Weak Cayley Table Groups of Wallpaper Groups

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Weak Cayley Table Groups of Wallpaper Groups

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Let $G$ be a group. A Weak Cayley Table mapping $\varphi : G \to G$ is a bijection such that $\varphi(g_1g_2)$ is conjugate to $\varphi(g_1)\varphi(g_2)$ for all $g_1, g_2$ in $G$. The set of all such mappings forms a group $W(G)$ under composition. We study $W(G)$ for the seventeen wallpaper groups $G$.

Keywords: wallpaper groups, weak cayley table isomorphisms
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Chapter 1. Introduction

Let $G$ and $H$ be groups. As defined in [JMS, JS, Hu], a weak Cayley table morphism (or weak Cayley table morphism) is a map $\varphi : G \to H$ such that $\varphi(g)\varphi(g')$ is conjugate to $\varphi(gg')$ for all $g, g' \in G$. A weak Cayley table isomorphism is a bijective weak Cayley table morphism.

In the situation where $G = H$, $\varphi$ is a generalization of an automorphism. The set of all such mappings $\varphi : G \to G$ forms the weak Cayley table group $\mathcal{W}(G) = \mathcal{W}$ under composition and we note that the automorphism group $\text{Aut}(G)$ of $G$ is a subgroup of $\mathcal{W}(G)$. The inverse map $\iota : g \mapsto g^{-1}$ is also an element in $\mathcal{W}$. We define $\mathcal{W}_0 = \langle \text{Aut}(G), \iota \rangle$, and if $\mathcal{W} = \mathcal{W}_0$ we say $\mathcal{W}(G)$ is trivial. Given a group $G$ one would like to determine whether or not $\mathcal{W}(G)$ is trivial and to find the non-trivial mappings in $\mathcal{W}\setminus\mathcal{W}_0$. Previous research has determined the weak Cayley table groups of various finite groups. For example, dihedral groups, symmetric groups, most finite irreducible Coxeter groups, most alternating groups, and $\text{PSL}(2, p^n)$ all have trivial weak Cayley table groups [Hu, HN, HN2]. Our goal has been to determine the weak Cayley table groups of certain infinite groups, namely the seventeen wallpaper groups.

We have found that thirteen wallpaper groups have trivial weak Cayley table groups while four have non-trivial weak Cayley table groups. For these four wallpaper groups we have completely determined $\mathcal{W}(G)$, its presentation and its structure.

A wallpaper group $G$ consists of discrete isometries of a pattern that tiles the Euclidean plane. These isometries form a group under composition. In 1891 Evgraf Fedorov proved that there were exactly seventeen such groups, up to isomorphism [F]. Elements of a wallpaper group are classified as translations, rotations, reflections or glide reflections. Wallpaper groups are two-dimensional crystallographic groups.

Our main result shows that with one exception, groups having glide reflections or reflections have trivial weak Cayley table groups, while groups having only translations and rotations of order at least three have non-trivial weak Cayley table groups. There are three such groups, and they are of the form $G = A \rtimes F$ where $A$ is the abelian subgroup of trans-
lations and $F$ is a cyclic group of order three, four or six. In the latter two cases, $\mathcal{W}(G)$ can be written as a semidirect product $\mathcal{W}(G) = N \rtimes \mathcal{W}_0$ where every nonidentity element of $N$ is a nontrivial weak Cayley table map.
Chapter 2. Wallpaper groups

The group formed by all bijections that map lines to lines in Euclidean $n$–space $\mathbb{E}^n$ is known as the **affine group**, $A(n)$. The subgroup of transformations that map the origin to the origin is the **similarity group**, $S(n)$. The subgroup of the similarity group consisting of all maps that preserve distances in $\mathbb{E}^n$, called **isometries**, is known as the Euclidean group $E(n)$. One type of isometry, known as a translation, has the form $v \mapsto v + a$ for some fixed $a \in \mathbb{E}^n$. **Crystallographic groups** are discrete subgroups of $E(n)$ that contain a finite index subgroup $A$ of translations isomorphic to $\mathbb{Z}^n$ such that $\mathbb{E}^n/A$ is compact. Such a subgroup is called a lattice. In the $n = 3$ case we may think of the isometries of $\mathbb{E}^3$ as symmetries of the arrangement of atoms in a crystal. This gives us a three dimensional crystallographic group, also known as a space group. There are 219 three dimensional space groups. A thorough enumeration and description of them can be found in [CFHT]. When we have $n = 2$, and thus translation subgroup $A \cong \mathbb{Z}^2$, we have a plane crystallographic group, also known as a wallpaper group. In this case, we can think of the isometries in the group as symmetries of a repetitive two-dimensional pattern, such as a wallpaper pattern.

In addition to translations, there are three other types of group elements in a crystallographic group. Let $\sigma$ be a reflection across a line $\ell$ that contains the origin. Then $\sigma$ maps $v \in \mathbb{E}^2$ to $2(v \cdot u)u - v$ where $u$ is a unit vector in $\ell$. Note that for any vector $w \in \ell$, $(w \cdot u)u = w$ and so $\sigma(w) = 2(w \cdot u)u - w = w$; thus we see that $\sigma$ fixes any vector in the line $\ell$, its axis of reflection. Reflections are orientation reversing involutions. The other type of orientation reversing isometry is a glide reflection. A glide reflection $\gamma$ reflects across a line $\ell$ and then translates in a direction parallel to $\ell$. Thus, $\gamma : v \mapsto (2/|w|)(v \cdot w)w + w - v$ for some nonzero $w \in \ell$. Notice that $\gamma$ fixes $\ell$ but does not fix vectors in $\ell$. Also note that $\gamma^2(v) = v + 2w$ is a translation. Thus glide reflections, unlike reflections, do not have finite order. The product of two reflections with intersecting axes gives a fourth type of isometry. A rotation $\rho$ is a finite order, orientation preserving isometry that fixes one point. If $\rho \neq Id$
then this point, its center of rotation, is unique. Suppose \( \rho \) is rotation about the origin with turning angle \( \theta \). Then \( \rho(v) = w \) implies that \( |v| = |w| \) and \( \theta \) is the angle from \( v \) to \( w \). The crystallographic restriction is a well known fact [L, p. 63] that rotations in a two or three dimensional crystallographic group can only have order 1, 2, 3, 4 or 6. One consequence of this is that there are only seventeen wallpaper groups up to isomorphism. We will use Herrman-Maughn notation to name them.

We think of a wallpaper pattern corresponding to a wallpaper group \( G \) as a subset of \( \mathbb{E}^2 \). The action of the group elements on \( \mathbb{E}^2 \) leaves the pattern unchanged. For \( a \in A \), let \( v_a \) be a vector in \( \mathbb{E}^2 \) that corresponds to the translation action of \( a \). Now relative to the metric on \( \mathbb{E}^2 \) the vector \( v_a \) has length \( |a| \). The translations in \( A \) that are conjugate to \( a \) form the set \( a^G \) and they are vectors in \( \mathbb{E}^2 \) of length \( |a| \). We define a ball \( B_r \) of radius \( r = |a| \) centered at \((0, 0)\) in \( \mathbb{E}^2 \). The subgroup \( A \) is generated by translations \( x \) and \( y \), and thus there exist vectors \( v_x, v_y \in \mathbb{E}^2 \) that span a lattice \( \mathcal{L} \subset \mathbb{E}^2 \). It will be useful to identify \( A \) with the lattice \( \mathcal{L} \): \( x^iy^j \) corresponds to \( iv_x + jv_y \).

Suppose we have two wallpaper groups \( G_1, G_2 \) and suppose \( G_1 \) has generating translations \( x, y \) corresponding to vectors \( v_x, v_y \in \mathbb{E}^2 \). Then \( G_1 \cong G_2 \) if and only if there exists \( \alpha \in A \) such that \( G_1 = G_2^\alpha \) [L, p. 62]. In this situation, \( \alpha : (v_x, v_y) \mapsto (u_x, u_y) \) where \( u_x, u_y \) correspond to generating translations \( x, y \in G_2 \). In other words, if we can obtain one wallpaper group from another via a change in rectangular coordinates, then they are isomorphic.

The translation subgroup \( A \) is a finite index normal subgroup \( A \cong \mathbb{Z}^2 \). We will write \( A = \langle x, y \rangle \) and we may think of \( x \) as horizontal translation (to the right) and \( y \) as either vertical translation upwards or translation at an angle of \( \pi/3 \) from the horizontal (depending on the group.) The other generators of \( G \) will be denoted \( \rho \) (for a rotation), \( \sigma \) (for a reflection) and \( \gamma \) (for a glide reflection).

There is exactly one wallpaper group, \( p1 \), with \( G = A \). Since conjugacy implies equality in an abelian group, weak Cayley table isomorphisms of abelian groups are homomorphisms and so abelian groups always have trivial weak Cayley table groups. We will focus on the
remaining sixteen groups.

Let \( n \in \{1, 2, 3, 4, 6\} \) be the order of the highest order rotation in the group. Each coset in \( G/A \) is one of three types. Cosets of the form \( A\rho^k \) (for \( 0 < k < n \in \mathbb{Z} \)) we will call a rotation coset. Every element in a rotation coset is a rotation with order equal to the order of \( \rho^k \). Cosets of the form \( A\rho^k\sigma \) or \( A\rho^k\gamma \) contain no rotations but always contain glide reflections which have infinite order. If \( A\rho^k\sigma \) contains reflections as well, then we will call it a reflection coset. Otherwise, we call it a glide reflection coset. This distinction is important because reflection cosets contain elements of order two, but glide reflections do not.

Let \( F' = G/A \) and let \( F \) be a set of coset representatives for the cosets in \( F' \). We will choose 1 as the coset representative of \( A \). In twelve wallpaper groups, \( F \) can be chosen so that it is a subgroup isomorphic to \( F' \), and so these groups are semidirect products \( G = A \rtimes F \). When \( G \) does not contain reflections or glide reflections then \( G/A \) contains only rotation cosets and \( F \) is a cyclic group of order \( n \), which we will denote \( C_n \). There are four such wallpaper groups. When \( G \) contains both reflections and glide reflections and \( G/A \) does not contain any glide reflection cosets, then \( F \) is a dihedral group, \( D_{2n} \). There are eight such groups. When \( G/A \) has at least one glide reflection coset, then the coset representative for that coset must be a glide reflection. Recall that glide reflections do not have finite order; consequently, \( F \) is not a subgroup and \( G \) is not a semidirect product in these cases. There are four groups of this type and they satisfy \( F' \cong D_{2n} \). Thus all wallpaper groups \( G \) fit in the short exact sequence

\[
1 \rightarrow A \rightarrow G \rightarrow F' \rightarrow 1
\]

where \( F' \) is either cyclic or dihedral. Note that in most cases, \( F' \) does not uniquely determine the group \( G \). For example, there are four groups with \( F' \cong C_2 \) and four groups with \( F' \cong (C_2 \times C_2) \).

For the four types of groups that are not semidirect products, we will find it useful to assign a variable to the square of each glide reflection in \( F \).

For a group of type \( p1g1 \) let \( \beta_1 = \gamma^2 = x \).
For a group of type $p2mg$ let $\beta_1 = \sigma^2 = 1$; $\beta_2 = (\rho \sigma)^2 = y$.

For a group of type $p2gg$ let $\beta_1 = \gamma^2 = x$; $\beta_2 = (\rho \gamma)^2 = y$.

For a group of type $p4mg$ let $\beta_1 = \gamma^2 = x$; $\beta_2 = (\rho^2 \gamma)^2 = y$.

For any other group let $\beta_1 = \beta_2 = 1$. For the general case we will write a reflection or glide reflection in $F$ as $r$ and its square as $\beta$.

Now consider the conjugation action of elements of $G$ on elements in $A$. For any nontrivial rotation generator $\rho \in F$ the action of $\rho$ fixes only the identity. However for any reflection or glide reflection $r \in F$, the action of $r$ fixes a cyclic subgroup of $A$, its line of reflection, which we denote $L(r)$.

$$L(r) \leq A, L(r) = \{a \in A|a^r = a \text{ or } \langle a, a^r \rangle \cong \mathbb{Z}\}.$$ 

The line perpendicular to $L(r)$ meets $A$ in a cyclic subgroup that we denote by $L^\perp(r)$. Then we have:

**Lemma 2.1.** Suppose $r \in G$ is a reflection or a glide reflection. Let $r^2 = \beta$. Then for $a \in A$ we have $(ar)^2 = \beta$ if and only if $a \in L^\perp(r)$. In particular, if $\sigma \in F$ is a reflection, then $(a\sigma)^2 = 1$ if and only if $a \in L^\perp(\sigma)$.

**Proof.** Since $r$ is a reflection or glide reflection we see that $(ar)^2 \in A$. Then we have $(ar)^2 = \beta$ if and only if $ar \cdot rr^{-1} \cdot ar = ar^2 \cdot a^r = a\beta a^r = \beta$ if and only if $aa^r = 1$ if and only if $a \in L^\perp(r)$. 

An arbitrary reflection or glide reflection in $G$ is of the form $ar$ for $a \in A$, $r \in F$ a reflection or glide reflection. Note that if $ar$ is conjugate to $br$ then $(ar)^2$ is conjugate to $(br)^2$. Now the square of this group element can be written as $(ar)^2 = arar = a\beta r^{-1}ar = aa^r \beta$. Note that $aa^r \in \langle x^2, y^2 \rangle$. We can see that $(ar)^2 \in L(r)$, since $((ar)^2)^r = (aa^r \beta)^r = a^r a\beta = (ar)^2$.

We have proven

**Proposition 2.2.** (i) If two glide reflections are conjugate, then their squares are conjugate translations.
(ii) There are two distinct types of glide reflections. In groups where we have a reflection \( \sigma \in F \) with \( \sigma^2 = \beta = 1 \) we have \( (a\sigma)^2 \in L(\sigma) \cap \langle x^2, y^2 \rangle \). In groups where we have a glide reflection \( \gamma \in F \) such that \( \gamma^2 = \beta \notin \langle x^2, y^2 \rangle \) we have \( (a\gamma)^2 \in L(\gamma) \setminus \langle x^2, y^2 \rangle \).

Now for any wallpaper group \( G \) we define

\[
H = H(G) = \{ a \in A : |a^G| \neq |F| \}.
\]

Thus \( H \) consists of the translations that correspond to the union of all axes of symmetry for reflections and glide reflections in \( F \). Note that \( H \) is trivial for all groups that contain only rotations and translations.

We will write the commutator \( (g,h) = g^{-1}h^{-1}gh \). Let \( \rho_\theta \) be rotation by angle \( \theta \) about the origin. For \( f \in F \), we define

\[
K_f = \{(a,f) : a \in A\}.
\]

**Lemma 2.3.** Let \( t \in F \).

(i) \( K_t = \langle (x,t), (y,t) \rangle \leq A \).

(ii) \( [G : K_t] \) is finite if and only if \( t \) is a non-trivial rotation. Also \( K_t = \{1\} \) if and only if \( t = Id \).

(iii) \( K_t \) is a non-trivial cyclic group if and only if \( t \) is a reflection or glide reflection. In this case, \( K_t \leq L^\perp(t) \). If \( t, t' \in F \) are reflections or glide reflections, then \( K_t = K_{t'} \) if and only if \( t = t' \).

(iv) For all \( a \in A \) we have \( (at)^G \cap At = \bigcup_{t^* \in F^*} K_t(at)^{t^*} \) where \( F^* = \{t^* \in F | (t, t^*) \in A\} \). In the cases where \( G = A \times F \), this is equivalent to \( (at)^G \cap At = \bigcup_{t^* \in C} K_t a^{t^*} t \) where \( C \) is the centralizer of \( t \) in \( F \).
(v) If $t$ is rotation, then (iv) gives the following:

$$(at)^G \cap At = \begin{cases} 
  aK_t \cup a^\circ K_t & \text{if } t \text{ is rotation and } \rho_{\pi/2} \in G; \\
  aK_t \cup (a\beta_1 \beta_2)^{-1}K_t & \text{if } G \text{ is of type } p2mm, p2mg \text{ or } p2gg; \\
  aK_t \cup a^\sigma K_t & \text{if } G \text{ is of type } c2mm; \\
  aK_t \cup a\beta_1\beta_2\rho_1 K_t & \text{if } t = \rho_\pi \in G \text{ and } G \text{ is of type } p4mg; \\
  aK_t & \text{otherwise.} 
\end{cases}$$

If $t$ is a glide reflection or reflection, then (iv) implies that

$$(at)^G \cap At = \begin{cases} 
  aK_t \cup (a\beta_1 \beta_2)^{-1}K_t & \text{if } \rho_\pi \in G; \\
  aK_t & \text{otherwise.} 
\end{cases}$$

If $G$ has no glide reflection cosets and $t$ is a reflection we have

$$(at)^G \cap At = \begin{cases} 
  (aK_t \cup a^{-1}K_t & \text{if } \rho_\pi \in G \\
  aK_t & \text{otherwise.} 
\end{cases}$$

Proof.  (i) Since $A$ is normal in $G, (a, t) \in A$ for any $a \in A$. Now $A$ is abelian and so for $a, b \in A$ we have $(ab, t) = (a, t)(b, t)$ by the Witt-Hall identities ([MKS] p. 290). Thus $K_t$ is generated by the two commutators $(x, t)$ and $(y, t)$.

(ii) and (iii) If $t = \rho_\theta$, then $x^{-1}x^t$ is a translation in the direction with angle $(\pi + \theta)/2$. Similarly, $y^{-1}y^t$ is a translation in a direction with angle $\alpha + (\pi + \theta)/2$ where $\alpha$ is the angle of the direction of translation for $y$. Hence $(x, t)$ and $(y, t)$ will generate a lattice, and so $K_t$ is a finite index subgroup of $A$.

If $t$ is a reflection or a glide reflection, then for any $a \in A$, $a^{-1}a^t$ is a (possibly trivial) translation in a direction that is perpendicular to $L(t)$. Note that if $(x, t)$ is trivial then $(y, t)$ is not. Thus $(x, t)$ and $(y, t)$ generate a non-trivial cyclic group, and so $[G : K_t]$ is infinite.
Let \( b \in \{x, y\} \). Then \((b,t) \cdot t)^2 = (b^{-1}b't)(b^{-1}b't) = b^{-1}b'(b^{-1})tbt^2 = \beta\). Then by Lemma 2.1 \((x,t), (y,t) \in L^\perp(t)\), thus \(K_t \leq L^\perp(t)\). Since \(L^\perp(t) \neq L^\perp(t')\) for distinct \(t, t'\) this gives us \(K_t \neq K_{t'}\).

Now (iv) follows because conjugates of the element \(at\) result from conjugating finitely many times by something in \(A\) or something in \(F\). Conjugating by elements in \(A\) gives us \(K_{at}\). If \((t', t) \notin A\), then \((at)t' \notin At\). Thus we only need to conjugate by those \(t'\) that satisfy \((t', t) \in A\).

As an example, we consider the groups of type \(p4mm\) which has rotation \(\rho_{\pi/2}\). The conjugacy class of \(a\rho\) contains \((a\rho)^x = a(x, \rho^{-1})\rho\) and \((a\rho)^y = a(y, \rho^{-1})\rho\) as well as \((a\rho)^\rho = a^\rho \rho\). Since \((a\rho)^\sigma \notin A\) we see \((a\rho)^G \cap A\rho = aK_{\rho}\rho \cup a^\rho K_{\rho}\rho\). Likewise the conjugacy class of \(a\rho^2\) contains \((a\rho^2)^x = a(x, \rho^2)\rho^2\) and \((a\rho^2)^y = a(y, \rho^2)\rho^2\) as well as \((a\rho^2)^\rho = a^\rho \rho^2\). Since \((a\rho^2)^\sigma \in aK_{\rho}\rho^2\rho^2\) we have \((a\rho^2)^G \cap A\rho^2 = aK_{\rho}\rho^2 \cup a^\rho K_{\rho}\rho^2\), proving the first line of (v) for this case.

For a second example, suppose \(\gamma\) is a glide reflection in a group with generator \(\rho = \rho_\pi\). (This applies to groups of type \(p2mg\) and \(p2gg\).) Recall that we defined \(\beta_1 = \gamma^2\) and \(\beta_2 = (\rho\gamma)^2\). The conjugacy class of \(a\gamma\) will include \((a\gamma)^x = a(x, \gamma^{-1})\gamma\) and \((a\gamma)^y = a(y, \gamma^{-1})\gamma\), thus it includes \(aK_{\gamma}\gamma\). Note that \((a, \gamma) \in K_{\gamma}\) implies that \((a\gamma)^\gamma \in aK_{\gamma}\gamma\). So lastly we consider \((a\gamma)^\rho\). Since \(\gamma^\rho = \beta_1^{-1}\beta_2\gamma\), \((a\gamma)^\rho = (a\beta_1\beta_2)^{-1}\gamma\). But since \(\beta_2^2 \in K_{\gamma}\) we see that \((a\beta_1\beta_2)^{-1}K_{\gamma}\gamma \subseteq (a\gamma)^G\). Thus \((a\gamma)^G = aK_{\gamma}\gamma \cup (a\beta_1\beta_2)^{-1}K_{\gamma}\gamma\).

We now give an example of a wallpaper pattern corresponding to each type of wallpaper group and indicate the generators of the wallpaper group. Below each we list a presentation for the group, the coset representatives we have chosen to comprise \(F\), and the set \(H\) that corresponds to the group. We also give a complete list of conjugacy classes for the wallpaper group.
Wallpaper group $\textbf{p1}$:

$G = \langle x, y \rangle$.
$F = \{1\}$.

This group is abelian.
Wallpaper group \textbf{p1m1}:

\[ G = \langle x, y, \sigma \mid (x, y), x^\sigma = x, y^\sigma = y^{-1}, \sigma^2 \rangle. \]
\[ F = \langle \sigma \rangle. \]
\[ H = \langle x \rangle. \]

Conjugacy classes in \( G \):

\[ (x^iy^j)^G = \{ x^iy^{\pm j} \}; \]
\[ (a\sigma)^G = \langle y^2 \rangle a\sigma, \]
for \( a \in A, i, j \in \mathbb{Z} \).
Wallpaper group \textbf{p1g1}:

\[ G = \langle x, y, \gamma \mid (x, y), x^\gamma = x, y^\gamma = y^{-1}, \gamma^2 = x \rangle. \]

\[ F = \{1, \gamma\}. \]

\[ H = \langle x \rangle. \]

Conjugacy classes in \( G \):

\[(x^i y^j)^G = \{x^i y^{\pm j}\};\]

\[(a \gamma)^G = \langle y^2 \rangle a \gamma,\]

for \( a \in A, \ i, j \in \mathbb{Z} \).
Wallpaper group $\text{c1m1}$:

$$G = \langle x, y, \sigma | (x, y), x^\sigma = y, y^\sigma = x, \sigma^2 \rangle.$$  
$$F = \langle \sigma \rangle.$$  
$$H = \langle xy \rangle.$$  

Conjugacy classes in $G$:

$$(x^i y^j)^G = \{x^i y^j, x^j y^i\};$$  
$$(a \sigma)^G = \langle xy^{-1} \rangle a \sigma,$$

for $a \in A, i, j \in \mathbb{Z}$. 

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Wallpaper group $\textbf{p2}$:

\[ G = \langle x, y, \rho \mid (x, y), x^\rho = x^{-1}, y^\rho = y^{-1}, \rho^2 \rangle. \]
\[ F = \langle \rho \rangle. \]
\[ H = \{1\}. \]

Conjugacy classes in $G$:

\[ a^G = \{a, a^{-1}\}; \]
\[ (a\rho)^G = \langle x^2, y^2 \rangle a\rho, \]
for $a \in A$. 
Wallpaper group \textbf{p2mm}:

\[
G = \langle x, y, \rho, \sigma \mid (x, y), \rho^2, \sigma^2, (\rho, \sigma), x^\rho = x^{-1}, y^\rho = y^{-1}, x^\sigma = x, y^\sigma = y^{-1} \rangle.
\]
\[
F = \langle \rho, \sigma \rangle.
\]
\[
H = \langle x \rangle \cup \langle y \rangle.
\]

Conjugacy classes in $G$:

\[
(x^i y^j)^G = \{ x^{\pm i} y^{\pm j} \};
\]
\[
(a\rho)^G = \langle x^2, y^2 \rangle a\rho;
\]
\[
(a\sigma)^G = \langle y^2 \rangle a\sigma \cup \langle y^2 \rangle a^{-1}\sigma;
\]
\[
(a\rho\sigma)^G = \langle x^2 \rangle a\rho\sigma \cup \langle x^2 \rangle a^{-1}\rho\sigma,
\]
for $a \in A$, $i, j \in \mathbb{Z}$. 
Wallpaper group \( p2mg: \)

\[ G = \langle x, y, \rho, \sigma | (x, y), \rho^2, \sigma^2, x^\rho = x^{-1}, y^\rho = y^{-1}, x^\sigma = x, y^\sigma = y^{-1}, (\rho\sigma)^2 = y \rangle. \]
\[ F = \{1, \rho, \sigma, \rho\sigma\}. \]
\[ H = \langle x \rangle \cup \langle y \rangle. \]

Conjugacy classes in \( G: \)

\[ (x^i y^j)^G = \{x^{\pm i} y^{\pm j}\}; \]
\[ (a\rho)^G = \langle x^2, y \rangle a\rho; \]
\[ (a\sigma)^G = \langle y^2 \rangle a\sigma \cup \langle y^2 \rangle (ay)^{-1} \sigma; \]
\[ (a\rho\sigma)^G = \langle x^2 \rangle a\rho\sigma \cup \langle x^2 \rangle (ay)^{-1} \rho\sigma, \]

for \( a \in A, i, j \in \mathbb{Z}. \)
Wallpaper group **p2gg:**

\[ G = \langle x, y, \rho, \gamma | (x, y), \rho^2, \gamma^2 = x, x^\rho = x^{-1}, y^\rho = y^{-1}, x^\gamma = x, y^\gamma = y^{-1}, (\rho \gamma)^2 = y \rangle. \]

\[ F = \{1, \rho, \gamma, \rho \gamma\}. \]

\[ H = \langle x \rangle \cup \langle y \rangle. \]

Conjugacy classes in \( G \):

\[(x^i y^j)^G = \{ x^{\pm i} y^{\pm j} \}; \]

\[(a \rho)^G = \langle xy, xy^{-1} \rangle a \rho; \]

\[(a \gamma)^G = \langle y^2 \rangle a \gamma \cup \langle y^2 \rangle (xya)^{-1} \gamma; \]

\[(a \rho \gamma)^G = \langle x^2 \rangle a \rho \gamma \cup \langle x^2 \rangle (xya)^{-1} \rho \gamma, \]

for \( a \in A, i, j \in \mathbb{Z} \).
Wallpaper group \textbf{c2mm}: 

\[
G = \langle x, y, \rho, \sigma \mid (x, y), \rho^2, \sigma^2, x^\rho = x^{-1}, y^\rho = y^{-1}, x^\sigma = y, y^\sigma = x, (\rho \sigma)^2 \rangle.
\]

\[
F = \langle \rho, \sigma \rangle.
\]

\[
H = \langle xy \rangle \cup \langle xy^{-1} \rangle.
\]

Conjugacy classes in \( G \):

\[
(x^iy^j)^G = \{ (x^iy^j)^{\pm 1}, (x^iy^j)^{\pm 1} \};
\]

\[
(a\rho)^G = \langle x^2, y^2 \rangle a\rho, \text{ if } a \in \langle xy, xy^{-1} \rangle;
\]

\[
(a\rho)^G = \langle xy, xy^{-1} \rangle a\rho, \text{ if } a \notin \langle xy, xy^{-1} \rangle;
\]

\[
(a\sigma)^G = \langle xy^{-1} \rangle a\sigma \cup \langle xy^{-1} \rangle a^{-1}\sigma;
\]

\[
(a\rho\sigma)^G = \langle xy \rangle a\rho\sigma \cup \langle xy \rangle a^{-1}\rho\sigma,
\]

for \( a \in A, i, j \in \mathbb{Z} \).
Wallpaper group \textbf{p4}:

\[ G = \langle x, y, \rho | (x, y), x^\rho = y, y^\rho = x^{-1}, \rho^4 \rangle. \]
\[ F = \langle \rho \rangle. \]
\[ H = \{ 1 \}. \]

Conjugacy classes in \( G \):

\[ (x^iy^j)^G = \{(x^iy^j)^{\pm1}, (x^{-j}y^i)^{\pm1}\} \]
\[ (a\rho)^G = \langle xy, xy^{-1} \rangle a\rho; \]
\[ (a\rho^2)^G = \langle x^2, y^2 \rangle a\rho; \text{ if } a \in \langle xy, xy^{-1} \rangle \]
\[ (a\rho^2)^G = \langle xy, xy^{-1} \rangle a\rho; \text{ if } a \notin \langle xy, xy^{-1} \rangle \]
\[ (a\rho^3)^G = \langle xy, xy^{-1} \rangle a\rho^3, \]
\[ \text{for } a \in A, i, j \in \mathbb{Z}. \]
Wallpaper group \textbf{p4mm}: 

\[
G = \langle x, y, \rho, \sigma \mid (x, y), \rho^4, \sigma^2, x^\rho = y, y^\rho = x^{-1}, x^\sigma = x, y^\sigma = y^{-1}, (\rho\sigma)^2 \rangle.
\]

\[
F = \langle \rho, \sigma \rangle.
\]

\[
H = \langle x \rangle \cup \langle xy \rangle \cup \langle y \rangle \cup \langle xy^{-1} \rangle.
\]

Conjugacy classes in \( G \):

\[
(x^i y^j)^G = \{ x^{\pm i} y^{\pm j}, x^{\pm i} y^{\pm j} \};
\]

\[
(\rho \sigma)^G = \langle xy, xy^{-1} \rangle \rho \sigma \cup \langle xy, xy^{-1} \rangle \rho \sigma^3;
\]

\[
(x^2, y^2)^G = \langle x^2, y^2 \rangle \rho^2 \quad \text{if } a \in \langle xy, xy^{-1} \rangle;
\]

\[
(\rho^2)^G = \langle xy, xy^{-1} \rangle \rho^2 \quad \text{if } a \notin \langle xy, xy^{-1} \rangle;
\]

\[
(a \sigma)^G = \langle y^2 \rangle a \sigma \cup \langle y^2 \rangle a^{-1} \sigma \cup \langle x^2 \rangle a^\rho \rho^2 \sigma \cup \langle x^2 \rangle (a^{-1})^\rho \rho^2 \sigma;
\]

\[
(a \rho \sigma)^G = \langle xy \rangle a \rho \sigma \cup \langle xy \rangle a^{-1} \rho \sigma \cup \langle xy \rangle a^\rho \rho^3 \sigma \cup \langle xy \rangle (a^{-1})^\rho \rho^3 \sigma,
\]

for \( a \in A, i, j \in \mathbb{Z} \).
Wallpaper group \textbf{p4mg}:

\[
G = \langle x, y, \rho, \gamma \mid (x, y), \rho^4, \gamma^2 = x, x^\rho = y, y^\rho = x^{-1}, x^\gamma = x, y^\gamma = y^{-1}, (\rho\gamma)^2 \rangle.
\]

\[
F = \{1, \rho, \rho^2, \rho^3, \gamma, \rho\gamma, \rho^2\gamma, \rho^3\gamma\}.
\]

\[
H = \langle x \rangle \cup \langle xy \rangle \cup \langle y \rangle \cup \langle xy^{-1} \rangle.
\]

Conjugacy classes in \(G\):

\[
(x^iy^j)^G = \{x^{\pm i}y^{\pm j}, x^{\pm i}y^{\pm j}\};
\]

\[
(\rho)^G = \langle x, xy^{-1} \rangle \rho \cup \langle xy, xy^{-1} \rangle ax\rho^3;
\]

\[
(\rho^2)^G = \langle xy, xy^{-1} \rangle \rho^2;
\]

\[
(\rho^3)^G = \langle xy, xy^{-1} \rangle \rho^3 \cup \langle xy, xy^{-1} \rangle ax\rho;
\]

\[
(\gamma)^G = \langle y^2 \rangle \gamma \cup \langle y^2 \rangle a^{-1}x^{-1}y \gamma \cup \langle x^2 \rangle a^\rho x \rho^2 \gamma \cup \langle x^2 \rangle a^\rho^3 y^{-1} \rho^2 \gamma;
\]

\[
(\rho\gamma)^G = \langle xy \rangle \rho a \gamma \cup \langle xy \rangle a^{-1} \rho \gamma \cup \langle xy^{-1} \rangle y \gamma \cup \langle xy^{-1} \rangle y^{-1} a \rho \gamma \cup \langle xy^{-1} \rangle x \gamma \cup \langle x^{-1} \rangle \gamma a^{-1} \rho \gamma;
\]

\[
(\rho^2\gamma)^G = \langle x^2 \rangle a^\rho^2 \gamma \cup \langle x^2 \rangle a^{-1}x \gamma \rho^2 \gamma \cup \langle y^2 \rangle a^\rho x \gamma \cup \langle y^2 \rangle a^\rho y \gamma \cup \langle y^2 \rangle a^\rho x \gamma \cup \langle y^2 \rangle a^\rho x \gamma;
\]

\[
(\rho^3\gamma)^G = \langle xy^{-1} \rangle a^\rho \gamma \cup \langle xy^{-1} \rangle x^{-1}a^{-1} \rho^3 \gamma \cup \langle xy \rangle xa^\rho^3 \rho \gamma \cup \langle xy \rangle x a^\rho \rho \gamma;
\]

for \(a \in A, i, j \in \mathbb{Z}\).
Wallpaper group \textbf{p3}:

\[ G = \langle x, y, \rho \mid (x, y), x^\rho = x^{-1} y, y^\rho = x^{-1}, \rho^3 \rangle. \]

\[ F = \langle \rho \rangle. \]

\[ H = \{1\}. \]

Conjugacy classes in \( G \):

\[ (x^i y^j G) = \{ x^i y^j, x^{-i-j} y^i, x^j y^{-i-j} \}; \]

\[ (a \rho G) = \langle xy, x^{-2} y \rangle a \rho; \]

\[ (a \rho^2 G) = \langle xy, x^{-2} y \rangle a \rho^2, \]

for \( a \in A, i, j \in \mathbb{Z} \).
Wallpaper group **p31m**: 

\[ G = \langle x, y, \rho, \sigma \mid (xy), \rho^3, \sigma^2, (\rho \sigma)^2, x^\rho = x^{-1} y, y^\rho = x^{-1}, x^\sigma = x, y^\sigma = xy^{-1} \rangle. \]

\[ F = \langle \rho, \sigma \rangle. \]

\[ H = \langle x \rangle \cup \langle y \rangle \cup \langle x^{-1} y \rangle. \]

Conjugacy classes in \( G \):

\[ (x^i y^j)^G = \{ x^i y^j, x^{i+j} y^{-j}, x^{-i-j} y^j, x^{-j} y^{-i}, x^i y^{-i-j}, x^{-i} y^{i+j} \}; \]

\[ (a \rho)^G = \langle xy, x^{-2} y \rangle a \rho \cup \langle xy, x^{-2} y \rangle a^\sigma \rho^2; \]

\[ (a \rho^2)^G = \langle xy, x^{-2} y \rangle a \rho^2 \cup \langle xy, x^{-2} y \rangle a^\sigma \rho; \]

\[ (a \sigma)^G = \langle xy^{-2} \rangle a \sigma \cup \langle xy \rangle a^\rho \rho \sigma \cup \langle x^{-2} y \rangle a^\rho^2 \rho^2 \sigma; \]

\[ (a \rho \sigma)^G = \langle xy \rangle a \rho \sigma \cup \langle x^{-2} y \rangle a^\rho \rho^2 \sigma \cup \langle xy^{-2} \rangle a^\rho^2 \sigma; \]

\[ (a \rho^2 \sigma)^G = \langle x^{-2} y \rangle a \rho^2 \sigma \cup \langle xy^{-2} \rangle a^\rho \sigma \cup \langle xy \rangle a^\rho^2 \rho \sigma, \]

for \( a \in A, i, j \in \mathbb{Z} \).
Wallpaper group $p3m1$:

$G = \langle x, y, \rho, \sigma \mid (x, y), \rho^3, \sigma^2, (\rho \sigma)^2, x^\rho = x^{-1} y, y^\rho = x^{-1}, x^\sigma = y, y^\sigma = x \rangle$.

$F = \langle \rho, \sigma \rangle$.

$H = \langle xy \rangle \cup \langle x^{-1} y^2 \rangle \cup \langle x^{-2} y \rangle$.

Conjugacy classes in $G$:

$(x^i y^j)^G = \{ x^i y^j, x^i y^j, x^i y^{-i-j}, x^{-i-j} y^i, x^i y^{-i-j}, x^{-i-j} y^i \}$;

$(a \rho)^G = \langle xy, x^{-2} y \rangle a \rho \cup \langle xy, x^{-2} y \rangle a^\sigma \rho^2$;

$(a \rho^2)^G = \langle xy, x^{-2} y \rangle a \rho^2 \cup \langle xy, x^{-2} y \rangle a^\sigma \rho$;

$(a \sigma)^G = \langle xy^{-1} \rangle a \sigma \cup \langle y \rangle a^\sigma \rho \sigma \cup \langle x \rangle a^\rho^2 \rho^2 \sigma$;

$(a \rho \sigma)^G = \langle y \rangle a \rho \sigma \cup \langle x \rangle a^\rho^2 \rho \sigma \cup \langle x y^{-1} \rangle a^\rho^2 \sigma$;

$(a \rho^2 \sigma)^G = \langle x \rangle a \rho^2 \sigma \cup \langle x y^{-1} \rangle a^\rho^2 \sigma \cup \langle y \rangle a^\rho^2 \rho \sigma$,

for $a \in A, i, j \in \mathbb{Z}$. 
Wallpaper group **p6**: 

$$G = \langle x, y, \rho \mid (x, y), x^\rho = y, y^\rho = x^{-1}y, \rho^6 \rangle.$$ 

$$F = \langle \rho \rangle.$$ 

$$H = \{1\}.$$ 

Conjugacy classes in **G**: 

$$(x^i y^j)^G = \{x^i y^j, x^{-j} y^{i+j}, x^{-i-j} y^i, x^{-i} y^{-j}, x^j y^{-i-j}, x^{i+j} y^{-i}\};$$ 

$$(a\rho)^G = A\rho;$$ 

$$(a\rho^2)^G = \langle xy, x^{-2}y \rangle a\rho^2;$$ 

$$(a\rho^3)^G = \langle x^2, y^2 \rangle a\rho^3;$$ 

$$(a\rho^4)^G = \langle xy, x^{-2}y \rangle a\rho^4;$$ 

$$(a\rho^5)^G = A\rho^5,$$ 

for $a \in A$, $i, j \in \mathbb{Z}$. 

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Wallpaper group **p6mm:**

\[ G = \langle x, y, \rho, \sigma \mid (x, y), \rho^6, \sigma^2, x^\rho = y, y^\rho = x^{-1}y, x^\sigma = x, y^\sigma = xy^{-1}, (\rho\sigma)^2 \rangle. \]

\[ F = \langle \rho, \sigma \rangle. \]

\[ H = \langle x \rangle \cup \langle xy \rangle \cup \langle y \rangle \cup \langle x^{-1}y^2 \rangle \cup \langle x^{-1}y \rangle \cup \langle x^{-2}y \rangle. \]

Conjugacy classes in \( G \):

\[ (x^iy^j)^G = \{ x^iy^j, x^{i+j}y^{-j}, x^{-j}y^{i+j}, x^iy^{i-j}, x^{-i-j}y^i, x^{-i}y^{i-j}, x^{i-j}y^{-i}, x^{-i}y^{i-j}, x^{i-j}y^{-i}, x^iy^{-j}, x^{-i-j}y^i \}. \]

\[ (a\rho)^G = (a\rho^5)^G = A\rho \cup A\rho^5; \]

\[ (a\rho^2)^G = \langle xy, x^{-2}y \rangle a\rho^2 \cup \langle xy, x^{-2}y \rangle a\sigma \rho^4; \]

\[ (a\rho^3)^G = \langle x^2, y^2 \rangle a\rho^3; \]

\[ (a\rho^4)^G = \langle xy, x^{-2}y \rangle a\rho^4 \cup \langle xy, x^{-2}y \rangle a\sigma \rho^2; \]

\[ (a\sigma)^G = \langle xy^{-2} \rangle a\sigma \cup \langle xy^{-2} \rangle a^{-1} \sigma \cup \langle xy \rangle a\rho^2 \rho^2 \sigma \cup \langle xy \rangle a\rho^5 \rho^2 \sigma \cup \langle x^{-2}y \rangle a\rho^4 \sigma \cup \langle x^{-2}y \rangle a\rho^4 \rho^4 \sigma; \]

\[ (a\rho\sigma)^G = \langle y \rangle a\rho \sigma \cup \langle y \rangle a^{-1} \rho \sigma \cup \langle x \rangle a\rho^2 \rho^3 \sigma \cup \langle x \rangle a\rho^5 \rho^3 \sigma \cup \langle x^{-1}y \rangle a\rho^4 \rho^5 \sigma \cup \langle x^{-1}y \rangle a\rho^4 \rho^5 \sigma, \]

for \( a \in A, i, j \in \mathbb{Z}. \)
Lemma 2.4. The following maps define automorphisms:

\begin{align*}
&c1m1 \quad \psi : (x, y, \sigma) \mapsto (x^{-1}, y^{-1}, \sigma^{-1}); \\
&p1m1 \quad \psi : (x, y, \sigma) \mapsto (x^{-1}, y^{-1}, \sigma^{-1}); \\
&p1g1 \quad \psi_y : (x, y, \gamma) \mapsto (x, y, y\gamma) \text{ and also } \psi : (x, y, \gamma) \mapsto (x^{-1}, y^{-1}, \gamma^{-1}); \\
&c2mm \quad \psi_{u,v,i,j} : (x, y, \rho, \sigma) \mapsto (x, y, x^uy^v\rho, x^iy^j\sigma) \text{ where } u + j = i + v; \\
&\text{also } \psi : (x, y, \rho, \sigma) \mapsto (x^{-1}, y^{-1}, \rho, \sigma); \\
&p2mm \quad \psi_x : (x, y, \rho, \sigma) \mapsto (x, y, x\rho, \sigma); \text{ also } \psi_y : (x, y, \rho, \sigma) \mapsto (x, y, y\rho, y\sigma) \\
&\text{and } \psi_1 : (x, y, \rho, \sigma) \mapsto (y, x, \rho\sigma); \\
&p2mg \quad \psi_x : (x, y, \rho, \sigma) \mapsto (x, y, x\rho, \sigma); \text{ also } \psi_y : (x, y, \rho, \sigma) \mapsto (x, y, y\rho, y\sigma) \\
&\text{and } \psi : (x, y, \rho, \sigma) \mapsto (x^{-1}, y^{-1}, y^{-1}\rho, \sigma); \\
&p2gg \quad \psi_x : (x, y, \rho, \gamma) \mapsto (x, y, x\rho, \gamma) \text{ and } \psi_y : (x, y, \rho, \gamma) \mapsto (x, y, y\rho, y\gamma); \\
&\text{also } \psi : (x, y, \rho, \gamma) \mapsto (y, x, \rho\gamma) \text{ and } \psi : (x, y, \rho, \gamma) \mapsto (x^{-1}, y^{-1}, \rho, x^{-1}y\gamma); \\
&p3m1 \quad \psi_1 : (x, y, \rho, \sigma) \mapsto (x, y, y^{-1}\rho, \sigma); \\
&p4 \quad \psi_1 : (x, y, \rho) \mapsto (x, y^{-1}, \rho^{-1}); \\
&p4mg \quad \psi_1 : (x, y, \rho, \gamma) \mapsto (x, y, x\rho, y^{-1}\gamma); \psi_2 : (x, y, \rho, \gamma) \mapsto (x^{-1}, y^{-1}, y^{-1}\rho, x^{-1}\gamma); \\
&p4mm \quad \psi_1 : (x, y, \rho, \sigma) \mapsto (x, y, y\rho, y\sigma).
\end{align*}

For any group we also have inner automorphisms. These will be denoted \( I_g, g \in G \).

Proof. These maps satisfy the relations in the presentation of their respective groups. \( \square \)

Lemma 2.5. The following are non-trivial weak Cayley table isomorphisms:

\begin{align*}
&p2mm \quad \text{Define } \tau : \left\{ \begin{array}{ll}
&g \mapsto g \quad \text{for } g \in A \cup A\sigma; \\
&g \mapsto g^\sigma \quad \text{for } g \in A\rho \cup A\sigma.
\end{array} \right.
\end{align*}

\begin{align*}
&p3 \quad \text{Define } \tau : \left\{ \begin{array}{ll}
&g \mapsto g \quad \text{for } g \in A; \\
&g \mapsto g^\sigma \quad \text{for } g \in A\rho \cup A\rho^2.
\end{array} \right.
\end{align*}

\begin{align*}
&p4 \quad \text{For } h \in \{x, y, \rho^2\} \text{ define } \tau_h : \left\{ \begin{array}{ll}
&g \mapsto g \quad \text{for } g \notin A\rho^2; \\
&g \mapsto g^h \quad \text{for } g \in A\rho^2.
\end{array} \right.
\end{align*}
For $h \in \{x, y, \rho\}$ define $\mu_h : \begin{cases} 
g \mapsto g; & \text{for } g \in A \cup A\rho^2; 
g \mapsto g^h & \text{for } g \notin A \cup A\rho^2. 
\end{cases}$

$p6$ For $h \in \{xy, xy^{-2}, \rho^2\}$ define $\tau_h : \begin{cases} 
g \mapsto g & \text{for } g \in A\rho \cup A\rho^3 \cup A\rho^5; 
g \mapsto g^h & \text{for } g \notin A\rho \cup A\rho^3 \cup A\rho^5. 
\end{cases}$

For $h \in \{x^2, y^2, \rho^3\}$ define $\mu_h : \begin{cases} 
g \mapsto g & \text{for } g \notin A \cup A\rho^3; 
g \mapsto g^h & \text{for } g \in A \cup A\rho^3. 
\end{cases}$

Proof. This follows from the relations in the group presentations. \hfill \square

Corollary 2.1. Groups of type $p2mm$, $p3$, $p4$ and $p6$ have non-trivial weak Cayley table groups. \hfill \square
Chapter 3. Properties of weak Cayley table isomorphisms

The following properties of weak Cayley table isomorphisms in $W$ will be helpful as we determine $W(G)$. We will write $g$ is conjugate to $h$ as $g \sim h$.

**Proposition 3.1.** Let $\varphi : G \to H$ be a bijective weak Cayley table isomorphism. Then

(i) $\varphi$ maps the identity in $G$ to the identity in $H$: $\varphi(1_G) = 1_H$.

(ii) $\varphi$ respects inverses: $\varphi(g^{-1}) = \varphi(g)^{-1}$.

(iii) $\varphi$ maps conjugacy classes to conjugacy classes: If $g \sim h$ then $\varphi(g) \sim \varphi(h)$.

(iv) For the centers of groups we have $\varphi(Z(G)) \supseteq Z(H)$.

(v) $\varphi$ maps involutions to involutions: $g^2 = 1$ implies $\varphi(g)^2 = 1$.

(vi) $\varphi$ maps normal subgroups to normal subgroups: $N \triangleleft G$ implies $\varphi(N) \triangleleft H$.

(vii) If $N \triangleleft G$ and $\varphi(N) = M$, then $\varphi$ maps cosets of $N$ to cosets of $M$: $\varphi(gN) = \varphi(g)\varphi(N)$.

(viii) Let $N \triangleleft G$. Then $\varphi$ induces a map $\bar{\varphi} : G/N \to H/\varphi(N)$ which is also a bijective weak Cayley table isomorphism.

(ix) If $H = G$ then we have $g \sim h$ if and only if $\varphi(g) \sim \varphi(h)$, thus $\varphi(Z(G)) = Z(G)$. In this situation, $\varphi^{-1}$ is also a bijective weak Cayley table isomorphism.

**Proof.** (i) Let $\varphi(\alpha) = 1$. Then $\varphi(\alpha \cdot \alpha) \sim 1$. Since $\varphi$ is a bijection this implies $\alpha^2 = \alpha$, thus $\alpha = 1$.

(ii) Here $1 = \varphi(g \cdot g^{-1}) \sim \varphi(g)\varphi(g^{-1})$ implies $\varphi(g^{-1}) = \varphi(g)^{-1}$.

(iii) Here we have $\varphi(g) = \varphi(x \cdot x^{-1}g) \sim \varphi(x)\varphi(x^{-1}g) \sim \varphi(x^{-1}g)\varphi(x) \sim \varphi(x^{-1}g \cdot x)$.

(iv) Let $g \in G \setminus Z(G)$ and suppose we have $\varphi(g) = z \in Z(H)$. Since $g$ is not central, there exists $h \neq g$ such that $h \sim g$. By (iii) we must have $\varphi(h) \sim \varphi(g) = z$, and since $\varphi$ is a bijection, $\varphi(h) \neq \varphi(g)$. Since $z \in Z(H)$ this is a contradiction.
(v) Here \( 1 = g^2 \) implies \( \varphi(1) = \varphi(g \cdot g) \sim \varphi(g)\varphi(g) \).

(vi) From (i) and (iii) we see that \( \varphi \) will map a normal subgroup to a union of conjugacy classes that contains \( 1 \). By the definition of a weak Cayley table map this union is closed under the group operation and by (ii) we have inverses, thus it is a normal subgroup.

(vii) Let \( xN = yN \) and so \( xy^{-1} \in N \). Then \( \varphi(xy^{-1}) \in M \) implies \( \varphi(x)\varphi(y^{-1}) = \varphi(x)\varphi(y)^{-1} \in M \) and thus \( \varphi(x)M = \varphi(y)M \).

(viii) Using (vii) we define \( \bar{\varphi}(g) : gN \mapsto \varphi(g)\varphi(N) \). Note that if \( g \) is conjugate to \( h \) in \( G \) then \( gN \) is conjugate to \( hN \) in \( G/N \). Then \( \varphi(gh) \sim \varphi(g)\varphi(h) \) implies that \( \varphi(gh)\varphi(N) \sim \varphi(g)\varphi(h)\varphi(N) \). Thus

\[
\bar{\varphi}(gN \cdot hN) = \bar{\varphi}(ghN) = \varphi(gh)\varphi(N) \sim \varphi(g)\varphi(h)\varphi(N) = \bar{\varphi}(gN)\bar{\varphi}(hN),
\]

which shows that \( \bar{\varphi} \) is a weak Cayley table map.

(ix) The first two statements follow from (iii) and the bijectivity of \( \varphi \). Now we have

\[
\varphi(\varphi^{-1}(g)\varphi^{-1}(h)) \sim \varphi(\varphi^{-1}(g))\varphi(\varphi^{-1}(h)) = gh.
\]

Since \( \varphi^{-1} \) preserves conjugacy classes we may apply it to both sides of the above relation which gives \( \varphi^{-1}(g)\varphi^{-1}(h) \sim \varphi^{-1}(gh) \) and we are done.

\[\square\]

**Theorem 3.2.** The set of all weak Cayley table bijections \( \varphi : G \to G \) forms a group \( \mathcal{W} \) under composition.

**Proof.** Let \( \varphi_1, \varphi_2 \in \mathcal{W} \) and \( g, h \in G \). Since \( \varphi_2(g \cdot h) \sim \varphi_2(g)\varphi_2(h) \), Proposition 3.1 (iii) gives

\[
\varphi_1 \circ \varphi_2(g \cdot h) \sim \varphi_1(\varphi_2(g)\varphi_2(h)) \sim \varphi_1 \circ \varphi_2(g) \cdot \varphi_1 \circ \varphi_2(h).
\]

Thus \( \mathcal{W} \) is closed under composition. By Proposition 3.1 (ix) we have \( \varphi^{-1} \in \mathcal{W} \). Clearly composition is associative and the identity map is in \( \mathcal{W} \), hence \( \mathcal{W} \) is a group.

\[\square\]
Lemma 3.1. The inverse map $\iota : g \mapsto g^{-1}$ is in the center of $\mathcal{W}(G)$.

Proof. First we show $\iota \in \mathcal{W}$: For $g_1, g_2 \in G$,

$$g_2^{-1}g_1^{-1} = (g_1g_2)^{-1} = \iota(g_1g_2) \sim \iota(g_1)(g_2) = g_1^{-1}g_2^{-1}.$$ 

The fact that $\iota$ commutes with any $\varphi \in \mathcal{W}$ follows from Proposition 3.1 (ii): $(\varphi \circ \iota)(g) = \varphi(g^{-1}) = \varphi(g)^{-1} = (\iota \circ \varphi)(g)$. \hfill \qed

An anti-automorphism is a bijective map $\alpha : G \to G$ that satisfies $\alpha(gh) = \alpha(h)\alpha(g)$ for all $g, h \in G$. Let $\alpha$ be an anti-automorphism of $G$ and let $\psi = \iota \circ \alpha$. Then for any $g, h \in G$

$$\psi(gh) = \alpha(gh)^{-1} = (\alpha(h)\alpha(g))^{-1} = \alpha(g)^{-1}\alpha(h)^{-1} = \psi(g)\psi(h).$$

Thus $\psi$ is an automorphism. It follows that any anti-automorphism is the composition of an automorphism with the inverse map. By Theorem 3.2 an anti-automorphism is a weak Cayley table isomorphism. This shows that $\mathcal{W}_0(G)$ is the set of all automorphisms and anti-automorphisms of $G$.

Theorem 3.3. Let $G$ be a group that can be written as a direct product of two nonabelian subgroups $H$ and $K$. Then $\mathcal{W}(G)$ is nontrivial.

Proof. Let $\psi_1$ be an automorphism of $H$ that is not an anti-automorphism of $K$ and let $\psi_2$ be an anti-automorphism of $K$ that is not an automorphism of $H$. For any $g \in G$ we can write $g = hk$ for some $h \in H$ and $k \in K$. Define $\tau(hk) = \psi_1(h)\psi_2(k)$.

We show $\tau \in \mathcal{W}(G)$:

$$\tau(h_1k_1 \cdot h_2k_2) = \tau(h_1h_2k_1k_2) = \tau(h_1h_2)\tau(k_1k_2)$$

$$= \tau(h_1)\tau(h_2)\tau(k_2)\tau(k_1) = \tau(k_2)\tau(h_1)\tau(k_1)\tau(h_2)$$

$$\sim \tau(h_1)\tau(k_1)\tau(h_2)\tau(k_2) = \tau(h_1k_1)\tau(h_2k_2).$$
Now $\tau$ is not an automorphism of $G$ since for some $k_1, k_2 \in K$ we have $\psi_2(k_1k_2) = \psi_2(k_2)\psi_2(k_1) \neq \psi_2(k_1)\psi_2(k_2)$, so that $\tau(k_1k_2) = \tau(k_2)\tau(k_1) \neq \tau(k_1)\tau(k_2)$ and $\tau$ is not an anti-automorphism since for any $h_1, h_2 \in H$ we have $\psi_1(h_1) = \psi_1(h_1)\psi_1(h_2) \neq \psi_1(h_2)\psi_1(h_1)$, so that $\tau(h_1h_2) = \tau(h_1)\tau(h_2) \neq \tau(h_2)\tau(h_1)$. Therefore $\tau$ is nontrivial.

**Corollary 3.2.** The wallpaper group $G$ of type $\text{p}2\text{mm}$ has nontrivial $W(G)$.

**Proof.** We have subgroups $\langle x, \rho\sigma \rangle \cong D_\infty$ and $\langle y, \sigma \rangle \cong D_\infty$. Since $x$ and $\rho\sigma$ commute with $y$ and $\sigma$ and since the intersection of these subgroups is trivial, we see that $G$ is a direct product: $G = D_\infty \times D_\infty$. By Theorem 3.3 this wallpaper group has nontrivial $W(G)$.

Conjugacy in $G$ plays a key role in determining $W(G)$ since weak Cayley table maps by definition respect multiplication up to conjugacy. By Proposition 3.1 (iii) and bijectivity, $\varphi$ maps conjugacy classes of size less than $[G : A]$ to conjugacy classes of size less than $[G : A]$. Thus for any $\varphi \in W(G)$ we have $\varphi(H) = H$. Similarly, because elements of $A$ have finite conjugacy classes but all other elements in $G$ have infinite conjugacy classes, $\varphi$ must map $A$ to $A$. This is the beginning of the first step in determining $W(G)$. The steps we take are as follows:

Step One: For an arbitrary weak Cayley table isomorphism $\varphi$, we show that $\varphi|_A$ is an automorphism.

Step Two: We show that we may compose $\varphi$ with trivial weak Cayley table mappings so that $\varphi|_A$ is the identity.

Step Three: We show that we may compose again with trivial weak Cayley table maps so that $\varphi$ fixes each coset in $G/A$.

Step Four: We show that we may compose with known elements of $W(G)$ so that we have $\varphi(t) = t$ for $t \in F$.

Step Five: We show that for $t \in F$ there is an $f \in F$ such that $\varphi(at) = a^ft$ for $a \in A$. 32
Step Six: We show that we may compose with known elements of $\mathcal{W}(G)$ so that we have $\varphi = Id$.

Most non-trivial maps appear in Step Six though in two groups we compose with non-trivials in Step Four.
Chapter 4. Steps One and Two

Step One

Let $A$ be the translations subgroup of a wallpaper group $G$. We want to show that $\varphi|_A$ is an automorphism of $A$.

Let $\varphi \in \mathcal{W}$, and let $a, b \in A$. The only elements of $G$ that have finite conjugacy classes are elements of $A$, hence $\varphi(A) = A$. Because $A$ is abelian, in this chapter we will denote the group operation additively. Also, to denote the conjugation action of $g \in G$ on $a \in A$ we will write $g(a)$.

Since $\varphi$ respects inverses, we know that

$$\varphi(a + b) \sim \varphi(a) + \varphi(b); \quad (4.1)$$
$$\varphi(b) = \varphi(a + b - a) \sim \varphi(a + b) - \varphi(a); \quad (4.2)$$
$$\varphi(a) = \varphi(a + b - b) \sim \varphi(a + b) - \varphi(b). \quad (4.3)$$

Equation (4.1) implies that there exists some $g_1 \in G$ such that $\varphi(a + b) = g_1(\varphi(a) + \varphi(b))$.

Similarly, equation (4.2) implies that there exists some $g_2 \in G$ such that $g_2 \varphi(b) = \varphi(a + b) - \varphi(a)$ and equation (4.3) implies that there exists some $g_3 \in G$ such that $g_3 \varphi(a) = \varphi(a + b) - \varphi(b)$. This gives us the following three equations:

$$\varphi(a + b) = g_1(\varphi(a) + \varphi(b)); \quad (4.4)$$
$$= \varphi(a) + g_2 \varphi(b); \quad (4.5)$$
$$= g_3 \varphi(a) + \varphi(b). \quad (4.6)$$

Recall that we may think of the elements of $A$ as their corresponding points in the lattice $\mathbb{L}$ and that all elements that are conjugate to $a$ translate the wallpaper pattern in $\mathbb{E}^2$ by $|a| = r$, thus each $g(a), g \in G, a \in A$, is in the boundary of $B_r$. Thus equation (4.4) corresponds to
the requirement that $\varphi(a+b)$ must lie on a circle centered at the origin of radius $|\varphi(a)+\varphi(b)|$. We will denote this circle $C_1$. Equation (4.5) tells us that $\varphi(a+b)$ must lie on a circle centered at $\varphi(a)$ of radius $|\varphi(b)|$. We will denote this circle $C_2$. Similarly, equation (4.6) requires that $\varphi(a+b)$ lie on a circle centered at $\varphi(b)$ of radius $|\varphi(a)|$. We will denote this circle $C_3$.

Each $(g_1, g_2, g_3)$ triple that satisfies equations (4.4), (4.5), and (4.6) corresponds to a point in $C_1 \cap C_2 \cap C_3$. Certainly the trivial triple $1 = g_1 = g_2 = g_3$ satisfies the three equations. The assertion that $\varphi|_A = Id$ is equivalent to the statement that this is the only $(g_1, g_2, g_3)$ combination that works. This is equivalent to the assertion that $C_1, C_2$ and $C_3$ intersect at only one point, $\varphi(a) + \varphi(b)$.

Suppose that is not the case. Then $C_1 \cap C_2$ consists of at most two points, $p_1$ and $p_2$. Now suppose $C_3$ also contains $p_1$ and $p_2$. This situation is pictured in Figure 4.1. Notice that the centers of all three circles are collinear, i.e. both $\varphi(a)$ and $\varphi(b)$ lie on a line, $L$ say, that goes through the origin. Then $\varphi(a)+\varphi(b)$ is also on line $L$. Notice that the line $L$ is the perpendicular bisector of the line segment $p_1p_2$. See Figure 4.2. Thus $p_1$ and $p_2$ must be equidistant from line $L$. However, we know that $\varphi(a)+\varphi(b) \in \{p_1, p_2\}$, since one of these two points must correspond to the trivial $(g_1, g_2, g_3)$ combination. Thus $p_1 = p_2 = \varphi(a) + \varphi(b)$ and the three circles only intersect at one point. Therefore $\varphi|_A$ is a homomorphism.
Our goal here is to show that we can compose with trivial weak Cayley table mappings so that the restriction to $A$ of the resulting map (which we still call $\varphi$) is the identity on $A$.

Since $\varphi|_A$ is a homomorphism, it can be represented as a matrix $M(\varphi)$ acting on $\mathbb{Z}$ with basis $v_x$ and $v_y$. Then $M(\varphi)$ must have integer entries but also be invertible. Thus the determinant of $M(\varphi)$ must be $\pm 1$. Let $G$ be any wallpaper group except $p2$. Choose two elements $a$ and $b \neq a^{-1}$ from an arbitrary conjugacy class in $A$. (One can check that for any group $G$ except $p2$ we can find an $a, b$ pair that satisfies this.) Thinking of elements of $A$ as points on $\mathfrak{L}$ we see that $a$ and $b$ lie on the boundary of the ball $B_{|a|}$ and that $\varphi(a), \varphi(b)$ lie on the boundary of the ball $B_{|\varphi(a)|}$. Given any $N \in \mathbb{N}$, there are only finitely many points of $\mathfrak{L}$ inside $B_N$, hence if we suppose that $\varphi$ has infinite order, then there must be some $k \in \mathbb{Z}$ such that $\varphi^k(a), \varphi^k(b)$ lie outside of $B_N$. The points $\{0, a, b\}$ and $\{0, \varphi^k(a), \varphi^k(b)\}$ form triangles (since $b \neq a^{-1}$) and the determinant of $M(\varphi) = \pm 1$ implies that the area of the triangle with vertices $\{0, a, b\}$ must be the same as the area of the triangle formed by $\{0, \varphi^k(a), \varphi^k(b)\}$. Since we can make $N$ arbitrarily large, this is a contradiction. Thus $\varphi|_A$ must have finite order.

The following can be checked:
Lemma 4.1. Given a wallpaper pattern in $\mathbb{E}^2$ and its corresponding group $G$, if $\omega$ is an isometry of the wallpaper pattern then there exists $\psi \in W_0(G)$ such that the action of $\psi$ on $A$ is equal to the action of $\omega$ on $\mathbb{E}^2$. \hfill \qed

We have shown that $M(\varphi)$ is an integer matrix of finite order that preserves $\mathfrak{L}$. By the crystallographic restriction we know it must have order 1, 2, 3, 4, or 6. It thus corresponds to a rotation or a reflection of $\mathfrak{L}$, and so by Lemma 4.1 $M(\varphi)$ corresponds to some $\psi \in W_0(G)$. We may compose with $\psi$ to get a new $\varphi$ that satisfies $\varphi|_A = \text{Id}$.

Now we consider the $\mathfrak{p}2$ case. In this group the action of $\rho$ on elements of $A$ is inversion. Homomorphisms respect inverses, and hence any mapping $\psi : \rho \mapsto \rho, x \mapsto a_1, y \mapsto a_2$ for some $a_1, a_2 \in A$ that give $\det M(\psi) = \pm 1$ will define a homomorphism. Thus we see that there exists $\psi \in W_0(G)$ that allows us to compose to get a $\varphi|_A = \text{Id}$. We have proven

Proposition 4.2. Given any wallpaper group $G$, for $\varphi \in W(G)$ there exists $\psi \in W_0$ such that $(\varphi \circ \psi)|_A = \text{Id}_A$. \hfill \qed

Examples:

The group $\mathfrak{p}2\text{gg}$ has one conjugacy class that lies on $B_1 : x^G = \{x^{\pm 1}, y^{\pm 1}\} = A \cap B_1$. We can assume $\varphi(x^G) = x^G$ since $M(\varphi)$ has determinant $\pm 1$. If $\varphi(x) \in \{y^{\pm 1}\}$ we may compose with $\psi$ as defined in Lemma 2.4 which trades $x$ and $y$. If $\varphi$ inverts $x$ we compose with the inner automorphism $I_\rho$ so that we have $\varphi(x) = x$. Then if necessary we can compose with $I_\gamma$ so that we have $\varphi(y) = y$. In this example, the original matrix $M(\varphi)$ corresponds to some combination of the following isometries: rotation by $\pi$, reflection about $\langle x \rangle$, or reflection about $\langle xy \rangle$. The corresponding automorphisms are $I_\rho, \psi$, and $I_\gamma$.

The group $\mathfrak{p}4$ has one conjugacy class that lies on $B_1 : x^G = \{x^{\pm 1}, y^{\pm 1}\} = A \cap B_1$. Again, $M(\varphi)$ has determinant $\pm 1$ implies that $\varphi(x^G) = x^G$. By composing with some power $I_\rho$ we can arrange $\varphi(x) = x$. Now if $\varphi(y) = y^{-1}$ we can compose with $\psi_1$ as defined in Lemma 2.4 to get $\varphi(y) = y$, and now $\varphi|_A = \text{Id}$. Here, the original matrix $M(\varphi)$ corresponds to some combination of the following isometries: rotation by $\pi/2$, reflection about $\langle x \rangle$. The corresponding automorphisms are $I_\rho$ and $\psi_1$. 37
The group \textbf{p31m} has two conjugacy classes that lie on $B_1$: $x^G = \{x, y^{-1}, x^{-1}y\}$ and $y^G = \{x^{-1}, y, xy^{-1}\}$. Here $x^G \cup y^G = A \cap B_1$. Since $M(\varphi)$ has determinant $\pm 1$ we must have $\varphi(x^G) \in \{x^G, y^G\}$. If we have $\varphi(x^G) = y^G$ then we may compose with the inverse map $\iota$ and now we have each conjugacy class going to itself. Now we need only compose with inner automorphisms to get each element in $x^G$ to map to itself and we’re done. In this example, the original matrix $M(\varphi)$ corresponds to some combination of the following isometries: rotation by $2\pi/3$, reflection about $\langle x \rangle$, or reflection about $\langle x^{-1}y \rangle$. The corresponding nontrivial elements of $W_0$ are $I_\rho, I_\sigma$, and $\iota$. 
Chapter 5. Steps Three and Four

Step Three

We need to show that we may compose if necessary to obtain a new $\varphi$ that maps each coset of $A$ to itself while preserving $\varphi|_A = Id$. Throughout this chapter we will assume that $\varphi|_A = Id$.

We know by Proposition 3.1 (vii) that $\varphi$ maps cosets of $A$ to cosets of $A$. We will show that in most situations $\varphi$ maps cosets to themselves and otherwise, we may compose so that this is case. There are four wallpaper groups with $|G/A| = 2$ so we only need to consider the other twelve groups. We begin by considering the cosets that contain involutions.

Lemma 5.1.

(i) Recall that $\rho_{\pi}$ is rotation by $\pi$ radians. Let $G$ be a wallpaper group containing $\rho_{\pi}$. Then $\varphi(A\rho_{\pi}) = A\rho_{\pi}$.

(ii) Let $\sigma \in F$ be a reflection so that $A\sigma$ is a reflection coset. Then $\varphi(A\sigma) = A\sigma$.

Proof. (i) Every element in $A\rho_{\pi}$ has order two and this is the unique coset in $G$ with this property. By Proposition 3.1 (v) $\varphi$ maps involutions to involutions so we are done.

(ii) Reflection cosets contain elements that are not involutions but they also contain infinitely many involutions. This is never true of the other types of cosets of $A$. By Proposition 3.1 (v), (vii) $\varphi(A\sigma)$ must be a reflection coset. If $A\sigma$ is the only reflection coset in the group then we are done. Otherwise, suppose that $\varphi(A\sigma) = A\sigma_2$ where $A\sigma_2 \neq A\sigma$. By Lemma 2.1 $\varphi(\sigma) = b\sigma_2$ implies that $b \in L^\perp(\sigma_2)$. There exists $a \in L^\perp(\sigma)$ such that $ab \notin L^\perp(\sigma_2)$. But then $\varphi(a \cdot \sigma) \sim \varphi(a)\varphi(\sigma) = ab\sigma_2$ which is not possible since $a\sigma$ is order two and $ab\sigma_2$ is not.

An immediate consequence of this is

Corollary 5.1. If $G$ is a group of type $c2mm$, $p2mm$ or $p2mg$ then $\varphi$ maps every coset of $A$ to itself.
Now if $G$ is a group without reflections or glide reflections, then $G/A$ is isomorphic to a cyclic group of order 2, 3, 4, or 6. Otherwise, $G/A$ is dihedral; it is one of $C_2, C_2 \times C_2 = D_4, S_3 = D_6, D_8,$ or $D_{12}$. By Proposition 3.1 (viii), $\varphi : G \to G$ induces a weak Cayley table map $\bar{\varphi} : G/A \to G/A$. It is known that dihedral groups have trivial weak Cayley table groups ([Hu]) and of course abelian groups have trivial weak Cayley table groups and thus we know $\bar{\varphi}$ is an automorphism or an anti-automorphism. This will be helpful as we consider where $\varphi$ sends rotation cosets.

**Lemma 5.2.** Let $G$ be a wallpaper group having no glide reflection cosets. The map $\iota_A : (x, y, \rho, \sigma) \mapsto (x^{-1}, y^{-1}, \rho, \sigma)$ defines an automorphism of $G$, and the anti-automorphism $\iota_A \circ \iota \in W_0(G)$ allows us to compose if necessary so that $\bar{\varphi}(A\rho^k) = A\rho^k$ for every rotation coset $A\rho^k \in G/A$.

**Proof.** We can see that the automorphism $\iota_A$ satisfies the relations in the presentation of $G$ because each relation that involves $x$ and $y$ is defined by the action of conjugation of an element in $F$, and conjugation respects inverses. So then the anti-automorphism $\iota_A \circ \iota$ fixes $A$ but maps $\rho$ to $\rho^{-1}$. Since $\bar{\varphi}$ is an automorphism or anti-automorphism of $G/A$, we know that $\bar{\varphi}$ maps rotation cosets to rotation cosets of the same order. Since the rotations must be of order 2, 3, 4 or 6, this implies that $\bar{\varphi}(A\rho^k) \in \{A\rho^k, A\rho^{-k}\}$. Composing with $\iota_A \circ \iota$ we have $\varphi(A\rho) = A\rho$. Note that in groups with rotations of order six, $\bar{\varphi}(A\rho) = A\rho$ implies that $\bar{\varphi}(A\rho^2) \neq A\rho^4$ (because $\bar{\varphi} \in W_0(G/A)$). Hence after composing we have $\varphi$ sending every rotation coset to itself. 

**Corollary 5.2.** After composing with $\iota_A \circ \iota$ if necessary, we can assume $\bar{\varphi} = Id$ for the wallpaper groups $p3, p4, p6$. Applying Lemma 5.1 as well we also have this result for $p4mm, p3m1, p31m$ and $p6mm$. 

It remains to show that $\bar{\varphi} = Id$ for $p2gg$ and $p4mg$. The following Lemma will be applied in each case.
Lemma 5.3. Let \( G \) be a wallpaper group having glide reflections \( \gamma_1, \gamma_2 \in F \) such that

\[
\gamma_1^2 = x, \gamma_2^2 = y, (x, \gamma_1), (y, \gamma_2), x\gamma_2 = x^{-1}, y\gamma_1 = y^{-1}.
\]

Then \( \varphi(A\gamma_1) \neq A\gamma_2 \).

Proof. Assume to the contrary that \( \varphi(\gamma_1) = b\gamma_2 \) for some \( b \in A \). Then for any \( k \in \mathbb{Z} \), \( \varphi(x^k \cdot \gamma_1) \sim x^kb\gamma_2 \). Squaring both sides of this relation gives

\[
x^{2k+1} = \varphi(x^{2k+1}) = \varphi(x^k \gamma_1 \cdot x^k \gamma_1) \sim \varphi(x^k \gamma_1) \varphi(x^k \gamma_1) \sim x^kb\gamma_2 x^kb\gamma_2 = x^k b x^{-k} \gamma_2 b\gamma_2 = (b\gamma_2)^2
\]

which cannot be true for any \( k \in \mathbb{Z} \). \( \square \)

\( \text{p2gg} \) This group has three cosets: two glide reflection cosets and one rotation coset consisting of rotations of order two. Lemma 5.1 tells us \( \varphi(A\rho) = A\rho \). This group satisfies the hypothesis of Lemma 5.3 so we must have \( \bar{\varphi} = \text{Id} \).

\( \text{p4mg} \) For this group we have \( G/A \cong D_8 \). We have three rotation cosets: \( A\rho, A\rho^2, A\rho^3 \); two reflection cosets: \( A\rho\gamma, A\rho^2\gamma \); and two glide reflection cosets: \( A\gamma, A\rho^2\gamma \). Since \( \rho^2 \) has order two, Lemma 5.1 (i) gives \( \varphi(A\rho^2) = A\rho^2 \). Lemma 5.1 (ii) gives us \( \varphi(A\rho\gamma) = A\rho\gamma \), and \( \varphi(A\rho^3\gamma) = A\rho^3\gamma \).

Now we will use the fact that \( \bar{\varphi} \) is a trivial weak Cayley table map. This tells us that we must have \( \varphi(A\rho), \varphi(A\rho^3) \in \{A\rho, A\rho^3\} \) and \( \varphi(A\gamma), \varphi(A\rho^2\gamma) \in \{A\gamma, A\rho^2\gamma\} \) since an automorphism or an anti-automorphism cannot send a rotation coset to a glide reflection coset. If we have \( \varphi(A\rho) = A\rho^3 \) we may compose with the anti-automorphism \( \iota \circ \psi_2 \). (The automorphisms \( \psi_2 \) is defined in Lemma 2.4.) This preserves the identity map on \( A \) while interchanging the two order four rotation cosets. This group satisfies the hypothesis of Lemma 5.3 so we apply it and we are done.

We have shown

Theorem 5.3. Composing with an anti-automorphism if necessary, we can assume that \( \varphi(At) = At \) for all cosets \( At \) in any wallpaper group. \( \square \)
Step Four

We will show that we can compose so that we have a map that satisfies \( \varphi(t) = t \) for every \( t \in F \). As before, we compose with maps that maintain the identity on \( A \). In most cases, we will compose with an automorphism defined in Lemma 2.4. However, in the \( p4 \) case and in the \( p6 \) case, we will compose with nontrivial weak Cayley table maps to obtain this. These maps are defined in Lemma 2.5. Throughout the remainder of this chapter we will assume that \( \bar{\varphi} = \text{Id} \) (see Theorem 5.3).

First we consider the four wallpaper groups that have only rotation cosets, namely \( p2, p3, p4, \) and \( p6 \). We can assume that \( \varphi(\rho) = a\rho \) for some \( a \in A \). Note that the map \( \psi_a : (x, y, \rho) \mapsto (x, y, a\rho) \) is an automorphism of \( G \). This is true because every element of \( A\rho \) is a rotation of order \( |\rho| \) and thus the images of the generators under this map satisfy all the relations in the presentation of \( G \). Composing with the inverse of this map, we now have \( \varphi(\rho) = \rho \) and since \( \varphi \) respects inverses, we also have \( \varphi(\rho^{-1}) = \rho^{-1} \). Then for \( p2 \) and \( p3 \) we are done.

Now \( p4 \) has one more element of \( F \) to consider. Suppose that \( \varphi(\rho^2) = a\rho^2 \) for some \( a \in A \). Since \( \varphi(\rho) = \rho \) implies that \( \varphi(\rho^2) \sim \varphi(\rho)^2 = \rho^2 \), we can write \( \varphi(\rho^2) = (\rho^2)^b, b = x^iy^j \in A \). Composing with the nontrivial maps \( \tau_x^{-i} \) and \( \tau_y^{-j} \) we have \( \varphi(\rho^2) = \rho^2 \) as desired.

For \( p6 \) we likewise have \( \varphi(\rho^2) \sim \varphi(\rho)^2 = \rho^2 \). This implies that \( \varphi(\rho^2) = (xy)^k(xy^{-2})^m \rho^2 \) for some \( k, m \in \mathbb{Z} \). Then we may compose \( \varphi \) with the nontrivial weak Cayley table mappings \( \tau_{xy}^{-k}\tau_{xy^{-2}}^{-m} \) so that we have \( \varphi(\rho^2) = \rho^2 \) and then by inverses \( \varphi(\rho^4) = \rho^4 \). Using this result we have \( \varphi(\rho^3) \sim \varphi(\rho)\varphi(\rho^2) = \rho^3 \) therefore \( \varphi(\rho^3) = x^{2i}y^{2j} \rho^3 \) for some \( i, j \in \mathbb{Z} \). Composing with nontrivial maps \( \mu_{x^2}^{-i} \) and \( \mu_{y^2}^{-j} \) will give us \( \varphi(\rho^3) = \rho^3 \).

We will need a few lemmas as we consider \( \varphi(t) \) for \( t \in F \) in the remaining groups.

**Lemma 5.4.** Suppose that \( \varphi|_A = \text{Id}_A \) and that \( \varphi(\rho) = \rho \) for a rotation \( \rho \). Suppose that \( \sigma, \sigma \rho \) are distinct reflections in \( G \) with \( \varphi(A\sigma) = A\sigma \). Then \( \varphi(\sigma) = \sigma \).

**Proof.** Since \( \varphi(A\sigma) = A\sigma \) we can write \( \varphi(\sigma) = a\sigma, a \in A \). Since \( \sigma^2 = 1 \) we must have \( a \in L^+(\sigma) \). Now \( \varphi(\rho \cdot \sigma) \sim \rho \sigma \sigma = a^{\rho^{-1}} \rho \sigma \) and so \( a^{\rho^{-1}} \in L^+(\rho \sigma) \). But then \( a \in L^+(\sigma \rho) \).
Since $\sigma \neq \sigma \rho$ we have $a \in L^\perp(\sigma) \cap L^\perp(\sigma \rho) = \{1\}$, and we are done. \hfill \Box

**Lemma 5.5.** Let $\varphi|_A = \text{Id}|_A$ and assume $\varphi(A\gamma) = A\gamma$ for some glide reflection coset $A\gamma$. Then $\varphi(\gamma) = a\gamma$, $a \in A$, implies that $a \in L^\perp(\gamma)$.

**Proof.** Let $\beta = \gamma^2$. Squaring both sides of the equation $\varphi(\gamma) = a\gamma$ gives us $\beta \sim \beta a\gamma$. Since $\beta$ and $aa\gamma$ both lie on the line fixed by the action of $\gamma$, this implies that $\beta a\gamma = \beta^{\pm 1}$.

Then we have two possibilities: either $aa\gamma = 1$ which implies that $a \in L^\perp(\gamma)$, or $aa\gamma = \beta^{-2}$ in which case $a \in \beta^{-1}L^\perp(\gamma)$. Now let $L^\perp(\gamma) = \langle \alpha \rangle$ and suppose we have $a = \beta^{-1}\alpha^k$. Then $\varphi(\beta \cdot \gamma) \sim \alpha^k\gamma$. Squaring both sides gives $\beta^3 \sim \beta$ which is a contradiction. Thus we must have $a \in L^\perp(\gamma)$.

**Lemma 5.6.** Let $G$ be a wallpaper group with a reflection or glide reflection coset $Ar$. Assume $\varphi|_A = \text{Id}$ and $\varphi(Ar) = Ar$. Suppose that for $ar \in Ar$, we have $\varphi(ar) \sim ar$. Then $\varphi(ar) \in K_r ar$. In particular, $ar = \varphi(r) \sim r$ implies that $a \in K_r$.

**Proof.** If $\rho_x \notin G$, then $(ar)^G \cap Ar = K_r ar$ and we are done, so suppose $\rho_x \in G$. Let $\lambda$ be the commutator $(\rho_x, r^{-1})$, so that $r^{\rho_x} = \lambda r$. Let $\beta = r^2$ so then $r = \beta r^{-1}$. (If $r$ is a reflection then $\beta$ is trivial.) Thus

$$(ar)^G \cap Ar = K_r ar \cup K_r (ar)^{\rho_x} = K_r ar \cup K_r a^{-1}\lambda r.$$

First we note that if $\lambda = 1$ and $a^2 \in K_r$ we have nothing to prove, so suppose that is not the case. Since $\varphi(Ar) = Ar$ it suffices to show that $\varphi(ar) \notin K_r (ar)^r$. Let $K_r = \langle \alpha \rangle$. (We know $K_r$ is a line so it is generated by a single element in $A$.) Suppose to the contrary that $\varphi(ar) = \alpha^i(ar)^{\rho_x}$ for some $\alpha^i \in K_r$. Then for any $b \in A$

$$\varphi(b \cdot ar) \sim ba^i a^{-1}\lambda r.$$
Squaring both sides we have

\[ \varphi(ba\beta ba^r) = ba\beta ba^r \sim (ba^t a^{-1} \lambda \beta)(ba^t a^{-1} \lambda)^r. \]

Reordering the elements this becomes

\[ \beta aa^r bb^r \sim \beta \alpha^i (\alpha^i)^r a^{-1} (a^{-1})^r bb^r \lambda \lambda^r. \] \hspace{1cm} (5.1)

Recall that \( L(r) \) is the line fixed by the action of \( r \). Note that \( aa^r \) is in \( L(r) \) (since \( (aa^r)^r = a^r a = aa^r \)). Similarly \( bb^r, \lambda \lambda^r \in L(r) \). Recall that \( \alpha \in K_r \) which is perpendicular to \( L(r) \), and thus the conjugation action of \( r \) on \( \alpha \) inverts \( \alpha \). So we know \( \alpha^i (\alpha^i)^r = 1 \). Also, \( \beta \in L(r) \) (since \( \beta^r = (r^2)^r = r^2 = \beta \)). Thus equation (5.1) states that two elements of \( L(r) \) are conjugate to each other. This implies that they are equal or they are inverses. If they are inverses of each other, equation (5.1) becomes \( b^{-2}(b^{-2})^r = \beta^2 \lambda \lambda^r \), which can’t be true for all \( b \in B \). If they are equal to each other, we have \( a^2(a^2)^r = \lambda \lambda^r \). We can assume \( \lambda \) is not trivial in this case because then \( a^2 \in K_r \) and we assumed both of those cannot be true together. So then assuming \( \lambda \neq 1 \), we must have \( a^2(a^2)^r \in \langle x^4, y^4 \rangle \) but (looking at the three different \( \lambda s \) \( \lambda \lambda^r \notin \langle x^4, y^4 \rangle \)). This is a contradiction. Thus \( \varphi(ar) \notin K_r(ar)^r \). \( \square \)

**p1m1** and **c1m1** cases: We require \( \varphi(\sigma) = a\sigma \) to be an involution. Note that for any \( a \in A \) the map \((x, y, \sigma) \mapsto (x, y, a\sigma)\) is an automorphism. Composing we obtain \( \varphi(\sigma) = \sigma \).

**p1g1** case: Let \( \varphi(\gamma) = a\gamma \) for some \( a \in A \). By Lemma 5.5 \( a \in L^+(\gamma) = \langle \gamma \rangle \). Then we may compose with a power of the automorphism \( \psi_y \) and we are done.

**p2mm** case: Suppose that \( \varphi(\rho) = x^u y^v \rho \). We can compose \( \varphi \) with \((\psi_x)^{-u} \) and \((\psi_y)^{-v} \) to get \( \varphi(\rho) = \rho \). To get \( \varphi(\sigma) = \sigma \) and \( \varphi(\rho \sigma) = \rho \sigma \) we now apply Lemma 5.4 and we are done.

**p2mg** case: Suppose that \( \varphi(\rho) = x^u y^v \rho \). Composing with a power of \( \psi_y \) and then a power of \( \psi_x \) we can arrange that \( v = 0 \) and \( u = 0 \). Let \( \varphi(\sigma) = a\sigma \). The fact that \( \sigma^2 = 1 \) gives \( a \in L^+(\sigma) \). In this group this implies \( \varphi(\sigma) \sim \sigma \) and we may apply Lemma (5.6) and now we have \( a \in K_\sigma \). Let \( \varphi(\rho \sigma) = b\rho \sigma \), and so by Lemma (5.5) \( b \in L^+(\rho \sigma) \). Since \( b\rho \sigma = \varphi(\rho \cdot \sigma) \sim a^o \rho \sigma \),
we know \( a^\rho \) lies on \( L^+(\rho \sigma) \cup y^{-1}L^+(\rho \sigma) \). Thus \( a \in K_{\sigma} \cap (L^+(\rho \sigma) \cup yL^+(\rho \sigma)) = \{1\} \). This gives us \( \varphi(\rho \sigma) \sim \rho \sigma \) and so by Lemma (5.6) we know \( b \in K_{\rho \sigma} \). Now \( \sigma = \varphi(\rho \cdot \rho \sigma) \sim b^\rho \sigma \) which tells us \( b \in L^+(\sigma) \), and so \( b \) is also trivial.

**p2gg** case: Suppose that \( \varphi(\rho) = x^uy^v \rho \). By using \( \psi_x, \psi_y \) we can assume that \( u = v = 0 \). Let \( \varphi(\gamma) = a\gamma, \varphi(\rho \gamma) = b\rho \gamma \). By Lemma 5.5 \( a \in L^+(\gamma) \). Note that \( L^+(\gamma) \) contains elements from only two conjugacy classes, namely \( (\gamma)^G \) and \( (y\gamma)^G \). If \( a\gamma \sim y\gamma \), then we may compose with \( \psi \circ \iota \). So now we have \( \varphi(\gamma) \sim \gamma \) and so by Lemma 5.6 \( a \in K_{\gamma} \). Lemma 5.5 gives us \( b \in L^+(\rho \gamma) \) and thus \( b\rho \gamma = \varphi(\rho \cdot \gamma) \sim a^{-1}\rho \gamma \) implies that \( a \in L^+(\rho \gamma) \cup yL^+(\rho \gamma) \). This intersects \( K_{\gamma} \) trivially, hence \( \varphi(\gamma) = \gamma \). Now \( \varphi(\rho \cdot \gamma) \sim \rho \gamma \) so we may apply Lemma 5.6 which gives us \( b \in K_{\rho \gamma} \). We also know \( \gamma = \varphi(\rho \cdot \rho \gamma) \sim b^{-1} \gamma \) which tells us \( b \in K_{\gamma} \cup xyK_{\gamma} \). The only possibility then is that \( b = 1 \).

**c2mm** case: Suppose that \( \varphi(\rho) = x^u y^v \rho, \varphi(\sigma) = x^i y^j \sigma \). Then the fact that \( (\rho \sigma)^2 = 1 \) implies that

\[
1 = \varphi((\rho \sigma)^2) = \varphi(\rho \sigma)^2 = (\varphi(\rho)\varphi(\sigma))^2 = (x^u y^v \rho x^i y^j \sigma)^2,
\]

which is only true when \( u + j = i + v \), so that we can compose \( \varphi \) with the inverse of \( \psi_{u,v,i,j} \) to get \( \varphi(\rho) = \rho, \varphi(\sigma) = \sigma \). To get \( \varphi(\rho \sigma) = \rho \sigma \) we now apply Lemma 5.4 and we are done.

**p4mm** case: Suppose that \( \varphi(\rho) = a\rho, \varphi(\sigma) = b\sigma \). Then acting by an automorphism in \( \langle I_x, I_y \rangle \), we can assume \( \varphi(\rho) \in \{\rho, y\rho\} \). If we have \( \varphi(\rho) = y\rho \), then acting by the automorphism \( \psi_1^{-1} \) we see that \( \varphi(\rho) = \rho \); thus \( \varphi(\rho^3) = \rho^3 \). Now Lemma 5.4 gives \( \varphi(\rho^k \sigma) = \rho^k \sigma, 0 \leq k \leq 3 \). Now let \( a \in A \) and suppose we have \( \varphi(\rho^2) = a\rho^2 \). Then for any reflection \( t \in \{\sigma, \rho \sigma, \rho \sigma^2 \rho \sigma^3 \} \) we have

\[
\rho^2 t = \varphi(\rho^2 t) \sim \varphi(\rho^2) \varphi(t) = a\rho^2 t,
\]

which tells us that \( a \in L^+(\rho^2 t) \) by Lemma 2.1. However this must be true for all four reflections \( t \in F \), consequently \( a \) must be trivial. Thus we have \( \varphi(t) = t \) for all \( t \in F \) in this group.
**p3m1** case: Suppose that \( \varphi(\rho) = a\rho \). Composing with an element of \( \langle \psi_1, I_x \rangle \) we can arrange that \( \varphi(\rho) = \rho \). Thus \( \varphi(\rho) = \rho, \varphi(\rho^2) = \rho^2 \). We now apply Lemma 5.4 so that we have \( \varphi(\rho^k\sigma) = \rho^k\sigma, k = 0, 1, 2 \), and we are done.

**p31m** case: In this group, the \( A\rho \) coset is one conjugacy class. Thus, composing with an inner automorphism in \( \langle I_x, I_y \rangle \) we can arrange to have a map that obeys \( \varphi(\rho) = \rho \) and thus \( \varphi(\rho^2) = \rho^2 \). We apply Lemma 5.4 to get \( \varphi(\rho^k\sigma) = \rho^k\sigma, k = 0, 1, 2 \), as required.

**p6mm** case: In this group, any two elements of the \( A\rho \) coset are conjugate, so we can compose with an inner automorphism in \( \langle I_x, I_y \rangle \) to arrange so that we have \( \varphi(\rho) = \rho \). We apply Lemma 5.4 to get \( \varphi(\rho^k\sigma) = \rho^k\sigma, 0 \leq k < 5 \). Now assume \( \varphi(\rho^2) = a\rho^2 \) for some \( a \in A \). By the same argument used for **p4mm**, \( a \) must be contained in the union of the six \( L^\perp(t) \) lines, thus it is trivial. This completes this case and concludes the proof of Step Four. □
Step Five

**Proposition 6.1.** Let \( t \in F \) so that \( At \) is a coset of \( A \) in the wallpaper group \( G \). Assume \( \varphi|_A = \text{Id} \) and that \( \varphi(t) = t \). Then there exists a \( \delta \in F \) such that \( \varphi(at) = a^\delta t \) for all \( at \in At \). \( \square \)

**Proof.** We begin with a lemma:

**Lemma 6.2.** Let \( t \in G \) and suppose \( \varphi(t) = t \). If \( a, b \in A \) satisfy \( \varphi(at) = bt \), then \( a \sim b \).

**Proof.** Since \( \varphi|_A = \text{Id} \) and \( \varphi(t) \) we have:

\[
a \sim a^t = \varphi(a^t) = \varphi(t^{-1} \cdot at) \sim \varphi(t^{-1}) \cdot \varphi(at) = t^{-1}bt \sim b. \quad \square
\]

This lemma tells us that for \( t \in F \), \( \varphi(at) = a^\alpha t \) for some \( \alpha \in F \). Define \( r_a, r_b \in F \) to satisfy \( \varphi(at) = a^{r_a}t \) and \( \varphi(bt) = b^{r_b}t \). We have

\[
ab^{-1} = at \cdot (bt)^{-1} = \varphi(at \cdot (bt)^{-1}) \sim a^{r_at} \cdot t^{-1}(b^{r_b})^{-1} = a^{r_a(b^\alpha)^{-1}}.
\]

Hence \( ab^{-1} = (a^{r_a}(b^{r_b})^{-1})^f \) for some \( f \in F \). Letting \( \alpha = r_a f \) and \( \beta = r_b f \) gives us

\[
(a^{-1})^\alpha a = (b^{-1})^\beta b. \tag{6.1}
\]

For \( a \in A \), let \( C_a \) denote the circle in \( \mathbb{E}^2 \) with center at \( v_a \) and having a radius of \( |v_a| \). Now let \( S_a = \{v_{a(a^{-1})^f} : f \in F \} \). Then \( S_a \) consists of \( |a^G| \) points that lie on \( C_a \) and we note that \( (0, 0) \in S_a \) for all \( a \in A \). Since two distinct circles can meet in at most two points, equation (6.1) gives \( |S_a \cap S_b| \in \{1, 2\} \).

**Lemma 6.3.** (i) If \( |S_a \cap S_b| = 1 \) then there exists \( \delta \in F \) satisfying \( a^{r_a} = a^\delta \) and \( b^{r_b} = b^\delta \).

(ii) If \( (0, 0), v_a, v_b \) are collinear, then there exists \( \delta \in F \) satisfying \( a^{r_a} = a^\delta \) and \( b^{r_b} = b^\delta \).
Proof. (i) If $|S_a \cap S_b| = 1$ then if $S_a \cap S_b = \{(0,0)\}$ and so equation (6.1) gives $a(a^{-1})^\alpha = b(b^{-1})^\beta = 1$ so we must have $a = a^\alpha$, $b = b^\beta$. Then $a^{f^{-1}} = a^{r_a}, b^{f^{-1}} = b^{r_b}$. Let $\delta = f^{-1}$ and we have proven (i).

(ii) We may assume that $a \neq b$. The points $v_a$ and $v_b$ are the centers of the circles $C_a$ and $C_b$. Thus by hypothesis the centers of these two circles are on a line containing $(0,0)$. Since $a \neq b$ we have $C_a \neq C_b$. Since $(0,0)$ is common to $C_a$ and $C_b$ we see that $C_a \cap C_b = \{(0,0)\}$ and thus $S_a \cap S_b = \{(0,0)\}$. Then by (i) we are done.

\[ \]

**Lemma 6.4.** Let $b \notin H$. Then for all $a \in A$ there is a unique $f = f_{a,b} \in F$ such that $\varphi(at) = a^f t, \varphi(bt) = b^f t$.

**Proof.** Recall that $b \notin H$ means that $|b^G| = |F|$. This means that the action on $b$ of each element in $F$ yields a unique conjugate of $b$. Then for any $a, b$ pair, it suffices to prove the existence of $f = f_{a,b}$; uniqueness comes from the fact that $b \notin H$.

If $|S_a \cap S_b| = 1$ then Lemma 6.3 (i) gives us the existence of $f$ and if $(0,0), v_a, v_b$ are collinear then Lemma 6.3 (ii) gives us the existence of $f$. So then assume that $|S_a \cap S_b| = 2$ and $(0,0), v_a, v_b$ are not collinear.

Lemma 6.3 (ii) tells us that there exists $f \in F$ such that for all $k \in \mathbb{N}$, $(b^k)^{r_b} = (b^k)^f$. Since $|S_a|$ is finite, there must be some $k \in \mathbb{N}$ such that $|S_a \cap S_{b^k}| = 1$. This situation is depicted in Figure 6.1.

Thus we have $h, h' \in F$ satisfying $b^{h^k} = b^h$, $(b^{h^k})^{r_b} = (b^{h^k})^f$ as well as $b^{h'^k} = (b^{h'})^h, a^{r_a} = a^{h'}$. However we must have $h = h'$ because of the uniqueness that comes from $b \notin H$. Then $a^{r_a} = a^h$ and $b^{h^k} = b^h$.

Now let $b, c, d \in A, b \notin H$. By Lemma 6.4 there exists unique $f, h \in F$ satisfying $\varphi(bt) = b^f t, \varphi(ct) = c^f t, \varphi(bt) = b^h t, \varphi(dt) = d^h t$. However because $b \notin H$ we know that $b^f = b^h$ implies that $f = h$. Then by fixing $c$ and varying $d$ it follows that for all $a \in A, \varphi(at) = a^f t$ where $f$ is unique and completely determined by $b$. \[ \]

**Step Six**
For this final step we need to identify weak Cayley table isomorphisms that we may compose with $\varphi$ to get $\varphi = Id$. In most cases we will compose with an element in $W_0$ but for groups of type $p2mm$, $p3$, $p4$ and $p6$ we will compose with the nontrivial weak Cayley table mappings defined in Lemma 2.5.

For the remainder of this chapter we will assume that we have $\varphi|_A = Id$ and $\varphi(t) = t$ for all $t \in F$. We will also assume that for every coset $At$, there exists $f \in F$ such that $\varphi(at) = af t$ for all $a \in A$.

The groups of type $c1m1$, $p1m1$, $p1g1$, $p2$

Let $F = \{1, t\}$, or in other words, let $t$ be the non-translation generator in the group presentation. The map $\psi : (x, y, t) \mapsto (x^{-1}, y^{-1}, t^{-1})$ defines an automorphism of the group. (Note that in the $p2$ case, $\psi$ is the inner automorphism $I_\rho$.) Now we know $\varphi(at) = af t$ for some $f \in F$. If $f = 1$ we are done and if $f = t$ then we may compose with $\psi \circ \iota$ and we are done. We have proved

**Corollary 6.1.** The groups of type $c1m1$, $p1m1$, $p1g1$, $p2$ have trivial weak Cayley table groups.

Before we consider groups having $|F| > 2$ we will prove a few lemmas that narrow the
possibilities for values of $f$.

**Lemma 6.5.** Let $r \in F$ be a reflection or a glide reflection. Then for all $a \in A$, $\varphi(ar) \in \{ar, a^r r\}$.

**Proof.** Since $ar \sim \varphi(a \cdot r)$ we may apply Lemma 5.6. Then $\varphi(ar) = a^r r \in K_r ar$ for some $t \in F$. This implies that $a^{-1}a^t \in K_r$ for all $a \in A$. But $a^{-1}a^t = (a, t)$ and $K_t = \{(a, t) : a \in A\}$ shows that $K_t \subset K_r$. From Lemma 2.3 (iii) it follows that either $K_t = \{1\}$ or $K_t = K_r$, from which we see that either $t = 1$ or $t = r$.

The following applies to the groups of type $p4mm$, $p4mg$, $p31m$, $p3m1$, and $p6mm$.

**Theorem 6.6.** Let $G$ be a wallpaper group having a nontrivial rotation $\rho \neq \pi$ radians, and distinct reflection or glide reflection cosets $Ar$ and $A\rho r$. Then $\varphi|_{Ar} = Id$.

**Proof.** Recall that for any group that is one of these five types,

$$(ar)^G \cap Ar \subseteq K_r ar \cup K_r(a\beta_1, \beta_2)^{-1}r,$$

where $\beta_1\beta_2 = xy$ when $G$ is of type $p4mg$ but is trivial otherwise. Suppose to the contrary that $\varphi|_{Ar} \neq Id_{Ar}$. Then by Lemma 6.5 we must have $\varphi(ar) = a^r r$ for all $a \in A$. Now $\rho r$ is also a reflection, and so we have:

$$a^r \cdot \rho r \sim \varphi(a \cdot \rho r) = \varphi(a \cdot r) \sim ar\rho.$$

Thus for all $a \in A$ we must have (i) $a^r a^{-1} \in K_{\rho r}$ or (ii) $a^r a \in K_{\rho r}(\beta_1\beta_2)^{-1}$. Since $r$ and $\rho r$ are distinct reflections, case (i) is only true for $a$ which satisfy $a^r a^{-1} = 1$. Any such $a \in A$ must lie on $L(r)$. Now since $\rho$ is not rotation by $\pi$ radians, $L(r)$ and $K_{\rho r}$ do not coincide nor are they parallel. Hence $L(r) \cap K_{\rho r}(\beta_1\beta_2)^{-1}$ is exactly one point. (If $G$ is of type $p4mg$ then when $r = \rho \gamma$ this point is $xy^{-1}$, when $r = \rho^2 \gamma$ it is $y^{-2}$, and when $r = \rho^3 \gamma$ it is $x^{-1}y^{-1}$. In all other cases, this point is the identity.) Thus case (ii) is only true for $a \in L^\perp(r) y^{-1}$ or in $L^\perp(r)$. This shows that the only $a \in A$ that satisfy $a^r \rho r \sim ar\rho$ are in the union of two lines.
(L(r) ∪ L⊥(r) or L(r) ∪ L⊥(r)y^{-1}) which is not all of A. Thus \( \varphi(ar) \neq ar \) is not possible.

We conclude that \( \varphi|_A = Id. \)

\[ \square \]

**Corollary 6.2.** The groups of type \( p4mm, p3m1, p31m, p6mm \) have trivial weak Cayley table groups.

\[ \square \]

The following will be applied to the four group types \( c2mm, p2mm, p2mg, p2gg \).

**Theorem 6.7.** Let \( G \) be a wallpaper group having at least two cosets that are reflection or glide reflection cosets. Suppose we have \( \varphi|_A = Id \) for all reflection cosets \( A_\sigma \) as well as \( \varphi|_{A_\gamma} = Id \) for all glide reflection cosets \( A_\gamma \). Then \( \varphi = Id. \)

Let \( Ar \) denote a reflection or glide reflection coset. Let \( A\rho^k \) be a rotation coset. We know here is some \( t \in F \) such that \( \varphi(a\rho^k) = a't^k \) for all \( a \in A \). By hypothesis we have \( \varphi(ra^{-1}) = ra^{-1} \) which gives

\[ r\rho^k \sim \varphi(ra^{-1} \cdot a\rho^k) \sim ra^{-1} \cdot a't^k = (a^{-1}a't)^r \cdot r\rho^k. \]

(The last equality is true because the action of \( r \) is the same as the action of \( r^{-1} \).) This gives \( (a^{-1}a't)^r \in K_{r\rho^k} \cup K_{r\rho^k}(\beta_1\beta_2)^{-1} \) for all \( a \in A \). Write \( \beta_r = ((\beta_1\beta_2)^{-1})^{r^{-1}} \) so that \( (a,t) \in K_{\rho^k} \cup K_{\rho^k}\beta_r \) for all \( a \in A \). Now the element \( t \in F \) depended upon \( k \) and not on \( r \). Then the fact that we have at least two reflection or glide reflection cosets, say \( r, r' \in F \) means that

\[ (a,t) \in (K_{\rho^k} \cup K_{\rho^k}\beta_r) \cap (K_{\rho^{k'}} \cup K_{\rho^{k'}}\beta_{r'}). \]

This intersection is trivial if neither \( r \) nor \( r' \) are glide reflections. Otherwise, this intersection is finite and therefore bounded. Thus there exists a \( t \in F \) such that for all \( a \in A \), we must have \( (a,t) \) in this small intersection. This can only be true if \( t \) is trivial. So \( a' = a \) and \( \varphi(a\rho^k) = a\rho^k \). This completes the proof. \[ \square \]

The groups of type \( c2mm \)
By Lemma 6.5 we know that \( \varphi(a\sigma) = a^s\sigma \), and \( \varphi(a\rho\sigma) = a^u\rho\sigma \) for some \( s \in \{1, \sigma\}, u \in \{1, \rho\sigma\} \), for all \( a \in A \). If we have \( u = \rho\sigma \), we may compose with \( \psi \circ \iota \) so that we have \( u = 1 \).

If we assume \( s = \sigma \) we have

\[
x y \rho \sim \varphi(xy \cdot \rho) = \varphi(x\sigma \cdot x\rho\sigma) \sim y\sigma x\rho\sigma = y^2 \rho.
\]

Since \( x y \rho \sim y^2 \rho \), this is not possible so we must have \( s = 1 \). We apply Theorem 6.7 and now \( \varphi = \text{Id} \).

**The groups of type p2mm**

By Lemma 6.5 we know that \( \varphi(a\sigma) = a^s\sigma \), and \( \varphi(a\rho\sigma) = a^u\rho\sigma \) for some \( s \in \{1, \sigma\}, u \in \{1, \rho\sigma\} \), for all \( a \in A \). If we have \( s = \sigma \), we may compose with \( \tau_\sigma \) so that we have \( s = 1 \). If we have \( u = \rho\sigma \) we may compose with \( \psi_1 \circ \tau_\sigma \circ \psi_1 \). Then Theorem 6.7 will give us \( \varphi = \text{Id} \).

Thus, **Corollary 6.3.** A group of type p2mm has a non-trivial weak Cayley table group generated by \( \mathcal{W}_0 \) and \( \tau_\sigma \).

**The groups of type p2mg**

By Lemma 6.5 we know that \( \varphi(a\sigma) = a^s\sigma \), and \( \varphi(a\rho\sigma) = a^u\rho\sigma \) for some \( s \in \{1, \sigma\}, u \in \{1, \rho\sigma\} \), for all \( a \in A \). Now if \( s = \sigma \), then

\[
\rho \sigma = \varphi(\rho\sigma) = \varphi(y\sigma \cdot \rho) \sim y^8 \sigma \rho = y^8 y^{-1} \rho \sigma = y^{-2} \rho \sigma,
\]

which contradicts Lemma 2.3. Thus \( s = 1 \). Lastly, if \( u = \rho\sigma \), then

\[
x \sigma = \varphi(x\sigma) = \varphi(x\sigma \rho \cdot \rho) = \varphi(xy^{-1}\rho\sigma \cdot \rho) \sim (xy^{-1})^u \rho \sigma \rho = (xy^{-1})^u y \sigma,
\]

so that \( (xy^{-1})^u y = x^{-1} y^{-1} y = x^{-1} \) has the form \( xy^{2k} \) or \( x^{-1} y^{2k} y^{-1} \), both of which are not possible. Thus \( u = 1 \). Now Theorem 6.7 gives us \( \varphi = \text{Id} \).

**The groups of type p2gg**
We know there are \( s, u \in F = \{1, \rho, \gamma, \rho\gamma\} \) such that \( \varphi(a\gamma) = a^s\gamma \), and \( \varphi(a\rho\gamma) = a^s\rho\gamma \) for all \( a \in A \). By Lemma 6.5 we know \( s \in \{1, \gamma\}, u \in \{1, \rho\gamma\} \).

Let \( X = \langle x^2 \rangle, Y = \langle y^2 \rangle \). We have

**Lemma 6.8.** Let \( G \) be a group of type \( \mathfrak{p}2\mathfrak{g}\) and let \( a, b \in A \). Then

(i) \( a\gamma \sim b\gamma \) if and only if \( ab^{-1} \in Y \) or \( ab \in x^{-1}yY \).

(ii) \( a\rho\gamma \sim b\rho\gamma \) if and only if \( ab^{-1} \in X \) or \( ab \in xy^{-1}X \).

(iii) If \( v \in F \) and \( a \in A \), then \( a^s \in x^{-1}yY \cup xy^{-1}X \).

**Proof.** Parts (i) and (ii) follow from the presentation. For (iii) note that for \( a = x^iy^j \) we have \( a^\rho = a^{-1}, a^\gamma = x^iy^{-j}, a^{\rho\gamma} = x^{-i}y^j \), which gives (iii).  

Now for all \( a \in A \) we have:

\[
(a^{-1})^{\gamma u} \rho \gamma = a^{\rho u} \rho \gamma = \varphi(a^{\rho \gamma} \rho \gamma) = \varphi((a^{\rho \gamma})^{\rho \gamma}) \sim \varphi(a^{\rho \gamma}) \sim \varphi(\rho \cdot a) = \varphi(\rho a^{\gamma \gamma}) = \rho(a^{\gamma})^{\rho \gamma} = a^{\gamma s \rho \gamma} = (a^{-1})^{\gamma s \rho \gamma}.
\]

From Lemma 6.8 we thus have \( a^s(a^{-1})^u \in X \) for all \( a \in A \). Let \( a = x^iy^j \), and recall that \( u \in \{1, \rho\gamma\} \). We now show that the above implies that \( s = 1 \), for if \( s = \gamma \), then the possibility \( u = 1 \) gives \( a^s(a^{-1})^u = (x^iy^{-j})(x^{-i}y^{-j}) = y^{-2j} \notin X \); while the possibility \( u = \rho\gamma \) gives \( a^s(a^{-1})^u = (x^iy^{-j})(x^{-i}y^{-j}) = x^{2i}y^{-2j} \notin X \). Thus we now have \( s = 1 \).

Similarly, for all \( a \in A \) we have:

\[
a^{\gamma \gamma} = \varphi(a^{\gamma \gamma}) = \varphi(\rho \cdot a^{\rho \gamma} \rho \gamma) \sim \rho a^{\rho \gamma u} \rho \gamma = a^{\gamma s \rho \gamma},
\]

from which we obtain \( (a^{-1})^\gamma a^{\gamma u} \in Y \) for all \( a \in A \); or \( a^{-1}a^u \in Y \) for all \( a \in A \). However, if \( u = \rho\gamma \), then \( x^{-1}x^{\rho\gamma} = x^{-2} \), a contradiction. Thus \( u = 1 \). Then by Theorem 6.7 we have \( \varphi = Id. \)

We have shown that
Corollary 6.4. The groups of type \texttt{c2mm, p2mg, p2gg} have trivial weak Cayley table groups.

The groups of type \texttt{p3}

We have \( \varphi(a \rho) = a^f \rho \) for \( f \in \{1, \rho, \rho^2\} \). If \( f = \rho \) or \( \rho^2 \) we may compose with a power of \( \tau \) so that \( f = 1 \). Since \( \varphi \) respects inverses we have \( \varphi = Id \) on the other coset as well. Hence we know that

Corollary 6.5. The groups of type \texttt{p3} have a non-trivial weak Cayley table group generated by \( \mathcal{W}_0 \) and \( \tau \).

The groups of type \texttt{p4}

Assume \( \varphi(a \rho) = a^u \rho, \varphi(a \rho^2) = a^v \rho^2 \). If \( u = \rho^k \) then composing with \( \mu_{\rho^k} \) gives us a new \( \varphi \) satisfying \( \varphi|_{A_{\rho}} = Id \). Again, since \( \varphi \) respects inverses we also have \( \varphi|_{A_{\rho^2}} = Id \). Now suppose that \( v = \rho \). Then \( \varphi(x^2 \rho^2) = (x^2)^\rho \rho^2 = y^2 \rho^2 \) but then \( \varphi(x \cdot x \rho^2) \sim x \cdot x^\rho \rho^2 = xy \rho^2 \). This is a contradiction because \( y^2 \rho^2 \) is not conjugate to \( xy \rho^2 \). By a similar argument we cannot have \( v = \rho^3 \). If we have \( v = \rho^2 \) then we compose with \( \tau_{\rho^2} \) and now we have the identity on the remaining coset. Recall that in Step Four we composed with non-trivial maps \( \tau_x \) and \( \tau_y \).

Corollary 6.6. The groups of type \texttt{p4} have a non-trivial weak Cayley table group generated by \( \mathcal{W}_0, \tau_x, \tau_y, \mu_{\rho}, \tau_{\rho^2} \).

The groups of type \texttt{p6}

Let \( s, u, v \in F \) satisfy

\[
\varphi(a \rho) = a^s \rho, \varphi(a \rho^2) = a^u \rho^2, \varphi(a \rho^3) = a^v \rho^3.
\]

By composing with an element of \( \langle \tau_{\rho^2} I_{\rho^3}, \mu_{\rho^2} I_{\rho^3} \rangle \) we can reduce to the case where \( s = 1 \) while preserving \( \varphi|_A = Id \) and \( \varphi(f) = f \) for all \( f \in F \).

Lemma 6.9. Suppose that for some \( t \in F = \langle \rho \rangle \) we have \( a^{-1} a^t \in K_{\rho^2} \cap K_{\rho^3} \) for all \( a \in A \).

Then \( t = 1 \).
Proof. Recall that for a group of type \textbf{p6} we have the subgroups $K_{\rho^2} = \langle xy, x^{-2}y \rangle$ and $K_{\rho^3} = \langle x^2, y^2 \rangle$. Then $K_{\rho^2} \cap K_{\rho^3} = \langle x^2y^2, x^4y^{-2} \rangle$. Now when $a = y$ then

$$\{a^{-1}a^t : t \in F\} = \{1, x^{-1}, x^{-1}y^{-1}, y^{-2}, xy^{-2}, xy^{-1}\}.$$  

The intersection of this set with $K_{\rho^2} \cap K_{\rho^3}$ contains only the identity. This proves the Lemma. \hfill \square

Now for all $a \in A$ we have

\begin{align*}
a^u \rho^2 &= \varphi(a \rho \cdot \rho) \sim a \rho^2 \text{ which gives } a^{-1}a^u \in K_{\rho^2}; \quad (6.2) \\
a^v \rho^2 &= \varphi(a \rho^3 \cdot \rho^{-1}) \sim a^v \rho^2 \text{ which gives } (a^{-1})^u a^v \in K_{\rho^2}; \quad (6.3) \\
a^v \rho^3 &= \varphi(a \rho \cdot \rho^2) \sim a \rho^3 \text{ which gives } a^{-1}a^v \in K_{\rho^3}; \quad (6.4) \\
a^v \rho^3 &= \varphi(a \rho^2 \cdot \rho) \sim a^v \rho^3 \text{ which gives } a^u(a^{-1})^v \in K_{\rho^3}. \quad (6.5)
\end{align*}

Combining equations (6.2) and (6.3) we have $a^{-1}a^v \in K_{\rho^2}$ for all $a \in A$. Together with equation (6.4) this tells us $a^{-1}a^v \in K_{\rho^2} \cap K_{\rho^3}$ for all $a \in A$. Then by Lemma 6.9 we must have $v = 1$. Similarly, equations (6.2) and (6.5) together with Lemma 6.9 imply $u = 1$. This gives us $\varphi|_{A_{\rho^2}} = Id$ and $\varphi|_{A_{\rho^3}} = Id$. We know $\varphi$ respects inverses so we have the identity on the remaining cosets as well. Recall that in Step Four we composed with the non-trivial maps $\tau_{xy}, \tau_{xy^{-2}}, \mu_{x^2}, \mu_{y^2}$. Thus we have shown that

**Corollary 6.7.** A group of type \textbf{p6} has a non-trivial weak Cayley table group generated by $W_0, \tau_{xy}, \tau_{xy^{-2}}, \mu_{x^2}, \mu_{y^2}, \mu_{\rho^3}$. \hfill \square
We have shown that we may assume $\varphi|_A = \text{Id}$ and that $\varphi(At) = (At)$ for all $t \in G$. Let

$$U = \langle xy, x^2 \rangle, \ X = \langle x^2 \rangle, \ Y = \langle y^2 \rangle, \ V = \langle xy \rangle, \ W = \langle xy^{-1} \rangle.$$ 

In the next two results we list the conjugacy classes and involutions:

**Lemma 7.1.** Let $G$ be a group of type $\text{p4mg}$ and let $a = x^iy^j \in A$. Then

$$\begin{align*}
(x^iy^j)^G &= \{x^{\pm i}y^{\pm j}, x^{\pm i}y^{\pm j}\}; \\
(a\rho)^G &= UA \rho \cup Uax\rho^3; \\
(a\rho^2)^G &= UA \rho^2; \\
(a\rho^3)^G &= UA \rho^3 \cup Uax\rho; \\
(a\gamma)^G &= YA \gamma \cup YA^{-1}x^{-1}y\gamma \cup Xa^{\rho}x\rho^2\gamma \cup Xa^{\rho^3}y^{-1}\rho^2\gamma; \\
(a\rho\gamma)^G &= VA \rho\gamma \cup VA^{-1}x^{-1}y\rho\gamma \cup Wy^{-1}a^{\rho}x\rho^3\gamma \cup Wx^{-1}(a^{-1})^{\rho}\rho^3\gamma; \\
(a\rho^2\gamma)^G &= Xa^{\rho^2}\gamma \cup Xa^{-1}x^{-1}y\rho^2\gamma \cup YA^{\rho^3}y\gamma \cup YA^{\rho}x^{-1}y^{-1}; \\
(a\rho^3\gamma)^G &= WA^{\rho^3}\gamma \cup WX^{-1}y^{-1}a^{-1}x^{-1}y\rho^3\gamma \cup VXa^{\rho}x\rho\gamma \cup VX^{-1}a^{\rho}\rho\gamma. \quad \square
\end{align*}$$

**Corollary 7.1.** Any involution in $G$ is in one of the sets:

$$(i)A\rho^2; \quad (b)V\rho\gamma; \quad (c)x^{-1}WP\rho^3\gamma. \quad \square$$

Now assume that $\varphi(\rho) = a\rho, a = x^iy^j \in A$. Note that $I_x(\rho) = (xy)^{-1}\rho, I_y(\rho) = xy^{-1}\rho$, so that by acting by some $I_b, b \in A$, we can assume that $\varphi(\rho) \in \{\rho, x\rho\}$. If we have $\varphi(\rho) = x\rho$, then acting by the automorphism $\psi_1$ defined in Lemma 2.4 we see that $\varphi(\rho) = \rho$. Now let $\varphi(\gamma) = a\gamma$. Lemma 5.5 gives $a \in \langle y \rangle$. Note that $(\rho\gamma)^2 = 1$ implies that $\varphi(\rho\gamma) \sim \rho\gamma$. So Corollary 7.1 (b) with $\varphi(\rho \cdot \gamma) \sim \rho a\gamma$ tells us $a^{\rho^3} \in V$ so $a \in W$. We conclude that $a$ is trivial.

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Thus we now have $\varphi(\rho) = \rho, \varphi(\gamma) = \gamma$, so that $\varphi(\rho^3) = \rho^3$.

Now let $\varphi(\rho^2 \gamma) = b\rho^2 \gamma$. Lemma 5.5 gives $b \in \langle x \rangle$. Then using Corollary 7.1 (b) again, $(\rho \gamma)^2 = 1$ and $\varphi(\rho \gamma) = \varphi(\rho^{-1} \cdot \rho^2 \gamma)$ gives $b^\rho \in V$, hence $b \in W$. Thus $\varphi(\rho^2 \gamma) = \rho^2 \gamma$.

Assume that $\varphi(\rho^3 \gamma) = c\rho^3 \gamma$. By Lemma 5.5 we know $c \in W$. Note that $$\gamma = \varphi(\rho \cdot \rho^3 \gamma) \sim c^\rho \rho^3 \gamma \text{ implies } c^\rho \in Y \cup Yx^{-1} \text{ implies } c \in X \cup Xxy^{-1}.$$ But also, $$\rho^2 \gamma = \varphi(\rho^3 \cdot \rho^2 \gamma) \sim c^\rho \rho^2 \gamma \text{ implies } c^\rho \in X \cup Xxy^{-1} \text{ and thus } c \in Y \cup Yx^{-1}y.$$ As $c$ must be in the intersection of all three sets it must be trivial and so $\varphi(\rho^3 \gamma) = \rho^3 \gamma$.

The above shows that $\varphi(v) = v$ for all $v \in F \setminus \{\rho^2, \rho \gamma\}$. Thus by Proposition 6.1 there is $t_v \in F \setminus \{\rho^2, \rho \gamma\}$ such that for all $a \in A$ we have $\varphi(av) = a^t_v v$. Let $t, u, w \in F$ satisfy $\varphi(a \rho) = a^t \rho, \varphi(a \gamma) = a^u \gamma$, and $\varphi(a \rho^{-1} \gamma) = a^w \rho^{-1} \gamma$. By Lemma 6.5 we have $u \in \{1, \gamma\}$ and $w \in \{1, \rho^{-1} \gamma\}$.

Now let $\varphi(\rho \gamma) = a \rho \gamma$. Then Lemma 7.1 (b) gives $a \in V$, while $\varphi(\rho \cdot \rho \gamma) \sim a^\rho \rho^2 \gamma$ tells us $a \in Y \cup Yxy$. Therefore we know $a \in \{1, xy\}$. However if we write $t^\rho \gamma = s$, and assume $\varphi(\rho \gamma) = xy \rho \gamma$, then

$$x^2 \gamma \sim \varphi(x^3 \cdot x^{-1} \gamma) = \varphi(x^3 \rho \gamma \rho \gamma^{-1}) = \varphi(\rho \gamma \cdot (x^3)^\rho \gamma \rho)$$

$$\sim xy \rho \gamma (x^3)^\rho \gamma \rho = xy(x^3)^s \rho \gamma \rho = xy(x^3)^s \gamma^{-1} = y(x^3)^s \gamma.$$ 

This gives a contradiction unless $s \in \{\rho^2, \rho^2 \gamma\}$ which corresponds to $t \in \{\rho^2, \gamma\}$. But since $(y^{-1})^\rho = (y^{-1})^\gamma = y$ and $(x^{-1})^\gamma = x^{-1}$ for any combination of $t \in \{\rho^2, \gamma\}$ and $u \in \{1, \gamma\}$ we have $\rho \gamma \sim \varphi(\rho \gamma) = \varphi(y^{-1} \rho \cdot x^{-1} \gamma) \sim (y^{-1})^t \rho (x^{-1})^u \gamma = y \rho (x^{-1}) \gamma = y^2 \rho \gamma$, which is a contradiction. Thus $\varphi(\rho \gamma) = \rho \gamma$. 

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Now we have $\varphi(\rho^k\gamma) = \rho^k\gamma$ for all $k \in \{0, 1, 2, 3\}$ and we may apply Lemma 6.6 four times so that $\varphi = Id$ on both reflection cosets and both glide reflection cosets.

Let $\varphi(\rho^2) = b\rho^2$. Since $\varphi(\rho^2 \cdot \gamma) \sim b\rho^2\gamma$ we see $b \in X \cup Xxy^{-1}$. By considering $\varphi(\rho^2 \cdot \rho^2\gamma) \sim b\gamma$ we know $b \in Y \cup Yx^{-1}y$. Thus $b \in \{1, x^{-1}y^{-1}\}$. Assume $b = x^{-1}y^{-1}$. Since $\rho^2\gamma = x^{-1}y\gamma\rho^2$

this gives

$$\rho^2\gamma = \varphi(x^{-1}y\gamma \cdot \rho^2) \sim x^{-1}y\gamma x^{-1}y^{-1}\rho^2 = x^{-2}y^2\gamma\rho = x^{-1}y\rho^2\gamma$$

which is a contradiction according to Lemma 7.1. Thus $\varphi(\rho^2) = \rho^2$. By Lemma 6.7 we have $\varphi = Id$ on all of $G$. Thus the groups of type $\text{p4mg}$ have trivial weak Cayley table group.  

\qed
We have found all the generators of $W(G)$ for each of the wallpaper groups and determined that four groups have $W(G) \neq W_0(G)$. Here we will give presentations for $W(G)$ for these four cases. For the other cases, presentations can be found in [GW].

The groups of type p2mm

For this group we have the inner automorphisms $I_\rho$, $I_\sigma$, as well as the previously defined automorphisms $\psi_x, \psi_y$, and $\psi_1$. Here $I_x = \psi_x^{-2}$ and $I_y = \psi_y^{-2}$. We have the nontrivial weak Cayley table map $\tau_\sigma$. We have also composed with the map $\tau_\rho \sigma = \tau_{\psi_1}$. (Note that $\text{Aut}(G)$ contains an element of order four, namely $\psi_1 I_\sigma$. It is easy to check that $(\psi_1 I_\sigma)^2 = I_\rho$ and that $(\psi_1 I_\sigma)^{-1} = (\psi_1 I_\sigma)^{-1}$.) So we have

$$\text{Aut}(G) = \langle \psi_x, \psi_y \rangle \times \langle I_\rho, I_\sigma, \psi_1 \rangle \cong \mathbb{Z}^2 \rtimes D_8,$$

and thus $W_0 \cong (\mathbb{Z}^2 \rtimes D_8) \times C_2$. We find that

$$W = \left( \left( \langle \psi_x, \psi_y \rangle \rtimes_\theta \langle \tau_\sigma, \tau_\rho \sigma \rangle \right) \times_\theta \langle I_\rho, I_\sigma, \psi_1 \rangle \right) \times \langle \iota \rangle \cong \left( (\mathbb{Z}^2 \rtimes (C_2 \times C_2)) \rtimes D_8 \right) \times C_2,$$

where the semidirect products are given by

$$\theta_1(\tau_\sigma) : (\psi_x, \psi_y) \mapsto (\psi_x, \psi_y^{-1});$$

$$\theta_1(\tau_\rho \sigma) : (\psi_x, \psi_y) \mapsto (\psi_x^{-1}, \psi_y);$$

$$\theta_2(I_\rho) : (\psi_x, \psi_y, \tau_\sigma, \tau_\rho \sigma) \mapsto (\psi_x^{-1}, \psi_y^{-1}, \tau_\sigma, \tau_\rho \sigma);$$

$$\theta_2(I_\sigma) : (\psi_x, \psi_y, \tau_\sigma, \tau_\rho \sigma) \mapsto (\psi_x, \psi_y^{-1}, \tau_\sigma, \tau_\rho \sigma);$$

$$\theta_2(\psi_1) : (\psi_x, \psi_y, \tau_\sigma, \tau_\rho \sigma) \mapsto (\psi_y, \psi_x, \tau_\rho \sigma, \tau_\sigma).$$
The groups of type p3

Here we have the automorphisms $\psi_x : (x, y, \rho) \mapsto (x, y, x\rho)$, $\psi_y : (x, y, \rho) \mapsto (x, y, y\rho)$, $I_\rho$, as well as the non-trivial weak Cayley table map $\tau$ defined in Chapter 2. Note that $I_x = \psi_x^{-1} \psi_y^{-1}$, $I_y = \psi_x, \psi_y^{-2}$. We find that

$$W_0 = \langle (\psi_x, \psi_y) \rangle \rtimes \langle I_\rho \rangle \times \langle \iota \rangle \cong (\mathbb{Z}^2 \rtimes C_3) \times C_2,$$

and

$$W = \langle (\psi_x, \psi_y) \rangle \rtimes_{\theta_1} \langle \tau \rangle \times \langle I_\rho \rangle \times \langle \iota \rangle \cong ((\mathbb{Z}^2 \rtimes C_3) \rtimes C_3) \times C_2,$$

where $\theta_1(\tau) : (\psi_x, \psi_y) \mapsto (\psi_y^{-1}, \psi_x \psi_y^{-1})$ and $\theta_2(I_\rho) : (\psi_x, \psi_y, \tau) \mapsto (\psi_y^{-1}, \psi_x \psi_y^{-1}, \tau)$.

The groups of type p4

This group has automorphisms $\psi_x : (x, y, \rho) \mapsto (x, y, x\rho)$ and $\psi_y : (x, y, \rho) \mapsto (x, y, y\rho)$ as well as the inner automorphism $I_\rho$. We also have the maps $\psi_1, \tau_x, \tau_y, \tau_\rho^2, \mu_x, \mu_y, \mu_\rho$ that we defined previously. Note that $I_x = \psi_x^{-1} \psi_y^{-1}$, $I_y = \psi_x, \psi_y^{-1}$. We have

$$W_0 = \langle (\psi_x, \psi_y) \rangle \rtimes (I_\rho, \psi_1) \times \langle \iota \rangle \cong (\mathbb{Z}^2 \rtimes D_8) \times C_2.$$

We find that $\langle \tau_x, \tau_y, \tau_\rho^2 \rangle \triangleleft W$, $\langle \mu_x, \mu_y, \mu_\rho \rangle \triangleleft W$, and that we have a semidirect product:

$$W = \left(\left(\langle \tau_x, \tau_y \rangle \rtimes_{\theta_1} \langle \tau_\rho^2 \rangle \right) \times \left(\langle \mu_x, \mu_y \rangle \rtimes_{\theta_2} \langle \mu_\rho \rangle \right) \times \langle \iota \rangle \right) \times \langle \iota \rangle \cong \left(\left((\mathbb{Z}^2 \rtimes C_2) \times (\mathbb{Z}^2 \rtimes C_4) \right) \times (\mathbb{Z}^2 \rtimes D_8) \right) \times C_2.$$
where the semidirect products are given by

\[
\begin{align*}
\theta_1(\tau_\rho) : (\tau_x, \tau_y) & \mapsto (\tau_x^{-1}, \tau_y^{-1}); \\
\theta_2(\mu_\rho) : (\mu_x, \mu_y) & \mapsto (\mu_y^{-1}, \mu_x); \\
\theta_3(\psi_x) : (\tau_x, \tau_y, \tau_\rho, \mu_x, \mu_y, \mu_\rho) & \mapsto (\tau_x, \tau_y, \tau_\rho \tau_x^{-1}, \mu_x, \mu_y, \mu_\rho \mu_x^{-1}); \\
\theta_3(\psi_y) : (\tau_x, \tau_y, \tau_\rho, \mu_x, \mu_y, \mu_\rho) & \mapsto (\tau_x, \tau_y, \tau_\rho \tau_x \tau_y^{-1}, \mu_x, \mu_y, \mu_\rho \mu_y^{-1}); \\
\theta_3(I_\rho) : (\tau_x, \tau_y, \tau_\rho, \mu_x, \mu_y, \mu_\rho) & \mapsto (\tau_y^{-1}, \tau_x, \tau_\rho \mu_y^{-1}, \mu_x, \mu_y); \\
\theta_3(\psi_1) : (\tau_x, \tau_y, \tau_\rho, \mu_x, \mu_y, \mu_\rho) & \mapsto (\tau_y^{-1}, \tau_\rho \mu_y^{-1}, \mu_x, \mu_y); \\
\theta_4(I_\rho) : (\psi_x, \psi_y) & \mapsto (\psi_y^{-1}, \psi_x); \\
\theta_4(\psi_1) : (\psi_x, \psi_y) & \mapsto (\psi_y, \psi_x).
\end{align*}
\]

The groups of type \textbf{p6}

Here \(\mathcal{W}\) is generated by the automorphism \(\psi : (x, y, \rho) \mapsto (y, x, \rho^{-1})\), three inner automorphisms, as well as six non-trivial weak Cayley table maps. We have

\[
\text{Aut}(G) = \langle I_x, I_y \rangle \rtimes \langle I_\rho, \psi \rangle \cong \mathbb{Z}^2 \rtimes D_{12}
\]

and so \(\mathcal{W}_0 \cong (\mathbb{Z}^2 \rtimes D_{12}) \rtimes C_2\). We find that the non-trivals that we found generate normal subgroups:

\[
\langle \mu_x^2, \mu_y^2, \mu_\rho^3 \rangle \triangleleft \mathcal{W}, \langle \tau_{xy}, \tau_{xy^{-2}}, \tau_\rho^2 \rangle \triangleleft \mathcal{W},
\]

and that we have a semidirect product:

\[
\mathcal{W} = \left( (\langle \mu_x^2, \mu_y^2 \rangle \times \theta_1 \langle \mu_\rho^3 \rangle) \times (\langle \tau_{xy}, \tau_{xy^{-2}} \rangle \times \theta_2 \langle \tau_\rho^2 \rangle) \right) \times \theta_3 \langle I_x, I_y \rangle \times \theta_4 \langle I_\rho, \psi \rangle \times \langle \iota \rangle \\
\cong \left( (\mathbb{Z}^2 \rtimes C_2) \times (\mathbb{Z}^2 \rtimes C_3) \right) \times (\mathbb{Z}^2 \rtimes D_{12}) \rtimes C_2,
\]
where the semidirect products are given by

\[ \theta_1(\mu^3) : (\mu x^2, \mu y^2) \mapsto (\mu^{-1} x^2, \mu^{-1} y^2); \]
\[ \theta_2(\tau) : (\tau x y, \tau x y^-2) \mapsto (\tau x y^{-1} x y^-2, \tau x y); \]
\[ \theta_3(I) : (I x, I y) \mapsto (I x I^{-1} y, I x); \]
\[ \theta_4(\psi) : (I x, I y) \mapsto (I y, I x); \]
\[ \theta_3(I) : (\mu x^2, \mu y^2, \mu^{\rho^3}, \tau x y, \tau x y^-2, \tau^{\rho^2}) \mapsto (\mu x^2, \mu y^2, \mu^{\rho^3} \mu x^2, \tau x y, \tau x y^-2, \tau^{\rho^2} \tau x y); \]
\[ \theta_3(I) : (\mu x^2, \mu y^2, \mu^{\rho^3}, \tau x y, \tau x y^-2, \tau^{\rho^2}) \mapsto (\mu x^2, \mu y^2, \mu^{\rho^3} \mu y^2, \tau x y, \tau x y^-2, \tau^{\rho^2} \tau x y \tau x y^-2); \]
\[ \theta_3(I) : (\mu x^2, \mu y^2, \mu^{\rho^3}, \tau x y, \tau x y^-2, \tau^{\rho^2}) \mapsto (\mu x^2, \mu y^2, \mu^{\rho^3} \mu^{-1} y^2, \tau x y, \tau x y^-2, \tau^{-1} \tau x y \tau x y^-2, \tau^{\rho^2}); \]
\[ \theta_3(I) : (\mu x^2, \mu y^2, \mu^{\rho^3}, \tau x y, \tau x y^-2, \tau^{\rho^2}) \mapsto (\mu y^2, \mu x^2, \mu^{\rho^3}, \tau x y, \tau x y^{-1} x y^-2, \tau^{\rho^2}). \]

We have shown the following:

**Theorem 8.1.** Let \( G \) be a wallpaper group. Then \( \mathcal{W}(G) \neq \mathcal{W}_0(G) \) if and only if either

(i) \( G \) is a direct product of two non-abelian groups, or

(ii) \( G \) contains only translations and rotations, with some rotation of order greater than two.

If \( G \) has rotations of order greater than three then there is a normal subgroup \( \mathcal{N}(G) \) all of whose non-identity elements are non-trivial, such that

\[ \mathcal{W}(G) = \mathcal{N}(G) \rtimes \mathcal{W}_0(G). \]
Here we give results for groups that are similar to wallpaper groups in some respect. We begin with a finite group that is related to wallpaper groups of type \textbf{p31m}. One can generalize plane crystallographic groups to spherical crystallographic groups or hyperbolic crystallographic groups [M]. The group

\[
G = \langle x, y, \rho, \sigma | x^6, y^6, \rho^3, \sigma^2, (x, y), x^\rho = x^{-1}y, y^\rho = x^{-1}, x^\sigma = x, y^\sigma = xy^{-1}, (\rho \sigma)^2 \rangle
\]

is a spherical crystallographic group of genus one. Note that its presentation is very similar to that of groups of type \textbf{p31m}. In fact it is a quotient of a group of type \textbf{p31m}. The difference is the addition of the relations \(x^6 = y^6 = 1\), which makes this group finite. As before, let \(A = \langle x, y \rangle\) denote the ‘translation’ subgroup, which is normal in \(G\). We find that this group has a non-trivial weak Cayley table group. One non-trivial map is

\[
\tau : \begin{cases} 
  g \mapsto g & \text{for } g \in A; \\
  g \mapsto g^x & \text{for } g \in A\rho \cup A\rho^2; \\
  g \mapsto g^{y^{-2}} & \text{for } g \in A\sigma; \\
  g \mapsto g^{x^{-1}} & \text{for } g \in A\rho\sigma; \\
  g \mapsto g^x & \text{for } g \in A\rho^2\sigma.
\end{cases}
\]

One might expect that \(\tau\) would similarly define a nontrivial map in the infinite case \textbf{p31m}. However, we have shown that groups of type \textbf{p31m} have \(W = W_0\) so this is not the case. In the infinite case, the map above does not satisfy \(\tau(g)\tau(h) \sim \tau(gh)\) for all \(g, h\) in that group.

Recall that for any wallpaper group that has reflection or glide reflection cosets, there are infinitely many conjugacy classes contained in those cosets. In the finite case, there are only six conjugacy classes of elements in the reflection cosets. Thus the requirement that
\( \varphi(gh) \sim \varphi(g)\varphi(h) \) is a less restrictive requirement.

**Groups of the form** \( A \rtimes_{\theta} C_p \), \( (p \text{ an odd prime}) \)

We will prove

**Theorem 9.1.** Let \( G = A \rtimes_{\theta} C_p \) with \( \theta \neq 1 \) where \( A \) is an abelian group and \( p \) is an odd prime. Then \( W(G) \neq W_0(G) \).

We will use the following Witt-Hall identities [MKS, p. 290] to prove a lemma:

\[
(a, b) \cdot (b, a) = 1. \tag{9.1}
\]

\[
(a, b \cdot c) = (a, c) \cdot (a, b) \cdot ((a, b), c). \tag{9.2}
\]

\[
(a \cdot b, c) = (a, c) \cdot ((a, c), b) \cdot (b, c). \tag{9.3}
\]

**Lemma 9.1.** (i) Let \( 1 \leq k < p \). Then every \( g \in G' \) has the form \( (a, r^k) \) for some \( a \in A \).

(ii) For any \( 1 \leq k < p \) and \( g \in G' \) there is \( a \in A \) such that \( (r^k)^a = gr^k \).

**Proof** (i) Let \( ar^k, br^m \in G, a, b \in A \). Using equation (9.3), then (9.2), then (9.1), we have:

\[
(ar^k, br^m) = (a, br^m) \cdot ((a, br^m), r^k) \cdot (r^k, br^m)
\]

\[
= (a, r^m) \cdot ((a, br^m), r^k) \cdot (r^k, r^m) \cdot (r^k, b) \cdot ((r^k, b), br^m)
\]

\[
= (a, r^m) \cdot ((a, br^m), r^k) \cdot (b, r^k)^{-1} \cdot ((r^k, b), r^m),
\]

which shows that each element of \( G' \) is a product of commutators of the form \( (a, r^j), a \in A \).

By the Witt-Hall identity 9.2 we have

\[
(a, r^k) = (a, r \cdot r^{k-1}) = (a, r^{k-1}) \cdot (a, r) \cdot ((a, r, r^{k-1}), r^m)
\]

which shows inductively that each element of \( G' \) is a product of commutators of the form \( (a, r), a \in A \).
Now using identity 9.3 we have (for every $1 \leq k < p$)

$$(ab, r^k) = (a, r^k) \cdot ((a, r^k), b) \cdot (b, r^k) = (a, r^k) \cdot (b, r^k),$$

which shows (by taking $k = 1$) that each element of $G'$ is a commutator of the form $(a, r)$.

Now for every $1 \leq k < p$ there is $u$ such that $ku \equiv 1 \bmod p$. Combining this with the following consequence of identity 9.2

$$(a, r) = (a, r^{ku}) = (a, r^k, r^{k(u-1)}) \cdot (a, r^k) \cdot ((a, r^k), r^{k(u-1)}),$$

we see that any $(a, r)$ can be written as a product of powers of commutators of the form $(a, r^k)$, and so is also a commutator of the form $(b, r^k)$. This does (i).

(ii) Now let $g \in G'$ and fix $1 \leq k < p$. Then by the above there is $a \in A$ such that

$$g = (a, r^{-k}) = a^{-1} r^k a r^{-k} \text{ so that } gr^k = (r^k)^a.$$ 

This concludes the proof of Lemma 9.1. \qed

**Theorem 9.2.** Define the map

$$\tau : \begin{cases} 
  a \mapsto a^r & \text{for all } a \in A; \\
  g \mapsto g & \text{for all } g \in G \setminus A.
\end{cases}$$

Then $\tau$ is a nontrivial weak Cayley table isomorphism.

**Proof.** Since $\theta$ is non-trivial, there exists $a \in A$ such that $a^r \neq a$. Then $\tau(ar) = ar \neq a^r r = \tau(a)\tau(r)$. Thus $\tau$ is not a homomorphism. Next we show that $\tau$ is not an anti-homomorphism: Note $rar \notin A$ since $|r|$ is odd. So $\tau(rar) = rar \neq ar \cdot r = \tau(ar)\tau(r)$.

Now we show that $\tau$ is a weak Cayley table isomorphism. There are five cases we need to consider:
Case 1: \( g_1, g_2 \in A \): then \( \tau(g_1 g_2) = (g_1 g_2)^r = g_1^r g_2^r = \tau(g_1) \tau(g_2) \).

Case 2: \( g_1 \in A, g_2 \notin A \): Let \( g_1 = a, g_2 = br^k \) for some \( a, b \in A, 1 \leq k < p \). Then

\[
\tau(g_1 g_2) = abr^k \quad \text{and} \quad \tau(g_1) \tau(g_2) = a^r br^k.
\]

Note that \( (r, a^{-1}) = r^{-1} ar b^{-1} a^{-1} = (a^r b)(ab)^{-1} \in G' \), so by Lemma 9.1, there exists \( c \in A \) such that \( (r^k)^c = (a^r b)(ab)^{-1}r^k \). This gives us

\[
\tau(g_1 g_2)^c = (abr^k)^c = ab(r^k)^c = ab \cdot (a^r b)(ab)^{-1}r^k
\]

\[
= (a^r b)(ab)^{-1} \cdot r^k = (a^r b)r^k = \tau(g_1) \tau(g_2).
\]

A similar argument works for Case 3: \( g_1 \notin A, g_2 \in A \).

Case 4: \( g_1, g_2 \notin A, g_1 g_2 \notin A \): We have \( \tau(g_1 g_2) = g_1 g_2 = \tau(g_1) \tau(g_2) \).

Case 5: \( g_1, g_2 \notin A, g_1 g_2 \in A \): Then \( \tau(g_1 g_2) = (g_1 g_2)^r \sim g_1 g_2 = \tau(g_1) \tau(g_2) \).

This proves that \( \tau \in \mathcal{W}(G) \), and proves Theorem 9.1.

Groups of the form \( \mathbb{Z}^n \rtimes C_2 \)

Let \( H = A \rtimes C_2 \), where \( A = \langle a_1, \ldots, a_n \rangle \cong \mathbb{Z}^n \) and \( \theta \in \text{Aut}(A) \). Let \( C_2 = \langle r \rangle \) and \( Z = Z(H) \). We write elements of \( H \) as \( ar^\varepsilon, a \in A, \varepsilon \in \{0, 1\} \). We also assume that \( H \) is not abelian. This allows us to assume that \( a_i \notin Z(H) \) for all \( 1 \leq i \leq n \), since if \( a_i \in Z(H) \) and \( a_j \notin Z(H) \), then we just replace \( a_i \) by \( a_i a_j \). We further note that \( a \in A \) is in \( Z(H) \) if and only if \( a^r = a \).

Let \( A_i = \langle a_i \rangle, 1 \leq i \leq n \).

Lemma 9.2. (i) For \( a \in A \) we have \( a^H = \{ a, a^r \} \).

(ii) For \( h = ar \in H \setminus A \) we have \( h^H = H'h \).

Proof. (i) is clear since \( A \) is abelian of index 2.

For (ii) we note that \( r^a = a^{-1} a^r r \), so that for \( b \in A \) we have \( (br)^a = b(a^{-1} a^r)^m r \) for all
\( m \in \mathbb{Z} \). Let
\[
\beta_i = a_i^{-1}a_i^r \in H', \quad 1 \leq i \leq n.
\]

Let \( B_i = \langle \beta_i \rangle \leq H', 1 \leq i \leq n \), and note that
\[
(a_i^{-1}a_i^r)^r = (a_i^r)^{-1}a_i = (a_i^{-1}a_i^r)^{-1} \in B_i.
\]

Also for \( a \in A \) we have \( (ar)^A = aB_ir \) and so \( (ar)^A = a\langle B_1, \ldots, B_n \rangle r \).

Also
\[
(ar)^{Ar} = \langle B_1, \ldots, B_n \rangle^r = \langle B_1, \ldots, B_n \rangle a^r r = \langle B_1, \ldots, B_n \rangle a^r r
\]
\[
= \langle B_1, \ldots, B_n \rangle (a^ra^{-1}) \cdot ar.
\]

Here we note that \( a^ra^{-1} \in H' \) and the result will follow upon showing that \( H' = \langle B_1, \ldots, B_n \rangle \). Now for \( b \in A \) we have
\[
(a_i, br) = a_i^{-1}rb^{-1}a_i br = a_i^{-1}ra_i r = a_i^{-1}a_i^r \in B_i,
\]
and
\[
(a_i r, a_j r) = ra_i^{-1}ra_j^{-1}a_i r a_j r = (a_i^r)^{-1}a_i \cdot a_j^{-1}a_j^r \in B_i B_j,
\]
shows that \( H' \leq \langle B_1, \ldots, B_n \rangle \), and it follows that \( H' = \langle B_1, \ldots, B_n \rangle \).

**Lemma 9.3.** Let \( \varphi \in \mathcal{W}(H) \). Then \( \varphi(A) = A \) and if \( \varphi(a_i) = \alpha_i, 1 \leq i \leq n \), then \( \varphi(a_1^{\lambda_1} \ldots a_n^{\lambda_n}) = \alpha_1^{\lambda_1} \ldots \alpha_n^{\lambda_n} \) for all \( \lambda \in \mathbb{Z} \).

**Proof.** From Lemma 9.2 we see that \( (ar)^H \) is infinite for any \( a \in A \). Thus only the elements of \( A \) have finite conjugacy classes, and so \( \varphi(A) = A \). Now from \( a^H = \{a, a^r\} \) we have \( \varphi(a) \sim \varphi(a^r) = \varphi(a)^r \). Thus \( \varphi\{a, a^r\} = \{\varphi(a), \varphi(a)^r\} \).

**Lemma 9.4.** For all \( n \in \mathbb{Z} \) and \( a \in A \) we have \( \varphi(a^n) = \varphi(a)^n \).
Proof. We only need to prove the cases $n \geq 0$. We have $a^H = \{a, a^r\}$. Let $\varphi(a) = \alpha$, so that $\varphi(a^r) = \alpha^r$. We then have $\varphi(a^\varepsilon) = \alpha^\varepsilon$ for $\varepsilon = 0, \pm 1$. So now assume (inductively) that $\varphi(a^i) = \alpha^i$ for $|i| < n$, for some $n \geq 2$, and that $\varphi(a^n) \neq \alpha^n$. Then

$$\varphi(a^n) = \varphi(a \cdot a^{n-1}) \sim \alpha \alpha^{n-1} \in \{\alpha^n, (\alpha^n)^r\},$$

which shows that $\varphi(a^n) = (\alpha^n)^r$. In particular we have $\varphi(a^n) = (\alpha^n)^r \neq \alpha^n$.

Now

$$\alpha = \varphi(a) = \varphi(a^n \cdot a^{1-n}) \sim (\alpha^n)\alpha^{1-n}.$$

Thus we must have either

(a) $(\alpha^n)\alpha^{1-n} = \alpha$; or

(b) $(\alpha^n)\alpha^{1-n} = \alpha^r$.

If we have (a), then $(\alpha^n) = \alpha^n$, from which we get (since $A$ is a free abelian group) $\alpha^r = \alpha$, giving a contradiction.

If we have (b), then we get $(\alpha^n)^{n-1} = \alpha^{n-1}$, which again gives $\alpha^r = \alpha$, since $n \geq 2$. This gives another contradiction and we have completed the proof.

For any $a \in A$ we can write $a = a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n}, \lambda_i \in \mathbb{Z}$. We define a length function on $A$ as follows:

$$|a| = |\lambda_1| + |\lambda_2| + \cdots + |\lambda_n|.$$

Thus we now know that $\varphi(a) = \prod_{i=1}^n \alpha_i^{\lambda_i}$ for all $a \in A$ with $|a| \leq 1$.

So now assume (inductively) that there is $n \geq 2$ such that $\varphi(a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n}) = \alpha_1^{\lambda_1}\alpha_2^{\lambda_2} \ldots \alpha_n^{\lambda_n}$ for all $a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n} \in A$ with $|a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n}| < n$, and that there is $a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n}$ such that $|a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n}| = n$ and $\varphi(a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n}) \neq a_1^{\lambda_1}\alpha_2^{\lambda_2} \ldots \alpha_n^{\lambda_n}$.

Without loss we can assume that $\lambda_1 \geq 1$. 

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Further, since \( \varphi(a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n}) \neq a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n} \), and since

\[
\varphi(a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n}) = \varphi(a_1 \cdot a_1^{-1}a_2^{\lambda_2} \ldots a_n^{\lambda_n}) \sim \varphi(a_1)\varphi(a_1^{-1}a_2^{\lambda_2} \ldots a_n^{\lambda_n}) = a_1 \cdot a_2^{\lambda_2} \ldots a_n^{\lambda_n} = a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n}.
\]

we must have \( \varphi(a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n}) = (a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n})^r \).

By induction we have \( \varphi(a_2^{\lambda_2} \ldots a_n^{\lambda_n}) = a_2^{\lambda_2} \ldots a_n^{\lambda_n} \), however, using Lemma 9.4, we also have

\[
\varphi(a_2^{\lambda_2} \ldots a_n^{\lambda_n}) = \varphi(a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n} \cdot a_1^{-\lambda_1}) \sim (a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n})^r \cdot a_1^{-\lambda_1}.
\]

Thus we have \( a_2^{\lambda_2} \ldots a_n^{\lambda_n} \sim (a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n})^r \cdot a_1^{-\lambda_1} \).

This gives two cases:

(a) \( a_2^{\lambda_2} \ldots a_n^{\lambda_n} = (a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n})^r \cdot a_1^{-\lambda_1} \); or

(b) \( a_2^{\lambda_2} \ldots a_n^{\lambda_n} = (a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n}) \cdot (a_1^{-\lambda_1})^r \).

Here possibility (a) gives \( (a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n})^r = a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n} \), which is a contradiction since

\( \varphi(a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n}) = (a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n})^r \neq a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_n^{\lambda_n} \).

But (b) gives \( (a_1^{\lambda_1})^r = a_1^{\lambda_1} \); this shows that \( \varphi(a_1) = a_1 \in Z(H) \), which in turn shows that \( a_1 \in Z(H) \), which is a contradiction to the choice of the \( a_i \). This concludes the proof of Lemma 9.3.

Thus we see that \( \varphi|_A : A \to A \) is an automorphism of \( A \).

**Lemma 9.5.** (1) For any \( a \in A \) such that \( ar \) has order 2 the homomorphism determined by

\[ b \mapsto b, (b \in A), r \mapsto ar \]

is an automorphism of \( H \).

(2) Let \( \psi \in \text{Aut}(A) \) where \( \psi \) commutes with \( \theta \). Then the the homomorphism determined by

\[ a \mapsto \psi(a), (a \in A), r \mapsto r \]

is an automorphism of \( H \).

**Proof.** (1) is clear since \( r \) and \( ar \) have order 2 and \( b^{ar} = b^r \) for all \( b \in A \).

(2) The relations in \( H \) have the form \( a^r = a^\theta \) and \( r^2 = 1 \). We have \( \psi(a^r) = \psi(\theta(a)) = \theta(\psi(a)) = (\psi(a))^r \), and we are done.
Since $\varphi(Ar) = Ar$ we can assume that $\varphi(r) = ar$ for some $a \in A$. Further, we know that $ar$ has order 2, since $\varphi$ is a weak Cayley table map.

By Lemma 9.5 (1) we can assume (by composing $\varphi$ with the inverse of such an automorphism) that $a = 1$ i.e. that $\varphi(r) = r$.

**Lemma 9.6.** Suppose that $a \in A$. Then $\varphi(a^r) = \varphi(a)^r$.

**Proof.** We know that $\varphi(a^r) \in \{\varphi(a), \varphi(a)^r\}$; however if $\varphi(a^r) = \varphi(a)$, then $a^r = a$ (so that $a = a^r$ and $\varphi(a) = \varphi(a^r)$ are central), and otherwise we have $\varphi(a^r) = \varphi(a)^r$. \qed

This lemma shows that the automorphism of $A$ determined by the action of $\varphi$ commutes with the action of $\theta$. Since $\varphi|_A$ is a automorphism of $A$ we can now use Lemma 9.5 (2) and Lemma 9.6 so as to be able to assume that $\varphi|_A = Id|_A$.

Thus we can now assume: $\varphi|_A = Id|_A$ and $\varphi(r) = r$.

By Lemma 6.2 we know that if $a \in A$ and $\varphi(ar) = br, b \in A$, then $b \in \{a, a^r\}$. Thus if $\varphi(ar) \neq ar$, then $\varphi(ar) = a^r r = ra$.

Now assume that there are $a, b \in A \setminus \{1\}$ such that $\varphi(ar) = ar \neq a^r r, \varphi(br) = b^r r \neq br$. Then

$$ ba^r = \varphi(ba^r) = \varphi(br \cdot ar) \sim \varphi(br)\varphi(ar) = b^r rar = b^r a^r. $$

It follows that either $ba^r = b^r a^r$ or $ba^r = ab$, each possibility giving a contradiction.

Thus there is $\varepsilon \in \{0, 1\}$ such that $\varphi(ar) = a^{\varepsilon r}$ for all $a \in A$. If $\varepsilon = 0$, then $\varphi$ is the identity map on $H$.

So now assume that $\varepsilon = -1$, so that $\varphi$ satisfies

$$ \varphi(a) = a, \quad \varphi(ar) = a^r r, \quad \text{for all } a \in A. $$

Now by Lemma 9.5 composing with $I_r$ reduces $\varphi$ to the identity, and we have proved

**Theorem 9.3.** The semidirect product $\mathbb{Z}^n \rtimes C_2, n \geq 0$, has trivial weak Cayley table group. \qed
Bibliography


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