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Spaces of Weakly Holomorphic Modular Forms in Level 52

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ABSTRACT

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Let $M_k^\sharp(52)$ be the space of weight k level 52 weakly holomorphic modular forms with poles only at infinity, and $S_k^\sharp(52)$ the subspace of forms which vanish at all cusps other than infinity. For these spaces we construct canonical bases, indexed by the order of vanishing at infinity. We prove that the coefficients of the canonical basis elements satisfy a duality property. Further, we give closed forms for the generating functions of these basis elements.

Keywords: modular forms, Zagier duality, weakly holomorphic

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Chapter 1. Introduction

Modular forms have applications in many areas, from Fermat's last theorem to partitions, the Monster Group, and sums of squares. These forms are even used in computing entropy in black holes and string theory.

We say that a function $f : \mathcal{H} \to \mathbb{C}$ is a modular form of weight $k \in \mathbb{Z}$ and level $N \in \mathbb{N}$ if the following conditions hold. First, f is holomorphic on $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ together with the set of cusps for level N. Second, f satisfies the equation

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \tag{1.1}$$

where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \mod N \right\}.$$

When equation 1.1 is applied to the matrices $\left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, we note that k must be even and that f is periodic of period 1. Since f is periodic, f has a Fourier expansion. Let $q = e^{2\pi i z}$; then $f(z) = \sum_{n \geq n_0} a_n q^n$ where n_0 is the order of vanishing of f at infinity.

If f vanishes at each cusp, then f is said to be a cusp form. We write $M_k(N)$ for the space of all modular forms of weight k and level N, and $S_k(N)$ for the space of cusp forms of weight k and level N. Both of these spaces are finite-dimensional vector spaces over $\mathbb{C}$.

By relaxing the definitions of the forms in $M_k(N)$ to allow functions be meromorphic at the cusps, we get the space $M_k^!(N)$. Further adjusting to the functions of $M_k^!(N)$ to allow the functions to be holomorphic at all cusps, except possibly at infinity, creates the space $M_k^\sharp(N)$. Of the functions in $M_k^\sharp(N)$, the functions that vanish at each cusp other than infinity form the subspace $S_k^\sharp(N)$. These last two spaces of weakly holomorphic forms are the topic of this thesis.

In their paper Duke and Jenkins [3] focus on $M_k^\sharp(1)$. From this paper we point out the following features.

(i) There is a canonical basis whose elements are in the form $f_{k,m}(z) = q^{-m} + \sum_{n \geq \ell+1} a_k(m,n) q^n$, where ℓ is the greatest order of vanishing at infinity of any form in $M_k^\sharp(1)$.

(ii) The generating function $\sum_{m\geq\ell} f_{k,m}(\tau)q^m$ has the closed form $\sum_{m\geq\ell} f_{k,m}(\tau)q^m = \frac{f_{k,-\ell}(z)f_{2-k,1+\ell}(\tau)}{j(\tau)-j(z)}$.

(iii) The coefficients of the basis elements satisfy Zagier duality: $a_k(m,n) = -a_{2-k}(n,m)$.

(iv) The coefficients of the basis elements satisfy the divisibility $n^{k-1}|a_k(m,n)$ for $(m,n)=1$ and $k \in \{4,6,8,10,14\}$.

They also show, for certain values of m, that the zeros of $f_{k,m}(z)$ are on the unit circle.

Many of these results have been generalized in other levels of genus 0. In levels 2 and 3, Garthwaite and Jenkins [5] look at the zeros of the basis elements of $M_k^\sharp(2)$ and $M_k^\sharp(3)$, giving a lower bound for the number of zeros on the lower boundary of the fundamental domain. Garthwaite and Jenkins utilize the generating function from El-Guindy [4] for their calculations concerning where the zeros lie. For levels 2 and 3, Andersen and Jenkins [1] prove several congruences for the weight zero basis element coefficients. Andersen and Jenkins also give a basis of the same form as (i) for levels 2, 3, 5, and 7. Level 4 has similar results by Haddock and Jenkins [6]. They give a generating function and show that the coefficients of the basis elements satisfy Zagier duality. Similar to the other levels they also give bounds for the zeros on the lower boundary of the fundamental domain. Jenkins and Thornton [8] add coefficient congruences to Andersen and Jenkins' work. In [7] Jenkins and Thornton list basis elements in the form (i) above for levels 8, 9, 16, and 25 and give congruences. Work is also underway for the other levels of genus zero. We seek to address the points (i)-(iv) in higher levels under a few specific conditions.

In the mentioned levels, all of the canonical basis elements have integral coefficients. However, this is not the case in all levels; for instance, levels 52, 56, 63, 66, 70, and 78 have a canonical basis with rational Fourier coefficients. This implies that there is a congruence between Hecke eigenforms modulo primes dividing the denominators of the rational coefficients. This initial difference raises the question of whether the same techniques applied in lower levels will generalize.

In this paper we examine level 52, which is the lowest level where $M_k(52)$ is spanned by eta-quotients and its basis elements have rational Fourier coefficients. We expect that the results of this paper could be easily generalized to give similar results for levels 56, 63, and 70 since, like level 52, they are of genus 5, spanned by eta-quotients, and have rational Fourier coefficients. These results may also generalize to levels 66 and 78 which have higher genus, but are spanned by eta-quotients and have rational Fourier coefficients.

This paper is outlined as follows: In Chapter 2 we construct canonical bases for the spaces $M_k^\sharp(52)$ and $S_k^\sharp(52)$, which are given in Theorems 2.10 and 2.11 respectively. Two cases of the main result are listed here.

Theorem 2.1. *$S_2^\sharp(52)$ has a canonical basis of the form $\{g_{2,m}(z) | m \geq -5,\ m \neq 0\}$ where*

$$g_{2,m}(z) = q^{-m} + a_2(m,0) + \sum_{n=6}^{\infty} a_2(m,n)q^n.$$

Theorem 2.2. *$M_2^\sharp(52)$ has a canonical basis $\{f_{2,m}(z) | m \geq -13,\ m \neq -9, \ldots, -12\}$ where*

$$f_{2,m}(z) = q^{-m} + \sum_{n=9}^{12} a_2(m,n)q^n + \sum_{n=14}^{\infty} a_2(m,n)q^n.$$

In Chapter 3 we prove a duality result similar to that of Zagier's work in [12].

Theorem 3.3. *Let the weight k be given. Let $f_{k,n}(\tau) \in M_k^\sharp(52)$ and $g_{2-k,n}(z) \in S_k^\sharp(52)$. Then we have*

$$a_k(m,n) = -b_{2-k}(n,m).$$

In Chapter 4 we give generating functions connecting the basis elements in Theorem 4.4.

Definition 4.1. Let $f_{k,m}(z)$ be basis elements as found in Theorem 2.10. Then the generating function $F_k(z,\tau)$ is given by

$$F_k(z,\tau) = \sum_{n=-n_0}^{\infty} f_{k,n}(\tau)q^n$$

where n_0 is the greatest order of vanishing at infinity of any form in $M_k^\sharp(52)$.

One case of the main result of that chapter is

Theorem 4.3. *Let $k = 2+12l$ and $y = 7k-1$. Let $F_k(z,\tau) = f_{k,-y}(\tau)q^{-y}+\sum_{n=5-y}^{\infty} f_{k,n}(\tau)q^n$. Then*

$$\begin{aligned}
F_k(z,\tau) =& \Big(\varphi_6(z) - \varphi_6(\tau)\Big)^{-1} \cdot \Big(f_{k,-y}(\tau)g_{2-k,y+6}(z) + 2f_{k,-y}(\tau)g_{2-k,y+4}(z) \\
&+ f_{k,-y}(\tau)g_{2-k,y+2}(z) + 2f_{k,5-y}(\tau)g_{2-k,y-1}(z) + f_{k,5-y}(\tau)g_{2-k,y-3}(z) \\
&+ f_{k,5-y}(\tau)g_{2-k,y+1}(z) + 2f_{k,6-y}(\tau)g_{2-k,y-2}(z) + f_{k,6-y}(\tau)g_{2-k,y-4}(z) \\
&+ f_{k,7-y}(\tau)g_{2-k,y-1}(z) + 2f_{k,7-y}(\tau)g_{2-k,y-3}(z) + f_{k,8-y}(\tau)g_{2-k,y-2}(z) \\
&+ 2f_{k,8-y}(\tau)g_{2-k,y-4}(z) + f_{k,9-y}(\tau)g_{2-k,y-3}(z) + f_{k,10-y}(\tau)g_{2-k,y-4}(z) \\
&+ 2f_{k,-y}(\tau)g_{2-k,y-2}(z)\Big).
\end{aligned}$$

The main result, in entirety, is Theorem 4.4. We give proofs of two cases. The omitted proofs have their recurrence relation and revised generating functions given in Appendix B. In Chapter 5 we state further work to be accomplished and conjectures for higher levels.

Chapter 2. Canonical Bases

In constructing a canonical basis, we want a basis which allows us to easily and uniquely identify the linear combination of the basis elements used to create any other form in the space. To do this, we will take a basis and row reduce to annihilate as many terms as possible. This gives elements which start with a given leading term and whose next term has an exponent as large as possible. The goal is to have a basis for all weights similar to the bases for weight 2 in the following theorems, which will be constructed in Section 2.3.

Theorem 2.1. *$S_2^\sharp(52)$ has a canonical basis $\{g_{2,m}(z)\}$ where*

$$g_{2,m} = q^{-m} + a_2(m,0) + \sum_{n=6}^{\infty} a_2(m,n)q^n, \text{ for all } m \geq -5,\ m \neq 0.$$

Theorem 2.2. *$M_2^\sharp(52)$ has a canonical basis $\{f_{2,m}(z)\}$ where*

$$f_{2,m} = q^{-m} + \sum_{n=9}^{12} a_2(m,n)q^n + \sum_{n=14}^{\infty} a_2(m,n)q^n, \text{ for all } m \geq -13,\ m \neq -9, \ldots, -12.$$

In order to construct a basis for $S_2^\sharp(N)$, work in levels of genus zero makes use of the Hauptmodul, a weakly holomorphic modular form of weight 0 with a pole of order 1 at infinity. However, Hauptmoduln do not exist in levels of higher genus. To have a function with comparable effect, we will need to find a function in the space $M_0^\sharp(N)$ with pole of minimal order at infinity. We then multiply the weight zero form by the forms in $S_2(52)$ to create other forms that are meromorphic at the cusp at infinity but sill holomorphic elsewhere. Note that multiplying a weight 0 form by a weight k form will result in a weight k form. This introduces forms with leading term q^{-m} for $m \in \mathbb{N}$, with maybe some subset of $\mathbb{N}$ unaccounted for. We then subtract off the previous forms to give a gap or multiple gaps as described at the beginning of this chapter. As we will see, it is necessary to use a second form to get all possible orders of vanishing at infinity in the absence of a Hauptmodul. We use eta-quotients to find these two functions. To do this we need a deeper understanding about the structure of our space.

2.1 Eta-quotients

First we list the definition of the Dedekind eta-function [2].

Definition 2.3. The *eta-function* is defined by the formula $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$. The modular transformations of the eta-function take the form

$$\eta\left(\frac{az+b}{cz+d}\right) = \epsilon(a,b,c,d)(cz+d)^{1/2}\eta(z) \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } \epsilon(a,b,c,d)^{24} = 1.$$

The *eta-quotient* is defined by $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ with $r_\delta \in \mathbb{Z}$.

The following theorem from Kilford's text [9] lists when an eta-quotient is in the space $M_k^\sharp(N)$.

Theorem 2.4. *Let* $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ *with* $r_\delta \in \mathbb{Z}$. *If*

$$(i) \sum_{\delta|N} \delta r_\delta \equiv 0 \mod 24,$$

$$(ii) \sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \mod 24,$$

$$(iii) \prod_{\delta|N} \delta^{r_\delta} \textit{ is the square of a rational number,}$$

then $f(z)$ *is weakly modular of weight* $k = \frac{1}{2}\sum_{\delta|N} r_\delta$ *and level* N.

The next theorem of Ligozat [10] gives us a way to compute the order of vanishing at each cusp.

Theorem 2.5 (Ligozat's formula)**.** *The order of vanishing of an eta-quotient $f(z)$ at the cusp c/d, where $(c, d) = 1$, is given by*

$$\mathrm{ord}_{c/d}(f) = \frac{N}{24} \sum_{\delta | N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd(d, N/d) d\delta}.$$

This formula allows us to test whether a weakly modular eta-quotient is in the space $S_k^\sharp(N)$.

In [11], Rouse and Webb give a sharp bound on the exponents used in an eta-quotient. Let r_δ for $\delta | N$ be the exponents of the eta-functions $\eta(\delta z)$ in an eta-quotient. Then

$$\sum_{\delta | N} |r_\delta| \leq 2k \prod_{p | N} \left(\frac{p+1}{p-1} \right)^{\min\{\, 2, \mathrm{ord}_p(N)\, \}}.$$

This means that for a given level and weight, there are a finite number of possible eta-quotient modular functions in $M_k(N)$. We say that a space is spanned by eta-quotients if it has a basis of eta-quotients. Rouse and Webb also state that spaces having a sufficiently composite level should be spanned by eta-quotients.

2.2 The space $M_2(52)$

Using SAGE, we can quickly produce a basis of $M_k(52)$ for relatively low weights $k \in \{\, 2, 4, 6, 8\, \}$, yielding:

$$
\begin{aligned}
M_2(52) = \mathrm{span}\Big\{ & q - \frac{4}{3}q^7 - \frac{5}{3}q^9 + \frac{2}{3}q^{11} - \frac{1}{3}q^{13} - \frac{2}{3}q^{15} + O(q^{17}),\\
& q^2 - q^6 + q^8 - 2q^{10} - 2q^{12} + O(q^{17}),\\
& q^3 - \frac{2}{3}q^7 - \frac{7}{3}q^9 + \frac{4}{3}q^{11} + \frac{1}{3}q^{13} - \frac{4}{3}q^{15} + O(q^{17}),\\
& q^4 - 2q^6 + q^{10} - q^{12} + q^{14} + q^{16} + O(q^{17}),\\
& q^5 - \frac{1}{3}q^7 - \frac{2}{3}q^9 - \frac{4}{3}q^{11} - \frac{1}{3}q^{13} + \frac{1}{3}q^{15} + O(q^{17}),\\
& 1 + O(q^{17}),\\
& q + 4q^3 + 6q^5 + 8q^7 + 13q^9 + 12q^{11} + 24q^{15} + O(q^{17}),\\
& q^2 + 4q^6 - 2q^8 + 6q^{10} + 8q^{14} - 6q^{16} + O(q^{17})\\
& q^4 + 3q^8 + 4q^{12} + 7q^{16} + O(q^{17})\\
& q^{13} + O(q^{17})\Big\},
\end{aligned}
$$

where the first 5 forms are a basis for $S_2(52)$. Verifying that $M_2(52)$ is indeed spanned by eta-quotients is done by Rouse [11] where he references

`http://users.wfu.edu/rouseja/eta/etamake9.data`.

On this site, Rouse lists tuples containing the exponents r_δ from Theorem 2.4 to represent eta-quotients, where the subscripts are the divisors of 52. There tuples are structured as $(r_1, r_2, r_4, r_{13}, r_{26}, r_{52})$. The eta-quotients spanning $M_2(52)$ are represented below.

$$
\begin{aligned}
&(0,-4,8,0,0,0), (8,-4,0,0,0,0), (0,0,0,0,-4,8), (0,0,0,8,-4,0),\\
&(0,-2,4,0,-2,4), (4,-2,0,4,-2,0), (1,-2,3,3,-2,1), (3,-2,1,1,-2,3),\\
&(-3,7,-2,-1,1,2), (-2,7,-3,2,1,-1).
\end{aligned}
$$

The next theorem gives us more structure about the spaces $M_k(52)$.

Theorem 2.6. *The form*

$$h(z) = q^{84} - 2q^{86} + 3q^{88} - 6q^{90} + O(q^{92}) \in M_{12}(52) \tag{2.1}$$

has all of its zeros at infinity. Additionally, $k = 12$ is the smallest positive weight in which such a form exists.

Proof. Using SAGE we create bases for weights $2, 4, 6, 8, 10$ and 12 in level 52. After row reduction on the basis, comparing the leading terms of the forms with the respective Sturm bound, only $M_{12}(52)$ has a form with power of its leading term equal to the Sturm bound. This form of weight 12 has a leading term q^{84}; by the valence formula we can verify this has all of the zeros at infinity, since with $k = 12$ and $N = 52$, we have $\frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = 84$. $\square$

Thus, the space $M_k(52)$ has no form with all of its zeros at infinity for $k = 2, 4, 6, 8, 10$. We may also state another condition on $S_2(52)$.

Theorem 2.7. *Then $S_2^\sharp(52)$ has no form starting with a constant term.*

Proof. Since the function $h(z) = q^{84} - 2q^{86} + 3q^{88} - 6q^{90} + O(q^{92})$ (see 2.1) has all of its zeros at infinity, multiplying $h(z)$ by any form $g(z) \in S_2^\sharp(52)$ with a constant leading term will give a product $h(z)g(z)$ in $S_{14}(52)$ (since it vanishes at infinity) whose leading term is q^{84}. However, computing a basis for $S_{14}(52)$ in SAGE shows no such form exists. $\square$

We will now apply this result to find a form in $M_0^\sharp(52)$ with a pole of minimal order at infinity.

Theorem 2.8. *The weakly holomorphic modular form*

$$\varphi_6(z) = q^{-6} + 2q^{-4} + q^{-2} + 2q^2 + 3q^6 + 2q^8 + O(q^{12}) \in M_0^\sharp(52) \tag{2.2}$$

has a pole of minimal order at infinity.

Proof. Given a form in $M_k^\sharp(52)$ with minimal pole at infinity, multiplying this form by any cusp form in $S_2(52)$ results in a form in $S_2^\sharp(52)$. By Theorem 2.7 this product will not have a constant leading term. Since the basis elements of $S_2(52)$ have leading terms q^5, q^4, q^3, q^2, and q, this eliminates the possibilities $q^{-1}, q^{-2}, q^{-3}, q^{-4}$ or q^{-5} as leading terms of the form of weight zero. Using SAGE (see A.1), we first conduct a search for a form starting with q^{-6}. Using Theorem 2.4 we create tuples $(r_1, r_2, r_4, r_{13}, r_{26}, r_{52})$ where $r_{52} = -r_1 - r_2 - r_4 - r_{13} - r_{26}$, which guarantees the weight is zero. The $|r_i|$ are bounded to make computation reasonable. To satisfy conditions (i) and (ii) of Theorem 2.4 we verify that Ligozat's formula (2.5) gives a nonnegative integer for $d = 1, N$; we also need the order of vanishing to be a nonnegative integer at every cusp other than infinity. Condition (iii) is satisfied when $r_2 + r_{26} \equiv 0 \mod 2$ and $r_{13} + r_{26} + r_{52} \equiv 0 \mod 2$. Lastly, to get the order of vanishing at infinity to be -6 we need $\sum_{\delta|52} \delta^{r_\delta} = -144$.

With these conditions programmed into SAGE we found the form

$$\frac{\eta^4(4z)\eta^2(26z)}{\eta^2(2z)\eta^4(52z)} = q^{-6} + 2q^{-4} + q^{-2} + 2 + 2q^2 + 3q^6 + 2q^8 + O(q^{12}).$$

Since this form resides in $M_0(52)$ and $1 \in M_0(52)$, we may take a linear combination of these two to get

$$\varphi_6(z) = q^{-6} + 2q^{-4} + q^{-2} + 2q^2 + 3q^6 + 2q^8 + O(q^{12}).$$

□

2.3 Constructing the basis for $S_2^\sharp(52)$

We can now create all the canonical basis elements of $S_2^\sharp(52)$ with leading terms $q^5, q^4, q^3, ...$ and any leading term of negative power not congruent 0 modulo 6 by using the elements of $S_2(52)$ and $\varphi_6(z)^l$ for some integer $l \geq 0$. However, we should be able to create all negative powers for the leading term. Thus, we need another form of weight zero and a pole at infinity of order between 7 and 11 (we don't want 12 since we can achieve that with φ_6^2).

Theorem 2.9. *The weakly holomorphic modular form*

$$\psi(z) = q^{-7} + \frac{1}{3}q^{-5} + \frac{2}{3}q^{-3} + \frac{4}{3}q^{-1} + \frac{2}{3}q + \frac{2}{3}q^{3} + \frac{1}{3}q^{5} + \frac{5}{3}q^{7} + 2q^{9} + \frac{7}{3}q^{11} + \frac{1}{3}q^{13} + \frac{5}{3}q^{15} + O(q^{17})$$

is an element of $M_0^\sharp(52)$.

Proof. Again with similar criteria, we change the order of vanishing at infinity to -8 in our SAGE code (A.1) (we didn't find an eta quotient with leading term q^{-7}) and find two forms, which are listed as the tuples $(-3, 7, -2, -1, 5, -6)$ and $(3, -2, 1, 1, 2, -5)$. Using a linear combination of the corresponding forms to cancel out the first term gives us the form $\psi(z) = q^{-7} + \frac{1}{3}q^{-5} + \frac{2}{3}q^{-3} + \frac{4}{3}q^{-1} + \frac{2}{3}q + \frac{2}{3}q^{3} + \frac{1}{3}q^{5} + \frac{5}{3}q^{7} + 2q^{9} + \frac{7}{3}q^{11} + \frac{1}{3}q^{13} + \frac{5}{3}q^{15} + O(q^{17})$. $\square$

Using $f_{o,7}(z)$ with $\varphi_6(z)$ and the forms in $S_2(52)$, we are able to create all of the forms with any negative leading power in $S_2^\sharp(52)$.

Looking back at $S_2(52)$, we label the cusp forms as:

$$\begin{aligned}
g_{2,-5}(z) =& q^5 - \frac{1}{3}q^7 - \frac{2}{3}q^9 - \frac{4}{3}q^{11} - \frac{1}{3}q^{13} + \frac{1}{3}q^{15} + O(q^{17}) \\
g_{2,-4}(z) =& q^4 - 2q^6 + q^{10} - q^{12} + q^{14} + q^{16} + O(q^{17}) \\
g_{2,-3}(z) =& q^3 - \frac{2}{3}q^7 - \frac{7}{3}q^9 + \frac{4}{3}q^{11} + \frac{1}{3}q^{13} - \frac{4}{3}q^{15} + O(q^{17}) \\
g_{2,-2}(z) =& q^2 - q^6 + q^8 - 2q^{10} - 2q^{12} + O(q^{17}), \\
g_{2,-1}(z) =& q - \frac{4}{3}q^7 - \frac{5}{3}q^9 + \frac{2}{3}q^{11} - \frac{1}{3}q^{13} - \frac{2}{3}q^{15} + O(q^{17}).
\end{aligned}$$

As discussed we will multiply these terms by φ_6 and $\psi(z)$ to obtain leading terms with any arbitrary negative exponent. To get a basis element we then subtract off a linear combination of the previous terms to cancel out as many terms as possible. As a reminder, by Theorem 2.7 there is no form $g_{2,0}(z)$. Continuing with the basis elements, we have:

$$
\begin{aligned}
g_{2,1}(z) =& g_{2,-5}(z)\varphi_6(z) - \frac{5}{3}g_{2,-1}(z) + \frac{1}{3}g_{2,-3}(z) + 3g_{2,-5}(z) && = q^{-1} + O(q^6)\\
g_{2,2}(z) =& g_{2,-4}(z)\varphi_6(z) + 3g_{2,-2}(z) + g_{2,-4}(z) && = q^{-2} + O(q^6)\\
g_{2,3}(z) =& g_{2,-3}(z)\varphi_6(z) - 2g_{2,1}(z) - \frac{1}{3}g_{2,-1}(z) + \frac{11}{3}g_{2,-3}(z) + 2g_{2,-5}(z) && = q^{-3} + O(q^6)\\
g_{2,4}(z) =& g_{2,-2}(z)\varphi_6(z) - 2g_{2,2}(z) + g_{2,-2}(z) - g_{2,-4}(z) && = q^{-4} + O(q^6)\\
g_{2,5}(z) =& g_{2,-1}(z)\varphi_6(z) - 2g_{2,3}(z) - g_{2,1}(z) + \frac{4}{3}g_{2,-1}(z) + \frac{7}{3}g_{2,-3}(z) + 4g_{2,-5}(z) && = q^{-5} + O(q^6)\\
g_{2,6}(z) =& g_{2,-1}(z)\psi(z) - \frac{1}{3}g_{2,4}(z) - \frac{2}{3}g_{2,2}(z) + \frac{13}{9}g_{2,-2}(z) + \frac{1}{9}g_{2,-4}(z) && = q^{-6} + O(q^6)\\
\vdots & && \vdots
\end{aligned}
$$

Hence, $S_2^\sharp(52)$ has a canonical basis consisting of the functions

$$g_{2,m}(z) = q^{-m} + a_2(m,0) + \sum_{n=6}^{\infty} a_2(m,n)q^n,$$

which are defined for $m \geq -5,\ m \neq 0$. Note that Theorem 3.2 shows $a_2(m,0) = 0$ for all m.

Similarly we start with the eight forms in $M_2(52)$ and perform the same process to get a canonical basis for $M_2^\sharp(52)$ consisting of the functions

$$f_{2,m}(z) = q^{-m} + \sum_{n=9}^{12} a_2(m,n)q^n + \sum_{n=14}^{\infty} a_2(m,n)q^n,$$

which are defined for $m \geq -13,\ m \neq -9, -10, -11, -12$.

2.4 Challenges in increasing and decreasing the weight

In constructing bases in other weights we will use $h(z)$ (see equation 2.1), the form of weight 12 with all of its zeros at infinity. Multiplying by $h(z)$ increases weight by 12 and dividing by $h(z)$ decreases weight by 12. Multiplication by $h(z)^\ell$ for $\ell \in \mathbb{Z}$ introduces a possible pole only at infinity. Thus, to construct the general basis it is sufficient to know the structure of

$M_k(N)$ and $S_k(N)$ for $k = 2, 4, 6, 8, 10,$ and 12. These weights serve as a residue system for any weight modulo 12. Below is a table listing the exponents of the leading terms for all possible forms in the respective spaces.

Space	$M_2(52)$	$M_4(52)$	$M_6(52)$	$M_8(52)$	$M_{10}(52)$	$M_{12}(52)$
Exponent	0–8, 13	0–22, 26	0–36, 39	0–50, 52	0–65	0–78, 84

Space	$S_2(52)$	$S_4(52)$	$S_6(52)$	$S_8(52)$	$S_{10}(52)$	$S_{12}(52)$
Exponent	1–5	1–18	1–31, 33	1–44, 46, 47	1–57, 59–61	1–70, 72–75

Note that we can take different forms in $M_k(52)$ and multiply by powers of $\varphi_6(z)$ or $\psi(z)$ to get any negative leading exponent in $M_k^\sharp(52)$. With the information in this table we are able to construct a canonical basis for $M_k^\sharp(52)$ or $S_k^\sharp(52)$ for any weight.

2.5 Canonical bases: $M_k^\sharp(52)$ cases

In the preceding table we took the first 6 spaces for $M_k(52)$ and $S_k(52)$ and found what leading terms were in this space. By use of $\varphi_6(z)$ and $\psi(z)$ we can create a basis for $M_k^\sharp(52)$ and $S_k^\sharp(52)$ for $k = 2, 4, 6, 8, 10,$ and 12. To list the canonical basis for $M_k^\sharp(52)$ for any given weight k we apply the following steps. Take the basis $M_\ell^\sharp(52)$ where $\ell \equiv k \mod 12$. Multiply or divide the basis elements by a power of $h(z)$ (see equation 2.1) which increases or decreases the weight by a multiple of 12 to get back to a form of weight k. Row reduction will change the bases previously found into canonical bases. Noting the sets of numbers in the table and translating them by multiples of 84 will give the structure for the canonical bases.

Theorem 2.10. *For a general weight $k \in 2\mathbb{Z}$ we have the following cases for the canonical basis of $M_k^\sharp(52)$.*

Case 1: $k \equiv 0 \mod 12$**.**

Let $k = 12l$ and let $y = 7k$. Then $M_k^\sharp(52)$ has a basis of functions

$$f_{k,m}(z) = q^{-m} + \sum_{n=y-5}^{y-1} a_k(m,n)q^n + \sum_{n=y+1}^{\infty} a_k(m,n)q^n,$$

which are defined for all $m \geq -y$, with $f_{k,m}(z) = 0$ if $m = 5-y, \ldots, 1-y$.

Case 2: $k \equiv 2 \mod 12$**.**

Let $k = 12l+2$ and let $y = 7k-1$. Then $M_k^\sharp(52)$ has a basis of functions

$$f_{k,m}(z) = q^{-m} + \sum_{n=y-4}^{y-1} a_k(m,n)q^n + \sum_{n=y+1}^{\infty} a_k(m,n)q^n,$$

which are defined for all $m \geq -y$, with $f_{k,m}(z) = 0$ if $m = 4-y, \ldots, 1-y$.

Case 3: $k \equiv 4 \mod 12$**.**

Let $k = 12l+4$ and let $y = 7k-2$. Then $M_k^\sharp(52)$ has a basis of functions

$$f_{k,m}(z) = q^{-m} + \sum_{n=y-3}^{y-1} a_k(m,n)q^n + \sum_{n=y+1}^{\infty} a_k(m,n)q^n,$$

which are defined for all $m \geq -y$, with $f_{k,m}(z) = 0$ if $m = 3-y, 2-y$, or $1-y$.

Case 4: $k \equiv 6 \mod 12$**.**

Let $k = 12l+6$ and let $y = 7k-3$. Then $M_k^\sharp(52)$ has a basis of functions

$$f_{k,m}(z) = q^{-m} + \sum_{n=y-2}^{y-1} a_k(m,n)q^n + \sum_{n=y+1}^{\infty} a_k(m,n)q^n,$$

which are defined for all $m \geq -y$, with $f_{k,m}(z) = 0$ if $m = 2-y$ or $1-y$.

Case 5: $k \equiv 8 \mod 12$***.***

Let $k = 12l + 8$ and let $y = 7k - 4$. Then $M_k^\sharp(52)$ has a basis of functions

$$f_{k,m}(z) = q^{-m} + a_k(m, y-1)q^{y-1} + \sum_{n=y+1}^{\infty} a_k(m,n)q^n,$$

which are defined for all $m \geq -y$, with $f_{k,m}(z) = 0$ if $m = 1 - y$.

Case 6: $k \equiv 10 \mod 12$***.***

Let $k = 12l + 10$ and let $y = 7k - 4$. Then $M_k^\sharp(52)$ has a basis of functions

$$f_{k,m}(z) = q^{-m} + \sum_{n=y+1}^{\infty} a_k(m,n)q^n,$$

which are defined for all $m \geq -y$.

2.6 Canonical bases: $S_k^\sharp(52)$ cases

The same process of multiplying or dividing by $h(z)$ and row reducing applies to the space $S_k^\sharp(52)$. This gives the following theorem.

Theorem 2.11. *For a general weight $k \in 2\mathbb{Z}$ we have the following cases for the canonical basis of $S_k^\sharp(52)$.*

Case 1: $k \equiv 0 \mod 12$***.***

Let $k = 12l$ and let $y = 7k - 9$. Then $S_k^\sharp(52)$ has a basis of functions

$$g_{k,m}(z) = q^{-m} + b_k(m, y-4)q^{y-4} + \sum_{n=y+1}^{\infty} b_k(m,n)q^n,$$

which are defined if $m \geq -y$, with $g_{k,m}(z) = 0$ if $m = 4 - y$.

Case 2: $k \equiv 2 \mod 12$.

Let $k = 12l + 2$ and let $y = 7k - 9$. Then $S_k^\sharp(52)$ has a basis of functions

$$g_{k,m}(z) = q^{-m} + b_k(m, y-5)q^{y-5} + \sum_{n=y+1}^{\infty} b_k(m,n)q^n,$$

which are defined for all $m \geq -y$, with $g_{k,m}(z) = 0$ if $m = 5 - y$.

Case 3: $k \equiv 4 \mod 12$.

Let $k = 12l + 4$ and let $y = 7k - 10$. Then $S_k^\sharp(52)$ has a basis of functions

$$g_{k,m}(z) = q^{-m} + \sum_{n=y+1}^{\infty} b_k(m,n)q^n,$$

which are defined for all $m \geq -y$.

Case 4: $k \equiv 6 \mod 12$.

Let $k = 12l + 6$ and let $y = 7k - 9$. Then $S_k^\sharp(52)$ has a basis of functions

$$g_{k,m}(z) = q^{-m} + b_k(m, y-1)q^{y-1} + \sum_{n=y+1}^{\infty} b_k(m,n)q^n,$$

which are defined for all $m \geq -y$, with $g_{k,m}(z) = 0$ if $m = 1 - y$.

Case 5: $k \equiv 8 \mod 12$.

Let $k = 12l + 8$ and let $y = 7k - 9$. Then $S_k^\sharp(52)$ has a basis of functions

$$g_{k,m}(z) = q^{-m} + b_k(m, y-2)q^{y-2} + \sum_{n=y+1}^{\infty} b_k(m,n)q^n,$$

which are defined for all $m \geq -y$, with $g_{k,m}(z) = 0$ if $m = 2 - y$.

Case 6: $k \equiv 10 \mod 12$.

Let $k = 12l + 10$ *and let* $y = 7k - 9$. *Then* $S_k^\sharp(52)$ *has a basis of functions*

$$g_{k,m}(z) = q^{-m} + b_k(m, y-3)q^{y-3} + \sum_{n=y+1}^{\infty} b_k(m,n)q^n,$$

which are defined for all $m \geq -y$ *with* $g_{k,m}(z) = 0$ *if* $m = 3 - y$.

Zagier duality expresses a relationship between the coefficients of $f_{k,m}(z) \in M_k^\sharp(52)$ and of those in $g_{2-k,n}(z) \in S_{2-k}^\sharp(52)$. This relationship was first proved for certain half-integral weight modular forms in [12]. This result hinges on the fact that forms in $S_2^\sharp(52)$ have no constant terms. To show this we will note a few key facts.

3.1 The theta and slash operators commute

Recall that $\theta := q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{dz}$. Also, if $f \in M_0^!(N)$, then $\theta(f) \in M_2^!(N)$. Recall for $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$ that $f(z)|[\gamma]_k = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$. The first theorem of this section relates how the $|[\gamma]_k$ and the θ operator commute for weight 0, so that when we take the derivative of forms from $M_0^\sharp(52)$, we know what happens at the cusps.

Theorem 3.1. *Let $f(z) \in M_0(N)$. Let $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$. Then, we have*

$$\theta(f(z)|[\gamma]_0) = (\theta[f(z)])|[\gamma]_2.$$

Proof. Let $f(z)$ be a weakly modular function of weight 0. Let $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$. Note that since $\gamma \in \Gamma$, then $ad - bc = 1$. Then

$$\begin{aligned}\theta[(f|[\gamma]_0)(z)] &= \theta\left[(cz+d)^0 f\left(\frac{az+b}{cz+d}\right)\right] = \frac{1}{2\pi i} f'\left(\frac{az+b}{cz+d}\right)\left(\frac{acz+ad-caz-bc}{(cz+d)^2}\right) \\ &= \frac{1}{2\pi i}(cz+d)^{-2} f'\left(\frac{az+b}{cz+d}\right)(ad-bc) = \frac{1}{2\pi i}(cz+d)^{-2} f'\left(\frac{az+b}{cz+d}\right).\end{aligned}$$

Also note that θ increases the weight by 2. Hence,

$$(\theta[f(z)]|[\gamma]_2) = \left[\frac{1}{2\pi i} f'(z)\right]\Big|[\gamma]_2 = \frac{1}{2\pi i}(cz+d)^{-2} f'\left(\frac{az+b}{cz+d}\right).$$

$\square$

Now we can track the order of vanishing at the cusps from the derivatives. To do this we first look at $(\theta[f(z)])|[\gamma]_2 = \theta(f(z)|[\gamma]_0)$, where γ moves the cusp in question to where we want it. Then, applying θ will annihilate any constant term. Note that if $f(z)|\gamma$ is holomorphic at infinity, then $\theta(f(z)|\gamma) = (\theta f(z))|\gamma$ vanishes at infinity. Thus, if $f(z) \in M_0^\sharp(52)$, then $\theta f(z) \in S_2^\sharp(52)$.

3.2 Forms in $S_2^\sharp(52)$ have no constant terms

From SAGE (A.2) we can create the first few terms of our basis:

$$\begin{aligned} S_2^\sharp(52) = \Big\{ & q^5 - \frac{1}{3}q^7 + O(q^9), && q^4 - 2q^6 + O(q^9), \\ & q^3 - \frac{2}{3}q^7 + O(q^9), && q^2 - q^6 + q^8 + O(q^9), \\ & q - \frac{4}{3}q^7 + O(q^9), && q^{-1} - \frac{2}{3}q^7 + O(q^9), \\ & q^{-2} - 2q^6 - q^8 + O(q^9) && q^{-3} - \frac{2}{3}q^7 + O(q^9), \\ & q^{-4} - 2q^8 + O(q^9), && q^{-5} - \frac{1}{3}q^7 + O(q^9), \\ & q^{-6} - 3q^6 + q^8 + O(q^9), && q^{-7} - \frac{5}{3}q^7 + O(q^9), \dots \Big\}. \end{aligned}$$

There are no constant terms, which is a key point to proving duality.

Theorem 3.2. $S_2^\sharp(52)$ *contains no forms with a constant term.*

Proof. Take $f_{k,m}(z) \in M_k^\sharp(52)$ and $g_{2-k,n}(z) \in S_{2-k}^\sharp(52)$. Let $F(z) = f_{k,m}g_{2-k,n}$. Note that $F(z) \in S_2^\sharp(52)$, and the leading power of $F(z)$ is $-m-n$. Considering the basis elements of $S_2^\sharp(52)$, we have leading terms q^5, q^4, q^2, q^1, q^{-1}, and every other negative power (by Theorem 2.7 there is no form starting with a constant term). If $-m-n \geq -5$ we can verify with SAGE that there are no constant terms in the first 10 basis elements of $S_2^\sharp(52)$, and we know that $F(z)$ is a linear combination of these first 10 basis elements. Hence $F(z)$ has no constant term.

Now if $-m-n \leq -6$, we know that the functions $\theta(f_{0,\ell})$ for $\ell \leq -6$ have no constant terms and are in $S_2^\sharp(52)$ by Theorem 3.1. Thus, we can list $\{\theta(f_{0,\ell})\} \cap \{g_{2,m}(z) : -5 \geq m \geq -1\}$ as another basis for $S_2^\sharp(52)$. Therefore, we may form a linear combination of the $\theta(f_{2,\ell})$ along with the first few forms of $S_{2-k}^\sharp(52)$ to create $F(z)$. Thus no form in $S_2^\sharp(52)$ has a constant term. $\square$

3.3 Proof of duality

With the previous results we can give the following theorem.

Theorem 3.3. *Let the weight $k \in 2\mathbb{Z}$ be given. Let $f_{k,m}(z) \in M_k^\sharp(52)$ and $g_{2-k,n}(z) \in S_k^\sharp(52)$. Then we have*

$$a_k(m,n) = -b_{2-k}(n,m).$$

Proof. Let $f_{k,m}(z) \in M_k^\sharp(52)$ and $g_{2-k,n}(z) \in S_k^\sharp(52)$.

Case 1: $k \equiv 0 \mod 12$.

Let $k = 12l$, and $y = 7k = 84l$. For $S_{2-k}^\sharp$ we have $2-k = -12l+2$ and $84(l+1)+90) = 6-y$.

Then our basis elements are in the form

$f_{k,m}(z) = q^{-m} + \sum_{r=y-5}^{y-1} a_k(m,r)q^r + \sum_{r=y+1}^{\infty} a_k(m,r)q^r$, which is defined if $m \geq -y$ and is zero if $y-5 < m < y-1$, and

$g_{2-k,n}(z) = q^{-n} + b_{2-k}(n,-y)q^{-y} + \sum_{r=6-y}^{\infty} b_{2-k}(n,r)q^r$, which is defined if $n \geq 5-y$ and is zero if $n = y$.

Then $(f_{k,m}(z))(g_{2-k,n}(z)) =$

$$\left(q^{-m} + \sum_{r=y-5}^{y-1} a_k(m,r)q^r + \sum_{r=y+1}^{\infty} a_k(m,r)q^n\right)\left(q^{-n} + b_{2-k}(n,-y)q^{-y} + \sum_{r=6-y}^{\infty} b_{2-k}(n,r)q^r\right)$$

has constant term $a_k(m,n) + b_{2-k}(n,m)$. Since the indices match, there are no other terms which contribute to the constant term. By Theorem 3.2, $a_k(m,n) = -b_{2-k}(n,m)$.

The other cases are similar. See Theorem 2.10 and Theorem 2.11 for the structure of the basis elements in the omitted cases. $\square$

Chapter 4. Generating Functions

Recall the definition of the generating function.

Definition 4.1. Let $f_{k,n}(z)$ be basis elements for $M_k^\sharp(52)$ as found in Theorem 2.10. Then the generating function $F_k(z,\tau)$ is given by

$$F_k(z,\tau) = \sum_{n=-n_0}^{\infty} f_{k,n}(\tau)q^n$$

where n_0 is the greatest order of vanishing at infinity of any form in $M_k^\sharp(52)$.

By taking the function $\varphi_6(z) = q^{-6} + 2q^{-4} + q^{-2} + 2q^2 + O(q^6)$ (equation 2.2), we can multiply this weight 0 form by any basis element $f_{k,m}(z)$ and with the tool of duality in Theorem 3.3, we will create a recurrence relation that will be used to get a closed form of the generating function for our basis elements. Here we give an exposition on a generating function that behaves akin to those in lower levels.

Theorem 4.2. *Let $k = 12l + 10$ and let $y = 84l + 66$. Let $F_k(z,\tau) = \sum_{n=-y}^{\infty} f_{k,n}(\tau)q^n$. Then*

$$\begin{aligned}
&F_k(z,\tau) = \\
&(\varphi_6(z) - \varphi_6(\tau))^{-1} \cdot \Big(f_{k,5-y}(\tau)g_{2-k,y+1}(z) + f_{k,4-y}(\tau)g_{2-k,y+2}(z) + f_{k,3-y}(\tau)g_{2-k,y+3}(z) \\
&+ 2f_{k,3-y}(\tau)g_{2-k,y+1}(z) + f_{k,2-y}(\tau)g_{2-k,y+4}(z) + 2f_{k,2-y}(\tau)g_{2-k,y+2}(z) \\
&+ f_{k,1-y}(\tau)g_{2-k,y+5}(z) + 2f_{k,1-y}(\tau)g_{2-k,y+3}(z) + f_{k,1-y}(\tau)g_{2-k,y+1}(z) \\
&+ f_{k,-y}(\tau)g_{2-k,y+6}(z) + 2f_{k,-y}(\tau)g_{2-k,y+4}(z) + f_{k,-y}(\tau)g_{2-k,y+2}(z)\Big).
\end{aligned}$$

Proof. Let $k = 12l + 10$ and let $y = 84\ell + 66$. For the $2 - k$ weight, $2 - k = -12\ell - 8$ and $19 + 84(-\ell - 1) = -y$. Then

$f_{k,m}(z) = q^{-m} + \sum_{n=y+1}^{\infty} a_k(m,n)q^n$ is defined for all $m \geq -y$ and

$g_{2-k,m}(z) = q^{-m} + \sum_{n=-y}^{\infty} b_{2-k}(m,n)q^n$ is defined for all $m \geq y+1$.

To obtain the recurrence relation multiply $\varphi_6(z)$ by a general basis element $f_{k,m}(z)$.

$$\begin{aligned}\varphi_6(z)f_{k,m}(z) =& \left(q^{-6} + 2q^{-4} + q^{-2} + \sum_{r\geq 2} c_r q^r\right)\left(q^{-m} + \sum_{n=y+1}^{\infty} a_k(m,n)q^n\right) \\ =& q^{-6-m} + 2q^{-4-m} + q^{-2-m} + \sum_{r\geq 2} c_r q^{r-m} + \sum_{n=y+1}^{\infty} a_k(m,n)q^{n-6} \\ &+ 2\sum_{n=y+1}^{\infty} a_k(m,n)q^{n-4} + \sum_{n=y+1}^{\infty} a_k(m,n)q^{n-2} + \sum_{n=y+1}^{\infty} a_k(m,n)q^n \sum_{r=2}^{\infty} c_r q^r.\end{aligned}$$

Note that $\varphi_6(z)f_{k,m}(z) \in M_k^\sharp(52)$, and has leading term q^{-m-6}. Hence, we can write the product $\varphi_6(z)f_{k,m}(z)$ as the basis element $f_{k,m+6}$ and a linear combination of a few other basis elements. Using the canonical basis, q^{-n} is identified with $f_{k,n}(z)$. We then list these basis elements in a linear combination. For convenience we will omit the (z) part of the function notation.

$$\begin{aligned}f_{k,m+6} =& \varphi_6 f_{k,m} - 2f_{k,m+4} - f_{k,m+2} - \sum_{r=2}^{m+y} c_r f_{k,m-r} - \sum_{n=y+1}^{y+6} a_k(m,n) f_{k,6-n} \\ &- \sum_{n=y+1}^{y+4} 2a_k(m,n) f_{k,4-n} - \sum_{n=y+1}^{y+2} a_k(m,n) f_{k,2-n}.\end{aligned}$$

Further grouping of the basis elements gives

$$
\begin{aligned}
f_{k,m+6} = &\varphi_6 f_{k,m} - 2f_{k,m+4} - f_{k,m+2} \\
&- (a_k(m,y+1))f_{k,5-y} - (a_k(m,y+2))f_{k,4-y} \\
&- (a_k(m,y+3) + 2a_k(m,y+1))f_{k,3-y} \\
&- (a_k(m,y+4) + 2a_k(m,y+2))f_{k,2-y} \\
&- (a_k(m,y+5) + 2a_k(m,y+3) + a_k(m,y+1))f_{k,1-y} \\
&- (a_k(m,y+6) + 2a_k(m,y+4) + a_k(m,y+2))f_{k,-y} \\
&- \sum_{r=2}^{m+y} c_r f_{k,m-r}.
\end{aligned}
$$

Using the result of duality, Theorem 3.3, we have the following recurrence relation:

$$
\begin{aligned}
f_{k,m+6} = &\varphi_6 f_{k,m} - 2f_{k,m+4} - f_{k,m+2} \\
&+ b_{2-k}(y+1,m)f_{k,5-y} + b_{2-k}(y+2,m)f_{k,4-y} \\
&+ (b_{2-k}(y+3,m) + 2b_{2-k}(y+1,m))f_{k,3-y} \\
&+ (b_{2-k}(y+4,m) + 2b_{2-k}(y+2,m))f_{k,2-y} \\
&+ (b_{2-k}(y+5,m) + 2b_{2-k}(y+3,m) + b_{2-k}(y+1,m))f_{k,1-y} \\
&+ (b_{2-k}(y+6,m) + 2b_{2-k}(y+4,m) + b_{2-k}(y+2,m))f_{k,-y} \\
&- \sum_{r=2}^{m+y} c_r f_{k,m-r}.
\end{aligned}
$$

Now, we compute the generating function.

$$
\begin{aligned}
F_k(z,\tau) = &F_k = \sum_{n=-y}^{\infty} f_{k,n}(\tau)q^n = \sum_{n=-y-6}^{\infty} f_{k,n+6}(\tau)q^{n+6} \\
= &f_{k,-y}(\tau)q^{-y} + f_{k,1-y}(\tau)q^{1-y} + f_{k,2-y}(\tau)q^{2-y} + f_{k,3-y}(\tau)q^{3-y} + f_{k,4-y}(\tau)q^{4-y} \\
&+ f_{k,5-y}(\tau)q^{5-y} + \sum_{n=-y}^{\infty} f_{k,n+6}(\tau)q^{n+6},
\end{aligned}
$$

where we separated the first terms to get to a point where we could use the recurrence relation. By substituting in the recurrence relation and simplifying we have

$$
\begin{aligned}
F_k =& f_{k,-y}(\tau)q^{-y} + f_{k,1-y}(\tau)q^{1-y} + f_{k,2-y}(\tau)q^{2-y} + f_{k,3-y}(\tau)q^{3-y} \\
&+ f_{k,4-y}(\tau)q^{4-y} + f_{k,5-y}(\tau)q^{5-y} \\
&+ \sum_{n=-y}^{\infty} \Bigg[(\varphi_6)(\tau)f_{k,n}(\tau) - 2f_{k,n+4}(\tau) - f_{k,n+2}(\tau) \\
&+ (b_{2-k}(y+1,n))f_{k,5-y}(\tau) + (b_{2-k}(y+2,n))f_{k,4-y}(\tau) \\
&+ (b_{2-k}(y+3,n) + 2b_{2-k}(y+1,n))f_{k,3-y}(\tau) \\
&+ (b_{2-k}(y+4,n) + 2b_{2-k}(y+2,n))f_{k,2-y}(\tau) \\
&+ (b_{2-k}(y+5,n) + 2b_{2-k}(y+3,n) + b_{2-k}(y+1,n))f_{k,1-y}(\tau) \\
&+ (b_{2-k}(y+6,n) + 2b_{2-k}(y+4,n) + b_{2-k}(y+2,n))f_{k,-y}(\tau) \\
&- \sum_{i=2}^{n+y} c_i f_{k,n-i}(\tau)\Bigg] q^{n+6}.
\end{aligned}
$$

Summing over each term gives

$$
\begin{aligned}
F_k =& f_{k,-y}(\tau)q^{-y} + f_{k,1-y}(\tau)q^{1-y} + f_{k,2-y}(\tau)q^{2-y} + f_{k,3-y}(\tau)q^{3-y} \\
&+ f_{k,4-y}(\tau)q^{4-y} + f_{k,5-y}(\tau)q^{5-y} \\
&+ (\varphi_6)(\tau)q^6 \sum_{n=-y}^{\infty} f_{k,n}(\tau)q^n - 2q^6 \sum_{n=-y}^{\infty} f_{k,n+4}(\tau)q^n - q^6 \sum_{n=-y}^{\infty} f_{k,n+2}(\tau)q^n \\
&+ q^6 f_{k,5-y}(\tau) \sum_{n=_-y}^{\infty} b_{2-k}(y+1,n)q^n + q^6 f_{k,4-y}(\tau) \sum_{n=-y}^{\infty} b_{2-k}(y+2,n)q^n \\
&+ q^6 f_{k,3-y}(\tau) \sum_{n=-y}^{\infty} (b_{2-k}(y+3,n) + 2b_{2-k}(y+1,n))q^n \\
&+ q^6 f_{k,2-y}(\tau) \sum_{n=-y}^{\infty} (b_{2-k}(y+4,n) + 2b_{2-k}(y+2,n))q^n \\
&+ q^6 f_{k,1-y}(\tau) \sum_{n=-y}^{\infty} (b_{2-k}(y+5,n) + 2b_{2-k}(y+3,n) + b_{2-k}(y+1,n))q^n \\
&+ q^6 f_{k,-y}(\tau) \sum_{n=-y}^{\infty} (b_{2-k}(y+6,n) + 2b_{2-k}(y+4,n) + b_{2-k}(y+2,n))q^n \\
&- q^6 \sum_{n=-y}^{\infty} \sum_{i=2}^{n+y} c_i f_{k,n-i}(\tau)q^n.
\end{aligned}
$$

Now we identify the basis elements of $S^{\sharp}_{2-k}(52)$ and replace the sums above. We will also identify the generating function and replace it where necessary.

$$
\begin{aligned}
F_k =& f_{k,-y}(\tau)q^{-y} + f_{k,1-y}(\tau)q^{1-y} + f_{k,2-y}(\tau)q^{2-y} + f_{k,3-y}(\tau)q^{3-y} \\
&+ f_{k,4-y}(\tau)q^{4-y} + f_{k,5-y}(\tau)q^{5-y} \\
&+ \varphi_6(\tau)q^6 F_k - 2q^2\Bigg(F_k - \sum_{n=-y}^{4-y} f_{k,n}(\tau)q^n\Bigg) - q^4\Bigg(F_k - \sum_{n=-y}^{2-y} f_{k,n}(\tau)q^n\Bigg) \\
&+ q^6 f_{k,5-y}(\tau)(g_{2-k,y+1}(z) - q^{-y-1}) + q^6 f_{k,4-y}(\tau)(g_{2-k,y+2}(z) - q^{-y-2}) \\
&+ q^6 f_{k,3-y}(\tau)(g_{2-k,y+3}(z) - q^{-y-3}) + 2q^6 f_{k,3-y}(\tau)(g_{2-k,y+1}(z) - q^{-y-1}) \\
&+ q^6 f_{k,2-y}(\tau)(g_{2-k,y+4}(z) - q^{-y-4}) + 2q^6 f_{k,2-y}(\tau)(g_{2-k,y+2}(z) - q^{-y-2}) \\
&+ q^6 f_{k,1-y}(\tau)(g_{2-k,y+5}(z) - q^{-y-5}) + 2q^6 f_{k,1-y}(\tau)(g_{2-k,y+3}(z) - q^{-y-3}) \\
&+ q^6 f_{k,1-y}(\tau)(g_{2-k,y+1}(z) - q^{-y-1}) \\
&+ q^6 f_{k,-y}(\tau)(g_{2-k,y+6}(z) - q^{-y-6}) + 2q^6 f_{k,-y}(\tau)(g_{2-k,y+4}(z) - q^{-y-4}) \\
&+ q^6 f_{k,-y}(\tau)(g_{2-k,y+2}(z) - q^{-y-2}) \\
&- q^6 \sum_{n=-y}^{\infty} \sum_{i=2}^{n+y} c_i f_{k,n-i}(\tau) q^n.
\end{aligned}
$$

Note that $-q^6\Bigg(\displaystyle\sum_{n=-y}^{\infty} f_{k,n}(\tau)q^n\Bigg)\Bigg(\sum_{i=0}^{\infty} c_i q^i\Bigg) = -q^6\Bigg(F_k\Bigg)\Bigg(\varphi_6(z) - q^{-6} - 2q^{-4} - q^{-2}\Bigg)$. Substituting this after canceling a few items, we have the following.

$$
\begin{aligned}
F_k =& \varphi_6(\tau)q^6 F_k - 2q^2 F_k - q^4 F_k \\
&+ q^6 f_{k,5-y}(\tau) g_{2-k,y+1}(z) + q^6 f_{k,4-y}(\tau) g_{2-k,y+2}(z) \\
&+ q^6 f_{k,3-y}(\tau) g_{2-k,y+3}(z) + 2q^6 f_{k,3-y}(\tau) g_{2-k,y+1}(z) \\
&+ q^6 f_{k,2-y}(\tau) g_{2-k,y+4}(z) + 2q^6 f_{k,2-y}(\tau) g_{2-k,y+2}(z) \\
&+ q^6 f_{k,1-y}(\tau) g_{2-k,y+5}(z) + 2q^6 f_{k,1-y}(\tau) g_{2-k,y+3}(z) + q^6 f_{k,1-y}(\tau) g_{2-k,y+1}(z) \\
&+ q^6 f_{k,-y}(\tau) g_{2-k,y+6}(z) + 2q^6 f_{k,-y}(\tau) g_{2-k,y+4}(z) + q^6 f_{k,-y}(\tau) g_{2-k,y+2}(z) \\
&+ F_k - q^6 \varphi_6(z) F_k + 2q^2 F_k + q^4 F_k.
\end{aligned}
$$

Regrouping terms and factoring yields:

$$\begin{aligned}(\varphi_6(z)-\varphi_6(\tau))F_k =& f_{k,5-y}(\tau)g_{2-k,y+1}(z)+f_{k,4-y}(\tau)g_{2-k,y+2}(z)\\ &+f_{k,3-y}(\tau)g_{2-k,y+3}(z)+2f_{k,3-y}(\tau)g_{2-k,y+1}(z)\\ &+f_{k,2-y}(\tau)g_{2-k,y+4}(z)+2f_{k,2-y}(\tau)g_{2-k,y+2}(z)\\ &+f_{k,1-y}(\tau)g_{2-k,y+5}(z)+2f_{k,1-y}(\tau)g_{2-k,y+3}(z)+f_{k,1-y}(\tau)g_{2-k,y+1}(z)\\ &+f_{k,-y}(\tau)g_{2-k,y+6}(z)+2f_{k,-y}(\tau)g_{2-k,y+4}(z)+f_{k,-y}(\tau)g_{2-k,y+2}(z).\end{aligned}$$

Dividing by $(\varphi_6(z)-\varphi_6(\tau))$ gives the desired result. □

4.1 Challenges in other weights

The need for $h(z)=q^{84}+O(q^{86})$ (see equation 2.1) to move up and down weight has split most of our work into a residue system modulo 12. Likewise, the work on the generating functions is also split into cases for the respective weight modulo 12. Note that the recurrence relation in each case is similar, but the indexing on the bases elements $f_{k,i-y}$ is not consistent. This is due to how these basis elements are defined. Thus, the recurrence relation can change subtly. This same indexing of the basis elements causes the infinite sum in the definition of the generating function to be rewritten to match the indices where the basis elements are defined.

We will give the proof in one more case to give the reader a chance to view the recurrence relation being substituted back in to cancel terms, and the difference in the basis elements.

Theorem 4.3. *Let* $k = 2+12l$ *and* $y = 7k-1$. *Let* $F_k(z,\tau) = f_{k,-y}(\tau)q^{-y} + \sum_{n=5-y}^{\infty} f_{k,n}(\tau)q^n$. *Then*

$$\begin{aligned}F_k(z,\tau) =& \Big(\varphi_6(z) - \varphi_6(\tau)\Big)^{-1} \cdot \Big(f_{k,-y}(\tau)g_{2-k,y+6}(z) + 2f_{k,-y}(\tau)g_{2-k,y+4}(z) \\ &+ f_{k,-y}(\tau)g_{2-k,y+2}(z) + 2f_{k,5-y}(\tau)g_{2-k,y-1}(z) + f_{k,5-y}(\tau)g_{2-k,y-3}(z) \\ &+ f_{k,5-y}(\tau)g_{2-k,y+1}(z) + 2f_{k,6-y}(\tau)g_{2-k,y-2}(z) + f_{k,6-y}(\tau)g_{2-k,y-4}(z) \\ &+ f_{k,7-y}(\tau)g_{2-k,y-1}(z) + 2f_{k,7-y}(\tau)g_{2-k,y-3}(z) + f_{k,8-y}(\tau)g_{2-k,y-2}(z) \\ &+ 2f_{k,8-y}(\tau)g_{2-k,y-4}(z) + f_{k,9-y}(\tau)g_{2-k,y-3}(z) + f_{k,10-y}(\tau)g_{2-k,y-4}(z) \\ &+ 2f_{k,-y}(\tau)g_{2-k,y-2}(z)\Big).\end{aligned}$$

Proof. Let $k = 2+12l$ and $y = 7k-1$. For the $2-k$ weight, $2-k = -12\ell$ and $76+84(-\ell-1) = 5-y$. Then

$f_{k,m}(z) = q^{-m} + \sum_{n=y-4}^{y-1} a_k(m,n)q^n + \sum_{n=y+1}^{\infty} a_k(m,n)q^n$ is defined for all $m \geq 1-y$ and is 0 if $m = 4-y, \ldots, 1-y$, and

$g_{2-k,m}(z) = q^{-m} + b_{2-k}(m,-y)q^{-y} + \sum_{n=5-y}^{\infty} b_{2-k}(m,n)q^n$ is defined for all $m \geq y-4$ and is 0 if $m = y$.

Noting $\varphi_6(z) = q^{-6} + 2q^{-4} + q^{-2} + 2q^2 + 3q^6 + 2q^8 + 2q^{12} + 2q^{14} + O(q^{16})$, we have no terms of odd power for small powers of q. If $\varphi_6(z)f_{k,m}(z) = q^{-6} + 2q^{-4} + q^{-2} + \sum_{r\geq 2} c_r q^r$, then $c_4 = c_{10} = 0$.

Note that we will drop the (z) function notation for convenience.

To obtain the recurrence relation we have:

$$
\begin{aligned}
\varphi_6 f_{k,m} = & \left(q^{-6} + 2q^{-4} + q^{-2} + \sum_{r\geq 2} c_r q^r\right)\left(q^{-m} + \sum_{n=y-4}^{y-1} a_k(m,n)q^n + \sum_{n=y+1}^{\infty} a_k(m,n)q^n\right) \\
=& q^{-6-m} + \sum_{n=y-4}^{y-1} a_k(m,n)q^{n-6} + \sum_{n=y+1}^{\infty} a_k(m,n)q^{n-6} \\
& + 2q^{-4-m} + 2\sum_{n=y-4}^{y-1} a_k(m,n)q^{n-4} + 2\sum_{n=y+1}^{\infty} a_k(m,n)q^{n-4} \\
& + q^{-2-m} + \sum_{n=y-4}^{y-1} a_k(m,n)q^{n-2} + \sum_{n=y+1}^{\infty} a_k(m,n)q^{n-2} \\
& + a_k(m,y-4)\sum_{r\geq 2} c_r q^{r+y-4} + a_k(m,y-3)\sum_{r\geq 2} c_r q^{r+y-3} \\
& + a_k(m,y-2)\sum_{r\geq 2} c_r q^{r+y-2} + a_k(m,y-1)\sum_{r\geq 2} c_r q^{r+y-1} \\
& + \sum_{r=2}^{\infty} c_r q^{r-m} + \sum_{n=y+1}^{\infty} a_k(m,n)q^n \sum_{r\geq 2} c_r q^r.
\end{aligned}
$$

If $n \geq y+1$ we will remove the terms containing a q^n that are accounted for in the infinite sum in the general basis element. Also, using the fact that $c_2 = 2$, $c_r = 0$ for $r = 3, 4, 5$, and $f_{k,1-y}(z) = \cdots = f_{k,4-y}(z) = 0$, we rewrite the sums.

$$
\begin{aligned}
f_{k,m+6} =& \varphi_6 f_{k,m} - 2f_{k,m+4} - f_{k,m+2} \\
& - \sum_{n=y-4}^{y-1} a_k(m,n) f_{k,6-n} - a_k(m,y+1) f_{k,5-y} - a_k(m,y+6) f_{k,-y} \\
& - 2\sum_{n=y-4}^{y-1} a_k(m,n) f_{k,4-n} - 2a_k(m,y+4) f_{k,-y} \\
& - \sum_{n=y-4}^{y-3} a_k(m,n) f_{k,2-n} - a_k(m,y+2) f_{k,-y} \\
& - c_{m+y} f_{k,-y} - \sum_{r=2}^{y+m-5} c_r f_{k,m-r}.
\end{aligned}
$$

Further grouping gives

$$
\begin{aligned}
f_{k,m+6} =& \varphi_6 f_{k,m} - 2f_{k,m+4} - f_{k,m+2} \\
& - (a_k(m,y+6) + 2a_k(m,y+4) + a_k(m,y+2))f_{k,-y} \\
& - (2a_k(m,y-1) + a_k(m,y-3) + a_k(m,y+1))f_{k,5-y} \\
& - (2a_k(m,y-2) + a_k(m,y-4))f_{k,6-y} \\
& - (a_k(m,y-1) + 2a_k(m,y-3))f_{k,7-y} \\
& - (a_k(m,y-2) + 2a_k(m,y-4))f_{k,8-y} \\
& - a_k(m,y-3)f_{k,9-y} \\
& - a_k(m,y-4)f_{k,10-y} \\
& - a_k(m,y-2)c_2 f_{k,-y} - c_{m+y} f_{k,-y} - \sum_{r=2}^{y+m-5} c_r f_{k,m-r}.
\end{aligned}
$$

Using duality, we have the following recurrence relation.

$$
\begin{aligned}
f_{k,m+6} =& \varphi_6 f_{k,m} - 2f_{k,m+4} - f_{k,m+2} \\
& + (b_{2-k}(y+6,m) + 2b_{2-k}(y+4,m) + b_{2-k}(y+2,m))f_{k,-y} \\
& + (2b_{2-k}(y-1,m) + b_{2-k}(y-3,m) + b_{2-k}(y+1,m))f_{k,5-y} \\
& + (2b_{2-k}(y-2,m) + b_{2-k}(y-4,m))f_{k,6-y} \\
& + (b_{2-k}(y-1,m) + 2b_{2-k}(y-3,m))f_{k,7-y} \\
& + (b_{2-k}(y-2,m) + 2b_{2-k}(y-4,m))f_{k,8-y} \\
& + b_{2-k}(y-3,m)f_{k,9-y} \\
& + b_{2-k}(y-4,m)f_{k,10-y} \\
& + b_{2-k}(y-2,m)c_2 f_{k,-y} - c_{m+y} f_{k,-y} - \sum_{r=2}^{y+m-5} c_r f_{k,m-r}.
\end{aligned}
$$

Now, we compute the generating function, substitute in the recurrence relation, and simplify.

$$\begin{aligned}
F_k(z,\tau) =& F_k = f_{k,-y}(\tau)q^{-y} + \sum_{n=5-y}^{\infty} f_{k,n}(\tau)q^n = f_{k,-y}(\tau)q^{-y} + \sum_{n=-1-y}^{\infty} f_{k,n+6}(\tau)q^{n+6} \\
=& f_{k,-y}(\tau)q^{-y} + f_{k,5-y}(\tau)q^{5-y} + f_{k,6-y}(\tau)q^{6-y} + f_{k,7-y}(\tau)q^{7-y} + f_{k,8-y}(\tau)q^{8-y} \\
&+ f_{k,9-y}(\tau)q^{9-y} + f_{k,10-y}(\tau)q^{10-y} + \sum_{n=5-y}^{\infty} f_{k,n+6}(\tau)q^{n+6}.
\end{aligned}$$

$$\begin{aligned}
F_k =& f_{k,-y}(\tau)q^{-y} + f_{k,5-y}(\tau)q^{5-y} + f_{k,6-y}(\tau)q^{6-y} + f_{k,7-y}(\tau)q^{7-y} + f_{k,8-y}(\tau)q^{8-y} \\
&+ f_{k,9-y}(\tau)q^{9-y} + f_{k,10-y}(\tau)q^{10-y} \\
&+ \sum_{n=5-y}^{\infty} \Bigg[\varphi_6(\tau)f_{k,n}(\tau) - 2f_{k,n+4}(\tau) - f_{k,n+2}(\tau) \\
&+ (b_{2-k}(y+6,n) + 2b_{2-k}(y+4,n) + b_{2-k}(y+2,n))f_{k,-y}(\tau) \\
&+ (2b_{2-k}(y-1,n) + b_{2-k}(y-3,n) + b_{2-k}(y+1,n))f_{k,5-y}(\tau) \\
&+ (2b_{2-k}(y-2,n) + b_{2-k}(y-4,n))f_{k,6-y}(\tau) \\
&+ (b_{2-k}(y-1,n) + 2b_{2-k}(y-3,n))f_{k,7-y}(\tau) \\
&+ (b_{2-k}(y-2,n) + 2b_{2-k}(y-4,n))f_{k,8-y}(\tau) \\
&+ b_{2-k}(y-3,n)f_{k,9-y}(\tau) \\
&+ b_{2-k}(y-4,n)f_{k,10-y}(\tau) \\
&+ c_2 b_{2-k}(y-2,n)f_{k,-y}(\tau) - c_{n+y}f_{k,-y}(\tau) - \sum_{r=2}^{y+n-5} c_r f_{k,n-r}(\tau) \Bigg] q^{n+6}.
\end{aligned}$$

Summing over each term gives

$$
\begin{aligned}
F_k =& f_{k,-y}(\tau)q^{-y} + f_{k,5-y}(\tau)q^{5-y} + f_{k,6-y}(\tau)q^{6-y} + f_{k,7-y}(\tau)q^{7-y} + f_{k,8-y}(\tau)q^{8-y} \\
&+ f_{k,9-y}(\tau)q^{9-y} + f_{k,10-y}(\tau)q^{10-y} \\
&+ \varphi_6(\tau)\sum_{n=5-y}^{\infty} f_{k,n}(\tau)q^{n+6} - 2\sum_{n=5-y}^{\infty} f_{k,n+4}(\tau)q^{n+6} - \sum_{n=5-y}^{\infty} f_{k,n+2}(\tau)q^{n+6} \\
&+ f_{k,-y}(\tau)\sum_{n=5-y}^{\infty}(b_{2-k}(y+6,n) + 2b_{2-k}(y+4,n) + b_{2-k}(y+2,n))q^{n+6} \\
&+ f_{k,5-y}(\tau)\sum_{n=5-y}^{\infty}(2b_{2-k}(y-1,n) + b_{2-k}(y-3,n) + b_{2-k}(y+1,n))q^{n+6} \\
&+ f_{k,6-y}(\tau)\sum_{n=5-y}^{\infty}(2b_{2-k}(y-2,n) + b_{2-k}(y-4,n))q^{n+6} \\
&+ f_{k,7-y}(\tau)\sum_{n=5-y}^{\infty}(b_{2-k}(y-1,n) + 2b_{2-k}(y-3,n))q^{n+6} \\
&+ f_{k,8-y}(\tau)\sum_{n=5-y}^{\infty}(b_{2-k}(y-2,n) + 2b_{2-k}(y-4,n))q^{n+6} \\
&+ f_{k,9-y}(\tau)\sum_{n=5-y}^{\infty} b_{2-k}(y-3,n)q^{n+6} + f_{k,10-y}(\tau)\sum_{n=5-y}^{\infty} b_{2-k}(y-45,n)q^{n+6} \\
&+ c_2 f_{k,-y}(\tau)\sum_{n=5-y}^{\infty} b_{2-k}(y-2,n)q^{n+6} - \sum_{n=5-y}^{\infty} c_{n+y} f_{k,-y}(\tau)q^{n+6} - \sum_{n=5-y}^{\infty}\sum_{r=2}^{y+n-5} c_r f_{k,n-r}(\tau)q^{n+6}.
\end{aligned}
$$

Note that the first 2 terms of the double sum are 0 since the $f_{k,m}$ basis elements are 0 when $m = 5 - y, \dots, 2 - y$. Changing the double sum to a product gives

$$\begin{aligned}
\sum_{n=5-y}^{\infty} \sum_{r=2}^{y+n-5} c_r f_{k,n-r}(\tau) q^{n+6} &= \sum_{n=7-y}^{\infty} \sum_{r=2}^{y+n-5} c_r f_{k,n-r}(\tau) q^{n+6} \\
&= q^6 \left(\sum_{n=-y}^{\infty} f_{k,n-r}(\tau) q^n \right) \left(\sum_{r=2}^{\infty} c_r q^r \right) \\
&= q^6 \left(F_k - f_{k,-y} q^{-y} \right) \left(\varphi_6(z) - q^{-6} - 2q^{-4} - q^{-2} \right).
\end{aligned}$$

As for the last sum, we have

$$\begin{aligned}
f_{k,-y}(\tau) \sum_{n=5-y}^{\infty} c_{n+y} q^{n+6} &= f_{k,-y}(\tau) q^6 \left(\sum_{n=5-y}^{\infty} c_{n+y} q^n \right) = f_{k,-y}(\tau) q^{6-y} \left(\sum_{n=5}^{\infty} c_n q^n \right) \\
&= f_{k,-y}(\tau) q^{6-y} \left(\varphi_6(z) - q^{-6} - 2q^{-4} - q^{-2} - c_2 q^2 \right).
\end{aligned}$$

Now we identify the basis elements of $S^\sharp_{2-k}(52)$ and pieces of the generating function, then replace them. This gives us

$$\begin{aligned}
F_k =& f_{k,-y}(\tau)q^{-y} + f_{k,5-y}(\tau)q^{5-y} + f_{k,6-y}(\tau)q^{6-y} + f_{k,7-y}(\tau)q^{7-y} + f_{k,8-y}(\tau)q^{8-y} \\
&+ f_{k,9-y}(\tau)q^{9-y} + f_{k,10-y}(\tau)q^{10-y} \\
&+ \varphi_6(\tau)\left(F_k - f_{k,-y}(\tau)q^{-y}\right)q^6 - 2q^2\left(F_k - f_{k,-y}(\tau)q^{-y} - \sum_{n=5-y}^{8-y} f_{k,n}(\tau)q^n\right) \\
&- q^4\left(F_k - f_{k,-y}(\tau)q^{-y} - \sum_{n=5-y}^{6-y} f_{k,n}(\tau)q^n\right) \\
&+ f_{k,-y}(\tau)(g_{2-k,y+6}(z) - q^{-6-y} - b_{2-k}(y+6,-y)q^{-y})q^6 \\
&+ f_{k,-y}(\tau)(2g_{2-k,y+4}(z) - 2q^{-4-y} - 2b_{2-k}(y+4,-y)q^{-y})q^6 \\
&+ f_{k,-y}(\tau)(g_{2-k,y+2}(z) - q^{-2-y} - b_{2-k}(y+2,-y)q^{-y})q^6 \\
&+ f_{k,5-y}(\tau)(2g_{2-k,y-1}(z) - 2q^{1-y} - 2b_{2-k}(y-1,-y)q^{-y})q^6 \\
&+ f_{k,5-y}(\tau)(g_{2-k,y-3}(z) - q^{3-y} - b_{2-k}(y-3,-y)q^{-y})q^6 \\
&+ f_{k,5-y}(\tau)(g_{2-k,y+1}(z) - q^{-1-y} - b_{2-k}(y+1,-y)q^{-y})q^6 \\
&+ f_{k,6-y}(\tau)(2g_{2-k,y-2}(z) - 2q^{2-y} - 2b_{2-k}(y-2,-y)q^{-y})q^6 \\
&+ f_{k,6-y}(\tau)(g_{2-k,y-4}(z) - q^{4-y} - b_{2-k}(y-4,-y)q^{-y})q^6 \\
&+ f_{k,7-y}(\tau)(g_{2-k,y-1}(z) - q^{1-y} - b_{2-k}(y-1,-y)q^{-y})q^6 \\
&+ f_{k,7-y}(\tau)(2g_{2-k,y-3}(z) - 2q^{3-y} - 2b_{2-k}(y-3,-y)q^{-y})q^6 \\
&+ f_{k,8-y}(\tau)(g_{2-k,y-2}(z) - q^{2-y} - b_{2-k}(y-2,-y)q^{-y})q^6 \\
&+ f_{k,8-y}(\tau)(2g_{2-k,y-4}(z) - 2q^{4-y} - 2b_{2-k}(y-4,-y)q^{-y})q^6 \\
&+ f_{k,9-y}(\tau)(g_{2-k,y-3}(z) - q^{3-y} - b_{2-k}(y-3,-y)q^{-y})q^6 \\
&+ f_{k,10-y}(\tau)(g_{2-k,y-4}(z) - q^{4-y} - b_{2-k}(y-4,-y)q^{-y})q^6 \\
&+ c_2 f_{k,-y}(\tau)(g_{2-k,y-2}(z) - q^{2-y} - b_{2-k}(y-2,-y)q^{-y})q^6 \\
&- f_{k,-y}(\tau)q^{-y}q^6\left(\varphi_6(z) - q^{-6} - 2q^{-4} - q^{-2} - c_2q^2\right) \\
&- q^6\left(F_k - f_{k,-y}q^{-y}\right)\left(\varphi_6(z) - q^{-6} - 2q^{-4} - q^{-2}\right).
\end{aligned}$$

Using the recurrence relation on $f_{k,-y+6}(z)$, we note that the sum yields

$$\begin{aligned}
f_{k,-m+6}(\tau) =& \varphi_6(\tau) f_{k,-y}(\tau) - 2f_{k,4-y}(\tau) - f_{k,2-y}(\tau) \\
&+ (b_{2-k}(y+6,-y) + 2b_{2-k}(y+4,-y) + b_{2-k}(y+2,-y)) f_{k,-y}(\tau) \\
&+ (2b_{2-k}(y-1,-y) + b_{2-k}(y-3,-y) + b_{2-k}(y+1,-y)) f_{k,5-y}(\tau) \\
&+ (2b_{2-k}(y-2,-y) + b_{2-k}(y-4,-y)) f_{k,6-y}(\tau) \\
&+ (b_{2-k}(y-1,-y) + 2b_{2-k}(y-3,-y)) f_{k,7-y}(\tau) \\
&+ (b_{2-k}(y-2,-y) + 2b_{2-k}(y-4,-y)) f_{k,8-y}(\tau) \\
&+ b_{2-k}(y-3,-y) f_{k,9-y}(\tau) + b_{2-k}(y-4,-y) f_{k,10-y}(\tau) \\
&+ b_{2-k}(y-2,-y) c_2 f_{k,-y}(\tau).
\end{aligned}$$

Multiplying this by q^{6-y} and solving for $\varphi_6(\tau) f_{k,-y}(\tau) q^{6-y}$, we will now replace the extra terms above.

$$
\begin{aligned}
F_k = & f_{k,-y}(\tau)q^{-y} + f_{k,5-y}(\tau)q^{5-y} f_{k,7-y}(\tau)q^{7-y} + f_{k,8-y}(\tau)q^{8-y} \\
& + f_{k,9-y}(\tau)q^{9-y} + f_{k,10-y}(\tau)q^{10-y} + \varphi_6(\tau) f_{k,-y}(\tau)q^{6-y} \\
& + \varphi_6(\tau)\left(F_k - f_{k,-y}(\tau)q^{-y}\right) q^6 - 2q^2 \left(F_k - f_{k,-y}(\tau)q^{-y} - \sum_{n=5-y}^{8-y} f_{k,n}(\tau)q^n\right) \\
& - q^4 \left(F_k - f_{k,-y}(\tau)q^{-y} - \sum_{n=5-y}^{6-y} f_{k,n}(\tau)q^n\right) \\
& + f_{k,-y}(\tau)(g_{2-k,y+6}(z) - q^{-6-y})q^6 \\
& + f_{k,-y}(\tau)(2g_{2-k,y+4}(z) - 2q^{-4-y})q^6 \\
& + f_{k,-y}(\tau)(g_{2-k,y+2}(z) - q^{-2-y})q^6 \\
& + f_{k,5-y}(\tau)(2g_{2-k,y-1}(z) - 2q^{1-y})q^6 \\
& + f_{k,5-y}(\tau)(g_{2-k,y-3}(z) - q^{3-y})q^6 \\
& + f_{k,5-y}(\tau)(g_{2-k,y+1}(z) - q^{-1-y})q^6 \\
& + f_{k,6-y}(\tau)(2g_{2-k,y-2}(z) - 2q^{2-y})q^6 \\
& + f_{k,6-y}(\tau)(g_{2-k,y-4}(z) - q^{4-y})q^6 \\
& + f_{k,7-y}(\tau)(g_{2-k,y-1}(z) - q^{1-y})q^6 \\
& + f_{k,7-y}(\tau)(2g_{2-k,y-3}(z) - 2q^{3-y})q^6 \\
& + f_{k,8-y}(\tau)(g_{2-k,y-2}(z) - q^{2-y})q^6 \\
& + f_{k,8-y}(\tau)(2g_{2-k,y-4}(z) - 2q^{4-y})q^6 \\
& + f_{k,9-y}(\tau)(g_{2-k,y-3}(z) - q^{3-y})q^6 \\
& + f_{k,10-y}(\tau)(g_{2-k,y-4}(z) - q^{4-y})q^6 \\
& + c_2 f_{k,-y}(\tau)(g_{2-k,y-2}(z) - q^{2-y})q^6 \\
& - f_{k,-y}(\tau)q^{-y}q^6 \left(\varphi_6(z) - q^{-6} - 2q^{-4} - q^{-2} - c_2 q^2\right) \\
& - q^6 \left(F_k - f_{k,-y}q^{-y}\right)\left(\varphi_6(z) - q^{-6} - 2q^{-4} - q^{-2}\right).
\end{aligned}
$$

Canceling yields:

$$
\begin{aligned}
0 =& \varphi_6(\tau)\left(F_k\right)q^6 - q^6 F_k\left(\varphi_6(z)\right) + f_{k,-y}(\tau)g_{2-k,y+6}(z)q^6 + 2f_{k,-y}(\tau)g_{2-k,y+4}(z)q^6 \\
&+ f_{k,-y}(\tau)g_{2-k,y+2}(z)q^6 + 2f_{k,5-y}(\tau)g_{2-k,y-1}(z)q^6 + f_{k,5-y}(\tau)g_{2-k,y-3}(z)q^6 \\
&+ f_{k,5-y}(\tau)g_{2-k,y+1}(z)q^6 + 2f_{k,6-y}(\tau)g_{2-k,y-2}(z)q^6 + f_{k,6-y}(\tau)g_{2-k,y-4}(z)q^6 \\
&+ f_{k,7-y}(\tau)g_{2-k,y-1}(z)q^6 + 2f_{k,7-y}(\tau)g_{2-k,y-3}(z)q^6 + f_{k,8-y}(\tau)g_{2-k,y-2}(z)q^6 \\
&+ 2f_{k,8-y}(\tau)g_{2-k,y-4}(z)q^6 + f_{k,9-y}(\tau)g_{2-k,y-3}(z)q^6 + f_{k,10-y}(\tau)g_{2-k,y-4}(z)q^6 \\
&+ c_2 f_{k,-y}(\tau)g_{2-k,y-2}(z)q^6.
\end{aligned}
$$

Therefore, setting $c_2 = 2$ and solving for F_k gives the generating function. □

4.2 The generating functions

We will list all of the generating functions here but save the proofs for the other cases for the appendix. Again the key change in these generating functions is when we have undefined basis elements after the $f_{k,-y}$ term (cf. Theorem 2.10), since these determine the number of terms we must break off of the summation before we can use the recurrence relation.

Theorem 4.4. ***Case 1:*** $k \equiv 0 \mod 12$**.** *Let $k = 12l$, and $y = 7k = 84l$. Let $F_k(z,\tau) = f_{k,-y}(\tau)q^{-y} + \sum_{n=6-y}^{\infty} f_{k,n}(\tau)q^n$. Then*

$$
\begin{aligned}
F_k(z,\tau) =& \Big(\varphi_6(z) - \varphi_6(\tau)\Big)^{-1} \cdot \Big(f_{k,-y}(\tau)g_{2-k,y+6}(z) + 2f_{k,-y}(\tau)g_{2-k,y+4}(z) \\
&+ f_{k,-y}(\tau)g_{2-k,y+2}(z) + 2f_{k,6-y}(\tau)g_{2-k,y-2}(z) + f_{k,6-y}(\tau)g_{2-k,y-4}(z) \\
&+ f_{k,7-y}(\tau)g_{2-k,y-1}(z) + 2f_{k,7-y}(\tau)g_{2-k,y-3}(z) + f_{k,7-y}(\tau)g_{2-k,y-5}(z) \\
&+ f_{k,8-y}(\tau)g_{2-k,y-2}(z) + 2f_{k,8-y}(\tau)g_{2-k,y-4}(z) + f_{k,9-y}(\tau)g_{2-k,y-3}(z) \\
&+ 2f_{k,9-y}(\tau)g_{2-k,y-5}(z) + f_{k,10-y}(\tau)g_{2-k,y-4}(z) + f_{k,11-y}(\tau)g_{2-k,y-5}(z) \\
&+ 2f_{k,-y}(\tau)g_{2-k,y-2}(z)\Big).
\end{aligned}
$$

Case 2: $k \equiv 2 \mod 12$***.*** *Let* $k = 2 + 12l$ *and* $y = 7k - 1$*. Let* $F_k(z, \tau) = f_{k,-y}(\tau)q^{-y} + \sum_{n=5-y}^{\infty} f_{k,n}(\tau)q^n$*. Then*

$$
\begin{aligned}
F_k(z,\tau) =&\Big(\varphi_6(z) - \varphi_6(\tau)\Big)^{-1} \cdot \Big(f_{k,-y}(\tau)g_{2-k,y+6}(z) + 2f_{k,-y}(\tau)g_{2-k,y+4}(z) \\
&+ f_{k,-y}(\tau)g_{2-k,y+2}(z) + 2f_{k,5-y}(\tau)g_{2-k,y-1}(z) + f_{k,5-y}(\tau)g_{2-k,y-3}(z) \\
&+ f_{k,5-y}(\tau)g_{2-k,y+1}(z) + 2f_{k,6-y}(\tau)g_{2-k,y-2}(z) + f_{k,6-y}(\tau)g_{2-k,y-4}(z) \\
&+ f_{k,7-y}(\tau)g_{2-k,y-1}(z) + 2f_{k,7-y}(\tau)g_{2-k,y-3}(z) + f_{k,8-y}(\tau)g_{2-k,y-2}(z) \\
&+ 2f_{k,8-y}(\tau)g_{2-k,y-4}(z) + f_{k,9-y}(\tau)g_{2-k,y-3}(z) + f_{k,10-y}(\tau)g_{2-k,y-4}(z) \\
&+ 2f_{k,-y}(\tau)g_{2-k,y-2}(z)\Big).
\end{aligned}
$$

Case 3: $k \equiv 4 \mod 12$***.*** *Let* $k = 12l + 4$ *and* $y = 7k - 2$*. Let* $F_k(z, \tau) = f_{k,-y}(\tau)q^{-y} + \sum_{n=4-y}^{\infty} f_{k,n}(\tau)q^n$*. Then*

$$
\begin{aligned}
F_k(z,\tau) =&\Big(\varphi_6(z) - \varphi_6(\tau)\Big)^{-1} \cdot \Big(f_{k,-y}(\tau)g_{2-k,y+6}(z) + 2f_{k,-y}(\tau)g_{2-k,y+4}(z) \\
&+ f_{k,-y}(\tau)g_{2-k,y+2}(z) + f_{k,4-y}(\tau)g_{2-k,y+2}(z) + f_{k,4-y}(\tau)g_{2-k,y-2}(z) \\
&+ 2f_{k,5-y}(\tau)g_{2-k,y-1} + f_{k,5-y}(\tau)g_{2-k,y-3}(z) + f_{k,5-y}(\tau)g_{2-k,y+1}(z) \\
&+ 2f_{k,6-y}(\tau)g_{2-k,y-2}(z) + f_{k,7-y}(\tau)g_{2-k,y-1}(z) + 2f_{k,7-y}(\tau)g_{2-k,y-3} \\
&+ f_{k,8-y}(\tau)g_{2-k,y-2}(z) + f_{k,9-y}(\tau)g_{2-k,y-3}(z) + 2f_{k,-y}(\tau)g_{2-k,y-2}(z)\Big).
\end{aligned}
$$

Case 4: $k \equiv 6 \mod 12$. *Let* $k = 12l + 6$ *and* $y = 84l + 39$. *Let* $F_k(z,\tau) = f_{k,-y}(\tau)q^{-y} + \sum_{n=3-y}^{\infty} f_{k,n}(\tau)q^n$. *Then*

$$\begin{aligned} F_k(z,\tau) =& \Big(\varphi_6(z) - \varphi_6(\tau)\Big)^{-1} \cdot \Big(f_{k,-y}(\tau)g_{2-k,y+6}(z) + 2f_{k,-y}(\tau)g_{2-k,y+4}(z) \\ &+ f_{k,-y}(\tau)g_{2-k,y+2}(z) + f_{k,3-y}(\tau)g_{2-k,y+3}(z) + 2f_{k,3-y}(\tau)g_{2-k,y+1}(z) \\ &+ f_{k,3-y}(\tau)g_{2-k,y-1}(z) + f_{k,4-y}(\tau)g_{2-k,y+2}(z) + f_{k,4-y}(\tau)g_{2-k,y-2}(z) \\ &+ 2f_{k,5-y}(\tau)g_{2-k,y-1}(z) + f_{k,5-y}(\tau)g_{2-k,y+1}(z) + 2f_{k,6-y}(\tau)g_{2-k,y-2}(z) \\ &+ f_{k,7-y}(\tau)g_{2-k,y-1}(z) + f_{k,8-y}(\tau)g_{2-k,y-2}(z) + 2f_{k,-y}(\tau)g_{2-k,y-2}(z)\Big). \end{aligned}$$

Case 5: $k \equiv 8 \mod 12$. *Let* $k = 12l + 8$ *and* $y = 84l + 52$. $F_k(z,\tau) = f_{k,-y}(\tau)q^{-y} + \sum_{n=2-y}^{\infty} f_{k,n}(\tau)q^n$. *Then*

$$\begin{aligned} F_k(z,\tau) =& \Big(\varphi_6(z) - \varphi_6(\tau)\Big)^{-1} \cdot \Big(f_{k,-y}(\tau)g_{2-k,y+6}(z) + 2f_{k,-y}(\tau)g_{2-k,y+4}(z) \\ &+ f_{k,-y}(\tau)g_{2-k,y+2}(z) + f_{k,2-y}(\tau)g_{2-k,y+4}(z) + 2f_{k,2-y}(\tau)g_{2-k,y+2}(z) \\ &+ f_{k,3-y}(\tau)g_{2-k,y+3}(z) + 2f_{k,3-y}(\tau)g_{2-k,y+1}(z) + f_{k,3-y}(\tau)g_{2-k,y-1}(z) \\ &+ f_{k,4-y}(\tau)g_{2-k,y+2}(z) + 2f_{k,5-y}(\tau)g_{2-k,y-1}(z) + f_{k,5-y}(\tau)g_{2-k,y+1}(z) \\ &+ f_{k,7-y}(\tau)g_{2-k,y-1}(z)\Big). \end{aligned}$$

Case 6: $k \equiv 10 \mod 12$. *Let* $k = 12l + 10$ *and let* $y = 84l + 66$. *Let* $F_k(z,\tau) = \sum_{n=-y}^{\infty} f_{k,n}(\tau)q^n$. *Then*

$$\begin{aligned} F_k(z,\tau) =& (\varphi_6(z) - \varphi_6(\tau))^{-1} \cdot \Big(f_{k,5-y}(\tau)g_{2-k,y+1}(z) + f_{k,4-y}(\tau)g_{2-k,y+2}(z) \\ &+ f_{k,3-y}(\tau)g_{2-k,y+3}(z) + 2f_{k,3-y}(\tau)g_{2-k,y+1}(z) + f_{k,2-y}(\tau)g_{2-k,y+4}(z) \\ &+ 2f_{k,2-y}(\tau)g_{2-k,y+2}(z) + f_{k,1-y}(\tau)g_{2-k,y+5}(z) + 2f_{k,1-y}(\tau)g_{2-k,y+3}(z) \\ &+ f_{k,1-y}(\tau)g_{2-k,y+1}(z) + f_{k,-y}(\tau)g_{2-k,y+6}(z) + 2f_{k,-y}(\tau)g_{2-k,y+4}(z) \\ &+ f_{k,-y}(\tau)g_{2-k,y+2}(z)\Big). \end{aligned}$$

Chapter 5. Further work and questions

This research has extended some of the previous known facts from the lower levels into a level spanned by eta-quotients, and whose basis elements have fractional Fourier coefficients. The techniques used here should be applicable in other levels.

Some further work that could be done includes extending the techniques used in lower levels to study the divisibility of coefficients in spaces whose basis elements have fractional coefficients. Additionally, it would be interesting to identify which spaces are like level 52 (those that are spanned by eta-quotients and have fractional Fourier coefficients in their basis elements) which have forms for some weight that have all their zeros at infinity. Another natural question is to identify lower levels spanned by eta-quotients without fractional Fourier coefficients and determine if these techniques also apply there.

Further questions to consider include finding a condition to tell when a space contains a form with all of its zeros at infinity, studying whether the genus affects the weight where we will find this form, and listing the conditions when the weight zero form with a pole of minimal order can be found by using eta-quotients.

Though the authors of [1], [3], [6], [7], and [8] gave congruences for the coefficients in the lower levels, this thesis does not treat that particular subject. However, we will provide a few conjectures on the subject from our observations.

Conjecture 5.1. *For any basis element $f_{k,m}(z) \in M_k^\sharp(52)$, if m is odd, then $a_n(m,n) = 0$ for n even. Likewise if m is even then $a_n(m,n) = 0$ for all n odd.*

Conjecture 5.2. *Let $f_{k,m}(z) \in M_k^\sharp(52)$. If $f_{k,m}(z)$ has fractional coefficients, then m must be odd, and the denominator of all the non-leading term coefficients is the same.*

Appendix A. Computer codes

A.1 Eta-quotient finder

```
R.<q> = LaurentSeriesRing(QQ)
def eta_from_tuple(myTuple, N, prec): # N is the level
    w = qexp_eta(QQ[[q]],prec)
    e = 0
    prod = 1
    divList = divisors(N)
    for i in range(len(myTuple)):
        e += divList[i]*myTuple[i]
        prod *= w(q^divList[i])^myTuple[i]
    return q^(int(e/24))*prod
def order_vanishing(etaTuple, N, d): #d represents the denominator of the cusp
    if (d == 0):
    e = 0
    divList = divisors(N)
    for i in range(len(etaTuple)):
        e += divList[i]*etaTuple[i] # the leading term is q^(sum delta * r_delta)/24
        e /= 24
        return e
    else:
        sum = 0
        divList = divisors(N)
        g = gcd(d,N/d)
        for i in range(len(divList)):
            sum += (gcd(d,divList[i])^2*etaTuple[i]/(g*d*divList[i]))
```

```
        sum *= (N/24)
        return sum
# Note: order_vanishing() when d=0 is the leading power of the fourier expansion


    def eta_weight(myTuple): # this assumes that N is understood and therefore doesn't
depend on it.
    sum = 0
    for i in myTuple:
        sum += i
    return sum / 2


leadingPower = -6
MAX =7 # usually |leadingPower|+1
numbers = range(-MAX,MAX+1)
def checkNum(r1,r2,r4,r13,r26,r52):
    if (((r2+r26)%2)==1 or ((r13+r26+r52)%2)==1):
        return false
    myTuple = (r1,r2,r4,r13,r26,r52)
    if (order_vanishing(myTuple, 52, 0) != leadingPower):return false
    for d in [1,2,4,13,26]:
        ord = order_vanishing(myTuple, 52, d)
        if (ord<0 or ord>abs(leadingPower) or ord.is_integer() == false):
            return false
        return true
import time
t0 = time.time()
myList=[]
```

```
for r1 in numbers:
    print r1
    for r2 in numbers:
        for r4 in numbers:
            for r13 in numbers:
                for r26 in numbers:
                    r52 = -r1-r2-r4-r13-r26 #if not weight 0 do 2*weight minus all else
                    if (checkNum(r1,r2,r4,r13,r26,r52) == true):
                        myList.append((r1,r2,r4,r13,r26,r52))
                        print(r1,r2,r4,r13,r26,r52)
t1 = time.time()
t1-t0
```

A.2 Generating $S_2^\sharp(52)$

```
R.<q> = LaurentSeriesRing(QQ)
def eta_from_tuple(myTuple, N, prec): # N is the level
    w = qexp_eta(QQ[[q]],prec)
    e = 0
    prod = 1
    divList = divisors(N)
    for i in range(len(myTuple)):
        e += divList[i]*myTuple[i]
        prod *= w(q^divList[i])^myTuple[i]
    return q^(int(e/24))*prod


    def order_vanishing(etaTuple, N, d): #d represents the denominator of the cusp we are
looking at
    if (d == 0):
    e = 0
    divList = divisors(N)
    for i in range(len(etaTuple)):
        e += divList[i]*etaTuple[i] # the leading term is q^(sum delta * r_delta)/24
        e /= 24
        return e
    else:
        sum = 0
        divList = divisors(N)
        g = gcd(d,N/d)
        for i in range(len(divList)):
            sum += (gcd(d,divList[i])^2*etaTuple[i]/(g*d*divList[i]))
```

```
        sum *= (N/24)
        return sum
# Note: order_vanishing() when d=0 is the leading power of the fourier expansion
def eta_weight(myTuple): # Assumes that N is understood and doesn't depend on it.
    sum = 0
    for i in myTuple:
        sum += i
    return sum / 2
zeroTup=(0,-2,4,0,2,-4)# q^-6 eta-quotient weight 0
zztup=(3,-2,1,1,2,-5)# q^-8 eta-quotient weight 0
gzero=eta_from_tuple(zeroTup,52,30)-2# -2 since 1 is in $M_0^\sharp(52)$ and so is gzero+2
gzz=eta_from_tuple(zztup,52,30)
print gzero
print gzz
func=[] func.append((q^5 - 1/3*q^7 - 2/3*q^9 - 4/3*q^11 - 1/3*q^13 + 1/3*q^15 + 2*q^17
- 2*q^19 + 1/3*q^21 + 2*q^23 - 2/3*q^25 + 4/3*q^27 - 2/3*q^29 + O(q^30))*gzero*gzero)
func.append((q^2 - q^6 + q^8 - 2*q^10 - 2*q^12 + 2*q^18 + q^20 + 2*q^22 - q^24 + q^28
+ O(q^30))*gzz)
func.append((q - 4/3*q^7 - 5/3*q^9 + 2/3*q^11 - 1/3*q^13 - 2/3*q^15 + 2*q^17 - 2*q^19
- 2/3*q^21 + 4*q^23 + 1/3*q^25 - 8/3*q^27 + 10/3*q^29 + O(q^30))*gzero)
func.append((q^2 - q^6 + q^8 - 2*q^10 - 2*q^12 + 2*q^18 + q^20 + 2*q^22 - q^24 + q^28
+ O(q^30))*gzero)
func.append((q^3 - 2/3*q^7 - 7/3*q^9 + 4/3*q^11 + 1/3*q^13 - 4/3*q^15 + q^17 - 2*q^19
+ 2/3*q^21 + 2*q^23 + 5/3*q^25 + 5/3*q^27 + 2/3*q^29 + O(q^30))*gzero)
func.append((q^4 - 2*q^6 + q^10 - q^12 + q^14 + q^16 + 4*q^18 - 2*q^20 - 4*q^22 - 2*q^24
- q^26 + O(q^30))*gzero)
func.append((q^5 - 1/3*q^7 - 2/3*q^9 - 4/3*q^11 - 1/3*q^13 + 1/3*q^15 + 2*q^17 - 2*q^19
```

```
+ 1/3*q^21 + 2*q^23 - 2/3*q^25 + 4/3*q^27 - 2/3*q^29 + O(q^30))*gzero)
func.append(q - 4/3*q^7 - 5/3*q^9 + 2/3*q^11 - 1/3*q^13 - 2/3*q^15 + 2*q^17 - 2*q^19 -
2/3*q^21 + 4*q^23 + 1/3*q^25 - 8/3*q^27 + 10/3*q^29 + O(q^30))
func.append(q^2 - q^6 + q^8 - 2*q^10 - 2*q^12 + 2*q^18 + q^20 + 2*q^22 - q^24 + q^28
+ O(q^30))
func.append(q^3 - 2/3*q^7 - 7/3*q^9 + 4/3*q^11 + 1/3*q^13 - 4/3*q^15 + q^17 - 2*q^19
+ 2/3*q^21 + 2*q^23 + 5/3*q^25 + 5/3*q^27 + 2/3*q^29 + O(q^30))
func.append(q^4 - 2*q^6 + q^10 - q^12 + q^14 + q^16 + 4*q^18 - 2*q^20 - 4*q^22 - 2*q^24
- q^26 + O(q^30))
func.append(q^5 - 1/3*q^7 - 2/3*q^9 - 4/3*q^11 - 1/3*q^13 + 1/3*q^15 + 2*q^17 - 2*q^19
+ 1/3*q^21 + 2*q^23 - 2/3*q^25 + 4/3*q^27 - 2/3*q^29 + O(q^30))
M=Matrix(QQ,len(func),(17))
    for i in range(len(func)):
        for e in [0..16]:
            M[(i,e)]=func[i][e-7]
MM=M.rref()
newFunc=[]
for i in range(len(func)):
    f=0
    for e in [0..16]:
        f+=MM[i,e]*q^(e-7)
    newFunc.append(f)

for i in newFunc:
    print latex(i)
```

Appendix B. Generating Functions: Recurrence Relations for the Omitted Proofs

B.1 $k \equiv 0 \mod 12$

Let $k = 12l$, and $y = 7k = 84l$.

The recurrence relation:

$$
\begin{aligned}
f_{k,m+6} = &\varphi_6 f_{k,m} - 2f_{k,m+4} - f_{k,m+2} \\
&+ (b_{2-k}(y+6,m) + 2b_{2-k}(y+4,m) + b_{2-k}(y+2,m))f_{k,-y} \\
&+ (2b_{2-k}(y-2,m) + b_{2-k}(y-4,m))f_{k,6-y} \\
&+ (b_{2-k}(y-1,m) + 2b_{2-k}(y-3,m) + b_{2-k}(y-5,m))f_{k,7-y} \\
&+ (b_{2-k}(y-2,m) + 2b_{2-k}(y-4,m))f_{k,8-y} \\
&+ (b_{2-k}(y-3,m) + 2b_{2-k}(y-5,m))f_{k,9-y} \\
&+ b_{2-k}(y-4,m)f_{k,10-y} \\
&+ b_{2-k}(y-5,m)f_{k,11-y} \\
&+ b_{2-k}(y-2,m)c_2 f_{k,-y} - c_{m+y} f_{k,-y} - \sum_{r=2}^{y+m-6} c_r f_{k,m-r}.
\end{aligned}
$$

The generating function:

$$
\begin{aligned}
F_k(z,\tau) = &f_{k,-y}(\tau)q^{-y} + f_{k,6-y}(\tau)q^{6-y} + f_{k,7-y}(\tau)q^{7-y} + f_{k,8-y}(\tau)q^{8-y} + f_{k,9-y}(\tau)q^{9-y} \\
&+ f_{k,10-y}(\tau)q^{10-y} + f_{k,11-y}(\tau)q^{11-y} + \sum_{n=2-y}^{\infty} f_{k,n+6}q^{n+6}(\tau).
\end{aligned}
$$

B.2 $k \equiv 4 \mod 12$

Let $k = 12l + 4$ and $y = 7k - 2$.

The recurrence relation:

$$
\begin{aligned}
f_{k,m+6} =& \varphi_6 f_{k,m} - 2f_{k,m+4} - f_{k,m+2} \\
&+ (b_{2-k}(y+6,m) + 2b_{2-k}(y+4,m) + b_{2-k}(y+2,m))f_{k,-y} \\
&+ (b_{2-k}(y+2,m) + b_{2-k}(y-2,m))f_{k,4-y} \\
&+ (2b_{2-k}(y-1,m) + b_{2-k}(y-3,m) + b_{2-k}(y+1,m))f_{k,5-y} \\
&+ 2b_{2-k}(y-2,m)f_{k,6-y} \\
&+ (b_{2-k}(y-1,m) + 2b_{2-k}(y-3,m))f_{k,7-y} \\
&+ b_{2-k}(y-2,m)f_{k,8-y} \\
&+ b_{2-k}(y-3,m)f_{k,9-y} \\
&+ c_2 b_{2-k}(y-2,m)f_{k,-y} - c_{m+y}f_{k,-y} - \sum_{r=2}^{y+m-4} c_r f_{k,m-r}.
\end{aligned}
$$

The generating function:

$$
\begin{aligned}
F_k(z,\tau) =& f_{k,-y}(\tau)q^{-y} + f_{k,4-y}(\tau)q^{4-y} + f_{k,5-y}(\tau)q^{5-y} + f_{k,6-y}(\tau)q^{6-y} + f_{k,7-y}(\tau)q^{7-y} \\
&+ f_{k,8-y}(\tau)q^{8-y} + f_{k,9-y}(\tau)q^{9-y} + \sum_{n=2-y}^{\infty} f_{k,n+6}q^{n+6}(\tau).
\end{aligned}
$$

B.3 $k \equiv 6 \mod 12$

Let $k = 12l + 6$ and $y = 84l + 39$.

The recurrence relation:

$$\begin{aligned} f_{k,m+6} =& \varphi_6 f_{k,m} - 2f_{k,m+4} - f_{k,m+2} \\ &+ (b_{2-k}(y+6,m) + 2b_{2-k}(y+4,m) + b_{2-k}(y+2,m))f_{k,-y} \\ &+ (b_{2-k}(y+3,m) + 2b_{2-k}(y+1,m) + b_{2-k}(y-1,m))f_{k,3-y} \\ &+ (b_{2-k}(y+2,m) + b_{2-k}(y-2,m))f_{k,4-y} \\ &+ (b_{2-k}(y+1,m) + 2b_{2-k}(y-1,m))f_{k,5-y} \\ &+ 2b_{2-k}(y-2,m)f_{k,6-y} \\ &+ b_{2-k}(y-1,m)f_{k,7-y} \\ &+ b_{2-k}(y-2,m)f_{k,8-y} \\ &+ c_2 b_{2-k}(y-2,m)f_{k,-y} - c_{m+y}f_{k,-y} - \sum_{r=2}^{y+m-3} c_r f_{k,m-r}. \end{aligned}$$

The generating function:

$$\begin{aligned} F_k(z,\tau) =& f_{k,-y}(\tau)q^{-y} + f_{k,3-y}(\tau)q^{3-y} + f_{k,4-y}(\tau)q^{4-y} + f_{k,5-y}(\tau)q^{5-y} + f_{k,6-y}(\tau)q^{6-y} \\ &+ f_{k,7-y}(\tau)q^{7-y} + f_{k,8-y}(\tau)q^{8-y} + \sum_{n=2-y}^{\infty} f_{k,n+6}q^{n+6}(\tau). \end{aligned}$$

B.4 $k \equiv 8 \mod 12$

Let $k = 12l + 8$ and $y = 84l + 52$.

The recurrence relation:

$$
\begin{aligned}
f_{k,m+6} =& \varphi_6 f_{k,m} - 2f_{k,m+4} - f_{k,m+2} \\
& + (b_{2-k}(y+6,m) + 2b_{2-k}(y+4,m) + b_{2-k}(y+2,m))f_{k,-y} \\
& + (b_{2-k}(y+4,m) + 2b_{2-k}(y+2,m))f_{k,2-y} \\
& + (b_{2-k}(y+3,m) + 2b_{2-k}(y+1,m) + b_{2-k}(y-1,m))f_{k,3-y} \\
& + b_{2-k}(y+2,m)f_{k,4-y} \\
& + (b_{2-k}(y+1,m) + 2b_{2-k}(y-1,m))f_{k,5-y} \\
& + b_{2-k}(y-1,m)f_{k,7-y} \\
& - c_{m+y}f_{k,-y} - \sum_{r=2}^{y+m-2} c_r f_{k,m-r}.
\end{aligned}
$$

The generating function:

$$
\begin{aligned}
F_k(z,\tau) =& f_{k,-y}(\tau)q^{-y} + f_{k,2-y}(\tau)q^{2-y} + f_{k,3-y}(\tau)q^{3-y} + f_{k,4-y}(\tau)q^{4-y} + f_{k,5-y}(\tau)q^{5-y} \\
& + f_{k,6-y}(\tau)q^{6-y} + f_{k,7-y}(\tau)q^{7-y} + \sum_{n=2-y}^{\infty} f_{k,n+6}q^{n+6}(\tau).
\end{aligned}
$$

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