



2015-08-01

Student Evaluation of Mathematical Explanations in an Inquiry-Based Mathematics Classroom

Ashley Burgess Hulet

Brigham Young University - Provo

Follow this and additional works at: <https://scholarsarchive.byu.edu/etd>



Part of the [Science and Mathematics Education Commons](#)

BYU ScholarsArchive Citation

Hulet, Ashley Burgess, "Student Evaluation of Mathematical Explanations in an Inquiry-Based Mathematics Classroom" (2015). *All Theses and Dissertations*. 5715.

<https://scholarsarchive.byu.edu/etd/5715>

This Thesis is brought to you for free and open access by BYU ScholarsArchive. It has been accepted for inclusion in All Theses and Dissertations by an authorized administrator of BYU ScholarsArchive. For more information, please contact scholarsarchive@byu.edu, ellen_amatangelo@byu.edu.

Student Evaluation of Mathematical Explanations in an
Inquiry-Based Mathematics Classroom

Ashley Burgess Hulet

A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of
Master of Arts

Daniel Siebert, Chair
Steven R. Williams
Kate R. Johnson

Department of Mathematics Education

Brigham Young University

August 2015

Copyright © 2015 Ashley Burgess Hulet

All Rights Reserved

ABSTRACT

Student Evaluation of Mathematical Explanations in an Inquiry-Based Mathematics Classroom

Ashley Burgess Hulet
Department of Mathematics Education, BYU
Master of Arts

Students do not always evaluate explanations based on the mathematics despite their teacher's effort to be the guide-on-the-side and delegate evaluation to the students. This case study examined how the use of three features of the Discourse—authority, sociomathematical norms, and classroom mathematical practices—impacted students' evaluation and contributed to students' failure to evaluate. By studying three pre-service elementary school students' evaluation methods, it was found that the students applied different types of each of the features of the Discourse and employed them at different times. The way that the features of the Discourse were used contributed to some of the difficulties that the participants experienced in their evaluation of explanations. The results suggest that researchers in the field must come to believe that resistance to teaching methods is not the only reason for student failure to evaluate mathematical explanations and that authority is operating in the classroom even when the teacher is acting as the guide on the side. The framework developed for the study will be valuable for researchers who continue to use for their investigation of individual student's participation in mathematical activity.

Keywords: mathematics education, evaluation, authority, sociomathematical norms, classroom mathematical practices, inquiry-based teaching

ACKNOWLEDGEMENTS

First I would like to thank my committee chair Daniel Siebert for all of his help and support. I could never have finished without his countless hours of brainstorming, reading, writing, editing, and counseling. I would also like to thank my husband and my parents for their support and encouragement.

TABLE OF CONTENTS

LIST OF FIGURES	vii
Chapter 1: Introduction.....	1
Chapter 2: Background	6
Authority.....	6
Typical View of Authority.....	6
Authority Separated From Control	8
Authority Relationships with People and with Rules	10
Conflating Authority and Control.....	12
Separating Authority from Control.....	14
Sociomathematical Norms	18
Sociomathematical Norms and Explanation Evaluation.....	19
Classroom Mathematical Practices.....	21
Chapter 3: Methods.....	25
Data Collection	25
Initial Study.....	25
Current Study	27
Data Analysis	29
Pass I.....	29
Pass II.....	33
Pass III	34
Chapter 4: Results.....	37

Melissa	37
Background	37
Melissa’s Evaluation Method	38
Authority	38
Classroom Mathematical Practices	44
Implications for Evaluation	47
Model of Melissa’s evaluation methods	48
Claire	49
Background	49
Claire’s Evaluation Method	49
Sociomathematical norms	50
Classroom mathematical practices	53
Implications for evaluation	56
Model of Claire’s evaluation method	57
Lynn	57
Background	57
Lynn’s Evaluation Method	58
Sociomathematical norms	59
Correct answer means correct thinking	59
Evaluations are subjective	60
Good explanations are clear	61
Good explanations afford connections	62
Classroom mathematical practices	63

One role of authority.....	65
Implications for evaluation.....	67
Model of Lynn’s evaluation methods.....	68
Discussion of Models.....	68
Evaluation Difficulties.....	68
Sociomathematical Norms Comparisons.....	69
Classroom Mathematical Practices Comparisons.....	71
Authority Comparisons.....	73
Common Consent Authority.....	76
Discussion of Common Consent Authority.....	78
Chapter 5: Conclusion.....	80
Contributions.....	80
Implications.....	82
Limitations.....	83
References.....	86

LIST OF FIGURES

<i>Figure 1</i>	22
<i>Figure 2</i>	28
<i>Figure 3</i>	50

Chapter 1: Introduction

Part of the push in mathematics education reform is toward teaching students how to reason mathematically. Traditional teaching hones in on procedures and thus, the correct performance of procedures is seen as evidence of mathematics learning. Leaders of the reform movement such as NCTM believe that there is more to learning mathematics than being able to perform procedures. One of NCTM's (2000) *Standards for School Mathematics* is reasoning and proof, and therein NCTM states, "Being able to reason is essential to understanding mathematics" (p. 56). Thus, the leaders of the reform movement not only see sense making and reasoning as part of learning mathematics, but as one of the keys to building understanding. Evidence for this emphasis in teaching students how to reason mathematically can also be found in the *Common Core State Standards* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). These standards have a section dedicated to reasoning abstractly and quantitatively, which is more than simply performing procedures, but also includes understanding problems in context, using mathematical symbols to represent important relationships between quantities in the problems, and selecting appropriate operations and procedures to construct solutions. This reasoning standard shows that reasoning abstractly and quantitatively is much more than just symbol manipulation, and is an essential part of doing mathematics.

A way to help encourage and teach students to reason mathematically that is advocated in the reform movement is for teachers to allow students to evaluate mathematical explanations based on the mathematics (i.e., whether the explanation makes sense and is valid mathematically). In a traditional classroom, most often the teacher is the sole evaluator of the mathematics. Hiebert et al. (1997), leaders in the reform movement, advocate for the

responsibility of evaluation to be given to the students. Giving students' responsibility for evaluation pushes them to reason mathematically when making an evaluation or presenting an explanation for a solution. However, students are unlikely to assume the role of evaluator if teachers do not remove themselves as the authority on the mathematics (Hiebert et al, 1997). Confrey (1990) and others have described this yielding up of authority as becoming the "guide on the side."

If the teacher is no longer the only evaluator in the classroom, students should be questioning their own explanations as well as the explanations of others based on the mathematics. Unfortunately students do not always evaluate explanations based on the mathematics, but often turn to the teacher or a knowledgeable classmate to learn how to do a problem or to determine if an explanation is correct, despite the teacher's effort to be the "guide on the side." In these situations it is easy to blame the student when he consistently is unable to solve a problem or evaluate an explanation based upon the mathematics. For example, Jo Boaler (1998) depicts a group of students wanting to be told how to solve a problem because they "simply don't like it [reformed teaching]" (pg. 51). Boaler (1998) may at least be partially right. Some students may reject the role of evaluating mathematics on their own because they dislike it, but it is unlikely that their dislike is the only reason for which students choose not to participate in evaluation. For instance, some students may refrain from evaluating because they are unable to evaluate explanations. For these students, it may be less stigmatizing to admit dislike than admit failure.

What constitutes an explanation depends largely on the social setting in which the explanation is given. Levenson (2013) explains that in an inquiry-based mathematics classroom, "explanations communicate interpretations and mathematical activity to others in order to

convince others that solutions are legitimate” (p. 183). This way of characterizing explanation suggests two main components: description of the mathematics and justification by appealing to the mathematics. According to this definition, in order to have an appropriate explanation, a person has to communicate his thinking and his actions effectively so that others in the community can understand what he did and why his thinking and actions are mathematically valid. While these two components—description and justification—can be intertwined within an explanation, both must be present for a student’s response to be considered an explanation in an inquiry-based classroom. Because student evaluation of explanations is often envisioned in the reform movement as taking place in an inquiry-based classroom, explanations in this study will be defined as responses that contain both a description of and justification for a mathematical solution or result.

Evaluation of explanations based on the mathematics can be a very complex activity, considering the complexity of the components that make up an explanation. In order to evaluate an explanation based on the mathematics one must be able to understand the mathematical thinking of the author and determine if the mathematics presented in the explanation is sufficiently convincing or justified. Thus it requires more than developing an understanding of the mathematics, but also becoming fluent in the Discourse (Gee, 1989) of the mathematics classroom. Gee (1989) introduced the field to what he called "big D" discourse, which is a particular way of “saying (writing)—doing—being—valuing—believing” (pg. 6) that is unique to a certain community. Because evaluation involves knowing what is valued, what counts as evidence, what constitutes an adequate description, and what counts as a valid argument in a particular community, evaluation requires fluency in the Discourse of that community. Knowing mathematical content is only part of what is necessary to evaluate an explanation; the student

must also know how explanations are “done” in his classroom. Thus a student must learn the Discourse of his classroom in order to evaluate the explanations offered by others based on the mathematics.

Researchers such as Cobb and McClain (2001) have created constructs that fit well with the idea of Discourse. A particularly useful construct is *sociomathematical norms*, which are the taken-as-shared ways of participating in all mathematical activity in a classroom community (Bowers, Cobb, & McClain, 2009). Because writing explanations is a common type of mathematical activity, every mathematics classroom community has sociomathematical norms that capture features of the Discourse that outline how to write explanations in mathematics. For example, in one study of sociomathematical norms, Cobb and McClain (2001) identified sociomathematical norms that dictate what counts as an acceptable explanation. According to Cobb, sociomathematical norms such as those that dictate acceptable explanations are negotiated by the students as they participate in mathematical activity in the classroom. Furthermore, Cobb's work supports a common belief held by many mathematics educators, namely that students who are given sufficient time to participate in mathematical activity in the classroom will grow in their understanding of taken-as-shared sociomathematical norms, and will eventually be able to evaluate explanations (Hiebert et al., 1997) based on the mathematics.

As educators we know that our students, despite our efforts, do not all learn to evaluate explanations based on the mathematics. When students are not able to evaluate explanations based on mathematics, mathematics educators often cite two reasons for their failure to evaluate explanations: a lack of the necessary mathematical understanding to make sense of the explanation, or a resistance to reform-oriented instruction manifested in the refusal to accept the role of evaluator. Ko and Knuth (2013) in their study of undergraduate student's evaluation of

proofs and counterexamples claimed that the reason for students' inability to evaluate proofs correctly was because "students seemed to have an inadequate understanding of mathematical content presented in the argument" (pg. 25). Boaler (1998), as discussed above, claimed that one-fifth of the students she researched at Phoenix Park chose not to participate in class activities, which included solving problems and explaining their solutions, because they did not like "find[ing] things out for themselves" (p. 51).

While these two reasons for the inability to evaluate explanations seem reasonable, they may also be too simplistic. Draper and Siebert (2004) note that mathematical understanding and fluency in mathematical Discourse cannot be separated, because expressing or judging mathematical ideas requires fluency in the Discourse so that communicating and interpreting mathematical thinking is possible. Thus, when students struggle to create or evaluate mathematical explanations, it is impossible to determine if that struggle is caused by a lack of mathematical understanding, a lack of fluency in the Discourse, or both. Student resistance to reform instruction is equally problematic. When students resist evaluating explanations, it is typically impossible to determine whether that resistance is caused by a dislike of evaluating explanations or the actual inability to evaluate explanations. Thus, a more careful consideration of the reasons underlying the inability to evaluate explanations is warranted.

In order for a student to appropriately acknowledge the mathematical contributions of his classmates, he must be able to evaluate explanations on his own. Because of the complexity of mathematical explanations, evaluation requires not only coming to understand the mathematics, but also learning the Discourse. The purpose of this research is to better understand how features of the Discourse may be preventing students from evaluating mathematical explanations based on the mathematics.

Chapter 2: Background

There are three features of the Discourse in a mathematics classroom that may prevent students from evaluating mathematical explanations based on the mathematics: authority, sociomathematical norms, and classroom mathematical practices. In this chapter I define each of these factors, and then review and use the relevant literature to support my hypothesis that each of these factors impacts a student's evaluation of mathematical explanations.

Authority

A possible feature that may be preventing students from evaluating mathematical explanations based on the mathematics is the “authority” “exercised” by the teacher. Often the teacher is the main evaluator of mathematics presented in the class. So when a student gives an explanation or a solution, the teacher immediately gives feedback as to whether or not this explanation or solution is correct. Students often use this response from the teacher as a means through which they evaluate the explanation or solution, concluding that “the teacher said it is correct, so it is correct” (Wood, 1998). In fact, this “exercising” of the teacher's authority to evaluate arguments is what reformers advocate avoiding, when possible, in order to facilitate students' use of mathematics to evaluate arguments (Hiebert et al. 1997). Thus, authority is a useful construct to adopt to highlight ways that students evaluate arguments by identifying what methods of evaluation they may be privileging over the use of mathematics and their reasons for using these methods.

Typical View of Authority

In our society, we commonly equate authority with power and control. Authority can be seen as exerting control over how another group or person thinks and acts through forced obedience or intimidation. Benne (1970) claims that this notion of authority, i.e., control, was a

typical characteristic of instruction in the early 20th century. This view of authority came from a belief that education is about teaching the younger generation to fit in with the older, more experienced generation. This enculturation was one sided and did not allow for the younger generation to impact or influence the society they were being brought into. Thus, a teacher saw her job as teaching students the ways of thinking and acting set by society (Benne, 1970). We still see this in today's mathematics classrooms as teachers instruct students in how they should think about a problem and what procedures they should use to solve problems, in an effort to push students to adopt expert ways of thinking that are not always sensible to them (Carpenter & Lehrer, 1999). Enculturation happens through instruction where the teacher has control over what is seen as correct mathematics and how this correct mathematics is to be used. In this educational system the teacher is seen as the one who determines what the students should know and how they should learn it. She maintains this position or “authority” through assigning grades and enforcing classroom discipline. Students often choose to learn in this system because it allows them to avoid negative consequences and obtain rewards.

Progressive movements have pushed for education to be seen as a means through which society is continually reshaped mutually by the younger and older generations. Thus the teacher’s job is to help mediate the ideas of the younger and older generations (Benne, 1970). This can be seen in reform mathematics instruction through the teacher working to help students build on their understanding and experiences and develop an expert understanding (NCTM, 2000). What is considered to be expert understanding is open to negotiation, and not limited to the same understanding that was seen as expert when the teacher was going through school. Instead, reform mathematics educators are taking into account how students make sense of mathematics, and they are questioning what students need to know and how they should know it

(NCTM 1989; NCTM, 2000. With this new view of education, progressives have focused on reducing the power of the “authority figure” in the classroom as it pertains to the mathematics so that students can help to determine what mathematics is correct, what mathematics makes sense, and how ideas fit together. This allows students to influence the mathematics that the class develops. This progressive view of education has even caused some mathematics education reformers to strive to abolish authority entirely in the classroom rather than acknowledge the way that authority may be functioning within the new educational system (Hiebert et al., 1997).

Authority Separated From Control

Not all educational researchers equate authority with control. Benne (1970) explained that authority is always a part of our circumstances. He argued that authority is not the power vested in a person to have control over another, but is a means through which individuals can receive guidance. This authority is necessary within education in order for the next generation to gain access to the knowledge that has taken centuries for society to build, and for collaboration to occur. In order to help educators see the part that authority plays in a progressive educational system and allow for guidance to be accepted, Benne (1970) offered a definition for authority where he describes it as a “function” of society that

operates in situations in which a person or group, fulfilling some purpose, project, or need, requires guidance or direction from a source outside himself or herself...the individual or group grants obedience to another person or group (or to a rule, a set of rules, a way of coping, or a method) which claims effectiveness in mediating the field of conduct or belief as a condition of receiving assistance. p. 393

Benne’s definition presents authority in terms of a relationship between two people, the bearer and the grantor. It is the bearer’s guidance which the grantor obeys, thus granting authority to the bearer. This definition highlights the fact that the bearer of the authority gains that authority

because of the willful obedience of the grantor. When an authority relationship has been established, the bearer is seen by the grantor as having advice on specific topics. The grantor comes to the bearer of the authority with a desire to be assisted. The grantor of authority consents to acting or believing based on the advice or information given him from the bearer because he believes it will enable him to address the questions, problems, or ideas he was struggling with (Benne, 1970). Thus, through this obedience an authority relationship is established and authority is granted to the bearer. Note that the existence of an authority relationship does not mean that the bearer of authority is necessarily aware of the authority relationship. What it does mean is that the bearer knowingly or unknowingly offered guidance or direction and the grantor chose to be influenced by this advice. On the other hand, if someone consents to acting or believing based on the advice or information from another because of that person's control or power over the results of obedience, they have instead established a control relationship and have granted control.

The following vignette illustrates the difference between a control and an authority relationship. One of the students establishes a control relationship with the teacher and the other establishes an authority relationship.

Both Janice and Julie are students in the same class. When their teacher asks the class to write a paragraph describing what they learned that day and what they still have questions about, both girls dutifully work on the assignment. Janice works on the assignment because she knows that if she does not do the assignment, the teacher may give her a low grade, which will prevent her from being able to play in her basketball game on Saturday. Julie works on the

assignment because she believes that doing the assignment will help her understand the mathematics better and prepare for the next day's class.

In the above vignette both students, Janice and Julie, did what the teacher asked them to do and worked on the assignment. Looking only at their actions of completing the assignment, it is unclear whether they establish a control relationship or an authority relationship with their teacher. The relationship that they establish can be determined by examining the reason for their obedience. Janice's reason for working on the assignment is so that she does not get in trouble. Thus her reason for obedience stems from the control that her teacher has over the results of her obedience, specifically that the teacher can control whether or not Janice will get to play in the game that week. Thus she establishes a control relationship and grants her teacher control as she obeys. Julie's reason for working on the assignment is that she believes the assignment will help her to learn the material better. Thus her reason for obedience stems from her belief that her teacher is the expert and asks students to participate in activities that will help them learn. Because she believes that her teacher's assignment will help her to address questions, problems, or ideas she is struggling with, Julie establishes an authority relationship with her teacher and grants her authority as she obeys.

Authority Relationships with People and with Rules

The above example with Julie is a specific kind of authority relationship. It is what I will term an *authority relationship with people*. It is this kind of authority relationship because it is a person who is seen as being able to create order and help others and it is that person who is being granted authority, in this case specifically, the teacher. There are also situations in which individuals look to a system of rules for mediation, or obey a system of rules. This is an *authority relationship with rules*, which Benne (1970) termed rule authority. Benne (1970) gave an image

of what he means by rule authority by referring to a group of people playing the game of baseball. If the group is participating in a game of baseball then they are playing by rules or obeying the rules of baseball. These rules are what determine the unique game they are playing. If the group agrees on a different set of rules, then they are playing a different game. The bearer of the authority is what Benne called “the moving spirit of the whole group” (p.398), which is this agreed upon system of rules that is ever developing and changing with the group to represent their consensus. For simplicity sake, if we remember that the rules are not stagnant but are representative of the group consensus, we can talk about granting authority to the rules.

Specific to a mathematics classroom, “the moving spirit of the whole group” is similar to what Cobb called “sociomathematical norms” (Yackel & Cobb, 1996). Mathematics classrooms develop their own set of rules that become the “moving spirit” of their mathematics class. The students and the teacher then obey a system of mathematical rules when they reason about mathematics in order to solve problems, create explanations, and evaluate solutions and explanations of others. These rules, i.e., sociomathematical norms, are developed by the class and are continually evolving as the students and the teacher engage in doing mathematics (Yackel & Cobb, 1996; Cobb & Yackel, 1996; Bowers, Cobb, & McClain, 1999).

Granting authority to people or rules is necessary at different times as a class participates in learning and doing mathematics. The students will encounter situations where it is imperative that they grant authority to people. For example, as students are developing the idea of a function, they will need the teacher to introduce them to the standard notation for functions. Students may be able to come up with clever, very similar notation for functions themselves, but they should eventually adopt the same language and notation as the rest of the mathematics community, and cannot “discover” these conventions on their own. As students participate in

doing mathematics they will also need to grant authority to rules. For example, as students are working on developing a valid justification for why $\frac{2}{3}$ is the same as $\frac{4}{6}$, they must know the rules for establishing knowledge as well as what ideas have already been accepted in order to develop a convincing justification that the class as a whole would agree with, or would see as valid. Without invoking a rule system, the students do not have a foundation to base their explanation off of and cannot write an explanation that will be acceptable to the class.

Conflating Authority and Control

Conflating authority with control has often prevented mathematics education researchers from seeing the necessity of students granting authority to the teacher while doing mathematics. Rather, they have advocated for significant reduction of the teacher's control and thus their "authority." For example, several researchers have investigated what they call "shared authority" (Wagner & Herbel-Eisenmann, 2014; Wilson & Lloyd's, 2000; Depaepe, Corte, & Verschaffel, 2012). Shared authority refers to the teacher allowing her students to be an authority on the mathematics in the classroom, thus "sharing" the authority that she has as the teacher with her students. Unfortunately this term "shared authority" is used in a way that conflicts with Benne's (1970) definition; it is misleading because it refers not to a type of authority, but a type of control. The teacher cannot share (or give) her authority with (to) someone because authority is a relationship between the student (grantor) and another individual (bearer), where the grantor willfully obeys the bearer. Shared authority, in contrast, is used to characterize the teacher relinquishing control over who can contribute mathematically to the class, or who can make a claim about whether or not an explanation is correct.

An example of this type of research is Wilson and Lloyd's (2000) case study detailing the challenges for three high school teachers to "share" mathematical authority with their students or

to encourage students to be authorities. In their descriptions of the teacher's challenges, they used language that indicated that tyrant and authority are synonymous. This view was evidenced by comments such as, "[the teachers] all predicted that their transition from 'dictator' to 'facilitator' would become easier in subsequent years" (pg. 168). Thus the difficulties moving from the dictator to facilitator, power to autonomy, is what they were studying. Though it is valuable to understand how the teacher learns to relinquish some of the control over the mathematics, this should not be confused with a study of authority.

Depaepe, Corte, and Verschaffel (2012) also looked at the teacher's "sharing" of mathematical authority, but their study gives researchers insight into how this "shared" authority, or control, may be influencing who the students will grant authority to in the classroom. The authors investigated the relationship between the distribution of this shared authority and the student's perception of the distribution of shared authority. They identified the teacher's distribution of shared authority through coding instances where the teacher indicated to the students whether it was the teacher or their classmates to whom they were allowed to go to for help, answer questions, and evaluate responses. They found that the teachers backed themselves as the authority more often than advocating that students fill that role. To identify the students perceived distribution of shared authority, they surveyed and interviewed the students about sociomathematical norms of the classroom. The authors found that the teachers' actual distribution of shared authority and the students' perceived distribution of shared authority lined up. Thus the students identified that the teacher did not "give" students authority as often as they "gave" themselves authority. The authors further identified, through teacher interviews, how the teachers thought they had distributed authority in the classroom. Teachers believed that they often advocated for students to be the authority in the classroom, but the coded classroom data

said that they did not. Though in their study of shared authority these researchers were not studying authority but control, this study suggests a fertile ground for further research on authority. The results show that the students are able to pick up on who is “allowed” to contribute to the mathematics in the classroom based off of the teachers actions. A study such as this could be beneficial in leading researchers to investigate whether the actions of the teachers which “distribute authority” matches with whom the students are actually granting authority to in the classroom. However, because the setting of this research is not an inquiry-based classroom, it still leaves unanswered the question of who or what the students turn to for authority in an inquiry-based mathematics classroom, when the teacher does not want to be the authority.

Separating Authority from Control

A few mathematics education researchers have tried to separate control from authority in their research. In their study Gerson and Bateman (2010) made two valuable contributions, but left two important issues unaddressed. Their first contribution was their definition for authority that does not conflate authority with power. What makes their revised definition of authority valuable is that rather than saying that the bearers willfully obeyed, they talked about how the claim of the bearer influences the grantors to change. The change from the term obedience to the term influence helps us to accept a broader view of “obedience” that encompasses situations when this obedience is not necessarily an observable action in itself, such as changing how one thinks mathematically. Also, the term "obedience" can be construed as meaning that the grantor is copying the bearer’s way of thinking, or thinking in the exact same way as the bearer does. This way of doing mathematics is conducive to traditional teaching settings but not to reform instruction practices and beliefs. For this study I use the term influence instead of obedient

because it better describes how students develop mathematical understanding in an inquiry-based mathematics class.

The second contribution of Gerson and Bateman's (2010) study was the types of authority they identified. One of the types of authority they identified was expertise authority. Expertise authority is the type of authority granted to the bearer because he is considered an expert on the material. Expert authority can easily be confused with control based solely on observable action or when authority is misunderstood. For example, suppose that a teacher tells her class that they all must apply multiplication before subtraction or addition when evaluating an expression that contains a string of operations. If a student uses the correct order of operations because he sees the teacher as an expert in mathematics and believes she is presenting what is true in mathematics, then he has established an expert authority relationship with the teacher. This is an example of granting expertise authority because the student believes that the teacher is a source to turn to for mathematical information. If a student uses the correct order of operations mostly because he is afraid that if he does not, he will lose points on the assignments and tests, then he has established a control relationship. This is control because the student is influenced by the teacher's control over his grade. Thus, we cannot examine just the actions of the student to determine what type of relationship has been established—an authority relationship or a control relationship; we must also understand his reason for being influenced.

The example given in the previous paragraph suggests an important limitation in the methods used by Gerson and Bateman (2010) to identify authority. Gerson and Bateman (2010) identified types of authority through methods of grounded theory and axial coding of classroom data from an inquiry-based university calculus class. Because these researchers based their analysis on videotapes of the class, the instances in which they recognized a granting of authority

were limited to situations where the bearers of authority explicitly vocalized the information they presented to help the students and where grantors outwardly indicated that they were influenced. By looking at only classroom data, the researchers had to identify the types of authority relationships according to observable actions alone; yet authority relationship types are defined based on the legitimacy according to the perspective of the grantor. Thus, the validity of their analysis is questionable. Also, reasons are not always simple, so a student may establish several different types of authority relationships in one small episode, all influencing him in some way. Thus there may be multiple types of authority being granted at the same time or in the same episode by one person. Because Gerson and Bateman may have incorrectly attributed types of authority relationships, overlooked other instances where authority relationships were established, and missed additional types of authority relationships.

Another type of authority that Gerson and Bateman identified was mathematical authority. Mathematical authority is the type of authority granted to the bearer because the grantor recognizes the bearer's mathematical explanation as valid for supporting the claim. For example, suppose that Bob and Mike are working together on their homework. Bob is having trouble creating a good explanation that justifies the fraction multiplication algorithm, so he listens to Mike's explanation closely. Bob decides that Mike's explanation makes sense and adequately justifies the conclusion. In this scenario, we can identify that Bob granted Mike mathematical authority because it was Mike's mathematical explanation that convinced Bob to accept his solution, and thus gives evidence for the fact that it is influencing Bob. Gerson and Bateman (2010) acknowledged that this type of authority is very different from the other three because it is not legitimized based on a person's perceived status, but on his use of mathematics. Because of this difference, mathematical authority is not as easily mistaken for control and other

authority relationships, but it can still be incorrectly identified when there is no access to a student's reason for allowing someone to influence him.

Mathematical authority is very valuable to identify because it shows that students are convinced by someone's use of mathematics. However, the question of what made the explanation make sense to Bob is still present. One thing that needs to be further addressed by researchers is that the definition of mathematical authority does not give clear enough insight into the grantor's reason for being convinced by the bearer's use of the mathematics. Thus, the definition of mathematical authority is too broad to capture the important aspect of why the grantor granted authority; it only indicates that the mathematics influenced Bob, but does not give us insight into how this conclusion was determined by Bob. The examples of mathematical authority that Gerson and Bateman (2010) presented are similarly not descriptive as to why mathematical authority was granted, so another construct is needed to help us understand how mathematical authority is established.

Each of the types of authority presented by Gerson and Bateman (2010) fit into the category of an authority relationship with people, because the grantors are being influenced by their belief in a person's ability to offer help and/or create order in society, and are granting authority to a person. However, unlike expertise authority, mathematical authority has an emphasis on the believability of the mathematics being presented. Furthermore, to determine the believability of the explanation, the grantor must appeal to the rules for doing mathematics. This granting of mathematical authority is what we want our students to be doing in inquiry-based instruction, but the construct lacks enough detail to give insight into how mathematical authority is granted. To better understand how mathematical authority is being granted, we need to understand what rules for doing mathematics the student is appealing to and how he goes about

appealing to these rules. Thus we need to know what the student identifies as the sociomathematical norms for writing and evaluating explanations.

Sociomathematical Norms

The second feature of the Discourse that may be preventing students from evaluating mathematical explanations based on the mathematics is the system of sociomathematical norms that a student has identified as making up the rules for writing explanations. Sociomathematical norms are the taken-as-shared rules for participating in mathematical activity associated with all of the mathematical topics discussed in the community (Bowers, Cobb, & McClain, 2009). One mathematical activity that is a major part of an inquiry-based mathematics classroom is that of creating and evaluating explanations. Sociomathematical norms are established by the teacher and students in a class specifically for the creation and evaluation of explanations, and give direction to the mathematics community in the classroom for these activities. According to Cobb these sociomathematical norms are emergent, or in other words, are built by the teacher and students while participating in mathematical activity (Cobb, 1999; Cobb & Yackel, 1996). Thus no two sets of sociomathematical norms from two different classrooms will be exactly the same, because they are specific to a given community. Furthermore, because they are taken-as-shared, the set of sociomathematical norms that one student has accepted may not be the same as those another student has accepted. Thus, to understand difficulties that students have in their evaluation of explanations, it is important to look at what the individual student takes as being the sociomathematical norms, as well as when she uses those sociomathematical norms in her evaluation.

Sociomathematical Norms and Explanation Evaluation

Researchers have identified sociomathematical norms associated with creating explanations and making sense of solutions and explanations being presented in class (Cobb & McClain, 2001; Fukawa-Connelly, 2012; Yackel, Rasmussen, & King, 2000). For example, Cobb and McClain (2001) analyzed video data from the teaching experiment they conducted in a first grade classroom. From their analysis they identified three sociomathematical norms that informed the students' explanations of their own work and evaluations of their classmates' explanations. One of the sociomathematical norms identified was "acceptable explanations." An acceptable explanation in the class that they were studying required references to physical quantities and actions on those quantities. For example, when explaining what the sum of 2 and 2 is, the student would have to model the solution with manipulatives or talk in terms of actions on a quantity such as taking some away or adding more to an amount. Thus, an analysis of this study shows that there are sociomathematical norms that define what makes a "good explanation," but the study does not describe when the sociomathematical norms are used to evaluate arguments. It would be valuable to extend this research by analyzing when individual students evaluate explanations using sociomathematical norms.

Fukawa-Connelly (2012) worked to identify sociomathematical norms to describe how students made sense of arguments being presented in a university abstract algebra course. The sociomathematical norms were identified by analyzing the classroom video data and verified by identifying evidence that an explanation was accepted or rejected by the class because of the violation of that norm. For example, three of the sociomathematical norms that were identified are "justify new inferences based upon old ones," "explaining and defending your work," and "only using peer-validated knowledge." Each of the sociomathematical norms identified by

Fukawa-Connelly (2012) could look very different depending on the class in which they are enacted. For example, in both a traditional and an inquiry-based mathematics classroom, students may be expected to explain and defend their work, but what that actually looks like in each class may be starkly different. Thus these three sociomathematical norms that were identified by the author give us an idea for the types of sociomathematical norms that students may be using for evaluation, and thus give an idea for the types of sociomathematical norms that may show up in my data. Nevertheless the limited description given in the definition of the sociomathematical norms does not give enough detail to help us understand why the system of rules may not be affording the student success in evaluating an argument. For example, it is unclear what “explaining and defending your work” means to students in the class, and whether those different meanings account for variation in students’ success at evaluating explanations. When identifying the sociomathematical norms in this study, I will want to be very detailed in my descriptions so that I can better understand how the type of sociomathematical norms may be impacting the student’s evaluation. In particular, I will need to determine what sociomathematical norms study participants use, and their meanings for those norms.

Because these studies focused on identifying sociomathematical norms that are used by the class as a whole to evaluate explanations, the researchers did not get an in-depth look at what types of sociomathematical norms an individual student uses to evaluate explanations. Furthermore, these studies did not attempt to carefully articulate when the sociomathematical norms were used to evaluate an argument, just that they were used. Thus more needs to be done to understand what types of sociomathematical norms a student appeals to and when an individual student evokes sociomathematical norms as they are evaluating arguments on their own.

Classroom Mathematical Practices

Some sociomathematical norms guide how a student uses classroom mathematical practices in her evaluation. Classroom mathematical practices are “the taken-as-shared ways of reasoning, arguing, and symbolizing” (Cobb, Stephan, McClain, & Gravemeijer, 2001). Like sociomathematical norms, mathematical practices are ever developing and changing. But unlike sociomathematical norms, which apply to all mathematical topics in the classroom, classroom mathematical practices are specific to particular ideas or topics within mathematics. For example, Bowers, Cobb and McClain (1999) identified the mathematical practice of counting arrangements by 100’s, 10’s, and 1’s in a first grade classroom. Students were working in the context of a candy factory where the company sold individual candies, rolls of candies, and boxes of candies. It became taken-as-shared in the class that they could identify the number of candies in a configuration by operating on boxes (100’s), rolls (10’s), and pieces (1’s) as units. Operating on units is a classroom mathematical practice and not a sociomathematical norm because this way of thinking is specifically applicable to situations where the class is actively appealing to the base ten number system to solve and understand problems such as addition and subtraction, and would not be generally used for reasoning or talking about all mathematical content.

Classroom mathematical practices are built and accepted by the class as a whole and once established, students can use these practices in their explanations without justification. For example, operating on units was considered an established mathematical practice when students no longer had to justify why they were thinking of a roll as a unit of 10 and a box as a unit of 100 when explaining their thinking to their classmates. Thus, when reading a classmate’s explanation, a student will be required to have some knowledge of the mathematical practices so

that he can make sense of the argument being presented. Since arguments are made up of these mathematical practices that are used without justification, students may also evaluate arguments by appealing to the classroom mathematical practices of the class. The rules that a student appeals to in order to guide how she uses classroom mathematical practices in her evaluations are sociomathematical norms.

There are many different roles classroom mathematical practices could take in the evaluation of explanation, one of which is presented in the literature. Stylianides and Ball (2008) presented an example that shows a student rejecting a classmate’s argument because of a violation of a mathematical practice (although they did not describe the incidence this way). The class in the authors’ study was trying to establish whether or not an odd number plus an odd number is always an even number. The argument that one student presented was that it was not always true because she found an example where an odd number plus an odd number is odd. She then wrote up her example on the board (see Figure 1).

$$\begin{array}{r}
 101 \\
 + 1000007 \\
 \hline
 2010007
 \end{array}$$

Figure 1: Contradiction to a sum of odds always being even

The objection to the argument offered by the classmate was that she “did not line up the two numbers correctly, thus adding hundreds with millions” (pg. 19). Then the objecting student fixed the line-up of the numbers, placing 101 just over the 007 at the end of the second number. In the evaluation, she appealed to the classroom mathematical practice that when using the standard algorithm for addition, it is necessary to line up the numbers correctly. It is this

violation of the practice that the student used to reject her classmates explanation. Thus the classroom mathematical practice was treated as something that could not be deviated from to get a good or valid explanation. This example gives evidence that students do evaluate explanations by appealing to their understanding of the classroom mathematical practices and that in this case, the classroom mathematical practice was treated as something that could not be violated.

Because classroom mathematical practices and their application impact students evaluation, it will be useful to identify when a student is appealing to classroom mathematical practices to evaluate explanations as we work to understand why students are not successful at judging the validity of explanations based on mathematics. Because the classroom mathematical practices that a student picks up may be slightly different than those observed in the classroom, it will be important to identify what the individual student sees as classroom mathematical practices. Thus, to see how classroom mathematical practices influence the students evaluation of explanations we need to look from the perspective of the individual student, to identify what classroom mathematical practices she is using as well as when she is appealing to these practices in her evaluation.

In summary, authority, sociomathematical norms, and classroom mathematical practices as defined above are at least three features of the Discourse of a classroom that impact a student's evaluation of and explanation in mathematics. Thus these features may be preventing students from judging the quality of mathematical explanations based on the mathematics. In order to understand how these features may be impacting a student's evaluation of an explanation, we first need to identify when the student uses these features to evaluate explanations. The following study has been designed to answer the research question: How does the use of authority, sociomathematical norms, and classroom mathematical practices impact a

student's evaluation of explanations in an inquiry-based mathematics classroom? In the next chapter I will present the data that was collected and how it was analyzed in order to answer this research question.

Chapter 3: Methods

In this chapter I will present the methodology for my study. Because of the limited research on student evaluation of explanations in an inquiry-based mathematics classroom, it is appropriate to conduct a case study to gain some traction in understanding the difficulties that students have with evaluating explanations. In the following sections I describe and justify the setting, participants, and data. Then I describe the steps I took in data analysis to reach my results.

Data Collection

Initial Study

The data for this case study comes from a research project conducted by Daniel Siebert at Brigham Young University (BYU). The participants in Siebert's project were the teacher and students in a pre-service elementary school teacher mathematics course at BYU. This was the second course in a series of two courses on mathematics content that the students take in their preparation to become teachers. The instructor Annie was a female faculty member with a doctorate degree in mathematics education. This instructor was experienced in teaching inquiry-based mathematics courses for pre-service elementary school teachers and had taught this exact course twice before. The class consisted of about twenty students, one male and the rest female. The class met twice a week for two hour classes the entire semester. The content of the class included units on fractions, fraction operations, and probability and statistics. The students sat at tables in groups of about four to six students to facilitate collaboration during classwork activities.

Data collected in class consisted of field notes, classroom video recordings, video recordings of individual group work (the last half of the semester), and student work. The field

notes were recorded by a research assistant and included a general outline of the activity in class along with time stamps to help capture a summary of the data in the video. The classroom video data was collected from a control area in the corner of the room where a research assistant controlled the direction of the camera mounted to the ceiling in the center of the room, as well as the volume on the teacher's personal microphone and the three classroom microphones at the front of the room. The group video data was collected through the same video camera mounted to the ceiling and group microphones placed at the center of two of the group's tables. The research assistant adjusted the volume of the microphones and the focus of the camera to capture the teacher's interactions with the individual groups. Copies of assignments and student work (including classroom tasks and completed homework, tests, and classwork) were collected. This provided a record of each student's performance on assignments in class, as well as a record of what was being discussed in the lessons.

The second part of the data collection consisted of out-of-class interviews performed throughout the semester on four students. These students were the only four who volunteered to participate in interviews and so with this small number of volunteers, all were taken as interviews participants. There were four interviews conducted with each of the four female participants who volunteered to participate in the interviews. These interviews were audio recorded and written student work was collected. The interview subjects had varying levels of confidence and ability in their performance in the class and in mathematics overall. The first interview was conducted within the first two to three weeks of the class and the subsequent interviews were conducted every two to three weeks. These interviews were conducted by Siebert and the questions focused on how the students determined the validity of solutions and their accompanying explanations.

Current Study

The primary data that I used to answer my research question were the recordings and transcriptions of the interviews conducted on three out of the four participants: Melissa, Claire, and Lynn. Since the application of this framework in looking at student evaluation is still new, it was appropriate to have only three students for sufficient comparisons in my study. Melissa and Lynn were chosen out of the four because they expressed difficulty in evaluating explanations and Melissa very explicitly discussed how she applied authority in her evaluations. Claire and the fourth student both seemed to be successful in the course and in evaluation and so I chose only one of those two students as a comparison with the other two; I anticipated that the fourth student would not have added much to the findings of the study. Once student's evaluation of explanations is better understood, then the field would benefit from studies with a larger number of participants using my framework. Also, in order to perform an in depth case study on each student, a small sample size was necessary. The interview recordings and transcriptions were appropriate data to analyze because in the interviews each participant was asked how she evaluates explanations in general, and how she evaluated specific explanations presented in class. In the third interview, participants were asked to evaluate a series of explanations for $\frac{3}{4} \times \frac{2}{3}$. I refer to this series of explanations as the evaluation task. This task yielded particularly rich data from all participants about how they evaluated explanations. Four of the explanations that participants were required to evaluate are given below (see Figure 2). These particular explanations from the evaluation task are referred to frequently in chapter four. I have given these explanations titles for later reference in my results but the students were not given the explanations with the titles. These titles were chosen to help the reader more easily identify

which explanation I am referencing in my results.

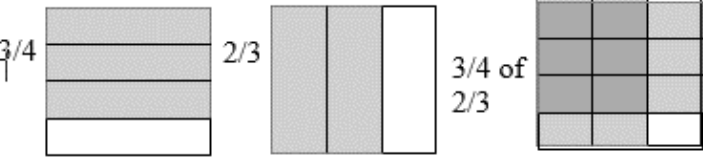
Explanation Title	Text Student Received	Expert Evaluation
Building up Explanation	To find $\frac{1}{4}$ of $\frac{2}{3}$, I took each third in my whole and divided it in half. Then there were two pieces in each third, so four equal pieces in the $\frac{2}{3}$ and six equal pieces in the whole. So $\frac{1}{4}$ is one of those pieces, or $\frac{1}{6}$. To find $\frac{3}{4}$ of $\frac{2}{3}$, I subtracted $\frac{1}{4}$ of $\frac{2}{3}$ from $\frac{2}{3}$. $\frac{2}{3}$ is $\frac{4}{6}$, minus $\frac{1}{4}$ of $\frac{2}{3}$, which is $\frac{1}{6}$, is $\frac{3}{6}$. So $\frac{3}{4}$ of $\frac{2}{3}$ is $\frac{3}{6}$, which is $\frac{1}{2}$.	Though the reasoning is roundabout, the final product results in the $\frac{2}{3}$ being split into 4 equal pieces and taking 3 of them. Thus, this is an adequate explanation.
Algorithm Explanation	$\frac{3}{4} \times \frac{2}{3} = \frac{3 \times 2}{4 \times 3} = \frac{6}{12} = \frac{1}{2}$	This explanation is not considered adequate because it is a direct application of the algorithm without a justification using pieces.
Overlapping Grid Explanation	<p>I started by making $\frac{3}{4}$ and $\frac{2}{3}$ in the same whole.</p>  <p>Then I put $\frac{3}{4}$ on top of $\frac{2}{3}$, and the double-shaded part is $\frac{3}{4}$ of $\frac{2}{3}$, which is $\frac{6}{12}$ or $\frac{1}{2}$.</p>	The strategy of overlapping the wholes split into the appropriate number of equal pieces has the result of splitting the $\frac{2}{3}$ into 4 equal pieces and those pieces happen to be represented in the double shaded section but this strategy would not work with fractional amounts larger than 1. So this is not a correct explanation.
Commutative Explanation	<p>$\frac{3}{4} \times \frac{2}{3}$ is $\frac{3}{4}$ of $\frac{2}{3}$. So split each fourth into three equal parts, and take two of those three equal parts. That's 3×2, and there are four groups of 3 equal parts in the whole, so that's 4×3, so $\frac{3}{4} \times \frac{2}{3}$ is $\frac{3 \times 2}{4 \times 3} = \frac{6}{12}$.</p>	This explanation is not correct because the solution describes taking $\frac{2}{3}$ of $\frac{3}{4}$, which corresponds to $\frac{2}{3} \times \frac{3}{4}$, not $\frac{3}{4} \times \frac{2}{3}$.

Figure 2: Evaluation Task

Individual interviews were needed in order to determine how each student appeals to authority, sociomathematical norms, and classroom mathematical practices in her evaluation. As was discussed above in chapter 2, authority cannot be identified solely by the class's reactions to the explanations presented in class, but must be investigated based upon the student's description of why she was influenced. Additionally, because students could use sociomathematical norms

or classroom mathematical practices that are not consistent with those emerging in the class, it is not enough to look at how the class as a group evaluates explanations to determine how an individual student is evaluating. I needed to look at the use of sociomathematical norms and classroom mathematical practices of each individual student as she explains the reason for her evaluation.

Data Analysis

The goal for my data analysis was to construct a model for each of the students that described in detail how she evaluated explanations, in particular, how she used authority, sociomathematical norms, and classroom mathematical practices to evaluate explanations. I analyzed one student's interview data and wrote up their evaluation model before moving on to the next student so that I would not confuse students' evaluation methods. Thus, each of the three passes described below were conducted on the interview data for one student before moving to the next student.

Pass I

In my first pass through the data for a student, I coded her responses to the evaluation task from the third interview. The reason I started with this segment of interview data was because it contained six individual evaluations where she actually evaluated and explained her evaluation for each explanation. This allowed me to generate some hypotheses of how she evaluated explanations before turning to the rest of the data, which consisted mostly of her talking about her evaluation in general or of specific explanations given in class.

I started this pass by chunking the data into evaluation instances. An evaluation instance is a segment of the data where the student is evaluating one single explanation. The reason I chunked the data in this way was to make comparisons across the student's evaluations of

explanations later on in the analysis. In order to answer my research question, I identified all of the reasons—the justifications for why an explanation was good or bad—in each evaluation instance, and then coded each reason for three features of Discourse: authority, classroom mathematical practices, and sociomathematical norms. Giving a reason in the evaluation instance one of these codes indicated that the student was using that feature of the Discourse in her evaluation. Reasons could be sentences or they could be paragraphs in length. Because the types of authority identified by Gerson and Bateman (2010) were defined based on the student’s reason for allowing someone to influence her, coding for reasons for the evaluation allowed me to then take these segments and categorize them into types of authority. My initial set of authority codes consisted of the types of authority identified by Gerson and Bateman. Additional authority codes were added as necessary.

Coding of sociomathematical norms and classroom mathematical practices was restricted to the reasons that had been coded as mathematical authority. It was only these reasons, where the student was discussing why she was convinced by the mathematics, where she might appeal to sociomathematical norms and classroom mathematical practices. When a reason was given the code classroom mathematical practice, it was because the student was drawing on specific ways of “reasoning, arguing, and symbolizing” (Cobb, Stephan, McClain, & Gravemeijer, 1999) that are tied to the particular mathematics content being addressed in the evaluation. In other words, segments coded as classroom mathematical practices were segments where a classroom mathematical practice was being appealed to.

Once coded for classroom mathematical practices, the mathematical authority reasons were then coded for sociomathematical norms, segments where the student communicates about what does or does not make an acceptable explanation in mathematics. I encountered some

difficulty initially in coding for sociomathematical norms. This difficulty came because sociomathematical norms are used to build classroom mathematical practices. When I encountered mathematical practices in the reason, I often could infer some of the sociomathematical norms that were used to guide the construction of that practice. For example, in Melissa's evaluation of one of the explanations, she said, "Cause if this is the way they are doing this every single time, are they fully grasping that really that it is a group of this?" At the end of this segment, Melissa is appealing to the classroom mathematical practices that multiplication of $a \times b$ means *a groups of size b*. This classroom mathematical practice is built on the sociomathematical norm that when justifying one must talk about quantities and operations on quantities. So it is tempting to also say that Melissa is appealing to that sociomathematical norm when she invokes this particular mathematical practice. But because the classroom mathematical practices are what is explicitly being appealed to, and not the sociomathematical norm, I only coded this reason with the code classroom mathematical practice. So I was careful not to code reasons such as these with sociomathematical norm, but only reasons where the student was directly appealing to a sociomathematical norm.

Initially I coded for classroom mathematical practices before sociomathematical norms because I thought these would be easier to identify in that they often refer to actions, and thus would be easier to hear in the student's critique. This was not necessarily true in my coding experience. I eventually found that the main distinction between these two codes that helped me to identify them was whether the student was appealing to something that was more specific to an explanation for that mathematics topic or if she were appealing to something that would equally apply across all explanations. This distinction can be thought of in terms of a continuum, for it is sometimes difficult to define a point from which an idea that is taken-as-shared jumps

from being specific to a particular topic to being general to enough topics that it should be classified as a sociomathematical norm instead of a classroom mathematical practice. Fortunately, most of the data fell clearly on one side of the continuum or the other. So I eventually transitioned to taking each reason coded with authority and asking if it were a classroom mathematical practice or a sociomathematical norm. Because I coded for authority by chunking the data into reasons for the evaluation, almost every reason coded as mathematical authority was associated with either the code sociomathematical norm or classroom mathematical practice. There were only a few instances where there were two codes to a mathematical authority segment. So for most of the data from the evaluation task, I coded for sociomathematical norms and classroom mathematical practices simultaneously. This was a much more efficient method because in coding for just one feature I was naturally attending to the other already, and so it allowed me to comb through the data once to attend to both features. After the small segment of interview three was coded for one student I recoded the same segment of data to make sure that my coding matched closely and to catch anything I may have missed.

With my coding completed I worked to find patterns in the types of authority, sociomathematical norms, and classroom mathematical practices appealed to by the student, as well as when the student appealed to each of these. Identifying these patterns allowed me to describe how the use of authority, sociomathematical norms, and classroom mathematical practices impact the student's evaluation. To find patterns in the types of each of the features of the Discourse I looked at the collection of codes corresponding to each feature and categorized them to look for the most common themes. For example, I took all of the reasons coded with sociomathematical norms and sorted them into groups where the sociomathematical norm that was being appealed to seemed similar. After some work identifying the similarities between

these reasons, I was able to attach titles to the sociomathematical norm the student was appealing to in that collection of reasons. To find patterns in the way they used sociomathematical norms and classroom mathematical practices, I looked at the evaluation instances and looked for patterns in which features appeared in the evaluation instance and in what order they appeared in. Once patterns were identified, I wrote up hypotheses for how the student evaluated an explanation using these three features of the Discourse, accompanied by support from the data for each hypothesis. For example, in looking for patterns in evaluation instances, I found that Melissa would rely primarily on classroom mathematical practices in her evaluation of each of the explanations. Thus I created the hypothesis that for Melissa it was important that the classroom mathematical practices were met in order to have a good evaluation.

Pass II

In my second pass through the data I coded the rest of the interview transcripts. If I encountered an evaluation instance, I coded it the same way I did the data in the first pass. If the student was instead discussing how she used a feature of the Discourse to evaluate explanations in general, I coded that paragraph or several paragraphs according to the feature of the Discourse she was discussing. The data where the student was describing how she used each feature of the Discourse helped me to interpret the evaluation instance data better in terms of how the student thought about evaluation so that I could describe her use of the three features of the Discourse with an interpretation that was more true to the student's intended meaning.

After all of the data for an individual student was coded according to the feature of the Discourse it corresponded to, I used this data to verify and refine my hypotheses. I looked at each segment and asked if it fit into the current model I had created with my hypotheses. If the segment did fit my hypotheses, I either added it to my results section as evidence for my

hypotheses if it gave more insight, or recorded the fit with my hypotheses in the transcript in case I needed to use it as support in my writing later. If the segment did not fit my hypotheses, I tried to modify the hypotheses to fit this new segment of data, while at the same time being sure that the modification fit the data I had already analyzed. If I had a really difficult time fitting a segment into my hypotheses, I set it aside as an example of a contradiction that needed to be fixed. If I was left with many contradictions after my modifications, I rejected the current hypotheses and tried to construct new ones with these contradictions in mind by redoing all or some of the passes over again so I could change the way I interpreted certain comments or moves. This helped me to come to interpret the data in a way that allowed me to create hypotheses that fit the data. After this second pass was through, I had a detailed model of the student's evaluation, which consisted of hypotheses for how the student used authority, classroom mathematical practices, and sociomathematical norms.

The model for each of the students consisted of a set of detailed hypotheses of her use of authority, classroom mathematical practices, and sociomathematical norms. This included what role expertise authority played in her evaluation and to whom she would grant expertise authority. It also included the types of sociomathematical norms and classroom mathematical practices, as well as when the norms and practices were being appealed to. These models were predictive, meaning that they could be used to predict how each student would typically evaluate an explanation, including the thought process the students took to make their conclusion.

Pass III

For the final pass through the data I had two tasks. One was to check the validity of my model for the student. So again the goal was to identify any segments that were not explained by my model. The second was to get a sense for the student's background with mathematics. To

accomplish the first goal, I reread my model for the student's evaluation. Then I quickly reread the transcripts for each of the student's interviews. As I was reading, I looked for violations of my model and took note of any I found. Once these were identified, I looked for ways my model could be modified so that it was consistent with previous analysis and these new violations. I was able to fit most of the data into my final model. The remaining data that did not fit the model consisted of two types. First, there were a few segments where the students would directly contradict themselves in the same discussion. Second, there were segments where the students were not very clear as to what they meant by what they were saying and there were little clues for making sense of the statements. Depending on the interpretation, these ambiguous statements could mean contradictory things. Because the problems with these statements seemed to be problems with the data and not the fit between the statements and the model, I did not try to further modify my model to accommodate these statements.

To accomplish the second task, I read the first and fourth interviews for each of the students. While reading the first interview, I focused on the conversation the student had about her past experiences in mathematics, and tried to summarize it with some quotes. While reading the fourth, I looked for any changes the students had in their relationship with or experience with mathematics. This analysis was used to form a mathematics background description for each of the students. The purpose of the background description was to understand how these students viewed mathematics. The way that each students viewed mathematics could then be used to tie my results back to the literature. This is important because the literature typically portrays students who are not successful at evaluating explanations as those who do not like learning mathematics through inquiry or do not have the mathematical knowledge necessary. I had to determine whether or not my students exhibited these characteristics.

Once I had completed these three passes through the data for a single student I had a final model for that student's evaluation methods using the three features of the Discourse, and a text summarizing their mathematics background. Once I completed one student's model, I set it aside and moved on to creating the next student's model until I had finished models for each of the students.

Chapter 4: Results

In this chapter I will first present my results on the evaluation model for each of three of our interview participants Melissa, Claire, and Lynn. Then I will discuss the comparisons between the students' models and the importance of these comparisons. Following this discussion I will present my results identifying a new type of authority and discuss its importance.

Melissa

Background

Melissa had almost always had positive experiences in mathematics. When asked about her experience in mathematics classes growing up she said, "I loved math all through elementary school, high school." The one bad experience she had in mathematics was in her high school AP Calculus class. She said, "I hated everything about it, because I don't get it." This was the first mathematics content that she had encountered that was difficult for her and it stuck out very strong in her mind as a difficult experience.

At the beginning of the course Melissa was uncomfortable with the way Annie taught, which teaching method she called "learning by experience." She liked Annie's way of teaching and did acknowledge that it had some benefits in terms of remembering content better over time, but wanted to be told what was right by her teacher because the teacher knew what was correct. In fact she said, "I think [the class is] beneficial for people who learn by experience. I'm not one of those people. I learn better by being just told, and then I remember it." Yet by the end of the course, Melissa disagreed with some of her classmates who had said, "I would rather she [Annie] would just tell us [the solution]." Melissa countered with, "You make more connections when

she [Annie] is like, why?" So by the end of the course Melissa found great value in inquiry-based teaching practices.

Melissa's Evaluation Method

There are two main characteristics of Melissa's evaluation method. The first is that she used her teacher to evaluate explanations by tracking Annie's reactions to explanations, using feedback from Annie on her written work, and appealing to classroom mathematical practices that Annie had sanctioned in class. The second characteristic is that she relied on classroom mathematical practices to make her evaluation in a way that shows she saw these practices as part of the rules for doing mathematics.

Authority. The first characteristic of Melissa's evaluation method is that she used her teacher to evaluate explanations, and thus granted expertise mathematical authority to Melissa. One way that Melissa directly used Annie to evaluate explanations was by looking at Annie's reactions to explanations given in class. When an explanation was presented, Melissa looked over at the teacher to see her reaction to the explanation. If Annie smiled and nodded, Melissa concluded that the explanation was good, and if Annie looked confused, Melissa concluded the explanation was bad. By monitoring Annie's facial expressions, she was able to get an expert opinion of the explanation. Melissa often did her best to evaluate an explanation before she looked at Annie to see if Annie understood or liked the explanation, but nonetheless would eventually look to Annie for approval.

Melissa described and showed evidence of her use of Annie's facial expressions throughout the interviews. The following was one of the many excerpts where she explained how she knew that an explanation was good.

I think it's the same lines as the explanation. If Annie, like, I can read Annie, and if Annie, like, smiles, or if I, if that's the answer I've kind of got, then I assume

they're mathematically correct. So, I think it's just an assumption. Like, I don't absolutely know until Annie is like, "OK. Good."

Melissa looked to her teacher to tell her or give her clues as to whether or not the explanation was good. Melissa did show some evidence, here and more in other parts of the interview, that she was willing to make some assumptions on her own as to whether or not the explanation was correct. But as she articulated well in this segment, she did not conclusively know whether it was a good explanation until Annie indicated it was by her facial expressions or her comments.

The second way that Melissa directly used Annie to evaluate explanations was in Annie's comments on Melissa's (or her group's) written explanations. She placed great weight on the feedback and scores she got on her homework. If an explanation was not marked wrong, it was seen by Melissa as a correct way to explain something. Thus, in Melissa's evaluation of her own work, she said that she knew her explanations were correct because she had given the same explanations on the homework and had not gotten it marked wrong. For example, when asked if she was confident in the explanations she had given Siebert in an interview, Melissa responded with, "Yeah. I think, I feel like they're what I wrote on the [homework] paper, and she hasn't marked them wrong yet. So, I feel like they're confident enough. I would write them on a test." So the reason that Melissa cited here for her positive evaluation of her own explanation was that she had written the same thing on her homework and it was not marked wrong. Thus Melissa's use of the feedback on her homework to evaluate her explanations was an instance of Melissa granting expertise authority to Annie. Annie did not mark it wrong so it must be correct. The same use of Annie's feedback held for explanations that Melissa wrote in class and got Annie to comment about.

The third way that Melissa used Annie to evaluate explanations was in her appeal to classroom mathematical practices. At first this may seem like an odd claim because according to Cobb and his colleagues, classroom mathematical practices are taken-as-shared, emergent, and negotiated between all members of the classroom rather than dictated by the teacher. But for Melissa, classroom mathematical practices seemed to be the particular practices that Melissa believed had been confirmed by Annie to be the accepted ways of reasoning, arguing, and symbolizing in mathematics. This confirmation by Annie took many different forms for Melissa. Sometimes Annie explicitly modeled a way of arguing about the mathematical concepts, whereas other times Melissa picked up on these classroom mathematical practices from Annie's participation in the class.

To illustrate, consider first an example of how Annie explicitly taught Melissa how to make sense of fraction multiplication problems. In the following segment Melissa was commenting about how the most recent unit was going for her and in doing so described some of the help that she has received from Annie.

Good. I went and saw Annie for help on homework, which was really helpful. It was, like, the ones where we had to write [fraction multiplication] problems with it. With, like, the two-fourths divided [*sic*] by this number. I had a lot of issues with it, but I went and talked to Annie, and she's like, "Remember, it's two-fourths of three-fourths. So, like, just remember that's how it works out." So, that helped me.

The above segment describes the classroom mathematical practice that Melissa received directly from Annie. The mathematical practice consisted of conceptualizing the multiplication problem of $a \times b$ as meaning a groups of size b , or the first number amount "of" the second number. Evidence that Melissa saw this way of reasoning as a classroom mathematical practice was that she used it to evaluate all but one of the explanations that Siebert gave her later on in the

same interview. Melissa was given six different explanations and asked to evaluate them out loud. The following was Melissa's evaluation of the commutative property explanation.

So they are starting with a different whole. So they are taking a fourth which means they would start off with a whole that was split into 4 equal pieces and then they split each into 3 parts. Which makes me think that they don't have that this is $\frac{3}{4}$ of $\frac{2}{3}$ even though they said it. Does that make sense? [reading aloud] and take 2 of those 3 equal parts, that's 3 times 2. 4 groups of 3. [To Siebert] okay I feel like they don't, they are not grasping this "of" thing. Even though they said it, they don't get it because they are starting off with lets draw the $\frac{3}{4}$ first and we'll split each fourth into 3 equal parts. It's kind of like, I don't know. I feel like they don't believe this [$\frac{3}{4} \times \frac{2}{3}$ means $\frac{3}{4}$ of $\frac{2}{3}$], but that it [$\frac{3}{4}$ of $\frac{2}{3}$], was needed so they wrote it. They didn't write it because they understood it. Which I have done multiple times.

In this evaluation Melissa used the classroom mathematical practice that $a \times b$ is a groups of b throughout the segment. She appealed to this classroom mathematical practice in the beginning where she said that the solution writer had stated that $\frac{3}{4} \times \frac{2}{3}$ means $\frac{3}{4}$ of $\frac{2}{3}$ but that his actions with the pieces were not consistent with that claim. He had violated the practice because he started with a whole split into 4 equal pieces rather than a whole split into three equal pieces.

Another way that Melissa acquired these classroom mathematical practices from Annie was by listening to how Annie talked about mathematics. According to Melissa the language that Annie used was the language that Annie wanted her students to adopt, and thus was seen by Melissa as the way the class talks about particular mathematical ideas, or creates good explanations. The following segment provides evidence for how Melissa picked up on these practices and norms.

So I think Annie, she'll like, let us know what she's looking for, but it's not direct. Does that make sense?...I think it's just the way she explains things. Like, when we do explanations as a class, like, she ... Instead of saying, "You're wrong," she'll be like, "Oh, so what you mean is ..." and she'll say it, like, when we go over questions in class, and she'll say it. That really helps me, 'cause I'm like, "Oh ..." I listen to her vocabulary she's using, and the language she uses.

And then that's like, "OK. Mine needs to sound similar to that, because that obviously is what she wants us to say."

Melissa easily picked up on Annie's ways of talking and believed her explanations should be like Annie's.

So we may question whether or not classroom mathematical practices had to be confirmed by Annie in order for them to be considered as such by Melissa. The following conversation between Siebert and Melissa clearly identified Melissa's need for classroom mathematical practices to be confirmed by Annie for her to consider them as such.

Siebert: So, I notice that in class, sometimes Annie will tell you if you're right or wrong, and then sometimes she won't. OK. So, have you ever picked up [on] a pattern of when she'll tell you and when she won't tell you?

Melissa: I think when it comes down to, like, a learning, like, you have to absolutely understand this. Like, you aren't getting it, we've tried the whole, like, "Oh, we'll learn it or maybe not learn it," thing. And then, when it comes down to it, and you, like, she thinks you get it, like, she'll be like, "That's right." Or, if she thinks you aren't really understanding it, like, "That's wrong. Let's work on it a different way...." When she's like, "Oh, I don't know," she wants us to still learn from it, and we haven't even got to the point where we need to grasp it yet. Does that make sense?

In this segment we read that Melissa had recognized that Annie would not let the class think about a problem incorrectly for too long. If the class was getting really confused and their ideas needed to be corrected, Annie would let them know that their thinking was incorrect or unproductive rather than let them continue to work with the wrong ideas. This reasoning suggests that Melissa believed that Annie confirms all the sociomathematical norms and classroom mathematical practices that are being developed by the class, either by telling them they are correct explicitly or by not objecting to ongoing practices. Melissa seemed confident that if students were thinking incorrectly, Annie would let them know.

In all three of these different ways that Melissa granted authority to Annie, there is evidence that she was granting expertise authority. When Melissa depended on Annie's facial expressions it is significant that she looked to Annie's facial expressions alone, not her fellow classmates. When Melissa depended on Annie's comments or feedback on explanations it is also significant the weight that she put on Annie's comments, as though what Annie said was fact. When Melissa granted authority to Annie in order to gain classroom mathematical practices she often actively sought Annie out though Annie was not explicitly stating the classroom mathematical practices and sociomathematical norms. This suggests that in all three of these situations Melissa was likely granting authority to Annie either because she was in charge or because she is the expert. The following segment provides further evidence that in most cases Melissa is granting authority to Annie because she sees her as an expert.

In this situation [in her class], Annie is my teacher for MathEd, but she's also a mentor in that aspect, as she knows how to teach children. So, I'm kind of like, "OK, you're the person that's going to be teaching me all of this stuff, what do I need to do? What am I supposed to say? How am I supposed to teach math to kids?" In my mind, there's got to be a right way, and Annie knows the right way, 'cause she's teaching it to us.

This quote suggests that when Melissa looked to Annie for help or to read Annie's facial expressions, it was because she considered Annie to be the expert. In Melissa's statement it is almost as though teacher is being equated with expert because she says that Annie is the person who is going to be teaching her and Annie must know the right way. Thus Annie was likely consistently seen as Melissa's mentor and knows what it was Melissa was trying to learn. Thus, Melissa was likely often granting Annie expertise authority.

In addition to gaining sociomathematical norms and classroom mathematical practices from Annie, Melissa was also able to gain confidence in her evaluations and become more

willing to participate in whole class activities. Melissa describes this benefit from reading Annie's facial expressions in the following segment.

Melissa: Um, I also like that she gives people the opportunity to come to the board. She doesn't just call people to go up there, which is really, that's scary for me at least. Sometimes I'm like, "I don't want to go to the board right now. I don't understand what's going on." But ...

Siebert: Yeah, but I've seen you go up to the board a couple of times.

Melissa: I know, because I understand it [know a correct explanation]. So, if I understand it, I go to the board, and if I don't understand it, I don't want to go to the board.

In this segment Melissa expresses that going to the board scares her, especially if she were to be called to the board. But she goes on to say that if she “understands” what a correct explanation would be then she is willing, and does go to the board. Thus, coming to know that she has a correct explanation is what changes her from being scared and unwilling to comfortable and willing.

Classroom Mathematical Practices. Melissa consistently appealed to classroom mathematical practices while evaluating explanations. For Melissa classroom mathematical practices had to be followed in an explanation for the explanation to be considered good. If an explanation violated the classroom mathematical practices then Melissa concluded the explanation was bad. For example, consider her evaluation of the building up explanation from the explanations task in interview three.

Melissa: [writing and talking to herself] each as 6 in the whole so $\frac{1}{4}$ is one of those pieces or $\frac{1}{6}$. To find...that's interesting. [To Siebert] This is really wordy and it like, from here one, I feel like it is just too much information. Like, I don't think this is mathematically correct either.

Siebert: Okay, the last part of it?

Melissa: Like the whole thing. Like they started off great, like, “Okay we are going to do this, we’re going to take it and divide each whole in half.” This is where I kind of get, where did you get the half from? Does that make sense? ‘Cause Annie is like, “Okay you have 4, you are making 4.” So I would say that this needs to be fourths. Then there would be 6 pieces, or no. Two. Four pieces, sorry in each third. So this, all of this would change. Cause I feel like here they kind of missed the importance of the fourth. That’s all I would say about that one. So this one I would say is not right as well, even though they get the right answer.

So one of Melissa’s main critiques of this explanation was that the author took the thirds from the two-thirds and split them each in half. The reason that Melissa took issue with this method was that she had identified the classroom mathematical practice of splitting each of the fraction pieces into the number of pieces indicated by the denominator of the first fraction. So since the explanation was for $\frac{3}{4}$ of $\frac{2}{3}$, according to Melissa’s classroom mathematical practice, each third needed to be split in fourths so that she had fourths of a third. Because Melissa believed this classroom mathematical practice must be followed, she failed to recognize that splitting each third in half was actually sufficient for solving this problem. Since we want to take $\frac{3}{4}$ of the $\frac{2}{3}$, breaking the $\frac{2}{3}$ into four equal pieces, or as the explanation says, “breaking each third into two equal pieces,” gives fourths of the two thirds and three one-fourths can be taken of $\frac{2}{3}$ to get $\frac{3}{4}$ of $\frac{2}{3}$. This example illustrates how Melissa used classroom mathematical practices as requirements for writing explanations.

In addition to classroom mathematical practices being requirements for a good explanation, the appeals to classroom mathematical practices were more conclusive in her evaluation of explanations than appeals to sociomathematical norms. In the above segment Melissa does appeal to a sociomathematical norm that *explanations need to be clear* in her statement that the explanation is wordy, but this critique seemed to be more superficial for her, or

the first thing that she could think to say. It is the classroom mathematical practice that she focuses on in her evaluation, showing that it is the main critique of the explanation.

These classroom mathematical practices were so strong for Melissa that she pieced them together to create what I refer to as *explanation templates*, templates for solutions to types of problems that consist of words and blanks in which numbers from a particular instance of a problem type can be inserted. These explanation templates were used by Melissa to create an explanation. The following is a segment that provided evidence of Melissa's creation and use of explanation template. Just before this segment begins, Melissa had given explanations for why two particular quantities are one-fifth and five-sixths.

Siebert: OK. It sounds like you've got it down.

Melissa: Yeah, it's like the same thing over and over again.

Siebert: Oh, OK.

Melissa: That's what I think that we're trying to work on.

So in this segment of the data Melissa was confident in her explanations for the one-fifth and the five-sixths and the reason she cited for this confidence was that the explanations for fractions are all essentially writing the same thing but with the numbers changed. These explanation templates eventually became so specific for Melissa that she even had difficulty changing up her wording slightly when writing explanations. She explained this difficulty in the following segment.

When I'm, like, explaining partitioning and iteration, I understand it, because I'm able to write it down. I'm in the habit of writing the same thing each time, though. Like, you do this, and then this, and then this. And she's [Annie] like, "No, I want you to use different words." [Siebert: Uh-huh.] So, that's hard for me.

In this segment, Melissa admitted that she had refined this explanation template and relied on it to the extent that she could not help herself but write the exact same words each time. She

acknowledged that Annie did not want them to use the exact same words in each explanation but that she had a very difficult time not doing so.

Implications for Evaluation. Though Melissa could effectively use her teacher to determine correctness and had much success applying her explanation templates, her method of evaluation still led to some difficulties. Specifically, the presentation of a lot of different student methods in class made it difficult for her to remember how to write a good explanation, and thus evaluate her own explanation. Her discussion of this point follows.

Melissa: Um, like I said earlier, I really have a hard time when [Annie says], "Oh, what do you think?" I don't care what a student who's taking the class with me thinks, as much as I care what you [Annie] think. Now, it's great that that person shared their, you know, opinions on what it is, but if it's not the right way, and I'm listening to them, or I'm listening to four or five different wrong ways, by the time you tell me what you're thinking, I'm not going to remember it.

Siebert: Mmm... OK. So, having so many different explanations out there is, like, it confuses?

Melissa: It confuses me. Especially when, like, I'm trying to tell my brain, "Remember the thing that's right." And then, it's like, "OK, we've heard six things." And like, it's happened multiple times where I go into a test and it's like, "OK, what's the answer to this question," and I am like, "OK, well, we talked about all four of these things in class, but only one of them is right. Which one did the professor say was right?" And like, I've sat in a class before and raised my hand, and been, like, "What? I need you to tell me what the right answer is because all of these kids are saying different things. Like, it's not working for me."

Melissa had difficulty recognizing and remembering which of the different ways that were presented in class were actually the most appropriate or accurate for an explanation. She remembered, in particular, trying to decide during a test, which of the ways they had talked about in class were the right ways.

Melissa's difficulty in remembering which of the multiple methods were correct is somewhat surprising considering Melissa was really good at reading her teacher. One might think that she could easily identify the correct explanations from watching Annie, and thus avoid confusion. However, when there were many explanations given for a particular problem, she nonetheless struggled. One possible reason for this struggle may have been that her ability to read the teacher reduced her need to understand what made the different explanations good or bad. Thus, when she encountered the test, she was left relying on her memory of which explanations Annie had sanctioned instead of being able to reason about the mathematics herself.

Model of Melissa's evaluation methods. Melissa's method of evaluating explanations is largely dependent upon the presence of an expert and Melissa's familiarity with the mathematical content and reasoning used in the explanation. When an expert is present and Melissa encounters an explanation containing content new enough that Melissa has a difficult time understanding the explanation, she looks to see what the expert thinks about the explanation and makes her evaluation of the explanation consistent with the expert's opinion. When an expert (specifically Annie) is around and Melissa encounters an explanation containing content that she has been working with for a while, she evaluates the explanation using classroom mathematical practices and sociomathematical norms, and accepts that Annie would let her know if the class's explanations were way off track. If an expert is not present, she relies primarily on the classroom mathematical practices (confirmed as being such by her teacher) to evaluate explanations. She rejects explanations that do not follow all of the classroom mathematical practices that she thinks should be used in the explanation. Thus for Melissa, classroom mathematical practices were part, if not the main part, of the system of rules for doing mathematics and had to be abided by. This reliance on Annie to confirm the mathematics in the classroom and appeal to classroom

mathematical practices actually helped Melissa become relatively successful in writing explanations in the class. However, her evaluation method did not come without limitations, for when asked to write an explanation on a test, she sometimes had difficulty deciding which of the explanations she heard in class were correct.

Claire

Background

Claire had only good experiences with school mathematics. She did well in all of the advanced mathematics courses that she took in junior high and high school, and participated in Mathletes at her school. She also said that she enjoyed mathematics and could see the usefulness of mathematics. When asked if she thought she was good at mathematics, she said, “I look and I think there are a lot of people that are way better at math. But I know that I am also, like I understand math a little bit more than other people.” She believed that once she became proficient at a mathematical process she could apply it to most other problems.

Claire enjoyed the inquiry-based teaching strategies of her mathematics education course from the beginning. When asked what she liked about the course she said, “I really like that she is like, ‘Okay I want you guys to do this and work in your groups and try to figure it out.’ And I know that that makes a lot of people mad because they say, ‘No, I still don’t understand it.’ But I just feel like being able to work with the problems, and get that experience helps us.” So this quote shows evidence that Claire actually likes and sees benefits from the teaching practices.

Claire’s Evaluation Method

There are two main characteristics of Claire’s evaluation of explanations. The first is that Claire’s main criterion for judging the quality of explanations was their fit with three sociomathematical norms: *explanations must justify the reasoning or steps, explanation must be*

clear, and explanations must be valid or specifically generalizable. The second is that the inclusion of classroom mathematical practices was not always required to write good explanations; classroom mathematical practices could be used, however, to write one of the many correct explanations.

Sociomathematical norms. Claire used three sociomathematical norms as the main determining factor of the quality of an explanation. The most frequent of the norms that Claire appealed to in her evaluation was that *a good explanation must justify all of the thinking and steps that are taken*. She talked about this norm in conjunction with explanations all throughout the interviews, which suggested that it was a major concern for her in writing explanations. One way she talked about the violation of this sociomathematical norm in her evaluation was by saying the explanation was “making assumptions.” For example, consider the explanation that Claire wrote for $\frac{3}{4} \times \frac{2}{3}$ (see Figure 3).

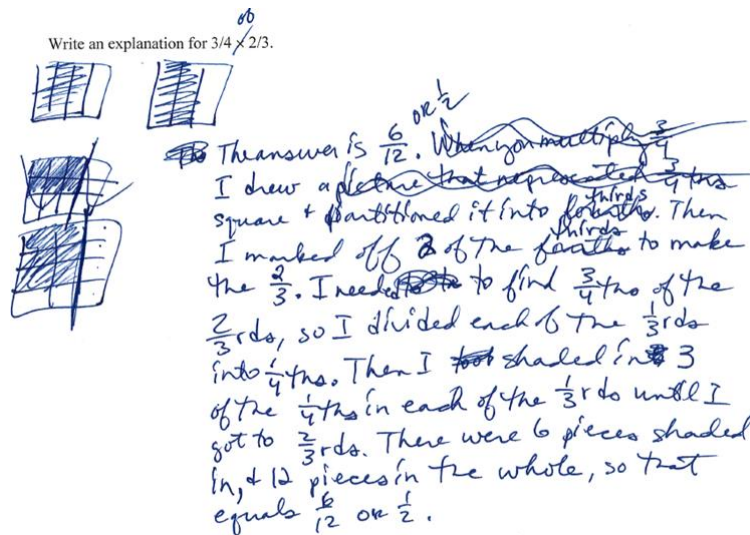


Figure 3: Claire’s Explanation for $\frac{3}{4} \times \frac{2}{3}$

After writing this explanation Claire was asked to evaluate her own explanation. In her evaluation she decided her explanation was just “okay” because it did not justify all of the steps she was taking.

Claire: It’s an okay explanation. [Siebert: okay] Because I explained what I did, I did make a few assumptions.

Siebert: Like what?

Claire: Um, well here I was like I needed to find $\frac{3}{4}$ of the $\frac{2}{3}$, so I divided each of the thirds into fourths. Someone else might be like, “Well why are you dividing the thirds into the fourths?” I didn’t really explain that. [Siebert: okay] So just stuff like that I guess.

In this example, Claire concluded that her explanation was just “okay” because she made “assumptions.” Her example of making an assumption suggests that she used this phrase to refer to a place where the author had not adequately explained or justified her steps or moves in the explanation.

Further evidence of her heavy reliance on this norm of justifying her thinking or steps can be found when she was discussing her participation in class. In the following segment Claire was explaining why she thought it was valuable for her fellow classmates to share their reasoning with the class.

Claire: Okay, so yes, I do think it is important that they share their reasoning. And it’s helpful, especially for those parts in the problem where we generally make assumptions. Like, when our group was working together on the problems that she told us to do in class, like, I was sitting and thinking about why, like why we were doing a certain thing. And I asked someone, I was like, “Wait, why? I feel like this is an assumption. Why did we do this?” and they were like, “Oh well I guess it is an assumption, but that’s okay.” You know, “Whatever just move on” kind of thing. So I feel like it’s good ‘cause that’s what the class is for. The class is to help us see why we do things and how we do things.

Claire believed that one of the primary reasons for sharing reasoning in class was so that the author could recognize and learn to explain places where she made assumptions. Since

explanations are ways of sharing one's reasoning, the same must be true of the purpose for writing explanations, they help the students identify assumptions. If explanations are the way reasoning is shared, then explanations are only good or fulfilling their purpose if they are not making assumptions.

The second sociomathematical norm that Claire appealed to in her evaluations was that *the explanation must be clear*. Evidence of this norm could be found in her comments, such as, "I mean, it's kind of hard to follow, so I don't think it's a very good explanation." This segment suggests that she believed that the explanation that she was critiquing was poor because she could not make sense of it. A specific feature of explanations that she believed made them more understandable, and thus better explanations, was having pictures to go along with the words. When evaluating the algorithm explanation from the explanations task in interview 3, Claire said that people would have a hard time understanding the explanation because it did not include a picture to help people understand what was going on.

Claire: So it's an okay explanation.

Siebert: It's Okay.

Claire: They'd have to like, I don't know, like I really do think drawing pictures is good and beneficial.

Siebert: So you would like them to have a picture to go along with this?

Claire: Yeah and to just kind of explain so that students are looking at this going, "Oh that's why they multiplied that times that and that times that."

Although Claire concluded that the explanation was okay, she felt it would have been a better explanation if it had included a picture. Claire seemed to feel that a picture could help clarify places in the written explanation where the writer had made assumptions or had not adequately

explained her reasoning. Claire felt that the inclusion of pictures with an explanation was a good tool to employ to help the explanation be clear, and thus be good.

The third norm that Claire used in her evaluation of explanations was that *for an explanation to be good, it needs to be generalizable*. Her evaluation of the overlapping grid explanation from the evaluation task in interview 3 provides an example of Claire appealing to this norm.

Siebert: Okay, so it's not enough that she got the right answer.

Claire: True. Yeah. Because I guess I, I'd want to see this and see if it works in other situations.

Before this segment Claire had criticized the explanation because the author had not adequately justified the steps in her solution. Her questioning the justifiability of the solution method naturally led her to question whether or not this explanation was generalizable, i.e., that it would work in all multiplication situations. Then she went on to try to figure out if the solution method did indeed work for all fraction multiplication problems. This example demonstrates that Claire used the norm that the explanation must be generalizable to evaluate explanations.

Classroom mathematical practices. The second characteristic of Claire's evaluation method was her flexibility with regards to whether or not explanations made use of classroom mathematical practices. Claire felt that classroom mathematical practices did not always have to be appealed to in an evaluation. By this I mean that failure to apply a classroom mathematical practice did not always constitute a bad explanation, but just a different explanation. Evidence that this was her view can be found in her evaluation of the building up explanation in the third interview. At the start of her evaluation, she struggled to understand the explanation. She decided to draw a picture to help her make sense of the explanation.

Claire: I just would probably have to draw a picture. Here let me—

Siebert: Yeah you are welcome to draw a picture.

Claire: [To herself, while drawing] find $1/4$ of $2/3$. I took each third in my whole and divided it in half. Then there were two pieces in each third so four equal pieces in the two thirds and six equal pieces in the whole. [To Siebert] Okay! So that's a...that is good so maybe it's just, I am a more visual learner. [To herself] So one fourth is one of the pieces or $1/6$. To find $3/4$ of $2/3$. [To Siebert] I guess when I am reading this I am thinking, "Okay you only found $1/2$." Oh! Okay. So I guess I was thinking about it in a different way and so it was confusing me.

Siebert: How were you thinking about it?

Claire: Well I wanted her to divide it [each third] again, 'cause it's $1/4$ of $2/3$. Like I wasn't counting those as, I was thinking about it wrong.

Siebert: So you wanted her to divide each third into four equal pieces, not two equal pieces? Is that what you were thinking?

Claire: Yeah. But her way. Okay. So yeah, I would say that it is both, mathematically correct and a good explanation.

In her evaluation of this explanation she started by appealing to the classroom mathematical practice that when taking a/b of a fraction, split each of the pieces of the fraction into b pieces. At first she was tempted to conclude that the explanation was incorrect because each of the pieces was split in half. Applying the classroom mathematical practice to this problem would suggest that each piece should be split in fourths. But Claire was willing to let the classroom mathematical practices be violated and then concluded that the explanation was good. This segment suggests that Claire was not limited to requiring that all the classroom mathematical practices to be met for an explanation to be good.

Further evidence for her acceptance that classroom mathematical practices are not required for a good explanation can be found in how she talked about her learning of these classroom mathematical practices. For example, Claire believed that the way of explaining fractions that they used in class was "one way of doing the problem." Claire seemed to believe

that using classroom mathematical practices can lead to correct justifications of the mathematics, but that other ways of reasoning and arguing could also be acceptable. This shows us that when Claire learned classroom mathematical practices, she accepted them as tools for creating a good explanation, but not the only tools.

Despite Claire's belief that the use of classroom mathematical practices is not required in a good explanation, she still created explanation templates as she solved problems on the homework assignments, usually cobbled together from classroom mathematical practices or practices that were later sanctioned in class as classroom mathematical practices. To show how these templates developed, consider the following segment where Claire was discussing her view of their classwork and homework. She had started the discussion by saying that doing several of the same type of problem on an assignment can be a waste of time, but started this following segment by taking back that claim.

Okay, not a waste. But like you know if we have four or five problems and we are explaining like the same thing over and over and over. Like that worksheet that we did yesterday, no on Tuesday, the blue one. I should still have it. Anyways there were a lot of problems on there. But like, they are all the same, the explanations were all going to be the same.

While the intent of the homework was to have students rethink and reword the solution to each problem, Claire bypassed this intent by forming explanation templates that resulted in solutions being virtually the same across the same problem type. Thus, classroom mathematical practices in the form of explanation templates played an important role in Claire's writing of explanations.

Claire also used explanation templates in her evaluation of explanations. In the above example where Claire struggled to make sense of the building up explanation, she seemed to be trying to match the explanation to her explanation template that included the mathematical practice of splitting each piece in c/d into b equal parts when finding $a/b \times c/d$. When the

explanation and the explanation template did not match up, she started her evaluation over again by trying to reason through the explanation instead of matching it to the explanation template. This allowed her to eventually realize that the explanation was good. The flexible use of explanation templates in evaluating explanations explains how Claire enacted the sociomathematical norm that a good explanation does not necessarily have to include classroom mathematical practices.

Implications for evaluation. While Claire evaluated most explanations she encountered correctly, her use of explanation templates did not always lead to successful evaluations. For example, in her evaluation of the commutative explanation, she said, “That is how I think about it.” Her comment suggests she checked the explanation against what she typically wrote for class to see if the explanation matched her explanation template for fraction multiplication. She did not recognize in the interview that the explanation was not for $\frac{3}{4} \times \frac{2}{3}$, but actually for $\frac{2}{3} \times \frac{3}{4}$.

A second type of difficulty Claire encountered during evaluation was in confidently identifying steps in the explanation that required more justification. Claire was able to identify places in explanations where she believed justification was missing, but often expressed concern about catching all of the missing justifications. For example, she read through the overlapping grid explanation quickly and said, “So like here that was a great explanation until, ‘the double shaded part is $\frac{3}{4}$ of $\frac{2}{3}$,’ because I am not sure.... I can only guess what she was thinking for that part.” So she noticed the missing justification by applying her explanation template. However, after she finished her evaluation of this explanation, she explained why it may be difficult for her to identify whether or not this is a good explanation. She said that, “It’s hard to say if that is a good explanation because that is how I think about it.” Explanations that were close to the way that Claire wrote her own explanations seemed hard for her to evaluate. Perhaps

this difficulty was caused by too quickly matching steps in the explanation template and the explanation without enough care being given to how closely the steps match. Or perhaps when the template and explanation match too closely, Claire is inclined to read more (i.e., her own thinking) into the explanation than what is actually written. Regardless, Claire's evaluation method was not fool proof even though Claire seemed very confident and proficient in her evaluations. There were still explanations which she had difficulty evaluating.

Model of Claire's evaluation method. The main feature of the Discourse that Claire used in her evaluation was sociomathematical norms. The sociomathematical norms that Claire used were also a distinguishing characteristic of her evaluation, and included that *explanations must justify the applied reasoning or steps*, *explanations must be clear*, and that *explanations must be generalizable*. Further if a classroom mathematical practice was violated in an explanation, Claire did not always conclude that the explanation was bad. She was able to see other ways of arguing as being good even if she had never encountered those arguments before. Claire did use classroom mathematical practices to evaluate explanations that were close to her explanation template by checking them against her template. Yet even when the explanation fit her explanation template closely, she made mistakes and was hesitant to conclude that the explanation was clear or adequately justified because she thought about the explanation in the same way.

Lynn

Background

Lynn had a relationship with mathematics that is very typical of many of our mathematics students in the public schools. She had hated mathematics since she was in junior high. She said that she was more of "an English person" and that "math just has never been a strength for me...I

think I am awful at it...It has just always been hard for me.” She felt that she had terrible mathematics teachers throughout junior high and high school, which made her relationship with mathematics even worse. To contrast this negative past experience, she told the interviewer that she really liked her first mathematics class in her elementary education program at the university and reported that “it was like the first time I actually liked a math class.” Because she liked her first class so much, she said that she participated a lot and felt like she learned a lot of mathematics that she would use in her future teaching.

One of the reasons that Lynn had really enjoyed this class was because the teacher was Annie. In the first interview she said, “I love Annie...So I like worked my schedule out so I could have her [again this semester]. I think she is a really good teacher.” Yet despite her good experience in the first class and her having Annie again as a teacher for the second course, she struggled in the second course. But despite her difficulties, Lynn valued the teaching methods. In the first interview Lynn was describing what they did in class by saying that they did a lot of analyzing and that she was actually good at analyzing. By analyzing she meant that in the class they worked to discover why the mathematical procedures worked. Lynn liked that they were working to discover why. In the interview she said, “I think that’s [thinking about why things work] what helps me. It's understanding the theory behind all the math problems.” She went on to add, “I think that’s why I like 305 [the first class]...This helped me like kind of fill in a lot of the gaps that I had.” So Lynn was not resistant to the teaching methods and was motivated to learn the content.

Lynn’s Evaluation Method

There were two main characteristics of Lynn’s evaluation method. The first was that the sociomathematical norms that she had accepted focused on how explanations are subjective,

leading her to conclude that there were lots of different ways to write good explanations, although all good explanations for the same problem will yield the same right numerical answer. The second was that classroom mathematical practices which outline a way of reasoning could be used as tools with which to check the sensibility of an explanation.

Sociomathematical norms. Sociomathematical norms that Lynn had accepted help to characterize her evaluation method. The two norms that Lynn used throughout her evaluation were *correct answers mean correct thinking* and *evaluations are subjective*. I describe and present evidence for each of these norms in turn.

Correct answer means correct thinking. One of the sociomathematical norms Lynn seemed to follow was that correct answers indicated correct mathematical thinking. One of the examples found in the data of Lynn appealing to this sociomathematical norm happened when she was describing how she would evaluate students' explanations when she became a teacher. She anticipated situations where “[the student] just took a completely different route than you would have or even than you would have liked them to, but they still got the right answer. Which means they have the train of thought, like to get there.” In this excerpt, Lynn argued that students cannot produce the correct answer unless they have a sufficiently accurate way of thinking to produce the answer. This is evidence of Lynn’s acceptance of the sociomathematical norm that correct answers are indications of correct thinking.

This sociomathematical norm was a very powerful tool for Lynn’s evaluation. With each of the explanations she was given in the third interview, she checked whether or not the explanation resulted in the correct answer and reported the correctness to the interviewer. A correct answer seemed to be an important criterion, if not the most important criterion, for a good explanation. Lynn also reported that when determining whether a solution and its accompanying

explanation were good, she would often “check it with an algorithm,” or consult with her classmates and “compare answers.” Lynn relied on the correctness of the answer fairly heavily in her evaluation of explanations.

Evaluations are subjective. While Lynn accepted the norm that correct answers imply correct reasoning, she also accepted the norm that an explanation that is good according to one person may not be good according to another, even if the explanation contained the correct answer. In one of the interviews where she was discussing her evaluation abilities and how explanations she saw as being good her classmates and teacher often did not see as adequate, Lynn stated, “I think it is a little bit subjective.” But she continued by explaining that her teacher knew more than she did, and concluded that she probably was not as good as her teacher at evaluating. Though she recognized that there may be differences in ability when it comes to evaluating explanations, it was still clear from other comments that she had accepted the norm that evaluations are subjective. For example, when evaluating the algorithm explanation in interview three, Lynn was hesitant to say that a record of procedural steps is actually not an explanation, and therefore not a good explanation:

So this one, it kind of just seems like they are using the algorithm, I mean maybe. But see it makes sense to me because I know the algorithm. It's just 3 times 2 equals 6, 4 times 3 equals twelve. $6/12$ is equal to $1/2$ when you reduce it. So, um, yeah, I mean it makes sense to me. As an explanation probably not so much. Mathematically, yes but as an explanation I think you need to say, maybe. Well I mean if you tell your students to just use the algorithm and to not like draw [a] picture or write an explanation, then the explanation speaks for itself, because you just followed an algorithm, and you followed it correctly. So I think it just would depend on what the directions were.

As Lynn began to say that the algorithm was not an explanation, she stopped herself and said that it depended on what the instructions were. I would interpret this to mean that whether or not we can consider the algorithm as an explanation and therefore a good explanation depended

on the sociomathematical norms for the community for which the explanation was written. She was recognizing here that in a setting other than the class she was in, such as a more traditional mathematics class, this algorithm written out in that way may have actually been seen as a sufficient explanation.

This subjectivity comes into play specifically when Lynn is considering two of what she claims are the main required criteria for a good explanation beside the requirement of the correct answer: *explanations must be clear* and explanations must afford connections. The interplay of these two sociomathematical norms with the sociomathematical norm that evaluations are subjective uniquely characterizes Lynn's evaluation. For Lynn, whether or not an explanation was clear or affords connections was subjective to the evaluator.

Good explanations are clear. There were two different meanings for Lynn when she used the phrase makes sense, each of which is an application of a different sociomathematical norm. Sometimes when Lynn said that something made sense to her, she used it to mean that the explanation was clear, which means that the explanation needs to be understandable to the reader, or that it is easy to picture for the reader what the author is doing and/or claiming throughout the explanation. There was evidence of Lynn's use of this sociomathematical norm in her evaluation of the overlapping grid explanation. When asked why she thought this explanation was a good one, one of the main reasons that Lynn used was that she could picture what the author was doing by reading the explanation.

Well I think they just drew out every picture so they had to say $3/2$, $2/3$. But when you say "So I put $3/4$ on top of $2/3$." And you are like, "Okay I see that." So they basically just, you know, made a rectangle, and then the double shaded part right here is $3/4$ of your $2/3$ that you had shaded, because 1,2,3,4 [points to each piece that is single shaded to form the $2/3$]. And then which is 6 out of the 12 or $1/2$. So yeah, I think that she [Annie] would check it and put a smiley face and say it was good.

So Lynn was appealing to the sociomathematical norm that good explanations are clear. This was evidenced by her comments such as “Okay I see that.” It was clear to Lynn because she was able to use the associated picture to easily picture what the author was doing, what steps he was taking. Because this sociomathematical norm is based on the readers’ experience with the explanation, whether or not an explanation is considered to be clear is subjective. Thus these two sociomathematical norms, that explanations need to be clear and evaluations are subjective, are consistent and often used together in Lynn’s evaluations

Good explanations afford connections. Often when Lynn said that an explanation made sense she meant that she could see a connection between her way of thinking and the argument contained in the explanation. This second meaning for makes sense is associated with the sociomathematical norm that *good explanations afford connections*. One example of Lynn appealing to this sociomathematical norm can be found in the following segment. Lynn was evaluating the building up explanation by appealing to the sociomathematical norm that good explanations afford connections. When appealing to this norm, she made comments about how the explanation did make sense to her now, but that the subtraction step was difficult to make sense of for her at first. She was able to connect the subtraction step with what she would have done in her explanation (take three of the fourths), but expressed that it was really difficult for her to make that connection.

It just makes sense, but adding in the subtraction, maybe it was easier for them to think, “Oh well, I have 4, like I’m going to just subtract 1 and be left with the 3.” Maybe subtraction just works for them, but it’s pretty confusing for me. So it makes sense. There, it’s kind of a not a great explanation, just because, but maybe it’s just not a great explanation to me because my brain doesn’t work that way.

In this excerpt, it is clear that Lynn accepted this explanation only because she was able to connect it to her own way of reasoning about the problem, suggesting that Lynn sees the criterion that *explanations afford connections* to be a sociomathematical norm.

In the above segment there is also evidence of Lynn appealing to the sociomathematical norm that *evaluations are subjective* when she said, “Maybe subtraction just works for them, but it’s pretty confusing for me,” and, “Maybe it’s just not a great explanation to me because my brain doesn’t work that way.” In these comments she acknowledged that whether or not the explanation afforded connections was dependent on her (the reader) and the way that she thought. Thus, this interplay between the two sociomathematical norms that *good explanations afford connections* and *evaluations are subjective* impacted her evaluation, making it difficult to make a final conclusion as to the quality of the explanation.

Classroom mathematical practices. As Lynn attempted to find connections between her ways of thinking and the explanations she was evaluating, she often drew upon what she perceived to be the taken-as-shared classroom mathematical practices. For example, Lynn used classroom mathematical practices to evaluate the commutative explanation in interview three. As Lynn worked to identify connections between her way of thinking and this explanation, she recognized that she was not able to make a connection because one of the classroom mathematical practices was directly violated. The classroom mathematical practice that was contradicted in the explanation was that multiplication $a \times b$ means *a groups of size b*, and so an explanation should start with one group of size *b* and then take *a* groups of that size. But in the following segment she described how she had identified that the explanation actually takes one groups of size *a* and takes *b* groups of that size.

Lynn: Well they got the right answer. And they did it, the only part I’m concerned about from the beginning it says, “So I split each fourth into 3 equal parts,” and it

seems like they were starting with the three, with the fourths instead of the $\frac{2}{3}$. Cause it should be $\frac{3}{4}$ of $\frac{2}{3}$ that's correct. So split each fourth into 3 equal parts. I'm not sure they can start that way, but maybe you can. This is where I'm really bad, 'cause I always don't know if can be.

Siebert: What is it exactly that you are struggling with? Which one to start with?

Lynn: Well no, no, I know that I always start with the second number, but um, like how they have, they split each fourth into 3 equal parts. I think that you have to start with the thirds, so I think that maybe the problem that they are solving is $\frac{2}{3}$ times $\frac{3}{4}$.

So Lynn had identified that the explanation was for a different multiplication problem based on her application of the sociomathematical norm that good explanations afford connections and the classroom mathematical practice for interpreting multiplication. Because this classroom mathematical practice had become Lynn's personal way of reasoning or thinking, and because this practice was violated in the explanation, Lynn was unable to construct a connection between her way of thinking and the way of thinking employed in the explanation. Thus, the explanation was seen as not being good.

Although the norm that *good explanations afford connections* was violated in this solution, Lynn experienced conflict because the explanation contained the correct answer, and thus met the norm that *correct answers indicate correct thinking*. Immediately following the excerpt above, Lynn hesitated, and then said the following:

But see this has always [been] really confusing to me because I'm like, those are the same, you get the same answer.... But I think that's what they are doing, is solving $\frac{2}{3}$ times $\frac{3}{4}$. So they start out with fourths and then divide each of those in thirds, take 2 of those. So yeah, I think they're solving $\frac{2}{3}$ times $\frac{3}{4}$ instead of $\frac{3}{4}$ times $\frac{2}{3}$. Um but they are going to get the same answer because those problems, like you know you get the same answer from them.

As Lynn continued to think about this problem, she seemed unable to decide if the explanation was good or not. After a little more deliberation, she declared that the explanation was

mathematically correct and very good. This statement was immediately followed by her claim that if a student in her class had written this explanation, she would make sure that the student understood that the explanation was for $\frac{2}{3} \times \frac{3}{4}$ and not $\frac{3}{4} \times \frac{2}{3}$. Lynn's simultaneous acceptance and rejection of this explanation suggests that Lynn is unable to evaluate explanations that contain correct answers but lack the appropriate mathematical practices.

One role of authority. Lynn developed these ways of thinking, or what she perceived to be classroom mathematical practices, that she uses in her evaluation from experts in the class by granting them expertise authority. When she had a difficult time writing her own explanation with the help of her classmates, she desired for Annie to help her create an explanation. Evidence of this was found in her interviews as she described when she listens to and when she tunes out presentations in class. She said,

If I haven't gotten that [an explanation for the problem], like if I have been confused by that [trying to write an explanation] then I definitely think that I am like, "TEACH ME!" But that's the point where I almost wish that I was the only one asking questions, so I could just be like, "Annie, can you do this problem, like walk us through [the explanation]. 'Cause I feel like I will ask a question, like, "Can you go through number six?" And she will be like, "Yeah, who got number 6?" And I'll be like, "I already asked someone in the class, they didn't explain it, that's why I want you to."

In this segment Lynn described how her classmates were often unable to present the explanation in a way that Lynn could understand. So she preferred for Annie to give her an example of the explanation.

When Lynn felt she had a good way of thinking about a problem, she liked it to be verified explicitly by her classmates. Once ways of thinking were confirmed by a classmate or given by the teacher, Lynn adopted them as if they were classroom mathematical practices. In

the following segment Lynn detailed how she relied on her classmates to confirm her ways of thinking.

With my group, it was always like, “Is it, am I thinking about this correctly?” And they would just kind of like glance at me like, “Yeah, you’re good.” And it was like, “Okay, then I am on the right track.” And I think once I had been, like once someone assured me just like one time that my pattern of thinking was correct, then I was more confident like the next time we had a problem similar to that, ‘cause I was like, “Oh yeah, I know how to do this, like I just, like I have thought about it in the correct way.”

This confirmation from her group helped Lynn to be confident and see her way of thinking as an acceptable way of thinking. Once she saw her way of thinking as a sanctioned, correct way of reasoning, she could then use it to create and evaluate explanations.

When Lynn was unable to solve a problem or write an explanation of her solution, she preferred to turn to Annie for help. Lynn felt that Annie’s explanations were the most helpful to her, detailed in the following segment.

Sometimes I wish she would just explain things, because how other people explain things I am like, I don’t really understand what they’re saying. But when she talks she’s very clear... Like I don’t want her to just give me the answers, but sometimes it helps me when she explains a concept instead of trying to be like, “You explain this, you explain this, you explain this,” because it’s like three different trains of thought that I am all trying to like get on board with, along with my own train of thought that I had originally. So it’s nice for me when she kind of explains things first so I can be like, “Okay, that makes sense” and then lets other students introduce, like, “Oh, I did it a little bit different, I did it a little bit different.” ‘Cause then it’s like I can compare them all.

As evidenced in this segment, when Lynn was having a hard time coming up with a good way of thinking about the mathematics on her own she wanted Annie to explain it to her. She clarified that she wanted Annie’s explanation not just so that she got the answer, but because it “helps [her] when she [Annie] explains a concept.” It gave her a way of thinking—a particular mathematical practice—that she could latch onto. Once she had the sanctioned, valid way of

thinking, she could compare others' reasoning to that one good way she had accepted. The sanctioned way of thinking could be used to decide if an explanation made sense, or to find connections between explanations and her way of thinking.

Implications for evaluation. In order for an explanation to satisfy the sociomathematical norm of *good explanations afford connections*, Lynn had to be able to identify connections between the explanations and what she perceived to be the classroom mathematical practices. A lack of applicable classroom mathematical practices could create difficulties for Lynn in evaluating her own explanations of new content. For example, when the class first started talking about fractions, Lynn had no access to classroom mathematical practices, because they had not yet emerged from students' explorations and discussions. All she had to rely on to build a sensible way of thinking about fractions was the standard fraction algorithms. In working to create an explanation, she had no practices to check her explanation against to decide if it made sense, and thus could not stop herself when she started to reason incorrectly. In fact, she had started to convince herself that her wrong way of thinking was actually correct once she has been working with that way of thinking for a while. Lynn described this experience in the following segment.

So I feel like it would help me if she [Annie] just kind of gave me a little bit more, like, scaffolding and guidance at the beginning instead of letting me get so far into it and then be wrong and have to start completely over. 'Cause I think she wants us to go through that process of making the mistakes in order for us to learn, but for me it just makes me so frustrated with myself that it's like, by the time I, like I have done this whole worksheet, I have done all these things and it's like, "Oh, I get it." But I get it completely wrong, and then I have to go back and start over.

This segment shows that this situation was frustrating to Lynn and that she wished in these situations that she would get more guidance from Annie. The situation was frustrating to her not

because it wasted a lot of her time; it was frustrating because she did an entire assignment with this wrong way of thinking, convincing herself along the way that her reasoning was valid.

Model of Lynn's evaluation methods. Lynn's unique interpretation of sociomathematical norms and application of classroom mathematical practices characterized the way that she evaluated explanations. The sociomathematical norms that she used were *correct answers mean correct thinking, evaluations are subjective, good explanations are clear, and good explanations afford connections*. The unique character of these sociomathematical norms led Lynn to accept most of the explanations that ended in the correct answer. Yet when certain classroom mathematical practices were directly violated in an explanation so that Lynn could not make connections between her way of thinking and the explanation, she rejected the explanation. This rejection was with reservation, though, when the answer was correct. Her reliance on the violation of a classroom mathematical practice to reject an explanation likely made it difficult to evaluate her own thinking and explanations about new content.

Discussion of Models

Evaluation Difficulties

All three of the research participants had difficulties or made mistakes when evaluating explanations. Melissa expressed difficulties remembering which of the arguments they had presented in class created good explanations. Claire missed the fact that the commutative explanation was actually a good explanation for the wrong problem. Lynn could not stop herself from going down the wrong path when she was creating an explanation for a new concept. One of the main reasons that researchers such as Boaler (1998) have cited for why students have difficulty in inquiry-based mathematics classes is because they resist the teaching methods. But as described in the background sections, each of the students either liked creating explanations

on their own or thought that doing so was beneficial to their learning. Also, two of the students, Melissa and Claire, had been successful and enjoyed mathematics growing up, and thus it is unlikely that their difficulty evaluating explanations was solely due to an impoverished understanding of mathematics. This case study adds to the literature by providing examples of students who struggle with inquiry-based mathematics instruction, particularly with evaluating explanations, but who are neither lazy nor resistant. These examples suggest that the field needs to more carefully consider what makes learning and being successful in an inquiry-based mathematics class difficult for students.

Sociomathematical Norms Comparisons

The types of sociomathematical norms used in evaluation were very important in defining the way that both Claire and Lynn evaluated explanations. Because Melissa relied mostly on classroom mathematical practices, sociomathematical norms were not as central to characterizing her evaluation. Claire and Lynn differed in the sociomathematical norms that they relied heavily on in their evaluations. In Claire's evaluation, the main sociomathematical norm that she appealed to was that *explanations must justify their steps* or not make assumptions. This was one sociomathematical norm that was very important to Claire's evaluation and was appealed to in almost every evaluation instance. When an explanation was presented, she questioned whether or not they were making any assumptions. By appealing to this norm, she demonstrated that she was highly sensitive to the fact that there was a purpose for the actions being taken in the explanations and that these purposes needed to be communicated, or the actions justified.

The sociomathematical norms that Lynn appealed to frequently in her evaluations were that *correct answers mean correct reasoning*, *evaluations are subjective*, *good explanations are clear*, and *good explanations afford connections*. These norms made it difficult for Lynn to reject

any explanation that had the correct answer, specifically because good explanations were so relative. The sociomathematical norms *good explanations afford connections* and *evaluations are subjective* when paired together created difficulties for Lynn making evaluation based on the mathematics. This difficulty came because everyone's way of reasoning is a little bit different, and Lynn acknowledged that there are lots of different ways of reasoning that are correct. Lynn has no way of definitively deciding whether or not an explanation affords connections because evaluations are subjective. She can identify if the explanation did or did not have connections to her own way of thinking, but she can never make a final conclusion as to whether or not the explanation is good because her identification of a connection may not be identified by someone else. Thus she can make a conclusion as to her own opinion about the quality of the explanation, but this opinion may differ from someone else's, who does not see a connection between his way of thinking and the explanation. This sociomathematical norm that *evaluations are subjective* seemed to contribute to the difficulty that Lynn has with evaluating explanations, such as in situations where she accepts them all, but her classmates and her teacher reject some of them. Students like Lynn may have similar difficulties, where the sociomathematical norms that they have accepted are such that norms do not allow them to evaluate explanations based on the mathematics. Thus, these students would likely need to turn to another source in order to evaluate explanations, such as their expert teacher, because they have limited means for evaluating explanations otherwise. .

In the literature of Cobb and his colleagues on sociomathematical norms, there is a focus on what rules for doing mathematics are seen as taken-as-shared from the class perspective. Once we have identified the sociomathematical norms of a class, we can decide if these norms are productive for evaluation. Researchers believe that if students are given time to participate in

a class where productive sociomathematical norms have been established, then they will learn these sociomathematical norms and will be able to use them to evaluate explanations. It was seen from an analysis of my data that Claire and Lynn both saw rules, such as *explanations should be clear*, as a sociomathematical norm. So they accepted some of the same sociomathematical norms. But the other sociomathematical norms that each student focused on were very different. Though these students were in the same class and spent lots of time talking about the mathematics and evaluating explanations, they had accepted different sociomathematical norms, which led to different evaluations of the same explanations. One likely reason for the acceptance of very different sociomathematical norms is that these norms are inferred by the students through their participation in class, and so the meaning invoked in each student by class activities and discussions is likely different. Since these students had constructed very different understandings of the sociomathematical norms from being in the same class, productive sociomathematical norms or not, it presents evidence against the belief that students will accept similar sociomathematical norms from simply participating in mathematical activity. This implies that participation in the activity of evaluating explanations alone, even extended participation, may be insufficient for students to learn how to evaluate explanations.

Classroom Mathematical Practices Comparisons

One of the major ways that the students' accepted sociomathematical norms differed was in the sociomathematical norms that dictated how to use classroom mathematical practices in their evaluation. Melissa needed classroom mathematical practices to be followed for a good explanation; Claire was flexible to them being violated by allowing different ways of reasoning to be acceptable for a good explanation; and Lynn needed to be able to construct connections between the classroom mathematical practices and the explanation for a good explanation. A

major importance of this result is actually a question: How should students be applying classroom mathematical practices to evaluate explanations? My analysis of the different ways that these students applied classroom mathematical practices suggests that we do not want students using Melissa's method of requiring them all to be met. This is a strategy that is detrimental to the student's success because it does not allow her to be flexible in her accepted ways of reasoning, and causes her to reject any reasoning that slightly deviates from the typical explanation in class. Yet, Melissa and Lynn both needed classroom mathematical practices to conclude that the commutative explanation was for a different problem. So we do not want students to completely ignore the classroom mathematical practices, because at times they can be helpful in identifying errors in explanations.

Melissa and Claire also used classroom mathematical practices to build explanation templates for each problem type. These explanation templates allowed them to easily create good explanations and offload some of the cognitive demand of writing explanations. But they both also used these explanation templates as a way of evaluating explanations. Melissa checked to see if the explanation fit the classroom mathematical practices in the explanation template. Claire checked if the explanation fit closely the explanation template. If it fit closely, she accepted the explanation as good, and if it did not fit her template, then she abandoned her explanation template during evaluation. This use of the explanation template led to Claire missing that the commutative explanation was for a different problem than what was being asked for. This application of explanation templates is very significant because it shows that sociomathematical norms for appealing to classroom mathematical practices that are productive for writing explanations may not be the most productive for evaluating explanations.

Authority Comparisons

The students in this study had a few different ways that they used expertise authority to evaluate explanations. Unique to Melissa was the way she read the teacher's facial expressions after having first evaluated the explanation on her own. She also used Annie's feedback on her written work to evaluate explanations; if Annie makes corrections on the explanation, then Melissa concludes it is wrong, while if Annie makes no corrections, Melissa concludes the explanation is correct and so are explanations similar to it. Thus, for Melissa, these explanations become teacher sanctioned ways of reasoning. Both Melissa and Lynn used the teacher to gain classroom mathematical practices. They both accepted practices that Annie stated explicitly. Lynn looked for Annie to give explanations in front of the class to learn accepted ways of reasoning when she was unable to build her own ways of reasoning. Melissa was able to gain them from Annie explicitly. Melissa would also gain them implicitly in that if Annie did not correct the class's ways of reasoning or arguing, then Melissa saw this as Annie sanctioning what was said, and she could then accept the ways of reasoning or arguing used in the explanation or discussion as classroom mathematical practices.

The three students also granted authority to their fellow classmates. Lynn granted expertise authority to her classmates in addition to her teacher in order to gain classroom mathematical practices. If she had a way of reasoning she felt was good, she would show it to classmates at her group and have them confirm it. Melissa and Claire also worked with their classmates in a similar way, but were not granting expertise authority. The way these participants granted authority to their classmates will be discussed later. Lynn also granted her classmates expertise authority when they were presenting solutions at the board. For Lynn, the benefit of having her classmates present solutions at the board was to help her when she was having a

difficult time constructing her own explanation. She would listen to others' reasoning as they presented explanations in front of the class and would grant them expertise authority by accepting those ways of reasoning as her own. Thus, Lynn accepted the explanations that were presented at the board as correct. Melissa, on the other hand, did not accept every explanation that was presented at the board. She would conduct her own evaluation based on the mathematics and then confirm her evaluation by looking at Annie's facial expressions, granting expertise authority. Thus, Melissa was not apt to accept all explanations, but instead reasoned mathematically about them before confirming her conclusion with the teacher. In comparison, Claire saw a very different purpose for the explanations being presented at the board than the other participants. The goal for Claire was not to evaluate their explanation, but to gain some insight into her own explanation. In particular, she used the presented explanation as a way of identifying assumptions she may be making in her own explanation. Thus, Claire did not grant expertise authority in these situations.

Reformers generally have a negative attitude toward authority being granted to the teacher in the mathematics classroom. Leaders in the field such as Hiebert et al. (1997) believe that if we want our students to be able to evaluate explanations based on the mathematics, we need to remove the teacher from a position of authority and have her play a "guide on the side" role (Confrey, 1990). The way that Melissa granted authority to her teacher contradicts this widespread belief. Annie's instructional practices included acting as a "guide on the side" as the research advocates and requiring her students to do the evaluating of the explanations offered in class. Yet by reading Annie's facial expressions, Melissa was able to cast Annie as a bearer of authority, granting her expertise authority. This evidences a contradiction to what leaders in the field believe, because it is unlikely that the teacher can avoid being granted authority in the

classroom even if she is the guide on the side. Additionally reformers may look down upon Melissa's granting of authority to her teacher in order to evaluate the explanation, but Melissa evaluated the explanations before looking to her teacher, which is what researchers want students to be doing. Though she has the ability to read her teacher's facial expressions throughout the class, Melissa chooses to make her evaluation based on the mathematics first. This shows a second contradiction to what researchers believe, namely that just because students are granting the teacher authority does not mean that they are not reasoning mathematically or developing their own ability to evaluate explanations. This granting of authority to the teacher acted as a check of what Melissa concluded based on the mathematics. Thus the granting of authority was at least beneficial for Melissa in that it helped her to gain confidence in her ability to evaluate explanations and helped encourage her to participate in class. The evidence of the confidence benefit to authority as well as the identification of the many different ways authority can be granted in an inquiry-based classroom warrants a closer investigation focused on the potential necessity of authority in evaluation.

There is potential for the identification of further benefits of authority in the classroom. For example, if a student is going to evaluate an explanation based on the mathematics, she must know what it means for an explanation to be good according to the mathematics. Thus, students must have accepted appropriate sociomathematical norms and classroom mathematical practices defining a good explanation. Melissa was able to gain these norms and practices from her teacher both explicitly and implicitly. Lynn could only gain them from her teacher if they were stated explicitly, and accepted those offered by her classmates when her teacher would not identify the norms and practices. So in the case of both of the participants it was through the granting of expertise authority that they gained the means necessary for evaluating explanations.

Though not all students may use expertise authority to gain practices and norms, they must grant some type of authority because sociomathematical norms and classroom mathematical practices are emergent. So her fellow classmates and/or teacher must be influencing what she comes to understand as the sociomathematical norms and classroom mathematical practices or else those ideas could not be considered widely accepted by the class. Thus she grants them some type of authority in order to gain the norms and practices. In addition to granting expertise authority to gain sociomathematical norms and classroom mathematical practices there are other authority types that students may be granting. One authority types that likely plays a significant role in gaining sociomathematical norms and classroom mathematical practices in an inquiry-based mathematics class is common consent authority, the new type of authority I have identified in my data.

Common Consent Authority

In my analysis of the data I came across a type of authority that has not yet been discussed in the literature, which I refer to as *common consent authority*. Common consent authority is the type of authority granted to the bearer, which is a group of people, because they all agree. This type of authority is not granted because the student believes that other people are more expert than she is, but simply because a group of people have the same answer or solution.

There is a clear example of common consent authority being granted in a segment from Claire's data. In the following segment Siebert has asked her how she decides if her solution is mathematically correct. Claire responds by saying, "Um, if I get the correct answer or if I compare it with like a good number of people and, or like even one or two other people, and we have like the same answer." In this segment the way that she decided if her solution was correct was by checking to see if others had the same answer as her. This is not acceptance based on the

mathematics or on another's expert status, but rather because they all agree. Thus, in situations like the one described above Claire is granting common consent authority. If several people share the same solution, Claire allowed this consensus to influence her acceptance of her solution.

Melissa has some segments in her interview as well where she describes how she uses common consent authority like Claire does. In the following segment Melissa is describing how she knows if an explanation presented at the board is mathematically correct. She first talks about the expertise authority of her teacher, particularly how she knows an answer is right if she can read Annie's facial expressions. Next, she discusses common consent authority by stating how if she and another person have the same solution or same reasoning, then the solution or reasoning is likely correct.

Melissa: I can read Annie, and if Annie, like, smiles, or if I, if that's the answer I've kind of got, then I assume they're mathematically correct. So, I think it's just an assumption. Like, I don't absolutely know until Annie is like, "OK. Good." Or when the class is like, "Yea!!!" [Laughs.]

Siebert: OK. Alright. But, if they're doing something the same way that you're doing it, then you would ...

Melissa: I would, I, I assume that it's the right way, because at least ... If I think if two people are doing it the same way, then you probably are on the same path, and it ... that path's going to lead to the right answer.

At the beginning of the segment, Melissa describes how she assumes that the solution is mathematically correct if the presented explanation and hers are the same. At the end of the segment, she explains why she assumes she is correct, namely that consensus between her solution and the one on the board increases the likelihood that both are correct. Even though data presented earlier in this chapter suggests that Melissa is not completely convinced unless ideas are confirmed by Annie, she nonetheless grants common consent authority, which influences her

conviction of the correctness of a solution. So this previously unidentified type of authority, common consent authority, is likely being appealed to by Claire and Melissa as they evaluate explanations in their inquiry-based mathematics class.

While common consent authority was typically used by the participants to evaluate explanations, Claire also used it to pick up classroom mathematical practices and sociomathematical norms. In the following segment Claire was discussing what had been the most helpful for learning how to write good explanations, and she said, “Watching other people do their explanations probably. Like I listen to their explanations and I think, ‘Oh that was really good way.’ And seeing how the class is like, ‘Oh yeah.’” So in this segment Claire explains that hearing an explanation that she thinks is good and then seeing her classmates indicating that the explanation is good has helped her learn to write good explanations. In particular, she talks about a “good way” of explaining, suggesting that she had identified either a sociomathematical norm or classroom mathematical practice for good explanations.

Discussion of Common Consent Authority

First, this new type of authority is significant because it is likely to occur in inquiry-based classrooms due to a lack of expert authorities with which to check explanations or solutions. One reason this type of authority may not have been identified by Gerson and Bateman (2010) in their study was because of the way they identified authority types. They inferred the type of authority present based on student’s reactions and comments in whole class discussions. My data suggests that authority is often granted in small group settings where the behavior which may indicate the type of authority is more readily observable. In contrast, during whole class discussions, students were shown to grant common consent authority without an outward indication, when silently comparing their solution with the solution that was being presented at the board. Because

Gerson and Bateman did not include small group work in their data, they likely did not have the opportunity to observe this type of authority.

Common consent authority is also significant because it is a type of authority that is not based on status. All of the types of authority that Gerson and Bateman identified except mathematical authority were based on the status of the bearer of the authority. For example, expertise authority was granted when the bearer was seen as having the status of an expert. However, common consent authority is granted based solely upon consensus and does not regard the position of the person or people in consent.

It is somewhat surprising that this type of authority had not been identified before, because the establishment of sociomathematical norms and mathematical practices involves at least in part common consensus. Cobb and his colleagues (Cobb & Yael, 1996; Cobb et al., 2001) describe these constructs at the classroom level as being taken-as-shared. In order for rules or ways of reasoning to be taken-as-shared, the class of the students and teacher must, in general, accept them as such. Thus, in building these sociomathematical norms and classroom mathematical practices, there would have to be some level of consensus in the class. Claire's experience with learning a sociomathematical norm or classroom mathematical practice for writing explanations is an example of this consensus.

Chapter 5: Conclusion

Students do not always evaluate explanations based on the mathematics despite their teacher's effort to be the guide-on-the-side and delegate evaluation to the students. The use of three features of the Discourse—authority, sociomathematical norms, and classroom mathematical practices—likely impacts students' evaluation and may be contributing to students' failure to evaluate. To investigate the role of each of these features of the Discourse in evaluation I conducted a case study on three pre-service elementary school students enrolled in an inquiry-based mathematics class. From studying these students' evaluation methods I found that the students applied different types of each of the features of the Discourse and employed them at different times. I found that the way that the features of the Discourse were used contributed to some of the difficulties that the participants experienced in their evaluation of explanations.

Contributions

The first contribution that my study makes to the research is that it presents a case study that challenges current conceptions of students' failure to participate in inquiry-based mathematics classrooms. Accounts in the literature about student failure in inquiry-based mathematics classrooms often blame this failure on the students themselves, claiming that the failure is due to lack of motivation or inadequate mathematical preparation (Boaler, 1998). This study gives an example of students who are neither unmotivated nor resistant to the teaching practices yet still have difficulties evaluating explanations. Yet it is important for the field to acknowledge that inquiry-based teaching methods are not without limitations. The results of my study suggest that some of the features of the Discourse which allow student participation in the mathematical activity of evaluation are not being taught adequately through inquiry-based teaching practices.

Another contribution that my study makes is the understanding that classroom mathematical practices play a large role in evaluating explanations. The use of classroom mathematical practices to evaluate explanations has not been discussed thus far in the literature, though researchers such as Stylianides and Ball (2008) have presented data in which we can identify students using classroom mathematical practices for evaluation. This study shows that for students, the application of classroom mathematical practices in evaluation is very significant, and in many cases pivotal, to identifying a good explanation. Through this case study, I have documented some of the different roles classroom mathematical practices can play in evaluation, which provides a starting place from which researchers can further investigate this feature of Discourse.

Another contribution this study makes is demonstrating that authority plays a major role in students coming to see ways of reasoning and acting as sociomathematical norms and classroom mathematical practices. Authority has previously been shunned from the classroom by reform leaders such as Hiebert et al. (1999). This study, however, shows that authority is not only operating in inquiry-based mathematics classrooms, even when the teacher takes the role of the guide on the side, but that students must rely on some type of authority to decide what is taken-as-shared in the classroom.

Another contribution to the field's view of authority is this study's identification of common consent authority. This new type of authority adds to the framework that Gerson and Bateman's (2010) presented for authority. The addition of this type of authority is significant for future research in that it adds a type of authority that could be used to explain a lot of how knowledge is widely accepted in inquiry-based mathematics classrooms. It identifies one way

that ideas can be taken as shared: through consensus. Also, this type of authority can help researchers in the field to come to see authority as useful in learning mathematics.

The final contribution of my study is the framework presented in chapter two for investigating student's evaluation of explanations and its suggested application. My results have shown that this framework consisting of three features of Discourse can be used to describe an individual student's evaluation of explanations and to identify when they are evaluating based on the mathematics or not. Before this study, researchers had been using frameworks for authority, sociomathematical norms, and classroom mathematical practices to investigate these features of Discourse at the level of the class, occasionally to study students' evaluations or arguments or explanations (Fukawa-Connelly, 2012; Cobb & Yackel, 1996). But this research study shows that the students coming from the same class are not always granting authority to the same people in the same situations, accept very different ideas as sociomathematical norms from participating in the same activities, and evoke mathematical practices differently during evaluation. Thus a framework that can help researchers investigate evaluation of individual students is warranted. Attending to these constructs at a class level is helpful, but not sufficient for understanding students' difficulties in evaluating explanations.

Implications

Based on these contributions the study has made to the field, there are several implications for researchers. First, because we now have a case study that shows that students can struggle with inquiry based teaching practices without being resistant, researchers must more carefully consider students' reasons for not being successful at evaluating explanations based on the mathematics. In particular, researchers must acknowledge that students' lack of success may be caused by their difficulties with features of mathematical Discourse.

Second, researchers need to acknowledge the constant presence of authority relationships rather than assuming that authority has no role in inquiry-based mathematics. By identifying the types of authority being granted, researchers may be able to better characterize how students participate in mathematical activity. In addition, acknowledging the presence of authority and coming to understand more about how authority impacts evaluation could help to tell us exactly what benefits different types of authority have in evaluating explanations.

Finally, research on student's participation in mathematical activity, particularly evaluation, cannot continue to be conducted solely by studying behavior at the classroom level. Studies at the classroom level are unable to identify important differences in students' participation, because they provide inadequate data for determining how students are making sense of and using important features of the Discourse. This study identified significant differences in student evaluation methods which were identifiable only through discussing evaluation with the students. Thus in order to understand why students are unsuccessful in participating in mathematical activities, research at the classroom level needs to be supplemented with data collected through interviews with individual participants.

Limitations

One of the main limitations in this study was the small sample size, yet little could be said about a large sample of participants until the careful analysis such as was done in this study had been completed. By first starting with a very detailed, careful analysis of a few participants, researchers can now test the framework on larger populations and with more direction for dealing with the large sets of data. Though we had a small sample size, the diversity in the students' evaluation methods helped to give needed insight into different methods for evaluation.

This study was also limited in that the interviews were not designed with the framework and research question decided; however, the construction of the framework would not have been possible without the data in mind to guide the way. Because the data were already collected, once the framework was constructed this allowed for immediate commencement of the analysis of the data, and kept the framework at the foreground of the researcher's mind during the analysis. Now that the framework is in place researchers can take the framework and design interviews and perhaps classroom activities that could focus on each component of the framework.

Another limitation to the study was that the only data used was interview data. The interview data was the most important to focus on for this study because researchers had yet to see authority, sociomathematical norms, and classroom mathematical practices from the individual's perspective. Now that we have conducted this study which gives us insight into the evaluation methods of individual students, researchers can conduct studies in which they look at how the student evaluates explanations in real time in the classroom setting and compare those with responses in interviews about the student's evaluations.

Writing explanations and evaluating these explanations based on the mathematics have become key features of inquiry-based mathematics instruction. Though evaluation is one of the main features of the inquiry-based mathematics classroom, students do not always learn to be successful evaluators of the explanations based on the mathematics. The purpose of this research was to better understand why students do not evaluate explanations based on the mathematics. In order to develop this understanding I sought to answer the question: How does the use of authority, sociomathematical norms, and classroom mathematical practices impact a student's evaluation of explanations in an inquiry-based mathematics classroom? This study has presented

the way that these features of the Discourse impacted three different student's evaluation methods through the presentation of detailed case studies. Though this work has been completed and gives insight into the research problem, more needs to be investigated in order to make greater strides toward solving the problem.

References

- Benne, K. D. (1970). Authority in education. *Harvard Educational Review*, 40, 385-410.
- Boaler, J. (1998). Open and closed mathematics: Student experiences and understandings. *Journal For Research in Mathematics Education*, 29, 41-62.
- Bowers, J., Cobb, P., & McClain, K. (1999). The evolution of mathematical practices: A case study. *Cognition and Instruction*, 17(1), 25-66.
- Carpenter, T. P., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 19 - 32). Mahwah, NJ: Erlbaum.
- Cobb, P. (1999). Individual and collective mathematical development: The case of statistical data analysis. *Mathematical Thinking and Learning*, 1(1), 5-43.
- Cobb, P., & McClain, K. (2001). An analysis of development of sociomathematical norms in one first-grade classroom. *Journal For Research in Mathematics Education*, 32, 236-266.
- Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *The Journal of the Learning Sciences*, 10(1&2), 113-163.
- Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of development research. *Educational Psychologist*, 31(3/4), 175-190.
- Confrey, J. (1990). What constructivism implies for teaching. In R. B. Davis, C. A. Maher & N. Noddings (Eds.), *Constructivist views on the learning and teaching of mathematics (JRME Monograph Number 4)* (pp. 107 - 122). Reston, VA: National Council of Teachers of Mathematics.

- Depaepe, F., Corte, E. D., & Verschaffel, L. (2012). Who is granted authority in the mathematics classroom? An analysis of the observed and perceived distribution of authority. *Educational Studies*, 38, 223-234.
- Draper, R. J., & Siebert, D. (2004). Different goals, similar practices: Making sense of the mathematics and literacy instruction in a Standards-based mathematics classroom. *American Educational Research Journal*, 41(4), 927-962.
- Fukawa-Connelly, T. (2012). Classroom sociomathematical norms for proof presentation in undergraduate in abstract algebra. *The Journal of Mathematical Behavior*, 31, 401-416.
doi: 0.1016/j.jmathb.2012.04.002
- Gee, J. P. (1989). Literacy, discourse, and linguistics: Introduction. *Journal of Education*, 171(1), 5-17.
- Gerson, H., & Bateman, E. (2010). Authority in an agency-centered, inquiry-based university calculus classroom. *The Journal of Mathematical Behavior*, 29, 195-206.
- Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., . . . Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.
- Levenson, E. (2013). Exploring one student's explanations at difference ages: The case of Sharon. *Educational Studies in Mathematics*, 83, 181-203.
- National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*. Washington, DC: Authors.

- Stylianides, A. J., & Ball, D. L. (2008). Understanding and describing mathematical knowledge for teaching: Knowledge about proof for engaging students in the activity of proving. *The Journal of Mathematics Teacher Education*, 11, 307–332.
- The National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: NCTM
- The National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.
- Wagner, D., & Herbel-Eisenmann, B. (2014). Identifying authority structures in mathematics classroom discourse: a case of a teacher’s early experience in a new context. *ZDM*, 46(6), 871-882.
- Wilson, M. S., & Lloyd, G. M. (2000). Sharing mathematical authority with students: The challenge for high school teachers. *Journal of Curriculum and Supervision*, 17(1), 146-169.
- Wood, T. (1998). Alternative patterns of communication in mathematics classes: Funneling or focusing? In H. Steinbring, M. G. Bartolini Bussi & A. Sierpiska (Eds.), *Language and communication in the mathematics classroom* (pp. 167-178). Reston, VA: National Council of Teachers of Mathematics.
- Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal For Research in Mathematics Education*, 27, 458-477.
- Yackel, E., Rasmussen, C., & King, K. (2000). Social and sociomathematical norms in an advanced undergraduate mathematics course. *The Journal of Mathematical Behavior*, 19, 275-287.