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eBay or Amazon? Time-Sensitive Retail Purchases via Auctions

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Abstract

We consider a population of buyers who have unit demand for a homogeneous good, and only differ in terms of how soon they need to purchase it. These buyers have access to a stochastic stream of second-price auctions, as well as a retail outlet that can be used at any time. We characterize the equilibrium bidding dynamics, showing that bidders steadily raise their reservation price as they approach their deadline. This market produces a considerable degree of dispersion in auction revenue. Surprisingly, extending the buyers' deadlines actually increases expected revenue and the fraction of buyers using the retail outlet.

JEL Classifications: C73, D44, D83

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1 Introduction

Standard static auction models depict bidders as having a single opportunity to acquire the good in question, receiving a payoff of zero upon losing. While this seems appropriate for unique one-of-a-kind treasures, the typical online auction site offers many auctions of common retail products. On e-Bay, for instance, most items are identified by their SKU, often sold new-in-box; with popular items, one can encounter dozens of identical offerings over the course of a week. This stream of homogeneous offerings is reminiscent of traditional retailers, except using a sequence of auctions to determine prices rather than a constant posted price.

We model retail purchases via auctions in a continuous time search environment. Buyers have unit demand for a homogeneous item and encounter auctions for the item

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at random intervals. In addition, they also have an option at any time to purchase the item at its retail price. Each buyer faces a private deadline for their purchase (for instance, as a birthday present).

Deadlines introduces non-stationarity into the search problem, allowing reservation prices to increase with the duration of search. Indeed, this feature can generate rich price dynamics. To demonstrate this, we consider the stark case in which all buyers are identical in their valuation and their deadline; however, differences arise among them ex-post because some take longer than others to win an auction. These differences will lead to an endogenous, non-degenerate distribution of instantaneous reservation prices, even though all buyers enjoy the same eventual utility from the good.

In equilibrium, a given buyer's reservation price increases as the deadline approaches, accelerating over time. This explains why losers of a prior auction would offer a higher price in subsequent auctions for an identical item. Buyers who have newly entered the market only win in the rare event that their auction had no other buyers; most of the winners are close to their deadlines. This is precisely when their reservation prices increase most rapidly; as a consequence, closing prices still have significant dispersion across auctions.

Most comparative statics are quite intuitive. For instance, more frequent auctions will benefit buyers by reducing search frictions; as a consequence, they shade their bids further and expected revenue falls. The surprising exception is when the deadline is extended for all bidders. One would expect this to also help a buyer by providing more opportunities before he must resort to the retail store. However, this also increases the average number of buyer in the market; as a consequence, each auction is more likely to be competitive and expected revenue increases! Moreover, a larger fraction of bidders are unsuccessful in the auctions and end up using the retail outlet.

This work contributes most directly to a nascent literature on infinite sequential auctions (Zeithammer, 2006; Ingster, 2009; Said, 2011, 2012; Backus and Lewis, 2012; Bodoh-Creed, 2012; Hendricks, *et al*, 2012).[1] These each consider a market with a steady flow of similar goods from independent sellers. These papers, as well as ours, focus on dynamics between auctions rather than within an auction, which occur instantaneously via a second-price sealed bidding.[2] Rather than truthfully revealing

[1] An early strand of literature, following Milgrom and Weber (2000), considers a finite sequence of auctions offered by a single seller.

[2] The exceptions are Hendricks, *et al* (2012), where new bidders move last to avoid disclosing their

the bidder's valuation, the optimal bid in sequential auctions is shaded down by the buyer's continuation value, reflecting the opportunity cost of forgoing future auctions.

The biggest distinction in our approach is the source of heterogeneity among bidders. Rather than assuming buyers fundamentally differ in their valuations, which are drawn from a known distribution, we assume all buyer are identical at the time they enter the market. This seems plausible in the context of auctioning standardized products, and is assumed in Hendricks, *et al* (2012); there, heterogeneity arises in an asymmetric strategy where newly-arrived bidders have an informational advantage and wait until all incumbent bidders (losers of prior rounds) have bid. In our environment, distinctions arise endogenously because some unlucky bidders search longer than others, and therefore are nearer their deadline and willing to pay more.

One abstraction in our model is that we assume that bidders do not infer any information about their rivals from prior rounds. Such information is unimportant if valuations are redrawn between each auction (as in Ingster, 2009; Said, 2011; Bodoh-Creed, 2012), but if valuations are persistent, this *leakage* of private information can hinder future success and leads to further shading of bids (Backus and Lewis, 2012; Said, 2012). In our context, it is the bidder's remaining time until deadline, rather than the valuation, that could be leaked. In a large market, though, the cost of recording hundreds of bidders past actions seems impractical; also, since not every bidder participates in every auction, the expected value of such information diminishes quickly. Zeithammer (2006) uses the same assumption and justification.

Methodologically, our work provides a unique perspective on sequential auctions by leveraging tools commonly used in the equilibrium search literature (surveyed in Rogerson, *et al*, 2005). This literature investigates the formation of wages or prices in a search environment, with particular attention to what model features can sustain a non-degenerate price distribution. Akin and Platt (2012) introduce deadlines as a source of wage dispersion, due to the eventual cutoff of unemployment insurance benefits. Akin and Platt (2013) investigate consumer search in the presence of deadlines and uncertain recall of past offers. Both articles use the assumption that workers or consumers are fundamentally homogeneous, with differences evolving only

arrival, and Said (2012), where each period features a multi-unit ascending auction. Within-auction dynamics have mostly been studied in the context of a single auction (Nekipelov, 2007; Ambrus and Burns, 2010) or concurrent auctions (Peters and Severinov, 2006; Ely and Hossain, 2009). In those environments, bidders can benefit from incremental bidding or waiting till the last minute (*sniping*) rather than submitting their true valuation as their only bid.

because of their search experience. The equilibrium price distribution can react in surprising ways to model parameters, as also seen in this model's comparative statics on time till deadline.

In addition to these insights from endogenous price dispersion, we use two modeling features that are common in the search literature but largely unexploited in the sequential auction literature: analysis in continuous time and steady state distributions. Specified in a continuous time framework, the model's equilibrium conditions can be translated into a solvable system of differential equations, which was the methodological innovation of Akin and Platt (2012). Only Ingster (2009) uses a continuous time model, but as bidder heterogeneity is exogenous there, our approach is unnecessary for the solution. Rather, the solution to the discrete time model in Said (2011) has more parallels to our method; there, the relevant state variable affecting bid shading is the (discrete) number of active bidders. The equilibrium conditions are then depicted as a first order system of difference equations.

We also restrict our analysis to steady state behavior; that is, the distribution of buyers (with respect to time until deadline) remains stable. This keeps population dynamics tractible yet consistent with flows of incoming buyers and outgoing winners. Most of the sequential auctions literature (with the exception of Zeithammer, 2006) includes at least the current number of bidders as a state variable which fluctuates over time, finding the Markov equilibrium over these states; some also include characteristics of the bidders in the state. Our focus on a steady state is best interpreted as long-run behavior in a market fairly thick with buyers,[3] which seems reasonable for the auctioning of retail items.

One of the benefits of employing these search-theoretic methods is greater tractability, allowing us to evaluate comparative statics readily, for instance. Most of the cited literature focuses primarily on the structural estimation of the model and do not investigate how their model reacts to parameter changes. The exceptions are Zeithammer (2006) and Said (2011), which feature comparative statics with respect to the discount rate, the time between auctions, and (in the latter) the expected number of competitors. Reservation prices respond to these in the same direction as in our model.

We proceed by developing the model in Section 2, and characterizing its solution

[3]For instance, Said (2011) shows that as the number of buyers increases, the marginal impact on bidding behavior diminishes.

in Section 3. We then close the model in Section 4 by introducing the supplier's problem and describing its equilibrium behavior. Section 5 concludes, with all proofs in the Appendix.

2 Model

2.1 Buyers

Consider the market for a homogeneous good in a continuous-time environment. Buyers randomly enter the market at Poisson rate δ, needing to purchase one unit of the good within T units of time. New second-price sealed-bid auctions occur at rate α; each of the active buyers in the market independently participates with exogenous probability τ, reflecting that buyers can be distracted by other commitments. Each participant submits a bid and immediately learns the auction outcome.

Alternatively, at any time, a buyer can obtain the good directly from a retail store (or Buy-it-now offering) at price z. If a buyer purchases (or wins) the good with s time remaining until his deadline, he pays z (or the second highest bid) immediately; but he discounts future utility x from the item, reflecting that it will not be used until the deadline: $e^{-\rho s}x$.

Every buyer shares the same utility x and deadline T on entering the market; but because they randomly enter the market, they will differ ex-post in their remaining time s. In any given auction, the number of bidders and their state s are private information. The endogenous distribution of bidders in the market, represented by cumulative distribution $F(s)$, is commonly known, as is the average number of buyers in the market, which is Poisson distributed with mean H.

The strategic question for buyers is what bid to submit and when to purchase from the retail store. This dynamic problem can be expressed recursively, letting $V(s)$ denote the discounted expected utility of a buyer with s time remaining until deadline. Each participant submits a bid $R(s)$ that depends on his remaining time until deadline.[4] In particular, each auction is independent, since it draws a new set of participants from the pool of active buyers.

[4]If a bidder were allowed to observe the number of participants, it would still be a weakly dominant strategy to follow the same strategy; thus, R would still not depend on n.

The optimal behavior is to bid one's reservation value, setting:

$$R(s) = e^{-\rho s}x - V(s); \tag{1}$$

that is, the present value of the item minus the opportunity cost of not waiting for future auctions. As in the standard second-price sealed-bid auction, this strategy is weakly dominant regardless of that employed by other participants.[5] We assume that $R(s)$ is decreasing in s, and later verify that this holds in equilibrium. The auctioneer is assumed to open the bidding at $R(T)$, which is relevant in the case that only one buyer participates.

In a given auction, the number of participants is Poisson distributed with mean $\lambda = \tau H$; moreover, Myerson (1998) demonstrates that in Poisson games, the players assess the distribution of other players the same as the external game theorist would assess the distribution for the whole game. Thus, from the perspective of a bidder who has just entered, the number n of *other* participants is distributed with density $e^{-\lambda}\lambda^n/(n)!$.

In light of this, a buyer's expected utility in state s can be expressed in the following Bellman equation:

$$\begin{aligned}\rho V(s) = -V'(s) + \tau\alpha\Bigg(& \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}(1-F(s))^n \left(e^{-\rho s}x - V(s)\right) - e^{-\lambda}R(T) \\ & - \sum_{n=1}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} \int_s^T R(t)n(1-F(t))^{n-1}F'(t)dt\Bigg).\end{aligned} \tag{2}$$

In this formulation, the term $-V'(s)$ reflects the fact that the state s is steadily decreasing as the deadline approaches. If an auction is encountered — which occurs at rate $\tau\alpha$ — the expected payoff depends on the number n of other participants, which is Poisson distributed with mean λ. The buyer in state s will only win (have the highest bid $R(s)$) if all n other participants have more than s time remaining; this occurs with probability $(1 - F(s))^n$. In such a case, the term $e^{-\rho s}x - V(s)$ gives the change in utility from winning.

[5] Suppose he instead bids price $p > R(s)$, and the second highest bid is q. This results in the same payoff whenever $q \leq R(s)$, but yields negative surplus when $q \in (R(s), p]$. Similarly, if he bids $p < R(s)$, he has the same payoff whenever $q \leq p$, but misses out on positive surplus when $q \in (p, R(s))$.

The remaining terms compute the expected cost of winning (*i.e.* the average second-highest bid times the probability of winning and thus paying it). If there are no other participants, the bidder pays the starting price $R(T)$. Otherwise, the sum provides the probability of n opponents and the integral computes the highest bid among those n opponents, which has distribution $n(1-F(t))^{n-1}F'(t)$.[6]

Buyers also have the option to purchase from the retail store at any time, receiving utility $e^{-\rho s}x - z$. However, a buyer in state s can get at least $e^{-\rho s}(x-z)$ by delaying this purchase until $s = 0$, which is strictly preferred due to the delay of payment z. This delay strategy has even greater payoff due to the possibility of winning an auction in the meantime. Thus, the retail option is exercised if and only if $s = 0$, and

$$V(0) = x - z. \tag{3}$$

2.2 Steady State Conditions

We next construct the steady state distribution of buyers in the market, as $F(s)$ is endogenously determined by the rate at which buyers win an auction. This law of motion is expressed as follows:

$$F''(s) = F'(s)\tau\alpha \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}(1-F(s))^n. \tag{4}$$

The density of buyers in state s is $F'(s)$, and such a buyer encounters an auction at rate $\tau\alpha$. However, the number of participants is random, and he only wins the auction if he is the closest of the n participants to his deadline. Combined, the right hand side of Equation 4 indicates the rate at which buyers in state s win and exit the market.

In addition, several boundary conditions are necessary.

$$F(0) = 0 \tag{5}$$
$$F(T) = 1 \tag{6}$$
$$\delta = H \cdot F'(T) \tag{7}$$

[6] Note that since the range of integration is only from s to T, $n(1-F(t))^{n-1}F'(t)$ would only sum to $(1-F(s))^n$ rather than 1. This coincides with the probability that this buyer's bid is higher than all opponents, as required.

The first two conditions rule out atoms at either end of the distribution. All buyers who reach state $s = 0$ immediately purchase from the retail store and exit, no longer participating in auctions. Hence, no stock of state 0 buyers can accumulate. Similarly, no stock of state T buyers can accumulate because as soon as they enter the market, their clock begins steadily counting down.

The third condition relates the entry rate to the total population. Buyers exogenously enter the market at rate δ in state T; on the other hand, $F'(T)$ is the relative density of bidders in state T. This must be scaled by the average number of active buyers, H.

2.3 Equilibrium Conditions

The preceding problems constitute a dynamic game; we define a *buyer steady-state equilibrium* of this game as a reservation price function $R^* : [0, T] \to \mathbb{R}$, a distribution of active buyers $F^* : [0, T] \to [0, 1]$, an average number of active buyers $H^* \in \mathbb{R}^+$, and an average number of participants per auction $\lambda^* \in \mathbb{R}^+$, such that:

1. The reservation prices R^* satisfies the Bellman Eqs. 1 through 3, taking F^* as given.

2. The distribution F^* is weakly increasing and satisfies the Steady State Eqs. 4 through 6.

3. The average number of active buyers H^* satisfies Steady State Eq. 7.

4. The average number of participants per auction satisfies $\lambda^* = \tau H^*$.

The first requirement requires buyers to use an optimal reserve price; the last three require buyers to have correct beliefs regarding the population of competitors they may face.

3 Equilibrium Characterization

We now present the unique equilibrium of this retail auction environment. We begin by examining the first three equilibrium conditions (on buyers), which we translate into two second-order differential equation regarding $F(s)$ and $R(s)$. The equations themselves have an analytic solution, but one boundary condition does not have a closed-form solution. Instead, we introduce a variable k^* which implicitly solves the boundary condition. If we define $\phi(k)$ as:

$$\phi(k) \equiv \alpha\tau - \lambda k - e^{-\lambda T k}\left(\alpha\tau - \lambda k e^{\lambda}\right), \tag{8}$$

then the boundary condition is equivalent to $\phi(k^*) = 0$. Thus, the final solution is expressed in terms of k^*.

First, the distribution of buyers over remaining time until deadline is:

$$F^*(s) = \frac{1}{\lambda} \ln\left(\frac{\alpha\tau - e^{\lambda s k^*}\left(\alpha\tau - \lambda k^*\right)}{\lambda k^*}\right). \tag{9}$$

Intuitively, the density function F' is strictly increasing in s, since some buyers win auctions and thus exit as the clock s counts down. The rate of increase depends on s, typically changing from convex to concave as s increases. This is because buyers rarely win at the beginning of their search, but increasingly do so as their reservation value falls. Those near their deadline win quite frequently, but few of them remain in the population, so their rate of exit decelerates.

The average number of buyers in the market is:

$$H^* = \frac{\delta\lambda}{\lambda k^* - \alpha\tau e^{-\lambda}}. \tag{10}$$

The population of active buyers increases with λ (including the implicit effect of λ on k^*), since more competitors makes winning a given auction less likely. An increase in α or τ has the opposite effect, as a given buyer is more likely to participate in available auctions.

Reservation prices are expressed as a function of the buyer's state, as follows:

$$R^*(s) = z\left(1 - \frac{\left(1 - e^{-\rho s}\right)\lambda k^*(\rho + \lambda k^* - \alpha\tau) + \alpha\tau\rho e^{-\rho s}\left(1 - e^{-\lambda s k^*}\right)}{\lambda k^*\left(\rho + \lambda k^* - \alpha\tau + \alpha\tau e^{-(\rho + \lambda k^*)T}\right)}\right). \tag{11}$$

The following result establish that this proposed solution is both necessary and sufficient for the three equilibrium requirements. The proof (provided in the appendix) is constructive, translating the necessary equilibrium conditions into differential equations of $F(s)$ and $R(s)$. These equations each have unique solutions, leading to the sufficiency result.

Proposition 1. *Equations 9 through 11 satisfy equilibrium conditions 1 through 3, and this equilibrium solution is unique.*

As previously conjectured, one can readily show that $R'(s) < 0$; that is, reservation prices increase as buyers approach their deadline. Moreover, this increase accelerates as the deadline approaches, since $R''(s) > 0$.

Proposition 2. *In equilibrium, $R'(s) < 0$ and $R''(s) > 0$.*

Finally, we need to ensure that equilibrium condition 4 is satisfied as well. After substituting in the solution for H^* in Equation 10, this becomes:

$$\lambda^* k^* - \alpha\tau e^{-\lambda^*} = \delta\tau. \tag{12}$$

Note that the solutions for F^*, R^*, and H^* hold for any λ; thus it only remains to demonstrate that $\phi(k^*) = 0$ and Equation 12 can simultaneously hold. Indeed, the solution is unique.

Proposition 3. *Assuming that $\delta > 2\alpha$, a solution (k^*, λ^*) exists for $\phi(k^*) = 0$ and Equation 12. A sufficient condition for uniqueness is $T\alpha\tau \leq 1$.*

The two sufficient conditions in the preceding result are much stronger than needed in practice. Indeed, we have not found any numerical counterexamples to existence, nor have we to uniqueness so long as $\delta > 2\alpha$. However, the condition $T\alpha\tau \leq 1$ is the simplest expression that avoids using endogenous variables.

3.1 Numerical Example

We now provide a numerical example which is representative of the typical equilibrium behavior. The retail price is normalized to $z = 1$. Consistent with a period being one month, we let $\rho = 0.005$ and give buyer half a year to search, $T = 6$. Buyers enter at rate $\delta = 100$ and auctions arrive at rate $\alpha = 85$. We set $\tau = 0.025$ so that buyers typically participate in one auction every two weeks. While these fall outside the sufficient conditions in Proposition 3, the equilibrium does appear to be unique.

In this environment, the average number of active buyers is $H^* = 524$, with $\lambda^* = 13.1$ of them showing up at any given auction. Fifteen percent of buyers are unsuccessful in purchasing via the auctions and eventually use the retail store. Also, we note that the average auction generates revenue equal to $0.982z$.

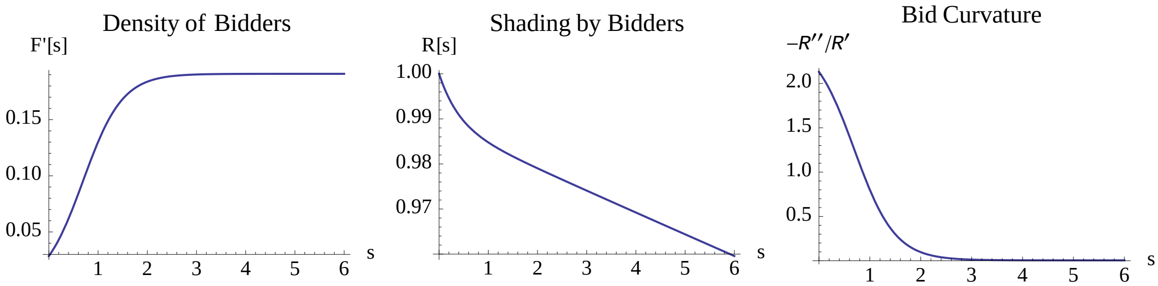

Figure 1: Numerical Example: the distribution of bidder states (left panel), the % by which bidders shade relative to z (center panel), and the curvature of the bid path (right panel).

The left panel of Figure 1 illustrates the equilibrium density of bidders. Note that $F'(s)$ is nearly constant from $s = 2.5$ to 6. Due to the large number of bidders per auction, those with lower valuations (hence longer time remaining) are highly unlikely to win. Ye the relative density cuts in half between $s = 2.5$ and $s = 0.8$, and then does so again before $s = 0$. Those closest to their deadline are far more likely to win and exit.

The center panel of Figure 1 depicts the equilibrium path of reservation prices. Since $z = 1$, these can be read as the factor by which bidders shade their bids below the retail price. Note that prices are dispersed across a range equal to 4% of the retail price. This becomes larger with higher discounting or longer deadlines. Initially (near $s = T$), the price path is more or less linear; but as the deadline approaches, greater curvature is introduced. To make this more visible, we compute curvature as $-R''(s)/R'(s)$, depicted in the right panel. This is essentially zero for $s > 2$, but quickly rises for S near the deadline.

Figure 2 provides another perspective on the realized bids in the auction. The left panel plots the density of submitted bids in the typical auction. That is, for any price p on the x-axis, the y value indicates the relative likelihood of that price being placed as a bid. Effectively, this is $F'\left(R^{-1}(p)\right)$, except that $R(s)$ is not invertible, so we obtain the figure via a parametric plot. This plot shares much in common with the plot of $F'(s)$ in the left panel of Figure 1, reversed in direction since the highest bids come from the bidders closest to their deadline.

The right panel of Figure 2 provides a similar parametric plot, but with the y-axis indicating the relative likelihood that p will be the value of the second-highest

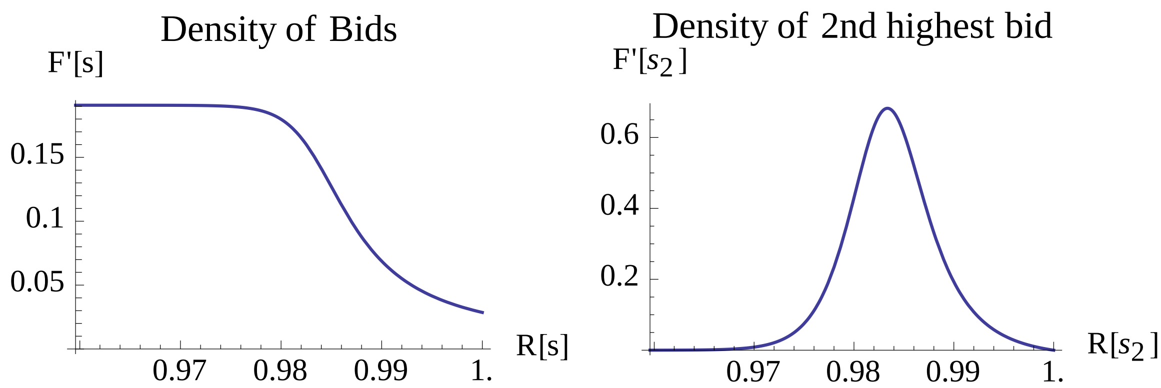

Figure 2: Price Density: the distribution of prices that a random bidder is willing to pay (left panel) and the distribution of the second highest bid (right panel).

submitted bid. Due to the large average number of participants, bidders with low valuations are unlikely to be the winner or even the second highest bidder. Rather, the modal closing price is 0.987, which is nearly the reservation price of a bidder with $s = 1$. This illustrates why expected revenues remain fairly close to z.

3.2 Comparative Statics

We next examine how the equilibrium behavior reacts to changes in the underlying parameters. Although our equilibrium has no closed form solution, these comparative statics can be obtained by implicit differentiation of the $\phi(k)$. Under the sufficient conditions of Proposition 3, we can sign the resulting derivatives. Numerical evaluation results in the same signs even when the sufficient conditions do not hold.

Table 1 reports the sign of the derivatives of four key statistics. First are reservation prices, $R(s)$; the reported sign holds across all s. Second is the expected revenue, which has a cumbersome analytic expression but is readily computed numerically. The last two indicate the fraction of buyers who must resort to the retail store, and the average number of buyers in the market.

Changes in α have a very intuitive impact. If auctions arrive more frequently, this reduces search frictions; that is, the value of continued search is greater as there are more opportunities to bid. Moreover, the increase in auctions will reduce the stock of bidders and hence the number of competitors per auction. Both of these effects lead to lower reservation prices and lower expected revenue.

Table 1: Comparative statics on key statistics: Buyer Equilibrium

	$\partial/\partial\alpha$	$\partial/\partial\tau$	$\partial/\partial\rho$	$\partial/\partial T$
Reservation Prices	$-$	$+$	$-$	$-$
Expected Revenue	$-$	$+$	$-$	$+$
% buying retail	$-$	$-$	0	$-$
Active Buyers	$-$	$+$	0	$+$

Changes in τ have nearly the reverse effect, though here there are opposing forces at work. Having a higher likelihood of participating also reduces the search friction of a given bidder, as he will participate in more of the existing auctions. On the other hand, all other bidders are more likely to participate as well. This greater number of competitors dominates the increased auction participation to reduce the value of search, increasing the reservation prices and expected revenue.

The discount rate has a rather substantial and obvious impact on reservation prices, as buyers become less interested in buying early. This leads to lower expected revenue. Surprisingly, ρ has no impact on the distribution or number of buyers.

The most intriguing case is when the deadline changes. Intuitively, one would expect that an increase in T would work to buyers' advantage. Indeed, reservation values are lower, yet at a given s the change is minor; the larger impact is because T is further from the deadline so $R(T)$ is lower. At the same time, the longer period of search allows for more bidders to accumulate (H^* increases). Thus, auctions are more likely to be competitive, causing expected revenue to increase.

It is surprising that sellers should gain from buyers having more time to shop, but it seems that much of this is driven by the likelihood of having anyone show up (which improves due to the higher stock of buyers). If one computes expected revenue conditional on having at least one bidder, it actually decreases with T due to the lower reservation values.

4 Seller Incentives

The interesting extensions to the model require us to also examine optimization by sellers, so that α is determined in equilibrium. We model this as a continuum of sellers who each produce the good at a constant marginal cost of $c < z$ at the time that the item is purchased. Sellers enter the market at the same rate δ as buyers. Since the buyers and sellers enter at the same rate, we also assume the stock of active sellers equals the stock of active buyers, H. Upon entry, each seller must decide whether to join the auction or retail sector.

The advantage of the auction sector is that they will conduct the auction immediately, though this comes at a disadvantage that the expected revenue is lower than the retail price z. This expected revenue θ is expressed as follows:

$$\theta \equiv \lambda e^{-\lambda} R(T) + \sum_{n=2}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} \int_0^T R(s) n(n-1) F(s)^{n-2}(1-F(s))F'(s)ds. \tag{13}$$

The first term, $\lambda e^{-\lambda} R(T)$, indicates the revenue earned if only one buyers participates. The sum handles cases when there are $n \geq 2$ simultaneous bidders. The integral computes the expected bid $R(s)$ of the second highest bidders. Note that this ignores the case in which no bidders arrive, which occurs with probability $e^{-\lambda}$. If this occurs, we assume that the unsuccessful seller exits the market (earning no revenue but incurring no cost) rather than making a repeated auction attempt. We make this assumption for convenience; in practice, λ is sufficiently high that this probability is negligible. Thus, the expected profit of an auction seller is:

$$\Pi_a \equiv \theta - \left(1 - e^{-\lambda}\right) c. \tag{14}$$

Note in the last term that the seller does not incur the cost of production if no buyers show up at the auction.

The retail sector consists of a pool of waiting sellers. Note that sellers entering the auction sector immediately exit, meaning there will never be a positive measure of active sellers in that sector; thus the measure of retail sellers always equals H. These sellers are more passive, awaiting the arrival of a buyer (occurring at rate $HF'(0)$), who is then randomly allocated to one of the sellers with equal probability. Thus, their discounted expected profit is recursively formulated as:

$$\rho \Pi_r \equiv \frac{HF'(0)}{H} \left(z - c - \Pi_r\right),$$

or after simple rearrangement,

$$\Pi_r = \frac{(z-c)\,F'(0)}{\rho + F'(0)}. \tag{15}$$

With the addition of the seller's problem, we augment an equilibrium with two conditions. A *market steady-state equilibrium* consists of a buyer equilibrium along with expected profits $\Pi_a^* \in \mathbb{R}^+$ and $\Pi_r^* \in \mathbb{R}^+$, a rate of auction arrival $\alpha^* \in \mathbb{R}^+$, and fraction of sellers who enter the auction sector, $\sigma \in \mathbb{R}^+$, such that:

1. The conditions for a buyer equilibrium are satisfied, given α^*.

2. The entering seller is indifferent between the two sectors: $\Pi_a^* = \Pi_r^*$, given α^*.

3. Auctions arrive at the same rate that sellers enter the auction sector: $\alpha^* = \delta\sigma^*$.

While these equilibrium conditions simplify remarkably, they still do not admit an analytic solution. Indeed, we now must numerically solve for α^* and λ^* simultaneously. In repeated calculations across a wide variety of parameters, we observed either one or two equilibria, depending primarily on the cost of production. When costs were rather low, a unique equilibrium emerges with α^* fairly close to δ. As costs increase, the number of auctions decrease; then eventually a second equilibrium emerges withα^* close to zero. If costs continue to increase, these two equilibria approach each other; eventually they coincide, and then for higher costs (*i.e.* when c is near z), no equilibrium exists. In this last circumstance, auctions are no longer profitable because enough early bidders could win at prices below cost; thus only the retail sector will operate.

To illustrate the outcome of this augmented model, we can refer back to our numerical example presented in Section 3.1. The same buyer behavior would emerge in the market equilibrium if $c = 88.165$, yielding an expected revenue of $\theta^* = 98.24$ and profit of $\Pi_a^* = 10.07$ from the auction. While a sale in the retail sector generates an extra \$1.76 dollars on average, this is offset in that only 2.9% of retail sellers are able to sell in a given period, requiring on average 35 periods to make that sale.

Having defined the sellers side of the market, we can also consider the important question of whether sellers would want to increase their reservation price above $R(T)$. We do this from the perspective of deviation by a single seller, with all buyers and sellers following the equilibrium strategy. That is, consider a seller decided to enter

Table 2: Comparative statics on key statistics: Market Equilibrium

	$\partial/\partial c$	$\partial/\partial\tau$	$\partial/\partial\rho$	$\partial/\partial T$
Auction rate	$-$	$+$	$-$	$+$
Reservation Prices	$+$	$-$	$-$	$-$
Expected Revenue	$+$	$-$	$+$	$-$
% buying retail	$+$	$-$	$+$	$-$
Active Buyers	$+$	$+$	$+$	$+$

the auction sector and set a minimum reserve price of $R(m)$ where $m < T$. For the buyers who happened to enter this deviating auction, their continuation value from continued search is unaffected, since all other sellers are behaving as before. Buyers with $s > m$ will be shut out from winning the auction, but are unwilling to bid more since they were indifferent between winning and continued search at their submitted bid. Those buyers with $s \leq m$ are more likely to win than in the equilibrium auction, but their reservation value of winning is still the same, determined by indifference with continued search. Thus, no buyer has an incentive to revise their reservation strategy.

A higher reservation price introduces two opposing effects. On the one hand, the buyers with the most time remaining and hence lowest willingness to pay are excluded; these low bidders are plentiful since our bidder density is strictly increasing in s. This has particular importance since the second-highest bid determines the price paid by the winner. On the other hand, a higher reservation price also increases the likelihood of ending the auction with no bidders above the reservation price. We investigate these effects through numerical integration of expected revenue; for all reasonable parameterizations, we find that the second effect strictly dominates — andy increase in the reservation price will decrease expected revenue for the seller. This is largely a consequence of λ^* being moderately large in equilibrium. Even with the large fraction of low bidders in the market, they are rarely the 1st or 2nd highest bidder in the market; and apparently, the possibility of losing them as the 1st highest bidder is worse than having them as the 2nd highest bidder.

5 Conclusion

This work reexamines the auction environment as a venue for selling retail goods. Our analysis leverages methods frequently used in search theory, which provide analytical tractability and plausibly match the auction/search setting. Standard auction theory ascribes all variation in bids as generated by exogenous differences in valuations; but this seems less compelling when there is a readily-available outside option for the same item. In our model, all buyers enjoy the same eventual utility from the good, but endogenously differ in how soon they must acquire it. This produces an increasing and accelerating path for an individual's reservation price, and a rich continuous distribution of closing prices across auctions.

In this vein, we anticipate future work in which we structurally estimate our model using eBay data and test its predictions. Moreover, it would be fruitful to compare what how the same data would be interpreted in a private value model such as in Backus and Lewis (2012).

The theory also is open to further development, particularly to move out of steady state analysis and instead allow the current number of bidders to be publicly known. Under this specification, buyers would still optimally bid $e^{-\rho s}x - V(s,n)$, but the continuation value would then depend on the current number of participants n, since this provides information on the likely number of future participants and hence the probability of winning. This setting (unlike one in which bidder identities are remembered between auctions) is likely to remain tractable.

A Proofs

Proof of Proposition 1. First, we note that the infinite sums in equations 2 and 4 can be readily simplified. In the case of the latter, it becomes:

$$F''(s) = \alpha\tau F'(s) e^{-\lambda F(s)}. \tag{16}$$

This differential equation has the following unique solution, with two constants of integration k and m.

$$F(s) = \frac{1}{\lambda} \ln\left(\frac{\alpha\tau - e^{\lambda k(s+m)}}{\lambda k}\right). \tag{17}$$

The constants are determined by our two boundary conditions. Applying Eq. 6, we obtain $m = \frac{1}{\lambda k} \ln\left(\alpha\tau - \lambda k e^{\lambda}\right) - T$. By substituting this into Eq. 17, one obtains the proposed solution for F^* in Eq. 9. The other boundary condition, Eq. 5, is equivalent to requiring that $\phi(k) = 0$.

We note here that for a given λ, there are always exactly two solutions where $\phi(k) = 0$. One of these occurs at $k = 0$, but if this is inserted in Eq. 9, we obtain $F(s) > 1$ for all s. Hence, this solution can be ignored.

There is no solution $\phi(k) = 0$ where $k < 0$. If $k < 0$, then $e^{-\lambda Tk} > 1$ and $e^{\lambda(1-Tk)} > 1$, so therefore $\phi(k) = \alpha\tau\left(1 - e^{-\lambda Tk}\right) - \lambda k\left(1 - e^{\lambda(1-Tk)}\right) < 0$.

However, there must exist a solution $\phi(k^*) = 0$ where $k^* > 0$. This is because $\phi'(0) = -\left(1 - e^{\lambda} - \alpha\tau T\right)\lambda > 0$, so $\phi(k) > 0$ for k near 0. At the same time, as $k \to \infty$, both exponential terms vanish and $\phi(k) < 0$. Since $\phi(k)$ is continuous and changes sign on $(0, \infty)$, then by the intermediate value theorem, there must exist a $k^* \in (0, \infty)$ such that $\phi(k^*) = 0$.

Furthermore, $\phi'(k)$ is also continuous and the slope of $\phi(k)$ is strictly positive at 0, but must have negative slope for some values since it eventually reach a value below zero as $k \to \infty$. Thus, by the intermediate value theorem, there exists a $\bar{k} \in (0, \infty)$ such that $\phi'(\bar{k}) = 0$.

Indeed, this $\bar{k}$ is unique. Note that $\phi'(k) = -\lambda e^{-\lambda Tk}\left(e^{\lambda Tk} + \lambda Tke^{\lambda} - \alpha\tau T - e^{\lambda}\right)$. The first term before the parentheses is negative for all values of k. Within the parentheses, the first two terms will increases as k increases, while the others are constant. Thus, if $k < \bar{k}$, then:

$$e^{\lambda Tk} + \lambda Tke^{\lambda} - \alpha\tau T - e^{\lambda} < e^{\lambda T\bar{k}} + \lambda T\bar{k}e^{\lambda} - \alpha\tau T - e^{\lambda} \quad \Longrightarrow \quad \phi'(k) > \phi'(\bar{k}) = 0.$$

The reverse holds if $k > \bar{k}$, so that $\phi'(k) < \phi'(\bar{k}) = 0$. Thus, there can only be one such $\bar{k}$.

Indeed, $\phi(k)$ can only equal 0 once in $(0, \infty)$, since it is positive and increasing initially, but once it starts decreasing it does so for all higher values of k. Thus k^* is unique.

We finally turn to the solution for reservation prices. Again, we start by simplifying the infinite sums in Eq. 2. The first sum is similar to that in Eq. 4. For the second, we first change the order of operation, to evaluate the sum inside the integral. This is permissible by the monotone convergence theorem, because $F(s)$ is monotone and $\sum \frac{e^{-\lambda}\lambda^n}{n!} R(t) n(1-F(t))^{n-1}$ converges uniformly on $t \in [0, T]$. After evaluating both sums, we obtain:

$$\rho V(s) = -V'(s) + \alpha\tau \left(e^{-\lambda F(s)} \left(e^{-\rho s} x - V(s) \right) - e^{-\lambda} R(T) - \int_s^T \lambda e^{-\lambda F(t)} R(t) F'(t) dt \right).$$

Next, by taking the derivative of $R(s) = -xe^{-\rho s} - V(s)$ (Eq. 1), we obtain $R'(s) = -\rho x e^{-\rho s} - V'(s)$. We use these two equations to substitute for $V(s)$ and $V'(s)$, obtaining:

$$(\rho + \alpha e^{-\lambda F(s)}) R(s) + R'(s) = \alpha\tau \left(e^{-\lambda} R(T) + \int_s^T \lambda e^{-\lambda F(t)} R(t) F'(t) dt \right). \tag{18}$$

This equation holds only if its derivative with respect to s also holds, which is:

$$(\rho + \alpha\tau e^{-\lambda F(s)}) R'(s) + R''(s) = 0.$$

After substituting for $F(s)$ solved above, this differential equation has the following unique solution, with two constants of integration a and b:

$$R(s) = a \cdot \left(\frac{e^{-\rho s} (\lambda k^* - \alpha\tau)}{\rho} + \frac{\alpha\tau e^{-s(\rho + \lambda k^*)}}{\rho + \lambda k^*} \right) + b. \tag{19}$$

This solves the differential equation; but to satisfy Eq. 18, a particular constant of integration must be used. We substitute $R(s)$ into Eq. 18 and solve for b. This can be done at any $s \in [0, T]$ with equivalent results, but is least complicated at $s = T$ since the integral disappears: $(\rho + \alpha\tau e^{-\lambda F(T)}) R(T) + R'(T) = \alpha\tau e^{-\lambda} R(T)$. After substituting $R(T)$, $R'(T)$, and $F(T)$, solving for b yields:

$$b = a \frac{\alpha\tau\lambda k^*}{\rho(\rho+\lambda k^*)} e^{-T(\rho+\lambda k^*)}. \tag{20}$$

The other constant of integration is determined by boundary condition Eq. 3. If we translate this in terms of $R(s)$ as we did for the interior of the Bellman Equation, we get $R(0) = z$. We then substitute for $R(0)$ using Eq. 19 evaluated at 0, and substitute for b using Eq. 20, then solve for a:

$$a = \rho z \cdot \frac{\rho+\lambda k^*}{\lambda k^* \left(\rho + \lambda k^* - \alpha\tau \left(1 - e^{-T(\rho+\lambda k^*)}\right)\right)}.$$

If the solutions for a and b are both substituted into Eq. 19, one obtains Eq. 11 with minor simplification. □

Proof of Proposition 2. The first derivative of $R(s)^*$ is:

$$R'(s) = -\frac{\rho z(\lambda k^* + \rho)\left(\lambda k^* - \alpha\tau + \alpha\tau e^{-\lambda s k^*}\right)}{\lambda k^* \left(\rho + \lambda k^* - \alpha\tau + \alpha\tau e^{-T(\rho+\lambda k^*)}\right)} e^{-s\rho}.$$

The equilibrium condition that $\phi(k^*) = 0$ rearranges to give $\lambda k^* = \alpha\tau \frac{1-e^{-\lambda T k^*}}{1-e^{\lambda-\lambda T k^*}}$. But since $e^\lambda > 1$, we know that $1 - e^{-\lambda T k^*} > 1 - e^{\lambda} e^{-\lambda T k^*}$. Therefore $\lambda k^* > \alpha\tau$. Hence, in the numerator, $\lambda k^* - \alpha\tau + \alpha\tau e^{-\lambda s k^*} > 0$ and in the denominator $\rho + \lambda k^* - \alpha\tau + \alpha\tau e^{-T(\rho+\lambda k^*)} > 0$. All other terms are clearly positive, so the negative sign in front ensures that $R'(s) < 0$.

The second derivative is:

$$R''(s) = \frac{\rho z(\lambda k^* + \rho)\left(\rho\left(\lambda k^* - \alpha\tau\right) + \alpha\tau\left(\rho + \lambda k^*\right) e^{-\lambda s k^*}\right)}{\lambda k^* \left(\rho + \lambda k^* - \alpha\tau + \alpha\tau e^{-T(\rho+\lambda k^*)}\right)} e^{-s\rho}.$$

Note the denominator is identical to that in $R'(s)$ and is thus positive, and the same reasoning as above ensures that the last term in the numerator is positive. Hence $R''(s) > 0$.

□

Proof of Proposition 3. To begin, we solve for k in Equation 12, obtaining $k = \frac{\tau}{\lambda}\left(\delta - \alpha e^{-\lambda}\right)$. Having assumed that $\delta > 2\alpha$, we ensure that $k > 0$ as was required in the proof of Proposition 1. We then substitute k into $\phi(k^*) = 0$ to obtain a single equilibrium condition $\psi(\lambda^*) = 0$, where:

$$\psi(\lambda) = \alpha\left(1 + e^{-\lambda}\right) - \delta - e^{\tau T(\alpha e^{-\lambda} - \delta)}(2\alpha - \delta e^{\lambda}).$$

Note that as $\lambda \to +\infty$, $\psi(\lambda) \to \infty$. Also, $\psi(0) = (2\alpha - \delta)\left(1 - e^{\tau T(\alpha - \delta)}\right)$. Since $\delta > 2\alpha$, $1 > e^{\tau T(\alpha-\delta)}$ and thus $\psi(0) < 0$. Since ψ is a continuous function, there exists a $\lambda^* \in (0, +\infty)$ such that $\psi(\lambda^*) = 0$.

We next turn to uniqueness. Define $\bar{\lambda}$ such that $\psi'(\bar{\lambda}) = 0$. That is, for any such $\bar{\lambda}$,

$$\alpha e^{\tau T\left(\delta - \alpha e^{-\bar{\lambda}}\right)} = 2\alpha^2 \tau T + \delta e^{\bar{\lambda}}\left(e^{\bar{\lambda}} - \alpha \tau T\right).$$

Next consider the second derivative of ψ:

$$\psi''(\lambda) = e^{\tau T\left(\alpha e^{-\lambda} - \delta\right) - \lambda}\left(\alpha e^{\tau T\left(\delta - \alpha e^{-\bar{\lambda}}\right)} + \delta e^{2\lambda} + \left(\delta - 2\alpha e^{-\lambda}\right)\alpha^2 \tau^2 T^2 - 2\alpha^2 \tau T - \alpha \delta e^{\lambda} \tau T\right).$$

If this is evaluated at $\bar{\lambda}$, then we can substitute for $\alpha e^{\tau T\left(\delta - \alpha e^{-\bar{\lambda}}\right)}$ using $\psi'(\bar{\lambda}) = 0$. This results in:

$$\psi''(\bar{\lambda}) = e^{\tau T\left(\alpha e^{-\bar{\lambda}} - \delta\right) - 2\bar{\lambda}}\left(\alpha^2 \tau^2 T^2\left(\delta e^{\bar{\lambda}} - 2\alpha\right) + \delta e^{\bar{\lambda}}\left(e^{\bar{\lambda}} + 1\right)\left(e^{\bar{\lambda}} - \alpha \tau T\right)\right).$$

By assumption, $2\alpha < \delta < \delta e^{\bar{\lambda}}$. Also, our sufficient condition yields $\alpha \tau T \leq 1 < e^{\bar{\lambda}}$. These ensure that the first and last parenthetical terms are each positive; all others are also clearly positive. Therefore $\psi''(\lambda) > 0$ for any λ where $\psi'(\lambda) = 0$. That is, any such point is a minimum, and hence there is at most one such point. Indeed, $\psi'(\lambda) > 0$ iff $\lambda > \bar{\lambda}$.

Recall that $\psi(0) < 0$. If $\psi'(0) < 0$, then $\psi'(\lambda) < 0$ for all $\lambda \in [0, \bar{\lambda})$, and hence $\psi(\lambda) < 0$ for all $\lambda \in [0, \bar{\lambda}]$. That is, λ^* cannot occur in that range. Rather, since it exists, λ^* must occur where $\phi'(\lambda^*) > 0$. But since ψ is continuous, there can only be one such occurrence.

If $\psi'(0) > 0$, then $\psi'(\lambda) > 0$ for all $\lambda > 0$, including λ^*. But again, this ensures a unique λ^*. $\square$

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