1}) \). Then to show that \( (\ell(U))_- = r(U_-) \), we must show that \( b(\ell(U)) = r(b(U)) \).

Let \( \overline{x} \in U \in \mathcal{U}_n \). Then

\[
\ell(U) = b(\overline{x}) = b(x_1/3, x_2, \ldots, x_n) = (1 - x_1/3, x_2, \ldots, x_n)
\]

and

\[
r(b(\overline{x})) = r(1 - x_1, x_2, \ldots, x_n) = ((1 - x_1 + 2)/3, x_2, \ldots, x_n) = (3 - x_1)/3, x_2, \ldots, x_n = (1 - x_1/3, x_2, \ldots, x_n),
\]

so \( b(\ell(U)) = r(b(U)) \) and therefore \( (\ell(U))_- = r(U_-) \).

The proof that \( (r(U))_- = \ell(U_-) \) is similar.
Bibliography


