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Subtraction Games: Range and Strict Periodicity

Bryce Emerson Blackham
Brigham Young University

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Bryce Emerson Blackham

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Rodney Forcade, Chair
Jasbir Chahal
Benjamin Webb

Department of Mathematics

Brigham Young University

ABSTRACT

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Bryce Emerson Blackham
Department of Mathematics, BYU
Master of Science

In this paper I introduce some background for subtraction games and explore the Sprague-Grundy functions defined on them. I exhibit some subtraction games where the functions are guaranteed to be strictly periodic. I also exhibit a class of subtraction games which have bounded range, and show there are uncountably many of these.

Keywords: Combinatorial Game Theory, Nim, Sprague-Grundy function, Periodicity, Subtraction Games

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Chapter 1. Basic definitions

In this paper, a **game** is a directed graph with the property that from any vertex v, called a **position**, there is a finite upper bound ℓ such that no path starting at v has length exceeding ℓ. Some graphs prohibited by this condition are those with cycles and those with infinite forward chains. To play a game defined this way, two players alternate turns moving a single marker from a position along an edge in the graph. The game ends when the marker rests on a position that is a sink and so cannot be moved. The player whose turn it is at that position loses. Anytime I refer to a losing position, I will mean a position from which the player to move has no winning strategy. Any game as defined above can be solved inductively as follows: losing positions are exactly those from which no legal move to a (descendant) losing position exists, and winning positions are those positions which are not losing positions. The set of losing positions in a game is sometimes referred to as the **kernel** for reasons which will be apparent.

A **subtraction game** is a game played on the nonnegative integers. Each subtraction game is defined by a set of positive integers, called the **subtraction set**. An edge from vertex n to m exists if $n - m$ belongs to the given subtraction set. A subtraction game is easily thought of as a game on a pile of objects, from which two players take turns removing an allowed number of objects. When there are insufficient number of objects to keep playing, the game is over and the player whose turn it is loses. In this paper I am mostly interested in subtraction games, though there are some applications to multiple-pile subtraction games which are a generalization of the game Nim (defined later). These games are extensively studied in [3], [4], [6], [9], and [12]. They appear easy to study and solve, and in individual cases are quite simple, but finding generalizations has proven to be far more difficult than one would think. For example, such games generated from subtraction sets with one or two elements have been completely solved, but for subtraction sets with three elements the problem is open and far from being completely solved (see [9]).

For each game with set of positions V, define the following function $G : V \to \mathbb{N} \cup \{0\}$ recursively as follows:

$$\begin{aligned} G(v) &= \min\{n \mid G(w) \neq n \text{ for all } w \text{ such that an edge from } v \text{ to } w \text{ exists}\} \\ &= \text{mex}\{G(w) \mid \text{an edge from } v \text{ to } w \text{ exists}\}. \end{aligned}$$

The function defined above is the **Sprague-Grundy function**. For ease of notation, *mex* is used to mean the minimum excluded element. It represents the smallest nonnegative integer excluded from the set it is applied to. Since the definition is easier to work with when thought of this way I will often mention "excluded elements" in this paper, and by that I mean the Spraque-Grundy values of positions accessible from the current position.

It is worth noting that positions in a game from which there are no legal moves are assigned the Sprague-Grundy value of 0, being the smallest nonnegative integer. This is the analog of a "lost" position being counted as a position from which the player whose turn it is has no winning strategy. The function is well-defined because of the finite upper bound condition for a game (which also serves as an upper bound for the Sprague-Grundy function for any given position). Some authors (see [5] and [7]) choose to use ordinals instead of nonnegative integers to avoid having to make this bounding restriction. For subtraction games, integers are more convenient to work with.

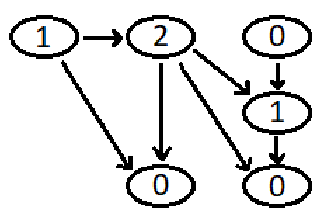

Figure 1.1: In this game I have labelled the vertices with their corresponding Sprague-Grundy values. Unsurprisingly the sinks in the graph have the value 0, but notice how the top-right vertex also has a value of 0 because no positions it can "access" are in the kernel. Similarly the top-left vertex has the value of 1, the smallest value not reachable from the position.

For subtraction games, I will notate the Sprague-Grundy function corresponding to the game with subtraction set $S \subset \mathbb{N}$ by G_S, and I will think of it as a sequence since it is a function on the nonnegative integers. Here, the accessible positions become numbers that are smaller by the right amounts. The function is defined as follows:

$$G_S(n) = \text{mex}\{G(n-\ell) \mid \ell \in S\}.$$

For example, if $S = \{1, 2, 3\}$ it is not too hard to see that

$$G_S = \{0, 1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3, ...\}$$

which is a **strictly periodic** sequence, meaning that there is a number p (here it is 4) such that $G_S(n + p) = G_S(n)$ for all $n \geq 0$.

And now I will explain what the Sprague-Grundy function says about the game being played which makes it worth studying. To begin with, the **kernel** $G^{-1}\{0\}$ of the Sprague-Grundy function is the set of losing positions, sometimes called zero-postions. This is pretty obvious since positions from which no moves are possible will be assigned the value of 0, and all other positions assigned to 0 are precisely those which have no legal moves to another zero-position. Positions assigned a nonzero value by the Sprague-Grundy function have at least one legal move to a zero-position because zero is not an excluded element in that case. The obvious question is of what there is to gain from distinguishing the different winning positions. I spend some time in this paper answering a question about the range of these Sprague-Grundy functions. Some discussion of range is given in [4] (there, functions with range $\{0, 1\}$ are called bipartite because they only return two values), but most study of subtraction games has focused on the periodicity rather than the range (see [4], [6], and [14]). What is interesting about the range of Sprague-Grundy functions? The most direct answer is the application to multiple-pile games. These games are played on a tuple of nonnegative integers called **piles** with the legal moves being subtraction by an element of S from one of the piles. **Nim** is the multiple-pile subtraction game with subtraction set $\mathbb{N}$.

More generally, any games can be combined in this way to create a **sum of games**, whose positions are the tuples of positions from each of the "summand" games. The easiest way to think of a sum of games is as a collection of graphs with a marker on a position from each. On a player's turn one marker may be moved along an edge. Figure 1.2 shows what this looks like for the postion (6,5,7) in the multiple-pile subtraction game for subtraction set $S = \{1, 2, 3\}$, along with how to "solve" it using the Sprague-Grundy theorem below.

Theorem 1.1. *(Sprague-Grundy)*
The Sprague-Grundy value of a position in a sum of n *games is as follows:*

$$G(v_1, v_2, ..., v_n) = \bigoplus_{i=1}^{n} G(v_i)$$

where the sum is the ***Nim-sum*** *given for integers* $m = \sum_{i=0}^{r} 2^i \alpha_i$ *and* $n = \sum_{i=0}^{r} 2^i \beta_i$ *written in binary form* (*so each* α_i *and* β_i *belong to* $\{0, 1\}$), *as follows:*

$$m \oplus n = \sum_{i=0}^{r} 2^i (\alpha_i \oplus_2 \beta_i).$$

Because of its usefulness in analyzing games similar to Nim, a Sprague-Grundy function is often referred to as a **Nim sequence**. When I refer to **Nim values** I mean the values of a Nim sequence.

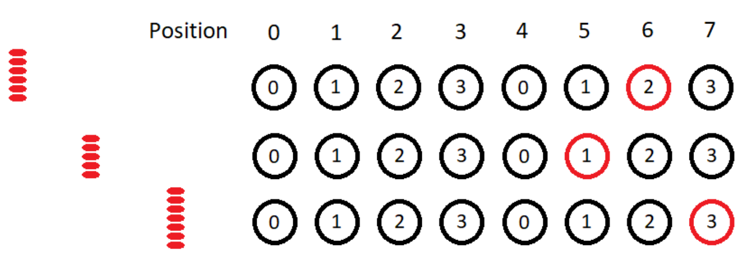

Figure 1.2: On the left is shown three piles to illustrate the position (6,5,7). A legal move is removing at least one stone from exactly one pile. On the right is shown the same position in a another way. A legal move is a move of one red marker at least one space to the left. The Nim values of each *pile* are given for the subtraction set $\{1, 2, 3\}$. Because $2 \oplus 1 \oplus 3 = 0$, this is a losing position.

The Nim-addition defined above is addition without carries in binary, sometimes called the *digital sum* or the *xor bitwise operator*. For example, the Nim-sum of 13 and 22 is most easily computed by writing their binary representations 01101 and 10110. Adding without carries, the sum is 11011 which is the number 27 in binary. So $13 \oplus 22 = 27$. Similarly $13 \oplus 27 = 22$ and $22 \oplus 27 = 13$ and the Nim-sum of any number with itself is zero. Here the set $\{0, 13, 22, 27\}$ is a group isomorphic to the Klein four-group under Nim-addition.

I won't provide a proof of the Sprague-Grundy theorem here, but it can be found on pages I-20 and I-21 of [12]. The idea of the proof is in three parts: (1) The Nim-sum of zeros is zero, (2) Given three numbers whose Nim-sum is nonzero, it is always possible to reduce one of the numbers so that their Nim-sum is zero (read left to right and find the first place where the sum isn't zero and change a 1 to a zero in one of the numbers, then change the digits to the right in that number to match the sum of the others), and (3) changing one number from a list always changes the Nim-sum of the list.

One of the applications of the range of the Nim sequence for subtraction games comes from the fact that the set of nonnegative integers under Nim-addition form an Abelian group isomorphic to the countable direct sum of $\mathbb{Z}/2\mathbb{Z}$. The range of a Nim sequence is either all nonnegative integers or the set $\{0, 1, 2, ..., r\}$ for some r. This range is a subgroup under Nim-addition precisely when r is one less than a power of 2. In this case it is always possible to add a third pile to a two-pile position that creates a losing position. Whether or not there are infinitely many ways to do this is partially dependent on whether or not the function is strictly periodic (see the example with $S = \{1, 4, 10\}$ in Table 1.1). This is a large part of why I am also interested in when these sequences are strictly periodic. Nim sequences are easiest to study when they are stricly periodic, and so there has been some study toward predicting which subtraction sets have strictly periodic Nim sequences (see [9]). It should be noted that there is also a multiplication operation discovered by John H. Conway explored in [7] that with Nim addition makes the nonnegative integers into a field. Applications of this field to games is explored in [5].

Before I continue I will mention one other application of the range. If the range does not contain 2, as is the case with any subtraction game with S a subset of the odd numbers containing 1, then all legal moves from a winning position are to losing positions (so it is impossible for the player with a winning strategy to fail to win). This makes the game quite trivial. In a game where the Nim sequence has outputs of 2 or more, it is possible to move from a 2-position to a 1-position (for example) and that means the game is somewhat interesting. Thus the range is related to the complexity of a game.

S	G_S (boldface font is used to show one period when it is periodic)
	0,0,0,...
1	**0,1**,0,1,0,1,...
1,2	**0,1,2**,0,1,2,0,1,2,...
2	**0,0,1,1**,0,0,1,1,...
1,4	**0,1,0,1,2**,0,1,0,1,2,...
2,3	**0,0,1,1,2**,0,0,1,1,2,...
1,2,3	**0,1,2,3**,0,1,2,3,...
1,2,6	**0,1,2,0,1,2,3**,0,1,2,0,1,2,3,...
1,4,10	0,1,0,1,2,0,1,0,1,2,3,2,3,0,1,3,**0,1,0,1,2,0,1,2,0,1,2**,0,1,0,1,2,0,1,2,0,1,2...
$\mathbb{N}$	0,1,2,3,4,5,6,7,...

Table 1.1: Table of Nim sequences

The periodicity (or lack thereof) is another way to describe the complexity of a subtraction game, because it is closely related to how much space is necessary to describe a winning strategy. Here I use the term **periodic** to describe sequences G for which there are positive integers n_0, p such that $G(n+p) = G(n)$ for all $n \geq n_0$. Above I have exhibited the Nim sequences for some simple subtraction games. Later I will provide proof of a lemma used in computing these. Observe that multiplying all the elements of a subtraction set by some number N replaces each term in the Nim sequence by a list of N copies of that term. Also observe that many of these games are periodic, even if they are not strictly periodic. This fact is true for all finite subtraction games, as stated in Theorem 1.2.

What is particularly interesting about the game given by $S = \{1, 4, 10\}$ as given in Table 1.1 is that its range is $\{0, 1, 2, 3\}$ but 3 only shows up in the pre-period, so that it appears only finitely many times in the range. This means if given two piles having Nim values of 1 and 2 respectively there are finitely many ways to add a third pile (specifically of 10, 12, or 15) to make a losing position in the three-pile game (here I use that the Nim-sum of 1, 2, and 3 is zero). Some immediate questions are: is it possible for the pre-period range be equal to the in-period range? Is it possible for the pre-period range to be greater than the in-period range? The answer to both questions is yes. I exhibit examples in Table 1.2.

S	G_S (boldface font indicates periodicity)
1,4,10	0,1,0,1,2,0,1,0,1,2,3,2,3,0,1,3,**0,1,0,1,2,0,1,2,0,1,2**,0,1,0,1,2,0,1,2,0,1,2,...
1,2,6,11	0,1,2,**0,1,2,3,0,1,2,0,1,2,3,4**,0,1,2,3,0,1,2,0,1,2,3,4,...
1,8,11	0,1,0,1,0,1,0,1,2,0,1,2,3,**2,3,2,0,1,0,1,2,0,1,0,1,0,1,2,0,1,0,1**,2,3,2,0,1,0,1,...

Table 1.2: Table of Nim sequences which are not strictly periodic

Theorem 1.2. *If S is finite, then G_S is periodic.*

I won't provide my proof, but this theorem is well-known and a discussion of it can be found on page 2 of [2]. The idea is simple. With only finitely many options there are also only finitely many possible ways to arrange reachable Nim values, and therefore only finitely many ways to make a finite list of such arrangements. Because a finite set is bounded, as soon as a list of appropriate size is repeated the excluded elements will follow the same pattern as before.

So, when are finite subtraction games not only periodic, but strictly periodic? I do not have the complete answer to this (this is an open question) but I do provide some cases where subtraction games are guaranteed to be strictly periodic. One case is quite easy to see. It is the case when S contains all the numbers from 1 to n and no multiples of $n+1$. The Nim sequence in this case is strictly periodic with period $n+1$, given by $\{0, 1, 2, ..., n\}$.

Example 1.3. If $S = \{1, 2, 3, 4, 5\} \cup \{\text{prime numbers}\}$ then

$$G_S = \{0, 1, 2, 3, 4, 5, 0, 1, 2, 3, 4, 5, 0, 1, 2, 3, 4, 5, ...\}$$

because appending nonmultiples of the period length 6 does not introduce to any position new reachable elements which were previously excluded.

I will hereafter show two less trivial cases. One is the case of symmetric sets, and the other is a collection of subtraction sets which each generate a Nim sequence with range $\{0, 1, 2\}$ which I explore in chapter 4.

Chapter 2. Symmetric sets

Here I describe one case in which subtraction games are somewhat well-behaved. The intuition behind this case comes from what happens if a game can be reorganized so that it is played on a finite undirected graph with a subset of its vertices declared "losing". For subtraction games, this happens when the subtraction set is symmetric.

Definition. A finite subset $S \subset \mathbb{N}$ is **symmetric** if there exists some number r so that $r - s \in S$ whenever $s \in S$.

For example, the set $S = \{1, 4, 5, 7, 9, 10, 13\}$ is symmetric with $r = 14$. Note that the number r above is unique and will hereafter be called the **modulus** of S. It is the sum of the kth and kth-to-last elements of S under the natural ordering, and this holds for any k. In particular, it is the sum of the first and last elements of S.

Theorem 2.1. *Let S be a symmetric set with modulus r. Then the Nim sequence of the subtraction game associated with S is strictly periodic with period dividing r.*

Proof. Let S be symmetric with modulus r. Let n be any nonnegative integer and let $i = G_S(n)$. Assume inductively that for any $h < i$ and m a nonnegative integer, $G_S(m) = h$ implies $G_S(m + r) = h$.

First I show $G_S(n + r) \leq G_S(n)$. For each $s \in S$,

$$G_S((n + r) - s) = G_S(n + (r - s)) \neq G_S(n) = i$$

because $r - s \in S$ by the symmetry of S. Since i is excluded from the collection of $G_S((n + r) - s)$ for all $s \in S$ it follows from the definition of G_S that $G_S(n + r)$, which is the smallest such excluded elements, is no bigger than i.

Now suppose by way of contradiction that the inequality is strict. Then $0 \leq h = G_S(n+r) < i$ for some h. Thus h is not excluded from the collection of $G_S(n-s)$ for $s \in S$. Let t be an element of S such that $G_S(n-t) = h$. By the inductive hypothesis, $G_S((n+r)-t) = G_S((n-t)+r) = G_S(n-t) = h$ so that h is also not excluded from the collection of $G_S((n+r)-s$ for $s \in S$. Thus $G_S(n+r) \neq h$ which is a contradiction.

Since n and i were arbitrary, it follows that G_S satisfies period r from the beginning of the sequence. Thus G_S is strictly periodic with period dividing r. □

Corollary 2.2. *For a finite subtraction set that is an arithmetic progression, the corresponding Nim sequence is strictly periodic with period dividing the sum of the first and last elements.*

Proof. If S is a finite arithmetic progression, then for some $n \in \mathbb{N}$, there exists some $a, d \in \mathbb{N}$ such that $S = \{a + kd : k \text{ is an integer between } 0 \text{ and } n\}$. The sum of the first and last elements of S is $a+(a+nd) = 2a+nd$. For $s \in S$, I have that $s = a+kd$ for some k between 0 and n. Then $(2a+nd)-s = 2a+nd-(a+kd) = a+(n-k)d \in S$ because $n-k$ is also an integer between 0 and n. Thus S is symmetric with modulus $2a+nd$. □

Corollary 2.3. *Any finite subtraction set S having a Nim sequence with period p can be extended to a set S' with no more than double the number of elements of S which is strictly periodic and also satisfies period p.*

Proof. Let $S' = S \cup \{p - s : s \in S\}$ which is the union of S and a set having the same cardinality as S, so it has no more than double the number of elements S has. It is also symmetric with modulus p. □

Another possible application of Theorem 2.1 could be to use symmetric closures in sequences of sets, so that no preperiods need be examined. Note, however, that (pointwise) limits of strictly periodic sequences need not be strictly periodic. Take the limit of the sequence 1,4,7,10,... and the corresponding Nim sequences for an example. Also note that any three-element set can be extended to a four-element symmetric set by appending only one element, so there could be some use of Theorem 2.1 in studying three-element sets.

Chapter 3. Counting Nim sequences

It is known that there are uncountably many sets of natural numbers, each of which can be used as a subtraction set for a game. However, many of these have the same Nim sequence. For example, if S is a subtraction set containing the integers from 1 to n, then S may (or may not) also contain anything which is not a multiple of $n+1$, and the Nim sequence is given by the periodic sequence $\{0, 1, 2, ..., n, 0, 1, 2, ..., n, ...\}$. A natural question to ask is *are there still uncountably many subtraction sets with distinct Nim sequences?* In other words, are there really uncountably many subtraction games?

Lemma 3.1. *If S is a finite subtraction set, then the kernel of G_S is infinite.*

Proof. It suffices to show that for any n there is a number $N > n$ such that $G_S(N) = 0$. Since S is finite is has a largest element t. If $G_S(n+i) = 0$ for some i between 1 and t, put $N = n+i$. Suppose $G_S(n+i) > 0$ for all i between 1 and t. Then $G_S(n+t+1) = 0$ because $n+t+1-\ell > n$ for all $\ell \in S$. Putting $N = n+t+1$ satisfies $G_S(N) = 0$. □

Theorem 3.2. *The set $\mathcal{G}$ of all Nim sequences of subtraction games is uncountable.*

Proof. I show there is an injection from the set of binary sequences to $\mathcal{G}$. Let Q be a binary sequence (assume its indices start at 1). Set $N_0 = 0$, $S_0 = \emptyset$, and for each $n > 0$ inductively define N_n to be the smallest integer in the kernel of $G_{S_{n-1}}$ which is larger than N_{n-1} (using Lemma 3.1), and define $S_n = S_{n-1}$ if $Q(n) = 0$ and $S_n = S_{n-1} \cup \{N_n\}$ if $Q(n) = 1$. Map Q to the limit of the S_n. Now suppose $Q_1 \neq Q_2$. I show that their corresponding subtraction sets (denote these $S^{(1)}$ and $S^{(2)}$) generate distinct Nim sequences, which will finish the proof. Let n be the first index where Q_1 and Q_2 differ. Without loss of generality, assume $Q_1(n) = 0$. For all $k < N_n$, the Nim sequences are the same because the relevant parts of the subtraction sets (only elements less than N_n) are the same. $G_{S^{(1)}}(N_n) = 0$ because N_n was chosen to be in the kernel for $S = \{s \in S^{(1)} \mid s < N_n\}$ and $S^{(1)}$ does not contain N_n (note here that the S_n are an ascending union). However $G_{S^{(2)}}(N_n) > 0$ because $N_n \in S^{(2)}$. Thus the Nim sequences differ. □

Finite subtraction sets give rise to periodic Nim sequences. If infinite subtraction sets are used the range of the corresponding Nim sequence may be infinite which precludes the possibility of a periodic Nim sequence. However, in many cases the discrete derivative of the Nim sequence is still periodic. For example, consider $S = \{n \in \mathbb{N} \mid n \not\equiv 2 \pmod 4\}$ whose corresponding Nim sequence follows the pattern: 0101 2323 4545 6767... which can be easily described using differences, since they are -1, 1, 1, 1 repeated for ever. In other words, though the Nim sequence has infinite range its discrete derivative has not only finite range but is periodic. Of course there are only countably many such functions, so how do I account for most of the Nim sequences? Do they typically have infinite range? Is it possible for a Nim sequence with finite range to be nonperiodic? I do not answer all of these questions in this paper, but I do answer the last question in the next chapter.

Chapter 4. Sprague-Grundy functions with Bounded Range

I begin with a useful lemma for computing Nim sequences. It is the lemma used to create the tables in chapter 1.

Lemma 4.1. *If N is the largest element of a finite subtraction set S, and if there exist p and n such that for all $0 \leq i < N$ the equality $G_S(n+i) = G_S(n+i+p)$ holds; then G_S satisfies period p and $G_S(n+i) = G_S(n+i+p)$ holds for all $i \geq 0$.*

This lemma gives an efficient way to compute periodic Nim sequences. I need only check that periodicity holds for as many consecutive pairs as the largest element in the subtraction set. For example, when computing the Nim sequence associated with $S = \{1, 4\}$ I get the sequence 0,1,0,1,2,0,1,0,1,... and these nine terms are enough to determine that the function is periodic satisfying period 5, and is 0,1,0,1,2 repeated indefinitely. This lemma can be found in page 2 of [9], though I provide a short proof here.

Proof. When computing $G_S(n+i+p)$ for $i \geq N$ the set $\{G_S(n+i+p-s) \mid s \in S\}$ used is inductively the same set as $\{G_S(n+i-s) \mid s \in S\}$ because $0 < s \leq N$ for all $s \in S$. Thus the minimum excluded elements of each are the same, so $G_S(n+i+p) = G_S(n+i)$. □

And now I will show that there is a way to build subtraction sets in such a way that maintains a small range while making the period arbitrarily long in a trivial way. The example $S = \{n \in \mathbb{N} \mid n \equiv 1 \mod 3\}$ is an example that shows it is not enough to simply increase period length to built a nonperiodic Nim sequence. The Nim sequence for the set $\{1, 4, ..., 3k+1\}$ is given by $0, 1$ followed by k repetitions of $0, 1, 2$ so that the larger the set the larger the period indeed, but the resulting function when using the full set S is still periodic, though not strictly periodic because the initial $0, 1$ is never visited again. Theorem 4.2 provides a way to build subtraction games where the pattern does not simplify in this way, and the corresponding infinite subtraction game (taking a limit) has a nonperiodic Nim sequence, all while the range is unchanged from that of the game with $S = \{1, 4\}$.

Theorem 4.2. *Suppose $S = \{s_1, ..., s_m\}$ has at least 2 elements, where $s_1 = 1$, $s_2 = 4$, and for each $n > 2$, $s_n > s_{n-1}$ and $s_n = 2s_{n-1} - s_{n-2} + k(s_{n-1}+1)$ for some nonnegative integer k. Then G_S is strictly periodic with period $s_m + 1$ and range $\{0, 1, 2\}$. More specifically, if $\{k_1, ..., k_{m-2}\}$ is a (possibly empty) sequence of k-values so that $s_n = 2s_{n-1} - s_{n-2} + k_{n-2}(s_{n-1}+1)$ for each $n > 2$, then inductively define $g\{k_1, ..., k_{m-2}\}$ in the following way:*

$$g\emptyset = \{0,1,0,1,2\}$$

$$g'\emptyset = \{0,1,2\}$$

and for $q > 0$ define

$$g\{k_1,...,k_q\} = \{g\{k_1,...,k_{q-1}\}^{k_q+1}, g'\{k_1,...,k_{q-1}\}\}$$

$$g'\{k_1,...,k_q\} = \{g\{k_1,...,k_{q-1}\}^{k_q}, g'\{k_1,...,k_{q-1}\}\}$$

where the superscripts indicate number of iterations in the sequence. G_S is strictly periodic, and the function is given by $g\{k_1,...,k_{m-2}\}$.

Proof. To begin, I will show that the statement about the period and range follows from the claim that G_S is given by $g\{k_1,...,k_{m-2}\}$, repeated. Since the range of these is obviously $\{0,1,2\}$ I only need to show that the length of $g\{k_1,...,k_{m-2}\}$ is $s_m + 1$ and that it satisfies no shorter period.

If $m = 2$ then $s_m = 4$ and $g\{k_1,...,k_{m-2}\} = g\emptyset$ has length $5 = s_m + 1$ as can be seen. If $m = 3$ then $s_m = 7 + 5k_1$ and $g\{k_1,...,k_{m-2}\} = g\{k_1\} = \{(g\emptyset)^{(k_1+1)}, g'\emptyset\}$ has length $5(k_1+1)+3 = 7+5k_1+1 = s_m+1$. For $m \geq 4$, assume inductively that the length $|g\{k_1,...,k_r\}|$ is $s_{r+2}+1$ for all values of r less than $m-2$. Then the length

$$
\begin{aligned}
|g\{k_1, ..., k_{m-2}\}| &= (k_{m-2}+1)|g\{k_1, ..., k_{m-3}\}| + |g'\{k_1, ..., k_{m-3}\}| \\
&= (k_{m-2}+1)|g\{k_1, ..., k_{m-3}\}| \\
&\quad + \ k_{m-3}|g\{k_1, ..., k_{m-4}\}| + |g'\{k_1, ..., k_{m-4}\}| \\
&= (k_{m-2}+1)|g\{k_1, ..., k_{m-3}\}| \\
&\quad + (k_{m-3}+1)|g\{k_1, ..., k_{m-4}\}| \\
&\quad + |g'\{k_1, ..., k_{m-4}\}| - |g\{k_1, ..., k_{m-4}\}| \\
&= (k_{m-2}+1)|g\{k_1, ..., k_{m-3}\}| \\
&\quad + |g\{k_1, ..., k_{m-3}\}| - |g\{k_1, ..., k_{m-4}\}| \\
&= (k_{m-2}+2)|g\{k_1, ..., k_{m-3}\}| - |g\{k_1, ..., k_{m-4}\}| \\
&= (k_{m-2}+2)(s_{m-1}+1) - (s_{m-2}+1) \\
&= 2s_{m-1} - s_{m-2} + k_{m-2}(s_{m-1}+1) + 1 \\
&= s_m + 1.
\end{aligned}
$$

Now I will show that for all q, the repetition of $g\{k_1, ..., k_{q-1}\}$ gives exactly the sequence $g\{k_1, ..., k_q\}$ up until the last term in $g\{k_1, ..., k_q\}$. This fact will be useful later in the proof, as well as for showing $g\{k_1, ..., k_{m-2}\}$ satisfies no smaller period than $s_m + 1$. I will show this inductively. First observe that $g\{k_1\} = \{g\emptyset^{k_1}, g'\emptyset\}$ is indeed repetition of $g\emptyset$ until the last term, being the last term of $g'\emptyset$, since $\{0, 1, 2\}$ is in agreement with $\{0, 1, 0, 1, 2\}$ until its final term. Assume $q \geq 2$ and observe how $g\{k_1, ..., k_q\} = \{g\{k_1, ..., k_{q-1}\}^{k_q+1}, g'\{k_1, ..., k_{q-1}\}\}$ implies that an equivalent proposition is that $g\{k_1, ..., k_{q-1}\}$ is in agreement with $g'\{k_1, ..., k_{q-1}\}$ until the last term of the latter (shorter) sequence. Now,

$$
\begin{aligned}
g\{k_1, ..., k_q\} &= \{g\{k_1, ..., k_{q-1}\}^{k_q+1}, g'\{k_1, ..., k_{q-1}\}\} \\
&= \{g\{k_1, ..., k_{q-1}\}^{k_q+1}, g\{k_1, ..., k_{q-2}\}^{k_{q-1}}, g'\{k_1, ..., k_{q-2}\}\}
\end{aligned}
$$

which by inductive hypothesis agrees (until the last term) with

$$\begin{aligned}&\{g\{k_1,...,k_{q-1}\}^{k_q+1}, g\{k_1,...,k_{q-2}\}^{k_{q-1}}, g\{k_1,...,k_{q-2}\}\}\\ =\ &\{g\{k_1,...,k_{q-1}\}^{k_q+1}, g\{k_1,...,k_{q-2}\}^{k_{q-1}+1}\}\end{aligned}$$

which being longer than the sequence it is to agree with until the last term of that sequence ($g\{k_1,...,k_q\}$), may be replaced by the longer $\{g\{k_1,...,k_{q-1}\}^{k_q+1}, g\{k_1,...,k_{q-2}\}^{k_{q-1}+1}, g'\{k_1,...,k_{q-2}\}\}$ which is equal to $\{g\{k_1,...,k_{q-1}\}^{k_q+1}, g\{k_1,...,k_{q-1}\}\} = g\{k_1,...,k_{q-1}\}^{k_q+2}$, an excessive repetition of $g\{k_1,...,k_{q-1}\}$. Thus the claim is proven.

In the case where $m = 2$ the period 5 is clearly minimal. Assume $m \geq 3$ and suppose $g\{k_1,...,k_{m-2}\}$ is periodic with period $r < s_m + 1$. Then $g\{k_1,...,k_{m-3}\}$ (here $m-3$ could be zero – just use the empty set in that case) which agrees with $g\{k_1,...,k_{m-2}\}$ for the first $s_m \geq r$ terms also satisfies period r. But this implies that r divides the lengths $s_m + 1$ and $s_{m-1}+1$ of both of the sequences. The greatest common divisor of $s_m+1 = 1+2s_{m-1}-s_{m-2}+k_{m-2}(s_{m-1}+1)$ and $s_{m-1}+1$ is the greatest common divisor of $-1-s_{m-2} = -(1+s_{m-2})$ and $1+s_{m-1}$. By an easy induction, this is $\gcd(s_1, s_2) = \gcd(1,4) = 1$. So r must be 1 and all of these g sequences are constant (obviously false). Thus the period s_m+1 is the smallest period satisfied by the sequence $g\{k_1,...,k_{m-2}\}$.

And now I will show that the Nim sequence of S is as claimed. If $S = \{1,4\}$ then the list of k-values is empty; and an easy computation gives the first 9 values as 0, 1, 0, 1, 2, 0, 1, 0, 1, which with Lemma 4.1 implies G_S is $\{0,1,0,1,2\} = g\emptyset$ repeated, as desired.

Now let $|S| = m \geq 3$ and assume inductively that for all smaller subtraction sets the theorem holds. Here, $S = \{s_1,...,s_m\}$ with the elements in ascending order as in the hypothesis, and the list of k-values is $\{k_1,...,k_{m-2}\}$. For each $n < s_m$ the element s_m is not used in computing $G_S(n)$, and so $G_S(n) = G_{\{s_1,...,s_{m-1}\}}(n)$. Thus the first s_m terms of G_S are the first s_m terms of the sequence given by repetition of $g\{k_1,...,k_{m-3}\}$. Recall from earlier that $g\{k_1,...,k_{m-2}\}$ is exactly this, up until its last term. So now I must show that $G_S(s_m) = 2$, the last term in the finite sequence $g\{k_1, ..., k_{m-2}\}$.

Clearly the second-to-last term in any of the $g\{k_1, .., k_q\}$ is 1. Because $1 \in S$ this means 1 is not an excluded element when calculating $G_S(s_m)$. And $s_m \in S$ implies zero is not an excluded element either. I now show that 2 is excluded from the set $\{G_S(s_m - \ell) \mid \ell \in S\}$ so that $G_S(s_m) = 2$ as claimed. Clearly $G_S(s_m - s_m) = G_S(0) = 0 \neq 2$. Since $s_m - s_{m-1} < s_m$, $G_S(s_m - s_{m-1}) = G_{\{s_1,...,s_{m-1}\}}(s_m - s_{m-1}) = G_{\{s_1,...,s_{m-1}\}}(2s_{m-1} - s_{m-2} + k_{m-2}(s_{m-1} + 1) - s_{m-1}) = G_{\{s_1,...,s_{m-1}\}}(s_{m-1} - s_{m-2} + k_{m-2}(s_{m-1} + 1)) = G_{\{s_1,...,s_{m-1}\}}(s_{m-1} - s_{m-2})$ by the periodicity of $G_{\{s_1,...,s_{m-1}\}}$. By inductive hypothesis this is not 2 (The range in previous cases would necessarily include 3 otherwise, so my base case from the main induction works here). Now I will show that there is a number $n < s_m$ for which $G_S(n) = 2$ and $G_S(n - \ell) = G_S(s_m - \ell)$ for each $\ell < s_{m-1}$ so that the set $\{G_S(s_m - \ell) \mid \ell \in S \text{ and } \ell < s_{m-1}\}$ cannot contain 2 (which will conclude the proof of the claim $G_S(s_n) = 2$).

There are two cases. Either $k_q = 0$ for all $q < m-2$ or there exists a number $0 < t < m-2$ for which $k_t > 0$.

Assume the first case. Then by an easy induction $g'\{k_1, ..., k_q\} = \{0, 1, 2\}$ for all $q < m - 2$. This implies that

$$\begin{aligned} g\{k_1, ..., k_{m-2}\} &= g\{k_1, ..., k_{m-3}\}^{k_{m-2}+1}, 0, 1, 2\} \\ &= \{g\{k_1, ..., k_{m-3}\}^{k_{m-2}}, g\{k_1, ..., k_{m-3}\}, 0, 1, 2\}. \end{aligned} \tag{4.1}$$

But by a similar induction

$$\begin{aligned} g\{k_1, ..., k_q\} &= \{g\{k_1, ..., k_{q-1}\}, g'\{k_1, ..., k_{q-1}\}\} \\ &= \{g\{k_1, ..., k_{q-1}\}, 0, 1, 2\} \\ &= ... \\ &= \{g\emptyset, \{0, 1, 2\}^q\} \\ &= \{0, 1, \{0, 1, 2\}^{q+1}\} \ \ \forall q < m - 2, \text{ so that} \end{aligned}$$

$$g\{k_1,...,k_{m-2}\} = \{g\{k_1,...,k_{m-3}\}^{k_{m-2}}, 0, 1, \{0,1,2\}^{m-2}, 0, 1, 2\} \\ = \{g\{k_1,...,k_{m-3}\}^{k_{m-2}}, 0, 1, \{0,1,2\}^{m-1}\}. \tag{4.2}$$

Let $n = s_m - 3$. Clearly $G_S(n) = 2$ and for all ℓ smaller than the length $|\{0,1,2\}^{m-2}| = 3(m-2)$, $G_S(s_m - \ell) = G_S(n - \ell)$ holds. However, $S = \{1, 4, 7, ..., 3m-8, 3m-5, 3m-2+k_{m-2}(3m-4)\}$ (here I use $2s_q - s_{q-1} = s_q + (s_q - s_{q-1})$) so that

$$s_{m-2} = 3m - 8 = 3(m-2) - 2 < 3(m-2).$$

Thus all elements $\ell \in S$ smaller than s_{m-1} satisfy

$$G_S(s_m - \ell) = G_S(n - \ell).$$

Now assume the second case and let t be the largest positive number smaller than $m-2$ such that $k_t > 0$ (note that this assumption forces $m \geq 4$). By another easy induction,

$$g'\{k_1,...,k_q\} = \{g\{k_1,...,k_{t-1}\}^{k_t}, g'\{k_1,...,k_{t-1}\}\} \tag{4.3}$$

for all q with $t \leq q \leq m-3$. Then

$$g\{k_1,...,k_{m-2}\} = \{g\{k_1,...,k_{m-3}\}^{k_{m-2}+1}, g'\{k_1,...,k_{m-3}\} \\ = \{g\{k_1,...,k_{m-3}\}^{k_{m-2}}, g\{k_1,...,k_{m-3}\}, \\ g\{k_1,...,k_{t-1}\}^{k_t}, g'\{k_1,...,k_{t-1}\}\}$$

and use of a similar induction shows

$$
\begin{aligned}
g\{k_1,...,k_q\} &= \{g\{k_1,...,k_{q-1}\}, g'\{k_1,...,k_{q-1}\}\} \\
&= \{g\{k_1,...,k_{q-1}\}, g\{k_1,...,k_{t-1}\}^{k_t}, g'\{k_1,...,k_{t-1}\}\} \\
&= ... \\
&= \{g\{k_1,...,k_{q-(q-t+1)}\}, \\
&\quad (\{g\{k_1,...,k_{t-1}\}^{k_t}, g'\{k_1,...,k_{t-1}\}\})^{q-t+1}\} \\
&= \{g\{k_1,...,k_{t-1}\}, \\
&\quad (\{g\{k_1,...,k_{t-1}\}^{k_t}, g'\{k_1,...,k_{t-1}\}\})^{q-t+1}\}
\end{aligned}
\tag{4.4}
$$

for all q satisfying $t \leq q \leq m-3$. Combining these,

$$
\begin{aligned}
g\{k_1,...,k_{m-2}\} &= \{g\{k_1,...,k_{m-3}\}^{k_{m-2}}, g\{k_1,...,k_{t-1}\}, \\
&\quad (\{g\{k_1,...,k_{t-1}\}^{k_t}, g'\{k_1,...,k_{t-1}\}\})^{m-3-t+1}, \\
&\quad g\{k_1,...,k_{t-1}\}^{k_t}, g'\{k_1,...,k_{t-1}\}\} \\
&= \{g\{k_1,...,k_{m-3}\}^{k_{m-2}}, g\{k_1,...,k_{t-1}\}, \\
&\quad (\{g\{k_1,...,k_{t-1}\}^{k_t}, g'\{k_1,...,k_{t-1}\}\})^{m-1-t}\}.
\end{aligned}
$$

It is clear from the $m-1-t$ iterations (as in the other case) of $\{g\{k_1,...,k_{t-1}\}^{k_t}, g'\{k_1,...,k_{t-1}\}\}$ that if $n = s_m - |\{g\{k_1,...,k_{t-1}\}^{k_t}, g'\{k_1,...,k_{t-1}\}\}|$ then $G_S(n) = 2$ and for each $\ell < (m-2-t)|\{g\{k_1,...,k_{t-1}\}^{k_t}, g'\{k_1,...,k_{t-1}\}\}|$ there holds $G_S(s_m - \ell) = G_S(n - \ell)$. To calculate $|\{g\{k_1,...,k_{t-1}\}^{k_t}, g'\{k_1,...,k_{t-1}\}\}|$ recall that this is

$$\begin{aligned}|g'\{k_1,...,k_{m-3}\}| &= |g\{k_1,...,k_{m-2}\}| - (k_{m-2}+1))|g\{k_1,...,k_{m-3}\}| \\ &= s_m + 1 - (k_{m-2}+1)(s_{m-1}+1) \\ &= 2s_{m-1} - s_{m-2} + k_{m-2}(s_{m-1}+1) \\ &\quad + 1 - (k_{m-2}+1)(s_{m-1}+1) \\ &= 2s_{m-1} - s_{m-2} + 1 - (s_{m-1}+1) \\ &= s_{m-1} - s_{m-2}.\end{aligned}$$

Now I will show $(m-2-t)(s_{m-1}-s_{m-2}) > s_{m-2}+1$. For each $0 < q \leq m-2-t$, I claim that $s_{m-1}-s_{m-2} = s_{m-1-q}-s_{m-2-q}+k_{m-2-q}(s_{m-1-q}+1)$. I will show this inductively. When $q = 1$ this is $s_{m-1}-s_{m-2} = -s_{m-2}+2s_{m-2}-s_{m-3}+k_{m-3}(s_{m-2}+1) = s_{m-2}-s_{m-3}+k_{m-3}(s_{m-2}+1)$. Assuming this is true for some fixed $q < m-2-t$, we see that

$$\begin{aligned}s_{m-1} - s_{m-2} &= s_{m-1-q} - s_{m-2-q} + k_{m-2-q}(s_{m-1-q}+1) \\ &= 2s_{m-2-q} - s_{m-3-q} + k_{m-3-q}(s_{m-2-q}+1) \\ &\quad - s_{m-2-q} + k_{m-2-q}(s_{m-1-q}+1) \\ &= s_{m-2-q} - s_{m-3-q} + k_{m-3-q}(s_{m-2-q}+1)\end{aligned}$$

because $k_{m-2-q} = 0$ when $q < m-2-t$. Now let q be an index variable ranging from 1 to $m-2-t$ and add all of these versions of $s_{m-1} - s_{m-2}$ to obtain

$$(m-2-t)(s_{m-1}-s_{m-2}) = \sum_{q=1}^{m-2-t} \left[s_{m-1-q} - s_{m-2-q} + k_{m-2-q}(s_{m-1-q}+1)\right]$$

which telescopes to $s_{m-2}-s_t+k_t(s_{t+1}+1)$ because all of the k_{m-2-q} are zero when $q < m-2-t$. And so $(m-2-t)(s_{m-1}-s_{m-2}) = s_{m-2}-s_t+k_t(s_{t+1}+1) \geq s_{m-2}-s_t+(s_{t+1}+1) > s_{m-2}+1$ since S is listed in increasing order. Thus for all $\ell < s_{m-1}$ there holds $G_S(s_m-\ell) = G_S(n-\ell)$.

I have shown $G_S(s_m) = 2$. To apply Lemma 4.1 I still need to show that for $s_m+1 \leq p \leq 2s_m$ that $G_S(p) = G_S(p-s_m-1)$. Since this is only the next s_m terms, and as shown earlier

$g\{k_1, ..., k_{m-2}\}$ is in agreement with repetition $g\{k_1, ..., k_{m-3}\}$ until the last term (being the $(s_m+1)^{th}$); it is enough to show that the finite sequence $\{G_S(p)\}_{p=s_m+1}^{2s_m}$ is just the repetition of $g\{k_1, ..., k_{m-3}\}$, which is the Nim sequence of the set $S - \{s_m\}$. Rewrite $p = s_m + 1 + \bar{p}$ and recall the two above cases:

In the first case $k_q = 0$ for all $q < m-2$, and in this case there holds

$$g\{k_1, ..., k_{m-2}\} = \{g\{k_1, ..., k_{m-3}\}^{k_{m-2}}, 0, 1, \{0, 1, 2\}^{m-1}\}.$$

Also in this case

$$g\{k_1, ..., k_{m-3}\} = \{0, 1, \{0, 1, 2\}^{m-2}\}$$

(applying equation 4.2 with $q = m-3$) so that there is agreement in the last $3(m-2) > s_{m-2}$ terms.

In the second case there is some (maximal) $t < m-2$ for which $k_t > 0$, and in this case there holds

$$\begin{aligned} g\{k_1, ..., k_{m-2}\} = \{ & g\{k_1, ..., k_{m-3}\}^{k_{m-2}}, g\{k_1, ..., k_{t-1}\}, \\ & (\{g\{k_1, ..., k_{t-1}\}^{k_t}, g'\{k_1, ..., k_{t-1}\}\})^{m-1-t}\} \\ = \{ & g\{k_1, ..., k_{m-3}\}^{k_{m-2}}, \\ & g\{k_1, ..., k_{t-1}\}, g'\{k_1, ..., k_{m-3}\}^{m-1-t}\} \end{aligned}$$

(applying equation 4.3). Also in this case

$$g\{k_1, ..., k_{m-3}\} = \{g\{k_1, ..., k_{t-1}\}, g'\{k_1, ..., k_{m-3}\}^{m-2-t}\}$$

(applying equation 4.4 with $q = m-3$ followed by equation 4.3) so that there is agreement in the last $(m-2-t)|g'\{k_1, ..., k_{m-3}\}| > s_{m+2}$ terms.

Let $S' = S - \{s_m\}$ and $S'' = S - \{s_{m-1}\}$. I will show that (in both cases) $\{G_S(p-\ell) \mid \ell \in S''\} = \{G_{S'}(s_{m-1} + 1 + \bar{p} - \ell) \mid \ell \in S'\}$. Then I will only need to show that $G_S(p - s_{m-1})$

belongs to this same set, so that this element does not perturb the calculation of the minimum excluded element. This will show (by known periodicity of $G_{S'}$) that G_S is $g\{k_1, ..., k_{m-3}\}$, repeated as claimed.

Assume inductively for all $s_m < q < p$ that $\{G_S(q - \ell) \mid \ell \in S''\} = \{G_{S'}(s_{m-1} + 1 + \bar{q} - \ell) \mid \ell \in S'\}$ is satisfied, and also for all $s_m - s_{m-2} < q < p$ that $G_S(q) = G_S(q - s_m - 1)$ (what is ultimately desired, and also note that the agreement shown above acts as the s_{m-2} base cases as necessary here). By inductive hypothesis $G_S(q - \ell) = G_{S'}(s_{m-1} + 1 + \bar{q} - \ell)$ for all $\ell \in S' \cup S''$. The only thing left to show here is that $G_S(q - s_m) = G_{S'}(s_{m-1} + 1 + \bar{q} - s_{m-1})$. The expression on the right is just $G_{S'}(1 + \bar{q}) = G_S(1 + \bar{q})$ because $q < p < 2s_m$ implies $\bar{q} < s_m$. And then $1 + \bar{q} = q - s_m$.

Finally I show $G_S(p - s_{m-1}) \in \{G_{S'}(s_{m-1} + 1 + \bar{p} - \ell) \mid \ell \in S'\}$. To do this, divide into two cases: $s_m < p \leq s_m + s_{m-1} - s_{m-2}$ and $s_m + s_{m-1} - s_{m-2} < p \leq 2s_m$. Assume the first case. Then

$$
\begin{aligned}
p - s_{m-1} \leq s_m - s_{m-2} < s_m& \\
\implies G_S(p - s_{m-1} &= G_{S'}(p - s_{m-1}) \\
&= G_{S'}(s_m + 1 + \bar{p} - s_{m-1}) \\
&= G_{S'}(2s_{m-1} - s_{m-2} + k_{m-2}(s_{m-1} + 1) \\
&\qquad + 1 + \bar{p} - s_{m-1}) \\
&= G_{S'}(s_{m-1} + 1 + \bar{p} + k_{m-2}(s_{m-1} + 1) - s_{m-2}) \\
&= G_{S'}(s_{m-1} + 1 + \bar{p} - s_{m-2})
\end{aligned}
$$

by periodicity of $G_{S'}$. Putting $\ell = s_{m-2}$ finishes this case. Now assume the latter case. Then $p - s_{m-1} > s_m - s_{m-2}$ where the inductive hypothesis $G_S(q) = G_S(q - s_m - 1)$ is applicable, so that $G_S(p - s_{m-1}) = G_S(p - s_{m-1} - s_m - 1) = G_S(\bar{p} - s_{m-1})$ and since $\bar{p} < s_m$ this is just $G_{S'}(\bar{p} - s_{m-1})$ (here in this case $\bar{p}$ is sufficiently large for this to make sense). Finally by periodicity of $G_{S'}$ I have $G_S(p - s_{m-1}) = G_{S'}(s_{m-1} + 1 + \bar{p} - s_{m-1})$, and putting $\ell = s_{m-1}$ finishes this case.

Thus the minimum excluded elements are the same and $G_S(p) = G_{S'}(\bar{p})$ for $s_m < p < 2s_m$. By Lemma 4.1, G_S satisfies the period s_m+1 and is given by the construction as claimed. $\square$

Table 4.1 shows the application of Theorem 4.2 for a few small cases. In it I omit commas to save space, and since the range for all of these subtraction games is $\{0, 1, 2\}$ there will be no ambiguity. None of them have a pre-period because they are all strictly periodic, so I only show the period in each case.

k-values	S	G_S (one period given)
	1,4	01012
0	1,4,7	01012 012
1	1,4,12	$\{01012\}^2 012$
2	1,4,17	$\{01012\}^3 012$
0, 0	1,4,7,10	$01\{012\}^3$
0, 1	1,4,7,18	$\{01012012\}^2 012$
0, 2	1,4,7,26	$\{01012012\}^3 012$
1, 0	1,4,12,20	$\{01012\}^2 012$ 01012 012
1, 1	1,4,12,33	$\{\{01012\}^2 012\}^2 01012$ 012
1, 2	1,4,12,46	$\{\{01012\}^2 012\}^3 01012$ 012
2, 0	1,4,17,30	$\{01012\}^3 012\{01012\}^2 012$
2, 1	1,4,17,48	$\{\{01012\}^3 012\}^2\{01012\}^2 012$
2, 2	1,4,17,66	$\{\{01012\}^3 012\}^3\{01012\}^2 012$
0, 0, 4	1,4,7,10,57	$\{01\{012\}^3\}^5 012$
1, 0, 1	1,4,12,20,49	$\{\{01012\}^2 012\ 01012\ 012\}^2 01012$ 012
1, 1, 1	1,4,12,33,88	$\{\{\{01012\}^2 012\}^2 01012\ 012\}^2\{01012\}^2 012$ 01012 012
2, 0, 0	1,4,17,30,43	$\{01012\}^3 012\{\{01012\}^2 012\}^2$
2, 0, 1	1,4,17,30,74	$\{\{01012\}^3 012\{01012\}^2 012\}^2\{01012\}^2 012$
2, 1, 0	1,4,17,48,79	$\{\{01012\}^3 012\}^2\{01012\}^2 012\{01012\}^3 012\{01012\}^2 012$
0,0,0,0,0,5	1,4,7,10,13,16,19,122	$\{01\{012\}^6\}^6 012$

Table 4.1: Table of Nim sequences built from k-values

Lemma 4.3. *If $\{S_n\}$ is an increasing sequence of subsets such that for every n, every element of $S_{n+1} - S_n$ is larger than every element of S_n, then*

$$G_{\lim_{n\to\infty} S_n} = \lim_{n\to\infty} (G_{S_n})$$

where the limit on the right side is the pointwise limit of the Nim sequences.

Proof. Since $\{S_n\}$ is monotone increasing (by inclusion), its limit is its union. By transitivity of $\geq$ (and an easy induction) every element of $S = \bigcup_{k=1}^{\infty} S_k$ not belonging to some particular S_n, is larger than all elements of S_n. Thus when computing the Grundy function G_S up to the largest element N of S_n none of the elements of S missing from S_n are used in exclusion sets, so G_S agrees with G_{S_n} up to N.

Fix N. There are two cases. Either S_n contains an element at least as large as N for some n, or all S_n are bounded above by N. In the first case $G_S(N) = \lim_{n\to\infty} G_{S_n}(N)$ because the sequence is eventually constant by the above argument. In the second case the sequence $\{S_n\}$ is eventually constant, so that there exists some m for which $S = S_m$ and the desired equality is simply $G_{S_m} = G_{S_m}$.

Since both sides are equal for all inputs, the functions are equal. □

Now applying Lemma 4.3 it is possible to extend to infinite sequences of k-values which build limits of subtraction sets in increasing order. Probably the easiest case of this is where the sequence of k-values is the all-zeros sequence. This builds the subtraction set $S = \{1, 4, 7, 10, ...\}$ of natural numbers which are congruent to 1 modulo 3. The Nim sequence corresponding to this set is 0,1,0,1,2,0,1,2,0,1,2, ... Here the sequence is no longer strictly periodic, but is still periodic with period length 3. A natural question to ask here would be "is it possible to build S in such a way that makes the Nim sequence nonperiodic while keeping the range $\{0, 1, 2\}$?" I will show that there are in fact uncountably many ways to do this.

Theorem 4.4. *There exists a set $S \subset \mathbb{N}$ such that G_S both has finite range and is nonperiodic.*

Proof. I show that two distinct (infinite) sequences of k-values generate subtraction sets with distinct Nim sequences. Let $\{k_n^{(1)}\}$ and $\{k_n^{(2)}\}$ be the sequences and let q be the index where the sequences first differ. Assume without loss of generality that $k_q^{(1)} < k_q^{(2)}$. The respective subtraction sets S_1 and S_2 agree up until the $(q+2)^{th}$ element by minimality of q. Let S_1' and S_2' be the sets of the first $q+2$ elements of S_1 and S_2 repectively. Let s and

t be the respective shared $(q+1)^{th}$ and q^{th} elements of S_1 and S_2. The next element of S_1 is $2s-t+k_q^{(1)}(s+1)$ and the next element of S_2 is $2s-t+k_q^{(2)}(s+1)$. Additions to the subtraction sets of larger elements (as given by the k-values) does not adjust either of the Nim sequences for the first terms up to the input $q+2$. By Lemma 4.3, the function G_{S_1} is equal to $G_{S_1'}$ for inputs 0 through $P = 2s-t+k_q^{(1)}(s+1)$ and G_{S_2} is equal to $G_{S_2'}$ for inputs 0 through $Q = 2s-t+k_q^{(2)}(s+1)$. P is smaller than Q by assumption. For inputs 0 through P, G_{S_2} is equal to $G_{S_1' \cap S_2'}$ and so

$$\begin{aligned} G_{S_2}(P) &= G_{S_2}(2s-t+k_q^{(1)}(s+1)) \\ &= G_{S_2}(s-t-1) \end{aligned}$$

because $G_{S_1' \cap S_2'}$ is strictly periodic with period $s+1$. If $t=1$ is the first element, then $s-t-1 = 4-1-1 = 2$ and

$$G_{S_2}(s-t-1) = G_{S_2}(2) = G_{\{1,4\}}(2) = 0.$$

Otherwise there is an element u of both S_1 and S_2 for which $s = 2t-u+k_{q-1}(t+1)$, and $s-t-1 = t-u-1+k_{q-1}(t+1)$ which being smaller than s has the same Nim value (for both functions) as $t-u-1$ again by periodicity. Continuing this inductively, $G_{S_2}(P) = 0$. But $G_{S_1}(P) \neq 0$ because $P \in S_1$ (specifically, the value is 2). Thus the two sequences are distinct.

The mapping from the uncountable set of infinite k-sequences to the Nim sequences of their corresponding subtraction sets is thus injective, so that there are uncountably many of these Nim sequences, all of which have range $\{0,1,2\}$. Only countably many periodic sequences exist, so there must be some nonperiodic sequence among them. $\square$

And of course, there are uncountably many of these finite-range non-periodic Nim sequences. Even better, here are some examples!

Chapter 5. Curious Examples

Example 5.1. The sequence of k-values {0,0,0,0,...} generates the subtraction set $\{1, 4, 7, 10, ... =$ the set of positive integers which are congruent to 1 modulo 3. The corresponding Nim sequence is

$$\{0, 1, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, ...\}$$

which is periodic with period length 3, but is not strictly periodic.

Example 5.2. The sequence of k-values $\{1, 1, 1, 1, ...\}$ generates the subtraction set $S = \{1, 4, 12, 33, 88, ...\} =$ is the sequence $F_{2n} - 1$ where F_n is the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, 13, ...\}$. The corresponding Nim sequence is

$$\{0, 1, 0, 1, 2, 0, 1, 0, 1, 2, 0, 1, 2, 0, 1, 0, 1, 2, 0, 1, 0, 1, 2, 0, 1, 2, 0, 1, 0, 1, 2, 0, 1, 2, ...\}.$$

Replacing the words "0,1" and "0,1,2" with "1" and "2" changes this sequence to the self-generating sequence

$$\{1, 2, 1, 2, 2, 1, 2, 1, 2, 2, 1, 2, 2, 1, 2, 1, 2, 2, 1, 2, 1, 2, 2, 1, 2, 2, 1, 2, 1, ...\}$$

which is sequence A001468 in the Online Encyclopedia of Integer Sequences [1]. The number of 2's appearing between consecutive 1's is given by the sequence itself. It is nonperiodic.

Example 5.3. The sequence of k-values $\{0, 1, 0, 1, 0, 1, ...\}$ generates the subtraction set $S = \{1, 4, 7, 18, 29, 70, 111, ...\}$ is a sequence whose every-other-element subsequences (using the even or the odd indices) have discrete derivatives that satisfy the recurrence $a_{n+2} = 4a_{n+1} - a_n$. Adding one to each term in either subsequence also generates a sequence that satisfies this recurrence. The corresponding Nim sequence is

$$\{0, 1, 0, 1, 2, 0, 1, 2, 0, 1, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 0, 1, 2, 0, 1, 2, 0, 1, ...\}.$$

Replacing the words "0,1" and "0,1,2" with "2" and "3" changes this sequence to the self-generating sequence

$$\{2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, ...\}$$

which is sequence A007538 in the Online Encyclopedia of Integer Sequences [11]. The number of 3's appearing between consecutive 2's is given by the sequence itself. It is nonperiodic.

Example 5.4. The sequence of k-values $\{1, 0, 1, 0, 1, 0, ...\}$ generates the subtraction set $S = \{1, 4, 12, 20, 49, 78, 186, ...\}$ and its corresponding Nim sequence is also related to the sequence

$$\{2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, ...\}$$

but this time by replacing the words "0,1,0,1,2" and "0,1,0,1,2,0,1,2" with "2" and "3". It is nonperiodic.

Example 5.5. The sequence of k-values $\{0, 0, 1, 0, 0, 1, 0, 0, 1, 0, ...\}$ generates the subtraction set $S = \{1, 4, 7, 10, 24, 38, 52, 119, 186, 253, ...\}$ and its corresponding Nim sequence is given by the self-generating sequence

$$\{3, 4, 4, 4, 3, 4, 4, 4, 4, 3, 4, 4, 4, 4, 3, 4, 4, 4, 4, 3, 4, 4, 4, 3, 4, 4, 4, 4, 3, 4, 4, 4, 4, 3, ...\}$$

(which is sequence A018244 in the Online Encyclopedia of Integer Sequences [8]) with "3" and "4" replaced by the words "0,1" and "0,1,2". It is nonperiodic.

More generally, if the k-sequence is periodic with period $\{0, 0, ..., 0, 1\}$ with r zeros, then the Nim sequence is given by the self-generating sequence as above, using $r + 1$ and $r + 2$ in the place of 3 and 4, and then replacing $r+1$ and $r+2$ with the words "0,1" and "0,1,2". None of these are periodic.

Example 5.6. The sequence of k-values $\{2, 2, 2, 2, ...\}$ generates the subtraction set $S = \{1, 4, 17, 66, 249, 932, ...\}$ and its corresponding Nim sequence is given by the self-generating sequence

$$\{3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, ...\}$$

(this is not quite A007538 and it is not an offset of it either, though it is constructed very similarly, with a different starting condition) with "3" and "2" replaced with the words "0,1,0,1,2" and "0,1,2". It is nonperiodic.

More generally, if the k-sequence is constant $= \{\text{k,...}\}$ then the corresponding Nim sequence can be built like above, but with $k+1$ and k in the stead of 3 and 2. Replace each $k+1$ with the word "0,1,0,1,2" and each k with the word "0,1,2". Note that this in indeed another way to describe the "all ones" case described above, with $k = 1$. In fact, the same rule holds for the "all zeros" k-sequence, though it's not the easiest way to think of it.

Example 5.7. The sequence of k-values $\{0, 0, 1, 1, 0, 0, 1, 1, ...\}$ generates the subtraction set $S = \{1, 4, 7, 10, 24, 63, 102, 141, 322, 826, ...\}$ and it should be no surprise that it takes some more effort to describe its Nim sequence. Taking the sequence and replacing the words "0,1" and "0,1,2" with "1" and "2" generates the sequence

$$\{1, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, 2, 2, 2, 1, ...\}$$

where the number of consecutive 2s is given by the sequence

$$\{3, 4, 3, 4, 4, 3, 4, 4, 3, 4, 4, 3, 4, 3, 4, 4, 3, 4, 4, 3, 4, 4, 3, 4, 4, 3, 4, 3, 4, 4, 3, 4, 4, 3, 4, 4, 3, 4, 3, ...\}$$

and the pattern of consecutive 4s is given again by the first sequence.

Chapter 6. Conjectures and Unanswered Questions

Conjecture. The infinite k-sequences that are not eventually zero generate nonperiodic subtraction games, while the finite k-sequences and the infinite k-sequences which are eventully zero generate periodic subtraction games.

Conjecture. The discrete derivative of the Nim sequence corresponding to a periodic subtraction set is periodic.

Question. Do these self-generating sequences show up in a predictable way? How does it relate to the periodicity of the k-sequence?

Question. In one of the above examples the Fibonacci sequence showed up. This was generated by the same sequence which gives the continued fraction for the golden ratio. The corresponding Nim sequence is also closely related to the Fibonacci Word (see A003849 in the Online Encyclopedia of Integer Sequences [10]). Is the continued fraction a coincidence, or is it related somehow? Does this generalize nicely for recurrances?

Question. Is there a way to use higher-order recurrences to generate Nim sequences with larger bounded range, like for example $\{0, 1, 2, 3\}$?

Question. Is there a way to use the strict periodicity of symmetric sets in limits to solve more infinite classes of subtraction games?

Question. Any three-element set of natural numbers can be extended to symmetric four-element set by appending only one element. Can this fact be used to make significant progress in solving three-element subtraction games? Three-element subtraction games are known to be deceptively simple but stubborn to solve, and have not yet been generally solved. See [9] for some further discussion.

Question. What other ways can be used to bound the range of Nim sequences? For example, if the smallest element of S is s_1 and $3s_1$ also belongs to S then the range contains 3 or does not contain 2. This is a direct consequence of Ferguson's Pairing Property, which is discussed in [3] and [13]. Are there any nice ways to generalize this fact enough to be useful?

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