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Existence of a Periodic Brake Orbit in the Fully Symmetric Planar Four Body Problem

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A thesis submitted to the faculty of Brigham Young University in partial fulfillment of the requirements for the degree of Master of Science

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We investigate the existence of a symmetric singular periodic brake orbit in the equal mass, fully symmetric planar four body problem. Using regularized coordinates, we remove the singularity of binary collision for each symmetric pair. We use topological and symmetry tools in our investigation.

Keywords: four body problem, brake orbits, binary collision, topological tools
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Contents

Abstract ii

Acknowledgements iii

List of Figures v

1 Introduction 1
  1.1 The Four Body Problem . . . . . . . . . . . . . . . . . . . . . . . . . . . 1

2 Main Results 4
  2.1 The Equations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5
  2.2 Introducing Regularized Coordinate . . . . . . . . . . . . . . . . . . . . 10
  2.3 Periodic Brake Orbit 121 . . . . . . . . . . . . . . . . . . . . . . . . . . . 15
  2.4 Generating the Full Periodic Orbit from Part of it Using Symmetry . . 16
  2.5 Zero Momentum Curve and Scaling Energy . . . . . . . . . . . . . . . . . 24
  2.6 Orbit Going Up . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 27
    2.6.1 Orbit that Starts at a Brake and Hits the $x_1$ Axis . . . . . . . . 28
    2.6.2 Orbit that Cross the $x_1 = x_2$ Line with $\dot{x}_2 > 0$ . . . . . . 33
    2.6.3 Set up for proving part 3 and 4 of the orbit going up . . . . . . 33
    2.6.4 Lower Bound on $x_1(0)$ and $x_2(0)$ . . . . . . . . . . . . . . . . . 34
    2.6.5 Lower and Upper bound for $\dot{x}_1^2(t_1)$ . . . . . . . . . . . . . 35
    2.6.6 Lower and Upper Bound for $\dot{x}_2^2(t)$ . . . . . . . . . . . . . . . 37
    2.6.7 Estimating Time Needed for $\dot{x}_2(t)$ to Go to Zero . . . . . . 37
    2.6.8 Estimating Time Elapsed from Crossing $x_1 = x_2$ to Hitting $x_2$ Axis 39
    2.6.9 Summary of Section 2.6 . . . . . . . . . . . . . . . . . . . . . . . . . 40
  2.7 Orbit Going Down . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 41
  2.8 Using Intermediate Value Theorem to Prove the Existence of the Periodic Solution . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 44
  2.9 Summary of Results . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 45

3 Future Work 46
List of Figures

1.1 Periodic brake orbit 121 ............................................. 2
2.1 Position of the four particles ..................................... 6
2.2 Periodic brake orbit 121 ............................................. 15
2.3 Periodic brake orbit 121 in four segments ..................... 16
2.4 Zero momentum curves ............................................. 25
2.5 Permissible and impermissible region ......................... 26
2.6 Four parts of showing orbit going up exist .................. 27
2.7 Numerical Results for Conjecture 2.10 ......................... 32
2.8 Times labeled ...................................................... 34
2.9 Orbit going down .................................................. 41
2.10 Using intermediate value theorem ............................ 44
3.1 Two other periodic brake orbit ................................. 47
3.2 Three periodic brake orbits on the same plot ................ 47
Chapter 1

Introduction

1.1 The Four Body Problem

In the classical four-body problem we study the motion of four bodies moving according to Newton’s law of gravitation. In the symmetric planar equal mass four body problem, the masses of the four bodies are all equal, and they lie on the same plane with their positions symmetric to each other along the horizontal and vertical direction. Chen [1] has proved existence of periodic orbits in the planar equal mass four body problem with parallelogram configuration. Bakker, Ouyang, Yan and Simmons [2] have proved the existence and determined stability of symmetric planar periodic orbits with simultaneous binary collision in the symmetric four-body problem.

Our main focus is on periodic brakes orbits in four-body problem. A brake orbit is a solution to the four-body problem for which, at some instant, all four bodies have zero velocity. Weinstein [3], Rabinowitz [4] and Chen [5] have studied periodic brake orbit under different set ups. Here we are looking for periodic brake orbit in the symmetric planar equal mass four-body problem.

Using numerical stimulations, we identified a potential candidate for a periodic brake orbit. Figure 1.1 is the periodic brake orbit that we are going to study, and we give it a name periodic brake orbit 121 because it hits the $x_1$ axis and the $x_2$ axis then the $x_1$ axis again in that order. We will give a rigorous definition of the periodic brake orbit 121 in the Main Results in Chapter 2.
In the periodic brake orbits that we study simultaneous binary collision (SBC) is involved, meaning that four bodies would collide with each other (with infinite speed and infinite acceleration at the collision), and bounce back to the opposite direction. The singularity at collision creates problems while trying to analyze the orbits. When trying to prove the existence of particular solutions to an n-body problem, the method of minimization of action (a variational technique) is often used. However the orbit we are trying to prove existence has a singularity at collision between our boundary conditions. Marchal [6] showed that for the method of minimization of action to be used, singularities can only occur at initial or final times but not intermediate times. Therefore the method of minimization of action cannot be used to prove existence of our target orbit. As a result we turn to topological techniques. We first need to regularize the simultaneous binary collisions. While binary collisions are well known to be regularizable, Simó[7] showed that simultaneous binary collision can be regularized too. After regularization we prove the existence of our target orbit using the shooting method. Topological techniques and regularization has been used in several instances to prove existence of solutions with regularizable collisions. Moeckel [8] used topological techniques to prove existence of a solution with regularizable collisions in the collinear three body problem. Duokui Yan [9] [10] used topological techniques to prove existence of a solution with regularizable collisions in a planar three body problem and a planar equal mass four body problem. Regina Martínez [11] gives topological proof of solutions with regularizable collisions in several even number n-body problems. Ouyang, Yan, and Simmons [12] used topological techniques to prove existences of periodic solutions with regularizable collisions in several even number n-body problems.
Besides using regularization, to apply the shooting method we used several differential inequalities to use get estimates of time intervals and information about the orbit between collisions. Symmetries are used to generate the full orbit from a part of the orbit. One part of the results has not been proven analytically so we provide only numerical results for that part. At the end we will discuss other SBC periodic brake orbits that we have found numerically. Techniques that we used here can likely be applied as we try proving the existence of those orbits too.
Chapter 2

Main Results

Our goal is to prove the existence of a symmetric periodic brake orbit with simultaneous binary collision (SBC) in the equal mass, fully symmetric planar four body problem. Section 2.1 lays out the set up for the fully symmetric planar four body problem and the resulting system of differential equations. In section 2.2 we transform the original coordinates into regularized coordinates and gives its corresponding system of differential equations. Section 2.3 describes the SBC periodic brake orbit we are trying to prove existence for. Section 2.4 describes the way of generating a full periodic orbit from part of it using symmetry. Section 2.5 shows how we can scale any solution to a particular energy level of the same sign. Section 2.6 and 2.7 proves the existence of an orbit going up and an orbit going down at \( x_2 \) axis collision. In section 2.8 we prove the existence of the periodic solution by Intermediate Value Theorem, using the results in all previous sections.
2.1 The Equations

We start with the equations of the equal mass planar four-body problem. Let $m \in \mathbb{R}^+$ and $\pi_k \in \mathbb{R}^2$, $k = \{1, 2, 3, 4\}$, be the masses and positions of three particles. The motions of the four particles are governed by the differential equations

$$m \frac{d^2 \pi_k}{dt^2} = \frac{\partial U}{\partial \pi_k}, \ k = 1, 2, 3, 4,$$

where

$$U(\pi_1, \pi_2, \pi_3, \pi_4) = \sum_{i<j} \frac{m^2}{|\pi_i - \pi_j|}, \ i, j \in 1, 2, 3, 4.$$

Dividing these equations through by $m$, and carrying out the partial derivatives gives

$$\frac{d^2 \pi_k}{dt^2} = \sum_{j<k} \frac{m(\pi_j - \pi_k)}{|\pi_j - \pi_k|^3}, \ j, k = 1, 2, 3, 4.$$

Writing these out we get

$$\frac{d^2 \pi_1}{dt^2} = \frac{m(\pi_2 - \pi_1)^3 + m(\pi_3 - \pi_1)^3 + m(\pi_4 - \pi_1)^3}{|\pi_2 - \pi_1|^3 |\pi_3 - \pi_1|^3 |\pi_4 - \pi_1|^3},$$

$$\frac{d^2 \pi_2}{dt^2} = \frac{m(\pi_1 - \pi_2)^3 + m(\pi_3 - \pi_2)^3 + m(\pi_4 - \pi_2)^3}{|\pi_1 - \pi_2|^3 |\pi_3 - \pi_2|^3 |\pi_4 - \pi_2|^3},$$

$$\frac{d^2 \pi_3}{dt^2} = \frac{m(\pi_1 - \pi_3)^3 + m(\pi_2 - \pi_3)^3 + m(\pi_4 - \pi_3)^3}{|\pi_1 - \pi_3|^3 |\pi_2 - \pi_3|^3 |\pi_4 - \pi_3|^3},$$

$$\frac{d^2 \pi_4}{dt^2} = \frac{m(\pi_1 - \pi_4)^3 + m(\pi_2 - \pi_4)^3 + m(\pi_3 - \pi_4)^3}{|\pi_1 - \pi_4|^3 |\pi_2 - \pi_4|^3 |\pi_3 - \pi_4|^3}.$$
We let $\pi_1 = (x_1, x_2)$. Through an Ansatz we impose a symmetry constraint, and set the coordinates of the four vectors to be respectively:

\[
\begin{align*}
\pi_1 &= (x_1, x_2) \\
\pi_2 &= (-x_1, x_2) \\
\pi_3 &= (-x_1, -x_2) \\
\pi_4 &= (x_1, -x_2)
\end{align*}
\]

for $x_1 \geq 0$ and $x_2 \geq 0$.

So $\pi_1$ is in the 1st quadrant, $\pi_2$ is in the 2nd quadrant, $\pi_3$ is in the 3rd quadrant, $\pi_4$ is in the 4th quadrant. (See Figure 2.1)

Now we can simplify the equations once we plug in the coordinates of each $\pi_i$.
\[ \frac{d^2 \pi_1}{dt^2} = m(-2x_1,0) + m(-2x_1,-2x_2) + m(0,-2x_2) \]
\[ \frac{d^2 \pi_2}{dt^2} = m(2x_1,0) + m(0,-2x_2) + m(2x_1,-2x_2) \]
\[ \frac{d^2 \pi_3}{dt^2} = m(2x_1,2x_2) + m(0,2x_2) + m(2x_1,0) \]
\[ \frac{d^2 \pi_4}{dt^2} = m(0,2x_2) + m((-2x_1,2x_2)) + m(2x_1,0) \]

If we let \( \frac{d^2 \pi_1}{dt^2} = (\ddot{x}_1, \ddot{x}_2) \) then

\[ \ddot{x}_1 = m \left( 1 \frac{x_1}{4x_1^2} \right) \] (2.1)
\[ \ddot{x}_2 = m \left( 1 \frac{x_2}{4x_2^2} \right) \] (2.2)

and we would have

\[ \frac{d^2 \pi_1}{dt^2} = (\ddot{x}_1, \ddot{x}_2) \]
\[ \frac{d^2 \pi_2}{dt^2} = (-\ddot{x}_1, \ddot{x}_2) \]
\[ \frac{d^2 \pi_3}{dt^2} = (-\ddot{x}_1, -\ddot{x}_2) \]
\[ \frac{d^2 \pi_4}{dt^2} = (\ddot{x}_1, -\ddot{x}_2). \]
Reduce to Studying One Particle Only

Since particles $\pi_2, \pi_3, \pi_4$ have their position (and thus acceleration) strictly determined by $\pi_1$, it is sufficient to study the behavior of particle $\pi_1$ alone.

**Theorem 2.1.** The mass $m$ in differential equations 2.1 and 2.2 can be scaled to one under appropriate time change.

**Proof.** Begin with differential equations 2.1 and 2.2:

\[
\ddot{x}_1(t) = m \left( -\frac{1}{4x_1^2(t)} - \frac{x_1(t)}{4(x_1^2(t) + x_2^2(t))^{3/2}} \right) \tag{2.3}
\]

\[
\ddot{x}_2(t) = m \left( -\frac{1}{4x_2^2(t)} - \frac{x_2(t)}{4(x_1^2(t) + x_2^2(t))^{3/2}} \right) \tag{2.4}
\]

introduce a time change $\sqrt{m}\tau = t$, then the differential equations become:

\[
\frac{\partial^2 x_1(\sqrt{m}\tau)}{\partial t^2} = m \left( -\frac{1}{4x_1^2(\sqrt{m}\tau)} - \frac{x_1(\sqrt{m}\tau)}{4(x_1^2(\sqrt{m}\tau) + x_2^2(\sqrt{m}\tau))^{3/2}} \right)
\]

\[
\frac{\partial^2 x_2(\sqrt{m}\tau)}{\partial t^2} = m \left( -\frac{1}{4x_2^2(\sqrt{m}\tau)} - \frac{x_2(\sqrt{m}\tau)}{4(x_1^2(\sqrt{m}\tau) + x_2^2(\sqrt{m}\tau))^{3/2}} \right)
\]

use chain rule to change the partial derivative from $t$ to $\tau$

\[
m \frac{\partial^2 x_1(\sqrt{m}\tau)}{\partial \tau^2} = m \left( -\frac{1}{4x_1^2(\sqrt{m}\tau)} - \frac{x_1(\sqrt{m}\tau)}{4(x_1^2(\sqrt{m}\tau) + x_2^2(\sqrt{m}\tau))^{3/2}} \right)
\]

\[
m \frac{\partial^2 x_2(\sqrt{m}\tau)}{\partial \tau^2} = m \left( -\frac{1}{4x_2^2(\sqrt{m}\tau)} - \frac{x_2(\sqrt{m}\tau)}{4(x_1^2(\sqrt{m}\tau) + x_2^2(\sqrt{m}\tau))^{3/2}} \right)
\]

therefore we have

\[
\frac{\partial^2 x_1(\sqrt{m}\tau)}{\partial \tau^2} = \frac{1}{4x_1^2(\sqrt{m}\tau)} - \frac{x_1(\sqrt{m}\tau)}{4(x_1^2(\sqrt{m}\tau) + x_2^2(\sqrt{m}\tau))^{3/2}}
\]

\[
\frac{\partial^2 x_2(\sqrt{m}\tau)}{\partial \tau^2} = \frac{1}{4x_2^2(\sqrt{m}\tau)} - \frac{x_2(\sqrt{m}\tau)}{4(x_1^2(\sqrt{m}\tau) + x_2^2(\sqrt{m}\tau))^{3/2}}
\]
Our system of differential equations become:

\begin{align}
\ddot{x}_1 &= -\frac{1}{4x_1^2} - \frac{x_1}{4(x_1^2 + x_2^2)^{3/2}} \\
\ddot{x}_2 &= -\frac{1}{4x_2^2} - \frac{x_2}{4(x_1^2 + x_2^2)^{3/2}}.
\end{align}

(2.5) (2.6)

on the open first quadrant \(x_1 > 0, x_2 > 0\).

From the system of differential equations above we can obtain a Hamiltonian

\begin{align}
H &= \frac{1}{2}(p_1^2 + p_2^2) + (x_1\dot{x}_1 + x_2\dot{x}_2) \\
&= \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{4x_1} - \frac{1}{4x_2} - \frac{1}{4(x_1^2 + x_2^2)^{1/2}}.
\end{align}

(2.7)

where \(p_1 = \dot{x}_1\) and \(p_2 = \dot{x}_2\) (velocity of the particle in the first quadrant).

Checking that \(H\) is a Hamiltonian:
\[ \frac{dH}{dp_1} = p_1 = \dot{x}_1 \]
\[ \frac{dH}{dp_2} = p_2 = \dot{x}_2 \]
\[ \frac{dH}{dx_1} = -\frac{1}{4x_1^2} - \frac{x_1}{4(x_1^2 + x_2^2)^{3/2}} = \dot{p}_1 \]
\[ \frac{dH}{dx_2} = -\frac{1}{4x_2^2} - \frac{x_2}{4(x_1^2 + x_2^2)^{3/2}} = \dot{p}_2. \]

Notice here that as \( x_1 \to 0 \) or \( x_2 \to 0 \) (near collision), both \( \ddot{x}_1, \ddot{x}_2 \) and our Hamiltonian \( H \) would blow up. In order to study the behavior of the particles near collision, we need a change of coordinates.

### 2.2 Introducing Regularized Coordinate

We introduce new coordinates \( Q_1, Q_2, P_1 \) and \( P_2 \), from which we apply a symplectic transformation which preserves the form of the Hamiltonian equations. As the first step of the symplectic transformation, we define a generating function

\[ F = p_1Q_1^2 + p_2Q_2^2. \]

Through our generating function we define

\[ x_1 = \frac{dF}{dp_1} = Q_1^2 \]
\[ x_2 = \frac{dF}{dp_2} = Q_2^2 \]
\[ P_1 = \frac{dF}{dQ_1} = 2p_1Q_1 \]
\[ P_2 = \frac{dF}{dQ_2} = 2p_2Q_2. \]
Therefore the relationship between the old coordinate and the regularized coordinate is:

\[
\begin{align*}
    x_1 &= Q_1^2 \\
    x_2 &= Q_2^2 \\
    p_1 &= \frac{P_1}{2Q_1} \\
    p_2 &= \frac{P_2}{2Q_2}.
\end{align*}
\]  

(2.8)

Now replace the old coordinates by the regularized coordinates into our Hamiltonian \( H \) (equation 2.7) to obtain \( \hat{H} \)

\[
\hat{H} = \frac{1}{2} \left( \frac{P_1^2}{4Q_1^2} + \frac{P_2^2}{4Q_2^2} \right) - \frac{1}{4} \left( \frac{1}{Q_1^2} + \frac{1}{Q_2^2} + \frac{1}{(Q_1^4 + Q_2^4)^{1/2}} \right).
\]  

(2.9)

In order to get rid of the singularity when \( Q_1 \to 0 \) or \( Q_2 \to 0 \), we introduce a time change from \( t \) to \( s \). As a particle approaches binary collision, we would observe that \( p_i \to \infty \) as \( Q_i \to 0 \). Defining \( s \) by \( \frac{dt}{ds} = Q_1^2Q_2^2 \) means that as \( Q_i \to 0 \), a small change in \( t \) would correspond to a big change in \( s \), resulting in a time dilation in \( s \) near collision. Through which the ‘velocity’ in \( s \), \( P_i \) is made finite at collision.

Multiply \( \hat{H} \) by \( Q_1^2Q_2^2 \) to get

\[
\frac{dt}{ds} \hat{H} = \frac{1}{8} \left( P_1^2Q_2^2 + P_2^2Q_1^2 \right) - \frac{1}{4} \left( Q_2^2 + Q_1^2 + \frac{Q_1^4Q_2^4}{(Q_1^4 + Q_2^4)^{1/2}} \right).
\]

Define a Hamiltonian \( \Gamma \) to be

\[
\Gamma = \frac{dt}{ds}(\hat{H} - E)
\]

and we get

\[
\Gamma = \frac{1}{8} \left( P_1^2Q_2^2 + P_2^2Q_1^2 \right) - \frac{1}{4} \left( Q_2^2 + Q_1^2 + \frac{Q_1^4Q_2^4}{(Q_1^4 + Q_2^4)^{1/2}} \right) - EQ_1^2Q_2^2,
\]  

(2.10)

a Hamiltonian defined in the extended phase space.
**Proposition 2.2.** $\Gamma$ is $C^1$ in the whole plane.

**Proof.** Looking at equation 2.10

$$
\Gamma = \frac{1}{8} \left( P_1^2 Q_2^2 + P_2^2 Q_1^2 \right) - \frac{1}{4} \left( Q_2^2 + Q_1^2 + \frac{Q_2^2 Q_1^2}{(Q_1^4 + Q_2^4)^{1/2}} \right) - EQ_1^2 Q_2^2,
$$

we readily see that $\Gamma$ as a function of $P_1, P_2, Q_1$ and $Q_2$ has no discontinuity except possible when $Q_1 \to 0$ and $Q_2 \to 0$.

Consider $$\left| \frac{Q_1^2 Q_2^2}{(Q_1^4 + Q_2^4)^{1/2}} \right|$$ when $Q_1 \to 0$ and $Q_2 \to 0$. Because

$$
\left| \frac{Q_1^2 Q_2^2}{(Q_1^4 + Q_2^4)^{1/2}} \right| \leq \frac{Q_1^2 Q_2^2}{(Q_1^4)^{1/2}} \leq |Q_2^2| \to 0,
$$

there is no singularity for $\Gamma$ when $Q_1 \to 0$ and $Q_2 \to 0$ or at any point in the whole plane. Therefore $\Gamma$ is $C^1$ in the whole plane.

We are assuming a system with no energy lost (perfectly elastic collision). Therefore $E$ does not vary with time. Under this set up, the level curve given by $\Gamma = 0$ would give us a system of differential equations in the regularized coordinates.

$\Gamma = 0$ is the new Hamiltonian system given by

$$
-\dot{P}_1 = \frac{\partial \Gamma}{\partial Q_1},
-\dot{P}_2 = \frac{\partial \Gamma}{\partial Q_2},
\dot{Q}_1 = \frac{\partial \Gamma}{\partial P_1},
\dot{Q}_2 = \frac{\partial \Gamma}{\partial P_2}.
$$
Therefore
\[
\begin{align*}
\dot{Q}_1 &= \frac{1}{4} P_1 Q_2^2 \\
\dot{Q}_2 &= \frac{1}{4} P_2 Q_1^2 \\
\dot{P}_1 &= -\frac{1}{4} P_2^2 Q_1 + \frac{1}{2} Q_1 + \frac{1}{2} \frac{Q_1 Q_6^2}{(Q_1^4 + Q_2^4)^{3/2}} + 2EQ_1 Q_2^2 \\
\dot{P}_2 &= -\frac{1}{4} P_1^2 Q_2 + \frac{1}{2} Q_2 + \frac{1}{2} \frac{Q_2 Q_6^2}{(Q_1^4 + Q_2^4)^{3/2}} + 2EQ_2 Q_1^2.
\end{align*}
\]

(2.12)

It is important to point out that in the regularized coordinate system we have a $C^1$ system of differential equations away from the origin, even at simultaneous binary collision.

**Theorem 2.3.** Under regularized coordinate, the differential equation is real analytic except possibly the origin, even through simultaneous binary collision.

**Proof.** As long as $Q_1$ and $Q_2$ are not zero at the same time (which covers away from collision and binary collision), there is no singularity in our differential equation. Therefore our differential equation is continuous away from the origin. \qed

**Proposition 2.4.** $P_1 = \pm \sqrt{2}$ when $Q_1 = 0$ and $Q_2 \neq 0$; $P_2$ equals to $\pm \sqrt{2}$ when $Q_2 = 0$ and $Q_1 \neq 0$.

**Proof.** We begin with Hamiltonian $\Gamma$ (equation 2.10) and set $\Gamma = 0$.

\[
0 = \frac{1}{8} \left( P_1^2 Q_2^2 + P_2^2 Q_1^2 \right) - \frac{1}{4} \left( Q_2^2 + Q_1^2 + \frac{Q_1 Q_2^2}{(Q_1^4 + Q_2^4)^{1/2}} \right) - EQ_1^2 Q_2^2.
\]

Set $Q_1 = 0$ and we get

\[
0 = \frac{1}{8} P_1^2 Q_2^2 - \frac{1}{4} Q_2^2.
\]

Since $Q_2 \neq 0$,

\[
P_1^2 = 2
\]

\[
P_1 = \pm \sqrt{2}.
\]
The result for $P_2$ equals to $\pm \sqrt{2}$ when $Q_2 = 0$ and $Q_1 \neq 0$ can be obtained similarly.

Note that $P_2$ equals to $\pm \sqrt{2}$ for multiple solution that has binary collision at the same point does not violates the uniqueness of solution. Two different solution can have all four coordinates the same in the extended regularized plane because $Q_1 = 0$ forces $P_1 = \pm \sqrt{2}$ or $Q_2 = 0$ forces $P_2 = \pm \sqrt{2}$. However this does not violates the uniqueness of solution because the energy $H$ is still different for the two different solutions.

**Proposition 2.5.** For each $(P_1, P_2) \in \mathbb{R}^2$, the point $(Q_1 = 0, Q_2 = 0, P_1, P_2)$ is an equilibrium in the regularized coordinate of the system 2.12.

**Proof.** Consider $\left| \frac{Q_2 Q_1^6}{(Q_1^4 + Q_2^4)^{3/2}} \right|$ and $\left| \frac{Q_2 Q_1^6}{(Q_1^4 + Q_2^4)^{3/2}} \right|$ when $Q_1 \to 0$ and $Q_2 \to 0$. Now

$$\left| \frac{Q_2 Q_1^6}{(Q_1^4 + Q_2^4)^{3/2}} \right| \leq \left| \frac{Q_2 Q_1^6}{(Q_1^4)^{3/2}} \right| \leq |Q_2| \to 0$$

$$\left| \frac{Q_1 Q_2^6}{(Q_1^4 + Q_2^4)^{3/2}} \right| \leq \left| \frac{Q_1 Q_2^6}{(Q_2^4)^{3/2}} \right| \leq |Q_1| \to 0.$$

As a result for an fixed $P_1$ and $P_2$, when $Q_1 \to 0$ and $Q_2 \to 0$ we have

$$\dot{Q}_1 = \frac{1}{4} P_1 Q_2^2 \to 0$$

$$\dot{Q}_2 = \frac{1}{4} P_2 Q_1^2 \to 0$$

$$\dot{P}_1 = -\frac{1}{4} P_2^2 Q_1 + \frac{1}{2} Q_1 + \frac{1}{2} \frac{Q_1 Q_2^6}{(Q_1^4 + Q_2^4)^{3/2}} + 2EQ_1 Q_2^2 \to 0$$

$$\dot{P}_2 = -\frac{1}{4} P_1^2 Q_2 + \frac{1}{2} Q_2 + \frac{1}{2} \frac{Q_2 Q_1^6}{(Q_1^4 + Q_2^4)^{3/2}} + 2EQ_2 Q_1^2 \to 0.$$

Therefore for any fixed $(P_1, P_2) \in \mathbb{R}^2$, the point $(Q_1 = 0, Q_2 = 0, P_1, P_2)$ is an equilibrium in the regularized coordinate system.
2.3 Periodic Brake Orbit 121

Our main goal is to prove the existence of a SBC periodic brake orbit. Using numerical stimulations, we have generated a solution that is close to satisfying the the SBC brake orbit criteria. The two plots below are the numerical stimulated solution in the original coordinates and the regularized coordinates.

Figure 2.2: Periodic brake orbit 121

Figure 2.2 plotted the SBC periodic brake orbit we are looking for on both $x_1$-$x_2$ plane and $Q_1$-$Q_2$ plane. Looking that the left plot ($x_1$-$x_2$ plane), the particle starts at a brake, first collides on the $x_1$ axis, then hits the $x_2$ axis and bounces back under the exact same track, hits the $x_1$ axis at the same spot, and finally reach a brake again at the starting position. To explain the relationship between $x_i$ and $Q_i$, notice that as $x_i \to 0$, $Q_i \to 0$ at the same time. While in the $x_1$-$x_2$ plane the particle bounces back from the axis after collision, in the $Q_1$-$Q_2$ plane the particle would pass through the axis in a $C^1$ manner.

From now on we are going to refer to this particular SBC periodic brake orbit as periodic brake orbit 121 (121 tells us the order of the binary collisions against $x_1$ and $x_2$ axis between two brakes).

In Chapter 3 Future Works we will talk about other possible SBC periodic brake orbit with a different set of collisions, and show the numerical stimulations for those potential SBC periodic brake orbits.
2.4 Generating the Full Periodic Orbit from Part of it Using Symmetry

In the regularized plane, periodic brake orbit 121 can be divided into four parts that are symmetric to each other. By symmetry and a proper time change we can generate the full periodic orbit from one part of the orbit.

Consider periodic brake orbit 121 as illustrated in Figure 2.3.

![Figure 2.3: Periodic brake orbit 121 in four segments](image)

We can describe periodic brake orbit 121 in terms of quadrants in the $Q_1-Q_2$ plane by the flow chart below: (I for 1st quadrant, II for 2nd quadrant, etc.)

brake at $I \rightarrow IV \rightarrow III \rightarrow$ brake at $II \rightarrow III \rightarrow IV \rightarrow$ brake at $I$

From the brake in the I quadrant to the brake in the II quadrant, we refer to the orbit in I and IV quadrants as the 1st segment, the orbit in III and II quadrants as the 2nd segment. From the brake in the II quadrant to the brake in the I quadrant, we refer to the orbit in III and II quadrants to as 3rd segment, the orbit in I and IV quadrants as the 4th segment.
Let $T$ denote the full period of the orbit.

In order for the 1st segment to be able to generate the other three segment in a continuous way such that the other three segments satisfy the differential equations 2.12 at the same time, we need to impose the restriction that $P_2 = 0$ when $Q_1 = 0$. (In other words, to use the argument below, our 1st segment orbit must satisfy the condition that $P_2 = 0$ when $Q_1 = 0$)

Suppose the 1st segment of the periodic orbit is given by

$$(Q_1, Q_2, P_1, P_2)(s) \quad 0 \leq s \leq \frac{T}{4}$$

then we define the others segments in the following way:

2nd segment:

$$(-Q_1, Q_2, P_1, -P_2)(\frac{T}{2} - s) \quad \frac{T}{4} \leq s \leq \frac{T}{2}$$

3rd segment:

$$(-Q_1, Q_2, -P_1, P_2)(s - \frac{T}{2}) \quad \frac{T}{2} \leq s \leq \frac{3T}{4}$$

4th segment:

$$(Q_1, Q_2, -P_1, -P_2)(T - s) \quad \frac{3T}{4} \leq s \leq T$$

Our claim is that the 2nd, 3rd and 4th segment defined in this way does combine with the 1st segment to form a periodic orbit.

First, we have to check that the head and tails of these segments do join together. Note that we have the following constraints:

$$Q_1(t) = 0 \quad \text{and} \quad Q_2(t) = 0 \quad \text{when} \quad t = \frac{T}{4} \text{ or } \frac{3T}{4}$$

$$P_1(t) = 0 \quad \text{and} \quad P_2(t) = 0 \quad \text{when} \quad t = 0 \text{ or } \frac{T}{2} \text{ or } T$$
End of 1st segment:

\[ (Q_1, Q_2, P_1, P_2)(\frac{T}{4}) = (0, Q_2(\frac{T}{4}), P_1(\frac{T}{4}), 0) \]

Beginning of 2nd segment:

\[ (-Q_1, Q_2, P_1, -P_2)(\frac{T}{4}) = (0, Q_2(\frac{T}{4}), P_1(\frac{T}{4}), 0) \]

End of 2nd segment:

\[ (-Q_1, Q_2, P_1, -P_2)(\frac{T}{2}) = (-Q_1(\frac{T}{2}), Q_2(\frac{T}{2}), 0, 0) \]

Beginning of 3rd segment:

\[ (-Q_1, Q_2, -P_1, P_2)(\frac{T}{2}) = (-Q_1(\frac{T}{2}), Q_2(\frac{T}{2}), 0, 0) \]

End of 3rd segment:

\[ (-Q_1, Q_2, -P_1, P_2)(\frac{3T}{4}) = (0, Q_2(\frac{3T}{4}), P_1(\frac{3T}{4}), 0) \]

Beginning of 4th segment:

\[ (Q_1, Q_2, -P_1, -P_2)(\frac{3T}{4}) = (0, Q_2(\frac{3T}{4}), P_1(\frac{3T}{4}), 0) \]

End of 4th segment:

\[ (Q_1, Q_2, -P_1, -P_2)(T) = (Q_1(0), Q_2(0), 0, 0) \]

Beginning of 1st segment:

\[ (Q_1, Q_2, P_1, P_2)(0) = (Q_1(0), Q_2(0), 0, 0) \]

So we see that the heads and tails of the segments do join together.

Now we have to check that, suppose segment 1 satisfies the system of differential equations 2.12, then segment 2, 3 and 4 also satisfy the same system of differential equations.

Suppose segment 1 satisfies the system of differential equations 2.12.
Segment 2:

Use $\hat{Q}_1, \hat{Q}_2, \hat{P}_1, \hat{P}_2$ to represent the the 4 coordinates of segment 2. Let

\[
\begin{align*}
\hat{Q}_1 &= Q_1\left(\frac{T}{2} - s\right) \\
\hat{Q}_2 &= Q_2\left(\frac{T}{2} - s\right) \\
\hat{P}_1 &= P_1\left(\frac{T}{2} - s\right) \\
\hat{P}_2 &= -P_2\left(\frac{T}{2} - s\right).
\end{align*}
\]

Then

\[
\frac{d\hat{Q}_1(s)}{ds} = \frac{d\left(-Q_1\left(\frac{T}{2} - s\right)\right)}{ds} = \hat{Q}_1\left(\frac{T}{2} - s\right)
\]

\[
= \frac{1}{4} P_1\left(\frac{T}{2} - s\right)Q_2^2\left(\frac{T}{2} - s\right)
\]

and

\[
\frac{d\hat{Q}_2(s)}{ds} = \frac{d\left(Q_2\left(\frac{T}{2} - s\right)\right)}{ds} = -\hat{Q}_2\left(\frac{T}{2} - s\right)
\]

\[
= -\frac{1}{4} P_2\left(\frac{T}{2} - s\right)Q_1^2\left(\frac{T}{2} - s\right)
\]

\[
= \frac{1}{4} \hat{P}_2(s)Q_1^2(s)
\]
and

\[
\frac{d\hat{P}_1(s)}{ds} = \frac{d(P_1(T/2 - s))}{ds} = \hat{P}_1(T/2 - s) = -\frac{1}{4} P_1^2(T/2 - s)Q_1(T/2 - s) - \frac{1}{2} Q_1(T/2 - s) - \frac{1}{2} \frac{Q_1(T/2 - s)Q_2(T/2 - s)}{(Q_1(T/2 - s) + Q_2(T/2 - s))^2} - 2EQ_1(T/2 - s)Q_2(T/2 - s)
\]

\[
\frac{d\hat{P}_2(s)}{ds} = \frac{d(-P_2(T/2 - s))}{ds} = \hat{P}_2(T/2 - s) = -\frac{1}{4} P_2^2(T/2 - s)Q_2(T/2 - s) - \frac{1}{2} Q_2(T/2 - s) + \frac{1}{2} \frac{Q_2(T/2 - s)Q_1(T/2 - s)}{(Q_2(T/2 - s) + Q_1(T/2 - s))^2} + 2EQ_2(T/2 - s)Q_1(T/2 - s)
\]

Therefore segment 2 also satisfies the system of differential equations 2.12.
Segment 3:

Use $\dot{Q}_1, \dot{Q}_2, \dot{P}_1, \dot{P}_2$ to represent the the 4 coordinates of segment 3. Let

$$
\begin{align*}
\dot{Q}_1 &= -Q_1(s - \frac{T}{2}) \\
\dot{Q}_2 &= Q_2(s - \frac{T}{2}) \\
\dot{P}_1 &= -P_1(s - \frac{T}{2}) \\
\dot{P}_2 &= P_2(s - \frac{T}{2}).
\end{align*}
$$

Then

$$
\begin{align*}
\frac{d\dot{Q}_1(s)}{ds} &= \frac{d(-Q_1(s - \frac{T}{2}))}{ds} \\
&= -\dot{Q}_1(s - \frac{T}{2}) \\
&= -\frac{1}{4}P_1(s - \frac{T}{2})Q_2^2(s - \frac{T}{2}) \\
&= \frac{1}{4}\dot{P}_1(s)\ddot{Q}_2^2(s)
\end{align*}
$$

and

$$
\begin{align*}
\frac{d\dot{Q}_2(s)}{ds} &= \frac{d(Q_2(s - \frac{T}{2}))}{ds} \\
&= \dot{Q}_2(s - \frac{T}{2}) \\
&= \frac{1}{4}P_2(s - \frac{T}{2})Q_1^2(s - \frac{T}{2}) \\
&= \frac{1}{4}\dot{P}_2(s)\ddot{Q}_1^2(s)
\end{align*}
$$
and

\[
\frac{d\tilde{P}_1(s)}{ds} = \frac{d(-P_1(s - \frac{T}{2}))}{ds} = -\dot{P}_1(s - \frac{T}{2})
\]

\[
= -\dot{P}_1(s - \frac{T}{2}) - \frac{1}{4}P_2^2(s - \frac{T}{2})Q_1(s - \frac{T}{2}) - \frac{1}{2}Q_1(s - \frac{T}{2}) - \frac{1}{2}Q_2(s - \frac{T}{2}) - \frac{1}{2}(\dot{Q}_1^4(s) + \dot{Q}_2^4(s)) - 2EQ_1(s - \frac{T}{2})Q_2^6(s - \frac{T}{2})
\]

\[
= -\frac{1}{4}\dot{P}_2^2(s)\dot{Q}_1(s) + \frac{1}{2}\dot{Q}_1(s) + \frac{1}{2}\dot{Q}_2(s) - \frac{1}{2}(\dot{Q}_1^4(s) + \dot{Q}_2^4(s)) + 2E\dot{Q}_1(s)\dot{Q}_2^2(s)
\]

and

\[
\frac{d\tilde{P}_2(s)}{ds} = \frac{d(P_2(s - \frac{T}{2}))}{ds} = \dot{P}_2(s - \frac{T}{2})
\]

\[
= \dot{P}_2(s - \frac{T}{2}) - \frac{1}{4}P_1^2(s - \frac{T}{2})Q_2(s - \frac{T}{2}) - \frac{1}{2}Q_2(s - \frac{T}{2}) - \frac{1}{2}Q_1(s - \frac{T}{2}) - \frac{1}{2}(\dot{Q}_1^4(s) + \dot{Q}_2^4(s)) - 2EQ_2(s - \frac{T}{2})Q_1^6(s - \frac{T}{2})
\]

\[
= -\frac{1}{4}\dot{P}_1^2(s)\dot{Q}_2(s) + \frac{1}{2}\dot{Q}_2(s) + \frac{1}{2}\dot{Q}_1(s) - \frac{1}{2}(\dot{Q}_1^4(s) + \dot{Q}_2^4(s)) + 2E\dot{Q}_2(s)\dot{Q}_1^2(s).
\]

Therefore segment 3 also satisfies the system of differential equations 2.12.

**Segment 4:**

Use \(\tilde{Q}_1, \tilde{Q}_2, \tilde{P}_1, \tilde{P}_2\) above to represent the the 4 coordinates of segment 4.
Let
\[ \tilde{Q}_1 = Q_1(T - s), \]
\[ \tilde{Q}_2 = Q_2(T - s), \]
\[ \tilde{P}_1 = -P_1(T - s), \]
\[ \tilde{P}_2 = -P_2(T - s). \]

Then
\[ \frac{d\tilde{Q}_1(s)}{ds} = \frac{d(Q_1(T - s))}{ds} \]
\[ = -\dot{Q}_1(T - s) \]
\[ = -\frac{1}{4} P_1(T - s) Q_2^2(T - s) \]
\[ = \frac{1}{4} \tilde{P}_1(s) \tilde{Q}_2^2(s) \]
and
\[ \frac{d\tilde{Q}_2(s)}{ds} = \frac{d(Q_2(T - s))}{ds} \]
\[ = -\dot{Q}_2(T - s) \]
\[ = -\frac{1}{4} P_2(T - s) Q_1^2(T - s) \]
\[ = \frac{1}{4} \tilde{P}_2(s) \tilde{Q}_1^2(s) \]
and
\[ \frac{d\tilde{P}_1(s)}{ds} = \frac{d(-P_1(T - s))}{ds} \]
\[ = \dot{P}_1(T - s) \]
\[ = -\frac{1}{4} P_2^2(T - s) Q_1(T - s) + \frac{1}{2} Q_1(T - s) + \frac{1}{2} \frac{Q_1(T - s) Q_2^2(T - s)}{(Q_1^4(T - s) + Q_2^4(T - s))^{3/2}} \]
\[ + 2E\tilde{Q}_1(T - s) Q_2^2(T - s) \]
\[ = -\frac{1}{4} \tilde{P}_2^2(s) \tilde{Q}_1(s) + \frac{1}{2} \tilde{Q}_1(s) + \frac{1}{2} \frac{\tilde{Q}_1(s) \tilde{Q}_2^2(s)}{(\tilde{Q}_1^4(s) + \tilde{Q}_2^4(s))^{3/2}} + 2E\tilde{Q}_1(s) \tilde{Q}_2^2(s) \]
and
\[
\frac{d\tilde{P}_2(s)}{ds} = \frac{d(-P_2(T - s))}{ds} = \tilde{P}_2(T - s)
\]
\[
= \frac{1}{4} P_1^2(T - s)Q_2(T - s) + \frac{1}{2} Q_2(T - s) + \frac{1}{2} \frac{Q_2(T - s)Q_6^5(T - s)}{(Q_4^1(T - s) + Q_5^2(T - s))^2}
\]
\[
+ 2E \tilde{Q}_2(T - s)\tilde{Q}_1^2(T - s)
\]
\[
= \frac{1}{4} \tilde{P}_1^2(s)\tilde{Q}_2(s) + \frac{1}{2} \tilde{Q}_2(s) + \frac{1}{2} \frac{\tilde{Q}_2(s)\tilde{Q}_1^6(s)}{(\tilde{Q}_4^1(s) + \tilde{Q}_5^2(s))^2} + 2E \tilde{Q}_2(s)\tilde{Q}_1^2(s).
\]

Therefore segment 4 also satisfies the system of differential equations 2.12.

Now since all 4 segments’ head and tail do join together, and they all satisfy differential equations 2.12, by uniqueness of solutions to the initial value problem, they must form a periodic orbit.

### 2.5 Zero Momentum Curve and Scaling Energy

In the coming sections we will consider solutions with different energy. It turns out that finding one of those solutions would mean that we can find it with all energy of the same sign. First we talk about what a zero momentum curve is. Then we show that any orbit can be scaled to a chosen energy level without changing the shape of the orbit, so we can scale one zero momentum curve to another.

While looking for a brake orbit, it is very difficult to specify the right initial position and velocity so that the orbit would momentarily reach a brake at some point. As a result we pick a family of orbits that starts stationary, and see whether a correctly chosen initial position would result in an orbit that returns back to the initial position at some time. Since energy is conserved, if the orbit returns to its initial stationary position it must reach a brake at that point (Having a non-zero velocity would mean it has extra kinetic energy that is not present at the initial position, contradicting energy conservation).
Now given any negative energy, we can identify a zero momentum curve for which under zero initial velocity, all the initial positions along the curve has the same energy. Figure 2.4 showed the zero momentum curves for a few different energy levels.

Now given energy $H_0 < 0$, the zero momentum curve divides the 1st quadrant into two regions, a permissible region and a impermissible region as illustrated in Figure 2.5. An orbit with energy $H_0$ has no way of reaching the impermissible region, and always stays within the permissible region (except possibly being stationary at the boundary of the permissible region).

**Theorem 2.6.** Any solution of our system of differential equations 2.5 and 2.6 can be scaled to any particular energy of the same sign.

**Proof.** Given $x_1(t)$, $x_2(t)$ a solution to our system of differential equations 2.5 and 2.6 with energy $H < 0$, for $a > 0$, we define a new position variable

\[ s_1(t) = ax_1(a^{-\frac{3}{2}}t) > 0 \]
\[ s_2(t) = ax_2(a^{-\frac{3}{2}}t) > 0. \]

Now $s_1(t)$ and $s_2(t)$ satisfy the exact same differential equation because
\[ \ddot{s}_1(t) = \frac{\dddot{x}_1(a^{-\frac{3}{2}}t)}{a^2} \]

\[ = \frac{1}{a^2} \left( \frac{1}{4x_1^2(a^{-\frac{3}{2}}t)} - \frac{x_1(a^{-\frac{3}{2}}t)}{4\left(x_1^2(a^{-\frac{3}{2}}t) + x_2^2(a^{-\frac{3}{2}}t)\right)^{3/2}} \right) \]

\[ = -\frac{1}{4a^2x_1^2(a^{-\frac{3}{2}}t)} - \frac{ax_1(a^{-\frac{3}{2}}t)}{4\left(a^2x_1^2(a^{-\frac{3}{2}}t) + a^2x_2^2(a^{-\frac{3}{2}}t)\right)^{3/2}} \]

\[ = \frac{1}{4s_1^2(t)} - \frac{s_1(t)}{4\left(s_1^2(t) + s_2^2(t)\right)^{3/2}} \]

\[ \ddot{s}_2(t) = \frac{\dddot{x}_2(a^{-\frac{3}{2}}t)}{a^2} \]

\[ = \frac{1}{a^2} \left( \frac{1}{4x_2^2(a^{-\frac{3}{2}}t)} - \frac{x_2(a^{-\frac{3}{2}}t)}{4\left(x_1^2(a^{-\frac{3}{2}}t) + x_2^2(a^{-\frac{3}{2}}t)\right)^{3/2}} \right) \]

\[ = -\frac{1}{4a^2x_2^2(a^{-\frac{3}{2}}t)} - \frac{ax_2(a^{-\frac{3}{2}}t)}{4\left(a^2x_1^2(a^{-\frac{3}{2}}t) + a^2x_2^2(a^{-\frac{3}{2}}t)\right)^{3/2}} \]

\[ = \frac{1}{4s_2^2(t)} - \frac{s_2(t)}{4\left(s_1^2(t) + s_2^2(t)\right)^{3/2}} \]
which matches our system of differential equations 2.5 and 2.6.

Substitute $s_1$ and $s_2$ into equation 2.7, The energy $H_a$ for $s_1$ and $s_2$ is given by

$$H_a = \frac{1}{2}(\ddot{s}_1^2(t) + \ddot{s}_2^2(t)) - \frac{1}{4s_1(t)} - \frac{1}{4s_2(t)} - \frac{1}{4(s_1^2(t) + s_2^2(t)))^{1/2}}$$

$$= \frac{1}{2a}(\ddot{x}_1^2(a^{-\frac{3}{2}}t) + \ddot{x}_2^2(a^{-\frac{3}{2}}t)) - \frac{1}{4ax_1(a^{-\frac{3}{2}}t)} - \frac{1}{4ax_2(a^{-\frac{3}{2}}t)} - \frac{1}{4a(x_1^2(a^{-\frac{3}{2}}t) + x_2^2(a^{-\frac{3}{2}}t)))^{1/2}}$$

$$= \frac{H}{a}.$$

By going from $x_1, x_2$ to $s_1, s_2$ under parameter $a$, we can scale the energy of our orbit by $\frac{1}{a}$. Thus when a particular orbit exists for one energy, it exists for all energy of the same sign.

\[ \square \]

### 2.6 Orbit Going Up

**Conjecture 2.7.** An orbit that starts at a brake, hits the $x_1$ axis, then collides on the $x_2$ axis with $\dot{x}_2 > 0$ exists.

![Figure 2.6: Four parts of showing orbit going up exist](image)

In this section we want to prove Conjecture 2.7: the existence of an orbit that starts at a brake, hits the $x_1$ axis, then hits the $x_2$ axis with $\dot{x}_2 > 0$. We will simply call it an orbit
going up. Proving the existence of an orbit going up involves 4 parts, as shown in Figure 2.6. In part 1 (section 2.6.1) we have to show that an orbit that starts at a brake on the right side of the $x_1 = x_2$ line, then hits the $x_1$ axis exists. In part 2 (section 2.6.2) we have to show that there exist an orbit that extends from part 1 and cross the $x_1 = x_2$ line with $\dot{x}_2 > 0$. In part 3 (section 2.6.3-2.6.7) we estimate the time it takes from the orbit crossing $x_1 = x_2$ to $x_2(t)$ going to zero. In part 4 (section 2.6.8) we estimate the time it takes for the orbit from crossing $x_1 = x_2$ to hitting $x_2$ axis. We have obtained analytical proof for part 1, 3 and 4; We rely on numerical results for part 2.

In the following sections the existence of orbits are proved under a certain energy, but again by Theorem 2.6 we know that we can scale the orbit to achieve all energy of the same sign.

### 2.6.1 Orbit that Starts at a Brake and Hits the $x_1$ Axis

In this section we want to show that an orbit that starts at a brake on the right side of the $x_1 = x_2$ line, then hits the $x_1$ axis exists. We begin with proving a lemma.

**Lemma 2.8.** Under differential equations 2.5 and 2.6,

\[
x_1 > x_2 \iff \ddot{x}_2 < \ddot{x}_1
\]

\[
x_1 < x_2 \iff \ddot{x}_2 > \ddot{x}_1.
\]

**Proof.** First we show that

\[
x_1 = x_2 \iff \dot{x}_1 = \dot{x}_2
\]

($\Rightarrow$)

If $x_1 = x_2$, then

\[
\frac{1}{4x^2_1} = \frac{1}{4x^2_2}
\]

and

\[
\frac{x_1}{4(x^2_1 + x^2_2)^{3/2}} = \frac{x_2}{4(x^2_1 + x^2_2)^{3/2}}
\]

so obviously $\ddot{x}_1 = \ddot{x}_2$. 

Now suppose $\bar{x}_1 = \bar{x}_2$. Then

$$\frac{1}{4x_1^3} - \frac{x_1}{4(x_1^2 + x_2^2)^{3/2}} = \frac{1}{4x_2^3} - \frac{x_2}{4(x_1^2 + x_2^2)^{3/2}}$$

$$(x_1^2 + x_2^2)^{3/2} + x_1^3 = \frac{(x_1^2 + x_2^2)^{3/2} + x_2^3}{x_1^2(x_1^2 + x_2^2)^{3/2}}$$

$$x_2^2[(x_1^2 + x_2^2)^{3/2} + x_1^3] = x_1^2[(x_1^2 + x_2^2)^{3/2} + x_2^3]$$

$$x_2^2(x_1^2 + x_2^2)^{3/2} + x_1^2x_2^2 = x_1^2(x_1^2 + x_2^2)^{3/2} + x_2^3x_1^2$$

$$(x_1^2 - x_1^3)(x_1^2 + x_2^2)^{3/2} = x_1^2x_2^2 - x_2^3x_1^2$$

$$(x_2 - x_1)(x_2 + x_1)(x_1^2 + x_2^2)^{3/2} = x_1^2x_2^2(x_2 - x_1)$$

$$0 = (x_2 - x_1)[(x_2 + x_1)(x_1^2 + x_2^2)^{3/2} - x_1^2x_2^2].$$

Now we want to show that

$$f(x_1, x_2) = (x_2 + x_1)(x_1^2 + x_2^2)^{3/2} - x_1^2x_2^2$$

is positive when $x_1 > 0$ and $x_2 > 0$

Assuming $x_1 > 0$ and $x_2 > 0$ we can use arithmetic and geometric mean inequality, to get the following two relationships:

$$\frac{x_1 + x_2}{2} > \sqrt{x_1x_2}$$

$$\frac{x_1^2 + x_2^2}{2} > x_1x_2.$$
when \( x_1 > 0 \) and \( x_2 > 0 \). Thus
\[
\ddot{x}_1 = \ddot{x}_2 \implies x_1 = x_2
\]

Now for \( x_1 >> x_2 \) we have
\[
\ddot{x}_1 \approx 0, \quad \ddot{x}_2 \approx -\frac{1}{4x_2^2} < 0,
\]
and for \( x_1 << x_2 \) we have
\[
\ddot{x}_2 \approx 0, \quad \ddot{x}_1 \approx -\frac{1}{4x_1^2} < 0.
\]
Thus
\[
x_1 > x_2 \iff \ddot{x}_1 < \ddot{x}_2
\]
\[
x_1 < x_2 \iff \ddot{x}_1 > \ddot{x}_2.
\]

**Theorem 2.9.** For solution \((x_1(t), x_2(t))\) with \( x_1(0) > x_2(0) \) and \( \dot{x}_1(0) = 0, \dot{x}_2(0) = 0 \), there exist \( \beta > 0 \) and \( t^* \in (0, \beta) \) such that \( x_1(t) < x_2(t) \) for all \( t \in [0, \beta] \) and \( x_2(t^*) = 0 \) with \( x_1(0) - x_2(0) \leq x_1(t^*) < x_1(0) \). Furthermore \( \dot{x}_1(t) < 0 \) for all \( t \in (0, \beta) \).

**Proof.** Let \((x_1(t), x_2(t))\) be the solution satisfying
\[
x_1(0) > x_2(0)
\]
\[
\dot{x}_1(0) = 0, \quad \dot{x}_2(0) = 0
\]
By continuity of the continuously extended solution (regularization) with respect to its initial conditions, there exist a maximal and energy preserving \( \beta > 0 \) such that \( x_1(t) > x_2(t) \) for all \( t \in [0, \beta) \). This implies that
\[
\lim_{t \to \beta} x_1(t) = \lim_{t \to \beta} x_2(t),
\]
oun{otherwise it contradicts} \( \beta \) being maximal. This implies that \( x_1(t) > 0 \) for all \( t \in [0, \beta) \). From our initial conditions we know that \( x_2(0) > 0 \). Let \( t^* \in (0, \beta] \) be the largest possible time such that \( x_2(t) > 0 \) for all \( t \in [0, t^*) \).

Claim: \( t^* < \beta \) and \( x_1(0) - x_2(0) \leq x_1(t^*) < x_1(0) \).
Since \( x_1(t) > 0 \) and \( x_2(t) > 0 \) for all \( t \in [0,t^*] \), we know that \( \dot{x}_1(t) \) and \( \dot{x}_2(t) \) are strictly decreasing on \([0,t^*]\), i.e.
\[
\dot{x}_1(t) < \dot{x}_1(0) = 0 \\
\dot{x}_2(t) < \dot{x}_2(0) = 0
\]
for all \( t \in (0,t^*) \).

Since \( x_1(t) > x_2(t) \) for all \( t \in [0,t^*] \), by Lemma 2.8 we know that
\[
\ddot{x}_2(t) < \ddot{x}_1(t)
\]
for all \( t \in [0,t^*] \). Integration of \( \dot{x}_2(t) < \dot{x}_1(t) \) gives
\[
\dot{x}_2(t) - \dot{x}_2(0) < \dot{x}_1(t) - \dot{x}_1(0)
\]
Since \( \dot{x}_1(0) = 0 \) and \( \dot{x}_2(0) = 0 \) we get
\[
\dot{x}_2(t) < \dot{x}_1(t) < 0
\]
for all \( t \in [0,t^*] \).

From \( \dot{x}_1(0) = 0 \) for all \( t \in (0,t^*) \), continuity of \( \dot{x}_1 \) and integration gives us
\[
x_1(t^*) < x_1(0).
\]

Now integration of \( \dot{x}_2(t) < \dot{x}_1(t) < 0 \) for \( t \in [0,t^*] \) gives,
\[
x_2(t) - x_2(0) < x_1(t) - x_1(0)
\]
for all \( t \in [0,t^*] \). So
\[
x_1(t) \geq x_2(t) + x_1(0) - x_2(0) > x_2(t) + k
\]
because \( x_1(0) - x_2(0) = k > 0 \). From the equation above we know that
\[
x_1(0) > x_1(t^*) \geq x_1(0) - x_2(0)
\]
If \( t^* = \beta \), then we have

\[
x_1(\beta) > x_2(\beta) + k
\]

for \( k > 0 \), contradicting \( \lim_{t \to \beta} x_1(t) = \lim_{t \to \beta} x_2(t) \).

Therefore it must be that \( t^* < \beta \). (Done with proof of our claim)

Theorem 2.9 essentially means that an orbit that starts at a brake on the right side of the \( x_1 = x_2 \) line, then hits the \( x_1 \) axis exists.

\[ \text{Figure 2.7: Numerical Results for Conjecture 2.10} \]
2.6.2 Orbit that Cross the $x_1 = x_2$ Line with $\dot{x}_2 > 0$

**Conjecture 2.10.** An orbit that starts at a brake, hits the $x_1$ axis, then crosses the $x_1 = x_2$ line with $\dot{x}_2 > 0$ exists.

We provide numerical results for Conjecture 2.10 in this section. Conjecture 2.10 is essentially part 2 of the orbit going up that we are trying to prove. Analytical results for this part have not been obtained yet. As shown in Figure 2.7, all four orbits here crosses the $x_1 = x_2$ line with $\dot{x}_2 > 0$. Numerical simulations suggest that orbits that starts close enough to the $x_1 = x_2$ line will cross the $x_1 = x_2$ line with $\dot{x}_2 > 0$.

In the Future Work section we will mention approaches to proving Conjecture 2.10 analytically.

2.6.3 Set up for proving part 3 and 4 of the orbit going up

Assuming Conjecture 2.10 is true, we now have an orbit that cross the $x_1 = x_2$ line with $\dot{x}_2 > 0$. Scale this orbit until the particle would cross the $x_1 = x_2$ line at $x_1 = x_2 = 1$, corresponding to energy $\hat{H}_0 > H_0$.

For any energy level higher than $\hat{H}_0$, we can always find a corresponding initial $x_1$ and $x_2$ such that the orbit would start at a brake, collide at $x_1$ axis then cross the $x_1 = x_2$ line at $x_1 = x_2 = 1$. The closer the negative energy is to 0, the closer the initial position approaches the $x_1 = x_2$ line.

Consider the case when energy level is negative and close to 0. Let $E = -\epsilon$ for some $\epsilon > 0$. We now define a set of times we are going to look at while proving part 3 and 4 of the orbit going up (refer to Figure 2.8).

Let 0 denote the starting time,

$t_1$ denote the time when $x_1 = x_2 = 1$,

$t_\frac{1}{2}$ denote the time when $x_1 = \frac{1}{2}$,

$t_0$ denote the time when $x_1 = 0$ (collision at $x_2$ axis),

$t_m$ denote the time when $x_2$ reaches a local maximum ($\dot{x}_2 = 0$).
2.6.4 Lower Bound on $x_1(0)$ and $x_2(0)$

We need to first obtain a lower bound on $x_1(0)$ and $x_2(0)$ using the energy constraint. Begin with the energy equation 2.7:

$$H = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \left(\frac{1}{4x_1} + \frac{1}{4x_2} + \frac{1}{4(x_1^2 + x_2^2)^{\frac{1}{2}}}\right).$$

Under $H = -\epsilon$, at the initial time when $\dot{x}_1(0) = 0$ and $\dot{x}_2(0) = 0$ we have,

$$-\epsilon = \frac{1}{2}(\dot{x}_1^2(0) + \dot{x}_2^2(0)) - \left(\frac{1}{4x_1(0)} + \frac{1}{4x_2(0)} + \frac{1}{4(x_1^2(0) + x_2^2(0))^{\frac{1}{2}}}\right)$$

$$-\epsilon = -\left(\frac{1}{4x_1(0)} + \frac{1}{4x_2(0)} + \frac{1}{4(x_1^2(0) + x_2^2(0))^{\frac{1}{2}}}\right)$$

$$\epsilon = \frac{1}{4x_1(0)} + \frac{1}{4x_2(0)} + \frac{1}{4(x_1^2(0) + x_2^2(0))^{\frac{1}{2}}}.$$ 

All these terms on the right are positive. We get the two inequalities,
\[
\epsilon \geq \frac{1}{4x_1(0)}, \quad \epsilon \geq \frac{1}{4x_2(0)}.
\]

Rearrange the terms to get

\[
x_1(0) \geq \frac{1}{4\epsilon}, \quad x_2(0) \geq \frac{1}{4\epsilon}.
\] (2.13)

### 2.6.5 Lower and Upper bound for \(\dot{x}_1^2(t_1)\)

Before crossing \(x_1 = x_2\) line, we can derive the following inequalities using \(x_2 \leq x_1\):

\[
x_1^2 + x_2^2 \leq 2x_1^2
\]

\[
-\frac{x_1\dot{x}_1}{4(x_1^2 + x_2^2)^{3/2}} \geq -\frac{x_1\dot{x}_1}{4(2x_1^2)^{3/2}}.
\] (2.14)

(The inequality sign flipped three times, the third time because \(\dot{x}_1 \leq 0\))

First we get a lower bound for \(\dot{x}_1(t_1)^2\) by integrating \(\ddot{x}_1\) (equation 2.5):

\[
\ddot{x}_1 = -\frac{1}{4x_1^2} - \frac{x_1}{4(x_1^2 + x_2^2)^{3/2}}
\]

\[
\dot{x}_1\ddot{x}_1 = -\frac{\dot{x}_1}{4x_1^2} - \frac{\dot{x}_1 x_1}{4(x_1^2 + x_2^2)^{3/2}},
\]

using inequality 2.14, we get

\[
\dot{x}_1\ddot{x}_1 \geq -\frac{\dot{x}_1}{4x_1^2} - \frac{x_1\dot{x}_1}{4(2x_1^2)^{3/2}}
\]

\[
\dot{x}_1\ddot{x}_1 \geq -\left(\frac{1}{4} + \frac{1}{8\sqrt{2}}\right) \frac{\dot{x}_1}{x_1^2}
\]

\[
\frac{1}{2} \dot{x}_1^2 \bigg|_{t_1}^{t_1} \geq \left(\frac{1}{4} + \frac{1}{8\sqrt{2}}\right) \frac{1}{x_1^2} \bigg|_{t_1}^{t_1},
\]
since $\frac{1}{x_1(0)} \leq 4\epsilon$ and $x_1(t_1) = 1$ and $\dot{x}_1(0) = 0,$

$$\frac{1}{2} \dot{x}_1(t_1)^2 \geq \left( \frac{1}{4} + \frac{1}{8\sqrt{2}} \right) \left( 1 - 4\epsilon \right)$$

$$\dot{x}_1(t_1)^2 \geq \frac{1}{2} + \frac{1}{4\sqrt{2}} - 2\epsilon - \frac{\epsilon}{\sqrt{2}}. \quad (2.15)$$

Now we get a upper bound for $\dot{x}_1(t_1)^2$:

$$\ddot{x}_1 = -\frac{1}{4x_1^2} - \frac{x_1}{4(x_1^2 + x_2^2)^{3/2}}$$

$$\dot{x}_1 \ddot{x}_1 = -\frac{\dot{x}_1}{4x_1^2} - \frac{\dot{x}_1 x_1}{4(x_1^2 + x_2^2)^{3/2}}$$

$$\dot{x}_1 \ddot{x}_1 \leq -\frac{\dot{x}_1}{4x_1^2} - \frac{x_1 \dot{x}_1}{4(x_1^2)^{3/2}} \quad \text{using } x_1^2 + x_2^2 \geq x_1^2$$

$$\dot{x}_1 \ddot{x}_1 \leq -\frac{1}{2x_1^2} \dot{x}_1$$

$$\frac{1}{2} \dot{x}_1(t_1)^2 \bigg|_0 \leq \frac{1}{2} \dot{x}_1(0)^2.$$

Since $x_1(t_1) = 1$ and $\dot{x}_1(0) = 0,$

$$\dot{x}_1^2(t_1) \leq 1 - \frac{1}{x_1(0)}$$

$$\dot{x}_1^2(t_1) \leq 1. \quad (2.16)$$

Combining with equation 2.15 to get

$$\frac{1}{2} + \frac{1}{4\sqrt{2}} - 2\epsilon - \frac{\epsilon}{\sqrt{2}} \leq \dot{x}_1^2(t_1) \leq 1. \quad (2.17)$$
2.6.6 Lower and Upper Bound for $\dot{x}_2^2(t)$

We go back to the energy constraint (Equation 2.7) and consider the time $t_1$ (the time when $x_1 = x_2 = 1$). Since $H = -\epsilon$ and $x_1(t_1) = x_2(t_1) = 1$, we have

$$-\epsilon = \frac{1}{2}(\dot{x}_1^2(t_1) + \dot{x}_2^2(t_1)) - \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4(1 + 1)^{\frac{1}{2}}}\right)$$

$$\frac{1}{2} + \frac{1}{4\sqrt{2}} - \epsilon = \frac{1}{2}(\dot{x}_1^2(t_1) + \dot{x}_2^2(t_1))$$

$$1 + \frac{1}{2\sqrt{2}} - 2\epsilon = \dot{x}_1^2(t_1) + \dot{x}_2^2(t_1). \quad (2.18)$$

Combine this with the bounds on $\dot{x}_1^2(t_1)$ (equation 2.17) to get

$$\frac{1}{2\sqrt{2}} - 2\epsilon \leq \dot{x}_2^2(t_1) \leq \frac{1}{2} + \frac{1}{4\sqrt{2}} + \frac{\epsilon}{\sqrt{2}}. \quad (2.19)$$

To summarize, we now have bounds for $\dot{x}_1^2(t_1)$ and $\dot{x}_2^2(t_1)$ from equation 2.17 and 2.19:

$$\frac{1}{2} + \frac{1}{4\sqrt{2}} - 2\epsilon - \frac{\epsilon}{\sqrt{2}} \leq \dot{x}_1^2(t_1) \leq 1$$

$$\frac{1}{2\sqrt{2}} - 2\epsilon \leq \dot{x}_2^2(t_1) \leq \frac{1}{2} + \frac{1}{4\sqrt{2}} + \frac{\epsilon}{\sqrt{2}}.$$ 

Using above inequalities we can get a upper bound on time elapsed from crossing $x_1 = x_2$ to hitting $x_2$ axis ($t_0 - t_1$), and also get a lower bound on the time needed for $\dot{x}_2(t_1)$ to go to zero ($t_m - t_1$).

2.6.7 Estimating Time Needed for $\dot{x}_2(t)$ to Go to Zero

Consider $t_1 \leq t \leq t_m$ (after crossing $x_1 = x_2$, before $\dot{x}_2(t)$ goes to zero). Since $\dot{x}_2(t)$ is positive in this time interval, we know that $x_2(t) \geq 1$.

Time needed for $\dot{x}_2(t)$ to start from $t_1$ and go to zero is given by $t_m - t_1$. 
Begin by estimating $\ddot{x}_2$ (equation 2.6):

$$
\ddot{x}_2(t) = -\frac{1}{4x_2^2(t)} - \frac{x_2}{4(x_2^2(t) + x_1^2(t))^{3/2}}
$$

$$
\ddot{x}_2(t) \geq -\frac{1}{4x_2^2(t)} - \frac{x_2}{4(x_2^2(t))^{3/2}} \quad \text{using } x_1^2 + x_2^2 \geq x_1^2
$$

because $x_2(t) \geq 1$,

$$
\ddot{x}_2(t) \geq -\frac{1}{2}
$$

$$
\dot{x}_2(t) \bigg|_{t_1}^{t_m} \geq -\frac{1}{2}(t_m - t_1)
$$

$$
\dot{x}_2(t_m) - \dot{x}_2(t_1) \geq -\frac{1}{2}(t_m - t_1)
$$

$$
\frac{1}{2}(t_m - t_1) + \dot{x}_2(t_m) \geq \dot{x}_2(t_1) \geq \sqrt{\frac{1}{2\sqrt{2}} - 2\epsilon} \quad \text{by equation (2.19)}.
$$

Since $\dot{x}_2(t_m) = 0$, we have

$$
t_m - t_1 \geq 2\sqrt{\frac{1}{2\sqrt{2}} - 2\epsilon}. \quad (2.20)
$$

In the case of $\epsilon = 0.01$ we would get

$$
t_m - t_1 \geq 1.155. \quad (2.21)
$$

This estimate gets bigger as $\epsilon \to 0$. 

2.6.8 Estimating Time Elapsed from Crossing $x_1 = x_2$ to Hitting $x_2$ Axis

Consider $t_1 \leq t \leq t_0$ (after crossing $x_1 = x_2$, before hitting $x_2$ axis).

Begin with estimating $\ddot{x}_1$ (equation 2.5)

$$\ddot{x}_1 = -\frac{1}{4x_1^2} - \frac{x_1}{4(x_1^2 + x_2^2)^{3/2}}$$

$$\ddot{x}_1 \leq -\frac{1}{4x_1^2}$$

$$\ddot{x}_1 \leq -\frac{1}{4}, \quad \text{because } x_1(t) \leq 1$$

now take $t_1 \leq t^* \leq t_2$

$$\ddot{x}_1(t) \bigg|_{t_1}^{t^*} \leq -\frac{1}{4} t \bigg|_{t_1}^{t^*}$$

$$\ddot{x}_1(t^*) - \ddot{x}_1(t_1) \leq -\frac{1}{4} (t^* - t_1)$$

$$\ddot{x}_1(t^*) \leq \ddot{x}_1(t_1) - \frac{1}{4} (t^* - t_1)$$

$$\ddot{x}_1(t^*) \leq -\sqrt{\frac{1}{2} + \frac{1}{4\sqrt{2}} - 2\epsilon - \frac{\epsilon}{\sqrt{2}} - \frac{1}{4}(t^* - t_1)}$$

$$x_1(t^*) \bigg|_{t_1}^{t_2} \leq -t^* \sqrt{\frac{1}{2} + \frac{1}{4\sqrt{2}} - 2\epsilon - \frac{\epsilon}{\sqrt{2}} - \frac{1}{4}(t^* - t_1)^2} \bigg|_{t_1}^{t_2}$$

$$x_1(t_2) - x_1(t_1) \leq -(t_2 - t_1) \sqrt{\frac{1}{2} + \frac{1}{4\sqrt{2}} - 2\epsilon - \frac{\epsilon}{\sqrt{2}} - \frac{1}{8}(t_2 - t_1)^2}$$

$$ (\frac{1}{2} - 1) \leq -(t_2 - t_1) \sqrt{\frac{1}{2} + \frac{1}{4\sqrt{2}} - 2\epsilon - \frac{\epsilon}{\sqrt{2}} - \frac{1}{8}(t_2 - t_1)^2}$$

$$0 \geq \frac{1}{8}(t_2 - t_1)^2 + (t_2 - t_1) \sqrt{\frac{1}{2} + \frac{1}{4\sqrt{2}} - 2\epsilon - \frac{\epsilon}{\sqrt{2}} - \frac{1}{2}}$$

(2.22)

Solving the quadratic inequality with $\epsilon = 0.01$, we get

$$-7.01826 \leq t_{\frac{1}{2}} - t_1 \leq 0.570.$$  (2.23)
Since \( \dot{x}_1 \) is going more and more negative as \( x_1 \to 0 \), the time needed to get from \( x_1 = 1 \) to \( x_1 = \frac{1}{2} \) is going to be more than the time needed to get from \( x_1 = \frac{1}{2} \) to \( x_1 = 0 \). Double the value of \((t_\frac{1}{2} - t_1)\) to get an upper bound for \((t_0 - t_1)\),

\[
t_0 - t_1 \leq 2 \cdot 0.570 = 1.14.
\]

Under \( H = -0.01 \), equation 2.21 gives us \( t_m - t_1 \geq 1.155 \), which in turn implies that \( t_0 - t_1 \leq t_m - t_1 \), (time it takes for the particle to hit \( x_2 \) axis is less than time needed for \( \dot{x}_2 \) to become negative) so the \( \dot{x}_2 \) is positive at collision.

A energy level closer to zero (less negative) would lead to tighter bound for both \( t_m - t_1 \) and \( t_0 - t_1 \) (a bigger \( t_m - t_1 \) and a smaller \( t_0 - t_1 \)). The inequality \( t_0 - t_1 \leq t_m - t_1 \) holds for all negative energy \( H > -0.01 \).

### 2.6.9 Summary of Section 2.6

The goal of section 2.6 is to prove Conjecture 2.7: existence of an orbit that starts at a brake, hits the \( x_1 \) axis, then hits the \( x_2 \) axis with \( \dot{x}_2 > 0 \). The orbit is divided into four parts. Theorem 2.9 gives us part 1 of the orbit. We provided numerical results for Conjecture 2.10, which would give us part 2 of the orbit if assumed to be true. After several time estimation, we showed that the time it takes from the orbit crossing \( x_1 = x_2 \) to \( \dot{x}_2(t) \) going to zero (part 3) is greater than the time it takes for the orbit from crossing \( x_1 = x_2 \) to hitting \( x_2 \) axis (part 4). Which means when the orbit hits the \( x_2 \) axis it must have \( \dot{x}_2 > 0 \).

With all four parts combined, assuming Conjecture 2.10 is true, then we know that Conjecture 2.7 is true; or that an orbit going up exist.
2.7 Orbit Going Down

We call an orbit that starts at a brake, hits the $x_1$ axis, then hits the $x_2$ axis with $\dot{x}_2 < 0$ an orbit going down. To prove the existence of an orbit going down, we are going to first prove the existence of an orbit that bounces off $x_1$ axis once, then ends in total collision. After that we apply a perturbation of the initial condition of the total collision solution to obtain an orbit going down.

**Lemma 2.11.** An orbit that bounces off the $x_1$ axis once, then ends in total collision exists.

**Proof.** We first show that we can begin with an orbit that bounces off $x_1$ axis many times before reaching $x_2$ axis.

Recall our energy equation 2.7:

$$H = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \left(\frac{1}{4x_1} + \frac{1}{4x_2} + \frac{1}{4(x_1^2 + x_2^2)^{1/2}}\right)$$

Given an energy level $H = H_0$. Set $\dot{x}_1(0) = \dot{x}_2(0) = 0$, we get

$$H_0 = -\left(\frac{1}{4x_1(0)} + \frac{1}{4x_2(0)} + \frac{1}{4(x_1^2(0) + x_2^2(0))^{1/2}}\right)$$
for any large enough $x_1(0)$ there is a corresponding $x_2(0)$ that would give our energy $H_0$ (We can show this by a simple implicit function theorem argument).

As $x_1(0) \to \infty$, we have $x_2(0) \to -\frac{1}{4H_0}$, a finite value.

Looking at $\ddot{x}_1$ and $\ddot{x}_2$,

$$\ddot{x}_1 = -\frac{1}{4x_1^2} + \frac{-x_1}{4(x_1^2 + x_2^2)\sqrt[3]{2}}$$

$$\lim_{x_1 \to \infty} \ddot{x}_1 = \lim_{x_1 \to \infty} -\frac{1}{4x_1^2} + \frac{-x_1}{4(x_1^2 + x_2^2)\sqrt[3]{2}} = 0$$

$$\ddot{x}_2 = -\frac{1}{4x_2^2} + \frac{-x_2}{4(x_2^2 + x_1^2)\sqrt[3]{2}}$$

$$\lim_{x_1 \to \infty} \ddot{x}_2 = \lim_{x_1 \to \infty} -\frac{1}{4x_2^2} + \frac{-x_2}{4(x_2^2 + x_1^2)\sqrt[3]{2}} = \lim_{x_1 \to \infty} -\frac{1}{4x_2^2} = -4H_0^2$$ which is bounded away from zero.

Pick a large enough $x_1(0)$, the acceleration along the $x_2$ direction would dominate the acceleration along the $x_1$ direction. Thus we can ensure that the particle would bounce off the $x_1$ axis at least twice before getting close to $x_2$ axis. Let $x_1 = c_2$ denote the $x_1$ coordinate of the second collision. Notice that since $\ddot{x}_2$ is strictly negative, $x_2(t)$ has a unique maximum between the first and the second collision at $x_1$ axis.

Now move the initial position to the left; That is, fix $x_2(0) = x_2^*$, and reduce $x_1(0)$. Notice that the energy of the system would change correspondingly. We want the show that the whole orbit before the second collision at $x_1$ axis would shift to the left.
Observe that magnitude of acceleration along the $x_1$ direction gets stronger as $x_1(t)$ gets smaller. We obtain this result by simply taking derivative of $\ddot{x}_1$ with respect to $x_1$.

$$\frac{\partial \ddot{x}_1}{\partial x_1} = \frac{3x_1^2}{4(x_1^2 + x_2^2)^{5/2}} - \frac{1}{4(x_1^2 + x_2^2)^{3/2}} + \frac{1}{2x_1^3} \geq \frac{3x_1^2}{4(x_1^2 + x_2^2)^{5/2}} - \frac{1}{4(x_1^2 + x_2^2)^{3/2}} + \frac{1}{2(x_1^2 + x_2^2)^{3/2}}$$

$$= \frac{3x_1^2}{4(x_1^2 + x_2^2)^{5/2}} + \frac{1}{4(x_1^2 + x_2^2)^{3/2}} > 0.$$

Note that $\ddot{x}_1$ is negative while $x_1$ is positive, so as $x_1$ decreases $\ddot{x}_1$ also decreases and becomes more negative, therefore the magnitude of acceleration would increase as $x_1(t)$ gets smaller.

Therefore as we move the initial position to the left, the acceleration of the whole orbit before the second collision would get stronger, leading to a smaller value of $c_2$. We keep moving the initial position to the left until $c_2$ exactly reaches zero. That gives us our orbit that bounces off $x_1$ axis once, then ends in total collision.

**Theorem 2.12.** An orbit that starts at a brake, hits the $x_1$ axis, then collides on the $x_2$ axis with $\dot{x}_2 < 0$ exists.

**Proof.** By Lemma 2.11 there exists an orbit that bounces off $x_1$ axis once, then ends in total collision. A perturbation of the orbit’s initial position to the left causes the second collision to happen at the $x_2$ axis. Continuity of $\dot{x}_2$ gives us that the particle would cross the $x_1 = x_2$ line with $\dot{x}_2 < 0$. Because $\ddot{x}_2$ is strictly negative and continuous until it reaches the second collision at the $x_1$ axis, we can guarantee that $\dot{x}_2$ is negative at collision.

Thus we have proved the existence of the orbit that goes down after collision at $x_2$ axis. □
Using Intermediate Value Theorem to Prove the Existence of the Periodic Solution

Theorem 2.13. The periodic brake orbit 121 exist. (An orbit that starts at a brake, first collides on the \( x_1 \) axis, then hits the \( x_2 \) axis and bounces back under the exact same track, hits the \( x_1 \) axis at the same spot, and finally reach a brake again at the starting position.)

Proof. Assuming Conjecture 2.7 is true, then we have the existence of an orbit that starts at a brake, hits the \( x_1 \) axis, then collides on the \( x_2 \) axis with \( \dot{x}_2 > 0 \). Theorem 2.12 gives us the existence of an orbit that starts at a brake, hits the \( x_1 \) axis, then collides on the \( x_2 \) axis with \( \dot{x}_2 < 0 \). By Theorem 2.6, we can fix an energy level and scale those two orbit to match the same energy level.

Now the initial position of these two orbits both lies on the zero momentum curve of the chosen energy. Let \( u(t) \) be the orbit going up and \( d(t) \) be the orbit going down. Since we have a \( C^1 \) system of differential equations under the regularized coordinate, under continuous dependence on initial conditions, as we slide along the zero momentum curve to get from \( u(0) \) to \( d(0) \), at some point \( \dot{x}_2 = 0 \) at the collision on the \( x_2 \) axis (Intermediate Value Theorem). Let this solution be \( (Q_1^*, Q_2^*, P_1^*, P_2^*) \) in the regularized coordinate. Looking at equation 2.8, we see that at the \( x_2 \) axis collision \( \dot{x}_2 = 0 \) implies that \( P_2^* = 0 \). Thus the segment of the solution \( (Q_1^*, Q_2^*, P_1^*, P_2^*) \) from the brake to the \( x_2 \) axis collision satisfies the condition required to generate a full periodic orbit from on part, as talked about in section 2.4. Using the result of section 2.4 we can obtain a full periodic orbit, that starts at a brake, collides on the \( x_1 \) axis, then hits the \( x_2 \) axis and
bounces back under the exact same track, hits the $x_1$ axis at the same spot, and finally reach a brake again at the starting position.

\section*{2.9 Summary of Results}

Using regularized coordinate we obtained a $C^1$ system of differential equations through binary collision. We obtained estimates on moving direction of the orbit under stationary initial conditions and through Intermediate Value Theorem we showed existence of the desired orbit in the 1st quadrant. Extending the orbit through symmetry to the whole plane, and we have given a analytic existence of a symmetric periodic brake orbit with simultaneous binary collision(SBC) in the equal mass, fully symmetric planar four body problem.
Chapter 3

Future Work

In Conjecture 2.5, we proposed that an orbit that starts at a brake, hits the \( x_1 \) axis, then collides on the \( x_2 \) axis with \( \dot{x}_2 > 0 \) exists. Although numerical simulations seems imply the statement is true, we are still working on finding an analytical prove of the result.

From the function \( I = x_1^2 + x_2^2 \), we can obtain that \( \ddot{I} = T + h \) (the Lagrange-Jacobi identity, see Meyer, Hall and Offing [13]). From that we know \( \ddot{I} \) is positive near the origin. If we can prove that \( \dot{I} = 0 \) at someone point before \( I \) reaches zero, then conjecture 2.5 can be proved to be true. Proving this requires more analysis on the function \( I \) and its derivatives.

The next step to studying periodic orbit 121 is to determine its stability. Techniques that Bakker, Ouyang, Yan and Simmons[2] has used in studying stability of symmetric planar periodic orbits with simultaneous binary collision in symmetric four-body problem might be useful.

Along with periodic orbit 121, we have also numerically found other plausible periodic brake orbits as illustrated in Figure 3.1. The orbit on the left starts stationary, hit the \( x_1 \) axis, then the \( x_2 \) axis, when it crosses the \( x_1 = x_2 \) line the second time, the remaining orbit is symmetric to the previous orbit. The orbit on the right starts stationary, hit the \( x_1 \) axis, then the \( x_2 \) axis, then the \( x_1 \) axis, and \( x_2 \) axis twice. After that it traces back its path until reaches a brake at the original position.
In Figure 3.2 we plotted all three periodic brake orbits the same plane. The black dotted line represents the constant energy level curve. It seems that as we move the initial position further to the right under the same energy level, we could discover more periodic brake orbits. While there are more potential periodic brake orbits undiscovered yet, numerical stimulation seems to suggest that a 1221 periodic brake orbit does not exist.
The two other orbits that we have not prove existence for has similarity to periodic brake orbit 121. First, there’s regularizable simultaneous binary collision, which direct us to using topological techniques. Second, there is useful symmetry embedded in the orbit. The shooting method and differential inequalities we used to prove existence of periodic brake orbit 121 will likely be useful as we explore how to prove existence of the remaining two orbits. In future research we can try to prove the existence of the remaining two orbits.
Bibliography


