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Weakly Holomorphic Modular Forms in Prime Power Levels of Genus Zero

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ABSTRACT

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Let $N \in \{8, 9, 16, 25\}$ and let $M_0^\sharp(N)$ be the space of level N weakly holomorphic modular functions with poles only at the cusp at infinity. We explicitly construct a canonical basis for $M_0^\sharp(N)$ indexed by the order of the pole at infinity and show that many of the coefficients of the elements of these bases are divisible by high powers of the prime dividing the level N. Additionally, we show that these basis elements satisfy an interesting duality property. We also give an argument that extends level 1 results on congruences from Griffin to levels 2, 3, 4, 5, 7, 8, 9, 16, and 25.

Keywords: modular forms, congruences, duality, weakly holomorphic

Contents

Chapter 1. Introduction

A modular form of level N and weight k is a function $f(z)$ which is holomorphic on the complex upper-half plane and holomorphic at cusps, satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

We will deal exclusively with genus zero congruence subgroups $\Gamma_0(N)$; that is, when $N \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25\}$. Here genus refers to topological genus. This thesis will particularly focus on $N = 8, 9, 16$, and 25 although reference will be made to $N = 1$ or $N = p$, where p is a prime in the set above. If we allow $f(z)$ to be meromorphic at the cusps of $\Gamma_0(N)$, then we say f is weakly holomorphic. Additionally, if f is weakly holomorphic of weight zero, we say f is a level N modular function. We denote by $M_k(N)$ the space of level N modular forms and by $M_k^!(N)$, the space of weakly holomorphic modular forms. As a special subspace of $M_k^!(N)$, we consider the space $M_k^\sharp(N)$ which includes all modular forms of weight k and level N which are holomorphic except possibly at the cusp at infinity. Any modular form must have a Fourier expansion $f(z) = \sum_{n=n_0}^{\infty} a(n)q^n$, where $q = e^{2\pi i z}$. Properties of the coefficients $a(n)$ of modular forms and functions have long been a topic of study.

1.1 The j-invariant

The modular function

$$j(z) = \frac{\left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n\right)^3}{q \prod_{n=1}^{\infty} (1-q^n)^{24}} = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n) q^n$$

is a weight 0 weakly holomorphic form of level 1. It has a simple pole at infinity. The coefficients $c(n)$ are all integers and play an important role in mathematics. For example, there is a special graded representation of the Monster group whose dimensions appear as these coefficients $c(n)$.

In 1949, Lehner proved [12],[13] that the coefficients $c(n)$ satisfy

$$c(2^a 3^b 5^c 7^d) \equiv 0 \pmod{2^{3a+8} 3^{2b+3} 5^{c+1} 7^d}.$$

This proves that many of the coefficients of the j-invariant are divisible by large powers of small primes. Lehner also showed that similar results hold for other level 1 modular functions, but he included a restriction on the size of the pole. Kolberg [10],[11] and Aas [1] refined Lehner's work to give more specific congruences modulo large powers of p for $p \in \{2, 3, 5, 7\}$. Griffin, in his paper [7], further extended the results of Lehner, Kolberg, and Aas by proving such congruences for every function in a canonical basis for $M_0^\sharp(1)$. He proved the following theorem:

Theorem 1.1 (Theorem 2.1 [7]). *Let* $f_{0,m}^{(1)}(z) = q^{-m} + \sum_{n>0} a_0^{(1)}(m,n) q^n$ *be a level 1 modular function. Write* $m = 2^\alpha m'$ *and* $2^\beta n'$ *where* $\alpha, \beta \geq 0$ *and* $m', n' \not\equiv 0 \pmod{p}$. *Then the following congruences hold.*

For $p = 2$*:*

$$\begin{aligned}
a_0^{(1)}(2^\alpha m', 2^\beta n') &\equiv -2^{3(\beta-\alpha)+8} 3^{\beta-\alpha-1} \cdot m' \sigma_7(m') \sigma_7(n') \pmod{2^{3(\beta-\alpha)+13}} \quad \textit{if } \beta > \alpha, \\
&\equiv -2^{4(\beta-\alpha)+8} 3^{\beta-\alpha-1} \cdot m' \sigma_7(m') \sigma_7(n') \pmod{2^{4(\beta-\alpha)+13}} \quad \textit{if } \alpha > \beta, \\
&\equiv 20 m' \sigma_7(m') \sigma_7(n') \pmod{2^7} \quad \textit{if } \alpha = \beta, m'n' \equiv 1 \pmod{8}, \\
&\equiv \frac{1}{2} m' \sigma(m') \sigma(n') \pmod{2^3} \quad \textit{if } \alpha = \beta, m'n' \equiv 3 \pmod{8}, \\
&\equiv -12 m' \sigma_7(m') \sigma_7(n') \pmod{2^8} \quad \textit{if } \alpha = \beta, m'n' \equiv 5 \pmod{8}.
\end{aligned}$$

For $p = 3$:

$$
\begin{aligned}
a_0^{(1)}(3^\alpha m', 3^\beta n') &\equiv \mp 3^{2(\beta-\alpha)+3} 10^{\beta-\alpha-1} \frac{\sigma(m')\sigma(n')}{n'} \pmod{3^{2(\beta-\alpha)+6}} \\
&\qquad \text{if } \beta > \alpha, m'n' \equiv \pm 1 \pmod 3, \\
&\equiv \mp 3^{3(\alpha-\beta)+3} 10^{\alpha-\beta-1} \frac{\sigma(m')\sigma(n')}{n'} \pmod{3^{3(\alpha-\beta)+6}} \\
&\qquad \text{if } \alpha > \beta, m'n' \equiv \pm 1 \pmod 3, \\
&\equiv 2 \cdot 3^3 \frac{\sigma(m')\sigma(n')}{n'} \pmod{3^7} \\
&\qquad \text{if } \alpha = \beta, m'n' \equiv 1 \pmod 3.
\end{aligned}
$$

For $p = 5$:

$$
\begin{aligned}
a_0^{(1)}(5^\alpha m', 5^\beta n') &\equiv -5^{\beta-\alpha+1} 3^{\beta-\alpha-1} (m')^2 n' \sigma(m')\sigma(n') \pmod{5^{\beta-\alpha+2}} \quad \text{if } \beta > \alpha, \\
&\equiv -5^{2(\alpha-\beta)+1} 3^{\alpha-\beta-1} (m')^2 n' \sigma(m')\sigma(n') \pmod{5^{2(\alpha-\beta)+2}} \quad \text{if } \alpha > \beta, \\
&\equiv 10 (m')^2 n' \sigma(m')\sigma(n') \pmod{5^2} \quad \text{if } \alpha = \beta, \left(\frac{m'n'}{5}\right) = -1.
\end{aligned}
$$

For $p = 7$:

$$
\begin{aligned}
a_0^{(1)}(7^\alpha m', 7^\beta n') &\equiv 7^{\beta-\alpha} 5^{\beta-\alpha-1} (m')^2 n' \sigma_3(m')\sigma_3(n') \pmod{7^{\beta-\alpha+1}} \quad \text{if } \beta > \alpha, \\
&\equiv 7^{2(\alpha-\beta)} 5^{\alpha-\beta-1} (m')^2 n' \sigma_3(m')\sigma_3(n') \pmod{7^{2(\alpha-\beta)+1}} \quad \text{if } \alpha > \beta, \\
&\equiv 2 (m')^2 n' \sigma_3(m')\sigma_3(n') \pmod 7 \quad \text{if } \alpha = \beta, \left(\frac{m'n'}{7}\right) = 1.
\end{aligned}
$$

One of the aims of this thesis to prove that similar congruences hold in higher levels.

1.2 Congruence subgroups of level p

For the congruence subgroups $\Gamma_0(p)$, $p \in \{2, 3, 5, 7\}$, Lehner showed that results similar to those obtained for $j(z)$ hold if these level p modular functions have integral Fourier coefficients and conform with a restriction on the size of the order of the pole at infinity. Specifically, he proved the following theorem:

Theorem 1.2 (Theorem 3 [13]). *Let $p \in \{2, 3, 5, 7\}$ and let $f(z)$ be a modular function on $\Gamma_0(p)$ having a pole at ∞ of order $< p$ and q-series of the form*

$$f(z) = \sum_{n=n_0}^{\infty} a(n)q^n,$$

$$f\left(-\frac{1}{pz}\right) = \sum_{m=m_0}^{\infty} b(m)q^m,$$

where $a(n), b(n) \in \mathbb{Z}$. Then the coefficients $a(n)$ satisfy the following congruence properties:

$$\begin{aligned}
a(2^a n) &\equiv 0 \pmod{2^{3a+8}} && \text{if } p = 2,\\
a(3^a n) &\equiv 0 \pmod{3^{2a+3}} && \text{if } p = 3,\\
a(5^a n) &\equiv 0 \pmod{5^{a+1}} && \text{if } p = 5,\\
a(7^a n) &\equiv 0 \pmod{7^{a}} && \text{if } p = 7.
\end{aligned}$$

Andersen and Jenkins [2] extended Lehner's theorem above to include all elements of canonical bases for $M_0^\sharp(p)$, thereby removing the restriction on the size of the order of the pole. For now, it is sufficient to indicate that $f_{0,m}^{(p)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_0^{(p)}(m,n)q^n$ is a unique element of our basis for $M_0^\sharp(p)$ where $m > 0$ represents the order of the pole at infinity. For the given basis, Andersen and Jenkins proved the following theorem which gives congruences for a portion of the Fourier coefficients, $a_0^{(p)}(m,n)$.

Theorem 1.3 (Theorem 2 [2]). *Let $p \in \{2, 3, 5, 7\}$, and let*

$$f_{0,m}^{(p)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_0^{(p)}(m,n)q^n$$

be an element of a canonical basis for $M_0^\sharp(p)$, with $m = p^\alpha m'$, $n = p^\beta n'$ and $(m', p) =$

$(n', p) = 1$. *Then for* $\beta > \alpha$,

$$
\begin{aligned}
a_0^{(2)}(2^\alpha m', 2^\beta n') &\equiv 0 \pmod{2^{3(\beta-\alpha)+8}} && \text{if } p = 2,\\
a_0^{(3)}(3^\alpha m', 3^\beta n') &\equiv 0 \pmod{3^{2(\beta-\alpha)+3}} && \text{if } p = 3,\\
a_0^{(5)}(5^\alpha m', 5^\beta n') &\equiv 0 \pmod{5^{(\beta-\alpha)+1}} && \text{if } p = 5,\\
a_0^{(7)}(7^\alpha m', 7^\beta n') &\equiv 0 \pmod{7^{(\beta-\alpha)}} && \text{if } p = 7.
\end{aligned}
$$

Using the above results, the author and Jenkins were able to show that stronger divisibility properties hold for those cases when $\alpha > \beta$. Specifically, they proved:

Theorem 1.4 (Theorem 1 [9])**.** *Let* $p \in \{2, 3, 5, 7, 13\}$ *and let* $f_{0.m}^{(p)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_0^{(p)}(m, n)q^n$ *be a weakly holomorphic modular form in* $M_0^\sharp(p)$. *Let* $m = p^\alpha m'$ *and* $n = p^\beta n'$ *with* m', n' *not divisible by* p. *Then for* $\alpha > \beta$, *we have*

$$
\begin{aligned}
a_0^{(2)}(2^\alpha m', 2^\beta n') &\equiv 0 \pmod{2^{4(\alpha-\beta)+8}} && \text{if } p = 2,\\
a_0^{(3)}(3^\alpha m', 3^\beta n') &\equiv 0 \pmod{3^{3(\alpha-\beta)+3}} && \text{if } p = 3,\\
a_0^{(5)}(5^\alpha m', 5^\beta n') &\equiv 0 \pmod{5^{2(\alpha-\beta)+1}} && \text{if } p = 5,\\
a_0^{(7)}(7^\alpha m', 7^\beta n') &\equiv 0 \pmod{7^{2(\alpha-\beta)}} && \text{if } p = 7,\\
a_0^{(13)}(13^\alpha m', 13^\beta n') &\equiv 0 \pmod{13^{\alpha-\beta}} && \text{if } p = 13.
\end{aligned}
$$

The reader will note that neither of the previous two theorems deals with the case when $\alpha = \beta$. This case will be treated in Chapter 7.

1.3 Level 4

Since level 4 is also genus zero, a natural question is whether results or methodology may be applied from level 2. Before citing some results, we must first define the U_p and V_p operators (see [3]). For a modular form $f(z) = \sum_{n=n_0}^{\infty} a(n)q^n \in M_k^!(N)$, we define

$$U_p(f(z)) = \sum_{n=n_0}^{\infty} a(pn)q^n \in M_k^!(pN),$$
$$V_p(f(z)) = \sum_{n=n_0}^{\infty} a(n)q^{pn} \in M_k^!(pN).$$

If $p|N$, then we actually have $U_p(f(z)) \in M_k^!(N)$, while if $p^2|N$, then $U_p(f(z)) \in M_k^!(N/p)$. Note that the operators are defined for any even weight k; however, we will be primarily concerned with their working when $k = 0$.

In a recent paper, Haddock and Jenkins [8] proved a theorem which will be of particular use in this thesis. It is stated here for convenience.

Theorem 1.5 (Theorem 3 [8]). *Let $f_{0,m}^{(4)}(z) = q^{-m} + \sum a_0^{(4)}(m,n)q^n$ be an element of a canonical basis for $M_0^\sharp(4)$. If $n \not\equiv m \pmod 2$, then $a_0^{(4)}(m,n) = 0$.*

This theorem proves that the powers of q with nonzero coefficient all have the same parity as the order of the pole at infinity. As a part of the proof, they prove that V_2U_2 preserves the space $M_0^\sharp(4)$, a fact to which we will later refer. Also of use will be the following two results from the previously cited paper published by the author and Jenkins [9].

Theorem 1.6 (Theorem 4 [9]). *For any nonnegative integer m, we have $U_2(f_{0,2m}^{(4)}(z)) = f_{0,m}^{(2)}(z)$ and $U_2(f_{0,2m+1}^{(4)}(z)) = 0$.*

Theorem 1.7 (Theorem 2). *Let $f_{0,m}^{(4)}(z) = q^{-m} + \sum a_0^{(4)}(m,n)q^n \in M_0^\sharp(4)$ be an element of the canonical basis. Write $m = 2^\alpha m'$ and $n = 2^\beta n'$ with m', n' odd. Then for $\alpha \neq \beta$, we have*

$$a_0^{(4)}(2^\alpha m', 2^\beta n') \equiv 0 \pmod{2^{4(\alpha-\beta)+8}} \qquad \text{if } \alpha > \beta,$$
$$a_0^{(4)}(2^\alpha m', 2^\beta n') \equiv 0 \pmod{2^{3(\beta-\alpha)+8}} \qquad \text{if } \beta > \alpha.$$

Theorem 1.6 proves that the U_2 operator sends basis elements in level 4 to basis elements in level 2 (equivalently, V_2 sends basis elements of level 2 to basis elements of level 4), and Theorem 1.7 uses this first fact and the results from [2] and [9] to obtain divisibility results in level 4.

Chapter 2. Level 8

While much has been done in levels 1, 2, 3, 4, 5, and 7, the theory of level 8 is relatively less studied. Since $\Gamma_0(8)$ is genus zero and 8 is a power of 2, like 4, it is interesting to know how level 8 relates to these lower levels. We begin first by constructing an infinite canonical basis indexed by the order of the pole at infinity. This basis will be analogous to those constructed in lower levels and it will allow us to use many of the previously mentioned techniques in order to prove some results about the coefficients of its elements. The methods and ideas from this chapter will easily extend to levels 9, 16, and 25 which we will include in Chapters 3, 4, and 5 respectively.

2.1 Building blocks

The congruence subgroup $\Gamma_0(8)$ has 4 cusps, which may be taken to be at $0, \frac{1}{2}, \frac{1}{4}, \infty$. Because $\Gamma_0(8)$ has genus zero, we may use a single form, known as a Hauptmodul, to generate the entire space of weight zero forms. For our purposes, we will let

$$\phi^{(8)}(z) = \frac{\eta^4(z)\eta^2(4z)}{\eta^2(2z)\eta^4(8z)} = q^{-1} - 4 + 4q + 2q^3 + \cdots$$

be the Hauptmodul for this space. Here as usual, $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta function. To compute the value of $\phi^{(8)}(z)$ at each of the cusps, we will need the following operator:

Definition 2.1. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in $GL_2^+(\mathbb{Q})$ and let f be a modular form of weight k. We define the *slash operator* to act as follows:

$$(f|[\gamma])(z) = \det(\gamma)^{k-1}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

Furthermore, if f is a modular form of level N, then $f|[\gamma] = f$ if $\gamma \in \Gamma_0(N)$.

Referring again to $\phi^{(8)}(z)$, it clearly has a simple pole at infinity (as given by the q^{-1} term). To calculate the value of $\phi^{(8)}(z)$ at the cusp at $\frac{1}{4}$, we take the limit as $z \to i\infty$ of $\phi^{(8)}(z)$ slashed with the matrix $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$. We will make use of the following functional equations for the η-function.

$$\eta(z+1) = e^{\pi i/12}\eta(z), \quad \eta\left(-\frac{1}{z}\right) = \sqrt{-iz}\;\eta(z), \quad \eta\left(z+\frac{1}{2}\right) = \frac{e^{\pi i/24}\eta^3(2z)}{\eta(z)\eta(4z)}.$$

Therefore we get

$$\begin{aligned}
\phi^{(8)}(z)\Big|\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} &= \frac{\eta^4(\frac{z}{4z+1})\eta^2(\frac{4z}{4z+1})}{\eta^2(\frac{2z}{4z+1})\eta^4(\frac{8z}{4z+1})} \\
&= \frac{\eta^4(\frac{-4z-1}{z})\left(\sqrt{\frac{-4z-1}{iz}}\right)^4 \eta^2(\frac{-4z-1}{4z})\left(\sqrt{\frac{-4z-1}{4iz}}\right)^2}{\eta^2(\frac{-4z-1}{2z})\left(\sqrt{\frac{-4z-1}{2iz}}\right)^2 \eta^4(\frac{-4z-1}{8z})\left(\sqrt{\frac{-4z-1}{8iz}}\right)^4} \\
&= \frac{\eta^4(\frac{-1}{z}-4)\eta^2(\frac{-1}{4z}-1)\frac{1}{(zi)^2}\frac{1}{4iz}}{\eta^2(\frac{-1}{2z}-2)\eta^4(\frac{-1}{8z}-\frac{1}{2})\frac{1}{2iz}\frac{1}{(8iz)^2}} \\
&= \frac{e^{-16\pi i/12}\eta^4(\frac{-1}{z})e^{-2\pi i/12}\eta^2(\frac{-1}{4z})\eta^4(\frac{-1}{8z})\eta^4(\frac{-1}{2z})\cdot 32}{e^{-4\pi i/12}\eta^2(\frac{-1}{2z})e^{-4\pi i/24}\eta^{12}(\frac{-1}{4z})} \\
&= -32\frac{\eta^4(\frac{-1}{z})\eta^4(\frac{-1}{8z})\eta^2(\frac{-1}{2z})}{\eta^{10}(\frac{-1}{4z})} \\
&= -32\frac{\eta^4(z)\eta^4(8z)\eta^2(2z)(\sqrt{\frac{z}{i}})^4\left(\sqrt{\frac{8z}{i}}\right)^4\left(\sqrt{\frac{2z}{i}}\right)^2}{\eta^{10}(4z)\left(\sqrt{\frac{4z}{i}}\right)^{10}} \\
&= -4\frac{\eta^4(z)\eta^4(8z)\eta^2(2z)}{\eta^{10}(4z)} \\
&= -4(1-4q+16q^3+\ldots)
\end{aligned}$$

When we take the limit as $z \to i\infty$, all of the positive powers of q go to 0; therefore only the constant term survives. We conclude that the value of $\phi^{(8)}(z)$ at this cusp is -4. Similarly, slashing $\phi^{(8)}(z)$ with the matrices $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 8 & 0 \end{pmatrix}$ shows that it has a value of -8 at the cusp at $\dfrac{1}{2}$ and it vanishes at 0.

2.2 Constructing the basis $f_{0,m}^{(8)}(z)$

The basis is constructed by taking successively higher powers of $\phi^{(8)}(z)$ and then subtracting off earlier basis elements. Each element can also be expressed as a polynomial in $\phi^{(8)}(z)$. The first few elements are

$$\begin{aligned}
f_{0,0}^{(8)}(z) &= (\phi^{(8)}(z))^0 = 1, \\
f_{0,1}^{(8)}(z) &= \phi^{(8)}(z) + 4 = q^{-1} + 4q + 2q^3 + \cdots, \\
f_{0,2}^{(8)}(z) &= (\phi^{(8)}(z))^2 + 8\phi^{(8)}(z) + 8 = q^{-2} + 20q^2 - 62q^6 + \cdots, \\
f_{0,3}^{(8)}(z) &= (\phi^{(8)}(z))^3 + 12(\phi^{(8)}(z))^2 + 36\phi^{(8)}(z) + 16 = q^{-3} + 6q + 64q^3 + \cdots.
\end{aligned}$$

The reader will notice that in the basis elements above, many of the coefficients are 0. This is not an accident. In fact, we have the following theorem:

Theorem 2.2. *Let $m > 0$ and let $f_{0,m}^{(8)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_0^{(8)}(m,n)q^n$ be an element of the described canonical basis. Then $m \not\equiv n \pmod 2$ implies that $a_0^{(8)}(m,n) = 0$.*

This theorem is analogous to Theorem 3 in [8] proven for $M_0^{\sharp}(4)$. Before presenting the proof, we require another theorem.

Theorem 2.3. *V_2U_2 preserves the space $M_0^{\sharp}(8)$.*

Proof. Suppose that $f \in M_0^!(8)$ is in the subspace $M_0^{\sharp}(8)$. Then $f|[\gamma]$ is holomorphic at ∞ for all $\gamma \in SL_2(\mathbb{Z}) \setminus \Gamma_0(8)$. Since $V_2U_2 f(z) = \dfrac{1}{2}\left(f(z) + f\left(z + \dfrac{1}{2}\right)\right)$, it suffices to show that $f|[\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}][\gamma]$ is holomorphic at ∞ for $\gamma \in SL_2(\mathbb{Z}) \setminus \Gamma_0(8)$. However, this can be written as $f|[\alpha][\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}]$ for some $\alpha \in SL_2(\mathbb{Z}) \setminus \Gamma_0(8)$. The result follows. $\square$

Proof of Theorem 2.2. Notice that the theorem may be restated as $V_2U_2f^{(8)}_{0,m}(z) = 0$ if m is odd and $V_2U_2f^{(8)}_{0,m}(z) = f^{(8)}_{0,m}(z)$ if m is even. If m is odd, an examination of the Fourier expansion of $V_2U_2f^{(8)}_{0,m}(z)$ gives that it must be holomorphic, vanishing to order at least 1. Therefore it must be identically 0. If m is even, then we consider the difference $V_2U_2f^{(8)}_{0,m}(z) - f^{(8)}_{0,m}(z)$. This cancels the pole at ∞. The resulting difference is a form in $M_0(8)$ which vanishes at ∞ to order greater than 1 and therefore must be identically zero. This proves the theorem. $\square$

Additionally, we have the following theorem, analogous to Theorem 4 in [9].

Theorem 2.4. *Let $m > 0$. Then $U_2(f^{(8)}_{0,2m}(z)) = f^{(4)}_{0,m}(z)$ and $U_2(f^{(8)}_{0,2m+1}(z)) = 0$.*

Proof. Let $f^{(8)}_{0,2m}(z) \in M_0^\sharp(8)$. By Theorem 2.3 above, V_2U_2 acts as the identity on f. Now $V_2(f^{(4)}_{0,m}(z))$ is a modular form in $M_0^\sharp(8)$ with a pole of order $2m$ at infinity. Therefore it must be that

$$V_2\left(U_2(f^{(8)}_{0,2m}(z)) - f^{(4)}_{0,m}(z)\right)$$

is a modular form in $M_0^\sharp(8)$ which vanishes at ∞ and therefore must be identically 0. This proves the first result. The second result follows immediately from Theorem 2.2 by recognizing that all coefficients of even powers of q in $f^{(8)}_{0,2m+1}(z)$ are zero. $\square$

2.3 Arbitrary weight k

A canonical basis as described can be constructed for any even weight k. Standard dimension formulas give that $\dim M_{k+2}(8) = \dim M_k(8) + 2$. Therefore, from the first weight k basis element, we may get the first basis element of weight $k+2$ (resp. $k-2$) by multiplying (resp. dividing) by a weight 2 form with all of its zeros at ∞ and with q-expansion beginning q^2. We will call this form $E^{(8)}(z)$ and it is given by:

$$E^{(8)}(z) := \frac{\eta^8(8z)}{\eta^4(4z)} = q^2 + 4q^6 + 6q^{10} + \cdots.$$

Thus with the Hauptmodul previously defined and this form $E^{(8)}(z)$ we may construct a canonical basis for arbitrary weight k.

Duke and Jenkins in [4] construct bases of arbitrary weight in level 1. In level 8, the idea is the same, and as such, we will only present a basic idea here. For a given weight k and $m > 0$, the basis element $f_{k,m}^{(8)}(z)$ is formed by taking $(E^{(8)}(z))^{k/2}$ (recall that k is even so that $k/2$ is an integer) as this will give the correct weight. Then we multiply by the appropriate power of $\phi^{(8)}(z)$ which yields a form beginning q^{-m}. The last step is to subtract off previous basis elements. Thus we define $f_{k,m}^{(8)}(z) = q^{-m} + \sum_{n=k+1}^{\infty} a_k^{(8)}(m,n)q^n$. Notice that the gap between the negative power of q and the first positive power of q is made to be as large as possible.

2.4 The basis $g_{k,m}^{(8)}(z)$

One of the reasons for calculating the value of $\phi^{(8)}(z)$ at the cusps of $\Gamma_0(8)$ is that by adding/subtracting the appropriate constant, c, we may force $\phi^{(8)}(z) + c$ to vanish at any cusp of our choosing. If we use each of these functions in the construction of the basis, then we will generate a set of basis elements that vanish at all the cusps except for infinity. Denote by $g_{k,m}^{(8)}(z) = q^{-m} + \sum_{n=k-2}^{\infty} b_k^{(8)}(m,n)q^n$ the element of this basis with a pole of order m at infinity. Explicitly, we may construct $g_{k,m}^{(8)}(z)$ by considering $(E^{(8)}(z))^{k/2}(\phi^{(8)}(z))(\phi^{(8)}(z) + 4)(\phi^{(8)}(z)+8)F(\phi^{(8)}(z))$. Here $\phi^{(8)}(z)$ vanishes at 0, $\phi^{(8)}(z)+4$ vanishes at $\frac{1}{4}$, $\phi^{(8)}(z)$ vanishes at $\frac{1}{2}$, and $F(\phi^{(8)}(z))$ is some Faber polynomial in $\phi^{(8)}(z)$ with integer coefficients that makes the gap between q^{-m} and the next power of q as large as possible.

2.5 Duality

The purpose of defining the $\{g_{k,m}^{(8)}(z)\}$ basis in the last section is to prove a result that stems from work done by El-Guindy [5]. In his paper, he provides criteria which, if satisfied, prove a duality result. In particular, if we consider $f_{k,m}^{(8)}(z)$ and $g_{2-k,n}^{(8)}(z)$, these two forms satisfy the conditions given by El-Guindy. Therefore the coefficients of these forms exhibit a nice duality property, as detailed in the following theorem:

Theorem 2.5. *Given the forms* $f_{k,m}^{(8)}(z) = q^{-m} + \sum_{n=k+1}^{\infty} a_k^{(8)}(m,n)q^n$ *and* $g_{2-k,n}^{(8)}(z) = q^{-n} + \sum_{m=-k}^{\infty} b_{2-k}^{(8)}(n,m)q^m$ *as described, then for all* m, n

$$a_k^{(8)}(m,n) = -b_{2-k}^{(8)}(n,m).$$

In particular, $a_0^{(8)}(m,n) = -b_2^{(8)}(n,m)$.

Proof. Consider the form $F(z) = f_{k,m}^{(8)}(z)g_{2-k,n}^{(8)}(z)$. It is weight 2 and vanishes at all cusps except ∞. Therefore it must have a pole at ∞ of at least order 1. The constant term of this form is $a_k^{(8)}(m,n) + b_{2-k}^{(8)}(n,m)$. Now the subspace of $M_2^\sharp(8)$ consisting of such forms as $F(z)$ is generated by derivatives $q\frac{d}{dq}f_{0,m}^{(8)}(z)$. Thus the constant term above must be 0 and the result follows. □

This result was proven in [4] for level 1, in [6] for levels 2 and 3, in [8] for level 4, and it was extended to levels 5, 7, and 13 in [9]. A consequence of this theorem is that results we prove for coefficients of weight zero functions in level 8 will extend easily to weight 2 forms. To illustrate, notice the duality in the following example (we include zero coefficients to aid in clarity):

$$f_{0,1}^{(8)}(z) = q^{-1} + 4q + 0q^2 + 2q^3 + 0q^4 + \cdots,$$
$$f_{0,2}^{(8)}(z) = q^{-2} + 0q + 20q^2 + 0q^3 + 0q^4 + \cdots,$$
$$f_{0,3}^{(8)}(z) = q^{-3} + 6q + 0q^2 + 64q^3 + 0q^4 + \cdots,$$
$$f_{0,4}^{(8)}(z) = q^{-4} + 0q + 0q^2 + 0q^3 + 276q^4 + \cdots,$$

$$g_{2,1}^{(8)}(z) = q^{-1} - 4q + 0q^2 - 6q^3 + 0q^4 + \cdots,$$
$$g_{2,2}^{(8)}(z) = q^{-2} + 0q - 20q^2 + 0q^3 + 0q^4 + \cdots,$$

$$g_{2,3}^{(8)}(z) = q^{-3} - 2q + 0q^2 - 64q^3 + 0q^4 + \cdots,$$

$$g_{2,4}^{(8)}(z) = q^{-4} + 0q + 0q^2 + 0q^3 - 276q^4 + \cdots.$$

For example, $a_0^{(8)}(3,1) = 6 = -b_2^{(8)}(1,3)$.

Chapter 3. Level 9

As mentioned at the beginning of Chapter 2, the results for level 8 may be proved in an analogous way for levels 9, 16, and 25.

The congruence subgroup $\Gamma_0(9)$ has 4 cusps, which may be taken to be at $0, \frac{1}{3}, -\frac{1}{3}, \infty$. A convenient Hauptmodul is given by

$$\phi^{(9)}(z) = \frac{\eta^3(z)}{\eta^3(9z)} = q^{-1} - 3 + 5q^2 - 7q^5 + \cdots.$$

Like $\phi^{(8)}(z)$, it has a simple pole at ∞, and using the same functional equations and techniques as in Section 2.1, we can calculate the values of $\phi^{(9)}(z)$ at the cusps. Using the matrix $\begin{pmatrix} 0 & 1 \\ 9 & 0 \end{pmatrix}$ we find that $\phi^{(9)}(z)$ has value 0 at the cusp at 0. With the matrices $\begin{pmatrix} 1 & 0 \\ \pm 3 & 1 \end{pmatrix}$, we calculate that $\phi^{(9)}(z)$ has values $\sqrt{27}\left(-\frac{\sqrt{3}}{2} \mp \frac{i}{2}\right)$ at the cusps at $\pm\frac{1}{3}$.

The process of forming the basis the same. Each element of the basis can be written as a polynomial in $\phi^{(9)}(z)$. The first few nonconstant basis elements are

$$f_{0,1}^{(9)}(z) = \phi^{(9)}(z) + 3 = q^{-1} + 5q^2 - 7q^5 + \cdots,$$

$$f_{0,2}^{(9)}(z) = (\phi^{(9)}(z))^2 + 6\phi^{(9)}(z) + 9 = q^{-2} + 10q + 11q^4 - \cdots,$$

$$f_{0,3}^{(9)}(z) = (\phi^{(9)}(z))^3 + 9(\phi^{(9)}(z))^2 + 27\phi^{(9)}(z) + 12 = q^{-3} + 54q^3 - 76q^6 - \cdots.$$

We have the following results which are analogous to those found in Section 2.2.

Theorem 3.1. *$V_3 U_3$ preserves the space $M_0^{\sharp}(9)$.*

Proof. Suppose that $f \in M_0^!(9)$ is in the subspace $M_0^\sharp(9)$. Then $f|[\gamma]$ is holomorphic at ∞ for all $\gamma \in SL_2(\mathbb{Z}) \setminus \Gamma_0(9)$. Since $V_3U_3f(z) = \frac{1}{3}\left(f(z) + f\left(z + \frac{1}{3}\right) + f\left(z + \frac{2}{3}\right)\right)$, we need only show that $f|[\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}][\gamma]$ and $f|[\begin{smallmatrix} 3 & 2 \\ 0 & 3 \end{smallmatrix}][\gamma]$ are holomorphic at ∞ for $\gamma \in SL_2(\mathbb{Z}) \setminus \Gamma_0(9)$. In both cases, this can be written as $f|[\alpha][\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}]$ for some $\alpha \in SL_2(\mathbb{Z}) \setminus \Gamma_0(9)$, and the result follows. □

Theorem 3.2. *Let $m > 0$ and let $f_{0,m}^{(9)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_0^{(9)}(m,n)q^n$ be an element of our canonical basis. If $m \not\equiv n \pmod 3$, then $a_0^{(9)}(m,n) = 0$.*

Proof. The proof of this theorem follows the same method as the proof of Theorem 2.2 and uses Theorem 3.1 above. □

Theorem 3.3. *For $m > 0$, $U_3(f_{0,3m}^{(9)}(z)) = f_{0,m}^{(3)}(z)$, $U_3(f_{0,3m+1}^{(9)}(z)) = U_3(f_{0,3m+2}^{(9)}(z)) = 0$.*

Proof. This holds by the same argument as the one found in the proof of Theorem 2.4. □

In order to construct a canonical basis of arbitrary even weight k, we need only the weight 2 form

$$E^{(9)}(z) := \frac{\eta^6(9z)}{\eta^2(3z)} = q^2 + 2q^5 + 5q^8 + \cdots$$

in addition to the Hauptmodul $\phi^{(9)}(z)$. This is because the dimension formulas show that $\dim M_{k+2}(9) = \dim M_k(9) + 2$, so we can get the first weight $k+2$ (resp. $k-2$) basis element from the first weight k basis element by multiplying (resp. dividing) by $E^{(9)}(z)$.

Thus, by taking an appropriate power of $E^{(9)}(z)$ (to get the correct weight), and multiplying by a polynomial in $\phi^{(9)}(z)$, we may form the basis $\{f_{k,m}^{(9)}(z)\}$. To construct the basis $\{g_{k,m}^{(9)}(z)\}$, we use the values of $\phi^{(9)}(z)$ at the cusps given above and proceed in the same way as described in Section 2.4.

Having defined the two bases, we may prove duality in level 9.

Theorem 3.4. *Given the forms $f_{k,m}^{(9)}(z) = q^{-m} + \sum_{n=k+1} a_k^{(9)}(m,n)q^n$ and $g_{2-k,n}^{(9)}(z) = q^{-n} +$*

$\sum_{m=-k} b_{2-k}^{(9)}(n,m)q^m$, *then for all* m, n

$$a_k^{(9)}(m,n) = -b_{2-k}^{(9)}(n,m).$$

Proof. Same as the proof of Theorem 2.5 with 8 replaced by 9. $\square$

As before this allows results in weight 0 to be easily applied to weight 2 forms.

Chapter 4. Level 16

There are 6 cusps in the congruence subgroup $\Gamma_0(16)$. They may be taken at $0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, -\frac{1}{4}$, and ∞. We take as our Hauptmodul

$$\phi^{(16)}(z) = \frac{\eta^2(z)\eta(8z)}{\eta(2z)\eta^2(16z)} = q^{-1} - 2 + 2q^3 - q^7 - \cdots.$$

It has a simple pole at ∞ and vanishes at 0. The values at the other cusps are as follows: -2 at $\frac{1}{8}$, -4 at $\frac{1}{2}$, and $-2 \mp 2i$ at $\pm\frac{1}{4}$. Note that the above values were calculated using the matrices $\begin{pmatrix} 0 & 1 \\ 16 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 8 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 4 & 1 \end{pmatrix}$.

The first few nonconstant basis elements are

$$\begin{aligned}
f_{0,1}^{(16)}(z) &= \phi^{(16)}(z) + 2 = q^{-1} + 2q^3 - q^7 - \cdots, \\
f_{0,2}^{(16)}(z) &= (\phi^{(16)}(z))^2 + 4\phi^{(16)}(z) + 4 = q^{-2} + 4q^2 + 2q^6 - \cdots, \\
f_{0,3}^{(16)}(z) &= (\phi^{(16)}(z))^3 + 6(\phi^{(16)}(z))^2 + 12\phi^{(16)}(z) + 8 = q^{-3} + 6q + 9q^5 - \cdots.
\end{aligned}$$

The following theorems hold, adapted as before to the present level. Their proofs are identical to those for Theorems 2.3, 2.2, and 2.4, changing 8 to 16.

Theorem 4.1. *V_2U_2 preserves the space $M_0^\sharp(16)$.*

Theorem 4.2. *Let $m > 0$ and let $f_{0,m}^{(16)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_0^{(16)}(m,n)q^n$ be an element of our canonical basis. If $m \not\equiv n \pmod 2$, then $a_0^{(16)}(m,n) = 0$.*

Theorem 4.3. *For $m > 0$, $U_2(f_{0,2m}^{(16)}(z)) = f_{0,m}^{(8)}(z)$, $U_2(f_{0,2m+1}^{(16)}(z)) = 0$.*

The dimension formulas give us that $\dim M_{k+2}(16) = \dim M_k(16) + 4$. By the same argument presented in Section 2.3, we may construct any even weight form by using the Hauptmodul and powers of the following weight 2 form:

$$E^{(16)}(z) := \frac{\eta^8(16z)}{\eta^4(8z)} = q^4 + 4q^{12} + 6q^{20} + \cdots.$$

The basis $\{g_{k,m}^{(16)}(z)\}$ is constructed as done previously.

Duality holds in level 16 as well.

Theorem 4.4. *Given the forms $f_{k,m}^{(16)}(z) = q^{-m} + \sum_{n=2k+1} a_k^{(16)}(m,n)q^n$ and $g_{2-k,n}^{(16)}(z) = q^{-n} + \sum_{m=-2k} b_{2-k}^{(16)}(n,m)q^m$, then for all m, n*

$$a_k^{(16)}(m,n) = -b_{2-k}^{(16)}(n,m).$$

Proof. Use the proof of Theorem 2.5 with 16 in place of 8. □

Chapter 5. Level 25

It turns out that level 25 is more complicated than levels 8, 9, or 16. A lot of the theory is the same, as seen below. However, there are instances when known results don't adapt quite as readily.

The subgroup $\Gamma_0(25)$ has 6 cusps, which may be taken at $0, \pm\frac{1}{5}, \pm\frac{2}{5}, \infty$. We use as our

Hauptmodul the function

$$\phi^{(25)}(z) = \frac{\eta(z)}{\eta(25z)} = q^{-1} - 1 - q + q^4 + \cdots.$$

Apart from vanishing at 0 and having a simple pole at ∞, its values at the cusps $\pm\frac{1}{5}$ are $\sqrt{5}\left(\frac{1-\sqrt{5}}{4} \mp i\sqrt{\frac{5+\sqrt{5}}{8}}\right)$ and at $\pm\frac{2}{5}$ are $\sqrt{5}\left(\frac{-1-\sqrt{5}}{4} \mp i\sqrt{\frac{5-\sqrt{5}}{8}}\right)$. Again these values are computed by slashing $\phi^{(25)}(z)$ with specific matrices; in this case we use the matrices $\begin{pmatrix} 0 & 1 \\ 25 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 5 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ \pm 5 & 1 \end{pmatrix}$.

The first few nonconstant basis elements are

$$\begin{aligned}
f_{0,1}^{(25)}(z) &= \phi^{(25)}(z) + 1 \\
&= q^{-1} - q + q^4 + q^6 - q^{11} - q^{14} + q^{21} + q^{24} - \cdots, \\
f_{0,2}^{(25)}(z) &= (\phi^{(25)}(z))^2 + 2\phi^{(25)}(z) + 3 \\
&= q^{-2} + q^2 + 2q^3 - 2q^7 + q^8 + 3q^{12} - 2q^{13} - 2q^{17} - 2q^{18} - q^{22} + 2q^{23} + \cdots, \\
f_{0,3}^{(25)}(z) &= (\phi^{(25)}(z))^3 + 3(\phi^{(25)}(z))^2 + 6\phi^{(25)}(z) + 4 \\
&= q^{-3} + 3q^2 - q^3 + 3q^7 + 3q^8 - 2q^{12} - 6q^{13} - 6q^{17} + 7q^{18} - 3q^{23} + \cdots, \\
f_{0,4}^{(25)}(z) &= (\phi^{(25)}(z))^4 + 4(\phi^{(25)}(z))^3 + 10(\phi^{(25)}(z))^2 + 12\phi^{(25)}(z) + 7 \\
&= q^{-4} + 4q + q^4 + 6q^6 - 4q^9 + 10q^{14} - 11q^{16} - 16q^{19} - 8q^{21} + 15q^{24} + \cdots.
\end{aligned}$$

Notice that the exponents in each basis element above are either all squares mod 5 or nonsquares mod 5. While this has not been proven, we make the following conjecture.

Conjecture 1. *Let $m > 0$ and let $f_{0,m}^{(25)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_0^{(25)}(m,n)q^n$ be an element of our canonical basis. If $\left(\frac{m}{5}\right) \neq \left(\frac{n}{5}\right)$ then $a_0^{(25)}(m,n) = 0$. Here $\left(\frac{\cdot}{5}\right)$ is the Legendre symbol.*

Theorem 5.1. *V_5U_5 preserves the space $M_0^\sharp(25)$.*

Proof. Suppose that $f \in M_0^!(25)$ is in the subspace $M_0^\sharp(25)$. Then $f|[\gamma]$ is holomorphic at

∞ for all $\gamma \in SL_2(\mathbb{Z}) \setminus \Gamma_0(25)$. Since

$$V_5U_5f(z) = \frac{1}{5}\left(f(z) + f\left(z+\frac{1}{5}\right) + f\left(z+\frac{2}{5}\right) + f\left(z+\frac{3}{5}\right) + f\left(z+\frac{4}{5}\right)\right),$$

it suffices to show that $f|\left[\begin{smallmatrix} 5 & 1 \\ 0 & 5 \end{smallmatrix}\right][\gamma]$, $f|\left[\begin{smallmatrix} 5 & 2 \\ 0 & 5 \end{smallmatrix}\right][\gamma]$, $f|\left[\begin{smallmatrix} 5 & 3 \\ 0 & 5 \end{smallmatrix}\right][\gamma]$, and $f|\left[\begin{smallmatrix} 5 & 4 \\ 0 & 5 \end{smallmatrix}\right][\gamma]$ are holomorphic at ∞ for $\gamma \in SL_2(\mathbb{Z}) \setminus \Gamma_0(25)$. In all cases, this can be written as $f|[\alpha]\left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right]$ for some $\alpha \in SL_2(\mathbb{Z}) \setminus \Gamma_0(25)$. The result follows. □

Theorem 5.2. *For $m > 0$, if $m = 5\ell$, then $U_5(f^{(25)}_{0,m}(z)) = f^{(5)}_{0,\ell}(z)$. If $5 \nmid m$, then $U_5(f^{(25)}_{0,m}(z)) = 0$.*

Proof. See Theorem 2.4. □

In order to work in arbitrary even weight, we need a weight 2 form and a weight 4 form. This is because $\dim M_{k+4}(25) = \dim M_k(25) + 10$. For the weight 2 form, as we did in Section 2.3, we let $E^{(25)}(z)$ be the unique weight 2 form in level 25 that has all of its zeros at ∞. Its q-expansion begins with q^4 and the first few terms are

$$E^{(25)}(z) := q^4 + q^6 + 2q^9 + 3q^{14} + 2q^{16} + 4q^{19} + 2q^{21} + 7q^{24} + \cdots.$$

Remark. Note that $E^{(25)}(z)$ is not given by a nice, closed formula in terms of the η-function. Rouse and Webb's work in [14] shows that the space of weight 2 eta quotients in level 25 has dimension 0. However, the form may be generated in SAGE by using the input

```
ModularForms(Gamma0(25),prec=50,weight=2).echelon_basis()[4].
```

Additionally, it should be noted that $E^{(25)}(z)$ is perhaps related to the form

$$\frac{\eta^4(25z)\eta(z)}{\eta(5z)} + \frac{\eta^5(25z)}{\eta(5z)} = q^4 - q^6 + 2q^9 + 3q^{14} - 2q^{16} + 4q^{19} - 2q^{21} + 7q^{24} - \cdots,$$

which is a form in $M_2(25,\chi)$. Here χ is a character defined by $\chi(d) = \left(\frac{s}{d}\right)$ with $s = \prod_{\delta|25} \delta^{r_\delta}$ and r_δ the exponent on $\eta(\delta z)$. This form is the same as $E^{(25)}(z)$ except with negative coefficients when the exponent is 1 mod 5.

The weight 4 form needed in this construction is given by

$$S^{(25)}(z) := \frac{\eta^{10}(25z)}{\eta^2(5z)} = q^{10} + 2q^{15} + 5q^{20} + 10q^{25} + \cdots.$$

In this level 25 case, write $k = 4\ell + k'$ where $k' \in \{0,2\}$. The basis element $f^{(25)}_{k,m}(z)$ is then formed by taking a power of $S^{(25)}(z)$ and possibly $E^{(25)}(z)$ to get the correct weight and then using $\phi^{(25)}(z)$ as before to get the correct order of the pole at infinity. The basis $\{g^{(25)}_{k,m}(z)\}$ is constructed the same as before by forcing each basis element to vanish at all the cusps (except ∞).

Theorem 5.3. *Given the forms* $f^{(25)}_{k,m}(z) = q^{-m} + \sum_{n=10\ell+2k'} a^{(25)}_k(m,n)q^n$ *and* $g^{(25)}_{2-k,n}(z) = q^{-n} + \sum_{m=-10\ell-2k'-1} b^{(25)}_{2-k}(n,m)q^m$*, then for all* m,n

$$a^{(25)}_k(m,n) = -b^{(25)}_{2-k}(n,m).$$

Proof. The proof is the same as the proof for Theorem 2.5 with 25 in place of 8. □

Chapter 6. Relating $j(z)$ and $\phi^{(N)}(z)$

Now that we have constructed the bases and laid the groundwork, we turn to some results that relate the j-invariant $j(z)$ to Hauptmoduln of various levels. This will allow us to apply many of the congruences satisfied by $j(z)$ to the modular functions in these higher levels $\Gamma_0(N)$. We deal first with the case when N is a power of $p = 2$ (that is levels, $2, 4, 8$, and

16) before moving to powers of $p = 3$, 5, and 7. For the remainder of the thesis, we will let

$$\psi^{(N)}(z) = \frac{1}{\phi^{(N)}(z)} = q + \cdots,$$

for $N \in \{1, 2, 3, 4, 5, 7, 8, 9, 16, 25\}$.

6.1 $N = 2^a$

Since we will be working with level 4, we begin by including an explicit expression for a Hauptmodul in level 4 as found in [8]. It is given as follows:

$$\phi^{(4)}(z) = \frac{\theta^4(z)}{F(z)} = q^{-1} + 8 + 20q - 62q^3 + \cdots,$$

where

$$\theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots \in M_{1/2}(4)$$

and

$$F(z) = \sum_{\text{odd } n>0} \sigma(n) q^n = q + 4q^3 + 6q^5 + 8q^7 + \cdots \in M_2(4).$$

Note that $\sigma(n)$ is the sum of divisors function and is discussed further in Chapter 9.

Now, this first result gives us a convenient way to relate $j(z)$ and $\phi^{(2)}(z)$.

Proposition 6.1. *Let* $\phi^{(2)}(z) = \dfrac{\eta^{24}(z)}{\eta^{24}(2z)}$ *be a Hauptmodul for level 2 and let* $\psi^{(2)}(z) = \dfrac{1}{\phi^{(2)}(z)}$*. Let $j(z)$ be the standard j-function. Then for $k \in \mathbb{N}$,*

$$(j(z))^k = (\phi^{(2)}(z))^k (1 + 2^8 \psi^{(2)}(z))^{3k}.$$

Proof. Since we know the behavior at the cusps and the order of the poles, it is an easy check using the Sturm bound that $\dfrac{j(z)}{\phi^{(2)}(z)} = (1 + 2^8\psi^{(2)}(z))^3$. The result follows. □

With this proposition, we prove another result and give a corollary that follows with minimal additional work. The reader will note that although the construction of a canonical basis in level 1 or level p is not explicitly included in this thesis, we freely use the existence of these bases in our results.

Proposition 6.2. $f_{0,1}^{(1)}(z) \equiv f_{0,1}^{(2)}(z) \pmod{2^{16}}$ *and* $f_{0,1}^{(1)}(z) \equiv f_{0,1}^{(4)}(z) \pmod{2^8}$.

Proof. Proposition 6.1 with $k = 1$ gives us that $j(z) = \phi^{(2)}(z) + 768 + 196608\psi^{(2)}(z) + 16777216(\psi^{(2)}(z))^2$ and one can easily check that $\phi^{(2)}(z) = \phi^{(4)}(z) - 32 + 256\psi^{(4)}(z)$. Therefore we have the following series of equations:

$$\begin{aligned} j(z) &= \phi^{(2)}(z) + 768 + 196608\psi^{(2)}(z) + 16777216(\psi^{(2)}(z))^2 \\ j(z) - 744 &= \phi^{(2)}(z) + 24 + 196608\psi^{(2)}(z) + 16777216(\psi^{(2)}(z))^2 \\ f_{0,1}^{(1)}(z) &= f_{0,1}^{(2)}(z) + 196608\psi^{(2)}(z) + 16777216(\psi^{(2)}(z))^2 \\ f_{0,1}^{(1)}(z) &\equiv f_{0,1}^{(2)}(z) \pmod{2^{16}} \end{aligned}$$

The congruence in the last step follows because $196608 = 2^{16} \cdot 3$ and $16777216 = 2^{24}$.

Furthermore we find that

$$\begin{aligned} \phi^{(2)}(z) &= \phi^{(4)}(z) - 32 + 256\psi^{(4)}(z) \\ \phi^{(2)}(z) + 24 &= \phi^{(4)}(z) - 8 + 256\psi^{(4)}(z) \\ f_{0,1}^{(2)}(z) &= f_{0,1}^{(4)}(z) + 256\psi^{(4)}(z) \\ f_{0,1}^{(2)}(z) &\equiv f_{0,1}^{(4)}(z) \pmod{2^8} \end{aligned}$$

and the second result follows. □

Corollary 6.3. $f_{0,1}^{(1)}(z) \equiv f_{0,1}^{(8)}(z) \pmod{2^4}$ *and* $f_{0,1}^{(1)}(z) \equiv f_{0,1}^{(16)}(z) \pmod{2^2}$.

Proof. It is easy to compute that $f_{0,1}^{(4)}(z) = f_{0,1}^{(8)}(z) + \dfrac{16}{f_{0,1}^{(8)}(z)}$ and $f_{0,1}^{(8)}(z) = f_{0,1}^{(16)}(z) + \dfrac{4}{f_{0,1}^{(16)}(z)}$. The results then follow from the previous proposition. □

Proposition 6.2 gives us that the first basis elements ($m = 1$) for levels 1 and 2 are congruent modulo 2^{16}. It is natural to want to extend this, if possible, to arbitrary m. We will prove the following theorem.

Theorem 6.4. *Let $m > 0$. Then $f_{0,m}^{(1)}(z) \equiv f_{0,m}^{(2)}(z) \pmod{2^{16}}$.*

Before proving this theorem, it is helpful to include a specific example, say when $m = 3$. For purposes of the example, we will let $\phi = \phi^{(2)}(z)$ and $\psi = \dfrac{1}{\phi}$. Now, by construction, any basis element in level 1 is a polynomial in $j(z)$. Thus $f_{0,3}^{(1)}(z) = j^3 + aj^2 + bj + c$ for some $a, b, c \in \mathbb{Z}$. Using Proposition 6.1 above, this gives

$$f_{0,3}^{(1)}(z) = \phi^3(1 + 2^8\psi)^9 + a\phi^2(1 + 2^8\psi)^6 + b\phi(1 + 2^8\psi)^3 + c.$$

Noting that $\phi\psi = 1$ and using the binomial theorem, we may rewrite this last equation as

$$\begin{aligned} f_{0,3}^{(1)}(z) = {} & \phi^3 + \phi^2(9 \cdot 2^8 + a) + \phi(36 \cdot 2^{16} + 6 \cdot 2^8 a + b) \\ & + (84 \cdot 2^{24} + 15 \cdot 2^{16} a + 3 \cdot 2^8 b + c) + 2^{16}(\text{stuff}). \end{aligned} \tag{6.1}$$

The exact content of "stuff" is unimportant although we will note that it has no poles. More particularly, it is made up of linear combinations of modular forms that all have integer coefficients themselves; hence, all the coefficients are integers. Continuing forward and again referring to the method of construction of our canonical basis in level 2, we may write $f_{0,3}^{(2)}(z)$ as a polynomial in ϕ, say $f_{0,3}^{(2)}(z) = \phi^3 + a'\phi^2 + b'\phi + c'$. Subtract this last equation from equation 6.1. We get

$$\begin{aligned} f_{0,3}^{(1)}(z) - f_{0,3}^{(2)}(z) = {} & \phi^2(9 \cdot 2^8 + a - a') + \phi(36 \cdot 2^{16} + 6 \cdot 2^8 a + b - b') \\ & + (84 \cdot 2^{24} + 15 \cdot 2^{16} a + 3 \cdot 2^8 b + c - c') + 2^{16}(\text{stuff}) \end{aligned} \tag{6.2}$$

Notice however, that the left hand side of equation 6.2 is a modular function with only positive powers of q. The pole at infinity has been cancelled. That means that there can be no negative powers of q on the right hand side nor constants. In particular, the coefficients of ϕ^2 and ϕ must be 0 and $84 \cdot 2^{24} + 15 \cdot 2^{16} a + 3 \cdot 2^8 b + c - c'$ must also equal 0. That leaves only the 2^{16}(stuff) part. The result is that $f_{0,3}^{(1)}(z) \equiv f_{0,3}^{(2)}(z) \pmod{2^{16}}$. We now prove the theorem.

Proof of Theorem 6.4. We may write $f_{0,m}^{(1)}(z)$ as a polynomial in $j(z)$, say $f_{0,m}^{(1)}(z) = j^m + a_{m-1}j^{m-1} + \cdots + a_0$. Using Proposition 6.1, we may write this as $f_{0,m}^{(1)}(z) = \phi^m + A\phi^{m-1} + \cdots + B + 2^{16}(\text{stuff})$. Note that "stuff" has no poles and comes from integer multiples of forms with integer coefficients. Additionally, we can write $f_{0,m}^{(2)}(z) = \phi^m + a'_{m-1}\phi^{m-1} + \cdots + a'_0$. Subtracting the two equations gives a modular function with only positive powers of q. The coefficients of ϕ^2 and ϕ as well as the constant must all be 0. Thus we are left with $f_{0,m}^{(1)}(z) \equiv f_{0,m}^{(2)}(z) \pmod{2^{16}}$. $\square$

In a similar way, we get the following theorem.

Theorem 6.5. *Let $m > 0$. Then $f_{0,m}^{(1)}(z) \equiv f_{0,m}^{(4)}(z) \pmod{2^8}$.*

Proof Outline. This proof proceeds almost identically to the previous one. However, in this case, we use the result from Proposition 6.2 that $f_{0,1}^{(1)}(z) \equiv f_{0,1}^{(4)}(z) \pmod{2^8}$. Since $\phi^{(2)}(z) = \phi^{(4)}(z) - 32 + 256\psi^{(4)}$, we find that $(j(z))^k = (\phi^{(4)}(z) - 32 + 256\psi^{(4)})^k(1 + 2^8\psi^{(2)}(z))^{3k}$. The proof then proceeds in the same manner. We write $f_{0,m}^{(1)}(z)$ as a polynomial in $j(z)$ and use the above equation to write it in terms of $\phi^{(4)}(z)$. We find that there are a large number of terms that have a factor of 2^8. We then write $f_{0,m}^{(4)}(z)$ as a polynomial in $\phi^{(4)}(z)$. The difference of these two functions must have only positive powers of q. The result is that all terms with negative powers of q cancel with each other and we're left with only terms that have a factor of 2^8. The result follows. $\square$

Corollary 6.6. *If $m > 0$, then $f_{0,m}^{(1)}(z) \equiv f_{0,m}^{(8)}(z) \pmod{2^4}$ and $f_{0,m}^{(1)}(z) \equiv f_{0,m}^{(16)}(z) \pmod{2^2}$.*

Proof. See the proof of Theorem 6.4 and use Corollary 6.3. $\square$

We will come back to these results in Chapters 7 and 8 in order to prove some congruence results in higher levels.

6.2 $N = 3^a$

The results of this section are in many ways analogous to those from the previous section. We will use many of the same techniques. We begin with the following result.

Proposition 6.7. *$f_{0,1}^{(1)}(z) \equiv f_{0,1}^{(3)}(z) \pmod{3^9}$ and $f_{0,1}^{(1)}(z) \equiv f_{0,1}^{(9)}(z) \pmod{3^3}$.*

Proof. It is an easy check that $j(z) = \phi^{(3)}(z) + 756 + 196830\psi^{(3)}(z) + 19131876(\psi^{(3)}(z))^2 + 387420489(\psi^{(3)}(z))^3$ from which we get the following:

$$\begin{aligned}
j(z) &= \phi^{(3)}(z) + 756 + 196830\psi^{(3)}(z) + 19131876(\psi^{(3)}(z))^2 + 387420489(\psi^{(3)}(z))^3 \\
j(z) - 744 &= \phi^{(3)}(z) + 12 + 196830\psi^{(3)}(z) + 19131876(\psi^{(3)}(z))^2 + 387420489(\psi^{(3)}(z))^3 \\
f_{0,1}^{(1)}(z) &= f_{0,1}^{(3)}(z) + 196830\psi^{(3)}(z) + 19131876(\psi^{(3)}(z))^2 + 387420489(\psi^{(3)}(z))^3 \\
f_{0,1}^{(1)}(z) &\equiv f_{0,1}^{(3)} \pmod{3^9}.
\end{aligned}$$

Like before, the congruence follows by looking at the factorizations of the coefficients. In this case 196830, 19131876, and 387420489 are all divisible by 3^9.

For the second part, we use the following equation relating the Hauptmoduln from levels 3 and 9:

$$\phi^{(3)}(z) = \phi^{(9)}(z) + \frac{54\phi^{(9)}(z)}{X} + \frac{243}{X} - 9 \tag{6.3}$$

where $X = (\phi^{(9)}(z))^2 + 9\phi^{(9)}(z) + 27$. Then

$$f_{0,1}^{(3)}(z) = \phi^{(3)} + 12 = \phi^{(9)}(z) + 3 + \frac{54\phi^{(9)}(z)}{X} + \frac{243}{X}.$$

Since $f_{0,1}^{(9)}(z) = \phi^{(9)}(z) + 3$ and 27 divides both 54 and 243, the result follows. $\square$

We remark that the expression X in the previous proof arises out of forcing the right hand side of 6.3 to have the same values at the cusps of $\Gamma_0(9)$ as does $\phi^{(3)}(z)$. In particular, we know that the values of $\phi^{(9)}(z)$ at $\pm 1/3$ are $\frac{\sqrt{27}}{2}(-\sqrt{3} \mp i)$. The sum of these two values is -9 and their product is 27.

The above proposition yields the following two theorems:

Theorem 6.8. $f_{0,m}^{(1)}(z) \equiv f_{0,m}^{(3)}(z) \pmod{3^9}$.

Theorem 6.9. $f_{0,m}^{(1)}(z) \equiv f_{0,m}^{(9)}(z) \pmod{3^3}$.

The proofs for both of these theorems use Proposition 6.7 and follow the method of Theorem 6.4.

6.3 $N = 5^a$

Proposition 6.10. $f_{0,1}^{(1)}(z) \equiv f_{0,1}^{(5)}(z) \pmod{5^5}$ *and* $f_{0,1}^{(1)}(z) \equiv f_{0,1}^{(25)}(z) \pmod{5}$.

Proof. We begin with

$$\begin{aligned} j(z) = {} & \phi^{(5)}(z) + 750 + 196875\psi^{(5)}(z) + 20312500(\psi^{(5)}(z))^2 \\ & + 615234375(\psi^{(5)}(z))^3 + 7324218750(\psi^{(5)}(z))^4 + 30517578125(\psi^{(5)}(z))^5. \end{aligned}$$

Then a calculation similar to the one done in Theorems 6.2 and 6.7 gives that $f_{0,1}^{(1)}(z) \equiv f_{0,1}^{(5)}(z) \pmod{5^5}$ since 5^5 divides the coefficient of each positive power of $\psi^{(5)}(z)$.

The second part follows in a manner similar to part 2 above. In this case, we set

$$X = (\phi^{(5)}(z))^4 + 5(\phi^{(5)}(z))^3 + 15(\phi^{(5)}(z))^2 + 25\phi^{(5)}(z) + 25.$$

Again this arises from the values of $\phi^{(25)}(z)$ at the cusps $\pm\frac{1}{5}, \pm\frac{2}{5}$. The equation we need then is

$$\phi^{(5)}(z) = \phi^{(25)}(z) - 5 + \frac{10(\phi^{(25)}(z))^3}{X} + \frac{50(\phi^{(25)}(z))^2}{X} + \frac{100\phi^{(25)}(z)}{X} + \frac{125}{X}.$$

This yields

$$f_{0,1}^{(5)}(z) = f_{0,1}^{(25)}(z) + \frac{10(\phi^{(25)}(z))^3}{X} + \frac{50(\phi^{(25)}(z))^2}{X} + \frac{100\phi^{(25)}(z)}{X} + \frac{125}{X},$$

from which we conclude that $f_{0,1}^{(5)}(z) \equiv f_{0,1}^{(25)}(z) \pmod{5}$ and the result follows. $\square$

This proposition yields the following two theorems:

Theorem 6.11. $f_{0,m}^{(1)}(z) \equiv f_{0,m}^{(5)}(z) \pmod{5^5}$.

Theorem 6.12. $f_{0,m}^{(1)}(z) \equiv f_{0,m}^{(25)}(z) \pmod{5}$.

Again, we apply the above proposition with the methodology of Theorem 6.4.

6.4 $N = 7$

Proposition 6.13. $f_{0,1}^{(1)}(z) \equiv f_{0,1}^{(7)}(z) \pmod{7^4}$.

Proof. We make use of the following equation

$$\begin{aligned} j(z) = {} & \phi^{(7)}(z) + 748 + 196882\psi^{(7)}(z) + 20706224(\psi^{(7)}(z))^2 + 695893835(\psi^{(7)}(z))^3 \\ & + 10976181104(\psi^{(7)}(z))^4 + 90957030178(\psi^{(7)}(z))^5 \\ & + 38756041628(\psi^{(7)}(z))^6 + 678223072849(\psi^{(7)}(z))^7. \end{aligned}$$

An argument analogous to those found in previous propositions shows that $f_{0,1}^{(1)}(z) = f_{0,1}^{(7)}(z) +$ 7^4(stuff) and we have our result. □

We conclude with the following theorem, the proof of which is analogous to that of Theorem 6.4:

Theorem 6.14. $f_{0,m}^{(1)}(z) \equiv f_{0,m}^{(7)}(z) \pmod{7^4}$.

Chapter 7. Congruences in level p

One of the main purposes of this thesis is to prove congruence results for the Fourier coefficients of modular functions in higher levels. We have now established enough of a foundation to proceed with these results. This chapter will introduce some congruences for levels 2, 3, 5, and 7. We will use the theorems presented in Chapter 6 to lift many of Griffin's congruences cited in Chapter 1 to these higher levels. This will complement the work done already in these levels in [2] and [9]. Chapter 8 will deal with levels 4, 8, 9, 16, and 25.

7.1 Level 2

Theorem 7.1. *Let* $f_{0,m}^{(2)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_0^{(2)}(m,n)q^n$ *be an element of our canonical basis. Write* $m = 2^\alpha m'$ *and* $n = 2^\beta n'$ *where* $(m',2) = (n',2) = 1$*. Then the following congruences hold.*

$$\begin{aligned}
a_0^{(2)}(2^\alpha m', 2^\beta n') &\equiv -2^{11} m'\sigma_7(m')\sigma_7(n') \pmod{2^{16}}, \quad \textit{if } \alpha = \beta - 1,\\
&\equiv 20m'\sigma_7(m')\sigma_7(n') \pmod{2^7}, \quad \textit{if } \alpha = \beta, m'n' \equiv 1 \pmod 8,\\
&\equiv \frac{1}{2} m'\sigma(m')\sigma(n') \pmod{2^3}, \quad \textit{if } \alpha = \beta, m'n' \equiv 3 \pmod 8,\\
&\equiv -12m'\sigma_7(m')\sigma_7(n') \pmod{2^8}, \quad \textit{if } \alpha = \beta, m'n' \equiv 5 \pmod 8.
\end{aligned}$$

Proof. All of the moduli above divide 2^{16}. By Theorem 6.4, we know that $f_{0,m}^{(1)}(z) \equiv f_{0,m}^{(2)}(z) \pmod{2^{16}}$. Therefore, we may use this result in conjunction with Theorem 1.1. □

7.2 Level 3

Theorem 7.2. *Let* $f_{0,m}^{(3)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_0^{(3)}(m,n)q^n$ *be an element of our canonical basis. Write* $m = 3^\alpha m'$ *and* $n = 3^\beta n'$ *where* $(m',3) = (n',3) = 1$*. Then we have the following:*

$$\begin{aligned}
a_0^{(3)}(3^\alpha m', 3^\beta n') &\equiv \mp 3^5 \frac{\sigma(m')\sigma(n')}{n'} \pmod{3^8}, \quad \textit{if } \alpha = \beta - 1, m'n' \equiv \pm 1 \pmod 3,\\
&\equiv \mp 3^6 \frac{\sigma(m')\sigma(n')}{n'} \pmod{3^9}, \quad \textit{if } \alpha = \beta + 1, m'n' \equiv \pm 1 \pmod 3,\\
&\equiv \frac{2 \cdot 3^3 \sigma(m')\sigma(n')}{n'} \pmod{3^7}, \quad \textit{if } \alpha = \beta, m'n' \equiv 1 \pmod 3.
\end{aligned}$$

Proof. Theorem 6.8 proves that $f_{0,m}^{(1)}(z) \equiv f_{0,m}^{(3)}(z) \pmod{3^9}$. The result follows by applying this to Theorem 1.1. □

7.3 Level 5

Theorem 7.3. *Let* $f_{0,m}^{(5)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_0^{(5)}(m,n)q^n$ *be an element of our canonical basis. Write* $m = 5^\alpha m'$ *and* $n = 5^\beta n'$ *where* $(m',5) = (n',5) = 1$*. Then we have the following:*

$$\begin{aligned} a_0^{(5)}(5^\alpha m', 5^\beta n') &\equiv -5^{\beta-\alpha+1}3^{\beta-\alpha-1}(m')^2 n'\sigma(m')\sigma(n') \pmod{5^{\beta-\alpha+2}}, \quad \text{if } 0<\beta-\alpha\le 3,\\ &\equiv -5^3(m')^2n'\sigma(m')\sigma(n') \pmod{5^4}, \quad \text{if } \alpha=\beta+1,\\ &\equiv 10(m')^2n'\sigma(m')\sigma(n') \pmod{5^2}, \quad \text{if } \alpha=\beta, \left(\frac{m'n'}{5}\right) = -1. \end{aligned}$$

Proof. We apply Theorem 6.11 together with Theorem 1.1 and notice that all the moduli above are powers of 5 less than or equal to 5^5. $\square$

7.4 Level 7

Theorem 7.4. *Let* $f_{0,m}^{(7)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_0^{(7)}(m,n)q^n$ *be an element of our canonical basis. Write* $m = 7^\alpha m'$ *and* $n = 7^\beta n'$ *where* $(m',7) = (n',7) = 1$.

$$\begin{aligned} a_0^{(7)}(7^\alpha m', 7^\beta n') &\equiv 7^{\beta-\alpha+1}5^{\beta-\alpha-1}(m')^2 n'\sigma_3(m')\sigma_3(n') \pmod{7^{\beta-\alpha+1}}, \quad \text{if } 0<\beta-\alpha\le 3,\\ &\equiv 7^2(m')^2(n')\sigma_3(m')\sigma_3(n') \pmod{7^3}, \quad \text{if } \alpha=\beta+1,\\ &\equiv 2(m')^2n'\sigma_3(m')\sigma_3(n') \pmod{7}, \quad \text{if } \alpha=\beta, \left(\frac{m'n'}{7}\right) = 1. \end{aligned}$$

Proof. All of the above moduli are powers of 7 less than or equal to 7^4 (see Theorem 6.14). Therefore, we use this and Theorem 1.1 to get the result. $\square$

7.5 Remarks

The reader will notice that no congruences exist for the case when $m'n' \equiv -1 \pmod{p}$. Griffin discusses this phenomenon in [7]. Additionally, in the theorems above, no provision is given for congruences when $|\alpha-\beta|$ grows larger than some bound. This stems from the particular method begin used. Empirical evidence suggests that the congruences appear to be true even if this bound on $|\alpha - \beta|$ is removed (see Conjecture 5 in Chapter 9).

Chapter 8. Congruences in level N

Recall some notation from our theorems. Given a form $f_{0,m}^{(N)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_0^{(N)}(m,n)q^n$, we write $m = p^\alpha m'$ and $n = p^\beta n'$ where p is the prime dividing N and m', n' are relatively prime to p. The results of this chapter will combine the theorems in Chapter 7 with the results in Chapters 2, 3, 4, and 5.

8.1 $p = 2$

Theorems 1.6, 2.4, and 4.3 prove that the U_2 and V_2 operators take basis elements to basis elements across levels ($N = 2, 4, 8, 16$). This fact gives us the following theorems:

Theorem 8.1. *Assume that $\alpha = \beta$ in level 4. Then*

$$\begin{aligned} a_0^{(4)}(2^\alpha m', 2^\beta n') &\equiv 20m'\sigma_7(m')\sigma_7(n') && \pmod{2^7}, \quad \text{if } m'n' \equiv 1 \pmod 8, \\ &\equiv \frac{1}{2}m'\sigma(m')\sigma(n') && \pmod{2^3}, \quad \text{if } m'n' \equiv 3 \pmod 8, \\ &\equiv -12m'\sigma_7(m')\sigma_7(n') && \pmod{2^8}, \quad \text{if } m'n' \equiv 5 \pmod 8. \end{aligned}$$

Proof. Since all the moduli are powers of 2 less than or equal to 2^8, Theorem 6.5 together with Theorem 1.1 gives us the result. $\square$

Theorem 8.2. *Assume that $\alpha = \beta - 1$ in level 4. Then*

$$a_0^{(4)}(2^\alpha m', 2^\beta n') \equiv -2^{11}m'\sigma_7(m')\sigma_7(n') \pmod{2^{16}}.$$

Proof. We apply the V_2 operator to the first congruence result of Theorem 7.1. $\square$

Now, with the V_2 operator and Corollary 6.6 we get the following theorems:

Theorem 8.3. *Assume that $\alpha \neq \beta$ in level 8. Then*

$$a_0^{(8)}(2^\alpha m', 2^\beta n') \equiv \begin{cases} 0 \pmod{2^{3(\beta-\alpha)+8}} \text{ if } \beta > \alpha, \\ 0 \pmod{2^{4(\alpha-\beta)+8}} \text{ if } \alpha > \beta. \end{cases}$$

Additionally, if $\alpha = \beta - 1$, then

$$a_0^{(8)}(2^\alpha m', 2^\beta n') \equiv -2^{11} m'\sigma_7(m')\sigma_7(n') \pmod{2^{16}}.$$

Proof. We apply the V_2 operator to basis elements in level 4 and use Theorems 1.7 and 8.2 □

Theorem 8.4. *Assume that $\alpha = \beta \neq 0$ in level 8. Then*

$$\begin{aligned} a_0^{(8)}(2^\alpha m', 2^\beta n') &\equiv 20m'\sigma_7(m')\sigma_7(n') && \pmod{2^7}, && \text{if } m'n' \equiv 1 \pmod 8, \\ &\equiv \frac{1}{2}m'\sigma(m')\sigma(n') && \pmod{2^3}, && \text{if } m'n' \equiv 3 \pmod 8, \\ &\equiv -12m'\sigma_7(m')\sigma_7(n') && \pmod{2^8}, && \text{if } m'n' \equiv 5 \pmod 8. \end{aligned}$$

Proof. This follows by applying what we know about the V_2 operator. □

Theorem 8.5. *Assume that $\alpha = \beta = 0$ in level 8 and that $m'n' \equiv 3 \pmod 8$. Then*

$$a_0^{(8)}(m, n) \equiv \frac{1}{2}m'\sigma(m')\sigma(n') \pmod{2^3}.$$

Proof. This result holds because $2^3 | 2^4$ (see Corollary 6.6). □

Theorem 8.6. *Assume that $\alpha \neq \beta$ in level 16. Then*

$$a_0^{(16)}(2^\alpha m', 2^\beta n') \equiv \begin{cases} 0 \pmod{2^{3(\beta-\alpha)+8}} \text{ if } \beta > \alpha, \\ 0 \pmod{2^{4(\alpha-\beta)+8}} \text{ if } \alpha > \beta. \end{cases}$$

Additionally, if $\alpha = \beta - 1$, then

$$a_0^{(16)}(2^\alpha m', 2^\beta n') \equiv -2^{11} m'\sigma_7(m')\sigma_7(n') \pmod{2^{16}}.$$

Proof. We use the V_2 operator in conjunction with Theorem 8.3 above. $\square$

Theorem 8.7. *Assume that $\alpha = \beta > 1$ in level 16. Then*

$$\begin{aligned}
a_0^{(16)}(2^\alpha m', 2^\beta n') &\equiv 20m'\sigma_7(m')\sigma_7(n') &&\pmod{2^7}, &&\text{if } m'n' \equiv 1 \pmod 8,\\
&\equiv \frac{1}{2}m'\sigma(m')\sigma(n') &&\pmod{2^3}, &&\text{if } m'n' \equiv 3 \pmod 8,\\
&\equiv -12m'\sigma_7(m')\sigma_7(n') &&\pmod{2^8}, &&\text{if } m'n' \equiv 5 \pmod 8.
\end{aligned}$$

Proof. Again, we use the appropriate theorem from level 8 (Theorem 8.4) and apply the V_2 operator. $\square$

Theorem 8.8. *Assume that $\alpha = \beta = 1$ in level 16 and that $m'n' \equiv 3 \pmod 8$. Then*

$$a_0^{(16)}(m, n) \equiv \frac{1}{2}m'\sigma(m')\sigma(n') \pmod{2^3}.$$

Proof. This comes straight from Theorem 8.5 together with the V_2 operator. $\square$

8.1.1 Summary

It is convenient at this point to summarize the known results for level $N = 2, 4, 8, 16$.

Level 2. Andersen and Jenkins proved their results in [2] for level 2 coefficients where $\alpha < \beta$. The reverse situation ($\alpha > \beta$) was proven in [9] by Jenkins and the author of this thesis. In Section 7.1, we proved congruences when $\alpha = \beta$ and $m'n' \not\equiv -1 \pmod 8$. We also proved that in the particular case when $\alpha = \beta - 1$, we can strengthen the divisibility result to a congruence.

Level 4. When $m = 2\ell$ is even, $f_{0,m}^{(4)}(z) = V_2(f_{0,\ell}^{(2)}(z))$ and we have the results from level 2. If m is odd, then so are all the powers of q in the series expansion (see Theorem 1.5). Therefore if m is odd, $\alpha = \beta = 0$. Congruences in this case are given above in Theorem 8.1.

Level 8. When $m = 2\ell$ is even, $f_{0,m}^{(8)}(z) = V_2(f_{0,\ell}^{(4)}(z))$ and we get to use all the results from level 4. If m is odd, then so is n (i.e. $\alpha = \beta$). In this case, we have congruence results if $m'n' \equiv 3 \pmod 8$ as given in Theorem 8.4.

Level 16. When $m = 4\ell$ is divisible by 4, then $f_{0,m}^{(16)}(z) = V_2(V_2(f_{0,\ell}^{(4)}(z)))$ and these congruences are already resolved. When $m = 4\ell + 2$ and $m'n' \equiv 12 \pmod{32}$, we have the result from level 8. No congruences have been proven in any other case when $\alpha = \beta$.

8.2 $p = 3$

We have the following theorems, which all come as a result of the V_3 operator.

Theorem 8.9. *Assume that $\alpha \neq \beta$ in level 9. Then*

$$a_0^{(9)}(3^\alpha m', 3^\beta n') \equiv \begin{cases} 0 \pmod{3^{2(\beta-\alpha)+3}} \text{ if } \beta > \alpha, \\ 0 \pmod{3^{3(\alpha-\beta)+3}} \text{ if } \alpha > \beta. \end{cases}$$

If $\alpha = \beta \pm 1$ then we have the following stronger congruences

$$\begin{aligned} a_0^{(9)}(3^\alpha m', 3^\beta n') &\equiv \mp 3^5 \frac{\sigma(m')\sigma(n')}{n'} \pmod{3^8}, \quad \textit{if } \alpha = \beta - 1, m'n' \equiv \pm 1 \pmod 3, \\ &\equiv \mp 3^6 \frac{\sigma(m')\sigma(n')}{n'} \pmod{3^9}, \quad \textit{if } \alpha = \beta + 1, m'n' \equiv \pm 1 \pmod 3. \end{aligned}$$

Proof. The first result follows by applying the V_3 operator to the corresponding level 3 results in [2] and [9]. The second part comes from Theorem 7.2, still applying V_3. □

Theorem 8.10. *Assume that $\alpha = \beta$ in level 9 and that $m'n' \equiv 1 \pmod 3$. Then*

$$a_0^{(9)}(3^\alpha m', 3^\beta n') \equiv \frac{2 \cdot 3^3 \sigma(m')\sigma(n')}{n'} \pmod{3^7}.$$

Proof. Apply the V_3 operator to Theorem 7.2. □

8.2.1 Summary

In summary, we have the following for levels 3 and 9.

Level 3. Andersen and Jenkins proved their results in [2] for level 3 coefficients where $\alpha < \beta$. The reverse situation ($\alpha > \beta$) was proven in [9] by Jenkins and the author. In Section 7.2, we proved congruences when $\alpha = \beta$ and $m'n' \not\equiv -1 \pmod{3}$. Additionally, we strengthened divisibility to congruences when $\alpha = \beta \pm 1$.

Level 9. When $m = 3\ell$, we know that $f_{0,m}^{(9)}(z) = V_3(f_{0,\ell}^{(3)}(z))$ so that the results from level 3 come up automatically. When $m = 3\ell+1$ or $3\ell+2$, then $\alpha = \beta$ and it turns out that $m'n' \equiv -1 \pmod{3}$ always. This is because when $3 \nmid m$, $m = m'$ and $n = n'$. Furthermore, $-m \equiv n \pmod{3}$. Since $3 \nmid m$ and $3 \nmid n$, it must be that one of m and n is $1 \pmod{3}$ and the other is $-1 \pmod{3}$. Therefore, there currently exist no congruence results to try and pull up to level 9.

8.3 $p = 5$

As in the previous section, we will use Theorem 5.2 to adapt the results in Section 7.3 to level 25.

Theorem 8.11. *Assume that $\alpha \neq \beta$ in level 25. Then*

$$a_0^{(25)}(5^\alpha m', 5^\beta n') \equiv \begin{cases} 0 \pmod{5^{\beta-\alpha+1}} \text{ if } \beta > \alpha, \\ 0 \pmod{5^{2(\alpha-\beta)+1}} \text{ if } \alpha > \beta. \end{cases}$$

If $0 < \beta - \alpha \leq 3$ or $\alpha = \beta + 1$ then we have the following stronger congruences

$$\begin{aligned} a_0^{(5)}(5^\alpha m', 5^\beta n') &\equiv -5^{\beta-\alpha+1}3^{\beta-\alpha-1}(m')^2 n'\sigma(m')\sigma(n') \pmod{5^{\beta-\alpha+2}}, \quad \textit{if } 0 < \beta - \alpha \leq 3, \\ &\equiv -5^3(m')^2 n'\sigma(m')\sigma(n') \pmod{5^4}, \quad \textit{if } \alpha = \beta + 1. \end{aligned}$$

Proof. The first result follows by applying the V_5 operator to the corresponding level 3 results in [2] and [9]. The second part comes from V_5 applied to Theorem 7.3. $\square$

Theorem 8.12. *Assume that $\alpha = \beta$ in level 25 and that $\left(\frac{m'n'}{5}\right) = -1$. Then*

$$a_0^{(25)}(5^\alpha m', 5^\beta n') \equiv 10(m')^2 n' \sigma(m') \sigma(n') \pmod{5^2}.$$

Proof. Apply the V_5 operator to Theorem 7.3. $\square$

8.3.1 Summary

Level 5. Andersen and Jenkins proved their results in [2] for level 5 coefficients where $\alpha < \beta$. Jenkins and the author proved the reverse situation ($\alpha > \beta$) in [9]. In Section 7.3, we proved congruences when $\alpha = \beta$ and $\left(\frac{m'n'}{5}\right) = -1$. In the case where $0 < \beta - \alpha \leq 3$ or $\alpha = \beta + 1$, we proved congruence results.

Level 25. When $m = 5\ell$, $f_{0,m}^{(25)}(z) = V_5(f_{0,\ell}^{(5)}(z))$ so we may apply the results from level 5. In all other cases, $\alpha = \beta$ (so that $m = m'$ and $n = n'$) and $\left(\frac{mn}{5}\right) = 1$. Like the problem encountered with level 9 above, there are currently no known congruence results to try and pull up to level 25.

Chapter 9. Conjectures

Throughout the process of researching these modular forms, there arose certain patterns and other nuances that seem too prevalent to be coincidence. This chapter will briefly discuss some conjectures that are supported by empirical evidence and/or other intuition. We first recall that Conjecture 1 was included in Chapter 5.

In general, it should be the case that the moduli of some (if not most or all) of the divisibility results included in this thesis can be improved. More particularly, this improvement appears to be entirely related to the presence of the sum of divisors function, $\sigma_k(n)$ in the congruences. The following conjecture attempts to strengthen some of the congruences to exact divisibility. This is a phenomenon that is mentioned by Griffin at the end of [7] and tested in a variety of cases by the author.

Conjecture 2. *Write* $m = 2^\alpha m'$. *Then* $f_{0,m}^{(1)}(z) \equiv f_{0,m}^{(2)}(z) \pmod{2^{4(\alpha+4)+v_2(\sigma(m'))}}$.

This improves the modulus of Theorem 6.4, making it dependent on the value of m chosen. We can, in fact, be more explicit with Conjecture 2 by focusing on a particular n, and it appears that the following is probably true:

Conjecture 3. *Write* $m = 2^\alpha m'$ *and* $n = 2^\beta n'$ *as usual. Then the difference* $a_0^{(1)}(m,n) - a_0^{(2)}(m,n)$ *is exactly divisible by* $2^{4(\alpha+4)+v_2(\sigma(m'))+3\beta+v_2(\sigma(n'))}$.

Conjecture 4. $v_2(a_0^{(2)}(2^\alpha m', 2^\beta n')) = 3(\beta-\alpha) + 8 + v_2(\sigma(m')\sigma(n'))$ *if* $\beta > \alpha$. *Otherwise if* $\alpha > \beta$, *replace* $3(\beta - \alpha)$ *with* $4(\alpha - \beta)$.

Partial Proof. This is true if $v_2(\sigma(m')\sigma(n')) < 5$ (use the congruences). □

In essence, these conjectures state that the extra powers of 2 that arise in the factorizations of the coefficients come from the sigma functions in the congruences. The utility of this conjecture (if proven true) is as follows: given any m and n, one could immediately know the exact power of 2 that divides $a_0^{(2)}(m,n)$. For example, if $m = 71$ and $n = 46$, it's known that $2^{11} | a_0^{(2)}(71, 46)$, but this conjecture claims that $2^{17} || a_0^{(2)}(71, 46)$ since $\sigma(71) = 72$ and $\sigma(23) = 24$ are both divisible by 2^3. And indeed, $a_0^{(2)}(71, 46)$ is a 155-digit integer exactly divisible by 2^{17}.

If Conjectures 2, 3, and 4 turn out to be true, it is reasonable to expect that analogous results could be developed for higher levels, at least in the case when $N = 2^a$.

With respect to Theorems 7.1 - 7.4, empirical evidence suggests restrictions on the size of $|\alpha - \beta|$ are unnecessary. Therefore the following should be true:

Conjecture 5. *Theorems 7.1, 7.2, 7.3, and 7.4 are true for any* α *and any* β. *In other words, the results from Theorem 1.1 should be exactly true in the appropriate level.*

Bibliography

[1] Hans-Fredrik Aas, *Congruences for the coefficients of the modular invariant $j(\tau)$*, Math. Scand. **14** (1964), 185–192. MR 0179137 (31 #3388)

[2] Nickolas Andersen and Paul Jenkins, *Divisibility properties of coefficients of level p modular functions for genus zero primes*, Proc. Amer. Math. Soc. **141** (2013), no. 1, 41–53. MR 2988709

[3] A. O. L. Atkin and J. Lehner, *Hecke operators on $\Gamma_0(m)$*, Math. Ann. **185** (1970), 134–160. MR 0268123 (42 #3022)

[4] W. Duke and Paul Jenkins, *On the zeros and coefficients of certain weakly holomorphic modular forms*, Pure Appl. Math. Q. **4** (2008), no. 4, Special Issue: In honor of Jean-Pierre Serre. Part 1, 1327–1340. MR 2441704 (2010a:11068)

[5] Ahmad El-Guindy, *Fourier expansions with modular form coefficients*, Int. J. Number Theory **5** (2009), no. 8, 1433–1446. MR 2582984 (2011i:11066)

[6] Sharon Garthwaite and Paul Jenkins, *Zeros of weakly holomorphic modular forms of levels 2 and 3*, Math. Res. Lett. **20** (2013), no. 4, 657–674.

[7] Michael Griffin, *Divisibility properties of coefficients of weight 0 weakly holomorphic modular forms*, Int. J. Number Theory **7** (2011), no. 4, 933–941. MR 2812644 (2012i:11048)

[8] Andrew Haddock and Paul Jenkins, *Zeros of weakly holomorphic modular forms of level 4*, Int. J. Number Theory **10** (2014), no. 2, 455–470.

[9] Paul Jenkins and DJ Thornton, *Congruences for coefficients of modular functions*, Ramanujan Journal **38** (2015), no. 3, 619–628.

[10] O. Kolberg, *The coefficients of $j(\tau)$ modulo powers of* 3, Arbok Univ. Bergen Mat.-Natur. Ser. **1962** (1962), no. 16, 7. MR 0158061 (28 #1288)

[11] ———, *Congruences for the coefficients of the modular invariant $j(\tau)$.*, Math. Scand. **10** (1962), 173–181. MR 0143735 (26 #1287)

[12] Joseph Lehner, *Divisibility properties of the Fourier coefficients of the modular invariant $j(\tau)$*, Amer. J. Math. **71** (1949), 136–148. MR 0027801 (10,357a)

[13] ———, *Further congruence properties of the Fourier coefficients of the modular invariant $j(\tau)$*, Amer. J. Math. **71** (1949), 373–386. MR 0027802 (10,357b)

[14] Jeremy Rouse and John J. Webb, *On spaces of modular forms spanned by eta-quotients*, Advances in Mathematics **272** (2015), 200–224.