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Isomorphisms of Landau-Ginzburg B-Models

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A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

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May 2016

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ABSTRACT

Isomorphisms of Landau-Ginzburg B-Models

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Landau-Ginzburg mirror symmetry predicts isomorphisms between graded Frobenius algebras (denoted $\mathcal{A}$ and $\mathcal{B}$) that are constructed from a nondegenerate quasihomogeneous polynomial W and a related group of symmetries G. In 2013, Tay proved that given two polynomials W_1, W_2 with the same quasihomogeneous weights and same group G, the corresponding $\mathcal{A}$-models built with (W_1,G) and (W_2,G) are isomorphic. An analogous theorem for isomorphisms between orbifolded $\mathcal{B}$-models remains to be found.

This thesis investigates isomorphisms between $\mathcal{B}$-models using polynomials in two variables in search of such a theorem. In particular, several examples are given showing the relationship between continuous deformation on the $\mathcal{B}$-side and isomorphisms that stem as a corollary to Tay's theorem via mirror symmetry. Results on extending known isomorphisms between unorbifolded $\mathcal{B}$-models to the orbifolded case are exhibited. A general pattern for $\mathcal{B}$-model isomorphisms, relating mirror symmetry and continuous deformation together, is also observed.

Keywords: Algebraic Geometry, Mirror Symmetry, FJRW Theory

Acknowledgments

I would like to thank Dr. Tyler Jarvis for his guidance and mentorship during the research and writing of this thesis. I would also like to thank the fellow members of the FJRW research group for many useful conversations and helpful meetings along the way.

And, as always, *soli Deo gloria*.

Contents

Chapter 1. Introduction

Physicists conjectured some time ago that that to each quasihomogeneous (weighted homogeneous) polynomial W with an isolated singularity at the origin, and to each admissible group of symmetries G of W, there should exist two different physical “theories,” (called the Landau-Ginzburg $\mathcal{A}$ and $\mathcal{B}$ models, respectively) consisting of graded Frobenius algebras (algebras with a nondegenerate pairing that is compatible with the multiplication). The $\mathcal{B}$-model theories have been constructed [6, 7, 8, 9, 10] and correspond to an “orbifolded Milnor ring.” The $\mathcal{A}$-model theories have also been constructed [4] and are a special case of what is often called “FJRW theory.” We will not address these in this thesis, but in many cases, these theories can be extended to whole families of Frobenius algebras, called *Frobenius manifolds.*

For a large class of these polynomials (called *invertible*) Berglund-Hübsch [3], Henningson [2], and Krawitz [10] described the construction of a dual (or transpose) polynomial W^T and a dual group G^T. The Landau-Ginzburg mirror symmetry conjecture states that the $\mathcal{A}$-model of a pair W, G should be isomorphic to the $\mathcal{B}$-model of the dual pair W^T, G^T. We denote this as $\mathcal{A}[W, G] \cong \mathcal{B}[W^T, G^T]$. This conjecture has been proved in many cases in papers such as [10] and [5], although the proof of the full conjecture remains open.

In 2013, Tay proved the following result for Landau-Ginzburg $\mathcal{A}$-models. It is a sufficient condition for $\mathcal{A}$-model isomorphisms, and is called the Group-Weights theorem.

Theorem 1.1 (Group-Weights, see Section 7.1 of [13])**.** *Let W_1 and W_2 be admissible polynomials which have the same weights. If $G \leq G^{max}_{W_1}$ and $G \leq G^{max}_{W_2}$, then $\mathcal{A}[W_1, G] \cong \mathcal{A}[W_2, G]$.*

This theorem shows that the $\mathcal{A}$-model is *deformation invariant.* That is, when the polynomial W_1 is continuously deformed to W_2 along a path in the coefficient space that avoids degenerate points, the respective graded Frobenius algebras along that path are all isomorphic.

No such theorem exists for $\mathcal{B}$-models, in part because the idea of deformation invariance is not true in general for $\mathcal{B}$-models (see Example 2.34). The purpose of this thesis is to investigate isomorphisms of $\mathcal{B}$-models using polynomials of two variables in search of an analogous theorem. One of the difficulties of verifying the mirror symmetry conjecture in general comes from a lack of understanding of the algebra structure in more difficult cases. So a general theorem such as this will not only be interesting as a result about graded algebras, but may also be useful in verifying mirror symmetry and investigating higher levels of mirror symmetry structure.

In Chapter 3, we will investigate the Group-Weights theorem and apply mirror symmetry to classify the isomorphisms of $\mathcal{B}$-models in two variables that stem as a corollary. This will catalog the isomorphisms that are already known by this previous result, and tell us when we have found new and interesting isomorphisms. In Chapter 4 we will introduce two algorithms used to determine when $\mathcal{B}$-models are isomorphic. In Chapters 5 and 6 we will give specific examples and classes of examples of new isomorphisms of $\mathcal{B}$-models built with polynomials in two variables that don't stem directly from Group-Weights. We will give further examples of when deformation invariance exists on the $\mathcal{B}$-side. Building on these results, we'll make the following conjecture about the relationship between $\mathcal{A}$-model and $\mathcal{B}$-model isomorphisms via mirror symmetry in Chapter 7.

Conjecture 7.1. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be any two Landau-Ginzburg $\mathcal{B}$-models such that $\mathcal{B}_1 \cong \mathcal{B}_2$. If this isomorphism is not the result of a continuous deformation, then there exists a finite chain of Landau-Ginzburg models $C_1, \ldots, C_n$ (either $\mathcal{A}$ or $\mathcal{B}$) such that*

$$\mathcal{B}_1 \longleftrightarrow C_1 \longleftrightarrow \ldots \longleftrightarrow C_n \longleftrightarrow \mathcal{B}_2,$$

where each arrow represents an isomorphism of graded Frobenius algebras that is either a continuous deformation or is the isomorphism predicted by mirror symmetry.

To further investigate this conjecture, we will conclude with one final result about extending isomorphisms of unorbifolded $\mathcal{B}$-models to their corresponding orbifolded models.

Chapter 2. Preliminaries

Here we will introduce some of the concepts needed to understand the theory of this thesis.

2.1 Admissible Polynomials

Definition 2.1. For a polynomial $W \in \mathbb{C}[x_1, \ldots, x_n]$, we say that W is *nondegenerate* if it has an isolated critical point at the origin.

Definition 2.2. Let $W \in \mathbb{C}[x_1, \ldots, x_n]$. We say that W is *quasihomogeneous* if there exist positive rational numbers $q_1, \ldots, q_n$ such that for any $c \in \mathbb{C}$, $W(c^{q_1}x_1, \ldots, c^{q_n}x_n) = cW(x_1, \ldots, x_n)$.

We often refer to the q_i as the *quasihomogeneous weights* of a polynomial W, or just simply the *weights* of W, and we write the weights in vector form $J = (q_1, \ldots, q_n)$.

Definition 2.3. $W \in \mathbb{C}[x_1, \ldots, x_n]$ is *admissible* if W is nondegenerate and quasihomogeneous with unique weights, having no monomials of the form $x_i x_j$ for $i \neq j$.

The condition that W have no cross-term monomials is necessary for constructing the $\mathcal{A}$-model. It is also interesting to note the following result about admissible polynomials.

Proposition 2.4 (Proposition 2.1.6 of [4])**.** *If $W \in \mathbb{C}[x_1, \ldots, x_n]$ is admissible, then the weights q_i are bounded above by $\frac{1}{2}$.*

Because the construction of $\mathcal{A}[W, G]$ requires an admissible polynomial, we will only be concerned with admissible polynomials in this paper. In order for a polynomial to be admissible, it needs to have at least as many monomials as variables. Otherwise its quasihomogeneous weights cannot be uniquely determined. We will now state the main subdivision of the admissible polynomials.

Definition 2.5. Let W be an admissible polynomial. We say that W is *invertible* if it has the same number of monomials as variables. If W has more monomials than variables, then it is *noninvertible*.

Admissible polynomials with the same number of variables as monomials are called invertible, since their associated exponent matrices (which we define in the next section) are square and invertible. The invertible polynomials can further be decomposed into sums of three types of polynomials, called the *atomic types*.

Theorem 2.6 (Theorem 1 of [11])**.** *Any invertible polynomial is the decoupled sum of polynomials in one of three atomic types:*

$$\begin{aligned} \textit{Fermat type:} \quad & W = x^a, \\ \textit{Loop type:} \quad & W = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_n^{a_n}x_1, \\ \textit{Chain type:} \quad & W = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_n^{a_n}. \end{aligned}$$

We also assume that the $a_i \geq 2$ to avoid terms of the form $x_i x_j$ for $i \neq j$.

2.2 Dual Polynomials

We will now introduce the idea of the transpose operation for invertible polynomials.

Definition 2.7. Let $W \in \mathbb{C}[x_1, \dots, x_n]$. If we write $W = \sum_{i=1}^m c_i \prod_{j=1}^n x_j^{a_{ij}}$ where the $c_i \neq 0$ for all i, then the associated *exponent matrix* is defined to be $A = (a_{ij})$.

From this definition we notice that n is the number of variables in W, and m is the number of monomials in W. Here A is an $m \times n$ matrix. Thus when W is invertible, we have $m = n$, which implies that A is square. One can show, without much work, that this square matrix is invertible if the polynomial W is quasihomogeneous with unique weights. When W is noninvertible, $m > n$, so A has more rows than columns.

Observe that if a polynomial is invertible, then we may rescale all nonzero coefficients to 1. So there is effectively a one-to-one correspondence between exponent matrices of invertible polynomials and the polynomials themselves (up to rescaling).

Definition 2.8. Let W be an invertible polynomial. If A is the exponent matrix of W, then we define the *transpose polynomial* to be the polynomial W^T resulting from A^T. By the classification in [11], W^T is again a nondegenerate, invertible polynomial.

2.3 Symmetry Groups and Their Duals

Definition 2.9. Let W be an admissible polynomial. We define the *maximal diagonal symmetry group* of W to be $G_W^{max} = \{(\zeta_1, \ldots, \zeta_n) \in (\mathbb{C}^\times)^n \mid W(\zeta_1 x_1, \ldots, \zeta_n x_n) = W(x_1, \ldots, x_n)\}$.

The proofs of Lemma 2.1.8 in [4] and Lemma 1 in [1] observe that G_W^{max} is finite and that each coordinate of every group element is a root of unity. The group operation $\circ$ in G_W^{max} is coordinate-wise multiplication. That is,

$$(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}) \circ (e^{2\pi i\phi_1}, \ldots, e^{2\pi i\phi_n}) = (e^{2\pi i(\theta_1+\phi_1)}, \ldots, e^{2\pi i(\theta_n+\phi_n)}).$$

Equivalently, in additive notation we can write $(\theta_1, \ldots, \theta_n)+(\phi_1, \ldots, \phi_n) = (\theta_1+\phi_1, \ldots, \theta_n+\phi_n) \mod \mathbb{Z}$. The map $(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}) \mapsto (\theta_1, \ldots, \theta_n) \mod \mathbb{Z}$ gives a group isomorphism. Using additive notation, we will often write $G_W^{max} = \{g \in (\mathbb{Q}/\mathbb{Z})^n \mid Ag \in \mathbb{Z}^m\}$, where A is the $m \times n$ exponent matrix of W.

Definition 2.10. In this notation, G_W^{max} is a subgroup of $(\mathbb{Q}/\mathbb{Z})^n$ with respect to coordinate-wise addition. For $g \in G_W^{max}$, we write $g = (g_1, \ldots, g_n)$ where each g_i is a rational number in the interval [0,1). The g_i are called the *phases* of g, and are uniquely determined by g.

The following definition of the transpose group is due to Krawitz and Henningson [10, 2].

Definition 2.11. Let W be an invertible polynomial, and let A be its associated exponent matrix. The *transpose group* of a subgroup $G \leq G_W^{max}$ is the set

$$G^T = \{g \in G_{W^T}^{max} \mid gAh^T \in \mathbb{Z} \text{ for all } h \in G\}.$$

The following is a list of properties of the transpose group.

Proposition 2.12 (Proposition 3 of [1]). *If W is an invertible polynomial with weights vector J, and $G \leq G_W^{max}$, then*

(1) $(G^T)^T = G$;

(2) $\{0\}^T = G_{W^T}^{max}$ *and* $(G_W^{max})^T = \{0\}$;

(3) $\langle J \rangle^T = G_{W^T}^{max} \cap SL(n, \mathbb{C})$ *where n is the number of variables in W;*

(4) if $G_1 \leq G_2$, then $G_2^T \leq G_1^T$ and $G_2/G_1 \cong G_1^T/G_2^T$.

2.4 Graded Frobenius Algebras

Landau-Ginzburg $\mathcal{A}$ and $\mathcal{B}$ models are algebraic objects that are endowed with many levels of structure. In this thesis, we will chiefly be concerned with their structure up to the level of graded Frobenius algebra. For the benefit of the reader, we will give a formal definition of a Frobenius algebra.

Definition 2.13. An *algebra* is a vector space A over a field of scalars F (in our case it is $\mathbb{C}$), together with a multiplication $\cdot : A \times A \to A$ that satisfies for all $x, y, z \in A$ and $\alpha, \beta \in F$

- Right distributivity: $(x + y) \cdot z = x \cdot z + y \cdot z$,
- Left distributivity: $x \cdot (y + z) = x \cdot y + x \cdot z$,
- Compatability with scalars: $(\alpha x) \cdot (\beta y) = (\alpha\beta)(x \cdot y)$.

We further require the multiplication to be associative and commutative, and for A to have a unity e such that $e \cdot x = x$ for all $x \in A$.

Definition 2.14. We also define a *pairing* operation to be a function $\langle \cdot, \cdot \rangle : A \times A \to F$ that is

- Symmetric: $\langle x, y \rangle = \langle y, x \rangle$,
- Linear: $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$,
- Nondegenerate: for every $x \in A$ there exists $y \in A$ such that $\langle x, y \rangle \neq 0$.

If the pairing further satisfies the *Frobenius property*, meaning that $\langle x \cdot y, z \rangle = \langle x, y \cdot z \rangle$ for all $x, y, z \in A$, then we call A a *Frobenius algebra.*

We will only develop the theory needed for this thesis. Interested readers may reference [4] for more details on the construction of the $\mathcal{A}$-model; references [5], [10], and [13] also contain more information on constructing $\mathcal{A}$ and $\mathcal{B}$ models, and related isomorphisms. We will start by discussing the $\mathcal{B}$-model.

2.5 Unorbifolded $\mathcal{B}$-models

Definition 2.15. $\mathcal{Q}_W = \mathbb{C}[x_1, \ldots, x_n]/(\frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_n})$ is called the *Milnor ring* of W (or *local algebra* of W).

We note that $\mathcal{Q}_W$ has a vector space structure with a basis consisting of monomials that aren't in the ideal generated by the partial derivatives of W. We define the standard scalar multiplication and addition operations for monomials, and further allow the standard quotient ring multiplication. We note the following result about the dimension of $\mathcal{Q}_W$.

Theorem 2.16 (Theorem 2.6 of [13])**.** *If W is admissible, then $\mathcal{Q}_W$ is finite dimensional.*

We will further think of the Milnor ring as a graded vector space over $\mathbb{C}$. The degree of a monomial in $\mathcal{Q}_W$ is given by $\deg(x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}) = 2\sum_i^n a_i q_i$, where the q_i are the quasihomogeneous weights of W. This defines a grading on the basis of $\mathcal{Q}_W$. We have the following results about the vector space structure of the Milnor ring. First, $\dim(\mathcal{Q}_W) = \prod_{i=1}^n \left(\frac{1}{q_i} - 1\right)$. Second, the highest degree of its graded pieces is $2\sum_{i=1}^n (1 - 2q_i)$. The number $\sum_{i=1}^n (1 - 2q_i)$ is called the *central charge*, and is denoted by $\widehat{c}$ (see Section 2.1 of [10]).

To make $\mathcal{Q}_W$ into a graded Frobenius algebra, we need to define its pairing functions. We have the following definition.

Definition 2.17. For an admissible polynomial W, let $m, n \in \mathcal{Q}_W$. We define the *pairing* $\langle m, n \rangle$ to be the complex number that satisfies

$$mn = \frac{\langle m, n \rangle}{\mu} \mathrm{Hess}(W) + \text{ terms of degree less than} \deg(\mathrm{Hess}(W)),$$

where μ is the dimension of $\mathcal{Q}_W$ as a vector space and $\mathrm{Hess}(W)$ is the *Hessian* of W—or the determinant of the matrix of second partial derivatives of W.

As noted by Krawitz [10], we can represent $\mathrm{Hess}(W)$ as a monomial in the Milnor ring. Further, the elements of highest degree in the Milnor ring form a one-dimensional subspace that is spanned by $\mathrm{Hess}(W)$.

One can also verify that the Milnor ring, together with the grading of the monomial basis and this paring function, forms a graded Frobenius algebra. This motivates our definition of the unorbifolded $\mathcal{B}$-model.

Definition 2.18. We define the *unorbifolded* $\mathcal{B}$-model $\mathcal{B}[W, \{0\}]$ by $\mathcal{B}[W, \{0\}] = \mathcal{Q}_W$.

2.6 Orbifolded $\mathcal{B}$-models

We'll now think about how to construct the orbifolded $\mathcal{B}$-model $\mathcal{B}[W, G]$, where G is a nontrivial group. We'll need the following definition.

Definition 2.19. Let $W \in \mathbb{C}[x_1, \ldots, x_n]$ be admissible, and let $g = (g_1, \ldots, g_n) \in G_W^{max}$. The *fixed locus* of the group element g is the set $\mathrm{fix}(g) = \{x \in \mathbb{C}^n \mid g(x) = 0\}$.

We now state how G acts on the Milnor ring.

Definition 2.20. Let W be an admissible polynomial, and let $g \in G_W^{max}$. We define the map $g^* : \mathcal{Q}_W \to \mathcal{Q}_W$ by $g^*(m) = \det(g) m \circ g$. (Here we think of g as being a diagonal map with multiplicative coordinates). This is the *group action* on the elements of $\mathcal{Q}_W$.

Definition 2.21. Let W be an admissible polynomial, and let $G \leq G_W^{max}$. The *G-invariant subspace* of $\mathcal{Q}_W$ is defined to be $\mathcal{Q}_W^G = \{m \in \mathcal{Q}_W \mid g^*(m) = m \text{ for each } g \in G\}$.

To construct an orbifolded $\mathcal{B}$-model, we restrict G to be a subgroup of $G_W^{max} \cap \mathrm{SL}(n, \mathbb{C})$.

Definition 2.22. Let W be an admissible polynomial, and $G \leq G_W^{max} \cap \mathrm{SL}(n, \mathbb{C})$ where n is the number of variables of W. We define $\mathcal{B}[W, G] = \bigoplus_{g \in G} \left(\mathcal{Q}_{W|_{\mathrm{fix}(g)}}\right)^G$, where $(\cdot)^G$ denotes all the G-invariants. This is called the $\mathcal{B}$-model *state space*.

The condition that $G \leq G_W^{max} \cap \mathrm{SL}(n, \mathbb{C})$ is required to construct the orbifolded $\mathcal{B}$-model. We will often denote the group $G_W^{max} \cap \mathrm{SL}(n, \mathbb{C})$ as $\mathrm{SL}(W)$.

Note that if we let $G = \{0\}$, then the formula yields the Milnor ring of W, as expected. We also note that the vector space basis of $\mathcal{B}[W, G]$ is made up of monomials from the basis of the Milnor ring, along with the group elements that preserve these monomials under the action given in Definition 2.20. We denote these basis elements $\lfloor m; g \rceil$, where m is a monomial and g is a group element corresponding to $m \in \left(\mathcal{Q}_{W|_{\mathrm{fix}(g)}}\right)^G$.

To make $\mathcal{B}[W, G]$ into a graded Frobenius algebra, we will define the grading, the multiplication and the pairing function. We'll start with the vector space grading.

Definition 2.23. Let W be an admissible polynomial with weights $(q_1, \dots, q_n)$. For a basis element $\lfloor m; (g_1, \dots, g_n) \rceil$ in the vector space basis for $\mathcal{B}[W, G]$, we define its *degree* to be

$$2p + \sum_{g_i \notin \mathbb{Z}} (1 - 2q_i),$$

where p is the weighted degree of m. That is, if $m = x_1^{a_1} \cdots x_n^{a_n}$, then $p = \sum_{i=1}^n a_i q_i$.

The definition of $\mathcal{B}$-model multiplication is due to Krawitz in [10].

Definition 2.24. The product of two elements $\lfloor m; g \rceil$ and $\lfloor n; h \rceil$ is given by

$$\lfloor m; g \rceil \star \lfloor n; h \rceil = \begin{cases} \lfloor \gamma nm; g + h \rceil & \text{if } \mathrm{fix}(g) \cup \mathrm{fix}(h) \cup \mathrm{fix}(g+h) = \mathbb{C}^n, \\ 0 & \text{otherwise,} \end{cases}$$

where γ is a monomial defined as

$$\gamma = \frac{\mu_{g \cap h} \mathrm{Hess}(W|_{\mathrm{fix}(g+h)})}{\mu_{g+h} \mathrm{Hess}(W|_{\mathrm{fix}(g) \cap \mathrm{fix}(h)})}.$$

Here $\mu_{g \cap h}$ is the dimension of the Milnor ring of $W|_{\mathrm{fix}(g) \cap \mathrm{fix}(h)}$, and μ_{g+h} is the dimension of the Milnor ring of $W|_{\mathrm{fix}(g+h)}$.

We note that Krawitz proved this multiplication to be associative in the case that W is an invertible polynomial (see Proposition 2.1 of [10]). We believe this to also always be associative when W is noninvertible polynomial, but it has never been proven in general. The multiplication structure for examples we compute in this thesis can be checked individually for associativity.

Finally, we have the pairing function.

Definition 2.25. Let $\lfloor m; g\rceil$ and $\lfloor n; h\rceil$ be two basis elements of $\mathcal{B}[W, G]$. We define the *pairing* as follows:

$$\langle \lfloor m; g\rceil, \lfloor n; h\rceil\rangle = \begin{cases} \langle m, n\rangle & \text{if } g = -h, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\langle m, n\rangle$ refers to the pairing on $\mathcal{Q}_{W|_{\text{fix}(g)}}$.

One can verify that the orbifolded $\mathcal{B}$-model $\mathcal{B}[W, G]$, as it has been defined, is a graded Frobenius algebra.

2.7 $\mathcal{A}$-models

We'll include here a few comments about $\mathcal{A}$-models. This will not be a full discussion of $\mathcal{A}$-model construction. For further treatment of this topic, we refer the reader to Sections 2.4 and 2.5 of [13].

To start, recall that the construction of the $\mathcal{B}$-model required the group G to be contained in $\text{SL}(W) = G_W^{max} \cap \text{SL}(n, \mathbb{C})$. From parts (3) and (4) of Proposition 2.12, the corresponding condition for the $\mathcal{A}$-model is that $\langle J\rangle \leq G$. This motivates the following definition for admissible groups for $\mathcal{A}$-models.

Definition 2.26. Let W be an admissible polynomial with weights vector $J = (q_1, \ldots, q_n)$, and let $G \leq G_W^{max}$. We say that G is *admissible* if $J \in G$.

We note that since W is quasihomogeneous, we have that $AJ^T = (1, \dots, 1)^T \in \mathbb{Z}^m$. Thus $J \in G_W^{max}$.

The state space of the $\mathcal{A}$-model $\mathcal{A}[W, G]$ is constructed in the same way the $\mathcal{B}$-model was constructed, but with the condition that G is an admissible group. However, the grading on the $\mathcal{A}$-model, differs from the $\mathcal{B}$-model grading.

Definition 2.27. The $\mathcal{A}$-model degree of a basis element $\lfloor m; g \rceil$ is defined to be $\deg(\lfloor m; g \rceil) = \dim(\text{fix}(g)) + 2\sum_{i=1}^n (g_i - q_i)$, where $g = (g_1, \dots, g_n)$ with the g_i chosen such that $0 \leq g_i < 1$ and $J = (q_1, \dots, q_n)$ is the vector of quasihomogeneous weights of W (see Section 2.1 of [10]).

Finally, we'll emphasize one comment about the Group-Weights theorem for $\mathcal{A}$-model isomorphisms. Note that one can give the $\mathcal{A}$-model a product and pairing such that $\mathcal{A}$ is a Frobenius algebra. The Group-Weights theorem then gives an isomorphism of Frobenius algebras, not just of graded vector spaces.

2.8 Isomorphisms of Graded Frobenius Algebras

We will begin with a formal definition of algebra isomorphisms.

Definition 2.28. Let A and B be two graded Frobenius algebras over a field F. A is *isomorphic* to B, written $A \cong B$, if there exists a bijective map $\phi : A \to B$ that satisfies for every $\alpha, \beta \in A$ and $t \in F$:

1. $\phi(\alpha +_A t\beta) = \phi(\alpha) +_B t\phi(\beta)$,
2. $\phi(\alpha \star_A \beta) = \phi(\alpha) \star_B \phi(\beta)$,
3. $\phi(1_A) = 1_B$,
4. $\deg_A(\alpha) = \deg_B(\phi(\alpha))$ for any homogeneous $\alpha \in A$,
5. $\langle \alpha, \beta \rangle_A = \langle \phi(\alpha), \phi(\beta) \rangle_B$.

We can now formally state the conjectured Landau-Ginzburg mirror symmetry correspondence.

Conjecture 2.29. *If W is an admissible polynomial and G is an admissible group, then*

$$\mathcal{A}[W, G] \cong \mathcal{B}[W^T, G^T].$$

To help us better understand Landau-Ginzburg mirror symmetry, we will focus on studying isomorphisms between Landau-Ginzburg $\mathcal{B}$-models. The following are some common results about isomorphisms between unorbifolded $\mathcal{B}$-models. We will refer back to these later on in the thesis. Note that we consider two polynomials to be equivalent if they define the same singularity at the origin. That is, we say that $f \sim g$ if there exists a diffeomorphism $h : \mathbb{C}^n \to \mathbb{C}^n$ such that $f = g \circ h$.

Theorem 2.30 (Theorem 2.2.8 of [12]). *If W_1 and W_2 are quasihomogeneous functions fixing the origin, then W_1 and W_2 are equivalent if and only if their Milnor rings are isomorphic.*

Theorem 2.31 (Theorem 5.1.1 of [12]). *If two nondegenerate quasihomogeneous polynomials are equivalent, then they have the same unordered set of weights.*

Theorem 2.32 (Webb's Theorem, Theorem 5.1.3 of [12]). *Let W_1 and W_2 be nondegenerate quasihomogeneous polynomials with the same (ordered) weights. If no elements in a basis for $\mathcal{Q}_{W_1}$ have weighted degree 1, then W_1 and W_2 are equivalent.*

These are all results about $\mathcal{B}$-model isomorphisms using the trivial group $\{0\}$. The following is a result includes orbifolded $\mathcal{B}$-models.

Proposition 2.33 (Proposition 2.3.2 of [5]). *Suppose W_1 and W_2 are nondegenerate, quasihomogeneous polynomials with no variables in common. If $G_1 \leq SL(W_1)$ and $G_2 \leq SL(W_2)$, then $G_1 \times G_2$ is contained in $SL(W_1 + W_2)$, fixes $W_1 + W_2$, and we have an isomorphism*

$$\mathcal{B}[W_1, G_1] \otimes \mathcal{B}[W_2, G_2] \cong \mathcal{B}[W_1 + W_2, G_1 \times G_2].$$

Note that Theorem 2.32 is a type of Group-Weights result on the $\mathcal{B}$-side. However, Group-Weights does not hold in general for $\mathcal{B}$-models as the next example demonstrates.

Example 2.34 (Example 5.1.4 of [12])**.** Let $W_1 = x^4 + y^4$ and $W_2 = x^3y + xy^3$. Both polynomials have weights $\left(\frac{1}{4}, \frac{1}{4}\right)$. The set $\{1, y, y^2, x, xy, xy^2, x^2, x^2y, x^2y^2\}$ is a basis for both $\mathcal{Q}_{W_1}$ and $\mathcal{Q}_{W_2}$. One can verify that any ring homomorphism from $\mathcal{Q}_{W_1}$ to $\mathcal{Q}_{W_2}$ will not be surjective, so we see that $\mathcal{B}[W_1, \{0\}] \ncong \mathcal{B}[W_2, \{0\}]$. But notice that x^2y^2 has weighted degree 1. We see that any choice of basis for $\mathcal{Q}_{W_1}$ or $\mathcal{Q}_{W_2}$ will contain a monomial of weighted degree 1. Therefore this does not contradict Webb's Theorem.

This example shows that Group-Weights is not sufficient for $\mathcal{B}$-model isomorphisms. This also shows that deformation invariance does not hold in general on the $\mathcal{B}$-side, since there is no way to deform $x^4 + y^4$ into $x^3y + xy^3$ while maintaining isomorphic Milnor rings. In Chapter 6 of this thesis we will investigate examples where one can continuously deform polynomials while maintaining isomorphic $\mathcal{B}$-models.

Chapter 3. Isomorphisms in Two Variables Stemming From Group-Weights

The purpose of this chapter is to investigate $\mathcal{B}$-model isomorphisms that stem from using the Group-Weights theorem for $\mathcal{A}$-models and mirror symmetry. We will focus on polynomials in two variables. This will help us to know when we have discovered new and interesting isomorphisms of $\mathcal{B}$-models—that is, ones that didn't already stem from this theorem.

3.1 Preliminaries

Proposition 3.1 (Proposition 2 of Section 3 in [1]). *(1) For a loop* $W = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1$, *then* $G_W^{max} = \langle(\phi_1, \ldots, \phi_n)\rangle$, *where*

$$\phi_1 = \frac{(-1)^n}{a_1 \cdots a_n + (-1)^{n+1}}, \quad \phi_i = \frac{(-1)^{n+1-i}a_1 \cdots a_{i-1}}{a_1 \cdots a_n + (-1)^{n+1}}, \; i \geq 2.$$

(2) For a chain $W = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$, *then* $G_W^{max} = \langle(\phi_1, \ldots, \phi_n)\rangle$, *where*

$$\phi_i = \frac{(-1)^{n+i}}{a_i \cdots a_n}.$$

Proposition 3.2. *If* W *is an invertible polynomial with weights vector* J, *then* $|\langle J\rangle| = \left[G_{W^T}^{max} : \langle J\rangle^T\right]$, *where* $\left[G_{W^T}^{max} : \langle J\rangle^T\right]$ *denotes the index of* $\langle J\rangle^T$ *in* $G_{W^T}^{max}$.

Proof. Consider $\{0\} \leq \langle J\rangle$. By property (4) of Proposition 2.12, $\langle J\rangle^T \leq \{0\}^T = G_{W^T}^{max}$ and $\langle J\rangle/\{0\} = \langle J\rangle \cong G_{W^T}^{max}/\langle J\rangle^T$. Hence

$$|\langle J\rangle| = \left|\frac{G_{W^T}^{max}}{\langle J\rangle^T}\right| = \frac{\left|G_{W^T}^{max}\right|}{|\langle J\rangle^T|} = \left[G_{W^T}^{max} : \langle J\rangle^T\right] \text{ by Lagrange's theorem.}$$

$\square$

3.2 Classification of Two-Variable Weight Systems

Landau-Ginzburg mirror symmetry is currently only defined for invertible polynomials. We therefore want to find admissible weight systems (q_1, q_2) that have at least two invertible polynomials. We can then use the Group-Weights theorem for $\mathcal{A}$-models and mirror symmetry to find isomorphic $\mathcal{B}$-models. The following results are due to my own calculations, but they can also be found in Section 3.1 of [13].

We first note that possible monomials are of the form x^a, y^a, $x^a y$, or xy^a. There are four possible types of invertible polynomials in two variables: $x^a + y^b$, $x^a y + y^b$, $xy^a + x^b$, or $x^a y + xy^b$.

Family 1. Let $n \in \mathbb{N}$, $n \geq 3$. $J = \left(\frac{1}{n}, \frac{1}{n}\right)$ fixes $x^n + y^n$, $x^{n-1}y + y^n$, $xy^{n-1} + x^n$, and $x^{n-1}y + xy^{n-1}$.

Proof. A Fermat monomial x^a is fixed by $\frac{1}{n}$ if and only if $\left(\frac{1}{n}\right) a = \frac{a}{n} = 1$ if and only if $a = n$. So J fixes x^n and y^n, and these are the only valid Fermat monomials. The monomial $x^a y$ is fixed by J if and only if $\frac{a}{n} + \frac{1}{n} = 1$ if and only if $\frac{a+1}{n} = 1$ if and only if $a = n - 1$. So J fixes $x^{n-1}y$, and similarly J fixes xy^{n-1}. Combining these monomials yields the four invertible polynomials as desired. □

We now note that in order for a weight system J to fix a Fermat monomial, one of its coordinates must be of the form $\frac{1}{n}$. Since three out of the four possible invertible types in two-variables contains a Fermat monomial, we will only need to consider weight systems that have a $\frac{1}{n}$ in one of its coordinates. We will proceed with choosing this to be the first coordinate. A similar case will result by swapping the two coordinates.

Family 2. Let $\alpha, n \in \mathbb{N}$ with $\alpha, n \geq 2$. $J = \left(\frac{1}{n}, \frac{1}{\alpha n}\right)$ fixes $x^n + y^{\alpha n}$ and $xy^{\alpha n - \alpha} + x^n$, and $J = \left(\frac{1}{\alpha n}, \frac{1}{n}\right)$ fixes $x^{\alpha n} + y^n$ and $x^{\alpha n - \alpha} y + y^n$.

Proof. First suppose that $\alpha \in \mathbb{R}$, $\alpha > 1$. We may assume that $\alpha n \in \mathbb{Z}$. Certainly J fixes x^n and $y^{\alpha n}$. Now consider xy^a. This is fixed by J if and only if $\frac{1}{n} + \frac{a}{\alpha n} = 1$ if and only if $\frac{a+\alpha}{\alpha n} = 1$ if and only if $a = \alpha n - \alpha$. Since $\alpha n \in \mathbb{Z}$, we must require $\alpha \in \mathbb{Z}$ to have $a \in \mathbb{Z}$.

Now consider the monomial $x^b y$. It is fixed by J if and only if $\frac{b}{n} + \frac{1}{\alpha n} = 1$ if and only if $\frac{1+\alpha b}{\alpha n} = 1$ if and only if $b = \frac{\alpha n - 1}{\alpha n}$. However, $b \in \mathbb{Z}$ if and only if $\alpha = 1$. Therefore there is no such monomial fixed by J.

After combining monomials, we find that the only invertible polynomials fixed by J are $x^n + y^{\alpha n}$ and $xy^{\alpha n - \alpha} + x^n$ where $\alpha > 1$, $\alpha \in \mathbb{Z}$.

Note that for $0 < \alpha < 1$ and $\alpha n \in \mathbb{Z}$, we can rewrite $J = \left(\frac{1}{n}, \frac{1}{\alpha n}\right)$ as $\left(\frac{1}{\beta m}, \frac{1}{m}\right)$, where $m = \alpha n < n$ and $\beta = \frac{1}{\alpha} > 1$ so that $\beta m = n$. The rest of the proof follows similarly to the above calculation. $\square$

We now consider when the weight system $J = \left(\frac{1}{n}, \frac{a}{b}\right)$ fixes both a chain and a loop polynomial. Obvious restrictions are $a, b \in \mathbb{N}$, $0 < \frac{a}{b} < \frac{1}{2}$, $\gcd(a, b) = 1$, $a < b$, etc. We know J fixes x^n, and we want to know when J fixes the monomials xy^α and $x^\beta y$.

For xy^α, we require $\frac{1}{n} + \alpha\frac{a}{b} = 1$ if and only if $\frac{b+\alpha an}{bn} = 1$ if and only if $b + \alpha an = bn$ if and only if $\alpha = \frac{b(n-1)}{an} \in \mathbb{N}$. For $x^\beta y$, we have that $\frac{\beta}{n} + \frac{a}{b} = 1$ if and only if $\frac{b\beta + an}{bn} = 1$ if and only if $b\beta + an = bn$ if and only if $\beta = \frac{n(b-a)}{b} \in \mathbb{N}$. We further require $\alpha, \beta \geq 2$.

Consider the case $b = n$. This yields $\alpha = \frac{n-1}{a}$ and $\beta = n - a$. So $a \mid (n-1)$. We will show that this is the only case to consider.

We first have that $\frac{1}{n} + \alpha\frac{a}{b} = 1$ if and only if $\frac{b+\alpha an}{bn} = 1$ if and only if $b + \alpha an = bn$ if and only if $\frac{b}{n} + \alpha a = b$, after dividing by n. Since αa and b are both integers, we must have that $\frac{b}{n}$ is an integer implies $n \mid b$.

We know that $\frac{\beta}{n} + \frac{a}{b} = 1$ if and only if $\beta + \frac{an}{b} = n$. Thus $\frac{an}{b} = n - \beta$ is an integer. Since $\gcd(a, b) = 1$ by hypothesis, we have that $b \mid n$. Since we know that both b and n are positive integers, we see that $b = n$ as desired.

A similar result follows when considering $J = \left(\frac{a}{b}, \frac{1}{n}\right)$. Therefore we have found all possible weight systems that have a $\frac{1}{n}$ in at least one of the coordinates. Hence, the following is the only other family we need to consider.

Family 3. Let $n \in \mathbb{N}$, $n \geq 2$. Let $a \in \mathbb{N}$ such that $1 < a < \frac{n}{2}$, $\gcd(a, n) = 1$, and $a \mid (n-1)$. $J = \left(\frac{1}{n}, \frac{a}{n}\right)$ fixes $x^n + xy^{\frac{n-1}{a}}$ and $x^{n-a}y + xy^{\frac{n-1}{a}}$, and $J = \left(\frac{a}{n}, \frac{1}{n}\right)$ fixes $x^{\frac{n-1}{a}}y + y^n$ and $x^{\frac{n-1}{a}}y + xy^{n-a}$.

3.3 Results

Now that we have found the weight systems in two variables that yield more than one invertible polynomial, we can write down all possible isomorphisms between $\mathcal{A}$-models in two variables that stem from the Group-Weights theorem. We can then apply the transpose operation to polynomials and groups to the find the corresponding $\mathcal{B}$-models that are also isomorphic via mirror symmetry. The following diagram illustrates the approach.

$$\begin{tikzcd}
\mathcal{A}[W_1, G_1] \arrow[r, leftrightarrow, "\text{Mirror Symmetry}"] \arrow[d, leftrightarrow, "\text{Group-Weights}"'] & \mathcal{B}[W_1^T, G_1^T] \arrow[d, leftrightarrow, dashed] \\
\mathcal{A}[W_2, G_2] \arrow[r, leftrightarrow, "\text{Mirror Symmetry}"] & \mathcal{B}[W_2^T, G_2^T]
\end{tikzcd}$$

On the $\mathcal{A}$-side we have invertible polynomials W_1 and W_2 that have the same weights, and groups $G_1 = G_2$ that fix W_1 and W_1. However, note that on the $\mathcal{B}$-side we may have $G_1^T \neq G_2^T$, since the transpose operation for the group depends upon the choice of polynomial. By our construction, the Group-Weights theorem gives us isomorphic $\mathcal{A}$-models. After using mirror symmetry, our diagram sets up three out of four isomorphisms in a square—thereby automatically yielding the fourth isomorphism between $\mathcal{B}$-models.

We now state the results of using this approach on polynomials in two variables. Our first theorem considers the invertible polynomials with a Family 1 weight system, using common subgroup $G = \langle J \rangle$.

Theorem 3.3. *For all $n \in \mathbb{N}$, $n \geq 3$,*

$$\begin{aligned}
\mathcal{B}\left[x^n + y^n,\ \left\langle\left(\frac{1}{n}, -\frac{1}{n}\right)\right\rangle\right] &\cong \mathcal{B}\left[x^{n-1} + xy^n,\ \left\langle\left(\frac{1}{n-1}, -\frac{1}{n-1}\right)\right\rangle\right] \\
&\cong \mathcal{B}\left[x^n y + y^{n-1},\ \left\langle\left(\frac{1}{n-1}, -\frac{1}{n-1}\right)\right\rangle\right] \\
&\cong \mathcal{B}\left[x^{n-1} y + xy^{n-1},\ \left\langle\left(\frac{1}{n-2}, -\frac{1}{n-2}\right)\right\rangle\right].
\end{aligned}$$

Proof. Let $n \in \mathbb{N}$, $n \geq 3$. Let $G = \left\langle\left(\frac{1}{n}, \frac{1}{n}\right)\right\rangle$.

Lemma 3.4. *Let $W_1 = x^n + y^n$. $W_1^T = x^n + y^n$ and as a subgroup of $G_{W_1}^{max}$, $G^T = \langle(\frac{1}{n}, -\frac{1}{n})\rangle$.*

Proof. Let $A_1 = \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix}$ be the exponent matrix of W_1. Since $A_1^T = A_1$, we have that $W_1^T = W_1 = x^n + y^n$. Now consider $G_{W_1^T}^{max} = \left\langle\left(\frac{1}{n}, 0\right), \left(0, \frac{1}{n}\right)\right\rangle$. We can uniquely represent all elements of $G_{W_1^T}^{max}$ in the form $\left(\frac{a}{n}, \frac{b}{n}\right)$ where $a, b \in \{0, 1, \ldots, n-1\}$. By part (3) of Proposition 2.12, $G^T = G_{W_1^T}^{max} \cap \mathrm{SL}(2, \mathbb{C})$. Hence

$$\begin{aligned}
G^T &= \left\{\left(\frac{a}{n}, \frac{b}{n}\right) \in G_{W_1^T}^{max} \mid a + b \equiv 0 \mod n\right\} \\
&= \left\{(0,0), \left(\frac{1}{n}, \frac{n-1}{n}\right), \left(\frac{2}{n}, \frac{n-2}{n}\right) \ldots, \left(\frac{n-1}{n}, \frac{1}{n}\right)\right\} \\
&= \left\langle\left(\frac{1}{n}, \frac{n-1}{n}\right)\right\rangle, \text{ under equivalence relations,} \\
&= \left\langle\left(\frac{1}{n}, -\frac{1}{n}\right)\right\rangle.
\end{aligned}$$

□

Lemma 3.5. *Let $W_2 = x^{n-1}y + y^n$. $W_2^T = x^{n-1} + xy^n$ and $G^T = \left\langle\left(\frac{1}{n-1}, -\frac{1}{n-1}\right)\right\rangle$.*

Proof. Let $A_2 = \begin{bmatrix} n-1 & 1 \\ 0 & n \end{bmatrix}$ be the exponent matrix of W_2. We then have $A_2^T = \begin{bmatrix} n-1 & 0 \\ 1 & n \end{bmatrix}$, so that $W_2^T = x^{n-1} + xy^n$.

By Proposition 3.1, $G^{max}_{W_2^T} = \left\langle \left(\frac{-1}{n-1}, \frac{1}{n(n-1)} \right) \right\rangle = \left\langle \left(\frac{n-2}{n-1}, \frac{1}{n(n-1)} \right) \right\rangle$. Let $H = \langle (\frac{1}{n-1}, -\frac{1}{n-1}) \rangle = \langle (\frac{1}{n-1}, \frac{n-2}{n-1}) \rangle$. We will first show that $H \leq G^{max}_{W_2^T} \cap \mathrm{SL}(2,\mathbb{C})$. Adding the coordinates of the generator for H yields $\frac{1}{n-1} + \frac{n-2}{n-1} = \frac{n-1}{n-1} = 1$, so $H \leq \mathrm{SL}(2,\mathbb{C})$. Now we'll multiply the generator of $G^{max}_{W_2^T}$ by the integer $n(n-2)$. This yields

$$\begin{aligned} n(n-2)\left(\frac{n-2}{n-1}, \frac{1}{n(n-1)} \right) &= \left(\frac{n(n-2)^2}{n-1}, \frac{n(n-2)}{n(n-1)} \right) \\ &= \left(\frac{1}{n-1} + n^2 - 3n + 1, \frac{n-2}{n-1} \right) \\ &= \left(\frac{1}{n-1}, \frac{n-2}{n-1} \right) \mod 1. \end{aligned}$$

Therefore $H \leq G^{max}_{W_2^T} \Rightarrow H \leq G^{max}_{W_2^T} \cap \mathrm{SL}(2,\mathbb{C})$, as desired.

Finally, we'll show that H must be G^T. To do this, we note by Proposition 3.2 that since $|G| = n$, we require $[G^{max}_{W_2^T} : G^T] = n$. Also, $G^T = G^{max}_{W_2^T} \cap \mathrm{SL}(2,\mathbb{C})$. Now $|H| = n-1$ and $|G^{max}_{W_2^T}| = n(n-1)$, so that $[G^{max}_{W_2^T} : H] = n$ and $H \leq G^{max}_{W_2^T} \cap \mathrm{SL}(2,\mathbb{C})$ which also has index n in $G^{max}_{W_2^T}$. Therefore $H = G^{max}_{W_2^T} \cap \mathrm{SL}(2,\mathbb{C}) = G^T$. □

Lemma 3.6. *Let* $W_3 = x^n + xy^{n-1}$. $W_3^T = x^n y + y^{n-1}$ *and* $G^T = \langle (\frac{1}{n-1}, -\frac{1}{n-1}) \rangle$.

Proof. This follows similarly as in Lemma 3.5 by relabeling the x and y variables. □

Lemma 3.7. *Let* $W_4 = x^{n-1}y + xy^{n-1}$. $W_4^T = x^{n-1}y + xy^{n-1}$ *and* $G^T = \langle (\frac{1}{n-2}, -\frac{1}{n-2}) \rangle$.

Proof. Let $A_4 = \begin{bmatrix} n-1 & 1 \\ 1 & n-1 \end{bmatrix}$ be the exponent matrix of W_4. Since $A_4^T = A_4$, we have that $W_4^T = W_4$. By Proposition 3.1, $G^{max}_{W_4^T} = \left\langle \left(\frac{1}{(n-1)^2-1}, \frac{-(n-1)}{(n-1)^2-1} \right) \right\rangle = \left\langle \left(\frac{1}{(n-1)^2-1}, \frac{(n-1)^2-n}{(n-1)^2-1} \right) \right\rangle$. Note that $|G^{max}_{W_4^T}| = (n-1)^2 - 1 = n(n-2)$. Let $H = \langle (\frac{1}{n-2}, -\frac{1}{n-2}) \rangle = \langle (\frac{1}{n-2}, \frac{n-3}{n-2}) \rangle$.

First notice that adding the coordinates of the generator for H yields $\frac{1}{n-2}+\frac{n-3}{n-2}=\frac{n-2}{n-2}=1$, so $H \leq \mathrm{SL}(2,\mathbb{C})$. Now multiply the generator of $G^{max}_{W_4^T}$ by the integer n. We obtain

$$\begin{aligned} n\left(\frac{1}{(n-1)^2-1}, \frac{(n-1)^2-n}{(n-1)^2-1}\right) &= \left(\frac{n}{n(n-2)}, \frac{n((n-1)^2-1)}{n(n-2)}\right) \\ &= \left(\frac{1}{n-2}, \frac{-1}{n-2}+n-1\right) \\ &= \left(\frac{1}{n-2}, \frac{n-3}{n-2}\right) \mod 1. \end{aligned}$$

Therefore $H \leq G^{max}_{W_4^T} \Rightarrow H \leq G^{max}_{W_4^T} \cap \mathrm{SL}(2,\mathbb{C})$. Since $|G|=n$, $|G^{max}_{W_4^T}|=n(n-2)$, and $|H|=n-2$, we see that $|G|=[G^{max}_{W_4^T}:H]=n$. Therefore $H=G^T$. $\square$

By the preceding four lemmas and by mirror symmetry, we have the following isomorphisms:

$$\begin{aligned} &\mathcal{A}[W_1,G] \cong \mathcal{B}\left[W_1^T, \left\langle\left(\tfrac{1}{n}, -\tfrac{1}{n}\right)\right\rangle\right], &&\mathcal{A}[W_2,G] \cong \mathcal{B}\left[W_2^T, \left\langle\left(\tfrac{1}{n-1}, -\tfrac{1}{n-1}\right)\right\rangle\right], \\ &\mathcal{A}[W_3,G] \cong \mathcal{B}\left[W_3^T, \left\langle\left(\tfrac{1}{n-1}, -\tfrac{1}{n-1}\right)\right\rangle\right], &&\mathcal{A}[W_4,G] \cong \mathcal{B}\left[W_4^T, \left\langle\left(\tfrac{1}{n-2}, -\tfrac{1}{n-2}\right)\right\rangle\right]. \end{aligned}$$

Since each W_i has weights $\left(\frac{1}{n}, \frac{1}{n}\right)$, each of these $\mathcal{A}$-models are isomorphic under the Group-Weights theorem. Hence each of these $\mathcal{B}$-models are also isomorphic, by mirror symmetry.

$\square$

Example 3.8 (Examples of Theorem 3.3)**.**

$$
\begin{aligned}
n=3:\ \mathcal{B}\left[x^3+y^3,\ \left\langle\left(\frac{1}{3},\ -\frac{1}{3}\right)\right\rangle\right] &\cong \mathcal{B}\left[x^2+xy^3,\ \left\langle\left(\frac{1}{2},\ \frac{1}{2}\right)\right\rangle\right] \\
&\cong \mathcal{B}\left[x^3y+y^2,\ \left\langle\left(\frac{1}{2},\ \frac{1}{2}\right)\right\rangle\right] \cong \mathcal{B}\left[x^2y+xy^2,\ \langle(0,\ 0)\rangle\right]. \\
n=4:\ \mathcal{B}\left[x^4+y^4,\ \left\langle\left(\frac{1}{4},\ -\frac{1}{4}\right)\right\rangle\right] &\cong \mathcal{B}\left[x^3+xy^4,\ \left\langle\left(\frac{1}{3},\ -\frac{1}{3}\right)\right\rangle\right] \\
&\cong \mathcal{B}\left[x^4y+y^3,\ \left\langle\left(\frac{1}{3},\ -\frac{1}{3}\right)\right\rangle\right] \\
&\cong \mathcal{B}\left[x^3y+xy^3,\ \left\langle\left(\frac{1}{2},\ \frac{1}{2}\right)\right\rangle\right]. \\
n=5:\ \mathcal{B}\left[x^5+y^5,\ \left\langle\left(\frac{1}{5},\ -\frac{1}{5}\right)\right\rangle\right] &\cong \mathcal{B}\left[x^4+xy^5,\ \left\langle\left(\frac{1}{4},\ -\frac{1}{4}\right)\right\rangle\right] \\
&\cong \mathcal{B}\left[x^5y+y^4,\ \left\langle\left(\frac{1}{4},\ -\frac{1}{4}\right)\right\rangle\right] \\
&\cong \mathcal{B}\left[x^4y+xy^4,\ \left\langle\left(\frac{1}{3},\ -\frac{1}{3}\right)\right\rangle\right].
\end{aligned}
$$

Our second theorem considers polynomials with a Family 2 weight system, again using common subgroup $G=\langle J\rangle$.

Theorem 3.9. *For all $n,\alpha\in\mathbb{N}$ with $n,\alpha\geq 2$,*

$$
\begin{array}{ccc}
\mathcal{B}\left[x^n+y^{\alpha n},\ \left\langle\left(\frac{1}{n},-\frac{1}{n}\right)\right\rangle\right] & \cong & \mathcal{B}\left[x^{\alpha n-\alpha}+xy^n,\ \left\langle\left(\frac{1}{n-1},-\frac{1}{n-1}\right)\right\rangle\right] \\
\cong & & \cong \\
\mathcal{B}\left[x^{\alpha n}+y^n,\ \left\langle\left(\frac{1}{n},-\frac{1}{n}\right)\right\rangle\right] & \cong & \mathcal{B}\left[x^ny+y^{\alpha n-\alpha},\ \left\langle\left(\frac{1}{n-1},-\frac{1}{n-1}\right)\right\rangle\right].
\end{array}
$$

Proof. Let $n,\alpha\in\mathbb{N}$ with $n,\alpha\geq 2$. First consider the weight system $J=\left(\frac{1}{n},\frac{1}{\alpha n}\right)$, and let $G=\langle J\rangle$.

Lemma 3.10. *Let $W_1=x^n+y^{\alpha n}$. $W_1^T=x^n+y^{\alpha n}$, and $G^T=\left\langle\left(\frac{1}{n},\ \frac{n-1}{n}\right)\right\rangle$.*

Proof. Since the exponent matrix of W_1 is symmetric, we have that $W_1^T = W_1$. We know that $G_{W_1^T}^{max} = \left\langle \left(\frac{1}{n}, 0\right), \left(0, \frac{1}{\alpha n}\right) \right\rangle$. Let $H = \left\langle \left(\frac{1}{n}, \frac{n-1}{n}\right) \right\rangle$. Certainly $H \leq \mathrm{SL}(2, \mathbb{C})$. Further, $\left(\frac{1}{n}, 0\right) + \alpha(n-1)\left(0, \frac{1}{\alpha n}\right) = \left(\frac{1}{n}, \frac{n-1}{n}\right)$, so $H \leq G_{W_1^T}^{max}$. So $H \leq G_{W_1^T}^{max} \cap \mathrm{SL}(2, \mathbb{C})$. Recall that $G = \left(\frac{1}{n}, \frac{1}{\alpha n}\right)$, and that $|G| = \alpha n$. Now $|G_{W_1^T}^{max}| = \alpha n^2$, and $|H| = n$. Therefore $[G_{W_1^T}^{max} : H] = |G_{W_1^T}^{max}/H| = \alpha n = |G|$. Thus $H = H \leq G_{W_1^T}^{max} \cap \mathrm{SL}(2, \mathbb{C}) = G^T$ by Proposition 2.12 and Proposition 3.2. □

Lemma 3.11. *Let $W_2 = x^n + xy^{\alpha n - \alpha}$. We can then represent W_2^T as either $x^n y + y^{\alpha n - \alpha}$ or $xy^n + x^{\alpha n - \alpha}$. In either case, $G^T = \left\langle \left(\frac{1}{n-1}, \frac{n-2}{n-1}\right) \right\rangle$.*

Proof. We can represent the exponent matrix of W_2 in two ways by interchanging the order of monomials:

$$A_1 = \begin{bmatrix} n & 0 \\ 1 & \alpha n - \alpha \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & \alpha n - \alpha \\ n & 0 \end{bmatrix}.$$

Transposing these matrices gives us

$$A_1^T = \begin{bmatrix} n & 1 \\ 0 & \alpha n - \alpha \end{bmatrix}, \quad A_2^T = \begin{bmatrix} 1 & n \\ \alpha n - \alpha & 0 \end{bmatrix},$$

which correspond to the polynomials $x^n y + y^{\alpha n - \alpha}$ and $xy^n + x^{\alpha n - \alpha}$. We will let $W_2^T = x^n y + y^{\alpha n - \alpha}$, with the other case following similarly by relabeling variables.

By Proposition 3.1, $G_{W_2^T}^{max} = \left\langle \left(\frac{-1}{n(\alpha n-\alpha)}, \frac{1}{\alpha n-\alpha} \right) \right\rangle$. Let $H = \left\langle \left(\frac{1}{n-1}, \ \frac{n-2}{n-1} \right) \right\rangle$. Certainly $H \leq \mathrm{SL}(2,\mathbb{C})$. Now multiply the generator of $G_{W_2^T}^{max}$ by $-\alpha n$:

$$\begin{aligned}
-\alpha n \left(\frac{-1}{n(\alpha n-\alpha)}, \frac{1}{\alpha n-\alpha} \right) &= \left(\frac{\alpha n}{n(\alpha n-\alpha)}, \frac{-\alpha n}{\alpha n-\alpha} \right) \\
&= \left(\frac{1}{n-1}, \frac{-n}{n-1} \right) \\
&= \left(\frac{1}{n-1}, \frac{-n+2(n-1)}{n-1} \right) \quad \text{mod } 1 \\
&= \left(\frac{1}{n-1}, \frac{n-2}{n-1} \right).
\end{aligned}$$

Therefore $H \leq G_{W_2^T}^{max} \Rightarrow H \leq G_{W_2^T}^{max} \cap \mathrm{SL}(2,\mathbb{C})$.

Now $|G_{W_2^T}^{max}| = n(\alpha n-\alpha) = \alpha n(n-1)$, and $|H| = n-1$. Therefore $[G_{W_2^T}^{max} : H] = \frac{\alpha n(n-1)}{n-1} = \alpha n = |G|$. Hence by Proposition 2.12 and Proposition 3.2, $H = G^T$. □

Now consider $J = \left(\frac{1}{\alpha n}, \frac{1}{n}\right)$, and let $G = \langle J \rangle$.

Lemma 3.12. *Let $W_3 = x^{\alpha n} + y^n$. $W_3^T = x^{\alpha n} + y^n$, and $G^T = \left\langle \left(\frac{1}{n}, \ \frac{n-1}{n} \right) \right\rangle$.*

Proof. This follows similarly as in Lemma 3.10 by relabeling the variables x and y. □

Lemma 3.13. *Let $W_4 = x^{\alpha n-\alpha}y + y^n$. We can then represent W_4^T as either $x^n y + y^{\alpha n-\alpha}$ or $xy^n + x^{\alpha n-\alpha}$. In either case, $G^T = \left\langle \left(\frac{1}{n-1}, \ \frac{n-2}{n-1} \right) \right\rangle$.*

Proof. We can represent the exponent matrix of W_4 in two ways by interchanging monomials:

$$A_1 = \begin{bmatrix} \alpha n-\alpha & 1 \\ 0 & n \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & n \\ \alpha n-\alpha & 1 \end{bmatrix}.$$

Transposing these matrices gives us

$$A_1^T = \begin{bmatrix} \alpha n-\alpha & 0 \\ 1 & n \end{bmatrix}, \quad A_2^T = \begin{bmatrix} 0 & \alpha n-\alpha \\ n & 1 \end{bmatrix},$$

which correspond to the polynomials $xy^n+x^{\alpha n-\alpha}$ and $x^ny+y^{\alpha n-\alpha}$. The proof of the transpose group follows similarly as in Lemma 3.11. $\square$

By Lemma 3.10, Lemma 3.11, and by mirror symmetry, we have that

$$\begin{aligned}
\mathcal{A}\left[x^n + y^{\alpha n},\ \left\langle\left(\frac{1}{n},\frac{1}{\alpha n}\right)\right\rangle\right] &\cong \mathcal{B}\left[x^n + y^{\alpha n},\ \left\langle\left(\frac{1}{n},\frac{n-1}{n}\right)\right\rangle\right],\\
\mathcal{A}\left[x^n + xy^{\alpha n-\alpha},\ \left\langle\left(\frac{1}{n},\frac{1}{\alpha n}\right)\right\rangle\right] &\cong \mathcal{B}\left[x^ny + y^{\alpha n-\alpha},\ \left\langle\left(\frac{1}{n-1},\frac{n-2}{n-1}\right)\right\rangle\right]\\
&\cong \mathcal{B}\left[xy^n + x^{\alpha n-\alpha},\ \left\langle\left(\frac{1}{n-1},\frac{n-2}{n-1}\right)\right\rangle\right].
\end{aligned}$$

By the Group-Weights theorem, these two $\mathcal{A}$-models are isomorphic. It follows that these $\mathcal{B}$-models are also isomorphic, by mirror symmetry.

By Lemma 3.12, Lemma 3.13, and by mirror symmetry, we have that

$$\begin{aligned}
\mathcal{A}\left[x^{\alpha n} + y^{n},\ \left\langle\left(\frac{1}{\alpha n},\frac{1}{n}\right)\right\rangle\right] &\cong \mathcal{B}\left[x^{\alpha n} + y^{n},\ \left\langle\left(\frac{1}{n},\frac{n-1}{n}\right)\right\rangle\right],\\
\mathcal{A}\left[x^{\alpha n-\alpha}y + y^{n},\ \left\langle\left(\frac{1}{\alpha n},\frac{1}{n}\right)\right\rangle\right] &\cong \mathcal{B}\left[x^ny + y^{\alpha n-\alpha},\ \left\langle\left(\frac{1}{n-1},\frac{n-2}{n-1}\right)\right\rangle\right]\\
&\cong \mathcal{B}\left[xy^n + x^{\alpha n-\alpha},\ \left\langle\left(\frac{1}{n-1},\frac{n-2}{n-1}\right)\right\rangle\right].
\end{aligned}$$

By the Group-Weights theorem, these two $\mathcal{A}$-models are isomorphic. It follows that these $\mathcal{B}$-models are also isomorphic.

Since W_2 and W_4 have the same transpose polynomial, we see that each of these $\mathcal{B}$-models are isomorphic. This proves the theorem. $\square$

Example 3.14. [Examples of Theorem 3.9, top row]

$$n=2,\alpha=2:\quad \mathcal{B}\left[x^2+y^4,\ \left\langle\left(\frac{1}{2},\ \frac{1}{2}\right)\right\rangle\right]\cong\mathcal{B}\left[x^2+xy^2,\ \langle(0,\ 0)\rangle\right]$$
$$n=2,\alpha=3:\quad \mathcal{B}\left[x^2+y^6,\ \left\langle\left(\frac{1}{2},\ \frac{1}{2}\right)\right\rangle\right]\cong\mathcal{B}\left[x^3+xy^2,\ \langle(0,\ 0)\rangle\right]$$
$$n=2,\alpha=4:\quad \mathcal{B}\left[x^2+y^8,\ \left\langle\left(\frac{1}{2},\ \frac{1}{2}\right)\right\rangle\right]\cong\mathcal{B}\left[x^4+xy^2,\ \langle(0,\ 0)\rangle\right]$$
$$n=3,\alpha=2:\quad \mathcal{B}\left[x^3+y^6,\ \left\langle\left(\frac{1}{3},\ -\frac{1}{3}\right)\right\rangle\right]\cong\mathcal{B}\left[x^4+xy^3,\ \left\langle\left(\frac{1}{2},\ \frac{1}{2}\right)\right\rangle\right]$$
$$n=3,\alpha=3:\quad \mathcal{B}\left[x^3+y^9,\ \left\langle\left(\frac{1}{3},\ -\frac{1}{3}\right)\right\rangle\right]\cong\mathcal{B}\left[x^6+xy^3,\ \left\langle\left(\frac{1}{2},\ \frac{1}{2}\right)\right\rangle\right]$$

This next result uses polynomials with a Family 3 weight system, and the common subgroup $G=\langle J\rangle$.

Theorem 3.15. *For all $n\in\mathbb{N}$, $n\geq 2$, and $a\in\mathbb{N}$ satisfying $1<a\leq\frac{n}{2}$, $a\mid(n-1)$, $\gcd(a,n)=1$, then*

$$\begin{array}{ccc}
\mathcal{B}\left[x^{\frac{n-1}{a}}+xy^n,\ \left\langle\left(\frac{a}{n-1},\ -\frac{a}{n-1}\right)\right\rangle\right] & \cong & \mathcal{B}\left[x^{n-a}y+xy^{\frac{n-1}{a}},\ \left\langle\left(\frac{a}{n-a-1},\ -\frac{a}{n-a-1}\right)\right\rangle\right]\\
\cong & & \cong\\
\mathcal{B}\left[x^{n}y+y^{\frac{n-1}{a}},\ \left\langle\left(\frac{a}{n-1},\ -\frac{a}{n-1}\right)\right\rangle\right] & \cong & \mathcal{B}\left[x^{\frac{n-1}{a}}y+xy^{n-a},\ \left\langle\left(\frac{a}{n-a-1},\ -\frac{a}{n-a-1}\right)\right\rangle\right].
\end{array}$$

Proof. Let $n\in\mathbb{N}$, and let $a\in\mathbb{N}$ satisfy the hypothesis of the theorem. Let $J=\left(\frac{1}{n},\frac{a}{n}\right)$, and $G=\langle J\rangle$. Notice that since $\gcd(a,n)=1$, we have that $|G|=n$.

Lemma 3.16. *Let $W_1=x^n+xy^{\frac{n-1}{a}}$. We can then represent W_1^T as either $x^{\frac{n-1}{a}}+xy^n$ or $x^ny+y^{\frac{n-1}{a}}$. In either case, $G^T=\left\langle\left(\frac{a}{n-1},\ -\frac{a}{n-1}\right)\right\rangle$.*

Proof. We can represent the exponent matrix of W_1 in two ways:

$$A_1 = \begin{bmatrix} n & 0 \\ 1 & \frac{n-1}{a} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & \frac{n-1}{a} \\ n & 0 \end{bmatrix}.$$

Transposing these matrices gives us

$$A_1^T = \begin{bmatrix} n & 1 \\ 0 & \frac{n-1}{a} \end{bmatrix}, \quad A_2^T = \begin{bmatrix} 1 & n \\ \frac{n-1}{a} & 0 \end{bmatrix},$$

which correspond to the polynomials $x^n y + y^{\frac{n-1}{a}}$ and $xy^n + x^{\frac{n-1}{a}}$. We will let $W_1^T = x^n y + y^{\frac{n-1}{a}}$, with the other case following similarly by relabeling variables.

By Proposition 3.1, $G^{max}_{W_1^T} = \left\langle \left(\frac{-a}{n(n-1)}, \frac{a}{n-1} \right) \right\rangle$. Let $H = \left\langle \left(\frac{a}{n-1}, -\frac{a}{n-1} \right) \right\rangle$. Certainly $H \leq \mathrm{SL}(2, \mathbb{C})$. Now multiply the generator of $G^{max}_{W_1^T}$ by $-n$:

$$\begin{aligned} -n \left(\frac{-a}{n(n-1)}, \frac{a}{n-1} \right) &= \left(\frac{a}{n-1}, \frac{-an}{n-1} \right) \\ &= \left(\frac{a}{n-1}, \frac{-an + a(n-1)}{n-1} \right) \quad \mod 1 \\ &= \left(\frac{a}{n-1}, -\frac{a}{n-1} \right). \end{aligned}$$

Therefore $H \leq G^{max}_{W_1^T} \Rightarrow H \leq G^{max}_{W_1^T} \cap \mathrm{SL}(2, \mathbb{C})$. Further, $|G^{max}_{W_1^T}| = n \left(\frac{n-1}{a} \right)$, and $|H| = \frac{n-1}{a}$, so $[G^{max}_{W_1^T} : H] = n = |G|$. By Proposition 2.12 and Proposition 3.2, $H = G^T$. $\square$

Lemma 3.17. *Let* $W_2 = x^{n-a} y + xy^{\frac{n-1}{a}}$. $W_2^T = x^{n-a} y + xy^{\frac{n-1}{a}}$, *and* $G^T = \left\langle \left(\frac{a}{n-a-1}, -\frac{a}{n-a-1} \right) \right\rangle$.

Proof. Since the exponent matrix of W_2 is symmetric, we have that $W_2^T = W_2$. By Proposition 3.1, $G^{max}_{W_2^T} = \left\langle \left(\frac{1}{(n-a)\left(\frac{n-1}{a}\right)-1}, \frac{-(n-a)}{(n-a)\left(\frac{n-1}{a}\right)-1} \right) \right\rangle$. Simplifying the denominator, we can write $G^{max}_{W_2^T} = \left\langle \left(\frac{1}{n\left(\frac{n-1}{a}-1\right)}, \frac{-(n-a)}{n\left(\frac{n-1}{a}-1\right)} \right) \right\rangle$. Let $H = \left\langle \left(\frac{1}{n-1}, \frac{n-2}{n-1} \right) \right\rangle$. Certainly $H \leq \mathrm{SL}(2, \mathbb{C})$.

Now multiply the generator of $G^{max}_{W_2^T}$ by n:

$$\begin{aligned} n\left(\frac{1}{n\left(\frac{n-1}{a}-1\right)}, \frac{-(n-a)}{n\left(\frac{n-1}{a}-1\right)}\right) &= \left(\frac{1}{\left(\frac{n-1}{a}\right)-1}, \frac{-n+a}{\left(\frac{n-1}{a}\right)-1}\right) \\ &= \left(\frac{1}{\left(\frac{n-1}{a}\right)-1}, \frac{-n+a+a\left(\frac{n-1}{a}-1\right)}{\left(\frac{n-1}{a}\right)-1}\right) \mod 1 \\ &= \left(\frac{1}{\left(\frac{n-1}{a}\right)-1}, \frac{-1}{\left(\frac{n-1}{a}\right)-1}\right) \\ &= \left(\frac{a}{n-a-1}, -\frac{a}{n-a-1}\right). \end{aligned}$$

Therefore $H \leq G^{max}_{W_2^T} \Rightarrow H \leq G^{max}_{W_2^T} \cap \mathrm{SL}(2,\mathbb{C})$. Now $|G^{max}_{W_2^T}| = n\left(\frac{n-1}{a}-1\right)$, and $|H| = \frac{n-1}{a}-1$, so $[G^{max}_{W_2^T} : H] = n = |G|$. Hence by Proposition 2.12 and Proposition 3.2, $H = G^T$. $\square$

Now let $J = \left(\frac{a}{n}, \frac{1}{n}\right)$, and $G = \langle J \rangle$.

Lemma 3.18. *Let $W_3 = x^{\frac{n-1}{a}}y + y^n$. We can then represent W_3^T as either $x^{\frac{n-1}{a}} + xy^n$ or $x^n y + y^{\frac{n-1}{a}}$. In either case, $G^T = \left\langle\left(\frac{a}{n-1}, -\frac{a}{n-1}\right)\right\rangle$.*

Proof. We can represent the exponent matrix of W_3 in two ways by interchanging monomials:

$$A_1 = \begin{bmatrix} \frac{n-1}{a} & 1 \\ 0 & n \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & n \\ \frac{n-1}{a} & 1 \end{bmatrix}.$$

Transposing these matrices gives us

$$A_1^T = \begin{bmatrix} \frac{n-1}{a} & 0 \\ 1 & n \end{bmatrix}, \quad A_2^T = \begin{bmatrix} 0 & \frac{n-1}{a} \\ n & 1 \end{bmatrix},$$

which correspond to the polynomials $x^{\frac{n-1}{a}} + xy^n$ and $x^n y + y^{\frac{n-1}{a}}$. The proof of the transpose group follows similarly as in Lemma 3.16. $\square$

Lemma 3.19. *Let $W_4 = x^{\frac{n-1}{a}}y + xy^{n-a}$. $W_4^T = x^{\frac{n-1}{a}}y + xy^{n-a}$, and $G^T = \left\langle\left(\frac{a}{n-a-1}, -\frac{a}{n-a-1}\right)\right\rangle$.*

Proof. This follows similarly as in Lemma 3.2 by relabeling the variables x and y. $\square$

By Lemma 3.16, Lemma 3.17, and by mirror symmetry, we have that

$$\begin{aligned}\mathcal{A}\left[x^n + xy^{\frac{n-1}{a}},\ \left\langle\left(\frac{1}{n}, \frac{a}{n}\right)\right\rangle\right] &\cong \mathcal{B}\left[x^{\frac{n-1}{a}} + xy^n,\ \left\langle\left(\frac{a}{n-1}, -\frac{a}{n-1}\right)\right\rangle\right]\\ &\cong \mathcal{B}\left[x^n y + y^{\frac{n-1}{a}},\ \left\langle\left(\frac{a}{n-1}, -\frac{a}{n-1}\right)\right\rangle\right],\\ \mathcal{A}\left[x^{n-a}y + xy^{\frac{n-1}{a}},\ \left\langle\left(\frac{1}{n}, \frac{a}{n}\right)\right\rangle\right] &\cong \mathcal{B}\left[x^{n-a}y + xy^{\frac{n-1}{a}},\ \left\langle\left(\frac{a}{n-a-1}, -\frac{a}{n-a-1}\right)\right\rangle\right].\end{aligned}$$

By the Group-Weights theorem, these two $\mathcal{A}$-models are isomorphic. It follows that these $\mathcal{B}$-models are also isomorphic, by mirror symmetry.

By Lemma 3.18, Lemma 3.19, and by mirror symmetry, we have that

$$\begin{aligned}\mathcal{A}\left[x^{\frac{n-1}{a}}y + y^n,\ \left\langle\left(\frac{a}{n}, \frac{1}{n}\right)\right\rangle\right] &\cong \mathcal{B}\left[x^{\frac{n-1}{a}} + xy^n,\ \left\langle\left(\frac{a}{n-1}, -\frac{a}{n-1}\right)\right\rangle\right]\\ &\cong \mathcal{B}\left[x^n y + y^{\frac{n-1}{a}},\ \left\langle\left(\frac{a}{n-1}, -\frac{a}{n-1}\right)\right\rangle\right],\end{aligned}$$

$$\begin{aligned}&\mathcal{A}\left[x^{\frac{n-1}{a}}y + xy^{n-a},\ \left\langle\left(\frac{a}{n}, \frac{1}{n}\right)\right\rangle\right]\\ &\qquad\cong \mathcal{B}\left[x^{\frac{n-1}{a}}y + xy^{n-a},\ \left\langle\left(\frac{a}{n-a-1}, -\frac{a}{n-a-1}\right)\right\rangle\right].\end{aligned}$$

By the Group-Weights theorem, these two $\mathcal{A}$-models are isomorphic. It follows that these $\mathcal{B}$-models are also isomorphic. Since W_1 and W_3 have the same transpose polynomial, we see that each of these $\mathcal{B}$-models are isomorphic. This proves the theorem. $\square$

Example 3.20 (Examples of Theorem 3.15, top row).

$$
\begin{aligned}
n=5, a=2: \ & \mathcal{B}\left[x^2+xy^5,\ \left\langle\left(\tfrac{1}{2},\ \tfrac{1}{2}\right)\right\rangle\right] && \cong \mathcal{B}\left[x^3y+xy^2,\ \langle(0,\ 0)\rangle\right].\\
n=7, a=2: \ & \mathcal{B}\left[x^3+xy^7,\ \left\langle\left(\tfrac{1}{3},\ -\tfrac{1}{3}\right)\right\rangle\right] && \cong \mathcal{B}\left[x^5y+xy^3,\ \left\langle\left(\tfrac{1}{2},\ \tfrac{1}{2}\right)\right\rangle\right].\\
n=7, a=3: \ & \mathcal{B}\left[x^2+xy^7,\ \left\langle\left(\tfrac{1}{2},\ \tfrac{1}{2}\right)\right\rangle\right] && \cong \mathcal{B}\left[x^4y+xy^2,\ \langle(0,\ 0)\rangle\right].\\
n=9, a=2: \ & \mathcal{B}\left[x^4+xy^9,\ \left\langle\left(\tfrac{1}{4},\ -\tfrac{1}{4}\right)\right\rangle\right] && \cong \mathcal{B}\left[x^7y+xy^4,\ \left\langle\left(\tfrac{1}{3},\ -\tfrac{1}{3}\right)\right\rangle\right].\\
n=9, a=4: \ & \mathcal{B}\left[x^2+xy^9,\ \left\langle\left(\tfrac{1}{2},\ \tfrac{1}{2}\right)\right\rangle\right] && \cong \mathcal{B}\left[x^5y+xy^2,\ \langle(0,\ 0)\rangle\right].
\end{aligned}
$$

3.4 A Complete Classification

So far, we have gone through each of the three families of weight systems and computed the resulting $\mathcal{B}$-model isomorphisms when the choice of group on the $\mathcal{A}$-side was $\langle J\rangle$—the smallest possible choice of common subgroup. In order to classify all possible isomorphisms between $\mathcal{B}$-models in two variables that stem from the Group-Weights theorem, we will need to check for other intermediate subgroups of $G_{W_1}^{max}\cap G_{W_2}^{max}$ on the $\mathcal{A}$-side where both W_1 and W_2 are invertible. A result proved by Tay states that $\langle J\rangle$ is the only possible intermediate subgroup.

Theorem 3.21 (Theorem 3.1 of [13]). *Let W_1 and W_2 be distinct invertible polynomials in two variables with the same weights. The only admissible subgroup of $G_{W_1}^{max}\cap G_{W_2}^{max}$ is $\langle J\rangle$.*

Since there are no other possible choices of common subgroup, and since the three families classify all cases in two variables where one weight system has more than one invertible polynomial, this shows us that we have uncovered all the $\mathcal{B}$-model isomorphisms in two variables that stem directly from the Group-Weights theorem for $\mathcal{A}$-models.

Chapter 4. Algorithms for $\mathcal{B}$-Model Isomorphisms

Now that we have a complete list of $\mathcal{B}$-model isomorphisms in two variables that stem from results we already know, we want to press forward to determine new isomorphisms between $\mathcal{B}$-models. To help us accomplish this goal, we will first develop some computational methods and algorithms for $\mathcal{B}$-model isomorphisms. These will assist us later on in the thesis as we explore various possibilities for isomorphic $\mathcal{B}$-models.

4.1 Computing and Verifying Algebra Isomorphisms

Here are two algorithms used in our computations of $\mathcal{B}$-model isomorphisms. They have been implemented in code, which can be found in Appendix A. Though our focus is on $\mathcal{B}$-model isomorphisms, there is nothing special about the choice of $\mathcal{B}$-models as input for these algorithms. These methods will also work the same between $\mathcal{A}$-models, and between $\mathcal{A}$ and $\mathcal{B}$ models.

4.1.1 Isomorphism Search. We take as input two $\mathcal{B}$-models, call them $\mathcal{B}_1$ and $\mathcal{B}_2$. We want to determine if $\mathcal{B}_1 \cong \mathcal{B}_2$. If so, then we want to compute an isomorphism $\phi : \mathcal{B}_1 \to \mathcal{B}_2$.

1. We first check the graded vector space structure. We can quickly determine if $\dim(\mathcal{B}_1) = \dim(\mathcal{B}_2)$, and if the grading on the basis elements line up. If this check fails, then $\mathcal{B}_1 \ncong \mathcal{B}_2$. Otherwise we proceed.

2. We now set up a possible isomorphism $\phi : \mathcal{B}_1 \to \mathcal{B}_2$, defined on the basis elements. Write $\mathcal{B}_1 = \mathrm{span}_{\mathbb{C}}\{1 = a_1, a_2, \ldots, a_n\}$, and $\mathcal{B}_2 = \mathrm{span}_{\mathbb{C}}\{1 = b_1, b_2, \ldots, b_n\}$ (ordered by degrees). We start with $\phi(1_{\mathcal{B}_1}) = 1_{\mathcal{B}_2}$. We then iterate over the basis elements of $\mathcal{B}_1$ from $k = 2$ to n.

- Case 1. If a_k has no product relations (that is, if there are no non-identity basis elements a_i, a_j such that $a_i \star a_j = ca_k$ for some $c \in \mathbb{C}$), then send a_k to some linear combination of basis elements of like degree in $\mathcal{B}_2$. There are various choices we can make for the particular linear combination. Currently implemented in code are diagonal blocks, upper/lower triangular blocks, and square (or full) blocks.

- Case 2. If a_k has one or more product relations, then we want to make sure that $\phi(a_i \star a_j) = \phi(a_i) \star \phi(a_j)$ for each combination of i, j that yields ca_k. We will then have $\phi(a_k) = \frac{1}{c}\phi(a_i \star a_k) = \frac{1}{c}\phi(a_i) \star \phi(a_j)$. Since we've ordered the basis elements by degree, we will have already set up the map for each a_i and a_j at this point (i.e. $i, j < k$). Each of these equations is equivalent, and we only need one of them to determine where a_k goes. We'll save the rest of the equations for later when it comes time to solve for the coefficients of our linear combinations.

3. Now that we have constructed our function ϕ, we need to decide if there are choices of the coefficients of the linear combinations that will make ϕ an isomorphism of graded Frobenius algebras. This will require us to create a system of several equations in several unknowns. We start by adding in the equations we get from the product structure of $\mathcal{B}_1$. That is, for each product $a_i \star a_j$, we want $\phi(a_i \star a_j) = \phi(a_i) \star \phi(a_j)$. Similarly, we obtain equations from the pairing structure by requiring $\langle a_i, a_j \rangle_{\mathcal{B}_1} = \langle \phi(a_i), \phi(a_j) \rangle_{\mathcal{B}_2}$. These equations will guarantee that ϕ respects the product and pairing structures. The other considerations (sending $1_{\mathcal{B}_1}$ to $1_{\mathcal{B}_2}$, linearity, and preserving degree) of Definition 2.28 have already been taken care of. So our problem reduces to solving a system of (generally) nonlinear equations, which we can set up and hand off to a computer algebra system to crunch on. If a solution is found, then we have an explicit construction of our isomorphism.

However, the biggest problem lies not in setting up this system of equations but in solving it! In many cases where the linear combinations get large, the computer is unable to solve the system in a reasonable amount of time. This restricts many of our computations to smaller cases that the computer can handle.

4.1.2 Isomorphism Verification. Given $\mathcal{B}$-models $\mathcal{B}_1$, $\mathcal{B}_2$, and a map $\phi : \mathcal{B}_1 \to \mathcal{B}_2$, we want to verify that ϕ is an isomorphism of graded Frobenius algebras. We can check that $\phi(1_{\mathcal{B}_1}) = 1_{\mathcal{B}_2}$, ϕ is linear, and ϕ is degree-preserving easily enough. Computationally, we can then verify $\phi(a_i \star a_j) = \phi(a_i) \star \phi(a_j)$ and $\langle a_i, a_j \rangle_{\mathcal{B}_1} = \langle \phi(a_i), \phi(a_j) \rangle_{\mathcal{B}_2}$ for all $1 \leq i \leq n$ and $i \leq j \leq n$ (like a double-loop, but we employ symmetry to reduce the number of computations). If this test passes, then ϕ is an isomorphism of graded Frobenius algebras as desired. This allows us to double-check our work in the first algorithm.

4.2 Showing That No Isomorphism Exists

Fix B, a $\mathcal{B}$-model based on some polynomial and group. We want to list some conditions that are easy to check to find other $\mathcal{B}$-models that are potentially isomorphic to B. Preferably, we want to narrow this down to a finite search-space.

First, B has a unique basis element of highest degree (see Lemma 4.3 for the two variable case). The degree of this element depends only on the weights of the polynomial, and is given by the familiar equation $\sum(1 - 2q_i)$. If we are looking at another weight system r_i, then we would require $\sum(1 - 2q_i) = \sum(1 - 2r_i)$ if we hope to find an isomorphic B-model. (Recall that this number is called the *central charge*, and is denoted by $\widehat{c}$).

Next, we know that the dimension of an unorbifolded $\mathcal{B}$-model is given by $\prod \left(\frac{1}{q_i} - 1\right)$. We can also observe that the dimension of an orbifolded $\mathcal{B}$-model is generally less than or equal to the dimension of its unorbifolded $\mathcal{B}$-model. However, there are exceptions as we can see in the following example.

Example 4.1 (A pathology)**.** The unorbifolded $\mathcal{B}$-model $\mathcal{B}[x^2+y^2, \{0\}] = \mathbb{C}[x,y]/(2x, 2y) \cong \mathbb{C}$ has one basis element. But the orbifolded $\mathcal{B}$-model $\mathcal{B}[x^2 + y^2, \langle(\frac{1}{2}, \frac{1}{2})\rangle]$ has two basis elements, one for each element in the symmetry group. However, this is a particularly special case since the original $\mathcal{B}$-model was one dimensional.

We can avoid these pathological examples by assuming that the central charge $\widehat{c} > 0$, which will force our unorbifolded $\mathcal{B}$-models to be at least two dimensional. Therefore, if we're looking at a potential weight system q_i, we need to have $\prod\left(\frac{1}{q_i}-1\right) \geq \dim(B)$. That way there *might* be some non-trivial group that we can orbifold with, and perhaps obtain an isomorphic $\mathcal{B}$-model. Further, the dimension must always be an integer. These conditions will narrow down our search-space.

This information, combined with the usual bounds, gives us the following test for $\mathcal{B}$-models in two variables:

Theorem 4.2 (Isomorphism Criteria)**.** *Fix B, a $\mathcal{B}$-model with dimension d and central charge $\widehat{c} > 0$. In order for a weight system (q_1, q_2) to have a polynomial and group that will yield an isomorphic $\mathcal{B}$-model to B, the following conditions must be satisfied:*

(1) $q_1, q_2 \in \mathbb{Q} \cap (0, \frac{1}{2}]$,

(2) $2 - 2q_1 - 2q_2 = \widehat{c}$,

(3) $\left(\frac{1}{q_1}-1\right)\left(\frac{1}{q_2}-1\right) = n \in \mathbb{N}_{\geq d}$.

So why should this yield a finite list of weight systems? The first condition limits us to rational coordinates of the box $(0, \frac{1}{2}] \times (0, \frac{1}{2}]$ in the coordinate plane. The second condition defines a line that cuts through this box. So these two conditions alone still yield an infinite number of possibilities. Finally, the third condition defines a hyperbola—part of which intersects our box and line. The inequality still gives us most of the line, but the fact that the hyperbola must equal an integer means that on a good day there will only be a finite number of cases before the solutions get too big and fall outside our box. This gives us an algorithm to compute these weight systems. However, sometimes this algorithm will not halt due to limiting conditions on how the line intersects the hyperbola. We'll classify these cases in Lemma 4.4.

Once we obtain a finite number of weight systems to look at, we can make a (slightly larger) finite list of polynomials and groups. We then will only have a finite number of $\mathcal{B}$-models to check for isomorphisms. If none of them match our initial $\mathcal{B}$-model, then we can definitively say that there cannot be any isomorphic $\mathcal{B}$-models in two variables. If necessary, we can also rule out the one variable cases.

4.2.1 Justifcation of the Criteria.

The following lemmas and conditions are needed to justify and explain parts of the algorithm listed above.

Lemma 4.3. *For a given W in two variables with $\widehat{c} > 0$ and a nontrivial group $G \leq \mathrm{SL}(W)$, the subspace highest degree in $\mathcal{B}[W, G]$ is spanned by the same element that spans the subspace of highest degree in $\mathcal{B}[W, \{0\}]$.*

Proof. As noted earlier, the subspace of highest degree of $\mathcal{B}[W, \{0\}]$ is spanned by $\mathrm{Hess}(W)$. Let m represent the monomial part of $\mathrm{Hess}(W)$, as we won't need its coefficient. We will examine the definition of $\mathcal{B}$-model multiplication to show that $\lfloor m; (0,0) \rceil$ is a basis element of $\mathcal{B}[W, G]$. Further, we will show that $\lfloor m; (0,0) \rceil$ is an element of highest degree in $\mathcal{B}[W, G]$ and that it is unique.

First, let's recall the definition of $\mathcal{B}$-model multiplication from Definition 2.24. The product of two elements $\lfloor m_1; g_1 \rceil$ and $\lfloor m_2; g_2 \rceil$ is given by

$$\lfloor m_1; g_1 \rceil \star \lfloor m_2; g_2 \rceil = \begin{cases} \lfloor \gamma m_1 m_2; g_1 + g_2 \rceil & \text{if } \mathrm{fix}(g_1) \cup \mathrm{fix}(g_2) \cup \mathrm{fix}(g_1 + g_2) = \mathbb{C}^n, \\ 0 & \text{otherwise.} \end{cases}$$

Here γ is a monomial defined by

$$\gamma = \frac{\mu_{g_1 \cap g_2} \mathrm{Hess}(W|_{\mathrm{fix}(g_1 + g_2)})}{\mu_{g_1 + g_2} \mathrm{Hess}(W|_{\mathrm{fix}(g_1) \cap \mathrm{fix}(g_2)})},$$

where $\mu_{g_1 \cap g_2}$ is the dimension of the Milnor ring corresponding to $W|_{\mathrm{fix}(g_1) \cap \mathrm{fix}(g_2)}$, and $\mu_{g_1 + g_2}$ is the dimension of the Milnor ring corresponding to $W|_{\mathrm{fix}(g_1 + g_2)}$.

Since we have assumed that our group G is nontrivial, there is an element $g \in G$ such that $g \neq (0,0)$. Since G is a group, there also exists $h \in G$ such that $g + h = (0,0)$. Fixing the coordinates of our group elements to be in $[0,1)$, noting that $G \leq \mathrm{SL}(W)$, and since g is not the identity element, we need the coordinates of g to sum to 1. Since they are both not zero, at least one is nonzero and is strictly between 0 and 1. The other coordinate then must be 1 minus the first coordinate, and therefore will also be strictly between 0 and 1. Hence $\mathrm{fix}(g)$ is trivial. The same argument applies to h.

We further note that $\lfloor 1; g \rceil$ and $\lfloor 1; h \rceil$ are basis elements of $\mathcal{B}[W, G]$. This follows since the group action on 1 is trivial. We then see that the dimension of $\mathcal{B}[W, G]$ is at least equal to $|G|$, but that's beside the point. We want to show that we get $\lfloor m; (0,0) \rceil$ in our basis, where m is the monomial part of $\mathrm{Hess}(W)$.

Consider $\lfloor 1; g \rceil \star \lfloor 1; h \rceil$. By the formula, we obtain $\lfloor \gamma; (0,0) \rceil$. When computing γ, we are not concerned about the coefficient—we just want to know the monomial. But notice that $\mathrm{fix}(g) \cap \mathrm{fix}(h)$ is trivial, and $\mathrm{fix}(g+h)$ is $\mathbb{C}^2$. So γ equals $\frac{\mathrm{Hess}(W)}{\mu}$, where μ is the dimension of $\mathcal{Q}_W$. Therefore γ is some nonzero coefficient times m, which is what we wanted. $\lfloor m; (0,0) \rceil$ is an element of $\mathcal{B}[W, G]$.

From the unorbifolded case, we know that $\deg \lfloor m; (0,0) \rceil = \widehat{c}$, which begs a question: Do we get any elements of higher degree in the orbifolded case? Fortunately, we only have two situations to check. Either we get some monomial from the basis of the Milnor ring and the identity element $(0,0)$, or we get 1 paired with some $g \in G$. We already know from the Milnor ring that $\lfloor m; (0,0) \rceil$ is unique. Now we just need to show that it has a higher degree than any $\lfloor 1; g \rceil$.

Recall that the $\mathcal{B}$-model degree of any basis element $\lfloor n; (\theta_1, \ldots, \theta_n) \rceil$ is given by $2p + \sum_{\theta_i \notin \mathbb{Z}} (1 - 2q_i)$ where p represents the quasihomogeneous degree of monomial n. In our case, we find that each $\lfloor 1; g \rceil$ has $\mathcal{B}$-model degree of $\widehat{c}$ and $\lfloor m; (0,0) \rceil$ has $\mathcal{B}$-model degree of $2\widehat{c}$. So for $\widehat{c} > 0$, which we have assumed in the hypothesis, we have our result. □

Lemma 4.4 (Halting Condition)**.** *The algorithm will halt provided that* $0 < \widehat{c} < 1$.

Proof. This is a simple geometric observation. We know that $0 < \widehat{c} < 2$. When $0 < \widehat{c} < 1$, the line $2 - 2q_1 - 2q_2 = \widehat{c}$ (defined in condition (2) of Theorem 4.2) has x and y intercepts that are greater than $\frac{1}{2}$. At $\widehat{c} = 1$, the intercepts are both precisely $\frac{1}{2}$, and for $1 \leq \widehat{c} < 2$, the intercepts are between 0 and $\frac{1}{2}$.

Now as we increase the possible dimension for our $\mathcal{B}$-model, the hyperbola $\left(\frac{1}{q_1}-1\right)\left(\frac{1}{q_2}-1\right) = n \in \mathbb{N}_{\geq d}$ (condition (3) of Theorem 4.2) will become steeper and the "bend" of the hyperbola will get closer to the origin. However, if the line has x and y intercepts between 0 and $\frac{1}{2}$, there is no possible way for the hyperbola to intersect the line outside of our "feasible region" $\mathbb{Q} \cap (0, \frac{1}{2}]$ (condition (1) of Theorem 4.2). $\square$

That being said, there seems to be a point at which the computer stops outputting rational-valued solutions even if $\widehat{c} \geq 1$. Further work must be done to figure out how far we need to go before we can reasonably say that we've found all possible weight systems.

4.2.2 Using the Criteria.

Example 4.5. Up to permutation of variables, $\mathcal{B}[x^3 + xy^3, \{0\}]$ is unique (for polynomials in two variables).

Proof. Let $B = \mathcal{B}[x^3 + xy^3, \{0\}]$. We have that $\deg(B) = 7$, and its $\widehat{c} = \frac{8}{9}$. Running the code, we obtain the following list of solutions:

Unorbifolded Dimension	Weight Systems
7	$(\frac{1}{3}, \frac{2}{9}), (\frac{2}{9}, \frac{1}{3})$
10	$(\frac{1}{9}, \frac{4}{9}), (\frac{4}{9}, \frac{1}{9})$
17	$(\frac{1}{18}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{18})$

The code halted since $\widehat{c} < 1$, so we conclude that these are all the possibilities. Our original polynomial has weights $(\frac{1}{3}, \frac{2}{9})$, and it is the only polynomial admitted by this weight system. Notice also that when there are two solutions, the second solution is just the result of permuting variables.

The weight system $(\frac{1}{9}, \frac{4}{9})$ has the polynomials $x^9 + xy^2$, $x^5y + xy^2$, and $x^9 + x^5y + xy^2$. The only group that works for any of the three is $\{0\}$, which gives us $\mathcal{B}$-models of dimension 10—too large to match with B.

The final weight system $(\frac{1}{18}, \frac{1}{2})$ yields $x^{18}+y^2$, x^9y+y^2, and $x^{18}+x^9y+y^2$. The groups $\{0\}$ and $\langle(\frac{1}{2}, \frac{1}{2})\rangle$ happen to work in each case. Of course $\{0\}$ yields dimension 17, which is too big. $\langle(\frac{1}{2}, \frac{1}{2})\rangle$ yields dimension 10, which is also too big. So we conclude that B is (in the sense that we defined above) unique. $\square$

Proposition 4.6. *The following is a complete list of isomorphic $\mathcal{B}$-models in two variables (up to permutation) involving the weight system $(\frac{1}{3}, \frac{1}{3})$.*

(1) $\mathcal{B}[x^3+y^3, \{0\}]$	(6) $\mathcal{B}[x^3+y^3+x^2y+xy^2, \{0\}]$
(2) $\mathcal{B}[x^3+xy^2, \{0\}]$	(7) $\mathcal{B}[x^3+y^3, \langle(\frac{1}{3}, -\frac{1}{3})\rangle]$
(3) $\mathcal{B}[x^2y+xy^2, \{0\}]$	(8) $\mathcal{B}[x^2+xy^3, \langle(\frac{1}{2}, \frac{1}{2})\rangle]$
(4) $\mathcal{B}[x^3+y^3+x^2y, \{0\}]$	(9) $\mathcal{B}[x^2+y^6, \langle(\frac{1}{2}, \frac{1}{2})\rangle]$
(5) $\mathcal{B}[x^3+x^2y+xy^2, \{0\}]$	(10) $\mathcal{B}[x^2+xy^3+y^6, \langle(\frac{1}{2}, \frac{1}{2})\rangle]$

Proof. In Theorem 5.1 we will prove that $\mathcal{B}$-models (1) through (10) are isomorphic. It now remains to show that there are no other possible isomorphic $\mathcal{B}$-models in two variables. Note that for the weight system $(\frac{1}{3}, \frac{1}{3})$ we have $\widehat{c} = \frac{2}{3} < 1$. So we are safe to run the code, which gives the following solutions:

Unorbifolded Dimension	Weight Systems
4	$(\frac{1}{3}, \frac{1}{3})$
5	$(\frac{1}{2}, \frac{1}{6}), (\frac{1}{6}, \frac{1}{2})$

We have listed all possible polynomials with weights $(\frac{1}{3}, \frac{1}{3})$, and all their possible $\mathcal{B}$-models. Interestingly, they all happen to be isomorphic. For the weight system $(\frac{1}{2}, \frac{1}{6})$ we have listed all possible polynomials with all their possible *orbifolded* $\mathcal{B}$-models (since the unorbifolded dimension is too big). These are also isomorphic to the other ones listed. Since there are no more potential weight systems, we conclude that this is a complete list. $\square$

Example 4.7. Let's investigate the weight system $(\frac{1}{4}, \frac{1}{4})$ and find all the possible isomorphisms. Applying the algorithm, we find these solutions:

Unorbifolded Dimension	Weight Systems
9	$(\frac{1}{4}, \frac{1}{4})$
10	$(\frac{1}{3}, \frac{1}{6}), (\frac{1}{6}, \frac{1}{3})$

However $\widehat{c} = 1$, so the algorithm does not halt. Checking up to dim $= 10000$ yields the previous results. We may proceed with some confidence that these are all the possibilities. This conclusion is also suggested by Group-Weights—Theorem 3.3 gives us isomorphisms between $\mathcal{B}$-models with polynomials of weight $(\frac{1}{4}, \frac{1}{4})$ and $\mathcal{B}$-models with polynomials of weight $(\frac{1}{3}, \frac{1}{6})$ and $(\frac{1}{6}, \frac{1}{3})$.

Applying the algorithms developed in the previous chapter, we compute new isomorphisms of Landau-Ginzburg $\mathcal{B}$-models. We start by choosing a weight system, listing the possible $\mathcal{B}$-models built with polynomials fixed by that weight system, and then proceed by searching for isomorphic $\mathcal{B}$-models.

5.1 Weight System $\left(\frac{1}{3}, \frac{1}{3}\right)$

The following is list of $\mathcal{B}$-models built with polynomials having weights $\left(\frac{1}{3}, \frac{1}{3}\right)$, along with other $\mathcal{B}$-models that are isomorphic to them. Recall that in Proposition 4.6 we showed that if these isomorphisms exist, then they form a complete list for this case.

Theorem 5.1. *Each of the following $\mathcal{B}$-models are isomorphic.*

(1) $\mathcal{B}[x^3 + y^3, \{0\}]$	(6) $\mathcal{B}[x^3 + y^3 + x^2y + xy^2, \{0\}]$
(2) $\mathcal{B}[x^3 + xy^2, \{0\}]$	(7) $\mathcal{B}[x^3 + y^3, \langle(\frac{1}{3}, -\frac{1}{3})\rangle]$
(3) $\mathcal{B}[x^2y + xy^2, \{0\}]$	(8) $\mathcal{B}[x^2 + xy^3, \langle(\frac{1}{2}, \frac{1}{2})\rangle]$
(4) $\mathcal{B}[x^3 + y^3 + x^2y, \{0\}]$	(9) $\mathcal{B}[x^2 + y^6, \langle(\frac{1}{2}, \frac{1}{2})\rangle]$
(5) $\mathcal{B}[x^3 + x^2y + xy^2, \{0\}]$	(10) $\mathcal{B}[x^2 + xy^3 + y^6, \langle(\frac{1}{2}, \frac{1}{2})\rangle]$

Proof. From the Group-Weights theorem, we already know that (3) $\cong$ (7) $\cong$ (8) and (2) $\cong$ (9) (see Theorem 3.3). By Theorem 2.32, we get (1) $\cong \ldots \cong$ (6). (8) $\cong$ (9) $\cong$ (10) will be shown in Theorem 6.6. We will show cases (1) $\cong$ (7) and (1) $\cong$ (2) directly in Example 5.2 and Example 5.3. $\square$

As a side note, we get the following $\mathcal{A}$-model isomorphisms that don't follow from Group-Weights:

(1) $\mathcal{A}\left[x^3+y^3, \langle(\frac{1}{3},0),(0,\frac{1}{3})\rangle\right]$ (3) $\mathcal{A}\left[x^2y+y^3, \langle(\frac{5}{6},\frac{1}{3})\rangle\right]$ (5) $\mathcal{A}\left[x^2+y^6, \langle(\frac{1}{2},\frac{1}{6})\rangle\right]$

(2) $\mathcal{A}\left[x^3+xy^2, \langle(\frac{1}{3},\frac{5}{6})\rangle\right]$ (4) $\mathcal{A}\left[x^2y+xy^2, \langle(\frac{1}{3},\frac{1}{3})\rangle\right]$

The following examples demonstrate how to use the code to compute isomorphisms. Here we will try to verify $\mathcal{B}\left[x^3+y^3,\{0\}\right] \cong \mathcal{B}\left[x^3+y^3, \langle(\frac{1}{3},-\frac{1}{3})\rangle\right]$. Running the code, we get the following.

```
sage: W1 = Singularity(x^3 + y^3)
sage: B1 = OrbMilnorRing(SymmetryGroup(W1,0))
Orbifold Milnor ring for x^3 + y^3 with group generated by <(0, 0)>.
Dimension:   4
Basis:
  [1]  b_(0, 0)      Degree: 0    (0, 0)
  [2]  yb_(0, 0)     Degree: 2/3  (1/3, 1/3)
  [3]  xb_(0, 0)     Degree: 2/3  (1/3, 1/3)
  [4]  x*yb_(0, 0)   Degree: 4/3  (2/3, 2/3)
sage: B = OrbMilnorRing(SymmetryGroup(W1,[[1/3,2/3]]))
Orbifold Milnor ring for x^3 + y^3 with group generated by <(1/3, 2/3)>.
Dimension:   4
Basis:
  [1]  b_(0, 0)       Degree: 0    (0, 0)
  [2]  b_(1/3, 2/3)   Degree: 2/3  (1/3, 1/3)
  [3]  b_(2/3, 1/3)   Degree: 2/3  (1/3, 1/3)
  [4]  x*yb_(0, 0)    Degree: 4/3  (2/3, 2/3)
sage: construct_map(B1,B)
Isomorphic as Graded Vector Spaces
using map:
[      1       0       0       0]
[      0      c0       0       0]
[      0       0      c1       0]
[      0       0       0 9*c0*c1]
Solving equations [True, (1/9) == c0*c1, 9*c0*c1 == 9*c0*c1]
Solution(s):  [
[c0 == 1/9/r1, c1 == r1]
]
```

The output of the code suggests our next result.

Example 5.2. $\mathcal{B}\left[x^3+y^3,\{0\}\right] \cong \mathcal{B}\left[x^3+y^3, \langle(\frac{1}{3},-\frac{1}{3})\rangle\right]$.

Proof. Let $\mathcal{B}_1 = \mathcal{B}\left[x^3+y^3,\{0\}\right]$ and $\mathcal{B}_2 = \mathcal{B}\left[x^3+y^3,\langle(\frac{1}{3},\frac{2}{3})\rangle\right]$. Choosing $c_1 = r_1 = 1$ in the computer's solution, we get the following map $\phi : \mathcal{B}_1 \to \mathcal{B}_2$:

$$\phi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{9} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We will verify directly that ϕ is indeed an isomorphism of graded Frobenius algebras. Certainly ϕ is a linear map, sending the identity of $\mathcal{B}_1$ to the identity of $\mathcal{B}_2$, and is bijective since the matrix is invertible. The following are the multiplication tables for $\mathcal{B}_1$ and $\mathcal{B}_2$:

$\star$	$[1]_1$	$[2]_1$	$[3]_1$	$[4]_1$
$[1]_1$	$[1]_1$	$[2]_1$	$[3]_1$	$[4]_1$
$[2]_1$		0	$[4]_1$	0
$[3]_1$			0	0
$[4]_1$				0

$\star$	$[1]_2$	$[2]_2$	$[3]_2$	$[4]_2$
$[1]_2$	$[1]_2$	$[2]_2$	$[3]_2$	$[4]_2$
$[2]_2$		0	$9[4]_2$	0
$[3]_2$			0	0
$[4]_2$				0

From the code, we have the following pairing matrices:

$$\eta_{B_1} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{9} \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & \frac{1}{9} & 0 & 0 \\ \frac{1}{9} & 0 & 0 & 0 \end{bmatrix} \qquad \eta_{B_2} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{9} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{9} & 0 & 0 & 0 \end{bmatrix}.$$

We'll check that ϕ respects the non-trivial products and pairings. There is only one non-trivial product to check:

$$\begin{aligned} \phi([2]_1 \star [3]_1) &= \phi([4]_1) = [4]_2, \\ \phi([2]_1) \star \phi([3]_1) &= \frac{1}{9}[2]_2 \star [3]_2 = \frac{1}{9}\left(9[4]_2\right) = [4]_2. \end{aligned}$$

This shows that ϕ respects the products. There are also two non-trivial pairing relations. First:

$$\langle [1]_1, [4]_1 \rangle = \frac{1}{9},$$
$$\langle \phi([1]_1), \phi([4]_1) \rangle = \langle [1]_2, [4]_2 \rangle = \frac{1}{9}.$$

Second:

$$\langle [2]_1, [3]_1 \rangle = \frac{1}{9},$$
$$\langle \phi([2]_1), \phi([3]_1) \rangle = \langle \frac{1}{9}[2]_2, [3]_2 \rangle = \frac{1}{9}\langle [2]_2, [3]_2 \rangle = \frac{1}{9}(1) = \frac{1}{9}.$$

So ϕ respects the pairing. Therefore ϕ does give us an isomorphism of graded Frobenius algebras. □

We'll now investigate $\mathcal{B}\left[x^3 + y^3, \{0\}\right]$ and $\mathcal{B}\left[x^3 + xy^2, \{0\}\right]$. Here the code yields the following output.

```
sage: W1 = Singularity(x^3 + y^3); W2 = Singularity(x^3 + x*y^2)
sage: B1 = OrbMilnorRing(SymmetryGroup(W1,0))
Orbifold Milnor ring for x^3 + y^3 with group generated by <(0, 0)>.
Dimension:   4
Basis:
  [1]  b_(0, 0)     Degree: 0    (0, 0)
  [2]  yb_(0, 0)    Degree: 2/3  (1/3, 1/3)
  [3]  xb_(0, 0)    Degree: 2/3  (1/3, 1/3)
  [4]  x*yb_(0, 0)  Degree: 4/3  (2/3, 2/3)
sage: B2 = OrbMilnorRing(SymmetryGroup(W2,0))
Orbifold Milnor ring for x^3 + x*y^2 with group generated by <(0, 0)>.
Dimension:   4
Basis:
  [1]  b_(0, 0)     Degree: 0    (0, 0)
  [2]  yb_(0, 0)    Degree: 2/3  (1/3, 1/3)
  [3]  xb_(0, 0)    Degree: 2/3  (1/3, 1/3)
  [4]  y^2b_(0, 0)  Degree: 4/3  (2/3, 2/3)
sage: construct_map(B1,B2, type="full")
Isomorphic as Graded Vector Spaces
using map:
[1   0   0                       0]
[0 c_0 c_1                       0]
```

```
[0 c_2 c_3                     0]
[0   0   0 c_0*c_2 - 1/3*c_1*c_3]
Solving equations [True, c2^2 - 1/3*c3^2 == 0, 0 == -1/2*c0^2 + 1/6*c1^2,
0 == -1/2*c2^2 + 1/6*c3^2, (1/9) == -1/2*c0*c2 + 1/6*c1*c3,
c0*c2 - 1/3*c1*c3 == c0*c2 - 1/3*c1*c3, c0^2 - 1/3*c1^2 == 0]
Solution(s):  [
[c_0 == -1/9/r1, c_1 == 1/9*sqrt(3)/r1, c_2 == r1, c_3 == sqrt(3)*r1],
[c_0 == -1/9/r2, c_1 == -1/9*sqrt(3)/r2, c_2 == r2, c_3 == -sqrt(3)*r2]
]
```

Since the computer was able to find a map, this output suggests that these $\mathcal{B}$-models are isomorphic.

Example 5.3. $\mathcal{B}\left[x^3+y^3,\{0\}\right] \cong \mathcal{B}\left[x^3+xy^2,\{0\}\right]$.

Proof. Let $\mathcal{B}_1 = \mathcal{B}\left[x^3+y^3,\{0\}\right]$ and $\mathcal{B}_2 = \mathcal{B}\left[x^3+xy^2,\{0\}\right]$. Choosing $r_1 = 1$ in the first solution from the computer, we get the following map $\phi : \mathcal{B}_1 \to \mathcal{B}_2$:

$$\phi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{9} & \frac{\sqrt{3}}{9} & 0 \\ 0 & 1 & \sqrt{3} & 0 \\ 0 & 0 & 0 & -\frac{2}{9} \end{bmatrix}.$$

We will verify directly that ϕ is indeed an isomorphism of graded Frobenus algebras. Certainly ϕ is a linear map, sending the identity of $\mathcal{B}_1$ to the identity of $\mathcal{B}_2$, and is bijective since the matrix is invertible. The following are the multiplication tables for $\mathcal{B}_1$ and $\mathcal{B}_2$:

$\star$	$[1]_1$	$[2]_1$	$[3]_1$	$[4]_1$
$[1]_1$	$[1]_1$	$[2]_1$	$[3]_1$	$[4]_1$
$[2]_1$		0	$[4]_1$	0
$[3]_1$			0	0
$[4]_1$				0

$\star$	$[1]_2$	$[2]_2$	$[3]_2$	$[4]_2$
$[1]_2$	$[1]_2$	$[2]_2$	$[3]_2$	$[4]_2$
$[2]_2$		$[4]_2$	0	0
$[3]_2$			$-\frac{1}{3}[4]_2$	0
$[4]_2$				0

From the code, we have the following pairing matrices:

$$\eta_{B_1} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{9} \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & \frac{1}{9} & 0 & 0 \\ \frac{1}{9} & 0 & 0 & 0 \end{bmatrix} \qquad \eta_{B_2} = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{bmatrix}.$$

We'll check that ϕ respects the non-trivial products and pairings. For the products:

$$\begin{aligned}
\phi([2]_1 \star [2]_1) &= \phi(0) = 0, \\
\phi([2]_1) \star \phi([2]_1) &= (-\frac{1}{9}[2]_2 + \frac{\sqrt{3}}{9}[3]_2) \star (-\frac{1}{9}[2]_2 + \frac{\sqrt{3}}{9}[3]_2) \\
&= \frac{1}{81}([2]_2 \star [2]_2) + \frac{3}{81}([3]_2 \star [3]_2) \\
&= \frac{1}{81}([4]_2) + \frac{3}{81}(-\frac{1}{3}[4]_2) \\
&= 0. \\
\phi([2]_1 \star [3]_1) &= \phi([4]_1) = -\frac{2}{9}[4]_2, \\
\phi([2]_1) \star \phi([3]_1) &= (-\frac{1}{9}[2]_2 + \frac{\sqrt{3}}{9}[3]_2) \star ([2]_2 + \sqrt{3}[3]_2) \\
&= -\frac{1}{9}([2]_2 \star [2]_2) + \frac{1}{3}([3]_2 \star [3]_2) \\
&= -\frac{1}{9}([4]_2) + \frac{1}{3}(-\frac{1}{3}[4]_2) \\
&= -\frac{2}{9}[4]_2. \\
\phi([3]_1 \star [3]_1) &= \phi(0) = 0, \\
\phi([3]_1) \star \phi([3]_1) &= ([2]_2 + \sqrt{3}[3]_2) \star ([2]_2 + \sqrt{3}[3]_2) \\
&= ([2]_2 \star [2]_2) + 3([3]_2 \star [3]_2) \\
&= [4]_2 + 3(-\frac{1}{3}[4]_2) \\
&= 0.
\end{aligned}$$

This shows that ϕ respects the products. Now for the pairings:

$$\langle [1]_1, [4]_1 \rangle = \frac{1}{9},$$
$$\langle \phi([1]_1), \phi([4]_1) \rangle = \langle [1]_2, -\frac{2}{9}[4]_2 \rangle = -\frac{2}{9}\langle [1]_2, [4]_2 \rangle = \left(-\frac{2}{9}\right)\left(-\frac{1}{2}\right) = \frac{1}{9}.$$

$$\begin{aligned}
\langle [2]_1, [2]_1 \rangle &= 0, \\
\langle \phi([2]_1), \phi([2]_1) \rangle &= \langle -\frac{1}{9}[2]_2 + \frac{\sqrt{3}}{9}[3]_2, -\frac{1}{9}[2]_2 + \frac{\sqrt{3}}{9}[3]_2 \rangle \\
&= -\frac{1}{9} \cdot -\frac{1}{9}\langle [2]_2, [2]_2 \rangle + \frac{\sqrt{3}}{9} \cdot \frac{\sqrt{3}}{9}\langle [3]_2, [3]_2 \rangle \\
&= \frac{1}{81} \cdot -\frac{1}{2} + \frac{3}{81} \cdot \frac{1}{6} \\
&= 0. \\
\langle [2]_1, [3]_1 \rangle &= \frac{1}{9}. \\
\langle \phi([2]_1), \phi([3]_1) \rangle &= \langle -\frac{1}{9}[2]_2 + \frac{\sqrt{3}}{9}[3]_2, [2]_2 + \sqrt{3}[3]_2 \rangle \\
&= -\frac{1}{9}\langle [2]_2, [2]_2 \rangle + \sqrt{3} \cdot \frac{\sqrt{3}}{9}\langle [3]_2, [3]_2 \rangle \\
&= -\frac{1}{9} \cdot -\frac{1}{2} + \frac{1}{3} \cdot \frac{1}{6} \\
&= \frac{1}{9}. \\
\langle [3]_1, [3]_1 \rangle &= 0, \\
\langle \phi([3]_1), \phi([3]_1) \rangle &= \langle [2]_2 + \sqrt{3}[3]_2, [2]_2 + \sqrt{3}[3]_2 \rangle \\
&= \langle [2]_2, [2]_2 \rangle + \sqrt{3} \cdot \sqrt{3}\langle [3]_2, [3]_2 \rangle \\
&= -\frac{1}{2} + 3 \cdot \frac{1}{6} \\
&= 0.
\end{aligned}$$

So ϕ respects the pairing. Therefore ϕ does give us an isomorphism of graded Frobenius algebras. $\square$

5.2 Weight System $\left(\frac{1}{4}, \frac{1}{4}\right)$

The following set of $\mathcal{B}$-models come from using polynomials with weight $\left(\frac{1}{4}, \frac{1}{4}\right)$ and a non-trivial group. Each of these match as graded vector spaces. Now do they match as graded Frobenius algebras? We use the computer to help us assemble the following table.

	Polynomial W	Symmetry Group G	
0.	$x^4 + y^4$	$\langle(\frac{1}{2}, \frac{1}{2})\rangle$	$\mathrm{SL}(W) = \langle(\frac{1}{4}, -\frac{1}{4})\rangle$
1.	$x^4 + xy^3$	$\mathrm{SL}(W) = \langle(\frac{1}{2}, \frac{1}{2})\rangle$	
2.	$x^3y + xy^3$	$\mathrm{SL}(W) = \langle(\frac{1}{2}, \frac{1}{2})\rangle$	
3.	$x^4 + x^3y + xy^3$	$\mathrm{SL}(W) = \langle(\frac{1}{2}, \frac{1}{2})\rangle$	
4.	$x^4 + x^3y + y^4$	$\mathrm{SL}(W) = \langle(\frac{1}{2}, \frac{1}{2})\rangle$	
5.	$x^4 + x^2y^2 + xy^3$	$\mathrm{SL}(W) = \langle(\frac{1}{2}, \frac{1}{2})\rangle$	
6.	$x^4 + x^2y^2 + y^4$	$\langle(\frac{1}{2}, \frac{1}{2})\rangle$	$\mathrm{SL}(W) = \langle(\frac{1}{4}, -\frac{1}{4})\rangle$
7.	$x^3y + x^2y^2 + xy^3$	$\mathrm{SL}(W) = \langle(\frac{1}{2}, \frac{1}{2})\rangle$	
8.	$x^3y + x^2y^2 + xy^3 + y^4$	$\mathrm{SL}(W) = \langle(\frac{1}{2}, \frac{1}{2})\rangle$	
9.	$x^4 + x^3y + x^2y^2 + y^4$	$\mathrm{SL}(W) = \langle(\frac{1}{2}, \frac{1}{2})\rangle$	
10.	$x^4 + x^3y + xy^3 + y^4$	$\mathrm{SL}(W) = \langle(\frac{1}{2}, \frac{1}{2})\rangle$	
11.	$x^4 + x^3y + x^2y^2 + xy^3 + y^4$	$\mathrm{SL}(W) = \langle(\frac{1}{2}, \frac{1}{2})\rangle$	

Here we let $\mathcal{B}_i$ denote the $\mathcal{B}$-model constructed with polynomial i and the smallest group listed, and we'll write $\mathcal{B}_i(\mathrm{SL})$ for the $\mathcal{B}$-model constructed with the larger group. Using a computer, we've been successful in computing the following isomorphisms:

$$\begin{aligned} \mathcal{B}_0(\mathrm{SL}) &\cong \mathcal{B}_2 \cong \mathcal{B}_3 \cong \mathcal{B}_5 \cong \mathcal{B}_6 \cong \mathcal{B}_6(\mathrm{SL}) \cong \mathcal{B}_7 \cong \mathcal{B}_8 \cong \mathcal{B}_9 \\ \mathcal{B}_1 &\cong \mathcal{B}_{11} \end{aligned}$$

These results were computed and verified with a computer, using the algorithms outlined in Chapter 4. The specific results of the computation can be found in Appendix B.

Chapter 6. Examples of $\mathcal{B}$-Model Deformation Invariance and Monodromy

Recall that an important property that yields $\mathcal{A}$-model isomorphisms is *deformation invariance*. Given polynomials W_1, W_2 with the same weights and a group G that preserves both W_1 and W_2, then at every point along a continuous path that transforms W_1 into W_2 we get isomorphic $\mathcal{A}$-models. We previously noted that deformation invariance doesn't exist in general for $\mathcal{B}$-models (see Example 2.34). In this chapter, we search for examples where $\mathcal{B}$-model deformation invariance does hold. To help us in our search, we use the algorithms developed in Chapter 4. In cases where $\mathcal{B}$-model deformation invariance does hold, we also exhibit examples of *monodromy*—cases where we get nontrivial automorphisms of a given $\mathcal{B}$-model by following a path from the polynomial to itself that goes around a point where the transformed polynomial is degenerate.

6.1 First Example

Theorem 6.1. *For $n \in \mathbb{N}$, $n > 3$, $2 \mid n$, $\mathcal{B}[x^n+y^n, SL(x^n+y^n)] \cong \mathcal{B}[x^n+y^n+(xy)^{n/2}, SL(x^n+y^n+(xy)^{n/2})]$.*

Proof. Consider $W_0 = x^n+y^n$ and $W_1 = x^n+y^n+x^{n/2}y^{n/2}$. Since $G^{max}_{W_0} = \langle \left(\frac{1}{n}, 0\right), \left(0, \frac{1}{n}\right)\rangle$, it follows that $\mathrm{SL}(W_0) = \langle \left(\frac{1}{n}, -\frac{1}{n}\right)\rangle$. We will check that $\mathrm{SL}(W_0)$ also fixes W_1. In multiplicative coordinates, the generator is $(e^{2\pi i(1/n)}, e^{2\pi i(-1/n)})$. Now notice $(e^{2\pi i(1/n)}x)^{n/2}(e^{2\pi i(-1/n)}y)^{n/2} = e^{\pi i}e^{-\pi i}x^{n/2}y^{n/2} = x^{n/2}y^{n/2}$. To see this another way, just consider $(\frac{n}{2}, \frac{n}{2}) \cdot (\frac{1}{n}, -\frac{1}{n}) = 2 - 2 = 0 \in \mathbb{Z}$. Therefore $\mathrm{SL}(W_0) = \mathrm{SL}(W_1)$.

Label $\mathcal{B}_0 = \mathcal{B}[W_0, \langle(\frac{1}{n}, -\frac{1}{n})\rangle]$ and $\mathcal{B}_1 = \mathcal{B}[W_1, \langle(\frac{1}{n}, -\frac{1}{n})\rangle]$. We'll begin by computing Milnor rings. We get that $\mathcal{Q}_{W_0} = \mathbb{C}[x,y]/(nx^{n-1}, ny^{n-1})$, with basis $\{1, y, \ldots, y^{n-2}\} \otimes \{1, x, \ldots, x^{n-2}\}$. Now we'll add in the group action g^*. So

$$g^*(x^a y^b) = 1 \cdot (e^{2\pi i/n}x)^a(e^{-2\pi i/n}y)^b = e^{(2\pi i/n)a-(2\pi i/n)b}x^a y^b.$$

In this case we want $\frac{a}{n} - \frac{b}{n} \in \mathbb{Z}$, so $n \mid (a-b)$ which is true if and only if $a = b$.

Now notice that $\mathcal{Q}_{W_1} = \mathbb{C}[x,y]/(nx^{n-1} + \frac{n}{2}x^{n/2-1}y^{n/2}, ny^{n-1} + \frac{n}{2}x^{n/2}y^{n/2-1})$. The basis is the same as $\mathcal{Q}_{W_0}$ with the further relations that $x^{n-1} = -\frac{1}{2}x^{n/2-1}y^{n/2}$ and $y^{n-1} = -\frac{1}{2}x^{n/2}y^{n/2-1}$. However, the group action is the same, and we obtain the same monomials in the orbifold Milnor ring as before.

For both $\mathcal{B}_0$ and $\mathcal{B}_1$, we obtain the state space

$$\{\lfloor 1; (0,0)\rceil, \lfloor xy; (0,0)\rceil, \ldots, \lfloor x^{n-2}y^{n-2}; (0,0)\rceil,$$
$$\lfloor 1; (\frac{1}{n}, -\frac{1}{n})\rceil, \lfloor 1; (\frac{2}{n}, -\frac{2}{n})\rceil, \ldots, \lfloor 1; (\frac{n-1}{n}, -\frac{(n-1)}{n})\rceil\}.$$

So $\dim(\mathcal{B}_0) = \dim(\mathcal{B}_1) = 2n-2$, as vector spaces.

We'll now compute degrees. For $0 \le a \le n-2$, the weighted degree of $x^a y^a$ is $\frac{a}{n} + \frac{a}{n} = \frac{2a}{n}$. So the degree of $\lfloor x^a y^a; (0,0)\rceil = \frac{4a}{n}$. Also, the degree of $\lfloor 1; (\frac{b}{n}, -\frac{b}{n})\rceil$ for $1 \le b \le n-1$ is $2 - \frac{4}{n} = \frac{2n-4}{n}$. In fact, these are equal when $4a = 2n-4$ if and only if $2a = n-2$. So we get the following:

$\lfloor 1; (0,0)\rceil$	Degree: 0
$\lfloor xy; (0,0)\rceil$	Degree: $\frac{4}{n}$
$\vdots$	
$\lfloor x^{n/2-1}y^{n/2-1}; (0,0)\rceil$	Degree: $\frac{2n-4}{n}$
$\lfloor 1; (\frac{1}{n}, -\frac{1}{n})\rceil$	Degree: $\frac{2n-4}{n}$
$\vdots$	
$\lfloor 1; (\frac{n-1}{n}, -\frac{(n-1)}{n})\rceil$	Degree: $\frac{2n-4}{n}$
$\lfloor x^{n/2}y^{n/2}; (0,0)\rceil$	Degree: $\frac{2n}{n} = 2$
$\vdots$	
$\lfloor x^{n-2}y^{n-2}; (0,0)\rceil$	Degree: $\frac{4n-8}{n}$

Products on $\mathcal{B}_0$. Here we have $a, b \in \{1, \ldots, n-1\}$.

$$\lfloor x^a y^a; (0,0) \rceil \star \lfloor 1; (\frac{b}{n}, -\frac{b}{n}) \rceil = 0.$$

$$\lfloor x^a y^a; (0,0) \rceil \star \lfloor x^b y^b; (0,0) \rceil = \begin{cases} \lfloor x^{a+b} y^{a+b}; (0,0) \rceil & \text{when } a+b \leq n-2, \\ 0 & \text{otherwise.} \end{cases}$$

$$\lfloor 1; (\frac{a}{n}, -\frac{a}{n}) \rceil \star \lfloor 1; (\frac{b}{n}, -\frac{b}{n}) \rceil = \begin{cases} n^2 \lfloor x^{n-2} y^{n-2}; (0,0) \rceil & \text{when } a+b = n, \\ 0 & \text{otherwise.} \end{cases}$$

Products on $\mathcal{B}_1$:

$$\lfloor x^a y^a; (0,0) \rceil \star \lfloor 1; (\frac{b}{n}, -\frac{b}{n}) \rceil = 0.$$

$$\lfloor x^a y^a; (0,0) \rceil \star \lfloor x^b y^b; (0,0) \rceil = \begin{cases} \lfloor x^{a+b} y^{a+b}; (0,0) \rceil & \text{when } a+b \leq n-2, \\ 0 & \text{otherwise.} \end{cases}$$

$$\lfloor 1; (\frac{a}{n}, -\frac{a}{n}) \rceil \star \lfloor 1; (\frac{b}{n}, -\frac{b}{n}) \rceil = \begin{cases} \frac{3}{4} n^2 \lfloor x^{n-2} y^{n-2}; (0,0) \rceil & \text{when } a+b = n, \\ 0 & \text{otherwise.} \end{cases}$$

Here we employed the relations $x^{n-1} = -\frac{1}{2} x^{n/2-1} y^{n/2}$ and $y^{n-1} = -\frac{1}{2} x^{n/2} y^{n/2-1}$ to line things up nicely. Note also that the products of $\mathcal{B}_0$ and $\mathcal{B}_1$ are nearly identical.

Nonzero/Nontrivial pairings on $\mathcal{B}_0$:

$$\langle \lfloor x^a y^a; (0,0) \rceil, \lfloor x^b y^b; (0,0) \rceil \rangle = \frac{1}{n^2}, \text{ given that } a+b = n-2.$$

$$\langle \lfloor 1; (\frac{a}{n}, -\frac{a}{n}) \rceil, \lfloor 1; (\frac{b}{n}, -\frac{b}{n}) \rceil \rangle = 1, \text{ given that } a+b = n.$$

Nonzero/Nontrivial pairings on $\mathcal{B}_1$:

$$\langle \lfloor x^a y^a; (0,0) \rceil, \lfloor x^b y^b; (0,0) \rceil \rangle = \frac{1}{\frac{3}{4} n^2} \text{ given that } a+b = n-2$$

$$\langle \lfloor 1; (\frac{a}{n}, -\frac{a}{n}) \rceil, \lfloor 1; (\frac{b}{n}, -\frac{b}{n}) \rceil \rangle = 1 \text{ given that } a+b = n$$

We're now ready to state and prove the isomorphism $\phi : \mathcal{B}_0 \to \mathcal{B}_1$. We define ϕ on basis elements. So

$$\begin{aligned}
\phi(\lfloor 1;(0,0)\rceil_{\mathcal{B}_0}) &= \lfloor 1;(0,0)\rceil_{\mathcal{B}_1},\\
\phi(\lfloor 1;(\frac{a}{n},-\frac{a}{n})\rceil_{\mathcal{B}_0}) &= \lfloor 1;(\frac{a}{n},-\frac{a}{n})\rceil_{\mathcal{B}_1},\\
\phi(\lfloor x^a y^a;(0,0)\rceil_{\mathcal{B}_0}) &= c^a\lfloor x^a y^a;(0,0)\rceil_{\mathcal{B}_1}.
\end{aligned}$$

Here we let $c \in \mathbb{C}$ satisfy $c^{n-2} = \frac{3}{4}$. In this case ϕ is a diagonal map with non-zero entries along the diagonal. So ϕ is a bijection. We'll now check that ϕ respects the product and and pairings.

First for the products.

$$\begin{aligned}
\phi(\lfloor x^a y^a;(0,0)\rceil_{\mathcal{B}_0} \star \lfloor x^b y^b;(0,0)\rceil_{\mathcal{B}_0}) &= \phi(\lfloor x^{a+b} y^{a+b};(0,0)\rceil_{\mathcal{B}_0})\\
&= c^{a+b}\lfloor x^{a+b} y^{a+b};(0,0)\rceil_{\mathcal{B}_1}\\
\phi(\lfloor x^a y^a;(0,0)\rceil_{\mathcal{B}_0}) \star \phi(\lfloor x^b y^b;(0,0)\rceil_{\mathcal{B}_0}) &= c^a\lfloor x^a y^a;(0,0)\rceil_{\mathcal{B}_1} \star c^b\lfloor x^b y^b;(0,0)\rceil_{\mathcal{B}_1}\\
&= c^{a+b}\lfloor x^{a+b} y^{a+b};(0,0)\rceil_{\mathcal{B}_1}
\end{aligned}$$

$$\begin{aligned}
\phi(\lfloor 1;(\frac{a}{n},-\frac{a}{n})\rceil_{\mathcal{B}_0} \star \lfloor 1;(\frac{b}{n},-\frac{b}{n})\rceil_{\mathcal{B}_0}) &= \phi(n^2\lfloor x^{n-2} y^{n-2};(0,0)\rceil_{\mathcal{B}_0})\\
&= n^2 c^{n-2}\lfloor x^{n-2} y^{n-2};(0,0)\rceil_{\mathcal{B}_1}\\
&= \frac{3}{4}n^2\lfloor x^{n-2} y^{n-2};(0,0)\rceil_{\mathcal{B}_1}\\
\phi(\lfloor 1;(\frac{a}{n},-\frac{a}{n})\rceil_{\mathcal{B}_0}) \star \phi(\lfloor 1;(\frac{b}{n},-\frac{b}{n})\rceil_{\mathcal{B}_0}) &= \lfloor 1;(\frac{a}{n},-\frac{a}{n})\rceil_{\mathcal{B}_1} \star \lfloor 1;(\frac{b}{n},-\frac{b}{n})\rceil_{\mathcal{B}_1}\\
&= \frac{3}{4}n^2\lfloor x^{n-2} y^{n-2};(0,0)\rceil_{\mathcal{B}_1}
\end{aligned}$$

Since these are all the non-trivial products, we have that ϕ respects multiplication.

Now for the pairings. Using values of a, b that make for nonzero pairings, we have

$$\begin{aligned}\langle\phi(\lfloor x^a y^a;(0,0)\rceil_{\mathcal{B}_0}),\phi(\lfloor x^b y^b;(0,0)\rceil_{\mathcal{B}_0})\rangle &= \langle c^a\lfloor x^a y^a;(0,0)\rceil_{\mathcal{B}_1}, c^b\lfloor x^b y^b;(0,0)\rceil_{\mathcal{B}_1}\rangle \\ &= c^{n-2}\langle\lfloor x^a y^a;(0,0)\rceil_{\mathcal{B}_1},\lfloor x^b y^b;(0,0)\rceil_{\mathcal{B}_1}\rangle \\ &= c^{n-2}\left(\frac{1}{\frac{3}{4}n^2}\right) = \frac{3}{4}\left(\frac{1}{\frac{3}{4}n^2}\right) = \frac{1}{n^2}.\end{aligned}$$

$$\langle\phi(\lfloor 1;(\frac{a}{n},-\frac{a}{n})\rceil_{\mathcal{B}_0}),\phi(\lfloor 1;(\frac{b}{n},-\frac{b}{n})\rceil_{\mathcal{B}_0})\rangle = \langle\lfloor 1;(\frac{a}{n},-\frac{a}{n})\rceil_{\mathcal{B}_1},\lfloor 1;(\frac{b}{n},-\frac{b}{n})\rceil_{\mathcal{B}_1}\rangle = 1.$$

We see that ϕ respects pairings. Therefore ϕ is indeed an isomorphism of graded Frobenius algebras. □

Theorem 6.2. *For $n \in \mathbb{N}$, $n > 3$, $2 \mid n$, and $\alpha \in \mathbb{C}$ with $\alpha \neq 2$, $\mathcal{B}[x^n + y^n, SL(x^n + y^n)] \cong \mathcal{B}[x^n + y^n + \alpha(xy)^{n/2}, SL(x^n + y^n + \alpha(xy)^{n/2})]$.*

Proof. By introducing a parameter $\alpha \in \mathbb{C}$, we can take this one step further. If $W_\alpha = x^n + y^n + \alpha(xy)^{n/2}$, we'll have $\nabla W_\alpha = (nx^{n-1} + \frac{\alpha n}{2}x^{n/2-1}y^{n/2}, ny^{n-1} + \frac{\alpha n}{2}x^{n/2}y^{n/2-1})$. We'll now check to see when W is nondegenerate. So we'll solve the system

$$\begin{aligned} nx^{n-1} + \frac{\alpha n}{2}x^{n/2-1}y^{n/2} &= 0, \\ ny^{n-1} + \frac{\alpha n}{2}x^{n/2}y^{n/2-1} &= 0. \end{aligned}$$

So $nx^{n-1} = -\alpha(\frac{n}{2})x^{n/2-1}y^{n/2} \Rightarrow x^{n/2} = -\frac{\alpha}{2}y^{n/2}$ when $x, y \neq 0$. Now substitute: $ny^{n-1} + \alpha(\frac{n}{2})(-\frac{\alpha}{2}y^{n/2})y^{n/2-1} = ny^{n-1} - \alpha^2(\frac{n}{4})y^{n-1} = (n - \alpha^2(\frac{n}{4}))y^{n-1} = 0$. So $n - \alpha^2(\frac{n}{4}) = 0 \Rightarrow n = \alpha^2(\frac{n}{4}) \Rightarrow 1 = \frac{\alpha^2}{4} \Rightarrow \alpha = \pm 2$. So W_α will be nondegenerate if and only if $\alpha \neq \pm 2$.

Let $\mathcal{B}_\alpha = \mathcal{B}[W_\alpha, \langle(\frac{1}{n}, -\frac{1}{n})\rangle]$. Many of our computations will be much the same as the work we did for $\mathcal{B}_1$ in the previous theorem. Note that the Milnor ring $\mathcal{Q}_{W_\alpha}$ has the same basis as $\mathcal{Q}_{W_1}$, but with relations $x^{n-1} = -\frac{\alpha}{2}x^{n/2-1}y^{n/2}$ and $y^{n-1} = -\frac{\alpha}{2}x^{n/2}y^{n/2-1}$. The $\mathcal{B}$-model state space is exactly the same.

Products on $\mathcal{B}_\alpha$:

$$\lfloor x^a y^a; (0,0) \rceil \star \lfloor 1; (\frac{b}{n}, -\frac{b}{n}) \rceil = 0.$$

$$\lfloor x^a y^a; (0,0) \rceil \star \lfloor x^b y^b; (0,0) \rceil = \begin{cases} \lfloor x^{a+b} y^{a+b}; (0,0) \rceil & \text{when } a+b \leq n-2, \\ 0 & \text{otherwise.} \end{cases}$$

$$\lfloor 1; (\frac{a}{n}, -\frac{a}{n}) \rceil \star \lfloor 1; (\frac{b}{n}, -\frac{b}{n}) \rceil = \begin{cases} \lfloor \gamma; (0,0) \rceil & \text{when } a+b = n, \\ 0 & \text{otherwise.} \end{cases}$$

We'll now compute what γ is. Let $g = (\frac{a}{n}, -\frac{a}{n})$, $h = (\frac{b}{n}, -\frac{b}{n})$. We then have $\mu_{g \cap h} = 1$, $\mu_{g+h} = (n-1)^2$ (this is the dimension of $\mathcal{Q}_{W_\alpha}$). $\text{Hess}((W_\alpha)_{|\text{fix} g \cap \text{fix} h}) = 1$. It remains to compute $\text{Hess}((W_\alpha)_{|\text{fix}(g+h)}) = \text{Hess}(W_\alpha)$. So we do some calculus:

$$\begin{aligned}
\frac{\partial W_\alpha}{\partial x} &= n x^{n-1} + \alpha \frac{n}{2} x^{n/2-1} y^{n/2} \\
\frac{\partial W_\alpha}{\partial y} &= n y^{n-1} + \alpha \frac{n}{2} x^{n/2} y^{n/2-1} \\
\frac{\partial^2 W_\alpha}{\partial x^2} &= n(n-1) x^{n-2} + \alpha (\frac{n}{2})(\frac{n}{2} - 1) x^{n/2-2} y^{n/2} \\
\frac{\partial^2 W_\alpha}{\partial x \partial y}, \frac{\partial^2 W_\alpha}{\partial y \partial x} &= \alpha (\frac{n}{2})^2 x^{n/2-1} y^{n/2-1} \\
\frac{\partial^2 W_\alpha}{\partial y^2} &= n(n-1) y^{n-2} + \alpha (\frac{n}{2})(\frac{n}{2} - 1) x^{n/2} y^{n/2-2}.
\end{aligned}$$

Now for a trip to the dentist. We will be using relations $x^{n-1} = -\frac{\alpha}{2} x^{n/2-1} y^{n/2}$ and $y^{n-1} = -\frac{\alpha}{2} x^{n/2} y^{n/2-1}$. Setting up the matrix of second partial derivatives and taking the determinant yields

$$\begin{aligned}
\text{Hess}(W_\alpha) = {} & [n(n-1) x^{n-2} + \alpha (\frac{n}{2})(\frac{n}{2} - 1) x^{n/2-2} y^{n/2}][n(n-1) y^{n-2} + \alpha (\frac{n}{2})(\frac{n}{2} - 1) x^{n/2} y^{n/2-2}] \\
& - [\alpha (\frac{n}{2})^2 x^{n/2-1} y^{n/2-1}]^2 \\
= {} & n^2 (n-1)^2 x^{n-2} y^{n-2} + \alpha n (n-1) (\frac{n}{2})(\frac{n}{2} - 1) x^{n/2+n-2} y^{n/2-2} \\
& + \alpha n (n-1) (\frac{n}{2})(\frac{n}{2} - 1) x^{n/2} y^{n/2+n-2} + \alpha^2 (\frac{n}{2})^2 (\frac{n}{2} - 1)^2 (xy)^{n-2} - \alpha^2 (\frac{n}{4})^4 (xy)^{n-2}.
\end{aligned}$$

Substitute:

$$\alpha n(n-1)(\frac{n}{2})(\frac{n}{2}-1)x^{n/2+n-2}y^{n/2-2} = \alpha n(n-1)(\frac{n}{2})(\frac{n}{2}-1)(-\frac{\alpha}{2}x^{n/2-1}y^{n/2})(x^{n/2-1}y^{n/2-2})$$
$$= -\frac{\alpha^2}{2}n(n-1)(\frac{n}{2})(\frac{n}{2}-1)(xy)^{n-2}.$$

Similarly, $\alpha n(n-1)(\frac{n}{2})(\frac{n}{2}-1)x^{n/2}y^{n/2+n-2} = -\frac{\alpha^2}{2}n(n-1)(\frac{n}{2})(\frac{n}{2}-1)(xy)^{n-2}$. We then obtain

$$[n^2(n-1)^2 - \alpha^2 n(n-1)(\frac{n}{2})(\frac{n}{2}-1) + \alpha^2(\frac{n}{2})^2(\frac{n}{2}-1)^2 - \alpha^2(\frac{n}{2})^4](xy)^{n-2}.$$

This is a mess, but it simplifies to $[n^2(n-1)^2 - \frac{\alpha^2}{4}(n^2(n-1)^2)]x^{n-2}y^{n-2}$. Now we can compute the value γ:

$$\gamma = \frac{1 \cdot [n^2(n-1)^2 - \frac{\alpha^2}{4}(n^2(n-1)^2)]x^{n-2}y^{n-2}}{(n-1)^2 \cdot 1} = \left(n^2 - \frac{\alpha^2}{4}(n^2)\right)x^{n-2}y^{n-2}.$$

We now investigate the pairing on $\mathcal{B}_\alpha$. As before, we see $\langle \lfloor 1; (\frac{a}{n}, -\frac{a}{n})\rceil, \lfloor 1; (\frac{b}{n}, -\frac{b}{n})\rceil \rangle = 1$ given that $a+b=n$. We'll now investigate $\langle \lfloor x^a y^a; (0,0)\rceil, \lfloor x^b y^b; (0,0)\rceil \rangle$ when $a+b=n-2$. We just need to compute $\langle x^a y^a, x^b y^b \rangle$ in $\mathcal{Q}_{W_\alpha}$. To that end,

$$(xy)^a(xy)^b = (xy)^{n-2} = \frac{\langle x^a y^a, x^b y^b \rangle}{(n-1)^2}[n^2(n-1)^2 - \frac{\alpha^2}{4}(n^2(n-1)^2)]x^{n-2}y^{n-2}$$
$$= \langle x^a y^a, x^b y^b \rangle (n^2 - \frac{\alpha^2}{4}(n^2))x^{n-2}y^{n-2}.$$

Equating coefficients yields $1 = \langle x^a y^a, x^b y^b \rangle (n^2 - \frac{\alpha^2}{4}(n^2))$, or $\langle x^a y^a, x^b y^b \rangle = \frac{1}{(n^2 - \frac{\alpha^2}{4}(n^2))} = \frac{1}{n^2(1-\alpha^2/4)}$. Therefore $\langle \lfloor x^a y^a; (0,0)\rceil, \lfloor x^b y^b; (0,0)\rceil \rangle = \frac{1}{n^2(1-\alpha^2/4)}$.

We're now ready to give the isomorphism. As before, we'll define the isomorphism $\phi : \mathcal{B}_0 \to \mathcal{B}_\alpha$ on basis elements. So

$$\begin{aligned}
\phi(\lfloor 1; (0,0)\rceil_{\mathcal{B}_0}) &= \lfloor 1; (0,0)\rceil_{\mathcal{B}_\alpha}, \\
\phi(\lfloor 1; (\frac{a}{n}, -\frac{a}{n})\rceil_{\mathcal{B}_0}) &= \lfloor 1; (\frac{a}{n}, -\frac{a}{n})\rceil_{\mathcal{B}_\alpha}, \\
\phi(\lfloor x^a y^a; (0,0)\rceil_{\mathcal{B}_0}) &= c^a \lfloor x^a y^a; (0,0)\rceil_{\mathcal{B}_\alpha}.
\end{aligned}$$

Here we let $c \in \mathbb{C}$ satisfy $c^{n-2} = 1 - \frac{\alpha^2}{4}$. Notice that this ϕ is really an extension of the map we had before. Again ϕ is a diagonal map with non-zero entries along the diagonal. So ϕ is a bijection. We can quickly check, as we did before, that ϕ respects the products and pairings.

First for the products.

$$\begin{aligned}
\phi(\lfloor x^a y^a; (0,0)\rceil_{B_0} \star \lfloor x^b y^b; (0,0)\rceil_{\mathcal{B}_0}) &= \phi(\lfloor x^{a+b} y^{a+b}; (0,0)\rceil_{\mathcal{B}_0}) \\
&= c^{a+b} \lfloor x^{a+b} y^{a+b}; (0,0)\rceil_{\mathcal{B}_\alpha} \\
\phi(\lfloor x^a y^a; (0,0)\rceil_{B_0}) \star \phi(\lfloor x^b y^b; (0,0)\rceil_{B_0}) &= c^a \lfloor x^a y^a; (0,0)\rceil_{\mathcal{B}_\alpha} \star c^b \lfloor x^b y^b; (0,0)\rceil_{\mathcal{B}_\alpha} \\
&= c^{a+b} \lfloor x^{a+b} y^{a+b}; (0,0)\rceil_{\mathcal{B}_\alpha}
\end{aligned}$$

$$\begin{aligned}
\phi(\lfloor 1; (\frac{a}{n}, -\frac{a}{n})\rceil_{\mathcal{B}_0} \star \lfloor 1; (\frac{b}{n}, -\frac{b}{n})\rceil_{\mathcal{B}_0}) &= \phi(n^2 \lfloor x^{n-2} y^{n-2}; (0,0)\rceil_{\mathcal{B}_0}) \\
&= n^2 c^{n-2} \lfloor x^{n-2} y^{n-2}; (0,0)\rceil_{\mathcal{B}_\alpha} \\
&= (1 - \frac{\alpha^2}{4}) n^2 \lfloor x^{n-2} y^{n-2}; (0,0)\rceil_{\mathcal{B}_\alpha} \\
\phi(\lfloor 1; (\frac{a}{n}, -\frac{a}{n})\rceil_{\mathcal{B}_0}) \star \phi(\lfloor 1; (\frac{b}{n}, -\frac{b}{n})\rceil_{\mathcal{B}_0}) &= \lfloor 1; (\frac{a}{n}, -\frac{a}{n})\rceil_{\mathcal{B}_\alpha} \star \lfloor 1; (\frac{b}{n}, -\frac{b}{n})\rceil_{\mathcal{B}_\alpha} \\
&= (1 - \frac{\alpha^2}{4}) n^2 \lfloor x^{n-2} y^{n-2}; (0,0)\rceil_{\mathcal{B}_\alpha}
\end{aligned}$$

Since these are all the non-trivial products, we have that ϕ respects multiplication.

Now for the pairings. For values of a, b that yield nontrivial parings, we have

$$\begin{aligned}
\langle \phi(\lfloor x^a y^a; (0,0)\rceil_{\mathcal{B}_0}), \phi(\lfloor x^b y^b; (0,0)\rceil_{\mathcal{B}_0})\rangle &= \langle c^a \lfloor x^a y^a; (0,0)\rceil_{\mathcal{B}_\alpha}, c^b \lfloor x^b y^b; (0,0)\rceil_{\mathcal{B}_\alpha}\rangle \\
&= c^{n-2}\langle \lfloor x^a y^a; (0,0)\rceil_{\mathcal{B}_\alpha}, \lfloor x^b y^b; (0,0)\rceil_{\mathcal{B}_\alpha}\rangle \\
&= c^{n-2}\left(\frac{1}{(1-\frac{\alpha^2}{4})n^2}\right) \\
&= (1-\frac{\alpha^2}{4})\left(\frac{1}{(1-\frac{\alpha^2}{4})n^2}\right) = \frac{1}{n^2}.
\end{aligned}$$

$$\langle \phi(\lfloor 1; (\frac{a}{n}, -\frac{a}{n})\rceil_{\mathcal{B}_0}), \phi(\lfloor 1; (\frac{b}{n}, -\frac{b}{n})\rceil_{\mathcal{B}_0})\rangle = \langle \lfloor 1; (\frac{a}{n}, -\frac{a}{n})\rceil_{\mathcal{B}_\alpha}, \lfloor 1; (\frac{b}{n}, -\frac{b}{n})\rceil_{\mathcal{B}_\alpha}\rangle = 1.$$

Thus ϕ respects pairings. Therefore ϕ is an isomorphism of graded Frobenius algebras. □

As an aside, notice that by setting $\alpha = 0$ we get $c^{n-2} = 1$ in the map. This gives us $n-2$ automorphisms of $\mathcal{B}_0 \to \mathcal{B}_0$. We further note that we can introduce more constants into the diagonal map. For every pair a, b such that $a + b = n$, we can send $\lfloor 1; (\frac{a}{n}, -\frac{a}{n})\rceil_{\mathcal{B}_1} \mapsto d_a \lfloor 1; (\frac{a}{n}, -\frac{a}{n})\rceil_{B_\alpha}$ and $\lfloor 1; (\frac{b}{n}, -\frac{b}{n})\rceil_{B_1} \mapsto d_b \lfloor 1; (\frac{b}{n}, -\frac{b}{n})\rceil_{B_\alpha}$ where $d_a \cdot d_b = 1$. This affords many different ways to construct our diagonal isomorphism.

6.1.1 A Case of Monodromy. We will consider the above example, still using the same map but with all the d's equal to 1. We will pick an initial value of $c^{n-2} = 1 - \frac{\alpha^2}{4}$ and a path in the complex plane to see how the isomorphism varies as we go around the "bad points" $\alpha = \pm 2$ where W_α is degenerate.

So first consider α as the path $\alpha(t) = 2(e^{it} - 1)$ for $t \in [0, 2\pi]$. This path is the circle of radius 2 centered at -2. The loop starts and ends at the origin, tracing the circle counter-clockwise (i.e., it is positively oriented). We now let $c^{n-2} = 1 - \frac{(\alpha(t))^2}{4} = 1 - (e^{it} - 1)^2 = e^{it}(2 - e^{it})$. Choose $c = e^{it/(n-2)}(2 - e^{it})^{1/(n-2)}$. Notice that at $t = 0$, $c = e^0(2-1)^{1/(n-2)} = 1$. At $t = 2\pi$, we get $c = e^{2\pi i/(n-2)}(2-1)^{1/(n-2)} = e^{2\pi i/(n-2)}$. If we repeat this process, we can go around the point -2 a total of $n-2$ times before we get back to $c = 1$. If we had chosen to go backwards (i.e. $\alpha(t) = 2(e^{-it} - 1)$), we would get the results $c = 1, e^{2\pi i(n-1)/(n-2)}, \ldots, e^{2\pi i/(n-2)}$ (since these are equivalent to $c = 1, e^{-2\pi i/(n-2)}, \ldots, e^{-2\pi i(n-1)/(n-2)}$).

Going around the point -2 on this circular loop generates a cyclic group of order $n-2$. What happens if we go around the point 2? Unfortunately, nothing much changes. For a positively-oriented circle, we get $\alpha(t) = -2(e^{it} - 1)$. But then $c^{n-2} = e^{it}(2 - e^{it})$ as before.

Borrowing some intuition from algebraic topology and the theory of multi-valued functions, for any loop in $\mathbb{C} - \{\pm 2\}$ based at 0 that doesn't go around a "bad point" we can define a branch cut that misses this loop. Hence every such loop is nullhomotopic, and we can conclude that homotopic paths will yield the same results for the monodromy in this example.

6.2 Second Example

Let $n \in \mathbb{N}$, $n \geq 2$. For the weight system $\left(\frac{1}{2}, \frac{1}{2n}\right)$, we have the following diagrams (where arrows represent isomorphisms, with the direction showing the way the map was constructed):

For all $n \geq 2$,

$$\mathcal{B}[x^2 + y^{2n}, \{0\}] \longrightarrow \mathcal{B}[x^2 + xy^n + y^{2n}, \{0\}] \longleftarrow \mathcal{B}[x^2 + xy^n, \{0\}]$$

In the special case $n = 2$ we get

$$\begin{tikzcd}
\mathcal{B}[x^2 + y^4, \langle(\frac{1}{2}, \frac{1}{2})\rangle] & & \\
\mathcal{B}[x^2 + y^4, \{0\}] \arrow[u] \arrow[r] & \mathcal{B}[x^2 + xy^2 + y^4, \{0\}] & \mathcal{B}[x^2 + xy^2, \{0\}] \arrow[l]
\end{tikzcd}$$

If n is odd,

$$\mathcal{B}[x^2 + y^{2n}, \langle(\tfrac{1}{2}, \tfrac{1}{2})\rangle] \longrightarrow \mathcal{B}[x^2 + xy^n + y^{2n}, \langle(\tfrac{1}{2}, \tfrac{1}{2})\rangle] \longleftarrow \mathcal{B}[x^2 + xy^n, \langle(\tfrac{1}{2}, \tfrac{1}{2})\rangle]$$

These are all the isomorphisms we get between $\mathcal{B}$-models built with polynomials in this weight system.

We will prove this result in the course of the next several theorems.

Theorem 6.3. *Let* $n \in \mathbb{N}_{\geq 2}$. $\mathcal{B}[x^2 + y^{2n}, \{0\}] \cong \mathcal{B}[x^2 + xy^n + y^{2n}, \{0\}] \cong \mathcal{B}[x^2 + xy^n, \{0\}]$.

Proof. The proof of this theorem follows by the following two lemmas.

Lemma 6.4. *If* $W_\alpha = x^2 + \alpha xy^n + y^{2n}$ *for* $\alpha \in \mathbb{C} - \{\pm 2\}$ *and* $n \in \mathbb{N}_{\geq 2}$*, then* $\mathcal{B}[W_0, \{0\}] \cong \mathcal{B}[W_\alpha, \{0\}]$.

Proof. Let $\mathcal{B}_\alpha = \mathcal{B}[W_\alpha, \{0\}]$. So $\mathcal{B}_0 = \mathbb{C}[x, y]/(2x, 2ny^{2n-1}) = \text{span}_\mathbb{C}\{1, y, \dots, y^{2n-2}\}$, which has dimension $2n - 1$. We also see that $\mathcal{B}_\alpha = \mathbb{C}[x, y]/(2x + \alpha y^n, n\alpha xy^{n-1} + 2ny^{2n-1}) = \text{span}_\mathbb{C}\{1, y, \dots, y^{2n-2}\}$, which has dimension $2n - 1$. $\mathcal{B}_\alpha$ has further relations $x = -\frac{\alpha}{2}y^n$ and $xy^{n-1} = -\frac{2}{\alpha}y^{2n-1}$. In each case $\deg(y^a) = \frac{a}{2n}$. The only possible map that can work in this case is diagonal.

When is W_α nondegenerate? We'll solve $2x + \alpha y^n = 0$, $n\alpha xy^{n-1} + 2ny^{2n-1} = 0$. Solving for x yields $x = -\frac{\alpha}{2}y^n$. Substitute: $-\frac{n\alpha^2}{2}y^{2n-1} + 2ny^{2n-1} = 0 \Rightarrow y^{2n-1}(-\frac{n\alpha^2}{2} + 2n) = 0$. So either $y = 0$, or $-\frac{n\alpha^2}{2} + 2n = 0 \Rightarrow \alpha^2 = 4 \Rightarrow \alpha = \pm 2$. So W_α is degenerate for $\alpha = \pm 2$.

Since we are working with unorbifolded $\mathcal{B}$-models, the product structure is relatively simple. We have that $y^a \star y^b = y^{a+b}$ if $a + b \leq 2n - 2$, and is equal to 0 otherwise.

To understand the pairing structure, we'll now compute the Hessian of W_α. We first compute $\frac{\partial^2 W_\alpha}{\partial x^2} = 2$, $\frac{\partial^2 W_\alpha}{\partial x \partial y} = \frac{\partial^2 W_\alpha}{\partial y \partial x} = n\alpha y^{n-1}$, and $\frac{\partial^2 W_\alpha}{\partial y^2} = n(n-1)\alpha xy^{n-2} + 2n(2n-1)y^{2n-2}$. So

$$
\begin{aligned}
\text{Hess}(W_\alpha) &= 2[n(n-1)\alpha xy^{n-2} + 2n(2n-1)y^{2n-2}] - (n\alpha y^{n-1})^2 \\
&= 2[(n^2 - n)\alpha(-\frac{\alpha}{2}y^n)y^{n-2} + (4n^2 - 2n)y^{2n-2}] - n^2\alpha^2 y^{2n-2} \text{ substituting for } x, \\
&= -\alpha^2(n^2 - n)y^{2n-2} + (8n^2 - 4n)y^{2n-2} - n^2\alpha^2 y^{2n-2} \\
&= [(-2\alpha^2 + 8)n^2 + (\alpha^2 - 4)n]y^{2n-2}.
\end{aligned}
$$

Plugging in 0 for α yields $\mathrm{Hess}(W_0) = (8n^2 - 4n)y^{2n-2}$. On $\mathcal{B}_0$ and $\mathcal{B}_\alpha$ we obtain a nonzero value for the paring $\langle y^a, y^b \rangle$ precisely when $a + b = 2n - 2$. For nonzero $\mathcal{B}_0$ pairings, we obtain

$$y^{2n-2} = \frac{\langle y^a, y^b \rangle}{2n-1}(8n^2 - 4n)y^{2n-2} \Rightarrow \frac{2n-1}{8n^2-4n} = \langle y^a, y^b \rangle \Rightarrow \frac{1}{4n} = \langle y^a, y^b \rangle.$$

For nonzero $\mathcal{B}_\alpha$ pairings, we obtain

$$y^{2n-2} = \frac{\langle y^a, y^b \rangle}{2n-1}[(-2\alpha^2 + 8) + (\alpha^2 - 4)n]y^{2n-2} \Rightarrow \langle y^a, y^b \rangle = \frac{2n-1}{(-2\alpha^2+8)n^2 + (\alpha^2-4)n}.$$

We'll now construct a map $\phi : \mathcal{B}_0 \to \mathcal{B}_\alpha$, defined by $\phi = \mathrm{diag}[1, c, c^2, \ldots, c^{2n-2}]$. We'll state what value c should be in just a moment. First we'll check that ϕ preserves the product structure.

$$\begin{aligned} \phi(y^a \star y^b) &= \phi(y^{a+b}) = c^{a+b}y^{a+b} \\ \phi(y^a \star y^b) &= \phi(y^a) \star \phi(y^b) = c^a y^a \star c^b y^b = c^{a+b}y^{a+b}. \end{aligned}$$

For ϕ to preserve pairings, we require (assuming $a + b = 2n - 2$):

$$\begin{aligned} \frac{1}{4n} = \langle y^a, y^b \rangle_{\mathcal{B}_0} &= \langle \phi(y^a), \phi(y^b) \rangle_{\mathcal{B}_\alpha} \\ &= \langle c^a y^a, c^b y^b \rangle_{\mathcal{B}_\alpha} = c^{2n-2}\langle y^a, y^b \rangle_{\mathcal{B}_\alpha} = c^{2n-2}\left(\frac{2n-1}{(-2\alpha^2+8)n^2 + (\alpha^2-4)n}\right). \end{aligned}$$

Therefore c is any complex number satisfying $c^{2n-2} = \frac{(-2\alpha^2+8)n^2+(\alpha^2-4)n}{4n(2n-1)} = \frac{(-2\alpha^2+8)n+(\alpha^2-4)}{4(2n-1)} = \frac{-2(\alpha^2-4)n+(\alpha^2-4)}{4(2n-1)} = \frac{-(2n-1)(\alpha^2-4)}{4(2n-1)} = -\frac{\alpha^2-4}{4}$. This gives us an isomorphism of graded Frobenius algebras.

Now we'll look for some monodromy in this example. We'll first circle around the point -2 by letting $\alpha = 2(e^{it}-1)$. Then

$$\begin{aligned} c^{2n-2} &= -\frac{1}{4}[(2(e^{it}-1))^2-4] \\ &= -\frac{1}{4}[4(e^{2it}-2e^{it}+1)-4] \\ &= -(e^{2it}-2e^{it}+1)+1 \\ &= -e^{2it}+2e^{it} \\ &= e^{it}(2-e^{it}). \end{aligned}$$

So $c = e^{it/(2n-2)}\left(2-e^{it}\right)^{1/(2n-2)}$. Starting at $t=0$, we get $c = e^0\left(2-1\right)^{1/(2n-2)} = (1)^{1/(2n-2)} = 1$, say. Increasing t by 2π yields $c = e^{2\pi i/(2n-2)}$, and so on. We cycle through the $2n-2$ roots of unity, each of which yields an automorphism of $\mathcal{B}_0$.

Letting $\alpha = -2(e^{it}-1)$ to go around the point 2 still yields $c^{2n-2} = e^{it}(2-e^{it})$, so the monodromy is the same as before. $\square$

Lemma 6.5. *If $W_\alpha = x^2 + xy^n + \alpha y^{2n}$ for $\alpha \in \mathbb{C} - \{\frac{1}{4}\}$ and $n \in \mathbb{N}_{\geq 2}$, then $\mathcal{B}[W_0,\{0\}] \cong \mathcal{B}[W_\alpha,\{0\}]$.*

Proof. If $\mathcal{B}_\alpha = \mathcal{B}[W_\alpha,\{0\}]$, then $\mathcal{B}_0 = \mathbb{C}[x,y]/(2x+y^n, nxy^{n-1}) = \mathrm{span}_{\mathbb{C}}\{1,y,\ldots,y^{2n-2}\}$ with relation $x = -\frac{1}{2}y^n$. Also, $\mathcal{B}_\alpha = \mathbb{C}[x,y]/(2x+y^n, nxy^{n-1}+2n\alpha y^{2n-1}) = \mathrm{span}_{\mathbb{C}}\{1,y,\ldots,y^{2n-2}\}$ with relations $x = -\frac{1}{2}y^n$ and $xy^{n-1} = -2\alpha y^{2n-1}$.

To find when W_α is nondegenerate, we solve the equations $2x+y^n = 0$, $nxy^{n-1} + 2n\alpha y^{2n-1} = 0$. We see that $x = -\frac{1}{2}y^n$. Substituting yields $n(-\frac{1}{2}y^n)y^{n-1} + 2n\alpha y^{2n-1} = 0$, so $(-\frac{n}{2}+2n\alpha)y^{2n-1} = 0$. Hence $-\frac{n}{2}+2n\alpha = 0$, which yields $\alpha = \frac{1}{4}$. This is our point of nondegeneracy.

The product structure behaves the same as the example in Lemma 6.4. So we proceed to compute $\frac{\partial^2 W_\alpha}{\partial x^2} = 2$, $\frac{\partial^2 W_\alpha}{\partial x \partial y} = \frac{\partial^2 W_\alpha}{\partial y \partial x} = ny^{n-1}$, and $\frac{\partial^2 W_\alpha}{\partial y^2} = n(n-1)xy^{n-2} + 2n(2n-1)\alpha y^{2n-2}$.

Therefore

$$\begin{aligned}
\mathrm{Hess}(W_\alpha) &= 2[n(n-1)xy^{n-2} + 2n(2n-1)\alpha y^{2n-2}] - (ny^{n-1})^2 \\
&= 2[-\frac{1}{2}n(n-1)y^{2n-2} + 2n(2n-1)\alpha y^{2n-2}] - n^2y^{2n-2} \\
&= [-n(n-1) + 4n(2n-1)\alpha - n^2]y^{2n-2} \\
&= [(8\alpha-2)n^2 + (-4\alpha+1)n]y^{2n-2}.
\end{aligned}$$

For $\mathcal{B}_0$ pairings, we find that

$$y^{2n-2} = \frac{\langle y^a, y^b\rangle}{2n-1}(-2n^2+n)y^{2n-2} \Rightarrow \langle y^a, y^b\rangle = \frac{2n-1}{-2n^2+n} = -\frac{1}{n}.$$

For $\mathcal{B}_\alpha$ pairings, we find that

$$y^{2n-2} = \frac{\langle y^a, y^b\rangle}{2n-1}((8\alpha-2)n^2 + (-4\alpha+1)n)y^{2n-2} \Rightarrow \langle y^a, y^b\rangle = \frac{2n-1}{(8\alpha-2)n^2 + (-4\alpha+1)n}.$$

(Noting, of course, that we use $a+b = 2n-2$). To define $\phi : \mathcal{B}_0 \to \mathcal{B}_\alpha$ that preserves the pairing structure, we'll need

$$\begin{aligned}
-\frac{1}{n} = \langle y^a, y^b\rangle_{\mathcal{B}_0} &= \langle c^a y^a, c^b y^b\rangle_{\mathcal{B}_\alpha} \\
&= c^{2n-2}\left(\frac{2n-1}{(8\alpha-2)n^2 + (-4\alpha+1)n}\right) \Rightarrow c^{2n-2} = -\frac{(8\alpha-2)n^2 + (-4\alpha+1)n}{n(2n-1)}.
\end{aligned}$$

Hence $c^{2n-2} = -\frac{(8\alpha-2)n+(-4\alpha+1)}{2n-1} = -\frac{(2n-1)(4\alpha-1)}{2n-1} = -4\alpha+1$. The map $\phi = \mathrm{diag}[1, c, c^2, \ldots, c^{2n-2}]$, which we checked before, gives us an isomorphism of graded Frobenius algebras.

Once again, we'll look at the monodromy in this example. Setting $\alpha = -(e^{it}-1)$ yields

$$c^{2n-2} = -4(-(e^{it}-1)) + 1 = 4e^{it} - 3 = e^{it}(4 - 3e^{-it}).$$

Then we can write $c = e^{it/(2n-2)} \left(4 - 3e^{-it}\right)^{1/(2n-2)}$. At $t = 0$, we get $c = e^0 \left(4 - 3\right)^{1/(2n-2)} = (1)^{1/(2n-2)}$, which we can choose to be just 1. Increasing t by 2π yields $c = e^{2\pi i/(2n-2)}$, and so on. Again we cycle through the $2n-2$ roots of unity, each of which yields an automorphism of $\mathcal{B}_0$. $\square$

Hence $\mathcal{B}[x^2 + y^{2n}, \{0\}] \cong \mathcal{B}[x^2 + xy^n + y^{2n}, \{0\}] \cong \mathcal{B}[x^2 + xy^n, \{0\}]$, which is what we wanted to show. The ambitious reader can investigate $W_{\alpha,\beta} = x^2 + \alpha xy^n + \beta y^{2n}$ with its corresponding $\mathcal{B}$-model, and compute the monodromy. Also, as a consequence of this result, we also get some isomorphisms of $\mathcal{A}$-models that don't follow from Group-Weights. We will catalog these shortly in Corollary 6.9. $\square$

Now we'll examine the special case $n = 2$. Besides the unorbifolded $\mathcal{B}$-models that we've already looked at, there is one more that exists for this weight system: $\mathcal{B}[x^2 + y^4, \langle(\frac{1}{2}, \frac{1}{2})\rangle]$. But this last isomorphism is a consequence of the Group-Weights theorem, as cataloged in Theorem 3.9 and Example 3.14

Theorem 6.6. *Let $n \in \mathbb{N}_{>2}$ be odd.* $\mathcal{B}[x^2 + y^{2n}, \langle(\frac{1}{2}, \frac{1}{2})\rangle] \cong \mathcal{B}[x^2 + xy^n + y^{2n}, \langle(\frac{1}{2}, \frac{1}{2})\rangle] \cong \mathcal{B}[x^2 + xy^n, \langle(\frac{1}{2}, \frac{1}{2})\rangle]$.

Proof. The proof of this theorem follows by the following two lemmas.

Lemma 6.7. *If $W_\alpha = x^2 + \alpha xy^n + y^{2n}$ for $\alpha \in \mathbb{C} - \{\pm 2\}$ and $n \in \mathbb{N}_{>2}$ is odd, then* $\mathcal{B}[W_0, \langle(\frac{1}{2}, \frac{1}{2})\rangle] \cong \mathcal{B}[W_\alpha, \langle(\frac{1}{2}, \frac{1}{2})\rangle]$.

Proof. Again we'll let $\mathcal{B}_\alpha = \mathcal{B}[W_\alpha, \langle(\frac{1}{2}, \frac{1}{2})\rangle]$. By our work in the previous lemmas, we know that the Milnor ring for W_α is $\text{span}_{\mathbb{C}}\{1, y, \ldots, y^{2n-2}\}$. To compute a basis for $\mathcal{B}_\alpha$, we need to know which basis elements are invariant under $g = (\frac{1}{2}, \frac{1}{2})$. We can see readily that these will precisely be the even powers of y. We also get the 1 paired up with both group elements, so obtain the following state space:

$$\begin{array}{ll}
\lfloor 1;(0,0)\rceil & \text{Degree: } 0 \\
\lfloor y^2;(0,0)\rceil & \text{Degree: } \frac{1}{n} \\
\vdots & \\
\lfloor y^{n-1};(0,0)\rceil & \text{Degree: } \frac{\left(\frac{n-1}{2}\right)}{n} \\
\lfloor 1;(\frac{1}{2},\frac{1}{2})\rceil & \text{Degree: } \frac{\left(\frac{n-1}{2}\right)}{n} \\
\lfloor y^{n+1};(0,0)\rceil & \text{Degree: } \frac{\left(\frac{n+1}{2}\right)}{n} \\
\vdots & \\
\lfloor y^{2n-2};(0,0)\rceil & \text{Degree: } \frac{n-1}{n}
\end{array}$$

The pairings on $\mathcal{B}_\alpha$ work the same as the unorbifolded case, with also $\langle \lfloor 1;(\frac{1}{2},\frac{1}{2})\rceil, \lfloor 1;(\frac{1}{2},\frac{1}{2})\rceil \rangle = 1$. For products on $\mathcal{B}_0$ we obtain $\lfloor y^a;(0,0)\rceil \star \lfloor y^b;(0,0)\rceil = \lfloor y^{a+b};(0,0)\rceil$ provided $a+b \leq 2n-2$, and $\lfloor 1;(\frac{1}{2},\frac{1}{2})\rceil \star \lfloor 1;(\frac{1}{2},\frac{1}{2})\rceil = \gamma \lfloor y^{2n-2};(0,0)\rceil$. Using the Hessians we have computed before, γ is given by $\frac{8n^2-4n}{2n-1} = 4n$.

For $\mathcal{B}_\alpha$ the same results hold, except $\lfloor 1;(\frac{1}{2},\frac{1}{2})\rceil \star \lfloor 1;(\frac{1}{2},\frac{1}{2})\rceil = \gamma \lfloor y^{2n-2};(0,0)\rceil$ where $\gamma = \frac{1}{2n-1}[(-2\alpha^2+8)n^2 + (\alpha^2-4)n]$. We want to define $\phi : \mathcal{B}_0 \to \mathcal{B}_\alpha$ by

$$\begin{aligned}
\phi : \lfloor y^a;(0,0)\rceil &\mapsto c^{a/2} \lfloor y^a;(0,0)\rceil \text{ for } a \in \{0,2,\ldots,2n-2\}, \\
\lfloor 1;(\frac{1}{2},\frac{1}{2})\rceil &\mapsto \lfloor 1;(\frac{1}{2},\frac{1}{2})\rceil.
\end{aligned}$$

To check that this map works, we need only verify

$$\begin{aligned}
\phi(\lfloor 1;(\frac{1}{2},\frac{1}{2})\rceil \star \lfloor 1;(\frac{1}{2},\frac{1}{2})\rceil) &= \phi(4n\lfloor y^{2n-2};(0,0)\rceil) = 4nc^{n-1}\lfloor y^{2n-2};(0,0)\rceil, \\
\phi(\lfloor 1;(\frac{1}{2},\frac{1}{2})\rceil) \star \phi(\lfloor 1;(\frac{1}{2},\frac{1}{2})\rceil) &= \lfloor 1;(\frac{1}{2},\frac{1}{2})\rceil \star \lfloor 1;(\frac{1}{2},\frac{1}{2})\rceil \\
&= \frac{(-2\alpha^2+8)n^2+(\alpha^2-4)n}{2n-1} \lfloor y^{2n-2};(0,0)\rceil.
\end{aligned}$$

So we choose $c^{n-1} = \frac{(-2\alpha^2+8)n+(\alpha^2-4)}{4(2n-1)} = -\frac{\alpha^2-4}{4}$. This gives us an isomorphism of graded Frobenius algebras.

Now we'll investigate the monodromy. Our bad points are ± 2. Surprisingly we have arrived at the same equation for monodromy as in Lemma 6.4, except we have c^{n-1} instead of c^{2n-2} as before. However, the power of root that we take is of little consequence. The result is the same, except that we find that our automorphisms form a cyclic group of order $n-1$. And, once again, it doesn't matter which circle we traverse—the result is the same. $\square$

Lemma 6.8. *Let* $W_\alpha = x^2+xy^n+\alpha y^{2n}$ *for* $\alpha \in \mathbb{C}-\{\frac{1}{4}\}$ *and* $n \in \mathbb{N}_{>2}$ *be odd.* $\mathcal{B}[W_0, \langle(\frac{1}{2}, \frac{1}{2})\rangle] \cong \mathcal{B}[W_\alpha, \langle(\frac{1}{2}, \frac{1}{2})\rangle]$.

Proof. Again we let $\mathcal{B}_\alpha = \mathcal{B}[W_\alpha, \langle(\frac{1}{2}, \frac{1}{2})\rangle]$. At this point we can rely on many of the computations we have already done. The Milnor rings and state spaces look exactly like those of Lemma 6.7. The pairings are the same as those computed in Lemma 6.5.

Using previous results, we compute the product structure for $\mathcal{B}_0$. We obtain

$$\begin{aligned}\lfloor y^a; (0,0)\rceil \star \lfloor y^b; (0,0)\rceil &= \lfloor y^{a+b}; (0,0)\rceil \text{ if } a+b \leq 2n-2,\\ \lfloor 1; (\frac{1}{2}, \frac{1}{2})\rceil \star \lfloor 1; (\frac{1}{2}, \frac{1}{2})\rceil &= \gamma \lfloor y^{2n-2}; (0,0)\rceil\end{aligned}$$

where $\gamma = \frac{-2n^2+n}{2n-1} = -n$. The products on $\mathcal{B}_\alpha$ are similar, except in that case $\gamma = \frac{(8\alpha-2)n^2+(-4\alpha+1)n}{2n-1}$.

We define our map $\phi : \mathcal{B}_0 \to \mathcal{B}_\alpha$ by

$$\begin{aligned}\phi : \lfloor y^a; (0,0)\rceil &\mapsto c^{a/2}\lfloor y^a; (0,0)\rceil \text{ for } a \in \{0, 2, \ldots, 2n-2\},\\ \lfloor 1; (\frac{1}{2}, \frac{1}{2})\rceil &\mapsto \lfloor 1; (\frac{1}{2}, \frac{1}{2})\rceil.\end{aligned}$$

As we've seen before, we need only set $c^{n-1} = -\frac{(8\alpha-2)n^2+(-4\alpha+1)n}{2n-1}$ to obtain an isomorphism of graded Frobenius algebras. Our monodromy example works almost exactly the same as what we did in Lemma 6.5. Here we get a cyclic group of order $n-1$ instead of order $2n-2$. □

To finish this result, we need to make a brief comment on why these are all the isomorphisms that exist between $\mathcal{B}$-models within this weight system. First note that the unorbifolded models have dimension $2n-1$, whereas the orbifolded models have dimension $n+1$. These match precisely when $2n-1=n+1$, which is true if and only if $n=2$. In the special case $n=2$ we get an isomorphism between an unorbifolded model and an orbifolded model.

When n is odd, each of the polynomials W are fixed by the group element $(\frac{1}{2},\frac{1}{2})$. When n is even, this is no longer the case. Only $W=x^2+y^{2n}$ is fixed by that group element, so there is only one orbifolded model. Since there are only three polynomials in this weight system, we have determined all possible isomorphisms. □

Corollary 6.9 ($\mathcal{A}$-model isomorphisms)**.** *For $n\in\mathbb{N}_{\geq 2}$, we have that*

$$\mathcal{A}[x^2+y^{2n},\langle(\frac{1}{2},0),(0,\frac{1}{2n})\rangle]\cong\mathcal{A}[x^2y+y^n,\langle(-\frac{1}{2n},\frac{1}{n})\rangle].$$

In the special case $n=2$ we obtain

$$\mathcal{A}[x^2+y^4,\langle(\frac{1}{2},\frac{1}{4})\rangle]\cong\mathcal{A}[x^2+y^4,\langle(\frac{1}{2},0),(0,\frac{1}{4})\rangle]\cong\mathcal{A}[x^2y+y^2,\langle(-\frac{1}{4},\frac{1}{2})\rangle].$$

For n odd, we also have

$$\mathcal{A}[x^2+y^{2n},\langle(\frac{1}{2},\frac{1}{2n})\rangle]\cong\mathcal{A}[x^2y+y^n,\langle(\frac{n-1}{2n},\frac{1}{n})\rangle].$$

Notice that these are new isomorphisms that don't follow by Group-Weights.

6.2.1 An Afterthought. At the risk of superseding what we have just accomplished, we will now analyze the polynomial $W_{\alpha,\beta}=x^2+\alpha xy^n+\beta y^{2n}$. We'll first solve for when

$W_{\alpha,\beta}$ is degenerate. We notice that

$$\nabla W_{\alpha,\beta} = (2x + \alpha y^n, n\alpha x y^{n-1} + 2n\beta y^{2n-1}).$$

So we solve $2x + \alpha y^n = 0 \Rightarrow x = -\frac{\alpha}{2}y^n$. Substituting into $n\alpha x y^{n-1} + 2n\beta y^{2n-1} = 0$ yields $n\alpha(-\frac{\alpha}{2}y^n)y^{n-1} + 2n\beta y^{2n-1} = 0 \Rightarrow [-\frac{\alpha^2 n}{2} + 2n\beta]y^{2n-1} = 0$. So $-\frac{\alpha^2 n}{2} + 2n\beta = 0 \Rightarrow -\alpha^2 + 4\beta = 0$, or $\beta = \frac{\alpha^2}{4}$. This is when $W_{\alpha,\beta}$ is degenerate. Points that we've seen before that satisfy this are $(\alpha, \beta) = (\pm 2, 1)$ and $(1, \frac{1}{4})$.

We now compute $\frac{\partial^2}{\partial x^2}(W_{\alpha,\beta}) = 2$, $\frac{\partial^2}{\partial x \partial y}(W_{\alpha,\beta}) = \frac{\partial^2}{\partial y \partial x}(W_{\alpha,\beta}) = n\alpha y^{n-1}$, and $\frac{\partial^2}{\partial y^2}(W_{\alpha,\beta}) = n(n-1)\alpha x y^{n-2} + 2n(2n-1)\beta y^{2n-2}$. Therefore

$$\begin{aligned}
\mathrm{Hess}(W_{\alpha,\beta}) &= 2[n(n-1)\alpha x y^{n-2} + 2n(2n-1)\beta y^{2n-2}] - (n\alpha y^{n-1})^2 \\
&= 2n(n-1)\alpha(-\frac{\alpha}{2}y^n)y^{n-2} + 4n(2n-1)\beta y^{2n-2} - n^2\alpha^2 y^{2n-2} \\
&= [-n(n-1)\alpha^2 + 4n(2n-1)\beta - n^2\alpha^2]y^{2n-2} \\
&= [(-n^2+n)\alpha^2 + (8n^2-4n)\beta - n^2\alpha^2]y^{2n-2} \\
&= [(-2n^2+n)\alpha^2 + (8n^2-4n)\beta]y^{2n-2}.
\end{aligned}$$

Consider $\mathcal{B}_{\alpha,\beta} = \mathcal{B}[W_{\alpha,\beta}, \{0\}]$. The pairings on $\mathcal{B}_{\alpha,\beta}$ become

$$\langle y^a, y^b \rangle = \frac{2n-1}{(-2n^2+n)\alpha^2 + (8n^2-4n)\beta} = \frac{2n-1}{n(2n-1)(-\alpha^2+4\beta)} = \frac{1}{n(-\alpha^2+4\beta)}.$$

Now we can define maps from our particular $\mathcal{B}$-model examples that we've looked at before. For instance, we can define $\phi : \mathcal{B}_{0,1} \to \mathcal{B}_{\alpha,\beta}$ by $\phi = \mathrm{diag}[1, c, \ldots, c^{2n-2}]$. We simply require

$$\frac{1}{4n} = c^{2n-2}\left(\frac{1}{n(-\alpha^2+4\beta)}\right) \Rightarrow c^{2n-2} = \frac{n(-\alpha^2+4\beta)}{4n} = \frac{-\alpha^2+4\beta}{4}.$$

Similarly, we can define $\phi : \mathcal{B}_{1,0} \to \mathcal{B}_{\alpha,\beta}$ with

$$-\frac{1}{n} = c^{2n-2}\left(\frac{1}{n(-\alpha^2+4\beta)}\right) \Rightarrow c^{2n-2} = -\frac{n(-\alpha^2+4\beta)}{n} = \alpha^2 - 4\beta.$$

Note also that if we used the nontrivial group $\langle(\frac{1}{2}, \frac{1}{2})\rangle$, we would get similar maps just by using c^{n-1} instead of c^{2n-2}. The monodromy still yields the same automorphisms that we found before.

6.2.2 A Related Result. As will be noted in just a moment, the following $\mathcal{B}$-models exist and are isomorphic (by Theorem 2.32) for each odd integer $n > 2$.

$$\mathcal{B}[x^n + xy^2, \{0\}] \longleftrightarrow \mathcal{B}[x^n + x^{\frac{n+1}{2}}y + xy^2, \{0\}] \longleftrightarrow \mathcal{B}[x^{\frac{n+1}{2}}y + xy^2, \{0\}]$$

We will investigate the monodromy in this example by letting the polynomials continuously deform from one to another.

Theorem 6.10. *Let for each odd $n \in \mathbb{N}_{>2}$, $\mathcal{B}[x^n + xy^2, \{0\}] \cong \mathcal{B}[x^n + x^{\frac{n+1}{2}}y + xy^2, \{0\}] \cong \mathcal{B}[x^{\frac{n+1}{2}}y + xy^2, \{0\}]$.*

Proof. We'll prove this by examining the monodromy in the following two lemmas.

Lemma 6.11. *Let $W_\alpha = x^n + \alpha x^{\frac{n+1}{2}}y + xy^2$ and let $\mathcal{B}_\alpha = \mathcal{B}[W_\alpha, \{0\}]$. $\mathcal{B}_0 \cong \mathcal{B}_\alpha$ for all $\alpha \in \mathbb{C} - \{\pm 2\}$.*

Proof. First, $\mathcal{B}_0 = \mathbb{C}[x,y]/(nx^{n-1} + y^2, 2xy) = \text{span}_{\mathbb{C}}\{1, y, y^2, x, \ldots, x^{n-2}\}$. Note that we have the relations $y^2 = -nx^{n-1}$ and $xy = 0$. We also compute $\mathcal{B}_\alpha = \mathbb{C}[x,y]/(nx^{n-1} + \alpha\left(\frac{n+1}{2}\right)x^{\frac{n+1}{2}}y + y^2, \alpha x^{\frac{n+1}{2}} + 2xy) = \text{span}_{\mathbb{C}}\{1, y, y^2, x, \ldots, x^{n-2}\}$. Here we get a relation $y = -\frac{\alpha}{2}x^{\frac{n-1}{2}}$. Substituting this into the relation $nx^{n-1} + \alpha\left(\frac{n+1}{2}\right)x^{\frac{n+1}{2}}y + y^2 = 0$ yields $y^2 = \left(\frac{\alpha^2(n+1)}{4} - n\right)x^{n-1}$.

We'll now list the degrees of the basis elements in the following table:

m	1	x	x^2	$\dots$	$x^{\frac{n-1}{2}}$	y	$x^{\frac{n+1}{2}}$	$\dots$	x^{n-2}	y^2
$\deg(m)$	0	$\frac{1}{n}$	$\frac{2}{n}$	$\dots$	$\frac{\left(\frac{n-1}{2}\right)}{n}$	$\frac{\left(\frac{n-1}{2}\right)}{n}$	$\frac{\left(\frac{n+1}{2}\right)}{n}$	$\dots$	$\frac{n-2}{n}$	$\frac{n-1}{n}$

In order to find out when W_α is nondegenerate, we must solve for α in the equations $nx^{n-1}+\alpha\left(\frac{n+1}{2}\right)x^{\frac{n+1}{2}}y+y^2=0$ and $\alpha x^{\frac{n+1}{2}}+2xy=0$. We know from before that $y=-\frac{\alpha}{2}x^{\frac{n-1}{2}}$. Therefore

$$\begin{aligned}
nx^{n-1}+\alpha(\frac{n+1}{2})x^{\frac{n+1}{2}}(-\frac{\alpha}{2}x^{\frac{n+1}{2}})+(-\frac{\alpha}{2}x^{\frac{n-1}{2}})^2 &= 0\\
nx^{n-1}-\frac{\alpha^2}{4}(n+1)x^{n-1}+\frac{\alpha^2}{4}x^{n-1} &= 0\\
[n-\frac{\alpha^2}{4}(n+1)+\frac{\alpha^2}{4}]x^{n-1} &= 0\\
n-\frac{\alpha^2}{4}n-\frac{\alpha^2}{4}+\frac{\alpha^2}{4} &= 0\\
n(1-\frac{\alpha^2}{4}) &= 0\\
1-\frac{\alpha^2}{4} &= 0.
\end{aligned}$$

So $\alpha^2=4\Rightarrow\alpha=\pm2$.

Now we'll look at the product structure of $\mathcal{B}_\alpha$. First, let's let $\gamma = \frac{\alpha^2(n+1)}{4} - n$ so that our relation becomes $y^2 = \gamma x^{n-1}$. We then get $x^{n-1} = \frac{1}{\gamma} y^2$, where $\frac{1}{\gamma} = \frac{4}{\alpha^2(n+1)-4n}$.

$$x^a \star x^b = \begin{cases} x^{a+b} & \text{if } a+b \leq n-2, \\ \frac{1}{\gamma} y^2 & \text{if } a+b = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

$$x^a \star y = \begin{cases} x^a y = -\frac{\alpha}{2} x^{\frac{n-1}{2}+a} & \text{if } \frac{n-1}{2}+a \leq n-2, \\ -\frac{\alpha}{2}(\frac{1}{\gamma}) y^2 & \text{if } \frac{n-1}{2}+a = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

$$x^a \star y^2 = \begin{cases} y^2 & \text{if } a = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$y \star y = y^2.$$

All other products on $\mathcal{B}_\alpha$ are zero.

Now for the pairing structure. We'll first compute $\text{Hess}(W_\alpha)$. To do so, we need

$$\frac{\partial^2 W_\alpha}{\partial x^2} = n(n-1)x^{n-2} + \alpha(\frac{n+1}{2})(\frac{n-1}{2})x^{\frac{n-3}{2}} y$$
$$\frac{\partial^2 W_\alpha}{\partial x \partial y} = \alpha(\frac{n+1}{2})x^{\frac{n+1}{2}} + 2y$$
$$\frac{\partial^2 W_\alpha}{\partial y^2} = 2x$$

Plugging in $\alpha = 0$ yields $\mathrm{Hess}(W_0) = 2n(n-1)x^{n-1} - 4y^2$. Applying the relation $x^{n-1} = -\frac{1}{n}y^2$, we obtain $\mathrm{Hess}(W_0) = [-2(n-1)-4]y^2 = -2(n+1)y^2$. Now for W_α we compute

$$\begin{aligned}
\mathrm{Hess}(W_\alpha) &= 2x[n(n-1)x^{n-2} + \alpha(\frac{n+1}{2})(\frac{n-1}{2})x^{\frac{n-3}{2}}y] - [\alpha(\frac{n+1}{2})x^{\frac{n+1}{2}} + 2y]^2 \\
&= 2n(n-1)x^{n-1} + \frac{\alpha}{2}(n+1)(n-1)x^{\frac{n-1}{2}}y \\
&\qquad\qquad - [\alpha^2(\frac{n+1}{2})^2x^{n-1} + 2(2y)(\alpha(\frac{n+1}{2})x^{\frac{n+1}{2}}) + 4y^2] \\
&= 2n(n-1)x^{n-1} + \frac{\alpha}{2}(n+1)(n-1)x^{\frac{n-1}{2}}(-\frac{\alpha}{2}x^{\frac{n-1}{2}}) - \alpha^2(\frac{n+1}{2})^2x^{n-1} \\
&\qquad\qquad - 4\alpha(\frac{n+1}{2})x^{\frac{n-1}{2}}(-\frac{\alpha}{2}x^{\frac{n-1}{2}}) - 4(\frac{\alpha^2(n+1)}{4} - n)x^{n-1} \\
&= [2n(n-1) - \frac{\alpha^2}{4}(n+1)(n-1) - \frac{\alpha^2}{4}(n+1)^2 + \alpha^2(n+1) - \alpha^2(n+1) + 4n]x^{n-1} \\
&= [2n(n-1) - \frac{\alpha^2}{4}(n+1)(n-1) - \frac{\alpha^2}{4}(n+1)^2 + 4n]x^{n-1}.
\end{aligned}$$

Changing x^{n-1} into y^2 yields $\mathrm{Hess}(W_\alpha) = -\frac{2(\alpha^2-4)n(n+1)}{\alpha^2(n+1)-4n}y^2$. Note that plugging in $\alpha = 0$ yields $\mathrm{Hess}(W_0) = -2(n+1)y^2$ as desired.

We will now proceed to compute the pairing structure, starting with $\mathcal{B}_0$. For $a+b = n+1$, we have that $x^{a+b} = x^{n-1} = -\frac{1}{n}y^2 = \frac{\langle x^a, x^b\rangle}{n+1}(-2)(n+1)y^2 \Rightarrow \frac{1}{2n}y^2 = \langle x^a, x^b\rangle y^2 \Rightarrow \frac{1}{2n} = \langle x^a, x^b\rangle$. Also, $y^2 = \frac{\langle y,y\rangle}{n+1}(-2)(n+1)y^2 \Rightarrow -\frac{1}{2} = \langle y, y\rangle$. And $y^2 = \frac{\langle 1,y^2\rangle}{n+1}(-2)(n+1)y^2 \Rightarrow -\frac{1}{2} = \langle 1, y^2\rangle$. Since $xy = 0$ in this case, these are all of our nonzero pairings.

Now for the pairings on $\mathcal{B}_\alpha$. We compute

$$\begin{aligned}
\langle y, y\rangle = \langle 1, y^2\rangle &= \frac{4n - \alpha^2(n+1)}{2n(\alpha^2-4)}, \\
\langle x^{\frac{n-1}{2}}, y\rangle &= \frac{\alpha}{n(\alpha^2-4)}, \\
\langle x^a, x^b\rangle &= -\frac{2}{n(\alpha^2-4)}.
\end{aligned}$$

We are now ready to construct the map $\phi : \mathcal{B}_0 \to \mathcal{B}_\alpha$. Define ϕ by

$$\begin{aligned} \phi &: x^a \mapsto c_0^a x^a \text{ for } a \in \{0, \dots, n-2\}, \\ \phi &: y \mapsto c_1 y + c_2 x^{\frac{n-1}{2}}, \\ \phi &: y^2 \mapsto \lambda c_0^{n-1} y^2. \end{aligned}$$

We'll choose the ordered pair (c_1, c_2) to equal either $(1, \frac{\alpha}{2})$ or $(-1, -\frac{\alpha}{2})$. The motivation for this choice comes from examples computed with the code, and we'll show that these choices do actually work. So we'll now solve for λ and c_0. In order for ϕ to respect the product structure, first note that

$$\begin{aligned} \phi(y^2) &= \phi(-n x^{\frac{n-1}{2}} \star x^{\frac{n-1}{2}}) \\ &= -n\phi(x^{\frac{n-1}{2}}) \star \phi(x^{\frac{n-1}{2}}) \\ &= -n(c_0^{\frac{n-1}{2}} x^{\frac{n-1}{2}}) \star (c_0^{\frac{n-1}{2}} x^{\frac{n-1}{2}}) \\ &= -\frac{n}{\gamma} c_0^{n-1} y^2. \end{aligned}$$

Hence $\lambda = -\frac{n}{\gamma}$. We also note that

$$\begin{aligned} \phi(y^2) &= \phi(y) \star \phi(y) \\ &= (c_1 y + c_2 x^{\frac{n-1}{2}}) \star (c_1 y + c_2 x^{\frac{n-1}{2}}) \\ &= c_1^2 y^2 + 2c_1 c_2 x^{\frac{n-1}{2}} \star y + c_2^2 x^{\frac{n-1}{2}} \\ &= y^2 + \alpha(-\frac{\alpha}{2})(\frac{1}{\gamma})y^2 + \frac{\alpha^2}{4}(\frac{1}{\gamma})y^2 \\ &= (1 - \frac{\alpha^2}{2\gamma} + \frac{\alpha^2}{4\gamma})y^2 \\ &= (1 - \frac{\alpha^2}{4\gamma})y^2. \end{aligned}$$

Since the previous two statements are both equal to $\phi(y^2)$, we can equate them and compare coefficients to obtain

$$\begin{aligned} -\frac{n}{\gamma}c_0^{n-1} &= 1 - \frac{\alpha^2}{4\gamma} \\ -nc_0^{n-1} &= \gamma - \frac{\alpha^2}{4} \\ c_0^{n-1} &= -\frac{\gamma}{n} + \frac{\alpha^2}{4n} = \frac{\alpha^2 - 4\gamma}{4n}. \end{aligned}$$

These equations completely determine ϕ. We note that ϕ is a bijection since it is upper-triangular and we can see that it has nonzero entries all along the main diagonal. It remains to show that ϕ respects the product structure and the pairing structure. By construction we have already verified that $\phi(y^2) = \phi(y \star y) = \phi(y) \star \phi(y)$. For the other products,

$$\begin{aligned} \phi(x^a \star x^b) &= \phi(x^{a+b}) = c_0^{a+b}x^{a+b} \text{ for } a+b \leq n-2, \\ \phi(x^a) \star \phi(x^b) &= c_0^a x^a \star c_0^b x^b = c_0^{a+b}x^{a+b}. \end{aligned}$$

$$\begin{aligned} \phi(x^a \star x^b) &= \phi(-\frac{1}{n}y^2) = -\frac{1}{n}(\frac{1}{\gamma})c_0^{n-1}y^2 = (\frac{1}{\gamma})c_0^{n-1}y^2 \text{ for } a+b = n-1, \\ \phi(x^a) \star \phi(x^b) &= c_0^a x^a \star c_0^b x^b = (\frac{1}{\gamma})c_0^{n-1}y^2. \end{aligned}$$

$$\begin{aligned} \phi(x^a \star y) &= \phi(x^a y) = \phi(0) = 0 \\ \phi(x^a) \star \phi(y) &= c_0^a x^a \star (c_1 y + c_2 x^{\frac{n-1}{2}}) = c_0^a c_1 x^a \star y + c_0^a c_2 x^a \star x^{\frac{n-1}{2}} \end{aligned}$$

If $\dfrac{n-1}{2} + a \leq n-2,$ we get $c_0^a c_1(-\dfrac{\alpha}{2}x^{\frac{n-1}{2}+a}) + c_0^a c_2 x^{\frac{n-1}{2}+a} = 0$ for either choice of (c_1, c_2).
If $\dfrac{n-1}{2} + a = n-1,$ we get $c_0^a c_1(-\dfrac{\alpha}{2})(\dfrac{1}{\gamma})y^2 + c_0^a c_2(\dfrac{1}{\gamma})y^2 = 0$ for either choice of (c_1, c_2).

Therefore ϕ respects the product structure. Now for the pairings:

$$\begin{aligned}
\langle 1, y^2\rangle_{\mathcal{B}_0} &= -\frac{1}{2},\\
\langle \phi(1), \phi(y^2)\rangle_{\mathcal{B}_\alpha} &= \langle 1, -\frac{n}{\gamma}c_0^{n-1}y^2\rangle_{\mathcal{B}_\alpha} = -\frac{n}{\gamma}\left(\frac{\alpha^2-4\gamma}{4n}\right)\langle 1, y^2\rangle_{\mathcal{B}_\alpha}\\
&= \left(\frac{4\gamma-\alpha^2}{4\gamma}\right)\left(\frac{4n-\alpha^2(n+1)}{2n(\alpha^2-4)}\right)\\
&= \left(\frac{\alpha^2(n+1)-4n-\alpha^2}{\alpha^2(n+1)-4n}\right)\left(\frac{4n-\alpha^2(n+1)}{2n(\alpha^2-4)}\right)\\
&= \left(\frac{n(\alpha^2-4)}{\alpha^2(n+1)-4n}\right)\left(\frac{-(\alpha^2(n+1)-4n)}{2n(\alpha^2-4)}\right) = -\frac{1}{2}.\\
\langle y, y\rangle_{\mathcal{B}_0} &= -\frac{1}{2},\\
\langle y, y\rangle_{\mathcal{B}_\alpha} &= \langle c_1 y + c_2 x^{\frac{n-1}{2}}, c_1 y + c_2 x^{\frac{n-1}{2}}\rangle_{\mathcal{B}_\alpha}\\
&= c_1^2\langle y, y\rangle_{\mathcal{B}_\alpha} + 2c_1c_2\langle y, x^{\frac{n-1}{2}}\rangle_{\mathcal{B}_\alpha} + c_2^2\langle x^{\frac{n-1}{2}}, x^{\frac{n-1}{2}}\rangle_{\mathcal{B}_\alpha}\\
&= \frac{4n-\alpha^2(n+1)}{2n(\alpha^2-4)} + \frac{\alpha^2}{n(\alpha^2-4)} - \frac{2\alpha^2}{4n(\alpha^2-4)}\\
&= \frac{8n-2\alpha^2(n+1)+4\alpha^2-2\alpha^2}{4n(\alpha^2-1)} = -\frac{1}{2}.\\
\langle x^{\frac{n-1}{2}}, y\rangle_{\mathcal{B}_0} &= 0,\\
\langle \phi(x^{\frac{n-1}{2}}), \phi(y)\rangle_{\mathcal{B}_\alpha} &= \langle c_0^{\frac{n-1}{2}}x^{\frac{n-1}{2}}, c_1 y + c_2 x^{\frac{n-1}{2}}\rangle_{\mathcal{B}_\alpha} = c_0^{\frac{n-1}{2}}c_1\langle x^{\frac{n-1}{2}}, y\rangle_{\mathcal{B}_\alpha} + c_0^{\frac{n-1}{2}}c_2\langle x^{\frac{n-1}{2}}, x^{\frac{n-1}{2}}\rangle_{\mathcal{B}_\alpha}\\
&= c_0^{\frac{n-1}{2}}\left[c_1\left(\frac{\alpha}{n(\alpha^2-4)}\right) - c_2\left(\frac{2}{n(\alpha^2-4)}\right)\right] = c_0^{\frac{n-1}{2}}\left(\frac{c_1\alpha-2c_2}{n(\alpha^2-4)}\right) = 0.\\
\langle x^a, x^b\rangle_{\mathcal{B}_0} &= \frac{1}{2n},\\
\langle \phi(x^a), \phi(x^b)\rangle_{\mathcal{B}_\alpha} &= \langle c_0^a x^a, c_0^b x^b\rangle_{\mathcal{B}_\alpha} = c_0^{n-1}\langle x^a, x^b\rangle_{\mathcal{B}_\alpha} = \left(\frac{\alpha^2-4\gamma}{4n}\right)\left(\frac{-2}{n(\alpha^2-4)}\right)\\
&= \frac{-2(\alpha^2+4n-\alpha^2(n+1)}{4n^2(\alpha^2-4)} = \frac{\alpha^2 n+\alpha^2-4n-\alpha^2}{2n^2(\alpha^2-4)} = \frac{n(\alpha^2-4)}{2n^2(\alpha^2-4)} = \frac{1}{2n}.
\end{aligned}$$

So ϕ respects the pairings. Hence ϕ is an isomorphism of graded Frobenius algebras. □

Before we exhibit the monodromy in this example, note that we can simplify the expression for c_0^{n-1}:

$$c_0^{n-1} = \frac{\alpha^2 - 4\gamma}{4n} = \frac{\alpha^2 - \alpha^2(n+1) + 4n}{4n} = \frac{-\alpha^2 n + 4n}{4n} = \frac{1}{4}(4 - \alpha^2).$$

To go around the bad points, we can choose $\alpha = \pm 2(e^{it} - 1)$ to go around either point in the positively-oriented direction. Now notice that

$$c_0^{n-1} = \frac{1}{4}(4 - \alpha^2) = \frac{1}{4}(4 - 4(e^{it} - 1)^2) = 1 - (e^{it} - 1)^2 = e^{it}(2 - e^{it}).$$

We've computed this case once before (see the monodromy computed in Lemma 6.4). We get a cyclic group of order $n-1$. Replacing t with $-t$ gives us the same cyclic group, but iterating in reverse order (as in Lemma 6.4).

Lemma 6.12. *Let $W_\alpha = \alpha x^n + x^{\frac{n+1}{2}}y + xy^2$ and let $\mathcal{B}_\alpha = \mathcal{B}[W_\alpha, \{0\}]$. $\mathcal{B}_0 \cong \mathcal{B}_\alpha$ for all $\alpha \in \mathbb{C} - \{\frac{1}{4}\}$.*

Proof. First note that $\mathcal{B}_0 = \mathbb{C}[x,y]/(\frac{n+1}{2}x^{\frac{n-1}{2}}y + y^2, x^{\frac{n+1}{2}} + 2xy) = \mathrm{span}_{\mathbb{C}}\{1, x, \ldots, x^{n-2}, y, y^2\}$. Also, $\mathcal{B}_\alpha = \mathbb{C}[x,y]/(\alpha n x^{n-1} + \frac{n+1}{2}x^{\frac{n-1}{2}}y + y^2, x^{\frac{n+1}{2}} + 2xy) = \mathrm{span}_{\mathbb{C}}\{1, x, \ldots, x^{n-2}, y, y^2\}$. The degrees of the monomials in this case are the same as we computed in Lemma 6.11.

To check the nondegeneracy of W_α, we'll set $\alpha n x^{n-1} + \frac{n+1}{2}x^{\frac{n-1}{2}}y + y^2 = 0$ and $x^{\frac{n+1}{2}} + 2xy = 0$. We then see that $y = -\frac{1}{2}x^{\frac{n-1}{2}}$, and

$$\begin{aligned}
\alpha n x^{n-1} + (\frac{n+1}{2})x^{\frac{n-1}{2}}(-\frac{1}{2}x^{\frac{n-1}{2}}) + (-\frac{1}{2}x^{\frac{n-1}{2}})^2 &= 0\\
[\alpha n - \frac{n+1}{4} + \frac{1}{4}]x^{n-1} &= 0\\
\alpha n - \frac{n+1}{4} + \frac{1}{4} &= 0\\
(4\alpha - 1)n &= 0.
\end{aligned}$$

So we require $\alpha \neq \frac{1}{4}$. Now consider the relations in the quotient ring of $\mathcal{B}_\alpha$. First we get $y = -\frac{1}{2}x^{\frac{n-1}{2}}$ as we noted before (and this is the same in $\mathcal{B}_0$). Also, $\alpha n x^{n-1} + \frac{n+1}{2}x^{\frac{n-1}{2}}y + y^2 = 0 \Rightarrow y^2 = \left[\frac{n+1}{4} - \alpha n\right]x^{n-1}$. Let $\gamma = \frac{n+1}{4} - \alpha n$, and note that on $\mathcal{B}_0$ this relation becomes $y^2 = \frac{n+1}{4}x^{n-1}$.

We'll now compute the products on $\mathcal{B}_\alpha$.

$$x^a \star x^b = \begin{cases} x^{a+b} & \text{if } a+b \leq n-2, \\ \frac{1}{\gamma}y^2 & \text{if } a+b = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

$$x^a \star y = \begin{cases} x^a y = -\frac{1}{2}x^{\frac{n-1}{2}+a} & \text{if } \frac{n-1}{2}+a \leq n-2, \\ -\frac{1}{2}(\frac{1}{\gamma})y^2 & \text{if } \frac{n-1}{2}+a = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

$$x^a \star y^2 = \begin{cases} y^2 & \text{if } a = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$y \star y = y^2.$$

To compute $\mathrm{Hess}(W_\alpha)$, we'll first need

$$\frac{\partial^2 W_\alpha}{\partial x^2} = \alpha n(n-1)x^{n-2} + (\frac{n+1}{2})(\frac{n-1}{2})x^{\frac{n-3}{2}}y$$
$$\frac{\partial^2 W_\alpha}{\partial x \partial y} = (\frac{n+1}{2})x^{\frac{n+1}{2}} + 2y$$
$$\frac{\partial^2 W_\alpha}{\partial y^2} = 2x$$

Then

$$
\begin{aligned}
\mathrm{Hess}(W_\alpha) &= 2x[\alpha n(n-1)x^{n-2} + (\frac{n+1}{2})(\frac{n-1}{2})x^{\frac{n-3}{2}}y] - [(\frac{n+1}{2})x^{\frac{n+1}{2}} + 2y]^2 \\
&= 2\alpha n(n-1)x^{n-1} + \frac{1}{2}(n+1)(n-1)x^{\frac{n-1}{2}}y - [(\frac{n+1}{2})^2x^{n-1} + 2(n+1)x^{\frac{n-1}{2}}y + 4y^2] \\
&= 2\alpha n(n-1)x^{n-1} - \frac{1}{4}(n+1)(n-1)x^{n-1} - (\frac{n+1}{2})^2x^{n-1} + (n+1)x^{n-1} - 4\gamma x^{n-1} \\
&= [2\alpha n(n-1) - \frac{1}{4}(n+1)(n-1) - (\frac{n+1}{2})^2 + (n+1) - (n+1) + 4\alpha n]x^{n-1} \\
&= [\frac{1}{2}(4\alpha-1)n(n+1)]x^{n-1}.
\end{aligned}
$$

Substituting $x^{n-1} = \frac{1}{\gamma}y^2$ yields $\mathrm{Hess}(W_\alpha) = -\frac{2(4\alpha-1)n(n+1)}{(4\alpha-1)n-1}y^2$. So $\mathrm{Hess}(W_0) = -2ny^2$. We can then compute the pairings on $\mathcal{B}_0$.

$$
\begin{aligned}
\langle x^a, x^b\rangle_{\mathcal{B}_0} &= -\frac{2}{n} \text{ for } a+b=n-1 \\
\langle 1, y^2\rangle_{\mathcal{B}_0} = \langle y, y\rangle_{\mathcal{B}_0} &= -\frac{n+1}{2n} \\
\langle x^{\frac{n-1}{2}}, y\rangle_{\mathcal{B}_0} &= \frac{1}{n}.
\end{aligned}
$$

For $\mathcal{B}_\alpha$, we get the following pairings.

$$
\begin{aligned}
\langle x^a, x^b\rangle_{\mathcal{B}_\alpha} &= \frac{2}{(4\alpha-1)n} \text{ for } a+b=n-1 \\
\langle 1, y^2\rangle_{\mathcal{B}_\alpha} = \langle y, y\rangle_{\mathcal{B}_\alpha} &= \frac{1-n(4\alpha-1)}{2n(4\alpha-1)} \\
\langle x^{\frac{n-1}{2}}, y\rangle_{\mathcal{B}_\alpha} &= -\frac{1}{(4\alpha-1)n}.
\end{aligned}
$$

We are ready to construct our map $\phi : \mathcal{B}_0 \to \mathcal{B}_\alpha$. As before, we'll define ϕ by

$$
\begin{aligned}
&\phi : x^a \mapsto c_0^a x^a \text{ for } a \in \{0, \ldots, n-2\}, \\
&\phi : y \mapsto c_1 y + c_2 x^{\frac{n-1}{2}}, \\
&\phi : y^2 \mapsto \lambda c_0^{n-1} y^2.
\end{aligned}
$$

First we notice that

$$\phi(y^2) = \phi(\frac{n+1}{4}x^{\frac{n-1}{2}} \star x^{\frac{n-1}{2}}) = \frac{n+1}{4}(c_0^{\frac{n-1}{2}} x^{\frac{n-1}{2}} \star c_0^{\frac{n-1}{2}} x^{\frac{n-1}{2}})$$
$$= (\frac{n+1}{4})(\frac{1}{\gamma})c_0^{n-1}y^2 = -\frac{n+1}{(4\alpha-1)n-1}c_0^{n-1}y^2.$$

Hence $\lambda = -\frac{n+1}{(4\alpha-1)n-1}$. We'll make a choice now and set $c_1 = 1$. From computed examples, we could have also set $c_1 = -1$. That being said, notice that

$$\begin{aligned}\lambda c_0^{n-1}y^2 = \phi(y^2) = \phi(y \star y) &= \phi(y) \star \phi(y)\\ &= (y + c_2 x^{\frac{n-1}{2}}) \star (y + c_2 x^{\frac{n-1}{2}})\\ &= y^2 + 2c_2(-\frac{1}{2})(\frac{1}{\gamma})y^2 + c_2^2(\frac{1}{\gamma})y^2\\ &= (1 - \frac{c_2}{\gamma} + \frac{c_2^2}{\gamma})y^2.\end{aligned}$$

Equating coefficients yields $\lambda c_0^{n-1} = 1 - \frac{c_2}{\gamma} + \frac{c_2^2}{\gamma}$. We also obtain the equation

$$\begin{aligned}c_0^{\frac{n-1}{2}} x^{\frac{n-1}{2}} = \phi(x^{\frac{n-1}{2}}) &= \phi(-2y)\\ &= -2[y + c_2 x^{\frac{n-1}{2}}]\\ &= -2(-\frac{1}{2})x^{\frac{n-1}{2}} - 2c_2 x^{\frac{n-1}{2}}\\ &= (1-2c_2)x^{\frac{n-1}{2}}.\end{aligned}$$

Now equate coefficients and square both sides to obtain $c_0^{n-1} = (1-2c_2)^2$. Substituting into our first equation yields

$$-\frac{n+1}{(4\alpha-1)n-1}(1-2c_2)^2 = 1 - \frac{c_2}{\gamma} + \frac{c_2^2}{\gamma}$$
$$-\frac{n+1}{(4\alpha-1)n-1}(1-4c_2+4c_2^2) = 1 + \frac{4}{(4\alpha-1)n-1}c_2 - \frac{4}{(4\alpha-1)n-1}c_2^2$$
$$-(n+1)[1-4c_2+4c_2^2] = (4\alpha-1)n - 1 + 4c_2 - 4c_2^2.$$

Then

$$\begin{aligned}
[-4(n+1)+4]c_2^2+[4(n+1)-4]c_2+[-(n+1)+1-(4\alpha-1)n]&=0\\
-4nc_2^2+4nc_2-4\alpha n&=0\\
c_2^2-c_2+\alpha&=0.
\end{aligned}$$

So $c_2 = \frac{1\pm\sqrt{1-4\alpha}}{2}$. We then obtain $c_0^{n-1} = [1-2(\frac{1\pm\sqrt{1-4\alpha}}{2})]^2 = [1-1\mp\sqrt{1-4\alpha}]^2 = 1-4\alpha$.

We now want to verify that ϕ respects the product structure. As in Lemma 6.11, we note that $\phi(x^a \star x^b) = \phi(x^a)\star\phi(x^b)$. (The computation is the same). By our construction, we have already forced $\phi(y\star y) = \phi(y)\star\phi(y)$. It now remains to verify that $\phi(x^a\star y) = \phi(x^a)\star\phi(y)$. To do this, note that

$$\phi(x^a\star y) = \begin{cases} \phi(-\frac{1}{2}x^{\frac{n-1}{2}}) = -\frac{1}{2}c_0^{\frac{n-1}{2}+a}x^{\frac{n-1}{2}+a} & \text{if } a < \frac{n-1}{2},\\ \phi(-\frac{1}{2}(\frac{4}{n+1})y^2) = \frac{2}{(4\alpha-1)n-1}c_0^{n-1}y^2 & \text{if } a = \frac{n-1}{2}.\end{cases}$$

Then,

$$\phi(x^a)\star\phi(y) = c_0^a x^a\star(y+c_2x^{\frac{n-1}{2}}) = c_0^a[x^a\star y + c_2x^a\star x^{\frac{n-1}{2}}].$$

If $a < \frac{n-1}{2}$, we obtain $c_0^a[-\frac{1}{2}x^{\frac{n-1}{2}+a} + c_2x^{\frac{n-1}{2}+a}] = -\frac{1}{2}c_0^a(1-2c_2)x^{\frac{n-1}{2}}$. Now since $c_0^{n-1} = (1-2c_2)^2$, we have that $c_0^{\frac{n-1}{2}} = 1-2c_2$. Therefore this equals $-\frac{1}{2}c_0^{\frac{n-1}{2}+a}x^{\frac{n-1}{2}+a}$ as desired.

If $a = \frac{n-1}{2}$, then we get $c_0^{\frac{n-1}{2}}[-\frac{1}{2\gamma}y^2 + \frac{c_2}{\gamma}y^2] = -\frac{1}{2\gamma}c_0^{\frac{n-1}{2}}(1-2c_2)y^2 = -\frac{1}{2\gamma}c_0^{n-1}y^2$. Now $-\frac{1}{2\gamma} = -\frac{1}{2}(-\frac{4}{(4\alpha-1)n-1}) = \frac{2}{(4\alpha-1)n-1}$, which is what we needed. Hence ϕ respects the product structure.

To finish, we'll now investigate if ϕ respects the pairing structure.

$$\begin{aligned}
\langle 1, y^2\rangle_{\mathcal{B}_0} &= -\frac{n+1}{2n}, \\
\langle \phi(1), \phi(y^2)\rangle_{\mathcal{B}_\alpha} &= \langle 1, \lambda c_0^{n-1} y^2\rangle_{\mathcal{B}_\alpha} \\
&= -\left(\frac{n+1}{(4\alpha-1)n-1}\right)(1-4\alpha)\left(\frac{1-n(4\alpha-1)}{2n(4\alpha-1)}\right) = -\frac{n+1}{2n}. \\
\langle y, y\rangle_{\mathcal{B}_0} &= -\frac{n+1}{2n}, \\
\langle y, y\rangle_{\mathcal{B}_\alpha} &= \langle y + c_2 x^{\frac{n-1}{2}}, y + c_2 x^{\frac{n-1}{2}}\rangle_{\mathcal{B}_\alpha} = \langle y, y\rangle_{\mathcal{B}_\alpha} + 2c_2\langle y, x^{\frac{n-1}{2}}\rangle_{\mathcal{B}_\alpha} + c_2^2\langle x^{\frac{n-1}{2}}, x^{\frac{n-1}{2}}\rangle_{\mathcal{B}_\alpha} \\
&= \frac{1-n(4\alpha-1)}{2n(4\alpha-1)} + 2c_2\frac{-1}{(4\alpha-1)n} + c_2^2\frac{2}{(4\alpha-1)n} \\
&= \frac{1}{2n(4\alpha-1)}[1-n(4\alpha-1)-4c_2+4c_2^2] \\
&= \frac{1}{2n(4\alpha-1)}[-n(4\alpha-1)+(1-2c_2)^2] \\
&= \frac{1}{2n(4\alpha-1)}[1-n(4\alpha-1)-(4\alpha-1)] = -\frac{n+1}{2n}. \\
\langle x^{\frac{n-1}{2}}, y\rangle_{\mathcal{B}_0} &= \frac{1}{n}, \\
\langle \phi(x^{\frac{n-1}{2}}), \phi(y)\rangle_{\mathcal{B}_\alpha} &= \langle c_0^{\frac{n-1}{2}} x^{\frac{n-1}{2}}, y + c_2 x^{\frac{n-1}{2}}\rangle_{\mathcal{B}_\alpha} = c_0^{\frac{n-1}{2}}\langle x^{\frac{n-1}{2}}, y\rangle_{\mathcal{B}_\alpha} + c_0^{\frac{n-1}{2}} c_2\langle x^{\frac{n-1}{2}}, x^{\frac{n-1}{2}}\rangle_{\mathcal{B}_\alpha} \\
&= (1-2c_2)\left(\frac{-1}{(4\alpha-1)n}\right) + (1-2c_2)c_2\left(\frac{2}{(4\alpha-1)n}\right) \\
&= \frac{1}{(4\alpha-1)n}(2c_2-1+2c_2-4c_2^2) \\
&= -\frac{1}{(4\alpha-1)n}(1-c_2)^2 = -\frac{1-4\alpha}{(4\alpha-1)n} = \frac{1}{n}. \\
\langle x^a, x^b\rangle_{\mathcal{B}_0} &= -\frac{2}{n}, \\
\langle \phi(x^a), \phi(x^b)\rangle_{\mathcal{B}_\alpha} &= c_0^{n-1}\langle x^a, x^b\rangle_{\mathcal{B}_\alpha} = (1-4\alpha)\frac{2}{(4\alpha-1)n} = -\frac{2}{n}.
\end{aligned}$$

So ϕ respects the pairing structure, and thus is an isomorphism of graded Frobenius algebras.

$\square$

To investigate monodromy, set $\alpha = -(e^{it}-1)$. We get $c_0^{n-1} = 1-4\alpha = 4e^{it}-3 = e^{it}(4-3e^{-it})$. Hence $c_0 = e^{it/(n-1)}(4-3e^{-it})^{1/(n-1)}$. At $t=0$, $c_0 = 1$. As we increase t by each 2π, we will cycle through the $(n-1)$th roots of unity.

Note that this also finishes the proof of the theorem. $\square$

It may be interesting at some point to investigate the continuous deformations of $W_{\alpha,\beta} = \alpha x^n + \beta x^{\frac{n+1}{2}}y + xy^2$ and its corresponding $\mathcal{B}$-model $\mathcal{B}_{\alpha,\beta} = \mathcal{B}[W_{\alpha,\beta}, \{0\}]$. For now this will be left to the avid reader.

6.2.3 A Complete Classification. Building on these results, we will now attempt to classify all possible isomorphisms of $\mathcal{B}$-models built using polynomials with weights $\left(\frac{1}{2}, \frac{1}{2n}\right)$. This will also involve the weight system $\left(\frac{1}{n}, \frac{n-1}{2n}\right)$. We will restrict our attention to polynomials in two variables, and only list isomorphisms up to permutation of variables.

For all $n \geq 2$,

$$\mathcal{B}[x^2+y^{2n}, \{0\}] \longrightarrow \mathcal{B}[x^2+xy^n+y^{2n}, \{0\}] \longleftarrow \mathcal{B}[x^2+xy^n, \{0\}]$$

For $n \geq 2$, if n is even then the following are isomorphic by Group-Weights:

$$\mathcal{B}[x^2+y^{2n}, \langle(\tfrac{1}{2}, \tfrac{1}{2})\rangle] \longleftrightarrow \mathcal{B}[x^n+xy^2, \{0\}]$$

For $n \geq 2$, if n is odd then the following are isomorphic:

$$\begin{tikzcd}
\mathcal{B}[x^2+y^{2n}, \langle(\tfrac{1}{2}, \tfrac{1}{2})\rangle] \arrow[r] \arrow[d, leftrightarrow, "\text{Group-Weights}"'] & \mathcal{B}[x^2+xy^n+y^{2n}, \langle(\tfrac{1}{2}, \tfrac{1}{2})\rangle] & \mathcal{B}[x^2+xy^n, \langle(\tfrac{1}{2}, \tfrac{1}{2})\rangle] \arrow[l] \arrow[d, leftrightarrow, "\text{Group-Weights}"] \\
\mathcal{B}[x^n+xy^2, \{0\}] \arrow[r, leftrightarrow, "\text{Webb}"'] & \mathcal{B}[x^n+x^{\frac{n+1}{2}}y+xy^2, \{0\}] \arrow[r, leftrightarrow, "\text{Webb}"'] & \mathcal{B}[x^{\frac{n+1}{2}}y+xy^2, \{0\}]
\end{tikzcd}$$

The special case $n=1$ is somewhat uninteresting. We get two solitary $\mathcal{B}$-models which are not isomorphic to each other or to anything else: $\mathcal{B}[x^2+y^2, \{0\}]$ and $\mathcal{B}[x^2+y^2, \langle(\frac{1}{2}, \frac{1}{2})\rangle]$.

In the special case $n = 2$ we get

$$\begin{tikzcd}
\mathcal{B}[x^2+y^4, \langle(\frac{1}{2},\frac{1}{2})\rangle] \arrow[d, leftrightarrow, "\text{Group-Weights}"'] & & \\
\mathcal{B}[x^2+y^4, \{0\}] \arrow[r] & \mathcal{B}[x^2+xy^2+y^4, \{0\}] & \mathcal{B}[x^2+xy^2, \{0\}] \arrow[l]
\end{tikzcd}$$

The special case $n = 3$ includes a few more isomorphisms. We get the following list:

(1) $\mathcal{B}[x^3+y^3, \{0\}]$	(6) $\mathcal{B}[x^3+y^3+x^2y+xy^2, \{0\}]$
(2) $\mathcal{B}[x^3+xy^2, \{0\}]$	(7) $\mathcal{B}[x^3+y^3, \langle(\frac{1}{3},-\frac{1}{3})\rangle]$
(3) $\mathcal{B}[x^2y+xy^2, \{0\}]$	(8) $\mathcal{B}[x^2+xy^3, \langle(\frac{1}{2},\frac{1}{2})\rangle]$
(4) $\mathcal{B}[x^3+y^3+x^2y, \{0\}]$	(9) $\mathcal{B}[x^2+y^6, \langle(\frac{1}{2},\frac{1}{2})\rangle]$
(5) $\mathcal{B}[x^3+x^2y+xy^2, \{0\}]$	(10) $\mathcal{B}[x^2+xy^3+y^6, \langle(\frac{1}{2},\frac{1}{2})\rangle]$

All other cases are given by the more general results. This is a list of all possible $\mathcal{B}$-models built using polynomials with weights $\left(\frac{1}{2},\frac{1}{2n}\right)$ and $\left(\frac{1}{n},\frac{n-1}{2n}\right)$. These are all the possible isomorphisms (up to the conditions stated previously).

Proof. Many of these isomorphisms come from the previous result. The Group-Weights isomorphisms come from Theorem 3.9 and Theorem 3.15. The isomorphisms denoted "Webb" come by noting that polynomials in the weight system $\left(\frac{1}{n},\frac{n-1}{2n}\right)$ have $\widehat{c} < 1$. Therefore the isomorphism follows by Theorem 2.32.

For the unorbifolded $\mathcal{B}$-models with weights $(\frac{1}{2},\frac{1}{2n})$, the algorithm only produces this solution for potential weight systems. This happens since the first coordinate is $\frac{1}{2}$. Any further solutions would necessarily increase one of the coordinates to have magnitude greater than $\frac{1}{2}$, which would be invalid. Since we have already classified the isomorphisms within this weight system for these unorbifolded $\mathcal{B}$-models, we conclude that they must be unique.

The question of uniqueness for the rest of the isomorphisms is a bit trickier. We want to show that for all $n \geq 2$, the only weight systems that work in the unorbifolded case are

$\left(\frac{1}{2}, \frac{1}{2n}\right)$ and $\left(\frac{1}{n}, \frac{n-1}{2n}\right)$. This reduces to asking the question: can there exist an unorbifolded $\mathcal{B}$-model $\widetilde{B}$ with $n+1 < \dim(\widetilde{B}) < 2n-1$ and $\widehat{c} = \frac{n-1}{n}$?

Let's look for a potential weight system (q_1, q_2). We require $2 - 2q_1 - 2q_2 = \frac{n-1}{n}$. Solving for q_1 yields $q_1 = \frac{n+1}{2n} - q_2$. Now consider $\left(\frac{1}{q_1} - 1\right)\left(\frac{1}{q_2} - 1\right) = \dim(\widetilde{B})$. Substituting for q_1 and simplifying yields

$$[-1 + \dim(\widetilde{B})]q_2^2 + [\frac{n+1}{2n} - \frac{n+1}{2n}\dim(\widetilde{B})]q_2 + [1 - \frac{n+1}{2n}] = 0.$$

We can now apply the quadratic formula to find solutions for q_2. For $n+1 < \dim(\widetilde{B}) < 2n-1$ we know that q_2 will be a real-valued solution in the interval $(0, \frac{1}{2})$. Since we are looking for rational-valued solutions, we can examine the discriminant of this quadratic equation. q_2 will be in $\mathbb{Q}$ if and only if the discriminant D is a square in $\mathbb{Q}$. After simplifying, we compute

$$D = \frac{(n+1)^2}{4n^2}(\dim(\widetilde{B}))^2 - \left[\frac{5n^2 - 2n + 1}{2n^2}\right]\dim(\widetilde{B}) + \frac{(3n-1)^2}{4n^2}.$$

By factoring out $\frac{1}{4n^2}$, we reduce our problem further to determining when

$$(n+1)^2(\dim(\widetilde{B}))^2 - 2(5n^2 - 2n + 1)\dim(\widetilde{B}) + (3n-1)^2$$

is a square in $\mathbb{Z}$. We know that the solutions $\dim(\widetilde{B}) = n+1$ and $\dim(\widetilde{B}) = 2n-1$ work.

How can we tell if anything in between will work too? We want our polynomial to factor over $\mathbb{Z}$. In order for it to be a square, it must have a repeated root. By a theorem in algebra, a polynomial P has a repeated root if and only if its discriminant is zero. Therefore we can check the polynomial discriminant (not to be confused with D itself) to determine when we get repeated roots. We'll first substitute $\dim(\widetilde{B}) = n + a$ for values of $a \in \mathbb{N}$. We obtain

$$(n^2 + 2n + 1)a^2 + (2n^3 - 6n^2 + 6n - 2)a + (n^4 - 8n^3 + 14n^2 - 8n + 1).$$

The polynomial discriminant is then

$$\Delta_n = 4096a^7 - 94208a^6 + 225280a^5 - 151552a^4 - 16384a^3 + 32768a^2.$$

Setting $\Delta_n = 0$ yields the solutions $a = 0, 1, 10 \pm 6\sqrt{3}$. Since we required $a \in \mathbb{N}$, the only solution that works is then $a = 1$. This corresponds to $\dim(\widetilde{B}) = n + 1$.

Now we'll substitute $\dim(\widetilde{B}) = 2n - a$ for values of $a \in \mathbb{N}$. We obtain

$$(n^2 + 2n + 1)a^2 + (-4n^3 + 2n^2 - 8n + 2)a + (4n^4 - 12n^3 + 21n^2 - 10n + 1).$$

The polynomial discriminant is then

$$\Delta_n = -16384(a^7 + 24a^6 - 53a^5 - 22a^4 + 103a^3 - 28a^2 - 51a + 26).$$

Setting $\Delta_n = 0$ yields the solutions $a = -26, -1, 1$. Since we required $a \in \mathbb{N}$, the only solution that works is then $a = 1$. This corresponds to $\dim(\widetilde{B}) = 2n - 1$. Hence these are the only two possible dimensions for $\dim(\widetilde{B})$ that work.

Within the weight system $\left(\frac{1}{n}, \frac{n-1}{2n}\right)$ we note that the only monomials we get are x^n and xy^2 when n is even, and $x^n, x^{(n+1)/2}y$, and xy^2 when n is odd. For any polynomial we choose in this weight system, $\mathrm{SL}(W)$ is trivial. So we have classified all possible $\mathcal{B}$-models.

The lists of isomorphisms in the special cases $n = 1, 2, 3$ have also been verified by direct computation. □

6.2.4 A Mirror Picture. Let's apply mirror symmetry and translate the above result to $\mathcal{A}$-model isomorphisms. For all $n \geq 2$ the result $\mathcal{B}[x^2 + y^{2n}, \{0\}] \leftrightarrow \mathcal{B}[x^2 + xy^n, \{0\}]$ becomes $\mathcal{A}[x^2 + y^{2n}, \langle(\frac{1}{2}, 0), (0, \frac{1}{2n})\rangle] \leftrightarrow \mathcal{A}[x^2 + xy^n, \langle(\frac{1}{n}, -\frac{1}{2n})\rangle]$. Also, for all $n \geq 2$ we have $\mathcal{A}[x^2 + y^{2n}, \langle(\frac{1}{2}, \frac{1}{2n})\rangle] \leftrightarrow \mathcal{A}[x^2 + xy^n, \langle(\frac{1}{2}, \frac{1}{2n})\rangle$ by Group-Weights. This is the analog of $\mathcal{B}[x^2 + y^{2n}, \langle(\frac{1}{2}, \frac{1}{2}\rangle] \leftrightarrow \mathcal{B}[x^n + xy^2, \{0\}]$.

Perhaps more interesting is the case $n > 2$ with n odd. To use mirror symmetry, we'll have to stick to invertible polynomials. Here is the $\mathcal{B}$-side picture first:

$$\begin{tikzcd}
\mathcal{B}[x^2 + y^{2n}, \langle(\frac{1}{2}, \frac{1}{2})\rangle] \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, "\text{Group-Weights}"'] & \mathcal{B}[x^2 + xy^n, \langle(\frac{1}{2}, \frac{1}{2})\rangle] \arrow[d, leftrightarrow, "\text{Group-Weights}"] \\
\mathcal{B}[x^n + xy^2, \{0\}] \arrow[r, leftrightarrow, "\text{Webb}"'] & \mathcal{B}[x^{\frac{n+1}{2}}y + xy^2, \{0\}]
\end{tikzcd}$$

On the $\mathcal{A}$-side, we get

$$\begin{tikzcd}
\mathcal{A}[x^2 + y^{2n}, \langle(\frac{1}{2}, \frac{1}{2n})\rangle] \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, "\text{Group-Weights}"'] & \mathcal{A}[x^n + xy^2, \langle(\frac{1}{n}, -\frac{1}{2n})\rangle] \arrow[d, leftrightarrow, "\text{Group-Weights}"] \\
\mathcal{A}[x^2 + xy^n, \langle(\frac{1}{2}, \frac{1}{2n})\rangle] \arrow[r, leftrightarrow] & \mathcal{A}[x^{\frac{n+1}{2}}y + xy^2, \langle(\frac{1}{n}, -\frac{1}{2n})\rangle]
\end{tikzcd}$$

Notice that in the $\mathcal{B}$-picture, vertical arrows represent "discrete" isomorphisms whereas horizontal arrows allow us to continuously deform from one $\mathcal{B}$-model to other. In the $\mathcal{A}$-picture, vertical arrows represent continuous deformations and horizontal arrows are discrete. This gives us the following "mirror-symmetric box" for each odd positive integer $n > 2$.

$$\begin{tikzcd}
& \mathcal{B}[x^2 + y^{2n}, \langle(\frac{1}{2}, \frac{1}{2})\rangle] \arrow[rr, leftrightarrow] \arrow[dl, leftrightarrow] \arrow[dd, leftrightarrow] & & \mathcal{A}[x^2 + y^{2n}, \langle(\frac{1}{2}, \frac{1}{2n})\rangle] \arrow[dl, leftrightarrow] \arrow[dd, leftrightarrow] \\
\mathcal{B}[x^2 + xy^n, \langle(\frac{1}{2}, \frac{1}{2})\rangle] \arrow[rr, leftrightarrow] \arrow[dd, leftrightarrow] & & \mathcal{A}[x^n + xy^2, \langle(\frac{1}{n}, -\frac{1}{2n})\rangle] \arrow[dd, leftrightarrow] & \\
& \mathcal{B}[x^n + xy^2, \{0\}] \arrow[rr, leftrightarrow] \arrow[dl, leftrightarrow] & & \mathcal{A}[x^2 + xy^n, \langle(\frac{1}{2}, \frac{1}{2n})\rangle] \arrow[dl, leftrightarrow] \\
\mathcal{B}[x^{\frac{n+1}{2}}y + xy^2, \{0\}] \arrow[rr, leftrightarrow] & & \mathcal{A}[x^{\frac{n+1}{2}}y + xy^2, \langle(\frac{1}{n}, -\frac{1}{2n})\rangle] &
\end{tikzcd}$$

6.3 Monodromy in Finite Cases

Here we will highlight some possible places to look for monodromy and deformation invariance among finite sets of $\mathcal{B}$-models. We will examine the weight systems $\left(\frac{1}{3}, \frac{1}{3}\right)$ and $\left(\frac{1}{4}, \frac{1}{4}\right)$ since we've already done some work in classifying the related isomorphisms.

For the weight system $\left(\frac{1}{3}, \frac{1}{3}\right)$, and restricting to invertible polynomials, we get the following picture:

$$
\begin{tikzcd}
 & & \mathcal{B}[x^3+y^3, \langle(\frac{1}{3}, -\frac{1}{3})\rangle] \arrow[d, leftrightarrow, "\text{G-W}"] \\
\mathcal{B}[x^3+y^3, \{0\}] \arrow[r, leftrightarrow, "\text{Webb}"'] & \mathcal{B}[x^3+xy^2, \{0\}] \arrow[r, leftrightarrow, "\text{Webb}"'] \arrow[d, leftrightarrow, "\text{G-W}"] & \mathcal{B}[x^2y+xy^2, \{0\}] \arrow[d, leftrightarrow, "\text{G-W}"] \\
 & \mathcal{B}[x^2+y^6, \langle(\frac{1}{2}, \frac{1}{2})\rangle] \arrow[r, leftrightarrow] & \mathcal{B}[x^2+xy^3, \langle(\frac{1}{2}, \frac{1}{2})\rangle]
\end{tikzcd}
$$

Here is the mirror picture:

$$
\begin{tikzcd}
 & & \mathcal{A}[x^3+y^3, \langle(\frac{1}{3}, \frac{1}{3})\rangle] \arrow[d, leftrightarrow, "\text{G-W}"] \\
\mathcal{A}[x^3+y^3, \langle(\frac{1}{3}, 0), (0, \frac{1}{3})\rangle] \arrow[r, leftrightarrow] & \mathcal{A}[x^2+xy^3, \langle(\frac{1}{2}, \frac{1}{6})\rangle] \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, "\text{G-W}"] & \mathcal{A}[x^2y+xy^2, \langle(\frac{1}{3}, \frac{1}{3})\rangle] \arrow[d, leftrightarrow, "\text{G-W}"] \\
 & \mathcal{A}[x^2+y^6, \langle(\frac{1}{2}, \frac{1}{6})\rangle] \arrow[r, leftrightarrow] & \mathcal{A}[x^3+xy^2, \langle(\frac{1}{3}, \frac{1}{3})\rangle]
\end{tikzcd}
$$

Notice that on the $\mathcal{B}$-side, horizontal arrows represent continuous deformations and vertical arrows represent "discrete" isomorphisms. On the $\mathcal{A}$-side the horizontal arrows represent "discrete" isomorphisms and vertical arrows represent continuous deformations.

For the weight system $(\frac{1}{4}, \frac{1}{4})$, we have the following picture so far (as much as we have been able to compute):

$$
\begin{tikzcd}
 & \mathcal{B}[x^4+y^4, \langle(\frac{1}{4}, -\frac{1}{4})\rangle] \arrow[d, leftrightarrow, "\text{G-W}"] & \\
 & \mathcal{B}[x^3+xy^4, \langle(\frac{1}{3}, -\frac{1}{3})\rangle] \arrow[d, leftrightarrow, "\text{G-W}"] & \mathcal{B}[x^3+y^6, \langle(\frac{1}{3}, -\frac{1}{3})\rangle] \arrow[d, leftrightarrow, "\text{G-W}"] \\
\mathcal{B}[x^4+y^4, \langle(\frac{1}{2}, \frac{1}{2})\rangle] & \mathcal{B}[x^3y+xy^3, \langle(\frac{1}{2}, \frac{1}{2})\rangle] & \mathcal{B}[x^4+xy^3, \langle(\frac{1}{2}, \frac{1}{2})\rangle]
\end{tikzcd}
$$

Here is the mirror picture:

$$\begin{tikzcd}
 & \mathcal{A}[x^4+y^4, \langle(\frac{1}{4},\frac{1}{4})\rangle] \arrow[d, leftrightarrow, "\text{G-W}"] & \\
 & \mathcal{A}[x^4+xy^3, \langle(\frac{1}{4},\frac{1}{4})\rangle] \arrow[d, leftrightarrow, "\text{G-W}"] & \mathcal{A}[x^3+y^6, \langle(\frac{1}{3},\frac{1}{6})\rangle] \arrow[d, leftrightarrow, "\text{G-W}"] \\
\mathcal{A}[x^4+y^4, \langle(0,\frac{1}{2}),(\frac{1}{4},\frac{3}{4})\rangle] & \mathcal{A}[x^3y+xy^3, \langle(\frac{1}{4},\frac{1}{4})\rangle] & \mathcal{A}[x^3+xy^4, \langle(\frac{1}{3},\frac{1}{6})\rangle]
\end{tikzcd}$$

Again notice that on the $\mathcal{B}$-side, horizontal arrows (if any were present) represent continuous deformations and vertical arrows are discrete isomorphisms. On the $\mathcal{A}$-side the horizontal arrows are discrete isomorphisms while the vertical arrows are continuous deformations. These results suggest a general pattern for $\mathcal{B}$-model isomorphisms, which we will explore in the next chapter.

From these results on $\mathcal{B}$-model continuous deformation, we conjecture that any isomorphism between $\mathcal{B}$-models can be decomposed into a finite chain of isomorphisms involving only mirror symmetry and deformation invariance between the $\mathcal{A}$ and $\mathcal{B}$ sides. We can state this formally.

Conjecture 7.1. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be any two Landau-Ginzburg $\mathcal{B}$-models such that $\mathcal{B}_1 \cong \mathcal{B}_2$. If this isomorphism is not the result of a continuous deformation, then there exists a finite chain of Landau-Ginzburg models $C_1, \ldots, C_n$ (either $\mathcal{A}$ or $\mathcal{B}$) such that*

$$\mathcal{B}_1 \longleftrightarrow C_1 \longleftrightarrow \ldots \longleftrightarrow C_n \longleftrightarrow \mathcal{B}_2,$$

where each arrow represents an isomorphism of graded Frobenius algebras that is either a continuous deformation or is the isomorphism predicted by mirror symmetry.

A similar result also likely holds for the $\mathcal{A}$ side, since the picture will be "flipped" by mirror symmetry as noted earlier. Though this is not the desired analog theorem to Group-Weights for $\mathcal{B}$-models, this does give us some more understanding on the deeper theoretical reasons for $\mathcal{B}$-model isomorphisms in the context of mirror symmetry.

This conjecture, if it holds as observed, fully explains the behavior of Landau-Ginzburg $\mathcal{B}$-model isomorphisms. Either we look for isomorphic $\mathcal{B}$-models within our current weight system, or we mirror over to the $\mathcal{A}$-side and use Group-Weights to get isomorphic $\mathcal{A}$-models that will mirror back to the $\mathcal{B}$-side in (possibly) different weight systems. This, we conjecture, will create a finite list of weight systems to choose from to look for isomorphic $\mathcal{B}$-models.

To assist us computationally in this classification, we give one more result on finding isomorphisms between $\mathcal{B}$-models.

7.1 An Isomorphism Extension Theorem

Given equivalent singularities W_1, W_2 with a common group $G \leq \mathrm{SL}(n, \mathbb{C})$ that fixes them both, we want to show that their corresponding $\mathcal{B}$-models $\mathcal{B}[W_1, G]$ and $\mathcal{B}[W_2, G]$ are also isomorphic (note the flavor of the Group-Weights theorem in this construction). To make progress on this result, we will need a few more definitions.

Definition 7.2. An isomorphism $\phi : \mathcal{Q}_{W_1} \to \mathcal{Q}_{W_2}$ is *equivariant* with respect to the group G if for all $g \in G$ and all monomials m in the basis of $\mathcal{Q}_{W_1}$ we have $\phi(g \cdot m) = g \cdot \phi(m)$. Here the operation $\cdot$ represents the group action of G on monomials of the Milnor ring.

Definition 7.3 (Property (*) of [5])**.** Let W be a nondegenerate, invertible polynomial, and let G be an admissible group of symmetries of W. The pair (W, G) has *Property (*)* if

(i) W can be decomposed as $W = \sum_{i=1}^{M} W_i$, where the W_i are themselves invertible polynomials having no variables in common with any other W_j.

(ii) For any element g of G whose associated sector $\mathcal{A}_g \subseteq \mathcal{A}[W, G]$ is nonempty, and for each $i \in \{1, \ldots, M\}$ the action of g fixes either all of the variables in W_i or none of them.

(iii) For any element g' of G^T whose associated sector of $\mathcal{B}_{g'} \subseteq \mathcal{B}[W^T, G^T]$ is nonempty, and for each $i \in \{1, \ldots, M\}$ the action of g' fixes either all of the variables in W_i^T or none of them.

Here the *sector* of an $\mathcal{A}$ or $\mathcal{B}$ model corresponding to a group element g refers to the subset of the vector space basis containing the elements of the form $\lfloor m; g \rceil$.

The condition imposed on the group G in the hypothesis of Theorem 7.4 is similar to Property (*) in [5]. For the following polynomials (see Remark 1.1.1 of [5]), any possible choice of group (that fixes the polynomial and is contained in $\mathrm{SL}(n, \mathbb{C})$) will satisfy the hypotheses of the theorem: fermats, loops in any number of variables, and any admissible polynomial in two variables.

Theorem 7.4. *Let W_1 and W_2 be admissible polynomials that are equivalent as singularities, with $\phi : \mathcal{Q}_{W_1} \to \mathcal{Q}_{W_2}$ an equivariant isomorphism of graded Frobenius algebras. If G is a group that preserves both W_1 and W_2 such that every $g \in G$ either fixes all or none of the variables of W_1 and W_2, then ϕ extends to an isomorphism $\psi : \mathcal{B}[W_1, G] \to \mathcal{B}[W_2, G]$.*

Consider the following diagram:

$$\begin{tikzcd}
\mathcal{B}[W_1, G] \arrow[r, "\psi"] & \mathcal{B}[W_2, G] \\
\mathcal{B}[W_1, \{0\}] \arrow[r, "\phi"] \arrow[u, dashed] & \mathcal{B}[W_2, \{0\}] \arrow[u, dashed]
\end{tikzcd}$$

The bottom horizontal arrow is the isomorphism we are given by hypothesis. The dashed vertical arrows represent an orbifolding (and, generally speaking, there won't exist an isomorphism going from bottom to top). The top horizontal arrow is the map that is conjectured to exist. In essence, we want to take the map ϕ that we are given, and use it to create an isomorphism of orbifolded $\mathcal{B}$-models.

Proof. By hypothesis, W_1 and W_2 are equivalent. We then know that $\mathcal{Q}_{W_1} \cong \mathcal{Q}_{W_2}$, so there exists an isomorphism $\phi : \mathcal{Q}_{W_1} \to \mathcal{Q}_{W_2}$. Also by hypothesis, we'll assume that ϕ is equivariant with respect to G. Suppose that a basis for $\mathcal{Q}_{W_1}$ is $\text{span}_{\mathbb{C}}\{m_1 = 1, \ldots, m_k\}$. We obtain a basis for $\mathcal{Q}_{W_2}$ with $\text{span}_{\mathbb{C}}\{\phi(m_1) = 1, \ldots, \phi(m_k)\}$.

Now we'll look at G-invariants. Suppose that $(\mathcal{Q}_{W_1})^G = \text{span}_{\mathbb{C}}\{p_1, \ldots, p_l\}$, where each $p_i = m_j$ for some j, and $l \leq k$. Recall that the group action $g \cdot m = \det(g)(m \circ g)$. For $p_i \in (\mathcal{Q}_{W_1})^G$, we have that $g \cdot p_i = p_i$. Now notice that since ϕ is equivariant, we have that $g \cdot \phi(p_i) = \phi(g \cdot p_i) = \phi(p_i)$. Therefore $\text{span}_{\mathbb{C}}\{\phi(p_1), \ldots, \phi(p_l)\} \subseteq (\mathcal{Q}_{W_2})^G$. But notice that if we take an m_i not preserved under the action of G, we get $g \cdot \phi(m_i) = \phi(g \cdot m_i) = \phi(cm_i) = c\phi(m_i)$ for some constant $c \neq 1$. Therefore $(\mathcal{Q}_{W_2})^G = \text{span}_{\mathbb{C}}\{\phi(p_1), \ldots, \phi(p_l)\}$.

Notice that the same process works even if we first restrict W to a fixed locus of a group element. So for $\left(\mathcal{Q}_{W_1|_{\text{fix}(g)}}\right)^G$, we can write it as $\text{span}_{\mathbb{C}}\{r_i\}$ where the r_i form a subset of the m_i. We see that $\left(\mathcal{Q}_{W_2|_{\text{fix}(g)}}\right)^G = \text{span}_{\mathbb{C}}\{\phi(r_i)\}$ as before. This gives us the following: there are (not necessarily distinct) group elements $h_1, \dots, h_l$ such that

$$\mathcal{B}[W_1, G] = \operatorname{span}_{\mathbb{C}}\{\lfloor p_1; h_1 \rceil, \dots, \lfloor p_l; h_l \rceil\},$$
$$\mathcal{B}[W_2, G] = \operatorname{span}_{\mathbb{C}}\{\lfloor \phi(p_1); h_1 \rceil, \dots, \lfloor \phi(p_l); h_l \rceil\}.$$

Now we have a reasonable grounding to define the map $\psi : \mathcal{B}[W_1, G] \to \mathcal{B}[W_2, G]$ by $\psi(\lfloor p_i; h_i \rceil) = \lfloor \phi(p_i); h_i \rceil$. Notice that we already have that ψ is a well-defined bijection that preserves the vector space bi-grading. If we say that $p_1 = 1$ and $h_1 = \mathbf{0}$, then we also readily see that ψ maps the identity to the identity.

That ψ preserves the pairing is also easy to show. Let $B_1 = \mathcal{B}[W_1, G]$ and $B_2 = \mathcal{B}[W_2, G]$. Using the properties of pairings, we have for $h_i + h_j = \mathbf{0}$,

$$\langle \lfloor p_i; h_i \rceil, \lfloor p_j; h_j \rceil \rangle_{B_1} = \langle p_i, p_j \rangle_{\mathcal{Q}_{W_1}} = \langle \phi(p_i), \phi(p_j) \rangle_{\mathcal{Q}_{W_2}} = \langle \lfloor \phi(p_i); h_i \rceil, \lfloor \phi(p_j); h_j \rceil \rangle_{B_2}.$$

Since all other pairings are zero, this shows that ψ respects the pairing.

Now for the products. For basis elements α, β of B_1, we want to show that $\psi(\alpha \star \beta) = \psi(\alpha) \star \psi(\beta)$. We'll consider the case where $\operatorname{fix}(h_i) \cup \operatorname{fix}(h_j) \cup \operatorname{fix}(h_i + h_j) = \mathbb{C}^n$. Otherwise, both products will be zero. First,

$$\begin{aligned}\psi(\alpha \star \beta) = \psi(\lfloor p_i; h_i \rceil \star \lfloor p_j; h_j \rceil) &= \psi(\lfloor \gamma_1 p_i p_j; h_i + h_j \rceil) \\ &= \lfloor \phi(\gamma_1 p_i p_j); h_i + h_j \rceil = \lfloor \phi(\gamma_1)\phi(p_i p_j); h_i + h_j \rceil.\end{aligned}$$

The last equality comes from considering γ_1 as a monomial in $\mathcal{Q}_{W_1}$. Here we have

$$\gamma_1 = \frac{\mu_{h_i \cap h_j} \operatorname{Hess}(W_1|_{\operatorname{fix}(h_i + h_j)})}{\mu_{h_i + h_j} \operatorname{Hess}(W_1|_{\operatorname{fix}(h_i) \cap \operatorname{fix}(h_j)})}.$$

Second, we have

$$\psi(\alpha) \star \psi(\beta) = \lfloor \phi(p_i); h_i \rceil \star \lfloor \phi(p_j); h_j \rceil = \lfloor \gamma_2 \phi(p_i)\phi(p_j); h_i + h_j \rceil = \lfloor \gamma_2 \phi(p_i p_j); h_i + h_j \rceil.$$

Finally, we have

$$\gamma_2 = \frac{\mu_{h_i \cap h_j} \mathrm{Hess}(W_2|_{\mathrm{fix}(h_i+h_j)})}{\mu_{h_i+h_j} \mathrm{Hess}(W_2|_{\mathrm{fix}(h_i)\cap \mathrm{fix}(h_j)})}.$$

Previously, we computed bases for the Milnor rings of W_1 and W_2 after restricting to fixed loci and taking G-invariants. Since the dimension remained the same between W_1 and W_2 after these operations, we see that $\mu_{h_i \cap h_j}$ for W_1 equals $\mu_{h_i \cap h_j}$ for W_2 and similarly for $\mu_{h_i+h_j}$. So it just remains to check how ϕ deals with the respective Hessians. That is, we will have $\lfloor \phi(\gamma_1)\phi(p_i p_j); h_i + h_j \rceil = \lfloor \gamma_2 \phi(p_i p_j); h_i + h_j \rceil$ if we can show $\phi(\gamma_1) = \gamma_2$. We'll consider the behavior of group elements, and break this down into cases.

Case 1: $h_i = h_j = \mathbf{0}$. Notice that W_i restricted to the fixed locus is just W_i again. So the Hessians divide each other, which shows that $\gamma_1 = \gamma_2$. Further, $\mu_{h_i \cap h_j} = \mu_{h_i+h_j}$, which shows that $\gamma_1 = \gamma_2 = 1$. Therefore $\phi(\gamma_1) = \gamma_2$.

Case 2: one of $h_i, h_j = \mathbf{0}$. Without loss of generality, we may assume $h_i = \mathbf{0}$. So $\gamma_1 = \dfrac{\mu_{h_j} \mathrm{Hess}(W_1|_{\mathrm{fix}(h_j)})}{\mu_{h_j} \mathrm{Hess}(W_1|_{\mathrm{fix}(h_j)})} = 1$. Similarly, $\gamma_2 = 1$. Therefore $\phi(\gamma_1) = \gamma_2$.

Case 3: Both h_i, h_j are nonzero. By hypothesis on the behavior of our group elements,we will have the fixed locus of h_i and h_j trivial. But $h_i + h_j$ must be $\mathbf{0}$ in order to get a nonzero product. Therefore $\gamma_1 = \dfrac{\mathrm{Hess}(W_1)}{\mu}$, $\gamma_2 = \dfrac{\mathrm{Hess}(W_2)}{\mu}$. We will have $\phi(\gamma_1) = \gamma_2$ if we can show that $\phi(\mathrm{Hess}(W_1)) = \mathrm{Hess}(W_2)$.

Lemma 7.5. *If $\phi : \mathcal{B}[W_1, \{0\}] \to \mathcal{B}[W_2, \{0\}]$ is a $\mathcal{B}$-model isomorphism, then $\phi(Hess(W_1)) = Hess(W_2)$.*

Proof. Let $B_1 = \mathcal{B}[W_1, \{0\}]$ and $B_2 = \mathcal{B}[W_2, \{0\}]$. Suppose m_1, m_2 are monomials in the basis of B_1 such that $m_1 m_2$ spans the sector of highest degree in B_1. Since ϕ is an isomorphism, we can write $\mathcal{B}_2 = \mathrm{span}_{\mathbb{C}}\{\phi(m) \mid m \text{ is a basis element of } B_1\}$. Also, we know that ϕ

preserves pairings:

$$\langle m_1, m_2 \rangle_{B_1} = \langle \phi(m_1), \phi(m_2) \rangle_{B_2}.$$

Recall that $m_1 m_2 = \dfrac{\langle m_1, m_2 \rangle_{B_1}}{\mu} \mathrm{Hess}(W_1)$, where $\mu = \dim(B_1)$. Since $B_1 \cong B_2$, we also have that $\mu = \dim(B_2)$. Now note that $\mathrm{Hess}(W_1) = \dfrac{\mu(m_1 m_2)}{\langle m_1, m_2 \rangle_{B_1}}$. Apply ϕ:

$$\phi(\mathrm{Hess}(W_1)) = \phi\left(\frac{\mu(m_1 m_2)}{\langle m_1, m_2 \rangle_{B_1}}\right) = \frac{\mu\phi(m_1 m_2)}{\langle m_1, m_2 \rangle_{B_1}} = \frac{\mu\phi(m_1 m_2)}{\langle \phi(m_1), \phi(m_2) \rangle_{B_2}}.$$

On the other hand, we know by the isomorphism that the element $\phi(m_1 m_2) = \phi(m_1)\phi(m_2)$ spans the sector of highest degree in B_2. Therefore $\phi(m_1)\phi(m_2) = \dfrac{\langle \phi(m_1), \phi(m_2) \rangle_{B_2}}{\mu} \mathrm{Hess}(W_2)$. So then

$$\mathrm{Hess}(W_2) = \frac{\mu\phi(m_1)\phi(m_2)}{\langle \phi(m_1), \phi(m_2) \rangle_{B_2}} = \frac{\mu\phi(m_1 m_2)}{\langle \phi(m_1), \phi(m_2) \rangle_{B_2}}.$$

This shows that $\phi(\mathrm{Hess}(W_1)) = \mathrm{Hess}(W_2)$, as desired. □

Back to the theorem now, we have by Lemma 7.5 the result we were seeking. So this verifies Case 3. And, we notice, that this is enough to prove the theorem. □

We can now generalize the result to sums of polynomials.

Corollary 7.6. *Let $W = W_1 + W_2$ and $V = V_1 + V_2$ be sums of admissible polynomials in distinct variables such that W_i is singularity equivalent to V_i, with $\phi_i : \mathcal{Q}_{W_i} \to \mathcal{Q}_{V_i}$ an equivariant isomorphism of graded Frobenius algebras. If G_i is a group that preserves both W_i and V_i for each i such that each group element of G_i fixes either all or none of the variables of W_i and V_i, then there exists isomorphism $\psi : \mathcal{B}[W, G] \to \mathcal{B}[V, G]$ where $G = G_1 \times G_2$.*

Proof. First we'll construct an isomorphism $\phi : \mathcal{B}[W, \{0\}] \to \mathcal{B}[V, \{0\}]$ using the ϕ_i.

Claim: By the tensor product structure (see Proposition 2.33), we know that any monomial m_i in the basis of $\mathcal{Q}_W$ can be written as $\alpha_i\beta_i$ where the α_i is in the basis of $\mathcal{Q}_{W_1}$ and the β_i is in the basis of $\mathcal{Q}_{W_2}$. We can define ϕ by $\phi : m_i \mapsto \phi_1(\alpha_i)\phi_2(\beta_i)$ and extend linearly.

Proof of Claim: It is easy to verify that ϕ is a bijection, is linear, sends the identity to the identity, and preserves degrees. To show that ϕ respects the pairing, we note that

$$\begin{aligned}
\langle \phi(m_i), \phi(m_j)\rangle_{\mathcal{Q}_V} &= \langle \phi_1(\alpha_i)\phi_2(\beta_i), \phi_1(\alpha_j)\phi_2(\beta_j)\rangle_{\mathcal{Q}_V} \\
&= \langle \phi_1(\alpha_i), \phi_1(\alpha_j)\rangle_{\mathcal{Q}_{V_1}} \langle \phi_2(\beta_i), \phi_2(\beta_j)\rangle_{\mathcal{Q}_{V_2}} \\
&= \langle \alpha_i, \alpha_j\rangle_{\mathcal{Q}_{W_1}} \langle \beta_i, \beta_j\rangle_{\mathcal{Q}_{W_2}} \\
&= \langle \alpha_i\beta_i, \alpha_j\beta_j\rangle_{\mathcal{Q}_W} \\
&= \langle m_i, m_j\rangle_{\mathcal{Q}_W}.
\end{aligned}$$

For the products, we note that

$$\begin{aligned}
\phi(m_im_j) = \phi(\alpha_i\beta_i\alpha_j\beta_j) = \phi(\alpha_i\alpha_j\beta_i\beta_j) = \phi_1(\alpha_i\alpha_j)\phi_2(\beta_i\beta_j) = \phi_1(\alpha_i)\phi_1(\alpha_j)\phi_2(\alpha_i)\phi_2(\alpha_j) \\
= \phi_1(\alpha_i)\phi_2(\beta_i)\phi_1(\alpha_j)\phi_2(\beta_j) = \phi(\alpha_i\beta_i)\phi(\alpha_j\beta_j) = \phi(m_i)\phi(m_j).
\end{aligned}$$

Therefore ϕ really is an isomorphism of graded Frobenius algebras. We further check that ϕ is equivariant: for $g \in G$, we have $g \cdot \phi(m) = g \cdot (\phi_1(\alpha)\phi_2(\beta) = (g \cdot \phi_1(\alpha))(g \cdot \phi_2(\beta))$, since α and β are in distinct variables, $= \phi_1(g \cdot \alpha)\phi_2(g \cdot \beta)$, since ϕ_1 and ϕ_2 are equivariant, $= \phi(g \cdot m)$.

Now given our map ϕ, we see that W and V are equivalent singularities. Construct map ψ as before, but with using ϕ as the base map. The only thing left to check is that ψ respects products for group elements with nontrivial fixed locus. First note that with the W_i in distinct variables, the block matrix structure of the second partial derivatives of W will give us $\mathrm{Hess}(W) = \mathrm{Hess}(W_1)\mathrm{Hess}(W_2)$. It follows that ϕ sends $\mathrm{Hess}(W_i)$ to $\mathrm{Hess}(V_i)$ by Lemma 7.5 and by construction. Now the group elements g, h have to fix all the variables in either W_1 or W_2 by the hypothesis of the symmetry group structure. This way any quotient of Hessians will reduce to either $\text{Hess}(W_1)$ or $\text{Hess}(W_2)$. This shows that ψ respects the products, and gives us the desired isomorphism. $\square$

To further generalize the results that we've found, we'll need the following definition.

Definition 7.7. A pair (W, G) is *well-behaved* if $W = \sum W_i$ where each W_i is an admissible polynomial in distinct variables, and $G = \bigoplus G_i$ where each $g \in G_i$ either fixes all or none of the variables of W_i for each i.

As mentioned before, a polynomial W that is a fermat, loop, or a 2-variable admissible polynomial, together with any symmetry group G of W, is guaranteed to be well-behaved. We can further admit arbitrary sums of fermat and loop polynomials in distinct variables, together with any of their symmetry groups (see Remark 1.1.1 of [5]). We now include a brief result on equivariant isomorphisms.

Lemma 7.8. *Suppose (W, G) and (V, G) are well-behaved. Then an isomorphism $\phi : \mathcal{Q}_W \to \mathcal{Q}_V$ is equivariant if and only if we have equivariant isomorphisms $\phi_i : \mathcal{Q}_{W_i} \to \mathcal{Q}_{V_i}$ for each i.*

Proof. ($\Rightarrow$) Suppose that $\phi : \mathcal{Q}_W \to \mathcal{Q}_V$ is an equivariant isomorphism of graded Frobenius algebras. We can write $W = W_1 + \cdots + W_n$ and $V = V_1 + \cdots + V_n$ where each W_i is in the same variables as V_i but W_i is in distinct variables from W_j for all $i \neq j$. We can also write $G = G_1 \times \cdots \times G_n$, where G_i preserves either all or none of the variables of W_i, V_i for each i. By Proposition 2.33, we can consider $\mathcal{Q}_W \cong \mathcal{Q}_{W_1} \otimes \cdots \otimes \mathcal{Q}_{W_n}$ and $\mathcal{Q}_V \cong \mathcal{Q}_{V_1} \otimes \cdots \otimes \mathcal{Q}_{V_n}$. From the tensor product structure, we find that there exists a basis of each $\mathcal{Q}_{W_i}$ that is a subset of a basis of $\mathcal{Q}_W$. By restricting ϕ to the variables of W_i, we obtain an equivariant isomorphism $\phi_i : \mathcal{Q}_{W_i} \to \mathcal{Q}_{V_i}$ for each i.

($\Leftarrow$) Conversely, suppose that we have equivariant isomorphisms $\phi_i : \mathcal{Q}_{W_i} \to \mathcal{Q}_{V_i}$ for each i. The argument in the proof of Corollary 7.6 shows how to construct an equivariant isomorphism $\phi : \mathcal{Q}_W \to \mathcal{Q}_V$ in the case that $n = 2$. Extending by induction gives us the result for all n. $\square$

We are now ready to obtain the main result of this section.

Theorem 7.9. *Let (W, G) and (V, G) be well-behaved. If $\phi : \mathcal{Q}_W \to \mathcal{Q}_V$ is an equivariant isomorphism of graded Frobenius algebras, then ϕ extends to an isomorphism $\psi : \mathcal{B}[W, G] \to \mathcal{B}[V, G]$.*

Proof. Given $\phi : \mathcal{Q}_W \to \mathcal{Q}_V$ an equivariant isomorphism of graded Frobenius algebras, we can apply Lemma 7.8 to obtain $\phi_i : \mathcal{Q}_{W_i} \to \mathcal{Q}_{V_i}$ that are also equivariant isomorphisms of graded Frobenius algebras. We can then extend Corollary 7.6 by induction in the case that $W = W_1 + \cdots + W_n$ and $V = V_1 + \cdots + V_n$ are sums of admissible polynomials in distinct variables such that each W_i is singularity equivalent to V_i, and G_i is a group that preserves both W_i and V_i for each i such that each group element of G_i fixes either all or none of the variables of W_i and V_i. $\square$

Since we have required an equivariant isomorphism for many of these results, we now offer a partial classification of such isomorphisms.

Definition 7.10. Suppose $\phi : \mathcal{B}_1 \to \mathcal{B}_2$ is an isomorphism of $\mathcal{B}$-models. Say that $\mathcal{B}_1$ has basis $\{a_1, \ldots, a_n\}$ and $\mathcal{B}_2$ has basis $\{b_1, \ldots, b_n\}$. We say that ϕ is *diagonal* if we can write $\phi(a_i) = c_i b_i$ for $c_i \in \mathbb{C}$ nonzero (possibly after reordering the basis elements).

Theorem 7.11. *Any diagonal isomorphism of Landau-Ginzburg $\mathcal{B}$-models is equivariant.*

Proof. Suppose $\phi : \mathcal{B}_1 \to \mathcal{B}_2$ is a diagonal isomorphism of $\mathcal{B}$-models. That is, if $\mathcal{B}_1$ has basis $\{a_1, \ldots, a_n\}$ and $\mathcal{B}_2$ has basis $\{b_1, \ldots, b_n\}$, then $\phi(a_i) = c_i b_i$ for $c_i \in \mathbb{C}$ nonzero (possibly after reordering the basis elements). Now notice the following. For any $g \in G$,

$$\phi(g \cdot a_i) = \phi(\det(g) a_i \circ g) = \det(g)\phi(a_i \circ g) = \det(g) c_i (b_i \circ g).$$

$$g \cdot \phi(a_i) = g \cdot c_i b_i = \det(g) c_i (b_i \circ g).$$

This happens since $a_i \circ g$ is really just a constant times a_i, etc. Because $\phi(g \cdot a_i) = g \cdot \phi(a_i)$ for each i, we see that ϕ is equivariant. $\square$

7.2 Examples

In the following examples, we will demonstrate how we can apply the results that we've just found.

Example 7.12. Consider the following isomorphisms (see Theorem 6.3 and Theorem 6.6):

$$
\begin{tikzcd}
\mathcal{B}[x^2+y^6, \langle(\frac{1}{2},\frac{1}{2})\rangle] \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, dashed] & \mathcal{B}[x^2+xy^3+y^2, \langle(\frac{1}{2},\frac{1}{2})\rangle] \arrow[d, leftrightarrow, dashed] \\
\mathcal{B}[x^2+y^6, \{0\}] \arrow[r, leftrightarrow] & \mathcal{B}[x^2+xy^3+y^2, \{0\}]
\end{tikzcd}
$$

Here $W_1 = x^2 + y^6$ and $W_2 = x^2 + xy^3 + y^2$ are equivalent singularities in two variables. Let $B_1 = \mathcal{B}[W_1, \{0\}]$ and $B_2 = \mathcal{B}[W_2, \{0\}]$. We have $B_1 = \text{span}_{\mathbb{C}}\{1, y, y^2, y^3, y^4\}$ and $B_2 = \text{span}_{\mathbb{C}}\{1, y, y^2, y^3, y^4\}$. We can define a map $\phi : B_1 \to B_2$ by $\phi(y^i) = c^i y^i$ for $i \in \{0, \ldots, 4\}$, where c satisfies $c^4 = \frac{3}{4}$. Since ϕ is diagonal, it is an equivariant isomorphism of unorbifolded $\mathcal{B}$-models.

Now consider $C_1 = \mathcal{B}[W_1, \langle(\frac{1}{2}, \frac{1}{2})\rangle]$ and $C_2 = \mathcal{B}[W_2, \langle(\frac{1}{2}, \frac{1}{2})\rangle]$. Both of these orbifolded $\mathcal{B}$-models have the state space

$$\text{span}_{\mathbb{C}}\{\lfloor 1; (0,0)\rceil, \lfloor 1; (\frac{1}{2}, \frac{1}{2})\rceil, \lfloor y^2; (0,0)\rceil, \lfloor y^4; (0,0)\rceil\}.$$

Theorem 7.4 now guarantees that the map $\psi : C_1 \to C_2$ given by $\psi : \lfloor y^i; (0,0)\rceil \mapsto \lfloor \phi(y^i); (0,0)\rceil = c^i \lfloor y^i; (0,0)\rceil$ where $c^4 = \frac{3}{4}$, and $\psi : \lfloor 1; (\frac{1}{2}, \frac{1}{2})\rceil \mapsto \lfloor \phi(1); (\frac{1}{2}, \frac{1}{2})\rceil = \lfloor 1; (\frac{1}{2}, \frac{1}{2})\rceil$ is an isomorphism of graded Frobenius algebras. But this is the same map that was separately computed in Lemma 6.7.

Example 7.13. More generally, recall from Theorem 6.3 that we can compute for all $n \geq 2$,

$$\mathcal{B}[x^2+y^{2n},\{0\}] \longleftrightarrow \mathcal{B}[x^2+xy^n+y^{2n},\{0\}] \longleftrightarrow \mathcal{B}[x^2+xy^n,\{0\}]$$

Label $B_1 = \mathcal{B}[x^2+y^{2n},\{0\}]$, $B_2 = \mathcal{B}[x^2+xy^n+y^{2n},\{0\}]$, and $B_3 = \mathcal{B}[x^2+xy^n,\{0\}]$. Each unorbifolded $\mathcal{B}$-model has basis $\mathrm{span}_{\mathbb{C}}\{1, y, \ldots, y^{2n-2}\}$. Previously, we defined a map $\phi_1 : B_1 \to B_3$ by $\phi_1(y^a) = c^a y^a$, where c is a complex number that satisfies $c^{2n-2} = \frac{3}{4}$ (see Lemma 6.4). We also defined a map $\phi_2 : B_2 \to B_3$ by $\phi_2(y^a) = c^a y^a$, where c is a complex number that satisfies $c^{2n-2} = -3$ (see Lemma 6.5). As verified before, ϕ_1 and ϕ_2 are isomorphisms of graded Frobenius algebras. And, since these are diagonal maps, they are equivariant.

If n is odd, then $G = \left\langle \left(\frac{1}{2}, \frac{1}{2}\right) \right\rangle$ fixes each polynomial. By Theorem 7.4, we have for all odd $n > 2$

$$\begin{tikzcd}
\mathcal{B}[x^2+y^{2n},G] \arrow[r, leftrightarrow] & \mathcal{B}[x^2+xy^n+y^{2n},G] \arrow[r, leftrightarrow] & \mathcal{B}[x^2+xy^n,G] \\
\mathcal{B}[x^2+y^{2n},\{0\}] \arrow[r, leftrightarrow] \arrow[u, dashed] & \mathcal{B}[x^2+xy^n+y^{2n},\{0\}] \arrow[r, leftrightarrow] \arrow[u, dashed] & \mathcal{B}[x^2+xy^n,\{0\}] \arrow[u, dashed]
\end{tikzcd}$$

Note that this is the same result obtained in Theorem 6.6. Applying mirror symmetry to $\mathcal{B}$-models built with invertible polynomials, we get the following mirror diagram.

$$\begin{tikzcd}
\mathcal{A}[x^2+y^{2n},\langle(\frac{1}{2},0),(0,\frac{1}{2n})\rangle] \arrow[r, leftrightarrow] \arrow[d, dashed] & \mathcal{A}[x^2y+y^n,\langle(-\frac{1}{2n},\frac{1}{n})\rangle] \arrow[d, dashed] \\
\mathcal{A}[x^2+y^{2n},\langle(\frac{1}{2},\frac{1}{2n})\rangle] \arrow[r, leftrightarrow] & \mathcal{A}[x^2y+y^n,\langle(\frac{n-1}{2n},\frac{1}{n})\rangle]
\end{tikzcd}$$

Note that this is the same result that we obtained in Corollary 6.9. Hence Theorem 7.4 can be used to obtain results that before were only able to be had after much difficult computation.

Example 7.14. Let $W = W_1+W_2$ and $V = V_1+V_2$ in $\mathbb{C}[x,y,z,w]$, where $W_1 = V_1 = x^3+y^3$, $W_2 = z^3+w^3$, and $V_2 = z^2w+zw^2$. We readily see that $\mathcal{Q}_{W_1} \cong \mathcal{Q}_{V_1}$ by letting $\phi_1 : \mathcal{Q}_{W_1} \to \mathcal{Q}_{V_1}$ be the identity map. By Theorem 2.32 we know that $\mathcal{Q}_{W_2} \cong \mathcal{Q}_{V_2}$. Let $\phi_2 : \mathcal{Q}_{W_2} \to \mathcal{Q}_{V_2}$ be any such isomorphism.

The symmetry group $\mathrm{SL}(W_1) = \langle(\frac{1}{3}, -\frac{1}{3})\rangle$. Since ϕ_1 is the identity map, it is equivariant with respect to this group. And we note that any choice of ϕ_2 will be equivariant with respect to the trivial group $\{0\}$. So we form $G = \mathrm{SL}(W_1) \times \{0\} = \langle(\frac{1}{3}, -\frac{1}{3}, 0, 0)\rangle$, and note that it is contained in both $\mathrm{SL}(W)$ and $\mathrm{SL}(V)$. Hence (W, G) and (V, G) are well-behaved, showing that $\mathcal{B}[W, G] \cong \mathcal{B}[V, G]$ by Corollary 7.6.

Appendix A. The Code

This code relies on other methods and classes developed for the FJRW research group. All computations in this thesis were done using Sage 6.8. Here is a summary of the classes and methods used in the following code.

Class *Singularity*. Used to create and manipulate polynomials. Commonly used methods and attributes for a Singularity object `W` include

- `W.q`—retrieves the quasihomogeneous weights of W.

Class *SymmetryGroup*. Used to create and manipulate symmetry groups. Commonly used methods and attributes for a SymmetryGroup object `G` include

- `G.poly`—retrieves the polynomial used to construct the symmetry group.

Class *OrbMilnorRing*. Used to create and manipulate $\mathcal{B}$-models. Commonly used methods and attributes for an OrbMilnorRing object `B` include

- `B[i]`—retrieves the i-th basis element of `B` (index starts at 1).
- `B[i].bi_degree`—retrieves the bi-degree of the i-th basis element. This is often just the degree of the basis element listed twice in a coordinate pair.
- `B[i].degree`—retrieves the degree of the i-th basis element.
- `B[i].index`—retrieves the index of the given basis element.
- `B.dimension()`—gives the dimension of the $\mathcal{B}$-model.
- `B.eta`—gives the matrix of pairing relations for `B`.
- `B.products()`—prints the nontrivial product relations for `B`.

```
"""
   The Isomorphism Finder (v. 2.0)
   Author:  Nathan Cordner, 2014-2015
"""

def construct_map(B1,B2, type="diagonal", mathematica=False):
    """
        Currently implemented types:
        'diagonal' -- default, constructs diagonal matrix
        'upper_triangular' -- constructs an upper triangular matrix
        'lower_triangular" -- constructs a lower triangular matrix
        'full' -- uses all possible linear combinations
```

```
        Setting 'mathematica' to True will make the computer print out
        code that will run in Wolfram Mathematica to solve the resulting
        system of equations. However, I currently do not know of any
        examples where Mathematica succeeded when Sage failed...
    """

    #First verify vector space isomorphism
    if not check_graded_vspace(B1,B2):
        return
    d = B1.dimension() #Let d be dimension (of both B1 and B2)

    sectors = find_graded_sectors(B1)
    product_relations = find_product_relations(B1)

    # create d^2 variables to use
    cc = list(var('c%d' % i) for i in range(0,d**2))
    counter = 0
    hom = matrix(SR,d) #SR is for Symbolic Ring

    if type == "upper_triangular":
        hom, counter = upper_triangular_hom(B2, d, cc, sectors, product_relations)
    elif type == "lower_triangular":
        hom, counter = lower_triangular_hom(B2, d, cc, sectors, product_relations)
    elif type == "full":
        hom, counter = full_hom(B2, d, cc, sectors, product_relations)
    else:
        hom, counter = diagonal_hom(B2, d, cc, sectors, product_relations)

    print "using map: "
    print str(hom)

    compute_isomorphism(B1,B2,hom,cc,counter,product_relations,mathematica)


#the following methods are subroutines for the isomorphism calculators
def check_graded_vspace(B1,B2):
    """
        Subroutine to verify that
        B1,B2 are isomorphic as graded
        vector spaces
    """
    d1 = B1.dimension()
    d2 = B2.dimension()
    if not (d1 == d2):
        print "Dimensions do not match"
        return False

    for i in range(1, d1+1):
        if not (B1[i].bi_degree == B2[i].bi_degree):
            print "Graded sectors do not match"
            return False
```

```
    print "Isomorphic as Graded Vector Spaces"
    return True


def find_graded_sectors(B1):
    """
    Here we partition B1 into graded pieces
    Basis elements of same degree are put together
    """
    d = B1.dimension()
    sectors = []
    i = 1
    while (i < d+1):
        grading = B1[i].degree
        cur_sector = []
        cur_sector.append(B1[i])
        while(i < d):
            if (B1[i+1].degree == grading):
                cur_sector.append(B1[i+1])
                i += 1
            else:
                break
        sectors.append(cur_sector)
        i += 1
    return sectors


def find_product_relations(B1):
    """
    We now organize the multiplication
    this information is stored in the list 'product_relations'
    where product_relations[i] stores a 3-tuple
    (a,b,c) where c * basis element i = basis element a * basis element b
    (using the order on the basis produced by the code)

    Note that product_relations[0] always stores 'None'
    For whatever reason, each B-model indexes by 1 instead of 0...
    """
    d = B1.dimension()
    product_relations = []
    for i in range(0, d+1):
        product_relations.append(None)
    for i in range(2,d+1):
        for j in range(i,d+1):
            elem = B1[i] * B1[j]
            if not (elem == 0):
                relation = (i,j,elem.coefficients()[0])
                num = elem.leading_monomial().index
                #print str(num) +": " + str(relation)
                if (product_relations[num] == None):
                    product_relations[num] = [relation]
                else:
                    product_relations[num].append(relation)
    return product_relations
```

```
    #Will now store a list of lists: each number contains a list of product relations


#METHOD 1: A DIAGONAL MATRIX
def diagonal_hom(B2, d, cc, sectors, product_relations):
    counter = 0
    hom = matrix(SR,d) #SR is for Symbolic Ring
    hom[0,0] = 1

    for i in range(1, len(sectors)):
        cur_sector = sectors[i]
        first_num = cur_sector[0].index
        for kk in range(len(cur_sector)):
            elem = cur_sector[kk]
            num = elem.index
            if (product_relations[num] == None):
                hom[num-1,num-1] = cc[counter]
                counter += 1
            else:
                #i.e. compute f(xi) = f(c*xj *xk) = c*f(xj)*f(xk)
                item = product_relations[num][0]
                product_row = mult_with_symbolic_ring(B2, hom[item[0]-1],hom[item[1]-1])
                #multiply by the constant c
                product_row[:] = [x*(1/item[2]) for x in product_row]
                hom[num-1] = product_row
    return hom, counter


#METHOD 2: AN UPPER TRIANGULAR MATRIX
def upper_triangular_hom(B2, d, cc, sectors, product_relations):
    counter = 0
    hom = matrix(SR,d) #SR is for Symbolic Ring
    hom[0,0] = 1

    for i in range(1, len(sectors)):
        cur_sector = sectors[i]
        first_num = cur_sector[0].index
        for kk in range(len(cur_sector)):
            elem = cur_sector[kk]
            num = elem.index
            if (product_relations[num] == None):
                for j in range(first_num + kk,first_num+len(cur_sector)):
                    hom[num-1,j-1] = cc[counter]
                    counter += 1
            else:
                #i.e. compute f(xi) = f(c*xj *xk) = c*f(xj)*f(xk)
                item = product_relations[num][0]
                product_row = mult_with_symbolic_ring(B2, hom[item[0]-1],hom[item[1]-1])
                #multiply by the constant c
                product_row[:] = [x*(1/item[2]) for x in product_row]
                hom[num-1] = product_row
    return hom, counter
```

```
#METHOD 3: A LOWER TRIANGULAR MATRIX
def lower_triangular_hom(B2, d, cc, sectors, product_relations):
    counter = 0
    hom = matrix(SR,d) #SR is for Symbolic Ring
    hom[0,0] = 1

    for i in range(1, len(sectors)):
        cur_sector = sectors[i]
        first_num = cur_sector[0].index
        for kk in range(len(cur_sector)):
            elem = cur_sector[kk]
            num = elem.index
            if (product_relations[num] == None):
                for j in range(first_num,first_num+kk+1):
                    hom[num-1,j-1] = cc[counter]
                    counter += 1
            else:
                #i.e. compute f(xi) = f(c*xj *xk) = c*f(xj)*f(xk)
                item = product_relations[num][0]
                product_row = mult_with_symbolic_ring(B2, hom[item[0]-1],hom[item[1]-1])
                #multiply by the constant c
                product_row[:] = [x*(1/item[2]) for x in product_row]
                hom[num-1] = product_row
    return hom, counter


#METHOD 4: ALL POSSIBLE LINEAR COMBINATIONS
def full_hom(B2, d, cc, sectors, product_relations):
    counter = 0
    hom = matrix(SR,d) #SR is for Symbolic Ring
    hom[0,0] = 1

    for i in range(1, len(sectors)):
        cur_sector = sectors[i]
        first_num = cur_sector[0].index
        for kk in range(len(cur_sector)):
            elem = cur_sector[kk]
            num = elem.index
            if (product_relations[num] == None):
                for j in range(first_num,first_num+len(cur_sector)):
                    hom[num-1,j-1] = cc[counter]
                    counter += 1
            else:
                #i.e. compute f(xi) = f(c*xj *xk) = c*f(xj)*f(xk)
                item = product_relations[num][0]
                product_row = mult_with_symbolic_ring(B2, hom[item[0]-1],hom[item[1]-1])
                #multiply by the constant c
                product_row[:] = [x*(1/item[2]) for x in product_row]
                hom[num-1] = product_row
    return hom, counter


def compute_isomorphism(B1,B2,hom,cc,counter,product_relations,mathematica):
    """
```

```
        set up equations and solve the isomorphism
    """
    d = B1.dimension()
    equations = set([])
    #set up equations that make product relations equal
    for i in range(2,d+1):
        for j in range(i, d+1):
            elem = B1[i] * B1[j]
            alt_prod = mult_with_symbolic_ring(B2, hom[i-1],hom[j-1])
            if elem == 0:
                eq = 0 == sum(alt_prod)
                equations.add(eq)
            else:
                coeff = elem.leading_coefficient()
                basis_num = elem.leading_monomial().index
                alt_prod[:] = [x*(1/coeff) for x in alt_prod]
                eq = sum(hom[basis_num-1]) == sum(alt_prod)
                equations.add(eq)

    for i in range(0,d):
        for j in range(i,d):
            #Set up equations that respect the pairing
            eq = B1.eta[i,j] == pair_with_symbolic_ring(B2, hom[i], hom[j])
            if not str(eq) == '0 == 0':
                equations.add(eq)

    list_equations = list(equations)

    if not mathematica:
        print 'Solving equations ' + str(list_equations)
        solution = solve(list_equations,cc[:counter])
        print 'Solution(s):  ' + str(solution)
    else:
        print_mathematica(str(list_equations), cc, counter)


def pair_with_symbolic_ring(B, row1, row2):
    """
    Computes pairing between two non-basis elements in B
    with scalar coefficients in the symbolic ring
    """
    eq = 0
    for i in range(0,len(row1)):
        for j in range(0,len(row2)):
            eq += row1[i]*row2[j]*B.eta[i,j]
    return eq


def mult_with_symbolic_ring(B, row1, row2):
    """
    Try to circumvent implementing full-blown multiplication
    in symbolic ring with sage. Maybe later...
    """
    new_row = [0] * len(row1)
```

```
    for i in range(0,len(row1)):
        for j in range(0,len(row1)):
            coeff = row1[i] * row2[j]
            if not coeff == 0:
                elem = B[i+1] * B[j+1]
                if not elem == 0:
                    num = elem.leading_monomial().index
                    new_row[num - 1] += elem.coefficients()[0] * coeff
    return new_row


def print_mathematica(str_equations, cc, counter):
    """
    Convert sage output into form recognizable by mathematica
    """
    str_equations = "Solve" + str_equations
    str_equations = str_equations.replace(","," &&")
    vars = cc[:counter]
    str_vars = str(vars)
    str_vars = str_vars.replace("[","{")
    str_vars = str_vars.replace("]","}")
    filler = ", " + str_vars + "]"
    str_equations = str_equations.replace("]", filler)
    print("Mathematica Code:")
    print(str_equations)


#Code for verifying isomorphisms
def verify_isomorphism(B1,B2,M):
    """
        Given B-models B1,B2 and a matrix M that defines a map
        from B1 to B2. This is an umbrella method to check
        that the map defined by M is an isomorphism of
        graded Frobenius algebras
    """

    #Our main concern is that M is invertible, and that
    # M respects products and pairings

    if not M.is_invertible():
        print "matrix is not invertible!!"

    print "checking products"
    respects_products(B1,B2,M)
    print "checking pairings"
    respects_pairings(B1,B2,M)


def respects_products(B1,B2,M):
    """
        Given B-models B1,B2 and a matrix M that defines a map
        from B1 to B2, this method checks that the map defined
        by M respects the product structure.
```

```
        i.e. show for all basis elements bi, bj in B1, that
        M(bi *_1 bj) = M(bi) *_2 M(bj)
    """

    n = B1.dimension()
    bi = B1[1]
    bj = B1[1]

    for i in range(n):
        bi = B1[i+1]
        for j in range(n):
            bj = B1[j+1]

            #Compute Left Hand Side
            prod = bi * bj    #may not be a basis element
            LHS = [0]*n

            if not prod == 0:
                p_index = prod.leading_monomial().index - 1
                p_coeff = prod.leading_coefficient()

                #Instead of worrying about what basis elements I'm getting,
                #I'll just look at coefficients
                for k in range(n):
                    LHS[k] = M[p_index][k] * p_coeff #fix row, vary over column

            #Compute Right Hand Side
            a1 = [0]*n
            a2 = [0]*n
            for k in range(n):
                a1[k] = M[bi.index-1][k]
                a2[k] = M[bj.index-1][k]
            RHS = mult_with_symbolic_ring(B2, a1, a2)

            if not LHS == RHS:
                print "Failed at " + str(i) + ", " + str(j)
                print "LHS: " + str(LHS)
                print "RHS: " + str(RHS)


def respects_pairings(B1,B2,M):
    """
        same idea as before, just with pairings!!!

        want to show that <bi, bj>_1 = <M(bi), M(bj)>_2
        for all i,j
    """
    n = B1.dimension()

    for i in range(n):
        for j in range(n):
            LHS = B1.eta[i,j]
            RHS = pair_with_symbolic_ring(B2, M[i], M[j])
```

```
        if not LHS == RHS:
            print "Failed at " + str(i + 1) + ", " + str(j + 1)
            print "LHS: " + str(LHS)
            print "RHS: " + str(RHS)


"""
    An algorithm to determine which weight systems could
    potentially have a B-model that is isomorphic to a
    given B-model. Note that the algorithm will halt on
    its own if the c-hat of the given B-model is less than 1.
    (two variables only)

    --Nathan Cordner, June 2015
"""

def potential_weights(G, interrupt = 100):
    """
        INPUT:  some "B-admissible" group G,
                a positive integer to halt the process from time to time
        OUTPUT:  a finite list of weight systems
    """

    #for some reason I can't recover the polynomial and group
    #from the OrbMilnorRing object

    B = OrbMilnorRing(G)
    weights = G.poly.q
    B_dim = B.dimension()
    highest_deg = 2 - 2*weights[0] - 2*weights[1]

    q1, q2 = var('q1,q2')
    i = B_dim

    while(True):
        solution = solve([2-2*q1-2*q2 == highest_deg,(1/q1-1)*(1/q2-1) == i],q1,q2)

        if not (0 <= float(abs(solution[0][0].rhs())) <= 0.5):
            break
        if not (0 <= float(abs(solution[0][1].rhs())) <= 0.5):
            break

        rational_value = solution[0][0].rhs() in QQ
        if(rational_value):
            print('Unorbifold dimension = ' + str(i))
            print(solution)

        if (i % interrupt == 0):
            response = input('We have reached ' + str(i) + '. Continue? (y/n)')
            if response == n:
                break
        i += 1
```

Appendix B. Computations

```
Investigation into B-model isomorphisms using polynomials of weight (1/4,1/4)
and non-trivial symmetry group. (April 2015)

-----------------------------------------------------------------------------

W0 = Singularity(x^4 + y^4)
W1 = Singularity(x^4 + x*y^3)
W2 = Singularity(x^3*y + y^3*x)
W3 = Singularity(x^3*y + y^3*x + x^4)
W4 = Singularity(x^4 + y^4 + x^3*y)
W5 = Singularity(x^4 + x^2*y^2 + x*y^3)
W6 = Singularity(x^4 + x^2*y^2 + y^4)
W7 = Singularity(x^3*y + x^2*y^2 + x*y^3)
W8 = Singularity(x^3*y + x^2*y^2 + x*y^3 + y^4)
W9 = Singularity(x^4 + x^3*y + x^2*y^2 + y^4)
W10= Singularity(x^4 + x^3*y + x*y^3 + y^4)          #(possibly degenerate)
W11= Singularity(x^4 + x^3*y + x^2*y^2 + x*y^3 + y^4)

B0SL = OrbMilnorRing(SymmetryGroup(W0,[[1/4,-1/4]]))
B0   = OrbMilnorRing(SymmetryGroup(W0,[[1/2,1/2]]))
B1   = OrbMilnorRing(SymmetryGroup(W1,[[1/2,1/2]]))
B2   = OrbMilnorRing(SymmetryGroup(W2,[[1/2,1/2]]))
B3   = OrbMilnorRing(SymmetryGroup(W3,[[1/2,1/2]]))
B4   = OrbMilnorRing(SymmetryGroup(W4,[[1/2,1/2]]))
B5   = OrbMilnorRing(SymmetryGroup(W5,[[1/2,1/2]]))
B6SL = OrbMilnorRing(SymmetryGroup(W6,[[1/4,-1/4]]))
B6   = OrbMilnorRing(SymmetryGroup(W6,[[1/2,1/2]]))
B7   = OrbMilnorRing(SymmetryGroup(W7,[[1/2,1/2]]))
B8   = OrbMilnorRing(SymmetryGroup(W8,[[1/2,1/2]]))
B9   = OrbMilnorRing(SymmetryGroup(W9,[[1/2,1/2]]))
B10  = OrbMilnorRing(SymmetryGroup(W10,[[1/2,1/2]]))
B11  = OrbMilnorRing(SymmetryGroup(W11,[[1/2,1/2]]))

Here's what I can either compute or verify by Group-Weights:
    Group 1. B0SL ~ B2 ~ B3 ~ B5 ~ B6 ~ B6SL ~ B7 ~ B8 ~ B9
    Group 2. B1 ~ B11

    Leftover:  B0, B4, B10 (note that code returns errors with B10)

COMPUTATIONS:

NOTE:  Group-Weights ==> B0SL ~ B2
       My Theorem (6.1) ==> B0SL ~ B6SL

B2 ~ B3

sage: B2.print_summary()
Orbifold Milnor ring for x^3*y + x*y^3 with group generated by <(1/2, 1/2)>.
Dimension:   6
Basis:
  [1]  b_(0, 0)      Degree: 0  (0, 0)
```

```
  [2]  y^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [3]  x*yb_(0, 0)   Degree: 1  (1/2, 1/2)
  [4]  x^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [5]  b_(1/2, 1/2)  Degree: 1  (1/2, 1/2)
  [6]  y^4b_(0, 0)   Degree: 2  (1, 1)
sage: B3.print_summary()
Orbifold Milnor ring for x^4 + x^3*y + x*y^3 with group generated by <(1/2, 1/2)>.
Dimension:   6
Basis:
  [1]  b_(0, 0)      Degree: 0  (0, 0)
  [2]  y^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [3]  x*yb_(0, 0)   Degree: 1  (1/2, 1/2)
  [4]  x^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [5]  b_(1/2, 1/2)  Degree: 1  (1/2, 1/2)
  [6]  y^4b_(0, 0)   Degree: 2  (1, 1)
sage: construct_map(B2,B3,type="upper_triangular")
Isomorphic as Graded Vector Spaces
using map:
[1  0  0  0  0                                                                 0]
[0 c0 c1 c2 c3                                                                 0]
[0  0 c4 c5 c6                                                                 0]
[0  0  0 c7 c8                                                                 0]
[0  0  0  0 c9                                                                 0]
[0  0  0  0  0 c0^2+1/7*c0*c1-1/21*c1^2-2/21*c0*c2-3/7*c1*c2+1/7*c2^2-62/21*c3^2]
Solving equations [(omitted)]
Solution(s):  [
[c0 == -1, c1 == 0, c2 == -1/186*sqrt(31)*(2*sqrt(31) - 31), c3 == 0, c4 == -1,
 c5 == (-3/2), c6 == 0, c7 == -1/2*sqrt(31), c8 == 0, c9 == 1],
... (15 other solutions)
]

To verify the isomorphism I will choose solution 1 and let
M = matrix([[1,0,0,0,0,0],[0,-1,0,-1/186*sqrt(31)*(2*sqrt(31) - 31),0,0],
    [0,0,-1,(-3/2),0,0],[0,0,0,-1/2*sqrt(31),0,0],[0,0,0,0,1,0],
    [0,0,0,0,0,1/7812*(2*sqrt(31) - 31)^2 - 1/1953*sqrt(31)*(2*sqrt(31) - 31) + 1]])

# c0^2 + 1/7*c0*c1 - 1/21*c1^2 - 2/21*c0*c2 - 3/7*c1*c2 + 1/7*c2^2 - 62/21*c3^2
#  == 1-(2/21)*(-1)*(-1/186*sqrt(31)*(2*sqrt(31)-31))
#      +(1/7)*(-1/186*sqrt(31)*(2*sqrt(31)-31))^2
#  == 1/7812*(2*sqrt(31) - 31)^2 - 1/1953*sqrt(31)*(2*sqrt(31) - 31) + 1

sage: verify_isomorphism(B2,B3,M)
checking products
checking pairings

B2 ~ B5

sage: B2.print_summary()
Orbifold Milnor ring for x^3*y + x*y^3 with group generated by <(1/2, 1/2)>.
Dimension:   6
Basis:
  [1]  b_(0, 0)      Degree: 0  (0, 0)
  [2]  y^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [3]  x*yb_(0, 0)   Degree: 1  (1/2, 1/2)
```

```
  [4]  x^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [5]  b_(1/2, 1/2)  Degree: 1  (1/2, 1/2)
  [6]  y^4b_(0, 0)   Degree: 2  (1, 1)
sage: B5.print_summary()
Orbifold Milnor ring for x^4 + x^2*y^2 + x*y^3 with group generated by <(1/2, 1/2)>.
Dimension:   6
Basis:
  [1]  b_(0, 0)      Degree: 0  (0, 0)
  [2]  y^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [3]  x*yb_(0, 0)   Degree: 1  (1/2, 1/2)
  [4]  x^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [5]  b_(1/2, 1/2)  Degree: 1  (1/2, 1/2)
  [6]  y^4b_(0, 0)   Degree: 2  (1, 1)
sage: construct_map(B2,B5,type="upper_triangular")
Isomorphic as Graded Vector Spaces
using map:
[1  0  0  0  0                                                                        0]
[0 c0 c1 c2 c3                                                                        0]
[0  0 c4 c5 c6                                                                        0]
[0  0  0 c7 c8                                                                        0]
[0  0  0  0 c9                                                                        0]
[0  0  0  0  0 c0^2-2/11*c0*c1+3/22*c1^2+3/11*c0*c2-9/22*c1*c2-1/22*c2^2-31/11*c3^2]
Solving equations [(omitted)]
Solution(s):  [
[c0 == 1, c1 == (2/3), c2 == -1/6*I*sqrt(31)*sqrt(3), c3 == 0, c4 == -1/3*I*sqrt(3),
c5 == 3/2*I*sqrt(3), c6 == 0, c7 == 1/2*I*sqrt(93), c8 == 0, c9 == 1],
... (15 others)
]

To verify the isomorphism I will choose solution 1 and let

M = matrix([[1,0,0,0,0,0],[0,1,2/3,-1/6*I*sqrt(31)*sqrt(3),0,0],[0,0,-1/3*I*sqrt(3),
    3/2*I*sqrt(3),0,0],[0,0,0,1/2*I*sqrt(93),0,0],[0,0,0,0,1,0],[0,0,0,0,0,93/88]])

# c0^2 - 2/11*c0*c1 + 3/22*c1^2 + 3/11*c0*c2 - 9/22*c1*c2 - 1/22*c2^2 - 31/11*c3^2
#  == 1 - (2/11)*(2/3) + (3/22)*(2/3)^2 + (3/11)*(-1/6*I*sqrt(31)*sqrt(3))
#     - (9/22)*(2/3)*(-1/6*I*sqrt(31)*sqrt(3)) - (1/22)*(-1/6*I*sqrt(31)*sqrt(3))^2
#  == 93/88

sage: verify_isomorphism(B2,B5,M)
checking products
checking pairings
Failed at 2, 4
LHS: 1/8
RHS: 1/744*sqrt(93)*sqrt(31)*sqrt(3)
Failed at 4, 2
LHS: 1/8
RHS: 1/744*sqrt(93)*sqrt(31)*sqrt(3)

#But notice that 1/744*sqrt(93)*sqrt(31)*sqrt(3) == 1/8, so we're good!

B2 ~ B7

sage: B2.print_summary()
```

```
Orbifold Milnor ring for x^3*y + x*y^3 with group generated by <(1/2, 1/2)>.
Dimension:   6
Basis:
  [1]  b_(0, 0)      Degree: 0  (0, 0)
  [2]  y^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [3]  x*yb_(0, 0)   Degree: 1  (1/2, 1/2)
  [4]  x^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [5]  b_(1/2, 1/2)  Degree: 1  (1/2, 1/2)
  [6]  y^4b_(0, 0)   Degree: 2  (1, 1)
sage: B7.print_summary()
Orbifold Milnor ring for x^3*y + x^2*y^2 + x*y^3 with group generated by <(1/2, 1/2)>.
Dimension:   6
Basis:
  [1]  b_(0, 0)      Degree: 0  (0, 0)
  [2]  y^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [3]  x*yb_(0, 0)   Degree: 1  (1/2, 1/2)
  [4]  x^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [5]  b_(1/2, 1/2)  Degree: 1  (1/2, 1/2)
  [6]  y^4b_(0, 0)   Degree: 2  (1, 1)
sage: construct_map(B2,B7,type="upper_triangular")
Isomorphic as Graded Vector Spaces
using map:
[1  0  0  0  0                                                                  0]
[0 c0 c1 c2 c3                                                                  0]
[0  0 c4 c5 c6                                                                  0]
[0  0  0 c7 c8                                                                  0]
[0  0  0  0 c9                                                                  0]
[0  0  0  0  0 c0^2 + 1/2*c0*c1 - 1/2*c1^2 - c0*c2 + 1/2*c1*c2 + c2^2 - 3*c3^2]
Solving equations [(omitted)]
Solution(s):  [
[c0 == -1, c1 == (-2/3), c2 == -1/12*sqrt(2)*(2*sqrt(2) + 3), c3 == 0,
  c4 == 1/3*sqrt(3)*sqrt(2), c5 == -1/12*sqrt(3)*sqrt(2), c6 == 0, c7 == 3/4*sqrt(2),
  c8 == 0, c9 == 1],
... (15 others)
]

------------
scratch

[c0 == -1, c1 == (-2/3), c2 == -1/4*sqrt(2) - 1/3, c3 == 0, c4 == 1/3*sqrt(3)*sqrt(2),
c5 == -1/12*sqrt(3)*sqrt(2), c6 == 0, c7 == 3/4*sqrt(2), c8 == 0, c9 == 1]

c0^2 + 1/2*c0*c1 - 1/2*c1^2 - c0*c2 + 1/2*c1*c2 + c2^2 - 3*c3^2
 == (-1)^2 + (1/2)*(-1)*(-2/3) - (1/2)*(-2/3)^2 - (-1)*(-1/4*sqrt(2) - 1/3)
    + (1/2)*(-2/3)*(-1/4*sqrt(2) - 1/3) + (-1/4*sqrt(2) - 1/3)^2
 == 1/144*(3*sqrt(2) + 4)^2 - 1/6*sqrt(2) + 8/9

M = matrix([[1,0,0,0,0,0], [0,-1,-2/3,-1/4*sqrt(2) - 1/3,0,0], [0,0,1/3*sqrt(3)*sqrt(2),
    -1/12*sqrt(3)*sqrt(2),0,0],[0,0,0,3/4*sqrt(2),0,0],[0,0,0,0,1,0],
    [0,0,0,0,0,1/144*(3*sqrt(2) + 4)^2 - 1/6*sqrt(2) + 8/9]])

sage: verify_isomorphism(B2,B7,M)
checking products
checking pairings
```

```
------------

B2 ~ B8

sage: B2.print_summary()
Orbifold Milnor ring for x^3*y + x*y^3 with group generated by <(1/2, 1/2)>.
Dimension:   6
Basis:
  [1]  b_(0, 0)      Degree: 0  (0, 0)
  [2]  y^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [3]  x*yb_(0, 0)   Degree: 1  (1/2, 1/2)
  [4]  x^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [5]  b_(1/2, 1/2)  Degree: 1  (1/2, 1/2)
  [6]  y^4b_(0, 0)   Degree: 2  (1, 1)
sage: B8.print_summary()
Orbifold Milnor ring for x^3*y+x^2*y^2+x*y^3+y^4 with group generated by <(1/2, 1/2)>.
Dimension:   6
Basis:
  [1]  b_(0, 0)      Degree: 0  (0, 0)
  [2]  y^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [3]  x*yb_(0, 0)   Degree: 1  (1/2, 1/2)
  [4]  x^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [5]  b_(1/2, 1/2)  Degree: 1  (1/2, 1/2)
  [6]  y^4b_(0, 0)   Degree: 2  (1, 1)
sage: construct_map(B2,B8,type="upper_triangular")
Isomorphic as Graded Vector Spaces
using map:
[1  0  0  0  0                                                    0]
[0 c0 c1 c2 c3                                                    0]
[0  0 c4 c5 c6                                                    0]
[0  0  0 c7 c8                                                    0]
[0  0  0  0 c9                                                    0]
[0  0  0  0  0 c0^2 - 4*c0*c1 + c1^2 + 2*c0*c2 + 5*c2^2 - 16*c3^2]
Solving equations [(omitted)]
Solution(s):  [
[c0 == -1/3*I*sqrt(5)*sqrt(3), c1 == -2/3*I*sqrt(5)*sqrt(3),
  c2 == 1/30*sqrt(5)*sqrt(3)*sqrt(2)*(I*sqrt(2) - 2), c3 == 0, c4 == -I*sqrt(2),
  c5 == 0, c6 == 0, c7 == 1/5*sqrt(6)*sqrt(5), c8 == 0, c9 == 1],
... (15 others)
]

To verify the isomorphism I will choose solution 1 and let
M = matrix([[1,0,0,0,0,0],[0,-1/3*I*sqrt(5)*sqrt(3),-2/3*I*sqrt(5)*sqrt(3),
    1/30*sqrt(5)*sqrt(3)*sqrt(2)*(I*sqrt(2) - 2),0,0],[0,0,-I*sqrt(2),0,0,0],
    [0,0,0,1/5*sqrt(6)*sqrt(5),0,0],[0,0,0,0,1,0],[0,0,0,0,0,1/6*(I*sqrt(2) - 2)^2
    - 1/3*I*sqrt(2)*(I*sqrt(2) - 2) + 5]])

# c0^2 - 4*c0*c1 + c1^2 + 2*c0*c2 + 5*c2^2 - 16*c3^2
#  == (-1/3*I*sqrt(5)*sqrt(3))^2 - 4*(-1/3*I*sqrt(5)*sqrt(3))
#     *(-2/3*I*sqrt(5)*sqrt(3)) + (-2/3*I*sqrt(5)*sqrt(3))^2 + 2*(-1/3*I*sqrt(5)
#     *sqrt(3))*(1/30*sqrt(5)*sqrt(3)*sqrt(2)
#     *(I*sqrt(2) - 2)) + 5*(1/30*sqrt(5)*sqrt(3)*sqrt(2)*(I*sqrt(2) - 2))^2
#  == 1/6*(I*sqrt(2) - 2)^2 - 1/3*I*sqrt(2)*(I*sqrt(2) - 2) + 5
```

```
sage: verify_isomorphism(B2,B8,M)
checking products
Failed at 1, 3
LHS: [0, 0, 0, 0, 0, -1/18*(I*sqrt(2) - 2)^2 + 1/9*I*sqrt(2)*(I*sqrt(2) - 2) - 5/3]
RHS: [0, 0, 0, 0, 0, 1/6*sqrt(6)*sqrt(3)*sqrt(2)*(I*sqrt(2) - 2) - 1/3*I*sqrt(6)*sqrt(3)]
Failed at 3, 1
LHS: [0, 0, 0, 0, 0, -1/18*(I*sqrt(2) - 2)^2 + 1/9*I*sqrt(2)*(I*sqrt(2) - 2) - 5/3]
RHS: [0, 0, 0, 0, 0, 1/6*sqrt(6)*sqrt(3)*sqrt(2)*(I*sqrt(2) - 2) - 1/3*I*sqrt(6)*sqrt(3)]
checking pairings
Failed at 2, 4
LHS: 1/8
RHS: -1/96*sqrt(6)*sqrt(3)*sqrt(2)*(I*sqrt(2) - 2) + 1/48*I*sqrt(6)*sqrt(3)
Failed at 4, 2
LHS: 1/8
RHS: -1/96*sqrt(6)*sqrt(3)*sqrt(2)*(I*sqrt(2) - 2) + 1/48*I*sqrt(6)*sqrt(3)

#Fortunately, -1/96*sqrt(6)*sqrt(3)*sqrt(2)*(I*sqrt(2) - 2)
# + 1/48*I*sqrt(6)*sqrt(3) == 1/8, and
# -1/18*(I*sqrt(2) - 2)^2 + 1/9*I*sqrt(2)*(I*sqrt(2) - 2) - 5/3 == -2,
# 1/6*sqrt(6)*sqrt(3)*sqrt(2)*(I*sqrt(2) - 2) - 1/3*I*sqrt(6)*sqrt(3) == -2.

B2 ~ B9

sage: B2.print_summary()
Orbifold Milnor ring for x^3*y + x*y^3 with group generated by <(1/2, 1/2)>.
Dimension:   6
Basis:
  [1]  b_(0, 0)      Degree: 0  (0, 0)
  [2]  y^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [3]  x*yb_(0, 0)   Degree: 1  (1/2, 1/2)
  [4]  x^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [5]  b_(1/2, 1/2)  Degree: 1  (1/2, 1/2)
  [6]  y^4b_(0, 0)   Degree: 2  (1, 1)
sage: B9.print_summary()
Orbifold Milnor ring for x^4 + x^3*y + x^2*y^2 + y^4 with group generated by <(1/2, 1/2)>.
Dimension:   6
Basis:
  [1]  b_(0, 0)      Degree: 0  (0, 0)
  [2]  y^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [3]  x*yb_(0, 0)   Degree: 1  (1/2, 1/2)
  [4]  x^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [5]  b_(1/2, 1/2)  Degree: 1  (1/2, 1/2)
  [6]  y^4b_(0, 0)   Degree: 2  (1, 1)
sage: construct_map(B2,B9,type="upper_triangular")
Isomorphic as Graded Vector Spaces
using map:
[1  0  0  0  0                                                                  0]
[0 c0 c1 c2 c3                                                                  0]
[0  0 c4 c5 c6                                                                  0]
[0  0  0 c7 c8                                                                  0]
[0  0  0  0 c9                                                                  0]
[0  0  0  0  0 c0^2-23/7*c0*c1-27/7*c1^2-54/7*c0*c2+52/7*c1*c2-6/7*c2^2-514/7*c3^2]
Solving equations [(omitted)]
Solution(s):  [
```

```
[c0 == -2/3*I*sqrt(3)*sqrt(2), c1 == -I*sqrt(3)*sqrt(2), c2 == -4/3*I*sqrt(3)*sqrt(2)
- 1/12*I*sqrt(514), c3 == 0, c4 == -1/2*I*sqrt(3), c5 == -13/6*I*sqrt(3), c6 == 0,
c7 == 1/4*I*sqrt(257)*sqrt(2), c8 == 0, c9 == 1], ... (15 others)
]

To verify the isomorphism I will choose solution 1 and let
M = matrix([[1,0,0,0,0,0],[0,-2/3*I*sqrt(3)*sqrt(2),-I*sqrt(3)*sqrt(2),
    -4/3*I*sqrt(3)*sqrt(2)-1/12*I*sqrt(514),0,0],[0,0,-1/2*I*sqrt(3),-13/6*I*sqrt(3),0,0],
    [0,0,0,1/4*I*sqrt(257)*sqrt(2),0,0],[0,0,0,0,1,0],[0,0,0,0,0,771/28]])

# c0^2 - 23/7*c0*c1 - 27/7*c1^2 - 54/7*c0*c2 + 52/7*c1*c2 - 6/7*c2^2 - 514/7*c3^2
#  == (-2/3*I*sqrt(3)*sqrt(2))^2 - (23/7)*(-2/3*I*sqrt(3)*sqrt(2))*(-I*sqrt(3)*sqrt(2))
#    - (27/7)*(-I*sqrt(3)*sqrt(2))^2 - (54/7)*(-2/3*I*sqrt(3)*sqrt(2))
#    *(-4/3*I*sqrt(3)*sqrt(2) - 1/12*I*sqrt(514)) + (52/7)*(-I*sqrt(3)*sqrt(2))
#    *(-4/3*I*sqrt(3)*sqrt(2) - 1/12*I*sqrt(514)) - (6/7)
#    *(-4/3*I*sqrt(3)*sqrt(2) - 1/12*I*sqrt(514))^2
#  == 4/21*I*sqrt(3)*sqrt(2)*(16*I*sqrt(3)*sqrt(2) + I*sqrt(514))
#    - 6/7*(-4/3*I*sqrt(3)*sqrt(2) - 1/12*I*sqrt(514))^2 + 706/21
#  == 771/28

sage: verify_isomorphism(B2,B9,M)
checking products
Failed at 1, 3
LHS: [0, 0, 0, 0, 0, -257/28]
RHS: [0, 0, 0, 0, 0, 1/56*I*sqrt(257)*sqrt(2)*(16*I*sqrt(3)*sqrt(2) + I*sqrt(514))
+ 4/7*sqrt(257)*sqrt(3)]
Failed at 3, 1
LHS: [0, 0, 0, 0, 0, -257/28]
RHS: [0, 0, 0, 0, 0, 1/56*I*sqrt(257)*sqrt(2)*(16*I*sqrt(3)*sqrt(2) + I*sqrt(514))
+ 4/7*sqrt(257)*sqrt(3)]
checking pairings
Failed at 2, 4
LHS: 1/8
RHS: -1/4112*I*sqrt(257)*sqrt(2)*(16*I*sqrt(3)*sqrt(2) + I*sqrt(514))
      - 2/257*sqrt(257)*sqrt(3)
Failed at 4, 2
LHS: 1/8
RHS: -1/4112*I*sqrt(257)*sqrt(2)*(16*I*sqrt(3)*sqrt(2) + I*sqrt(514))
      - 2/257*sqrt(257)*sqrt(3)

#But 1/56*I*sqrt(257)*sqrt(2)*(16*I*sqrt(3)*sqrt(2) + I*sqrt(514)) + 4/7*sqrt(257)*sqrt(3)
# == -257/28, and -1/4112*I*sqrt(257)*sqrt(2)*(16*I*sqrt(3)*sqrt(2) + I*sqrt(514))
#- 2/257*sqrt(257)*sqrt(3) == 1/8, so it all works!

B3 ~ B6

sage: B3.print_summary()
Orbifold Milnor ring for x^4 + x^3*y + x*y^3 with group generated by <(1/2, 1/2)>.
Dimension:   6
Basis:
  [1]  b_(0, 0)      Degree: 0  (0, 0)
  [2]  y^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [3]  x*yb_(0, 0)   Degree: 1  (1/2, 1/2)
  [4]  x^2b_(0, 0)   Degree: 1  (1/2, 1/2)
```

```
  [5]  b_(1/2, 1/2)  Degree: 1  (1/2, 1/2)
  [6]  y^4b_(0, 0)   Degree: 2  (1, 1)
sage: B6.print_summary()
Orbifold Milnor ring for x^4 + x^2*y^2 + y^4 with group generated by <(1/2, 1/2)>.
Dimension:   6
Basis:
  [1]  b_(0, 0)      Degree: 0  (0, 0)
  [2]  y^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [3]  x*yb_(0, 0)   Degree: 1  (1/2, 1/2)
  [4]  x^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [5]  b_(1/2, 1/2)  Degree: 1  (1/2, 1/2)
  [6]  y^4b_(0, 0)   Degree: 2  (1, 1)
sage: construct_map(B3,B6,type="upper_triangular")
Isomorphic as Graded Vector Spaces
using map:
[1  0  0  0  0                                        0]
[0 c0 c1 c2 c3                                        0]
[0  0 c4 c5 c6                                        0]
[0  0  0 c7 c8                                        0]
[0  0  0  0 c9                                        0]
[0  0  0  0  0 c0^2 - 2*c1^2 - 4*c0*c2 + c2^2 - 24*c3^2]
Solving equations [(omitted)]
Solution(s):  [
[c0 == -2/3*I*sqrt(3)*sqrt(2), c1 == 0, c2 == 1/93*sqrt(3)*(2*sqrt(93) - 124*I*sqrt(2)),
c3 == 0, c4 == -1/2*sqrt(3)*sqrt(2), c5 == 9/31*sqrt(31), c6 == 0, c7 == -6/31*sqrt(31),
c8 == 0, c9 == 1], ... (15 others)
]

To verify the isomorphism I will choose solution 1 and let
M = matrix([[1,0,0,0,0,0],[0,-2/3*I*sqrt(3)*sqrt(2),0,1/93*sqrt(3)*(2*sqrt(93)
 - 124*I*sqrt(2)),0,0], [0,0,-1/2*sqrt(3)*sqrt(2),9/31*sqrt(31),0,0],
 [0,0,0,-6/31*sqrt(31),0,0],[0,0,0,0,1,0], [0,0,0,0,0,252/31]])

# c0^2 - 2*c1^2 - 4*c0*c2 + c2^2 - 24*c3^2
#  == (-2/3*I*sqrt(3)*sqrt(2))^2 - 4*(-2/3*I*sqrt(3)*sqrt(2))*(1/93*sqrt(3)*(2*sqrt(93)
#     - 124*I*sqrt(2))) + (1/93*sqrt(3)*(2*sqrt(93) - 124*I*sqrt(2)))^2
#  == 252/31

sage: verify_isomorphism(B3,B6,M)
checking products
Failed at 1, 2
LHS: [0, 0, 0, 0, 0, 18/31]
RHS: [0, 0, 0, 0, 0, 3/961*sqrt(31)*sqrt(3)*(2*sqrt(93) - 124*I*sqrt(2))
+ 12/31*I*sqrt(31)*sqrt(3)*sqrt(2)]
Failed at 2, 1
LHS: [0, 0, 0, 0, 0, 18/31]
RHS: [0, 0, 0, 0, 0, 3/961*sqrt(31)*sqrt(3)*(2*sqrt(93) - 124*I*sqrt(2))
+ 12/31*I*sqrt(31)*sqrt(3)*sqrt(2)]
checking pairings

#But 3/961*sqrt(31)*sqrt(3)*(2*sqrt(93) - 124*I*sqrt(2))
#                  + 12/31*I*sqrt(31)*sqrt(3)*sqrt(2) == 18/31

B1 ~ B11
```

```
sage: B1.print_summary()
Orbifold Milnor ring for x^4 + x*y^3 with group generated by <(1/2, 1/2)>.
Dimension:   6
Basis:
  [1]  b_(0, 0)      Degree: 0  (0, 0)
  [2]  y^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [3]  x*yb_(0, 0)   Degree: 1  (1/2, 1/2)
  [4]  x^2b_(0, 0)   Degree: 1  (1/2, 1/2)
  [5]  b_(1/2, 1/2)  Degree: 1  (1/2, 1/2)
  [6]  y^4b_(0, 0)   Degree: 2  (1, 1)
sage: B11.print_summary()
Orbifold Milnor ring for x^4 + x^3*y + x^2*y^2 + x*y^3 + y^4
    with group generated by <(1/2, 1/2)>.
Dimension:   6
Basis:
  [1]  b_(0, 0)       Degree: 0  (0, 0)
  [2]  y^2b_(0, 0)    Degree: 1  (1/2, 1/2)
  [3]  x*yb_(0, 0)    Degree: 1  (1/2, 1/2)
  [4]  x^2b_(0, 0)    Degree: 1  (1/2, 1/2)
  [5]  b_(1/2, 1/2)   Degree: 1  (1/2, 1/2)
  [6]  x*y^3b_(0, 0)  Degree: 2  (1, 1)
sage: construct_map(B1,B11,type="upper_triangular")
Isomorphic as Graded Vector Spaces
using map:
[1  0  0  0  0                                              0]
[0 c0 c1 c2 c3                                              0]
[0  0 c4 c5 c6                                              0]
[0  0  0 c7 c8                                              0]
[0  0  0  0 c9                                              0]
[0  0  0  0  0 2*c0*c1 - 2*c1^2 - 4*c0*c2 + 2*c1*c2 - 20*c3^2]
Solving equations [(omitted)]
Solution(s):  [
[c0 == 1/3*I*sqrt(5)*sqrt(3), c1 == 2/3*I*sqrt(5)*sqrt(3), c2 == I*sqrt(5)*sqrt(3),
c3 == 0, c4 == -5/3/r44, c5 == -5/3/r44, c6 == 0, c7 == r44, c8 == 0, c9 == 1],
... (3 others)
]

To verify the isomorphism I will choose solution 1 and let
M = matrix([[1,0,0,0,0,0],[0,1/3*I*sqrt(5)*sqrt(3),2/3*I*sqrt(5)*sqrt(3),
 I*sqrt(5)*sqrt(3),0,0], [0,0,-5/3,-5/3,0,0],[0,0,0,1,0,0],[0,0,0,0,1,0],
 [0,0,0,0,0,20/3]])

# 2*c0*c1 - 2*c1^2 - 4*c0*c2 + 2*c1*c2 - 20*c3^2
#  == 2*(1/3*I*sqrt(5)*sqrt(3))*(2/3*I*sqrt(5)*sqrt(3)) - 2*(2/3*I*sqrt(5)*sqrt(3))^2
#     - 4*(1/3*I*sqrt(5)*sqrt(3))*(I*sqrt(5)*sqrt(3))
#     + 2*(2/3*I*sqrt(5)*sqrt(3))*(I*sqrt(5)*sqrt(3))
#  == 20/3

sage: verify_isomorphism(B1,B11,M)
checking products
checking pairings
```

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