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# Investigations into Non-Degenerate Quasihomogeneous Polynomials as Related to FJRW Theory

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# ABSTRACT

Investigations into Non-Degenerate Quasihomogeneous
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The motivation for this paper is a better understanding of the basic building blocks of FJRW theory. The basics of FJRW theory will be briefly outlined, but the majority of the paper will deal with certain multivariate polynomials which are the most fundamental building blocks in FJRW theory. We will first describe what is already known about these polynomials and then discuss several properties we proved as well as conjectures we disproved. We also introduce a new conjecture suggested by computer calculations performed as part of our investigation.

Keywords: FJRW theory, Landau-Ginzburg, algebraic geometry

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# Contents

# Chapter 1. Introduction

In this thesis we investigate some of the basic building blocks of FJRW theory. FJRW theory provides the A-model construction for Landau-Ginzburg mirror symmetry. Landau-Ginzburg mirror symmetry depends on two mathematical models, called the A-model and the B-model. These models are built from a polynomial $W$ and a group $G$, which should each have a "dual" or "transpose", $W^T$ and $G^T$ respectively. The Landau-Ginzburg mirror symmetry conjecture states that the A-model obtained from $W$ and $G$ (denoted by $\mathcal{A}_{W,G}$) is isomorphic in some way to the B-model obtained from $W^T$ and $G^T$ (denoted by $\mathcal{B}_{W^T,G^T}$).

Our present purpose is to provide better understanding about the polynomial $W$ needed to construct the FJRW A-model. Much of our work is exploring a very large list of weight systems (see definition 1) of these polynomials.

In Chapter 2 we will describe the requirements that $W$ must satisfy in order to be used in the FJRW construction, as well as provide pertinent facts about the polynomial that are already known. In Chapter 3 we will give some new results about the maximal group of diagonal symmetries (denoted $G^{max}$, the largest $G$ allowed in the FJRW construction) of a large class of polynomials, called invertible polynomials. Chapter 4 will detail our attempts to provide a new way of classifying which polynomials can be used to construct the A-model. Finally, in Chapter 5 we will discuss two important conjectures about invertible polynomials and present new evidence concerning them.

# Chapter 2. Non-Degenerate Quasihomogeneous Polynomials

The purpose of FJRW theory is to provide the construction of the A-model of Landau-Ginzburg mirror symmetry. The details of the full construction are found in [3] and [2], while a good overview can be found in [5]. The most basic building block of this model is a polynomial $W \in \mathbb{C}[x_1, \ldots, x_n]$. But only certain polynomials may be used; namely, they must satisfy the following two definitions.

**Definition 1:** Let $W \in \mathbb{C}[x_1, \ldots, x_n]$. We say that $W$ is **quasihomogeneous** if there exist positive rational numbers $(q_1, \ldots, q_n)$ such that for every $c \in \mathbb{C}$, $W(c^{q_1}x_1, \ldots, c^{q_n}x_n) = cW(x_1, \ldots, x_n)$. We call $(q_1, \ldots, q_n)$ the **weight system** or **weights** of $W$, and we say that each $x_i$ has weight $q_i$.

**Definition 2:** Let $W \in \mathbb{C}[x_1, \ldots, x_n]$. We say that $W$ is **non-degenerate** if

(i) $W$ has a critical point at the origin (i.e. all partial derivatives are zero there),

(ii) the origin is the only critical point of $W$,

(iii) $W$ has no term of the form $x_i x_j$ $(i \neq j)$, and

(iv) $W$ is quasihomogeneous and the weights of $W$ are unique (that is, there is only one way to choose $(q_1, \ldots, q_n)$ to satisfy definition 1).

We will demonstrate these conditions with a few examples.

**Example 1:** We will give an example of a non-degenerate quasihomogeneous polynomial and show that it satisfies the above definitions. We will follow this with several examples of polynomials that fail to be either quasihomogeneous or non-degenerate, along with an explanation of why it fails to meet one of the criteria listed above. Most of these non-examples are easily generalized, and we will point out how. This will give us a better idea of what non-degenerate quasihomogeneous polynomials look like, which will be useful in following the arguments through the rest of this thesis. We omit the proofs of the generalized non-examples as they follow the specific examples given below.

(i) Let $W = x^3 + y^4$. We will show that $W$ is a non-degenerate quasihomogeneous polynomial and we will find its weight system.

For $c \in \mathbb{C}$, suppose $W(c^{q_1}x, c^{q_2}y) = cW(x,y)$. This means that

$$\begin{aligned}
&(c^{q_1}x)^3 + (c^{q_2}y)^4 = cx^3 + cy^4 \\
\Rightarrow &c^{3q_1}x^3 + c^{4q_2}y^4 = cx^3 + cy^4 \\
\Rightarrow &c^{3q_1} = c \text{ and } c^{4q_2} = c \\
\Rightarrow &q_1 = \frac{1}{3} \text{ and } q_2 = \frac{1}{4}
\end{aligned}$$

so $W$ is quasihomogeneous with weight system $(\frac{1}{3}, \frac{1}{4})$. Also, this is clearly the only solution that will work for all $c$, so the weight system is unique.

Now we show that $W$ is non-degenerate. To do this we need to find the partial derivatives, set them equal to 0, and solve. If $(0,0)$ is the only solution, then $W$ is non-degenerate.

$$W_x = 3x^2 = 0 \qquad\qquad W_y = 4y^3 = 0$$

$$\Rightarrow x = 0 \qquad\qquad \Rightarrow y = 0$$

Since $(x,y) = (0,0)$ is the only solution, the origin is the only critical point of $W$. Therefore, $W$ is an example of a non-degenerate quasihomogeneous polynomial.

(ii) Let $W = x^3 + a$ where $a \in \mathbb{C}$, $a \neq 0$. Then for $c \in \mathbb{C}$,

$$W(c^{q_1}x) = c^{3q_1}x^3 + a \qquad\qquad cW = cx + ca$$

For any $c \neq 1$ these two equations cannot be equal, regardless of what $q_1$ is. So $W$ is not quasihomogeneous. In general, a polynomial with a nonzero constant term cannot be quasihomogeneous.

(iii) Let $W = x^3 + y$. It is easy to see that $W$ is quasihomogeneous with unique weight system $(q_1, q_2) = (\frac{1}{3}, 1)$. We will next look at the partial derivatives of $W$.

$$W_x = 3x^2 = 0 \qquad\qquad W_y = 1 \neq 0$$
$$\Rightarrow x = 0 \qquad\qquad \Rightarrow W \text{ is not non-degenerate}$$

In general, a quasihomogeneous polynomial with a nonzero linear term cannot be non-degenerate.

(iv) Let $W = x^3 + x^2y$. Again $W$ is quasihomogeneous, this time with weights $(q_1, q_2) = (\frac{1}{3}, \frac{1}{3})$. Again looking for critical points, we find

$$W_x = 3x^2 + 2xy = 0 \qquad\qquad W_y = x^2 = 0$$
$$\Rightarrow x(3x + 2y) = 0 \qquad\qquad \Rightarrow x = 0$$
$$\Rightarrow x = 0 \text{ or } x = -\frac{2}{3}y$$

Since $x = 0$ makes both partial derivatives 0, $y$ can roam freely. So $(0, 0)$ is a critical point of $W$, but there are also infinitely many other critical points all along the $y$ axis. So $W$ is not non-degenerate.

(v) Let $W = x^3 + y^2x + z^2x$. Then $W$ is quasihomogeneous with weight system $(q_1, q_2, q_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Examining the partial derivatives of $W$, we see that

$$W_x = 3x^2 + y^2 + z^2 = 0 \qquad W_y = 2yx = 0 \qquad W_z = 2zx = 0$$

Setting $x = 0$ makes both $W_y = 0$ and $W_z = 0$, leaving one equation in two variables. Thus there will be infinitely many solutions, so $W$ is not non-degenerate. In general, a quasihomogeneous polynomial in more than two variables where any one variable appears in every monomial cannot be non-degenerate.

(vi) Let $W = x^2y$. Then $W$ is quasihomogeneous with weight system $(q_1, q_2) = (\frac{1}{4}, \frac{1}{2})$.

But it could also have the weight system $(\frac{1}{3}, \frac{1}{3})$, or in general any weight system of the form $(q_1, 1 - 2q_1)$, $0 < q_1 < \frac{1}{2}$. So the weights are not unique. In general, a quasihomogeneous polynomial with fewer monomials than variables will not have unique weights, and therefore cannot be non-degenerate.

A straightforward argument like the ones given in the examples above shows that every non-degenerate quasihomogeneous polynomial has constant term 0, all linear terms 0, and at least as many nonzero terms as there are variables. Additionally, if there are more than two variables, then none of the variables appear in every term.

There is a large class of non-degenerate quasihomogeneous polynomials that are particularly useful and which we will be discussing in more detail throughout this paper. These are the invertible non-degenerate quasihomogeneous polynomials.

**Definition 3:** A non-degenerate quasihomogeneous polynomial is **invertible** if it has the same number of monomials as variables.

It is a simple exercise to show that any invertible non-degenerate quasihomogeneous polynomial can be rescaled so all coefficients are 1 via an invertible linear map. Such a rescaling does not affect the resulting theory at all, so it is common practice to always assume this has been done when discussing invertible polynomials. We will adopt this practice for the remainder of this paper.

Invertible polynomials are particularly well understood. In fact, we can give a complete list of the possible forms an invertible polynomial can have.

**Definition 4:** There are three **atomic types** of polynomials. These are:

- **Fermat** type: $x^{\alpha}$
- **loop** type: $x_1^{\alpha_1}x_2 + x_2^{\alpha_2}x_3 + \cdots + x_n^{\alpha_n}x_1$
- **chain** type: $x_1^{\alpha_1}x_2 + x_2^{\alpha_2}x_3 + \cdots + x_n^{\alpha_n}$

where each exponent $\alpha_i$ is an integer greater than one. We will refer to a polynomial that has one of these three types as an **atomic polynomial**.

The following proposition is based on the work of Kreuzer and Skarke [7], though it is not explicitly stated there. A proof is included in [6], although it was certainly known before that.

**Proposition 1:** *Let $W \in \mathbb{C}[x_1, \ldots, x_n]$. Then $W$ is an invertible non-degenerate quasihomogeneous polynomial if and only if it can be written as a sum of atomic polynomials having no variables in common (i.e. each $x_i$ appears in only one atomic polynomial).*

This is one of the features that make invertible polynomials easy to work with. The convenience of these atomic types will be demonstrated throughout this paper.

Besides being easier to work with than non-invertible polynomials, invertible polynomials are of particular interest for another reason. As mentioned in the introduction, our motivation for studying these polynomials is their place in Landau-Ginzburg mirror symmetry. While the FJRW construction (the A-model) makes sense for non-invertible polynomials, we do not currently know what the corresponding transpose polynomial should be for the B-model. However, we do know what the transpose should be for invertible polynomials. For this reason it is particularly important that we have a solid understanding of invertible polynomials.

When discussing non-degenerate quasihomogeneous polynomials, it is often helpful to consider the exponent matrix of the polynomial, which we define here.

**Definition 5:** Let $W = \sum_{j=1}^{m} a_j \prod_{i=1}^{n} x_i^{\alpha_{ij}} \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial. Then the

**exponent matrix** $A_W$ of $W$ is defined as

$$A_W = (\alpha_{ij}) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}.$$

That is, the rows represent the monomials in $W$ and the columns represent the variables.

This technical definition looks much complicated than it really is. We will demonstrate how simple the exponent matrix is with an example.

**Example 2:** Let $W = x^3y + y^4$. Then the exponent matrix of $W$ is

$$A_W = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}.$$

Notice that the exponents of the first monomial $x^3y$ appear in the first row (3 1). The first column corresponds to $x$, and the second column to $y$. Similarly, the second row gives us the second polynomial, $y^4$.

One thing to notice is that the exponent matrix includes no information about the coefficients in $W$. For invertible polynomials, this is irrelevant because, as mentioned, we always assume the coefficients have been rescaled to 1. The exponent matrix is most useful for analyzing invertible polynomials, and indeed we will only be using it for invertible polynomials in this paper, so there is no concern about losing information.

Finally, we cite one useful fact about the exponent matrix here.

**Lemma 1** ([4], Lemma 2)**:** *If $W$ is an invertible non-degenerate quasihomogeneous polynomial, then its exponent matrix $A_W$ is an invertible matrix.*

# Chapter 3. The Maximal Group of Diagonal Symmetries of an Invertible Polynomial

**Definition 6:** Let $W \in \mathbb{C}[x_1, \dots, x_n]$ be a non-degenerate quasihomogeneous polynomial. Then the **maximal group of diagonal symmetries of** $W$ is defined as

$$G_W^{max} = \{(c_1, \dots, c_n) \in (\mathbb{C}^*)^n \mid W(c_1x_1, \dots, c_nx_n) = W(x_1, \dots, x_n)\}$$

A simple proposition will show that this is in fact a group.

**Proposition 2:** *Let* $W = \sum_{j=1}^{m} a_j \prod_{i=1}^{n} x_i^{\alpha_{ij}} \in \mathbb{C}[x_1, \dots, x_n]$ *be a non-degenerate quasihomogeneous polynomial. Then* $G_W^{max}$ *forms a subgroup of* $(\mathbb{C}^*)^n$*.*

*Proof.* First, note that the identity $(1, \dots, 1) \in G_W^{max}$, so $G_W^{max}$ is not empty. We will show that it is also closed under multiplication and inverses.

Let $(c_1, \dots, c_n), (d_1, \dots, d_n) \in G_W^{max}$. Then their product is $(c_1d_1, \dots, c_nd_n)$. To see that this is in $G_W^{max}$, first notice that

$$\begin{aligned}
&W(c_1x_1, \dots, c_nx_n) = W(x_1, \dots, x_n) \\
\Rightarrow &\sum_{j=1}^{m} a_j \prod_{i=1}^{n} (c_ix_i)^{\alpha_{ij}} = \sum_{j=1}^{m} a_j \prod_{i=1}^{n} x_i^{\alpha_{ij}} \\
\Rightarrow &\sum_{j=1}^{m} a_j \prod_{i=1}^{n} c_i^{\alpha_{ij}} \prod_{i=1}^{n} x_i^{\alpha_{ij}} = \sum_{j=1}^{m} a_j \prod_{i=1}^{n} x_i^{\alpha_{ij}} \\
\Rightarrow &\prod_{i=1}^{n} c_i^{\alpha_{ij}} = 1 \text{ for each } j \text{ from } 1 \text{ to } m.
\end{aligned}$$

Similarly, $\prod_{i=1}^{n} d_i^{\alpha_{ij}} = 1$ for each $j$ from 1 to $m$. From this, we see that

$$\begin{aligned}
W(c_1d_1x_1, \ldots, c_nd_nx_n) &= \sum_{j=1}^{m} a_j \prod_{i=1}^{n} (c_id_ix_i)^{\alpha_{ij}} \\
&= \sum_{j=1}^{m} a_j \prod_{i=1}^{n} c_i^{\alpha_{ij}} \prod_{i=1}^{n} d_i^{\alpha_{ij}} \prod_{i=1}^{n} x_i^{\alpha_{ij}} \\
&= \sum_{j=1}^{m} a_j (1)(1) \prod_{i=1}^{n} x_i^{\alpha_{ij}} \\
&= \sum_{j=1}^{m} a_j \prod_{i=1}^{n} x_i^{\alpha_{ij}} \\
&= W(x_1, \ldots, x_n).
\end{aligned}$$

So by definition, $(c_1d_1, \ldots, c_nd_n) \in G_W^{max}$, and $G_W^{max}$ is closed under multiplication.

We now consider $(c_1, \ldots, c_n)^{-1} = (c_1^{-1}, \ldots, c_n^{-1})$.

$$\begin{aligned}
W(c_1^{-1}x_1, \ldots, c_n^{-1}x_n) &= \sum_{j=1}^{m} a_j \prod_{i=1}^{n} (c_i^{-1}x_i)^{\alpha_{ij}} \\
&= \sum_{j=1}^{m} a_j \prod_{i=1}^{n} (c_i^{\alpha_{ij}})^{-1} \prod_{i=1}^{n} x_i^{\alpha_{ij}} \\
&= \sum_{j=1}^{m} a_j (1) \prod_{i=1}^{n} x_i^{\alpha_{ij}} \\
&= \sum_{j=1}^{m} a_j \prod_{i=1}^{n} x_i^{\alpha_{ij}} \\
&= W(x_1, \ldots, x_n).
\end{aligned}$$

So by definition, $(c_1^{-1}, \ldots, c_n^{-1}) \in G_W^{max}$, and $G_W^{max}$ is closed under inverses. Therefore, $G_W^{max}$ is a subgroup of $(\mathbb{C}^*)^n$. $\square$

It is known (see [3], Lemma 2.1.8) that this group is finite. This allows us to prove the following corollary.

**Corollary 1:** *If $c = (c_1, \ldots, c_n) \in G_W^{max}$, then each $c_i$ is a root of unity.*

*Proof.* Since $G_W^{max}$ is finite, $c$ must have finite order. That is, for some $d \in \mathbb{N}$, $c^d = (c_1^d, \ldots, c_n^d) = (1, \ldots, 1)$, so each $c_i$ is a root of unity. $\square$

We will use corollary 1 to define some new notation. If we write $c_i = e^{2\pi i g_i}$, $g_i \in \mathbb{Q}/\mathbb{Z}$, then we can write $G_W^{max}$ as an additive subgroup of $(\mathbb{Q}/\mathbb{Z})^n$. So instead of writing $(e^{2\pi i g_1}, \ldots, e^{2\pi i g_n})$ for an element of $G_W^{max}$, we can simply write $(g_1, \ldots, g_n)$. For simplicity of our discussion, this latter notation is what we will use for the remainder of this paper.

Using this new notation, we have a convenient way of finding a set of generators for $G_W^{max}$.

**Proposition 3** ([8], Theorem 3.1.9)**:** *Let $W$ be an invertible non-degenerate quasihomogeneous polynomial. Then the group $G_W^{max}$ is generated by the columns of the inverse of the exponent matrix $A_W$.*

**Example 3:** Let $W = x^2y + y^3$. Notice that this polynomial is a chain, one of our three atomic types (see definition 4). So by proposition 1, $W$ is an invertible non-degenerate quasihomogeneous polynomial. Its exponent matrix is

$$A_W = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

which has inverse

$$A_W^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{6} \\ 0 & \frac{1}{3} \end{pmatrix}.$$

So, treating $G_W^{max}$ as a subgroup of $(\mathbb{Q}/\mathbb{Z})^2$, $G_W^{max} = \left\langle (\frac{1}{2}, 0), (-\frac{1}{6}, \frac{1}{3}) \right\rangle$. In our original (multiplicative) notation, this corresponds to $\left\langle (e^{2\pi i (\frac{1}{2})}, 1), (e^{2\pi i(-\frac{1}{6})}, e^{2\pi i(\frac{1}{3})}) \right\rangle$.

Let $g_1$ and $g_2$ be the two additive generators, respectively. Notice that $3g_2 = (-\frac{1}{2}, 1) = (\frac{1}{2}, 0) = g_1$ (recall that each coordinate can be reduced mod $\mathbb{Z}$), so in fact $G_W^{max}$ is generated by just $g_2$.

In the course of our investigations into the maximal symmetry group of a polynomial, we discovered a useful fact about $G_W^{max}$ when $W$ is invertible, which we prove here. We will first require a few lemmas about the generators of $G_W^{max}$.

**Lemma 2:** *Let $W = W_1 + \cdots + W_n$ be an invertible non-degenerate quasihomogeneous polynomial, where each $W_i$ corresponds to a single atomic polynomial in $W$. Then $G_W^{max} = G_{W_1}^{max} \times \cdots \times G_{W_n}^{max}$.*

*Proof.* Let us consider the exponent matrix $A_W$. By construction, $A_W$ is a block diagonal matrix with the exponent matrices of its atomic polymonials on the diagonal; that is,

$$A_W = \begin{pmatrix} A_{W_1} & 0 & \cdots & 0 \\ 0 & A_{W_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{W_n} \end{pmatrix}.$$

Since each $A_{W_i}$ is invertible, we see that

$$A_W^{-1} = \begin{pmatrix} A_{W_1}^{-1} & 0 & \cdots & 0 \\ 0 & A_{W_2}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{W_n}^{-1} \end{pmatrix}.$$

By proposition 3, $G_W^{max}$ is generated by the columns of $A_W^{-1}$. From here it is easy to see that $G_W^{max} = G_{W_1}^{max} \times \cdots \times G_{W_n}^{max}$. □

**Lemma 3** ([10])**:** *Let $W$ be an invertible non-degenerate quasihomogeneous polynomial with exponent matrix $A$. Then*

$$|G_W^{max}| = |\det(A)|.$$

**Lemma 4:** *Let $W$ be an atomic polynomial (i.e. $W$ is either a Fermat, chain, or loop). Then $G_W^{max}$ is cyclic.*

*Proof.* Let $A_W$ be the exponent matrix of $W$. We will prove the three cases separately.

The case of a Fermat is trivial since a Fermat consists of a single monomial, so $A_W$ has only one entry. Thus, $A_W^{-1}$ also has only one entry. By proposition 3, this one entry generates $G_W^{max}$, making $G_W^{max}$ cyclic.

Now consider the case where $W$ is a chain, say $W = x_1^{\alpha_1}x_2 + x_2^{\alpha_2}x_3 + \cdots + x_{n-1}^{\alpha_{n-1}}x_n + x_n^{\alpha_n}$. Then

$$A_W = \begin{pmatrix} \alpha_1 & 1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & 1 & \cdots & 0 \\ 0 & 0 & \alpha_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_n \end{pmatrix}.$$

This is an upper triangular matrix, so $|\det(A_W)| = \alpha_1\alpha_2\cdots\alpha_n$ (the product of the diagonal elements). By lemma 3, this is also the order of $G_W^{max}$. We will show that there is a column in $A_W^{-1}$ of order $\alpha_1\alpha_2\cdots\alpha_n$, proving that this column alone generates $G_W^{max}$.

The details of the calculation of the inverse matrix are rather tedious, but the result is simple to verify, so we give the inverse here:

$$A_W^{-1} = \begin{pmatrix} \frac{1}{\alpha_1} & \frac{-1}{\alpha_1\alpha_2} & \frac{1}{\alpha_1\alpha_2\alpha_3} & \cdots & \frac{(-1)^{n-1}}{\prod_{i=1}^n \alpha_i} \\ 0 & \frac{1}{\alpha_2} & \frac{-1}{\alpha_2\alpha_3} & \cdots & \frac{(-1)^{n-2}}{\prod_{i=2}^n \alpha_i} \\ 0 & 0 & \frac{1}{\alpha_3} & \cdots & \frac{(-1)^{n-3}}{\prod_{i=3}^n \alpha_i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\alpha_n} \end{pmatrix}.$$

Now let $\rho_n$ be the last column of $A_W^{-1}$. Recall that $G_W^{max} < (\mathbb{Q}/\mathbb{Z})^n$. Each $\alpha_i \in \mathbb{N}$, so $|\rho_n| = \prod_{i=1}^n \alpha_i$ since that is the largest denominator of any entry in $\rho_n$ and is a multiple of all other denominators (each numerator is $\pm 1$, so we only need to consider the denominators). This is the order of $G_W^{max}$, so $G_W^{max} = \langle \rho_n \rangle$, and $G_W^{max}$ is cyclic.

We now consider the final case where $W$ is a loop; that is, $W = x_1^{\alpha_1}x_2 + x_2^{\alpha_2}x_3 + \cdots +$

$x_{n-1}^{\alpha_{n-1}}x_n + x_n^{\alpha_n}x_1$. Then

$$A_W = \begin{pmatrix} \alpha_1 & 1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & 1 & \cdots & 0 \\ 0 & 0 & \alpha_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & \alpha_n \end{pmatrix}.$$

By expanding along the first column, we quickly find the determinant:

$$\det(A_W) = \alpha_1 \begin{vmatrix} \alpha_2 & 1 & 0 & \cdots & 0 \\ 0 & \alpha_3 & 1 & \cdots & 0 \\ 0 & 0 & \alpha_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_n \end{vmatrix} + (-1)^{n-1} \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_2 & 1 & 0 & \cdots & 0 \\ 0 & \alpha_3 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= \prod_{i=1}^{n} \alpha_i + (-1)^{n-1}$$

As in the case of chains, this tells us the order of $G_W^{max}$. We proceed as in that case, finding a column of $A_W^{-1}$ (i.e. an element of $G_W^{max}$) with this order.

Let $\rho_n = (a_{1n}, a_{2n}, \ldots, a_{nn})^T$ be the last column of $A_W^{-1}$. So

$$A_W\rho_n = \begin{pmatrix} \alpha_1 & 1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & 1 & \cdots & 0 \\ 0 & 0 & \alpha_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & \alpha_n \end{pmatrix} \begin{pmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ \vdots \\ a_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

This system of equations is also easily solved, giving the closed form solution

$$a_{mn} = \frac{(-1)^{n-m}\prod_{i=1}^{m-1}\alpha_i}{\prod_{i=1}^{n}\alpha_i + (-1)^{n-1}}.$$

In each entry of $\rho_n$, the numerator and denominator are relatively prime and the denominator is always the same. Thus the denominator again tells us the order of $\rho_n \in G_W^{max}$; that is, $|\rho_n| = \prod_{i=1}^n \alpha_i + (-1)^{n-1} = |G_W^{max}|$. Thus, $G_W^{max} = \langle \rho_n \rangle$, and $G_W^{max}$ is again cyclic. $\square$

**Corollary 2:** *Let $W_1$ and $W_2$ be two atomic polynomials (i.e. either a Fermat, chain, or loop), each having a different atomic type. Then $G_{W_1}^{max} \neq G_{W_2}^{max}$.*

**Remark:** Note that when we say $G_{W_1}^{max} \neq G_{W_2}^{max}$, we do not mean that they are not isomorphic. Indeed, since lemma 4 showed that every $G^{max}$ for atomic polynomials is cyclic, it is very easy to find examples where $G_{W_1}^{max} \cong G_{W_2}^{max}$. But recall that $G^{max}$ is a subgroup of $(\mathbb{Q}/\mathbb{Z})^n$, so when we say $G_{W_1}^{max} \neq G_{W_2}^{max}$, we mean they are not equal as subgroups of $(\mathbb{Q}/\mathbb{Z})^n$.

*Proof.* Again the case where either polynomial is a Fermat is trivial since Fermats have only one coordinate and chains and loops must have more than one. So we only need to consider the case where (without loss of generality) $W_1 = x_1^{\alpha_1}x_2 + x_2^{\alpha_2}x_3 + \cdots + x_{n-1}^{\alpha_{n-1}}x_n + x_n^{\alpha_n}$ is a chain and $W_2 = x_1^{\beta_1}x_2 + x_2^{\beta_2}x_3 + \cdots + x_{n-1}^{\beta_{n-1}}x_n + x_n^{\beta_n}x_1$ is a loop.

Consider the generators found in the proof of lemma 4 for $G_{W_1}^{max}$ and $G_{W_2}^{max}$. Let $\rho_1$ and $\rho_2$ be these generators, respectively. By examining the coordinates of $\rho_1 = (\frac{(-1)^{n-1}}{\prod_{i=1}^n \alpha_i}, \frac{(-1)^{n-2}}{\prod_{i=2}^n \alpha_i}, \ldots, \frac{1}{\alpha_n})$, we see that every coordinate except the first one will be zero for some nonidentity element of $G_{W_1}^{max}$ (recall that these coordinates are in $\mathbb{Q}/\mathbb{Z}$).

On the other hand, every coordinate of $\rho_2$ has the same denominator and is fully reduced. So each has the same order as elements of $\mathbb{Q}/\mathbb{Z}$, which is also the order of $G_{W_2}^{max}$. Thus, the only element of $G_{W_2}^{max}$ with any zero entries is the identity. This is sufficient to prove that $G_{W_1}^{max} \neq G_{W_2}^{max}$. $\square$

**Theorem 1:** *No two distinct invertible non-degenerate quasihomogeneous polynomials have the same $G^{max}$.*

*Proof.* Assume $G_1^{max} = G_2^{max}$, the maximal groups of diagonal symmetries for invertible polynomials $W_1$ and $W_2$ respectively. Let $G_1^{max} = \langle g_1, \ldots, g_m \rangle$ and $G_2^{max} = \langle h_1, \ldots, h_n \rangle$, where each of these generators corresponds to a single atomic polynomial in $W_1$ or $W_2$. Note that this means that each coordinate is nonzero in precisely one $g$ and one $h$.

For each $k$ from 1 to $m$, let $g_k = \alpha_1 h_{k_1} + \cdots + \alpha_l h_{k_l}$, where each $\alpha_j h_{k_j}$ is nonzero. This implies that $\langle g_k \rangle = \langle \alpha_1 h_{k_1} + \cdots + \alpha_l h_{k_l} \rangle \subseteq \langle h_{k_1}, \ldots, h_{k_l} \rangle$.

Suppose $\langle g_k \rangle \neq \langle h_{k_1}, \ldots, h_{k_l} \rangle$. Let $c_1, \ldots, c_{n_k}$ be the nonzero coordinates of $g_k$. If each $h_{k_j}$ is nonzero only on the coordinates $c_1, \ldots, c_{n_k}$, then there must be a linear combination $a = \sum_{j=1}^{l} \beta_j h_{k_j}$ that isn't a multiple of $g_k$ (i.e. there is an element of $\langle h_{k_1}, \ldots, h_{k_l} \rangle$ that isn't in $\langle g_k \rangle$). Since $a \in G_2^{max}$, it is also in $G_1^{max}$, and so we can write it as $a = \sum_{i=1}^{m} \gamma_i g_i$. We know that for $i \neq k$, every nonzero multiple of $g_i$ has a nonzero coordinate outside of $c_1, \ldots, c_{n_k}$. Since the only nonzero coordinates of $a$ are $c_1, \ldots, c_{n_k}$ (because $a \in \langle h_{k_1}, \ldots, h_{k_l} \rangle$), $\gamma_i g_i = 0$ for all $i \neq k$. So $a = \gamma_k g_k$, a contradiction since we chose $a$ to not be a multiple of $g_k$. So there exists an $h_{k_j}$ that is nonzero on a coordinate where $g_k$ is zero.

Let $s$ be a coordinate where $g_k$ is zero and $h_{k_j}$ is nonzero. As noted above, $h_{k_j}$ is the only generator of $G_2^{max}$ that is nonzero on that coordinate. But we know that $\alpha_j h_{k_j}$ is zero on $s$ and nonzero on at least one of $c_1, \ldots, c_{n_k}$. So $h_{k_j}$ is a generator for a single atomic type with a nonidentity multiple that has a zero in one of its coordinates. As mentioned in the proof of corollary 2, this generator cannot correspond to a Fermat or a loop, and must therefore be a chain.

Let $g_i$ be the unique generator of $G_1^{max}$ that is nonzero on $s$. Then we can write $g_i = \beta h_{k_j} + (\text{other terms})$, where $\beta h_{k_j}$ is nonzero on $s$. Since $\alpha_j h_{k_j}$ is zero on $s$ and nonzero on at least one of $c_1, \ldots, c_{n_k}$, $\beta h_{k_j}$ must also be nonzero on those same coordinates in $c_1, \ldots, c_{n_k}$. This claim is based on the structure of the generator for a chain: if any coordinate of a multiple is nonzero, then all preceding coordinates are nonzero, and if any coordinate of a multiple is zero, then all subsequent coordinates are zero. But this implies that $g_i$ is nonzero on at least one of $c_1, \ldots, c_{n_k}$, which were defined as precisely the coordinates where $g_k$ is nonzero. This means that two different generators must be nonzero on the same coordinate. But each generator corresponds to a distinct atomic polynomial, so this means that two different atomic polynomials have a variable in common, a contradiction.

So $\langle g_k \rangle = \langle h_{k_1}, \ldots, h_{k_l} \rangle$. A similar argument in the other direction shows that $\langle h_{k_1} \rangle = \langle g_k,\ \text{other terms}\ \rangle$. ($g_k$ must be a generator for this group since it is nonzero on a coordinate that $h_{k_1}$ is also nonzero on, and is the only $g_i$ that is nonzero on this coordinate.) This gives a chain of containments

$$\langle h_{k_1} \rangle \subseteq \langle h_{k_1}, \ldots, h_{k_l} \rangle = \langle g_k \rangle \subseteq \langle g_k,\ \text{other terms}\ \rangle = \langle h_{k_1} \rangle,$$

so $\langle h_{k_1} \rangle = \langle g_k \rangle$. By corollary 2, this implies that $\langle h_{k_1} \rangle$ and $\langle g_k \rangle$ correspond to the same atomic type. Since the exponents of an atomic polynomial completely and uniquely determine its $G^{max}$ (this can be seen by examining the formulas given in the proof of lemma 4), the corresponding atomic polynomials must be equal. Since $k$ was chosen arbitrarily, $W_1$ and $W_2$ must be composed of identical atomic polynomials, and are therefore equal. $\square$

This result was somewhat unexpected, but provides valuable insights into the relationship between the two ingredients for FJRW theory: the polynomial $W$ and the group $G$ (which must be a subgroup of $G_W^{max}$). This fact has already been helpful in understanding how different objects in mirror symmetry are related (see [8]).

# Chapter 4. Classification of Non-Degenerate Quasihomogeneous Polynomials

Our research also included looking for new ways to classify non-degenerate quasihomogeneous polynomials. To some degree, these polynomials have already been classified (Arnold worked extensively on this; see, for example, [1]). However, the existing classifications are not complete from the perspective of FJRW theory in that they are not rigid enough. That is, polynomials may be classified as equivalent that do not in fact yield the same results in FJRW theory.

Specifically, the classification of Arnold defines two polynomials to be equivalent if they differ by a smooth change of variables. But for FJRW theory, they are equivalent only if they differ by a permutation of variables, a much stricter condition. For this reason we continue to look for a compact, complete way to classify all non-degenerate quasihomogeneous polynomials, while providing more insight into the complete FJRW theory.

## 4.1 Quasihomogeneous dimension (QHDim)

One possible classification we've recently begun looking into is based on the quasihomogeneous dimension (or QHDim) of the polynomial.

**Definition 7:** Let $W \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-degenerate quasihomogeneous polynomial. Then **QHDim**$(W)$ is the number of monomials in $\mathbb{C}[x_1, \ldots, x_n]$ of weighted degree 1 with respect to the weight system of $W$. Since this is an invariant of the weight system itself, we will often refer to QHDim of a weight system.

Recall that a non-degenerate quasihomogeneous polynomial is composed of monomials of weighted degree 1. So QHDim of a weight system tells us how many monomials we have available to us to construct a polynomial with that weight system. This can be very helpful in determining which weight systems may give rise to non-degenerate quasihomogeneous polynomials because many potential weight systems will not have enough monomials of weighted degree 1 (recall that there must be at least as many monomials as there are variables).

Additionally, we are hoping that QHDim provides a good way of ordering non-degenerate quasihomogeneous polynomials so that we can compose a complete list of them, starting with those that have the smallest QHDim.

We began our investigation of QHDim by looking at ways to calculate QHDim for a given weight system. The one variable case is trivial as QHDim is either 0 or 1, depending on whether or not the numerator of the weight system divides the denominator (if we deal only with reduced weight systems, this is equivalent to the numerator being 1). The only one-variable non-degenerate quasihomogeneous polynomials are Fermats (having the form $x^n$), which correspond to the weight systems with QHDim $= 1$.

We therefore moved on to the two variable case and were able to come up with a formula for finding the quasihomogeneous dimension of any weight system. In order to establish this formula, we first require three lemmas.

**Lemma 5:** *Any two-variable, non-degenerate quasihomogeneous polynomial is the sum of an invertible polynomial in two variables and other monomials.*

*Proof.* Let $W$ be such a polynomial. Let $W = m_1 + \cdots + m_n$ where each $m_i$ is a monomial in $\mathbb{C}[x, y]$, i.e. $m_i = x^{a_i} y^{b_i}$. Note that for each $i$, $a_i \geq 2$ or $b_i \geq 2$.

Suppose that for each $i$, $a_i \geq 2$. Then $x$ appears in each monomial of each partial derivative of $W$. Thus if $x = 0$, $y$ can roam freely and the partials will still be 0, making $W$ degenerate, a contradiction. So there exists $k \in \mathbb{Z}$ such that $a_k \leq 1$ (implying $b_k \geq 2$).

Similarly, there exists $l \in \mathbb{Z}$ such that $b_l \leq 1$, meaning $a_l \geq 2$.

The pair $(a_k, b_l)$ has four possible values:

- $(0, 0) \Rightarrow m_k + m_l$ is a sum of Fermats.
- $(0, 1)$ or $(1, 0) \Rightarrow m_k + m_l$ is a chain.
- $(1, 1) \Rightarrow m_k + m_l$ is a loop.

Thus $W$ is a sum of a two-variable invertible polynomial $(m_k + m_l)$ and other monomials.

□

**Lemma 6:** *Let* $q = \left(\frac{r}{n}, \frac{s}{n}\right)$ *be a weight system with* $\gcd(r, s, n) = 1$ *and* $r \geq s$*. If* $q$ *corresponds to a non-degenerate quasihomogeneous polynomial, then* $\alpha_0 = \left\lfloor \frac{n}{r} \right\rfloor$ *is the largest exponent of* $x$ *(which has weight* $\frac{r}{n}$*) that gives rise to a monomial of weighted degree* 1*.*

*Proof.* If $r \mid n$, then $\alpha_0 = \frac{n}{r} \in \mathbb{Z}$. Then $x^{\alpha_0}$ has weighted degree 1, and any higher power of $x$ would have weighted degree greater than 1, so the lemma holds in this case.

Now assume that $r \nmid n$ and let $W$ be a two variable non-degenerate quasihomogeneous polynomial with weight $q$. Since $r \nmid n$, no power of $x$ alone can have weighted degree 1, so $W$ has no terms of the form $x^\alpha$. By lemma 5, $W$ must contain an invertible polynomial in $\mathbb{C}[x, y]$. Thus it must contain a term of the form $x^\alpha y$ (either as part of a loop or a chain). By definition,

$$\alpha r + s = n$$
$$\alpha r = n - s$$
$$\alpha = \frac{n}{r} - \frac{s}{r} \in \mathbb{Z}$$

because $\alpha$ is an integer. But $s \le r$, meaning $\frac{s}{r} \le 1$. Since $\frac{n}{r} \notin \mathbb{Z}$, the inequality must be strict, and by definition $\alpha = \frac{n}{r} - \frac{s}{r} = \left\lfloor \frac{n}{r} \right\rfloor = \alpha_0$. Obviously any higher power of $x$ would result in a weighted degree greater than 1, so we have the desired result. $\square$

**Lemma 7:** *Let* $q = \left(\frac{r}{n}, \frac{s}{n}\right)$ *be a weight system with* $\gcd(r, s, n) = 1$ *and* $r, s \le \frac{n}{2}$ *(so the weights are no more than* $\frac{1}{2}$*). Then* $q$ *corresponds to a non-degenerate quasihomogeneous polynomial if and only if*

*(i)* $n \equiv 0$ *or* $n \equiv r \mod s$, *and*

*(ii)* $n \equiv 0$ *or* $n \equiv s \mod r$.

*Proof.* ($\Longrightarrow$) Suppose $W \in \mathbb{C}[x, y]$ is a non-degenerate quasihomogeneous polynomial with weight system $q$. Then, by lemma 5, $W$ has a monomial $m$ of the form $x^{\alpha_0}$ or of the form $x^{\alpha_0} y$. Let $\delta$ be the power of $y$ in $m$ (so $\delta$ is either 0 or 1). Then, since $m$ has weighted degree

1, we have

$$\alpha_0 r + \delta s = n$$

$$\alpha_0 r = n - \delta s$$

$$\Rightarrow r | n - \delta s$$

$$\Rightarrow n \equiv \delta s \mod r$$

$$\Rightarrow n \equiv 0 \mod r \text{ or } n \equiv s \mod r$$

By a symmetric argument, $n \equiv 0 \mod s$ or $n \equiv r \mod s$.

($\Longleftarrow$) For this direction, we break into three cases:

Case 1 (corresponding to a sum of Fermats): $n \equiv 0$ mod $r$ and $n \equiv 0$ mod $s$

Let $W = x^{\frac{n}{r}} + y^{\frac{n}{s}}$. Then $W$ is a non-degenerate quasihomogeneous singularity with weight system $q$.

Case 2 (corresponding to a chain): WLOG $n \equiv 0$ mod $r$ and $n \equiv r$ mod $s$

By definition, $s | n - r$. Let $\beta = \frac{n-r}{s}$. Since $s, r \leq \frac{n}{2}$, we see that $\beta \geq \frac{n-r}{n/2} = \frac{n}{n/2} - \frac{r}{n/2} = 2 - \frac{r}{n/2} \geq 1$. However, since $\gcd(r, s, n) = 1$, $r$ and $s$ can't both be equal to $\frac{n}{2}$ (well, they could if the weights could both be equal to $\frac{1}{2}$, but we don't allow this, right?). So at least one of these inequalities is strict, meaning $\beta > 1$. Since $\beta$ is an integer, $\beta \geq 2$. Then $W = x^{\frac{n}{r}} + y^{\beta} x$ is a non-degenerate quasihomogeneous singularity with weight system $q$.

Case 3 (corresponding to a loop): $n \equiv s$ mod $r$ and $n \equiv r$ mod $s$

Let $\alpha = \frac{n-r}{s}$ and $\beta = \frac{n-s}{r}$. Similar to case 2, we know that $\alpha, \beta \geq 2$. Then $W = x^{\alpha} y + y^{\beta} x$ is a non-degenerate quasihomogeneous singularity with weight system $q$. □

**Theorem 2** (Two-variable QHDim formula)**:** *If $q = \left(\frac{r}{n}, \frac{s}{n}\right)$ is a weight system having $\gcd(r, s, n) = 1$ which corresponds to a non-degenerate quasihomogeneous polynomial $W$, then QHDim$(W) = \left\lfloor \frac{n}{rs} \right\rfloor + 1$.*

*Proof.* Assume WLOG that $r \geq s$. By lemma 6 the largest possible power of $x$ that can appear in a monomial of $W$ is

$$\alpha_0 = \left\lfloor \frac{n}{r} \right\rfloor = \begin{cases} \frac{n}{r} & \text{if } n \equiv 0 \mod r \\ \frac{n-s}{r} & \text{otherwise} \end{cases}$$

(lemma 7 tells us that these are the only two possibilities for $\lfloor \frac{n}{r} \rfloor$). Now suppose $x^{\alpha_0}y^{\beta_0}$ has degree 1. Since $\alpha_0$ was chosen as the largest possible power of $x$, $\beta_0$ must be either 0 or 1. We know that $\alpha_0 r + \beta_0 s = n$. To find all nonnegative solutions to the linear Diophantine equation $\alpha r + \beta s = n$, we simply find the linear combinations of our one known solution that give nonnegative values. Thus, all solutions are of the form $(\alpha_0 - ks)r + (\beta_0 + kr)s = n$, $k \in \{0, 1, \ldots, \lfloor \frac{\alpha_0}{s} \rfloor\}$ and each choice of k gives a unique solution. Thus the number of solutions is

$$\left\lfloor \frac{\alpha_0}{s} \right\rfloor + 1 = \left\lfloor \frac{\lfloor \frac{n}{r} \rfloor}{s} \right\rfloor + 1 = \left\lfloor \frac{n}{rs} \right\rfloor + 1$$

where the last equality comes from basic properties of the floor function. Thus, QHDim($W$) $= \lfloor \frac{n}{rs} \rfloor + 1$. □

There is much work remaining to determine whether or not QHDim will provide a useful way of classifying non-degenerate quasihomogeneous polynomials. The first step would be to continue working with the simplest case, that being two-variable polynomials. What information does QHDim give us about the resulting FJRW theory for these polynomials? How are different polynomials with the same QHDim related, if at all? The above formula for QHDim will make it easier to carry on these investigations and answer these important questions.

# Chapter 5. Invertible Representatives of Weight Systems

As we have demonstrated throughout this thesis, invertible polynomials are much easier to work with than non-invertible polynomials. For this reason, it would be nice if we could always work with invertible polynomials without losing any information. A long-standing conjecture which has recently been proven by Julian Tay provided hope that this could be done.

**Theorem 3** (Group-Weights Theorem [9])**:** *Let $W_1$ and $W_2$ be non-degenerate quasihomogeneous polynomials with the same weights. Suppose $G \leq G_{W_1}^{max}$ and $G \leq G_{W_2}^{max}$. Then*

$$\mathcal{A}_{W_1,G} \cong \mathcal{A}_{W_2,G}$$

*That is, the Landau-Ginzburg A-models (FJRW models) produced by the polynomials are isomorphic, as long as they both use the same group $G$.*

This theorem tells us that the polynomial doesn't actually matter when determining the structure of the A-model, only its weights and the chosen group. So we made a conjecture which, if true, would allow us to always work with invertible polynomials without losing any information in the theory:

**Conjecture 1:** *Let $W_1$ be a non-degenerate quasihomogeneous polynomial.*

*(i) There exists an invertible non-degenerate quasihomogeneous polynomial $W_2$ which has the same weight system as $W_1$.*

*(ii) If $W_2$ exists, it can be chosen such that $G_{W_1}^{max} \leq G_{W_2}^{max}$.*

We discuss each of these parts independently in the next two sections.

## 5.1 Weight systems with no invertible representatives

We first investigate part (i) of conjecture 1. An equivalent statement is that every weight system that corresponds to a non-degenerate quasihomogeneous polynomial corresponds to an invertible non-degenerate quasihomogeneous polynomial. It is certainly plausible because invertible polynomials are the smallest possible non-degenerate quasihomogeneous polynomials (meaning they have the minimal number of monomials). So for any non-invertible polynomial, we would hope that by eliminating some monomials we could make it invertible. We have already shown that this is in fact possible in two variables (see lemma 5).

However, it has been known for some time that this simple approach is not always possible in more than two variables. That is, there are non-invertible non-degenerate quasihomogeneous polynomials that are not made up of an invertible polynomial plus extra monomials. The following example demonstrates this.

**Example 4:** Let $W_1 = x^2y^2 + x^2z + y^4z + z^3$. Notice that there is no three-variable invertible polynomial contained in $W_1$. However, this polynomial is in fact non-degenerate quasihomogeneous with weights $(q_x,\ q_y,\ q_z) = (\frac{1}{3},\ \frac{1}{6},\ \frac{1}{3})$.

Although this polynomial is not an invertible plus other monomials, there are several invertible polynomials with the same weight system. The simplest example is a sum of three Fermats, $W_2 = x^3 + y^6 + z^3$.

The question remains whether every weight system has an invertible representative. In order to investigate this question, we turned to a paper by Hertling and Kurbel [6] that describes how to tell if a weight system corresponds to a non-degenerate quasihomogeneous polynomial without having to actually find the polynomial. This allowed them to compile a comprehensive list (up to a certain bound) of all possible weight systems in 2, 3, and 4 variables. We were then able to use this list to investigate which weight systems, if any, did not correspond to any invertible polynomials. We will first give a brief overview of their approach, then describe the computer code we used to examine the list of weight systems and the results of that examination.

**5.1.1 Hertling and Kurbel's approach.** Hertling and Kurbel detail a combinatorial characterization of weight systems of non-degenerate quasihomogeneous polynomials. Using this characterization, they determined an upper bound on $d$ (the greatest common denominator of the weights in a weight system) based on an invariant called the Milnor number of the weight system (defined below).

**Definition 8:** Let $W \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-degenerate quasihomogeneous polynomial with weight system $(q_1, \ldots, q_n)$. Then the **Milnor number** $\mu_W$ of $W$ is given by the formula

$$\mu_W = \prod_{i=1}^{n} \left( \frac{1}{q_i} - 1 \right).$$

Using their combinatorial characterization and the upper bound they proved, Hertling and Kurbel were able to generate a list of all weight systems up to rather large Milnor numbers for small numbers of variables (2, 3, 4). We were able to use their lists to investigate our conjecture. We will detail these investigations shortly.

While the work of Hertling and Kurbel has been very helpful, there are a few critical pieces of information that it does not provide. First, they do not give any way to easily look at a weight system and determine if it corresponds to a non-degenerate quasihomogeneous polynomial. Essentially you have to consider all possible monomials and check if they can be combined in any way to form a non-degenerate quasihomogeneous polynomial.

It is also unknown which weight systems have invertible representatives, and their work did not include symmetry groups in its scope.

#### 5.1.2 Analyzing Hertling and Kurbel's data.

Our first goal with the list of weight systems we received from Hertling and Kurbel was to generate examples and determine if there were any weight systems which did not correspond to any invertible polynomials (but did correspond to at least one non-invertible). We had never found such an example, and recall that part (i) of our conjecture states that no such examples exist.

Our approach in searching for counterexamples was essentially a brute force search. However, due to the size of the data provided, we had to develop some ways to optimize our search in order to analyze all of the weight systems (of which there were 884,543 in the lists). We will describe the general approach, including these optimizations.

First, by lemma 5 we know that every two-variable non-degenerate quasihomogeneous polynomial is composed of an invertible polynomial plus other monomials. Thus, part (i) is true in two variables because we can simply restrict any non-invertible polynomial to the invertible part to get an invertible polynomial with the same weight system. So there was no need to analyze the two-variable weight systems.

To understand the next optimization, we need another definition.

**Definition 9:** Let $x_1$ and $x_2$ be variables (not necessarily distinct) in a non-degenerate quasihomogeneous polynomial $W$. We say that $x_1$ **points to** or **points at** $x_2$ if the monomial $x_1^a x_2$ appears in $W$. We call $x_1$ a **pointer (for** $x_2$**)** and say that $x_2$ **has a pointer (at** $x_1$**)**.

The following lemma is a subcase of (C1), one of the combinatorial characterizations of non-degenerate quasihomogeneous polynomials given by Hertling and Kurbel, though they were not the first to discover it ([6] contains a short survey of sources for these characterizations). We will provide a short proof that covers the case necessary for our work.

**Lemma 8:** *Let $W \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-degenerate quasihomogeneous polynomial. Then every variable in $W$ is a pointer for at least one variable (possibly itself).*

*Proof.* By way of contradiction, suppose (without loss of generality) that $x_1$ is not a pointer. Let $m$ be a monomial in $W$. Since $x_1$ is not a pointer, $m$ is of the form $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ with $\sum_{j=2}^{n} \alpha_j \geq 2$.

Consider the partial derivatives of $m$, $\frac{\partial m}{\partial x_i} = \alpha_i x_i^{\alpha_i - 1} \prod_{j \neq i} x_j^{\alpha_j}$. At least one of the variables besides $x_1$ must still have a positive exponent, meaning $\frac{\partial m}{\partial x_i}(x_1, 0, \ldots, 0) = 0$. Since $m$ and $i$ were arbitrary, this is true for every monomial in $W$, meaning $W_{x_i}(x_1, 0, \ldots, 0) = 0$ for all $i$. In other words, $W$ has a critical point at all points on the $x_1$ axis, which contradicts our assumption that $W$ is non-degenerate. So every variable in $W$ must be a pointer for at least one variable, as desired. □

In the lists of weights we analyzed, Hertling and Kurbel provide for each weight system a map $p_1 p_2 \ldots p_n \colon \{x_1, \ldots, x_n\} \to \{x_1, \ldots, x_n\}$ by $x_i \mapsto x_{p_i}$. This map encodes the "type" of polynomial with that weight system by telling which variable each variable points to. Of course, most weight systems have many different choices for this map, so the given map is just one possible choice.

This map can be used to identify many weight systems that have an invertible representative because certain maps require the existence of an invertible polynomial. The following example demonstrates how this works. We will then give a lemma that will show how this technique applies in general. This will allow us to easily identify which maps correspond to invertible polynomials without having to actually find the polynomials.

**Example 5:** Consider the weight system $(\frac{1}{7}, \frac{1}{7}, \frac{2}{7})$. The accompanying map is 121, meaning $x_1$ points to itself, $x_2$ points to itself, and $x_3$ points to $x_1$. This means that there is a non-degenerate quasihomogeneous polynomial with this weight system that contains the monomials $x_1^7$, $x_2^7$, and $x_3^3x_1$ (these are the exponents needed to give the monomials weighted degree 1). Since these monomials together comprise a two-variable chain and a Fermat, no other monomials are needed to make the polynomial non-degenerate.

**Lemma 9:** *Let $p = p_1p_2 \dots p_n$ be a map giving the pointers in a non-degenerate quasihomogeneous polynomial, as described above. Then $p$ corresponds to an invertible polynomial if and only if no variable is pointed at by more than one other variable (i.e. besides itself).*

*Proof.* ($\Longrightarrow$) Suppose $p$ corresponds to the invertible polynomial $W = W_1 + \cdots + W_m$ where each $W_i$ is an atomic polynomial (i.e. either a loop, chain, or Fermat). Each atomic polynomial is in distinct variables, so any variable can only be pointed at by another variable from the same atomic polynomial. So it is sufficient to prove the statement for atomics. But this is obvious from the definitions of the different atomic types.

($\Longleftarrow$) Now suppose that no variable is pointed at by multiple other variables. We will construct an invertible polynomial $W$ with the map $p$.

In order to have the map $p$, a polynomial $W \in \mathbb{C}[x_1, \dots, x_n]$ must have the $n$ monomials $\{x_i^{\alpha_i}x_{p_i}\}$. Let $W$ be the sum of these monomials. We will show that $W$ is in fact a sum of loops, chains, and Fermats, making it invertible. We will do this by successively "removing" the three different atomic types, and showing that after doing so there are no monomials remaining.

First, let $x_{j_1}$ be a variable not pointed at by any variable (i.e. $p_i \neq j_1$ for all $i$), if it exists. Define $x_{j_{k+1}}$ recursively as the variable pointed at by $x_{j_k}$. This defines a polynomial $W_c = x_{j_1}^{\alpha_{j_1}} x_{j_2} + x_{j_2}^{\alpha_{j_2}} x_{j_3} + \cdots + x_{j_m}^{\alpha_{j_m}} x_{j_m}$ (notice the last variable is pointing to itself) contained in $W$. We know that $W_c$ must end with $x_{j_m}$ pointing to itself because our hypothesis is that no variable is pointed at by more than one other variable and $x_{j_1}$ isn't pointed at by anything. This means that $x_{j_m}$ can't point to any other variables in $W_c$, so either it points to a different variable in $W$ (in which case it isn't the "end" of $W_c$), or it must point to itself. By the same reasoning, no variable outside of $W_c$ can point to a variable in $W_c$ since each has already reached its allotment of pointers (0 for $x_{j_1}$, 1 for all other variables). So $W_c$ is a chain-type polynomial in $W$ decoupled from the rest of $W$.

Similarly, every variable in $W$ that does not have a pointer defines the beginning of a decoupled chain. So we can remove each of these chains and be left with a non-degenerate quasihomogeneous polynomial $W'$ in the remaining variables whose invertibility agrees with that of $W$.

Now every variable in $W'$ must have a pointer, and must be a pointer. By the pigeonhole principle, each variable must have exactly one pointer. This makes it obvious that $W'$ is a sum of Fermats and loops, making it invertible. Adding the chains back in shows that $W$ is also invertible, as desired. $\square$

With this lemma, we were able to determine that the vast majority of the weight systems in Hertling and Kurbel's lists correspond to invertible polynomials with minimal computational effort. We simply checked the map they gave to see if it met the conditions of lemma 9. If it did, we knew that an invertible representative had already been identified and we moved on.

Next we applied some of the combinatorial techniques that Hertling and Kurbel described for checking that a given polynomial is non-degenerate. These techniques involve analyzing the monomials that appear in the polynomial, without taking any derivatives or solving a system of equations (as is required to check the definition of non-degenerate directly).

Their is one small caveat to this approach: it considers the variables and exponents of the monomials, but not the coefficients. For invertible polynomials, this is irrelevant because we can always assume the coefficients have been rescaled to 1. However, for non-invertible polynomials, there may be a "bad" choice of coefficients that would make that polynomial degenerate. So if we ignore the coefficients, we can't truly be certain that any given polynomial is non-degenerate.

This caveat does not concern us for two reasons. The first is that the set of "bad" coefficients forms a closed set whose complement is dense in the space of all possible coefficients (that is, $(\mathbb{C}^*)^N$, where $N$ is the number of monomials in the polynomial). So almost all choices of coefficients will produce non-degenerate polynomials (we sometimes denote this by saying that the generic polynomial is non-degenerate). In the event that a certain choice of coefficients does produce a degenerate polynomial, a small change to the chosen coordinates will make the polynomial non-degenerate.

The other reason we can proceed with this approach is that our current objective is to determine only the existence of an invertible polynomial. For invertibles, there is no choice of coefficients that can make the polynomial degenerate unless all choices are degenerate.

The details of these techniques are found in [6], and we will give only a crude idea here. In that paper Hertling and Kurbel give five different equivalent conditions which, if satisfied by a quasihomogeneous polynomial, indicate that the polynomial is non-degenerate. The one we used in our calculations is referred to as (C1)'. Basically it says the following: Let $W \in \mathbb{C}[x_1, \dots, x_n]$ be a quasihomogeneous polynomial. Then $W$ is non-degenerate if and only if, for every subset $J \subseteq \{x_1, \dots, x_n\}$ with $|J| \leq \frac{n+1}{2}$, either

(i) there is a monomial in $W$ whose variables are in $J$, or

(ii) there are $|J|$ different monomials in $W$ that are "almost" in $J$. By "almost" in $J$, we mean that if you removed one variable of degree 1 from the monomial, then all of its variables would be in $J$. We also require that a different variable be removed from each of the $|J|$ monomials.

**Example 6:** Let $W = x^2y^2 + x^2z + y^4z + z^3$, which is quasihomogeneous with weights $(\frac{1}{3}, \frac{1}{6}, \frac{1}{3})$. We will show that $W$ satisfies (C1)' by listing each of the subsets $J$ up to order $\frac{3+1}{2} = 2$, along with which monomials $m$ from $W$ are needed to satisfy (C1)' for that $J$ and which condition ((i) or (ii)) is met by that monomial.

| $J$ | $\{x\}$ | $\{y\}$ | $\{z\}$ | $\{x, y\}$ | $\{x, z\}$ | $\{y, z\}$ |
|---|---|---|---|---|---|---|
| $m$ | $x^2z$ | $y^4z$ | $z^3$ | $x^2y^2$ | $x^2z$ | $y^4z$ |
| | (ii) | (ii) | (i) | (i) | (i) | (i) |

Since all of the subsets $J$ satisfy at least one of the two conditions, $W$ is non-degenerate.

Initially, this may not seem like an efficient technique for determining non-degeneracy because it requires analyzing half of a power set (the conditions must be satisfied by all possible $J$). However, in practice it has proven to be extremely efficient because we are generally working with small numbers of variables (in our current analysis, $n \leq 4$). This keeps the number of different $J$ to check very reasonable and turns out to be much faster than our earlier methods (which, as mentioned previously, required taking derivatives and solving systems of equations).

With these optimizations in place, we were able to effectively use a brute force approach to search for invertible representatives for each weight system. We first found all monomials of weighted degree 1, then tried every possible sum of $n$ different monomials and checked each to see whether or not it was non-degenerate. Once we found an invertible representative, we recorded it in a file and moved on to the next weight system.

Recall once again that part (i) of conjecture 1 was that we would find an invertible representative for every weight system. Sadly, this turned out not to be the case. Our search yielded the following counterexample.

**Theorem 4:** *The weight system* $(\frac{1}{7}, \frac{3}{14}, \frac{2}{7})$ *corresponds to the non-degenerate quasihomogeneous polynomial* $W = x^7 + y^4x + z^3x + y^2z^2$ *but not to any invertible non-degenerate quasihomogeneous polynomial.*

*Proof.* It is easy to verify that $W$ is quasihomogeneous with the given weight system. We will show that it is in fact non-degenerate directly. To do this we need to show that it has a unique critical point at the origin, i.e. that 0 is the only solution to the system of equations

$$\begin{aligned} W_x &= 7x^6 + y^4 + z^3 = 0 \\ W_y &= 4y^3x + 2yz^2 = 2y(2y^2x + z^2) = 0 \\ W_z &= 3z^2x + 2y^2z = z(3zx + 2y^2) = 0 \end{aligned}$$

This is a rather straightforward system to solve.

First, assume $y = 0$. Then our system is reduced to

$$\begin{aligned} W_x &= 7x^6 + z^3 = 0 \\ W_z &= 3z^2x = 0 \end{aligned}$$

If $x \neq 0$ then $W_z$ tells us $z = 0$. But that would leave us with $W_x = 7x^6 = 0 \Rightarrow x = 0$, a contradiction. So we must have $x = 0$, implying $W_x = z^3 = 0 \Rightarrow z = 0$. So if $y = 0$, then the only solution is the $x = y = z = 0$.

Now assume $y \neq 0$. Then our system is reduced to

$$\begin{aligned} W_x &= 7x^6 + y^4 + z^3 = 0 \\ W_y &= 2y^2x + z^2 = 0 \\ W_z &= z(3zx + 2y^2) = 0 \end{aligned}$$

If $z = 0$ then $W_y$ tells us $2y^2x = 0$, and since we are assuming $y \neq 0$ this means that

$x = 0$. But that would leave us with $W_x = 7x^6 + y^4 + z^3 = y^4 = 0 \Rightarrow y = 0$, a contradiction.

On the other hand, if $z \neq 0$, then we can solve $W_z$ for $x$, giving $x = -\frac{2y^2}{3z}$. Plugging this in to $W_y$ gives $2y^2(-\frac{2y^2}{3z}) + z^2 = 0 \Rightarrow 3z^3 = 4y^4$. Combining this with our formula for $x$ yields $x^2 = (-\frac{2y^2}{3z})^2 = \frac{4y^4}{9z^2} = \frac{3z^3}{9z^2} = \frac{1}{3}z$. Finally, plugging these two formulas in to $W_x$ gives $7(\frac{1}{3}z)^3 + \frac{3}{4}z^3 + z^3 = 0 \Rightarrow z = 0$, a contradiction. So there is no solution when $y \neq 0$, showing that $W$ is non-degenerate.

To show that there is no invertible polynomial with this weight system, we will refer again to lemma 9, which tells us that no variable in an invertible polynomial can be pointed at by multiple other variables. We will start by looking at $y$, which has weight $\frac{3}{14}$. $3 \nmid 14$, so $y$ cannot point to itself. Also, there is no way a monomial of the form $y^\alpha z$ can have weighted degree 1, so $y$ can't point to $z$ either. By lemma 8, every variable must be a pointer, so $y$ must point at $x$. But by the same reasoning, $z$ must also be a pointer for $x$ since it can't point to itself or to $y$. Thus, by lemma 9, the weight system $(\frac{1}{7}, \frac{3}{14}, \frac{2}{7})$ has no invertible representative because any non-degenerate quasihomogeneous polynomial would have $x$ being pointed at by both $y$ and $z$. This completes the proof. □

This counterexample can be easily extended to an arbitrary number of variables. To do this, we require one more straightforward lemma.

**Lemma 10:** *Let $x_1$ and $x_2$ be variables in a non-degenerate quasihomogeneous polynomial, with reduced weights $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$, respectively (by "reduced" we mean $\gcd(a_i, b_i) = 1$). If $x_1$ points to $x_2$, then $b_2 \mid b_1$.*

*Proof.* By definition, we know that $x_1^\alpha x_2$ has weighted degree 1, meaning $\frac{\alpha a_1}{b_1} + \frac{a_2}{b_2} = 1$. By rearranging, we see that $a_2 b_1 = b_1 b_2 - \alpha a_1 b_2 = b_2(b_1 - \alpha a_1)$, i.e. $b_2 \mid a_2 b_1$. However, since the weights are reduced, we know that $\gcd(a_2, b_2) = 1$, meaning $b_2 \mid b_1$, as desired. □

**Theorem 5:** *For $n \geq 3$ variables, there exists a weight system corresponding to a non-degenerate quasihomogeneous polynomial for which there are no invertible representatives.*

*Proof.* Consider the weight system $\left(\frac{1}{7}, \frac{3}{14}, \frac{2}{7}, \frac{1}{3}, \ldots, \frac{1}{3}\right)$ of length $n$. From before, we know that this weight system corresponds to the polynomial $x^7 + y^4x + z^3x + y^2z^2 + x_4^3 + \cdots + x_n^3$, which is a sum of two non-degenerate quasihomogeneous polynomials in distinct variables, and is therefore non-degenerate and quasihomogeneous as well.

Suppose $W$ is an invertible non-degenerate quasihomogeneous polynomial with this weight system. Since 3 and 7 are relatively prime, by lemma 10, none of the first three variables can point to or be pointed at by any of the remaining $n-3$ variables. This means that $W$ contains no atomic polynomials that contain one of the first three variables and one of the remaining variables, as either a loop or a chain would require at least one pointer between these two sets of variables (and of course Fermats only have one variable).

Thus, we can think of $W$ as a sum of two invertible non-degenerate quasihomogeneous polynomials, $W = W_1 + W_2$, where $W_1 \in \mathbb{C}[x, y, z]$ and $W_2 \in \mathbb{C}[x_4, \ldots, x_n]$. But this means that $W_1$ has weight $\left(\frac{1}{7}, \frac{3}{14}, \frac{2}{7}\right)$, which, as shown in theorem 4, has no invertible representatives, a contradiction. Thus, there is no invertible non-degenerate quasihomogeneous polynomial with the given weight system in $n$ variables. Since $n$ was arbitrary, we have the desired result. $\square$

#### 5.1.3 Statistics from analysis.

In analyzing the lists of weight systems, we found that there are in fact many such counterexamples. Out of the 596,879 weight systems analyzed in three variables, 6,414 (1.07%) did not correspond to any invertible polynomial. In four variables, there were 4,365 counterexamples out of 165,624 weight systems (2.64%). An abbreviated list of these counterexamples is provided in Appendix A.

While a detailed analysis of the distribution of these counterexamples was not performed, we did notice one interesting anomaly in the data. The lists were organized in increasing order by $d$ (the least common denominator of the weight system). The last counterexample found in three variables was $(\frac{3}{2995}, \frac{748}{2995}, \frac{749}{2995})$ which was the 267,383rd weight system in the list (less than halfway through). So the weight systems with no invertible representatives were skewed toward the beginning of the list (that is, those with smaller $d$).

There appears to be a connection between $d$ and the Milnor number $\mu$ that determines when a counterexample may be found. In our limited investigation, it seemed that in three variables, all weight systems with $d > \frac{\mu+1}{3}$ had an invertible representative. (This explains why none were found past $d = 3000$, since the list contained all weight systems up to $\mu = 9000$.) A similar limit seems to exist in four variables, with the bound being closer to $\frac{\mu}{2}$ (though there were some small discrepancies, suggesting that our proposed formula for the bound is incomplete). We can formalize this conjecture as follows:

**Conjecture 2:** *(i) Any three-variable weight system with greatest common denominator $d > \frac{\mu+1}{3}$ that corresponds to a non-degenerate quasihomogeneous polynomial has an invertible representative.*

*(ii) Any four-variable weight system with greatest common denominator $d \gtrapprox \frac{\mu}{2}$ that corresponds to a non-degenerate quasihomogeneous polynomial has an invertible representative.*

*(iii) A similar bound exists for larger numbers of variables.*

More investigation should be performed to determine exactly when an invertible non-degenerate quasihomogeneous polynomial is guaranteed to exist.

## 5.2 $G^{max}$ OF INVERTIBLE NOT ALWAYS THE LARGEST

We now move on to investigating part (ii) of conjecture 1. Even though part (i) turned out not to be true, part (ii) would still be useful for the weight systems that do have invertible representatives (which is the majority of them).

This second part again seems plausible, for the same reasons we thought part (i) might prove true. Invertible polynomials have the minimal number of monomials for a non-degenerate quasihomogeneous polynomial with a given number of variables. Therefore, it should in a sense be "easier" to fix all of them, meaning there should be more elements of

$G^{max}$. We will demonstrate with another example, using the same polynomial that we used in example 4.

**Example 7:** Again let $W_1 = x^2y^2 + x^2z + y^4z + z^3$ with weight system $q = (\frac{1}{3},\ \frac{1}{6},\ \frac{1}{3})$. Then $G^{max}_{W_1} = \left\langle\ \left(0, \frac{1}{2}, 0\right), \left(\frac{1}{6}, \frac{5}{6}, \frac{2}{3}\right)\ \right\rangle$, which has order 12.

We previously gave the sum of three Fermats, $W_2 = x^3 + y^6 + z^3$, as an example of an invertible with the same weight system as $W_1$, so $W_1$ matches the hypothesis of part (ii). However, $G^{max}_{W_2} = \left\langle\ \left(\frac{1}{3}, 0, 0\right), \left(0, \frac{1}{6}, 0\right), \left(0, 0, \frac{1}{3}\right)\ \right\rangle$, which has order 54, showing that $G^{max}_{W_1} \not\leq G^{max}_{W_2}$ since $12 \nmid 54$.

Consider instead $W_3 = x^2z + y^6 + z^3$, which also has weight system $q$. This is a sum of a two-variable chain and a Fermat. Using the formulas we found earlier, we find that $G^{max}_{W_3} = \langle g_1,\ g_2 \rangle = \left\langle\ \left(0, \frac{1}{6}, 0\right), \left(\frac{1}{6}, 0, \frac{2}{3}\right)\ \right\rangle$, which has order 36. With these generators, we see that

$$\begin{aligned} G^{max}_{W_1} &= \left\langle \left(0, \frac{1}{2}, 0\right), \left(\frac{1}{6}, \frac{5}{6}, \frac{2}{3}\right) \right\rangle \\ &= \langle 3g_1,\ 5g_1 + g_2 \rangle \\ &\leq G^{max}_{W_3} \end{aligned}$$

so this example agrees with part (ii).

Unfortunately, it turns out that there are also counterexamples to this conjecture, as the next theorem demonstrates.

**Theorem 6:** *The weight system $q = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ corresponds to the non-degenerate quasihomogeneous polynomial $W = x^4 + y^3x + z^3x + y^2z^2$ which has $G^{max}_W = \left\langle\ \left(\frac{1}{4}, \frac{11}{12}, \frac{7}{12}\right)\ \right\rangle$. $G^{max}_W$ is not a subgroup of $G^{max}_{W'}$ for any invertible polynomial $W'$ with this weight system.*

*Proof.* We will not give the details of the calculations here to show that $W$ is non-degenerate. It is easy to see that it has the given weight system, and it is also easy to see that the given generator $g = \left(\frac{1}{4}, \frac{11}{12}, \frac{7}{12}\right)$ does indeed fix $W$, so it is in $G^{max}_W$, even if it is not obvious that it generates $G_W^{max}$. This is sufficient for our present purpose.

The simplest way to see that no invertible polynomial with weight system $q$ is fixed by $g$ is to examine all of the possibilities. There are only eighteen invertible non-degenerate quasihomogeneous polynomials with weight system $q$. They are given in the following table, along with their respective $G^{max}$.

| Polynomial | $G^{max}$ |
|---|---|
| $x^3z + y^4 + z^4$ | $\langle (0, \frac{1}{4}, 0), (\frac{1}{12}, 0, \frac{3}{4}) \rangle$ |
| $x^3z + y^4 + yz^3$ | $\langle (\frac{1}{36}, \frac{1}{4}, \frac{11}{12}) \rangle$ |
| $x^3z + xz^3 + y^4$ | $\langle (0, \frac{1}{4}, 0), (\frac{1}{8}, 0, \frac{5}{8}) \rangle$ |
| $x^3z + xy^3 + z^4$ | $\langle (\frac{1}{12}, \frac{35}{36}, \frac{3}{4}) \rangle$ |
| $x^3z + xy^3 + yz^3$ | $\langle (\frac{1}{28}, \frac{9}{28}, \frac{25}{28}) \rangle$ |
| $x^3y + y^3z + z^4$ | $\langle (\frac{1}{36}, \frac{11}{12}, \frac{1}{4}) \rangle$ |
| $x^3y + y^4 + z^4$ | $\langle (0, 0, \frac{1}{4}), (\frac{1}{12}, \frac{3}{4}, 0) \rangle$ |
| $x^3y + xz^3 + y^3z$ | $\langle (\frac{1}{28}, \frac{25}{28}, \frac{9}{28}) \rangle$ |
| $x^3y + xz^3 + y^4$ | $\langle (\frac{1}{12}, \frac{3}{4}, \frac{35}{36}) \rangle$ |
| $x^3y + xy^3 + z^4$ | $\langle (0, 0, \frac{1}{4}), (\frac{1}{8}, \frac{5}{8}, 0) \rangle$ |
| $x^4 + y^3z + z^4$ | $\langle (\frac{1}{4}, 0, 0), (0, \frac{1}{12}, \frac{3}{4}) \rangle$ |
| $x^4 + y^3z + yz^3$ | $\langle (\frac{1}{4}, 0, 0), (0, \frac{1}{8}, \frac{5}{8}) \rangle$ |
| $x^4 + y^4 + z^4$ | $\langle (0, 0, \frac{1}{4}), (0, \frac{1}{4}, 0), (\frac{1}{4}, 0, 0) \rangle$ |
| $x^4 + y^4 + yz^3$ | $\langle (\frac{1}{4}, \frac{3}{4}, \frac{3}{4}), (0, \frac{3}{4}, \frac{1}{12}) \rangle$ |
| $x^4 + xz^3 + y^3z$ | $\langle (\frac{1}{4}, \frac{1}{36}, \frac{11}{12}) \rangle$ |
| $x^4 + xz^3 + y^4$ | $\langle (0, \frac{1}{4}, 0), (\frac{1}{4}, 0, \frac{11}{12}) \rangle$ |
| $x^4 + xy^3 + z^4$ | $\langle (0, 0, \frac{1}{4}), (\frac{1}{4}, \frac{11}{12}, 0) \rangle$ |
| $x^4 + xy^3 + yz^3$ | $\langle (\frac{1}{4}, \frac{11}{12}, \frac{1}{36}) \rangle$ |

A simple inspection of this table reveals that none of these groups could contain $g$. The easiest way to see this is by examining the last two coordinates. None of the groups listed could possibly have an element where both of the last two coordinates have order twelve in

$\mathbb{Q}/\mathbb{Z}$. Since $g$ does have this property, it is clearly not in any $G^{max}$ listed, and therefore $G_W^{max}$ cannot be a subgroup of any of them, as desired. □

As with theorem 4, this counterexample can be easily generalized.

**Theorem 7:** *For $n \geq 3$, there exists a non-invertible non-degenerate quasihomogeneous polynomial with an invertible representative such that $G^{max}$ of the non-invertible is not a subgroup of $G^{max}$ of any invertible representative.*

*Proof.* Consider the example from theorem 6, $W = x^4 + y^3x + z^3x + y^2z^2$. We will use it to construct the desired counterexample.

Define $W_n$ as follows: select integers $\alpha_4, \ldots, \alpha_n > 1$ such that the set $\{4, \alpha_4, \ldots, \alpha_n\}$ is pairwise relatively prime. Let $W_n = W + x_4^{\alpha_4} + \cdots + x_n^{\alpha_n}$. Then $G_{W_n}^{max} = G_W^{max} \times G_{x_4^{\alpha_4}}^{max} \times \cdots \times G_{x_n^{\alpha_n}}^{max}$.

Now let $W'$ be an invertible polynomial with the same weights as $W_n$, that is, $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{\alpha_4}, \ldots, \frac{1}{\alpha_n})$. By lemma 10, none of the last $n-3$ variables can point to any other variable. So the only atomic type they can have is Fermat. Thus, $W' = W_1 + x_4^{\alpha_4} + \cdots + x_n^{\alpha_n}$ where $W_1$ is an invertible polynomial with weight system $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. But theorem 6 tells us that $G_W^{max} \not\leq G_{W_1}^{max}$, so $G_{W_n}^{max} \not\leq G_{W'}^{max}$. Since $W'$ and $n$ were arbitrary, this gives the desired result. □

## Chapter 6. Conclusion

In this thesis we have proven some useful new facts about non-degenerate quasihomogeneous polynomials, which are the most fundamental building blocks of FJRW theory. We have shown that every invertible polynomial has a unique maximal group of diagonal symmetries, which has already proven to be useful in unexpected ways (see [8]). We began investigating a new way to classify non-degenerate quasihomogeneous polynomials and found promising results in two variables. And we disproved two conjectures about the relationship between invertible and non-invertible polynomials. In doing so, we have possibly discovered a new result about when a weight system is guaranteed to have an invertible representative.

## Appendix A. Lists of Weight Systems With No Invertible Representatives

Below are the first one hundred counterexamples to part (i) of conjecture 1 in three and four variables, ordered by increasing $d$ (the greatest common denominator of the weight system). These lists are not typeset as fractions so that they may be easily copied and inserted into computer code by future researchers.

| Three variables | Four variables |
|---:|:---|
| (1/7, 3/14, 2/7) | (1/7, 2/7, 2/7, 3/7) |
| (1/16, 3/16, 5/16) | (1/11, 2/11, 2/11, 5/11) |
| (3/19, 4/19, 5/19) | (1/13, 2/13, 2/13, 3/13) |
| (1/11, 2/11, 5/22) | (1/13, 3/13, 3/13, 4/13) |
| (2/23, 3/23, 7/23) | (1/13, 3/13, 4/13, 6/13) |
| (3/23, 4/23, 5/23) | (2/13, 3/13, 4/13, 5/13) |
| (1/25, 3/25, 8/25) | (1/14, 1/7, 3/14, 2/7) |
| (1/13, 3/26, 2/13) | (1/15, 2/15, 2/15, 7/15) |
| (3/28, 1/7, 2/7) | (1/15, 2/15, 4/15, 7/15) |
| (1/29, 4/29, 7/29) | (1/15, 1/5, 4/15, 2/5) |
| (3/29, 4/29, 5/29) | (1/5, 4/15, 1/3, 2/5) |
| (1/15, 2/15, 7/30) | (1/16, 1/8, 3/16, 5/16) |
| (1/31, 3/31, 5/31) | (2/17, 2/17, 3/17, 7/17) |
| (2/31, 3/31, 7/31) | (2/17, 3/17, 3/17, 5/17) |
| (3/31, 7/31, 8/31) | (2/17, 3/17, 4/17, 5/17) |
| (1/16, 3/32, 5/16) | (2/17, 3/17, 5/17, 6/17) |

| Three variables | Four variables |
|---:|:---|
| (1/16, 5/32, 3/16) | (3/17, 4/17, 5/17, 7/17) |
| (1/11, 5/33, 2/11) | (1/19, 2/19, 2/19, 9/19) |
| (1/34, 3/34, 11/34) | (1/19, 2/19, 3/19, 8/19) |
| (3/34, 2/17, 5/17) | (2/19, 3/19, 4/19, 5/19) |
| (2/17, 5/34, 3/17) | (2/19, 4/19, 5/19, 7/19) |
| (2/17, 3/17, 7/34) | (4/19, 5/19, 6/19, 7/19) |
| (3/35, 1/7, 2/7) | (1/10, 1/4, 3/10, 9/20) |
| (1/36, 5/36, 7/36) | (1/21, 2/21, 5/21, 8/21) |
| (1/19, 3/38, 2/19) | (1/21, 1/7, 4/21, 10/21) |
| (1/19, 2/19, 9/38) | (1/21, 4/21, 4/21, 5/21) |
| (3/38, 5/38, 7/38) | (1/21, 4/21, 5/21, 8/21) |
| (5/38, 3/19, 4/19) | (1/21, 4/21, 1/3, 10/21) |
| (1/13, 4/39, 3/13) | (2/21, 2/21, 1/7, 3/7) |
| (4/39, 5/39, 7/39) | (2/21, 1/7, 4/21, 3/7) |
| (1/41, 5/41, 8/41) | (2/21, 1/7, 2/7, 3/7) |
| (2/41, 3/41, 13/41) | (1/7, 4/21, 2/7, 3/7) |
| (5/41, 6/41, 7/41) | (1/22, 3/22, 3/22, 7/22) |
| (1/21, 2/21, 5/42) | (1/22, 3/22, 5/22, 7/22) |
| (1/21, 5/42, 4/21) | (1/22, 2/11, 3/11, 9/22) |
| (2/21, 1/7, 3/14) | (1/11, 3/22, 2/11, 5/22) |
| (1/43, 3/43, 14/43) | (1/11, 2/11, 5/22, 9/22) |
| (3/43, 4/43, 5/43) | (1/23, 2/23, 4/23, 11/23) |
| (3/43, 10/43, 11/43) | (1/23, 2/23, 6/23, 11/23) |
| (1/22, 3/44, 7/44) | (1/23, 4/23, 6/23, 11/23) |
| (1/22, 3/22, 7/44) | (2/23, 3/23, 4/23, 5/23) |
| (1/11, 5/44, 2/11) | (2/23, 3/23, 4/23, 7/23) |

| Three variables | Four variables |
|---:|:---|
| (1/45, 4/45, 11/45) | (2/23, 3/23, 5/23, 6/23) |
| (1/15, 2/15, 7/45) | (2/23, 3/23, 7/23, 10/23) |
| (1/46, 3/46, 5/46) | (1/25, 3/25, 1/5, 8/25) |
| (1/46, 5/46, 9/46) | (1/25, 3/25, 8/25, 11/25) |
| (1/23, 2/23, 11/46) | (1/25, 1/5, 6/25, 8/25) |
| (3/46, 2/23, 7/23) | (1/25, 1/5, 8/25, 12/25) |
| (2/23, 5/46, 3/23) | (2/25, 3/25, 7/25, 11/25) |
| (2/23, 3/23, 7/46) | (2/25, 4/25, 7/25, 9/25) |
| (2/47, 5/47, 9/47) | (3/25, 4/25, 1/5, 2/5) |
| (3/47, 4/47, 11/47) | (3/25, 4/25, 7/25, 9/25) |
| (3/47, 5/47, 7/47) | (4/25, 1/5, 7/25, 2/5) |
| (4/47, 5/47, 7/47) | (4/25, 6/25, 7/25, 9/25) |
| (5/47, 6/47, 7/47) | (1/13, 3/26, 3/26, 4/13) |
| (5/47, 7/47, 8/47) | (1/13, 3/26, 3/13, 4/13) |
| (1/16, 5/48, 3/16) | (1/13, 3/26, 4/13, 9/26) |
| (1/49, 3/49, 8/49) | (1/13, 2/13, 5/26, 3/13) |
| (3/49, 4/49, 5/49) | (1/13, 3/13, 4/13, 9/26) |
| (3/49, 1/7, 2/7) | (1/27, 1/9, 2/9, 8/27) |
| (4/49, 5/49, 11/49) | (2/27, 2/27, 5/27, 11/27) |
| (1/25, 3/50, 2/25) | (2/27, 4/27, 5/27, 11/27) |
| (1/25, 3/50, 8/25) | (2/27, 5/27, 8/27, 11/27) |
| (5/51, 2/17, 3/17) | (1/9, 2/9, 7/27, 8/27) |
| (2/17, 7/51, 3/17) | (1/28, 1/7, 3/14, 2/7) |
| (1/52, 3/52, 17/52) | (1/14, 3/28, 1/7, 2/7) |
| (3/52, 1/13, 2/13) | (3/28, 1/7, 5/28, 2/7) |
| (3/52, 5/52, 7/52) | (1/7, 5/28, 3/14, 2/7) |

| Three variables | Four variables |
|---:|:---|
| (3/53, 4/53, 5/53) | (1/7, 3/14, 1/4, 2/7) |
| (3/53, 5/53, 16/53) | (1/7, 3/14, 2/7, 11/28) |
| (5/53, 8/53, 9/53) | (1/29, 3/29, 4/29, 14/29) |
| (1/27, 2/27, 13/54) | (1/29, 4/29, 7/29, 11/29) |
| (2/27, 5/27, 11/54) | (2/29, 2/29, 3/29, 13/29) |
| (2/55, 3/55, 13/55) | (2/29, 2/29, 5/29, 9/29) |
| (3/55, 4/55, 13/55) | (2/29, 3/29, 4/29, 13/29) |
| (3/55, 4/55, 17/55) | (2/29, 3/29, 8/29, 13/29) |
| (3/55, 13/55, 14/55) | (2/29, 5/29, 6/29, 9/29) |
| (3/56, 1/7, 2/7) | (2/29, 5/29, 8/29, 9/29) |
| (1/57, 4/57, 7/57) | (2/29, 5/29, 9/29, 12/29) |
| (2/57, 5/57, 11/57) | (3/29, 7/29, 8/29, 13/29) |
| (5/57, 3/19, 4/19) | (1/15, 2/15, 1/6, 7/30) |
| (1/29, 2/29, 7/58) | (1/15, 2/15, 7/30, 13/30) |
| (1/29, 7/58, 4/29) | (1/31, 2/31, 8/31, 15/31) |
| (3/58, 4/29, 5/29) | (1/31, 3/31, 5/31, 14/31) |
| (2/29, 5/58, 9/29) | (1/31, 4/31, 6/31, 15/31) |
| (2/29, 3/29, 13/58) | (1/31, 5/31, 6/31, 10/31) |
| (2/59, 3/59, 19/59) | (1/31, 6/31, 10/31, 15/31) |
| (2/59, 5/59, 9/59) | (2/31, 3/31, 7/31, 8/31) |
| (3/59, 7/59, 8/59) | (2/31, 4/31, 9/31, 11/31) |
| (4/59, 5/59, 11/59) | (2/31, 5/31, 6/31, 13/31) |
| (1/15, 7/60, 2/15) | (3/31, 4/31, 4/31, 7/31) |
| (1/61, 3/61, 20/61) | (3/31, 4/31, 6/31, 7/31) |
| (1/61, 4/61, 15/61) | (3/31, 4/31, 7/31, 8/31) |
| (1/61, 5/61, 12/61) | (3/31, 4/31, 7/31, 9/31) |

| Three variables | Four variables |
|---:|:---|
| (1/31, 3/62, 2/31) | (3/31, 4/31, 7/31, 12/31) |
| (1/31, 3/62, 5/31) | (4/31, 5/31, 9/31, 11/31) |
| (1/31, 2/31, 5/62) | (4/31, 9/31, 10/31, 11/31) |
| (1/31, 2/31, 15/62) | (1/16, 5/32, 3/16, 9/32) |
| (1/31, 5/62, 3/31) | (1/16, 5/32, 3/16, 13/32) |
| (3/31, 7/62, 4/31) | (1/16, 3/16, 5/16, 11/32) |

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