A Volume Bound for Montesinos Links

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ABSTRACT

A Volume Bound for Montesinos Links

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The hyperbolic volume of a knot complement is a topological knot invariant. Futer, Kalfagianni, and Purcell have estimated the volumes of Montesinos link complements for Montesinos links with at least three positive tangles. Here we extend their results to all hyperbolic Montesinos links.

Keywords: knot, link, rational tangle, Montesinos, topology, manifold, hyperbolic, geometry
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Chapter 1. Introduction

An important problem in knot theory is to tell whether two knots are the same. See for example the two knots pictured in Figure 1.1. Is it possible to reconfigure knot (a), without cutting, until it looks like knot (b)? One powerful way to approach this problem is to use a knot invariant.

William Thurston proved in 1982 that the complement of any non-torus, non-satellite knot must admit a complete hyperbolic metric [17]. By the Mostow-Prasad rigidity theorem, this metric is unique up to isometry [12, 15]. Therefore, the volume of a hyperbolic knot complement is a knot invariant. Calculating or approximating this volume for classes of knots is very useful, although often difficult.

Using volume estimates from Perelman’s work on Ricci flow, Agol, Storm, and Thurston calculated the following lower bound on the volume of a hyperbolic manifold $M$:

**Theorem 1** (Agol, Storm, Thurston, [1]). Let $M$ be a closed, hyperbolic 3-manifold, and let $\Sigma \subset M$ be an incompressible surface. Then

$$vol(M) \geq v_8(\chi_-(\text{guts}(M \setminus \Sigma)))$$

where $v_8$ is the hyperbolic volume of a regular ideal octahedron, and $\chi_-(\cdot)$ is the negative Euler characteristic.

The guts of $M \setminus \Sigma$ are the pieces of $M \setminus \Sigma$ that admit a complete hyperbolic metric with totally geodesic boundary, after performing the annulus version of JSJ decomposition [7, 8].

Futer, Kalfagianni, and Purcell applied this result to knot complements. Under the hypothesis

Figure 1.1: Are these two knots equivalent?
of A-adequacy, they define an incompressible surface $S_A$ in the manifold $S^3 \setminus K$ and compute $\chi_-(guts((S^3 \setminus K) \setminus S_A))$.

**Theorem 2** (Futer, Kalfagianni, Purcell, [6]). Let $K$ be a hyperbolic knot with A-adequate diagram $D(K)$. Let $S_A$ be the essential spanning surface determined by $D(K)$. Then

$$\chi_-(guts((S^3 \setminus K) \setminus S_A)) = \chi_-(G'_A) - ||E_C||$$

where $\chi_-(\cdot)$ is the negative Euler characteristic, $G'_A$ is the reduced A-state graph, and $||E_C||$ is the number of essential product disks required to span the upper polyhedron of the knot complement.

Precise definitions for $\chi_-, G'_A$ and $||E_C||$ are given later (Definitions 26, 25, and Lemma 42). For now we’ll say only that these quantities can be read off the diagram of $K$ fairly easily. Combining Theorems 1 and 2 gives the following:

**Theorem 3** (Theorem 5.14, Futer, Kalfagianni, and Purcell, [6]). Let $D = D(K)$ be a prime A-adequate diagram of a hyperbolic link $K$. Then

$$\text{vol}(S^3 \setminus K) \geq v_8(\chi_-(G'_A) - ||E_C||),$$

where $\chi_-(\cdot)$, $G'_A$, and $||E_C||$ are as in the statement of Theorem 2 and $v_8 = 3.6638$ is the hyperbolic volume of a regular ideal octahedron.

We apply this result to Montesinos links. Montesinos links are a generalization of pretzel links, and are built out of rational tangles. Rational tangles themselves are objects of interest [4], and are particularly useful in understanding DNA topology (see for example [5]).

In order to use Theorem 3, we are interested in bounding both $\chi_-(G'_A)$ and $||E_C||$ for general Montesinos links. Of these two quantities, $||E_C||$ is more difficult to bound, and we spend the majority of our efforts on this. Futer, Kalfagianni, and Purcell proved that for Montesinos links with at least three positive tangles, $||E_C|| = 0$ [6, Proposition 8.16]. It is also true that $||E_C|| = 0$ for alternating links. In this paper we bound $||E_C||$ for a larger class of Montesinos links. From our bound on $||E_C||$ we obtain the following theorem.

**Theorem 4.** Let $K$ be a hyperbolic Montesinos link. Then $K$ has a reduced, admissible, A-adequate
\begin{equation*}
vol(S^3 \setminus K) \geq v_8(\chi_\cdot(\mathbb{G}_A') - 1)
\end{equation*}

where \(v_8 = 3.6638\ldots\) is the hyperbolic volume of a regular ideal octahedron, \(\chi_\cdot(\cdot)\) is the negative Euler characteristic, and \(\mathbb{G}_A'\) is the reduced A-state graph of \(D(K)\).

Again, the quantity \(\chi_\cdot(\mathbb{G}_A')\) is fairly easy to read off a knot diagram. However, another result in terms of twist number is even easier to use.

**Theorem 5.** Let \(K\) be a Montesinos link that is both A- and B-adequate, and let \(D(K)\) be a reduced, admissible diagram for \(K\). Then

\begin{equation*}
vol(S^3 \setminus K) \geq \frac{v_8}{4}(t(D) - \#K - 8)
\end{equation*}

where \(v_8 = 3.6638\ldots\) is the hyperbolic volume of a regular ideal octahedron, \(t(D)\) is the twist number of \(D(K)\), and \(\#K\) is the number of link components of \(K\).

Twist number is defined in Definition 23.

**Chapter 2. Montesinos Links**

We begin with a few preliminaries. We will use projection plane to denote the equatorial 2-sphere of \(S^3\). If \(K\) is a link in \(S^3\), then \(D(K) = D\) is the corresponding link diagram in the projection plane. We define positive and negative crossings in Figure 2.1. We will assume through the rest of this paper that \(D\) is connected.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{crossings.png}
\caption{(a) A positive crossing \hspace{1cm} (b) A negative crossing}
\end{figure}
2.1 Rational Tangles

Definition 6. Place four points on the surface of the ball $B^3$. Label the points NW, NE, SW, and SE. Connect these points by attaching two arcs $c_1$ and $c_2$ connecting NE to SE and NW to SW, as in Figure 2.2a. Let $f$ be a homeomorphism of $B^3$ that maps the set \{NW, NE, SW, SE\} to itself (although not necessarily as the identity map.) A rational tangle is the image of $c_1 \cup c_2$ under this homeomorphism.

The following material is classically known; for example, a good reference is [13]. A rational tangle may be built by the following process. Start with the ball $B^3$, the points NW, NE, SW, and SE on the surface of $B^3$, and the two arcs as in Figure 2.2a. Now, perform a homeomorphism of $B^3$ to rotate the points NW and NE a finite number of times, twisting the two arcs to create a vertical band of crossings. The crossings may be positive or negative. In Figure 2.2b, three positive crossings have been added. (After twisting, relabel the points NW, NE, SW, and SE in their original orientation.)

Next, perform a homeomorphism of $B^3$ to rotate NE and SE a finite number of times, adding positive or negative crossings in a horizontal band. See Figure 2.2c. Repeat this process finitely many times, alternating between adding crossings in vertical and horizontal bands (Figure 2.2d). The result is a rational tangle.

Alternatively, start by using two arcs to connect NW to NE and SW to SE, rather than NE to SE and NW to SW as we did before. In this case we add a horizontal band of crossings first, and then continue as before, alternating between horizontal and vertical bands finitely many times. This is shown in Figure 2.3.

Any rational tangle may be built by this process. We can summarize a rational tangle built in this way with a vector $(a_n, a_{n-1}, \ldots, a_1)$ which records the number and sign of crossings added.
at each step. For example, the vector corresponding to the tangle in Figure 2.2d is \((4, -1, -2, 3)\). As a convention, let us require that \(a_n\) always corresponds to a horizontal band of crossings. Thus if we build a rational tangle ending with a vertical band, as in figure 2.2b, we insert a 0 into the corresponding vector, representing a horizontal band of 0 crossings. For example, the vector corresponding to the tangle in Figure 2.2b is \((0, 3)\). This convention ensures that any vector of integers completely specifies a single rational tangle.

**Definition 7.** Given a rational tangle with corresponding vector \((a_n, a_{n-1}, \ldots, a_1)\), we define its slope to be the rational number produced by a continued fraction expansion

\[
[a_n, a_{n-1}, \ldots, a_1] = a_n + \frac{1}{a_{n-1} + \frac{1}{\ddots + \frac{1}{a_1}}}
\]

By convention, the slopes of the tangles in Figures 2.3a and 2.2a are 0 and \(\infty\) respectively. These two tangles, with no crossings, are called exceptional tangles.

It is well-known that rational tangles are in one-to-one correspondence with slopes in \(\mathbb{Q} \cup \{\infty\}\). This fact immediately supplies us with some important information about rational tangles. First, every positive rational number has a continued fraction expansion using only nonnegative integers. Thus, any rational tangle of positive slope can be described by a vector of only nonnegative integers, and therefore has a diagram with only positive crossings. Similarly, a rational tangle of negative slope has a diagram with only negative crossings. This proves that we may divide all non-exceptional rational tangles into two groups: *positive tangles* and *negative tangles*. In either case, the tangle has an alternating diagram.
Second, we may require for a continued fraction with \( n \) integers that \( a_i \neq 0 \) for all \( i < n \). This fact will be useful in proving some of the main results.

In the description of building rational tangles, we added vertical bands of crossings by rotating the points NW and NE, inserting the vertical band on the north of the tangle. Notice that we could have rotated SW and SE instead, adding a vertical band of crossings on the south of the tangle. These two methods are equivalent by a sequence of flypes (Figure 2.4). Likewise, we may add each horizontal band of crossings either on the west side of the tangle (by rotating the points NW and SW), or on the east side of the tangle (by rotating the points NE and SE). We would like to make a consistent choice. Therefore we give the following definition.

Figure 2.4: Here a flype moves a crossing from the east side to the west side of a tangle

**Definition 8.** (a) If \( T \) is a positive tangle, then an alternating diagram for \( T \) is admissible if all the vertical bands of crossings were added by rotating the points NW and NE, and all the horizontal bands of crossings were added by rotating the points NW and SW.

(b) If \( T \) is a negative tangle, then an alternating diagram for \( T \) is admissible if all the vertical bands of crossings were added by rotating the points NW and NE, and all the horizontal bands of crossings were added by rotating the points NE and SE.

By a sequence of flypes, any non-exceptional tangle has an admissible diagram.

### 2.2 Montesinos Links

**Definition 9.** Given two rational tangles \( T_1 \) and \( T_2 \) with slopes \( q_1 \) and \( q_2 \), we form their sum by connecting the NW and SW corners of \( T_1 \) to the NE and SW corners of \( T_2 \), respectively, with two disjoint arcs. If \( q_1 \in \mathbb{Z} \) or \( q_2 \), then the sum \( T_1 + T_2 \) is also a rational tangle; this is called a trivial sum.

The cyclic sum of \( T_1, \ldots, T_r \) is the numerator closure of the sum \( T_1 + \ldots + T_r \). A Montesinos link is the cyclic sum of a finite ordered list of tangles \( T_1, \ldots, T_r \) (see Figure 2.5).
A Montesinos link is determined by the integer $r$ and an $r$-tuple of slopes $q_1, \ldots, q_r$, with $q_i \in \mathbb{Q} \cup \infty$. Note that if $q_i = \infty$ for some $i$, the resulting knot is disconnected. Thus we assume that $q_i \notin \mathbb{Z} \cup \infty$, to avoid trivial or disconnected sums.

**Theorem 10** (Theorem 12.8 of [2]). Let $K$ be a Montesinos link obtained as the cyclic sum of $r \geq 3$ rational tangles whose slopes are $q_1, \ldots, q_r \in \mathbb{Q} \setminus \mathbb{Z}$. Then $K$ is determined up to isomorphism by the rational number $\sum_{i=1}^{r} q_i$ and the vector $((q_1 \mod 1), \ldots, (q_r \mod 1))$, up to dihedral permutation.

Note that this theorem gives isomorphism up to dihedral permutation; however we will only use isomorphism up to cyclic permutation. By Theorem 10, given $K$ as the cyclic sum of $T_1, \ldots, T_r$, we can “combine” the integer parts of $q_1, \ldots, q_r$. The following definition makes use of this fact.

**Definition 11.** A diagram $D(K)$ is called a reduced Montesinos diagram if it is the cyclic sum of the diagrams $T_i$; for each $i$, the diagram of $T_i$ has either all positive or all negative crossings; and either

1. all the $q_i$ have the same sign, or
2. $0 < |q_i| < 1$ for all $i$.

It is not hard to see that every Montesinos link with $r \geq 3$ has a reduced diagram. For example, if $q_i < 0$ while $q_j > 1$, one may add 1 to $q_i$ and subtract 1 from $q_j$. By Theorem 10, this does not change the link type. One may continue in this manner until condition (1) of Definition 11 is satisfied.

The next definition is simple.

**Definition 12.** A diagram $D(K)$ of the cyclic sum of $T_1, \ldots, T_r$ is an admissible Montesinos diagram if the diagram of $T_i$ is an admissible tangle diagram for each $i$. 

---

Figure 2.5: A Montesinos link with $r$ tangles
We will also need to know that Montesinos links have prime diagrams. We will prove this in the next few Lemmas.

**Definition 13.** A diagram $D$ of a link $K$ is said to be prime if any simple closed curve in the projection plane which intersects $D$ exactly twice contains no crossings on one side.

**Lemma 14.** Suppose $T_1, \ldots, T_n$ are diagrams of rational tangles such that for each $i$, either $\text{num}(T_i)$ or $\text{denom}(T_i)$ is prime. Then $\text{num}(T_1 + \ldots + T_n)$ is prime.

**Proof.** Let $D$ be the diagram produced by taking the numerator closure of the sum of the diagrams $T_1, \ldots, T_n$. Enclose each tangle $T_i$ with a simple closed curve $\alpha_i$; each $\alpha_i$ is a square with sides N, S, E, and W. Notice that the $\alpha_i$ together with the parts of $D$ outside the $T_i$ partition $P \setminus (T_1 \cup \ldots \cup T_r)$ into $r + 2$ disjoint regions (where $P$ is the projection plane). We label these as follows. Let $R_i$ be the region to the left of $T_i$ for $i = 1, \ldots, r$; let $R_{r+1}$ be the region to the north of the tangles; let $R_{r+2}$ the region to the south. Then $R_i$ is bounded by a simple closed curve $\beta_i$, where $\beta_i$ alternates running over $D$ and running over sides of the $\alpha_k$.

Let $\gamma$ be a simple closed curve in the projection plane that intersects $D$ exactly twice. We will show that $\gamma$ contains no crossings of $D$ on one side.

**Case 1.** $\gamma$ doesn’t intersect $D$ inside any tangles.

In this case, we may push $\gamma$ so that it doesn’t enter any tangles. Suppose $\gamma$ does contain crossings on both sides. Since all crossings of $D$ are inside tangles and $\gamma$ doesn’t enter any tangles, we must have a complete tangle $T_i$ on one side of $\gamma$ and another complete tangle $T_j$ on the other. Note that $T_i$ and $T_j$ both border $R_{r+1}$ on the north. Thus $\gamma$ separates $R_{r+1}$; so $\gamma$ must intersect $\beta_{r+1}$ twice. But $\gamma$ doesn’t intersect any $\alpha_k$, so $\gamma$ must intersect $\beta_{r+1}$ twice where $\beta_{r+1}$ runs along $D$.

Repeating the same argument for the region $R_{r+2}$, we find $\gamma$ must intersect $\beta_{r+2}$ twice where $\beta_{r+2}$ coincides with $D$. Thus we have 4 distinct intersections of $\gamma$ with $D$. Contradiction.

**Case 2.** $\gamma$ intersects $D$ twice inside a given tangle $T_i$.

We may push $\gamma$ so it does not enter any other tangles.
(i) Suppose $\gamma$ does not intersect $\alpha_i$. Suppose also that $\text{denom}(T_i)$ is prime; if this was not the case, we could replace $\text{denom}(T_i)$ with $\text{num}(T_i)$ in the following argument, and the argument would still hold.

Since $\gamma$ intersects $D$ twice inside $T_i$ and does not intersect $\alpha_i$, then $\gamma$ also intersects $\text{denom}(T_i)$ exactly twice. But $\text{denom}(T_i)$ is prime, so $\gamma$ contains no crossings of $T_i$ on one side. Suppose $\gamma$ contains no crossings of $T_i$ on the outside. See Figure 2.6a. Then $\text{denom}(T_i)$ must connect up at the points $a$ and $b$ and the four corners of $T_i$ without any crossings and without entering $\gamma$. But we will see this is impossible. Suppose $a$ and $b$ connect to the northeast and northwest corners. Then the southeast and southwest corners must be connected by one arc, as in Figure 2.6b. But this contradicts the assumption that $T_i$ is not trivial, in the definition of Montesinos links. The same argument holds if $a$ and $b$ connect to any adjacent corners of $T_i$. Now suppose $a$ and $b$ connect to opposite corners of $T_i$, as in in Figure 2.6c. Then there is no way the remaining two corners of $T_i$ can connect up without either intersecting $\gamma$ or introducing crossings in $\text{denom}(T_i)$ outside of $\gamma$. Therefore this case is impossible; so $\gamma$ must contain no crossings of $\text{denom}(T_i)$ on the outside. Then we are done; because then $\gamma$ contains no crossings of $D$ on the inside.

(ii) Suppose $\gamma$ intersects $\alpha_i$ on two different sides of $\alpha_i$. Then $T_i$ enters two distinct regions $R_j$ and $R_k$. In order to close itself off, $\gamma$ must exit $R_j$. In doing so $\gamma$ intersects $\beta_j$; but $\gamma$ doesn’t enter any other tangles and does not reenter $T_i$, so $\gamma$ intersects $\beta_j$ where $\beta_j$ coincides with $D$. But then $\gamma$ intersects $D$ at least three times. Contradiction.

(iii) Suppose $\gamma$ intersects $\alpha_i$ twice on the same side. Then $\gamma$ enters the region $R_j$; $\gamma$ cannot exit $R_j$ since it cannot intersect $\beta_j$ either through $D$ or through $\alpha_k$. So $\gamma$ closes off inside of $R_j$, containing no crossings in $R_j$. Then we can push $\gamma$ out of $R_j$ back into $T_i$. This reduces back to (i).

Case 3. $\gamma$ intersects $D$ exactly once inside a given tangle $T_i$.

(i) Suppose $\gamma$ does not intersect $\alpha_i$. Then all the intersections of $\gamma$ with $D$ are in $T_i$. But $\gamma$ is supposed to intersect $D$ twice. Contradiction.
(ii) Suppose \( \gamma \) exits \( T_i \) through two adjacent sides of \( \alpha_i \). Suppose again that \( \text{denom}(T_i) \) is prime; if this was not the case, we could replace \( \text{denom}(T_i) \) with \( \text{num}(T_i) \) in the following argument, and the argument would still hold.

Consider \( \text{denom}(T_i) \). Create a new simple closed curve \( \gamma' \) that coincides with \( \gamma \) inside \( T_i \). Just where \( \gamma \) exits \( T_i \), connect \( \gamma' \) so that \( \gamma' \) intersects \( \text{denom}(T_i) \) exactly twice (once in \( T_i \) and a new time outside \( T_i \)). Since \( \text{denom}(T_i) \) is prime, \( \gamma' \) contains no crossings on one side. Thus we can push \( \gamma' \) out of \( \text{denom}(T_i) \) without changing the number of crossings on each side of \( \gamma' \). Since \( \gamma \) coincides with \( \gamma' \) in \( T_i \), we can likewise push \( \gamma \) out of \( T_i \) without changing the number of crossings on each side of \( \gamma \).

(iii) Suppose \( \gamma \) exits \( T_i \) through the east and west sides of \( \alpha_i \). Again, consider \( \text{denom}(T_i) \). Create a new simple closed curve \( \gamma' \) that coincides with \( \gamma \) until \( \gamma \) exits \( T_i \). Then close off \( \gamma' \) without intersecting \( \text{denom}(T_i) \). But then we have a closed curve \( \gamma' \) that intersects \( \text{denom}(T_i) \) only once. Contradiction. Notice we do not need to assume that \( \text{denom}(T_i) \) is prime here.

(iv) Suppose \( \gamma \) exits \( T_i \) through the north and south sides of \( \alpha_i \). Repeat the same argument as in (iii), using \( \text{num}(T_i) \) in place of \( \text{denom}(T_i) \). We need not assume that \( \text{num}(T_i) \) is prime.

\begin{lemma}
Let \( T \) be a reduced admissible diagram of a non-exceptional rational tangle, such that either \( \text{num}(T) \) or \( \text{denom}(T) \) is prime. Let \( S \) be a reduced admissible diagram of a rational tangle with a single crossing. Then the diagram \( \text{num}(T + S) \) is prime.

\begin{proof}
The diagram \( D \) of \( \text{num}(T + S) \) is shown in Figure 2.7. There is a simple closed curve \( \alpha \) bounding \( T \), which intersects \( D \) four times. Let \( \gamma \) be a simple closed curve in the projection plane
\end{proof}
\end{lemma}
that intersects $D$ exactly twice transversely. We will show that $\gamma$ contains no crossings of $D$ on one side.

**Case 1.** Suppose $\gamma$ intersects $D$ twice inside $T$. We name these intersections $a$ and $b$. We may push $\gamma$ so that it does not exit $T$. Either $\text{num}(T)$ or $\text{denom}(T)$ is prime; suppose for example that $\text{denom}(T)$ is prime. (The following argument holds for $\text{num}(T)$ prime as well.) The curve $\gamma$ inside $\text{denom}(T)$ is shown in Figure 2.8a. Here $\gamma$ intersects $\text{denom}(T)$ exactly twice; these intersections are named $a$ and $b$. Since $\text{denom}(T)$ is prime, $\gamma$ contains no crossings of $\text{denom}(T)$ on one side. If $\gamma$ contains no crossings of $\text{denom}(T)$ on the inside, then $\gamma$ also contains no crossings of $D$ on the inside, and we are done. Suppose therefore that $\gamma$ does contain crossings of $\text{denom}(T)$ on the inside. Then $\gamma$ contains no crossings of $\text{denom}(T)$ on the outside.

![Figure 2.7](image)

![Figure 2.8](image)

Looking back at Figure 2.8a, we see that there must be an arc $\beta_1$ in $T$ that runs from $a$ to one of the four corners of $T$. Suppose for now that $\beta_1$ runs from $a$ to the NW corner. Similarly, there must be another arc $\beta_2$ in $T$ running from $b$ to another of the corners of $T$. Suppose for example that $\beta_2$ runs from $b$ to the NE corner. Then there must be a third arc, $\beta_3$, that runs from the SE to the SW corner of $T$. Note that $\beta_3$ may not intersect $\gamma$ or $\alpha$; and $\beta_3$ may not cross $\beta_1$ or $\beta_2$ because
\( \gamma \) contains no crossings of \( \text{denom}(T) \) on the outside. Then \( \beta_3 \) must run straight across the bottom of \( T \), like in Figure 2.8b. But this is impossible for a nontrivial tangle \( T \). So \( \beta_2 \) can’t run to the NW corner. If we try \( \beta_2 \) running from \( b \) to the SE corner, we get a similar contradiction; in that case we get \( \beta_3 \) running from the NE to SE corners along the east side of \( T \). Finally then we try \( \beta_2 \) running from \( b \) to the SE corner of \( T \), like in Figure 2.8c. But then the NE and SW corners must be connected by an arc \( \beta_3 \). The arc \( \beta_3 \) cannot intersect \( \gamma \) or \( \alpha \), and may not cross \( \beta_1 \) or \( \beta_2 \). This is clearly impossible.

This was all supposing that \( \beta_1 \) runs from \( a \) to the NW corner of \( T \). However, the diagram is rotationally symmetric inside \( \alpha \), so the same argument would hold no matter which corner \( \beta_1 \) runs to. Thus the whole situation is impossible. In other words, it must be the inside of \( \gamma \) that contains no crossings of \( \text{num}(T) \).

**Case 2.** Suppose \( \gamma \) intersects \( D \) twice outside of \( T \). We name these intersections \( a \) and \( b \). We may push \( \gamma \) so that it does not enter \( T \). Let \( R_i \) be regions of the projection plane as shown in Figure 2.9a. At the intersection \( a \), \( \gamma \) runs from one of the regions \( R_i \) into another of the \( R_i \). For example, suppose \( \gamma \) runs from \( R_1 \) to \( R_4 \). Now, since \( \gamma \) only intersects \( D \) twice, then \( \gamma \) also runs between \( R_1 \) and \( R_4 \) at the intersection \( b \). Then the rest of \( \gamma \) consists of an arc in \( R_1 \) from \( a \) to \( b \) and an arc in \( R_4 \) from \( a \) to \( b \). Figures 2.9b and 2.9c show two ways the arc in \( R_4 \) may run from \( a \) to \( b \). In either case, \( \gamma \) contains no crossings of \( D \) on either the inside or the outside.

![Figure 2.9](image)

If \( \gamma \) moves from \( R_1 \) to \( R_2 \) at \( a \), then the situation is simpler. By a similar argument as above, \( \gamma \) consists of an arc in \( R_1 \) running from \( a \) to \( b \) and an arc in \( R_2 \) running from \( a \) to \( b \). This situation is shown in Figure 2.10; here \( \gamma \) contains no crossings on the inside.

The cases where \( \gamma \) runs from \( R_3 \) to \( R_4 \) or \( R_3 \) to \( R_2 \) are analogous to the above two cases. Also,
\( \gamma \) can’t run between \( R_2 \) and \( R_4 \) or between \( R_1 \) and \( R_3 \), since those pairs are not adjacent. Thus we have analyzed all the cases for \( \gamma \). In each case, \( \gamma \) contains no crossings of \( D \) on one side.

**Case 3.** Suppose \( \gamma \) intersects \( D \) once inside \( T \) and once outside \( T \). Then \( \gamma \) moves from \( T \) to one of the regions \( R_i \) (where these regions are as in Figure 2.9a). Suppose for example \( \gamma \) enters \( R_1 \). From there, \( \gamma \) must intersect \( D \) to enter either \( R_2 \) or \( R_4 \). Suppose for example \( \gamma \) enters \( R_2 \). Now \( \gamma \) must enter \( T \) again, and an arc inside \( T \) turns \( \gamma \) into a simple closed curve. See Figure 2.11a. We can see here that on the inside, \( \gamma \) does not contain any crossings of \( D \) outside of \( T \). Thus we can push \( \gamma \) out of \( R_1 \) and \( R_2 \) so that it follows \( \alpha \) just inside \( T \) (Figure 2.11b). But now we are back to the case that \( \gamma \) intersects \( D \) twice inside \( T \). We’ve already proved the desired result for that case.

Here we assumed \( \gamma \) enters both \( R_1 \) and \( R_2 \). This argument clearly holds if \( \gamma \) enters \( R_2 \) and \( R_3 \) as well. However, if \( \gamma \) enters \( R_4 \) and one of \( R_1 \) or \( R_3 \), we may have the situation in Figure 2.12a. In this Figure \( \gamma \) enters \( R_4 \) and \( R_1 \); and on the inside, \( \gamma \) does contain a crossing of \( D \) outside of \( T \). We label the intersection of \( \gamma \) and \( D \) that occurs inside \( T \) as \( a \). We create a new simple closed curve \( \gamma' \) that follows \( \gamma \) along its original arc through \( T \). However, replace the arc of \( \gamma \) outside \( T \) with an arc that follows \( \alpha \) just inside \( T \). There are two choices for how to create this new arc of
α’; choose the one that only intersects $D$ once (at the northwest corner of $T$ in this case.) Now $\gamma'$ is as in Case 1. Therefore as we proved in Case 1, $\gamma'$ contains no crossings of $T$ on the inside. This shows that $\gamma$ contains no crossings of $D$ on the outside.

![Diagram](a)

![Diagram](b)

Figure 2.12

We assumed here that $\gamma$ entered $R_4$ and $R_1$. If $\gamma$ enters $R_4$ and $R_3$, a similar analysis holds. Therefore we have looked at each possibility for $\gamma$, and in each case $\gamma$ contains crossings of $D$ only on one side.

We conclude that $D$ is prime.

**Lemma 16.** If $T$ is a reduced admissible diagram of a rational tangle with at least two crossings, then either $num(T)$ or $denom(T)$ is prime.

**Proof.** We prove this by induction on the number of bands of crossings in $T$. Suppose $T$ has only one band of crossings (either a horizontal or vertical band). Then this band has $q \geq 2$ crossings. If it is a vertical band, then $num(T)$ is a $(2, q)$ torus knot which is prime by [11]. If it is a horizontal band, then $denom(T)$ is a $(2, q)$ torus knot.

Now assume that if $T$ is a reduced admissible diagram of a rational tangle with $k \in \mathbb{N}$ bands of crossings, and with at least two crossings, then either $num(T)$ or $denom(T)$ is prime. Let $R$ be a reduced admissible diagram of a rational tangle with $k + 1$ bands of crossings. Then $R$ may be obtained by taking a rational tangle with $k$ bands of crossings, $R_k$, and adding the last band of crossings $S$ either on the north, the east, or the west. Assume first that $R_k$ has at least two crossings, so that either $num(R_k)$ or $denom(R_k)$ is prime by inductive assumption.

If the last band of crossings of $R$ occurs on the east or west, then $R$ is obtained as a sum of tangles; either $R_k + S$, or $S + R_k$, which are equivalent by flyping. If the last band of crossings
occurs on the north, define $R'_k$ to be a rotation of $R_k$ $90^\circ$ clockwise. Then $R$ is obtained as the sum $R'_k + S$. See Figure 2.13. Notice that $R'_k$ still has either numerator or denominator closure prime. Notice also that the numerator closure of $R'_k + S$ is the same as the denominator closure of $R$.

![Figure 2.13](image)

Now, if $S$ contains at least two crossings, then either $\text{num}(S)$ or $\text{denom}(S)$ is prime by the base case. Then Lemma 14 implies that $R$ has prime numerator closure. If $S$ has only one crossing, then $R$ has prime numerator closure by Lemma 15.

This proves the inductive step under the assumption that $R_k$ has at least two crossings. Now suppose $R_k$ has only one crossing. If $S$ has at least two crossings, then either $\text{num}(S)$ or $\text{denom}(S)$ is prime by the base case. Then $R$ has prime numerator closure by Lemma 15. If $S$ has only one crossing, then our tangle sum is as in Figure 2.14. Notice that since the diagram is reduced, both of these crossings must be the same sign. This is equivalent to a single twist region with two crossings. It has either numerator or denominator closure prime by the base case.

![Figure 2.14](image)

**Corollary 17.** A reduced, admissible diagram of a Montesinos link which has at least two tangles is prime.
Proof. Let $D$ be a reduced, admissible diagram of a Montesinos link $K$ which is the numerator closure of the cyclic sum of tangles $T_1 + \ldots + T_n$, where $n \geq 2$. If each of $T_i$ contains at least two crossings, then each of $T_i$ has either numerator or denominator closure prime by Lemma 16. Then by Lemma 14, $D$ is prime.

Suppose none of the $T_i$ contain at least two crossings. Then $T_1 + \ldots + T_n$ is equivalent to a single horizontal band of $n$ crossings. (See Figure 2.14.) Then the numerator closure is a $(2,q)$ torus knot, which is prime by [11].

Otherwise, at least one of the $T_i$ contains at least two crossings, but at least one of the $T_i$ contains only one crossing. There exist $T_n$ and $T_m$ adjacent in the cyclic sum such that $T_n$ contains at least two crossings but $T_m$ contains a single crossing. We proved in Lemma 15 that $T_m + T_n$ has prime numerator closure. Then combine $T_m + T_n$ into a single “rational tangle” in the cyclic sum. This is not strictly a rational tangle, but it satisfies the properties needed in Lemma 14 and 15. That is, $T_m + T_n$ has prime numerator closure; and it is nontrivial in the sense that it may not have a single arc running straight along one side from one corner to an adjacent corner (as in Figures 2.8b for example). We can combine each of the $T_i$ which contains only one crossing into an adjacent tangle in this way, until we are back to the case that each of the “rational tangles” in the new cyclic sum has at least two crossings. Then we can apply Lemmas 16 and 14 to show that $D$ is prime.

2.2.1 Montesinos Links of Interest. Futer, Kalfagianni, and Purcell found a volume estimate for Montesinos links with at least three positive or three negative tangles. A Montesinos link with only one tangle has an alternating diagram; its volume is bounded by Lackenby [9]. Also, Champanerkar and Ording found that a Montesinos link with two tangles has a diagram with only one tangle [3]. Thus a Montesinos link with two tangles also has an alternating diagram, and its volume is already bounded. This means the following are the only types of Montesinos links whose volumes are not yet estimated:

(a) Montesinos links with two positive and one negative tangles

(b) Montesinos links with one positive and two negative tangles

(c) Montesinos links with two positive and two negative tangles
Notice that the mirror image of a type (b) link is a type (a) link; and taking the mirror will not change the volume of the complement. Thus we may ignore type (b) links in favor of type (a) links in our analysis. Notice also that there is only one “arrangement” of a type (a) link, up to cyclic permutation. However there are two arrangements of type (c) links, as seen in the definition below.

**Definition 18.** A $++-$ link is Montesinos link which is the cyclic sum of $T_a, T_b, T_c$, where $T_a$ and $T_b$ are positive tangles and $T_c$ is a negative tangle. A $+-+$ link is a Montesinos link which is the numerator closure of the sum $T_a + T_b + T_c + T_d$, where $T_a$ and $T_c$ are positive tangles and $T_b$ and $T_d$ are negative tangles. A $++--$ link is a Montesinos link which is the numerator closure of the sum $T_a + T_b + T_c + T_d$ where $T_a$ and $T_b$ are positive tangles and $T_c$ and $T_d$ are negative tangles.

Our goal is to find volume bounds for these three types of Montesinos links. We take Definition 18 as the definition not only of $++-$ and $+-+-$ links, but also of the tangles $T_a$, $T_b$, $T_c$ and $T_d$. Notice that the definitions of $T_a$, $T_b$, and $T_c$ change depending on whether we are talking about $++-$, $+-+-$, or $++--$ links.

Fortunately, by the following Lemma we do not need to study $+-++$ and $++--$ links separately.

**Lemma 19.** A $++--$ link is a mutation of a $+-++$ link.

**Proof.** We use terminology from Ruberman, [16].

![Figure 2.15: from [16]](image)

**Definition 20.** Let $K \subset S^3$ be a knot or link. A Conway sphere for $K$ is an embedded 2-sphere meeting $K$ transversely in 4 points. Call this set of four points $F$. 

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The surface $S^3 - F$ admits the orientation-preserving involutions depicted in Figure 2.15. That is, it can be rotated $180^\circ$ about one of three axes.

If $S$ is a Conway sphere, write $(S^3, K) = (B^3_+, K_+) \cup (B^3_-, K_-)$ where $K_\pm = B^3_\pm \cap K$.

**Definition 21.** Let $\mu$ be any of the involutions in Figure 2.15. Then the mutation of $K$ via $\mu$ is $(S^3, K^\mu) = (B^3_+, K_+) \cup \mu(B^3_-, K_-)$.

Consider a $++--$ diagram. This admits a Conway sphere that separates a positive and negative tangle. Its intersection with the projection plane is the red curve depicted in Figure 2.16a. Rotating this sphere $180^\circ$ in the direction indicated takes positive crossings to positive crossings, and vice versa. (You can convince yourself of this by viewing Figure 2.1 upside-down.) Therefore rotating the whole sphere results in Figure 2.16b.

![Figure 2.16](image)

(a) A $++--$ link  
(b) A $+-++$ link

Notice that reduced diagrams for $++-$ and $+-++$ tangles do not satisfy part (1) of Definition 11; therefore they must satisfy (2). This means that we can effectively “throw out” the integer parts of the slopes of each tangle. In other words: If $T_a$ has slope vector $(a_n, a_{n-1}, \ldots, a_1)$, then we may assume that $a_n = 0$. Recall from section 2.1 that $a_n$ corresponds to a horizontal band of crossings; so $a_{n-1}$ corresponds to a vertical band of crossings.

The most general form of a reduced admissible diagram for a $++-$ tangle is shown in Figure 2.17; for a $+-++$ tangle, in Figure 2.18. In these diagrams, $a_{n-1}$ must correspond to a vertical band of crossings. However, $a_1$ may correspond to a vertical or horizontal band. Also, $a_i \neq 0$ for $i < n$. The same holds true for the $b_i$, $c_i$, and $d_i$.
Recall from Theorem 1 that Agol, Storm, and Thurston give a lower bound for the volume of a closed hyperbolic manifold $M$ in terms of $\text{guts}(M \setminus \Sigma)$ where $\Sigma$ is an incompressible surface. For the special case where $M$ is a hyperbolic knot complement, Futer, Kalfagianni, and Purcell identify an incompressible surface $S_A$ and calculate $\text{guts}(M \setminus S_A)$. Here we briefly describe the incompressible surface $S_A$; all details may be found in [6, Chapter 2-3].
3.1 The Polyhedral Decomposition

Given a link diagram $D$ and a crossing $x$ of $D$, we define two new diagrams by replacing the crossing $x$ with either its $A$-resolution or its $B$-resolution. These resolutions replace the crossing with two disjoint arcs. The new diagrams have one fewer crossing than the original diagram $D$. See Figure 3.1.

![Figure 3.1](image)

**Definition 22.** A state $\sigma$ is a choice of $A$- or $B$-resolution at each crossing. Applying a state $\sigma$ to a diagram $D$ yields a crossing-free diagram called $s_\sigma$. This diagram is consists of disjoint simple closed curves called state circles. To $s_\sigma$ we attach edges as shown in red in Figure 3.1. This forms a trivalent graph $H_\sigma$. In $H_\sigma$, the state circles are connected by edges we call segments, which correspond to the red lines in Figures 3.1b and 3.1c.

We are concerned mainly with the all-$A$ state, which chooses the $A$-resolution at each crossing. We will occasionally mention the all-$B$ resolution as well. The graph $H_A$ is obtained by applying the all-$A$ state to $D$. See Figure 3.2.

**Definition 23.** Let $K$ be a link in $S^3$ and let $D$ be a diagram for $K$. Two crossings in $D$ are called twist equivalent if there is a simple closed curve in the projection plane that meets $D$ exactly at those two crossings. The twist number of $D$ is the number of equivalence classes of crossings. A twist region is a chain of crossings between two strands of $K$.

**Definition 24.** Let $R$ be a twist region in the diagram $D$, and assume $R$ contains $c_R > 1$ crossings. Either the $A$- or the $B$-resolution of $R$ contains $c_R$ parallel edges, only one of which survives in the reduced state graph (see Figure 3.3). This is called the short resolution of $R$. We say that $R$ is an $A$-region if the $A$-resolution is the short resolution.

We also make the following definition:
Definition 25. From $H_A$ we create the A-state graph $\mathbb{G}_A$ by shrinking each state circle to a single point. We obtain the reduced A-state graph $\mathbb{G'}_A$ by removing multiple edges between pairs of vertices in $\mathbb{G}_A$. See figure 3.2.

Recall that the term $\chi_-(\mathbb{G'}_A)$ appears in Theorem 2. We have just defined $\mathbb{G'}_A$, so now we give the following definition.

Definition 26. Let $Y$ be a compact cell complex, with connected components $Y_1, \ldots, Y_n$. Then the negative Euler characteristic of $Y$ is defined:

$$\chi_-(Y) = \sum_{i=1}^{n} \max\{-\chi(Y_i), 0\}$$

The following Definition and Theorem are by Lickorish and Thistlethwaite.
Definition 27. A link diagram $D(K)$ is called A-adequate if $G_A$ has no 1-edge loops, and B-adequate if $G_B$ has no 1-edge loops.

Theorem 28 (Lickorish and Thistlethwaite, [10]). Let $D(K)$ be a reduced Montesinos diagram with $r > 0$ positive tangles and $s > 0$ negative tangles. Then $D(K)$ is A-adequate if and only if $r \geq 2$ and B-adequate if and only if $s \geq 2$. Since $r + s \geq 3$ in a reduced diagram, $D$ must be either A- or B-adequate.

Also note that if $r = 0$ or $s = 0$ then $D(K)$ is alternating, in which case it is both A- and B-adequate. We will see shortly why adequacy is a desirable property.

From $H_A$ we may also obtain a surface as follows. The state circles of $H_A$ bound disjoint disks in the 3-ball below the projection plane. To these disks, attach a half-twisted band corresponding to each crossing in the original diagram (Figure 3.4). This forms a connected surface called the A-state surface, or simply $S_A$.

![Figure 3.4: The state surface $S_A$ corresponding to the link in figure 3.2](image)

Theorem 29 (Ozawa, [14]). Let $D$ be a (connected) diagram of a link $K$. Then the surface $S_A$ is essential in $S^3 \setminus K$ if and only if $D$ is A-adequate.

Define the manifold with boundary $M_A = S^3 \setminus S_A$, where $S^3 \setminus S_A$ is defined to be $S^3$ cut along (a regular neighborhood of) $S_A$.

Definition 30. Let $M = S^3 \setminus K$, and define the parabolic locus $P = \partial M_A \cap \partial M$. The parabolic locus consists of annuli.

Futer, Kalfagianni, and Purcell cut $M_A$ into ideal polyhedra. We won’t describe the details of this cutting here; we will concern ourselves only with the results. The cutting produces finitely
many polyhedra that lie below the projection plane; and a single polyhedron above, which we call the *upper polyhedron*. We only need to study the upper polyhedron for our purposes.

To visualize the upper polyhedron, start with the state graph $H_A$. Recall that $H_A$ lies in the projection plane, and is composed of state circles and segments. We call a given state circle $S$ *innermost* if $S$ bounds a region in the projection plane which doesn’t contain any edges of $H_A$. We shade each innermost disk a different color. These are the *shaded faces* of the upper polyhedron. See Figure 3.5a.

![Figure 3.5a: Building the upper polyhedron](image)

(a) Color the innermost circles
(b) Erase small holes
(c) Draw tentacles

Figure 3.5: Building the upper polyhedron

The faces extend from the innermost state circles as follows. Given a segment $s$ of $H_A$, rotate $H_A$ so that $s$ is vertical. There are two distinct ways to perform this rotation; the procedure that follows is independent of your choice. Once $s$ is vertical, erase a small part of the graph immediately northeast of $s$ and a small part immediately southwest of $s$. Repeat this rotation and erasing for each segment in the graph. See, for example, Figure 3.5b.

Finally, draw the “tentacles.” Choose a segment $s$ that meets one of the innermost state circles. The innermost state circle bounds a shaded face. Rotate $H_A$ so that $s$ is vertical with the shaded face on the top. The small hole to the northeast of $s$ acts as a “gate”, allowing the shaded face to run through the hole, forming a *tentacle*. The tentacle runs in a thin band along $H_A$ adjacent
to the initial segment, and across the top of the state circle to the south. It terminates when it runs into a segment on the same side of the state circle. However, the tentacle may run past other segments without terminating if they lie on the opposite side of the state circle. When this occurs, the tentacle spawns a new branch, running through the hole in $H_A$. These new branches are part of the tentacle. Each new branch also terminates when it hits a segment on the same side of the state circle, and also spawns other branches when it runs past a segment on the opposite side. Continue to draw the tentacle with all its appendages until each branch of the tentacle has terminated. Now one shaded face is complete. Repeat this process of tentacle-drawing for each innermost disk. Figure 3.5c shows a completed diagram of the upper polyhedron.

**Definition 31.** For a given face, or just the tentacles of the face, we will say the face originated in the innermost state circle which the tentacles came from. The place where a tentacle terminates by running into a segment is called a tail of the tentacle. The place where a tentacle spawns a new branch by running past a segment is called a head of the tentacle. We say an arc through a tentacle is running downstream if it runs from head to tail.

The diagram in Figure 3.5c depicts tails of $P_A$ rather roughly. See Figure 3.6 for a more accurate picture. We will often draw tentacles on top of $H_\sigma$ without drawing the small holes next to each segment (see for example Figure 3.7). We use diagrams such as Figure 3.7 as schematics, to demonstrate where the tentacles originating in a given state circle may end up.

![Figure 3.6: From [6]](image)

Futer, Kalfagianni, and Purcell prove that this process does indeed produce an ideal polyhedron in $S^3 \setminus K$, with ideal vertices on the parabolic locus. The ideal vertices correspond to remnants of the graph of $H_A$ after following the above process. Thus the parabolic locus consists of remnants of $H_A$. 

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Lemma 32 (Futer, Kalfagianni, and Purcell, 3.13 of [6]). Let $D(K)$ be an $A$-adequate link diagram. Then $P_A$ as described above is a checkerboard colored ideal polyhedron with 4-valent vertices.

Here “checkerboard” means that white faces never share an edge, and neither do colored faces. After erasing the small holes as in Figure 3.5b, each connected piece of the remnants of the knot is one ideal vertex of $P_A$. The fact that this ideal vertex is meets two white faces and two colored faces can be seen in Figure 3.6.

The careful reader may notice we have not discussed the non-prime arcs in the Futer, Kalfagianni, and Purcell polyhedral decomposition. This is because in Montesinos links, non-prime arcs only occur between adjacent negative tangles. (See Lemma 8.7, [6]). We focus mainly on $++-$ and $+-++$ links, which will not have any non-prime arcs.

The following two lemmas describe the upper polyhedron for $++-$ and $+-++$ tangles.

Lemma 33. Let $K$ be a $++-$ tangle. Then $K$ has a reduced, admissible diagram $D$, and $P_A$ is as in Figure 3.7. In particular, it has the following properties.

- There is an innermost disk, denoted $G$, between the two positive tangles $T_a$ and $T_b$.
- There are $a_{n-1} > 0$ horizontal segments at the north of $T_a$ which run all the way across $T_a$ from east to west. Similarly, there are $b_{n-1} > 0$ horizontal segments at the top of $T_b$ which run all the way across $T_b$ from east to west. We call these horizontal segments railroad tracks.
- If $T_a$ has no state circles, then all white faces in $T_a$ are bigons. Otherwise, there are $a_{n-1}$ bigons at the north of $T_a$, and non-bigons below these. The same holds for $T_b$.
- There is exactly one segment at the north of $T_c$. If $T_c$ has only one state circle, then all white faces in $T_c$ are bigons.

Proof. This follows from Figure 2.17, by applying the $A$-resolution to each crossing. Notice that in a positive tangle, all the segments added when applying the $A$-resolution are horizontal. In a negative tangle, all the segments are vertical. Also, in a positive tangle, a vertical band of crossings turns into a stack of horizontal segments; a horizontal band of crossings turns into a horizontal chain of state circles connected by segments. In a negative tangle, a vertical band of crossings turns into a vertical chain of state circles connected by segments; a horizontal band of crossings turns
into a stack of vertical segments. Once we have found the general form of $P_A$, it is easy to see where the non-bigon white faces occur.

Lemma 34. Let $K$ be a $+--+$ tangle. Then $K$ has a reduced, admissible diagram $D$, and $P_A$ is as in Figure 3.8. In particular, the following properties hold.

- There are $a_{n-1} > 0$ horizontal segments at the north of $T_a$ which run all the way across $T_a$ from east to west. Similarly, there are $c_{n-1} > 0$ horizontal stripes at the top of $T_c$. We call these horizontal segments railroad tracks.

- If $T_a$ has no state circles, then all white faces in $T_a$ are bigons. Otherwise, there are $a_{n-1}$ bigons at the north of $T_a$, and non-bigons below these. The same holds for $T_c$.

- There is exactly one segment at the north of $T_b$. If $T_b$ has only one state circle, then all white faces in $T_b$ are bigons. The same holds for $T_d$.

Proof. The proof is similar to that of the previous Lemma. We apply the A-resolution to each crossing in Figure 2.18.

Notice in these figures that not all the colored faces have been colored in. Most of the colored faces stay inside the tangle they originated in (see for example the yellow faces in both figures). We have colored the faces which may travel through multiple tangles.
3.2 Essential Product Disks

We are concerned with essential product disks in the upper polyhedron.

**Definition 35.** An essential product disk in the upper polyhedron (EPD for short) is a properly embedded essential disk in $P_A$ whose boundary meets the parabolic locus twice.

![Figure 3.9: A portion of the upper polyhedron](image)

(a) The boundary of an EPD $E$ in $P_A$  
(b) $\partial E$ pulled into a normal square

We shall generally only consider the boundary of an EPD, which we will represent in the diagram of $P_A$. Figure 3.9a depicts a portion of the diagram of $P_A$, with the boundary of an EPD. Given an EPD $E$ in $P_A$, we shall generally pull $\partial E$ into a normal square.

**Definition 36.** A normal square in $P_A$ is a disk $F \subset P_A$ such that

(i) $\partial F$ is a simple closed curve in $\partial P_A$;
(ii) $\partial F$ cannot enter and leave an ideal vertex through the same face of $P_A$;

(iii) $F$ intersects any face of $P_A$ in arcs, rather than simple closed curves;

(iv) no such arc can have endpoints in the same ideal vertex of a polyhedron, nor in a vertex and an adjacent edge;

(v) $\partial F$ consists of four such arcs.

We will only need to use the special normal squares described in the next Lemma.

**Lemma 37** (Lemma 6.1 of [6]). Let $D$ be a diagram of a Montesinos link, with an EPD $E$ embedded in the upper polyhedron. Then $\partial E$ can be pulled off the parabolic locus to give a normal square in $P_A$ such that:

1. Two opposite edges of the square run through shaded faces, which we color green and orange.

2. The other two edges of the square run through white faces, and cut off a single vertex of the white face. These are the white edges of the normal square.

3. The single vertex of the white face that is cut off by a white edge of $\partial E$ forms a triangle, so that when moving clockwise the edges of the triangle are colored green-white-orange.

Given $E$ an EPD in $P_A$, two edges of the square run through shaded faces; we may choose how we color these faces green and orange. Then by property (3) of Lemma 37, we may force the triangles formed by the white faces to be oriented accordingly. Notice that the orientation given in property (3) will force the white edges to occur at tails of green and heads of orange. Figure 3.9b depicts a normal square in $P_A$ oriented as in the above Lemma.

The careful reader may notice that in [6] the above Lemma is stated with *counter-clockwise* replacing *clockwise* in part (3). The two statements are equivalent, by simply swapping the coloring of the orange and green faces.

### 3.3 Complex Essential Product Disks

Recall that in Theorem 2, Futer, Kalfagianni and Purcell calculated the “guts” of the essential surface $S_A$ for an $A$-adequate knot. The term $|E_c|$ appears in this theorem; we can define it now.
We will need further information about essential product disks in $P_A$ and their relationships to each other.

**Definition 38.** Let $S$ be a surface in $P_A$. A parabolic compression disk for $S$ is an embedded disk $E$ in $P_A$ such that:

(i) $E \cap S$ is a single arc in $\partial E$;

(ii) The rest of $\partial E$ is an arc in $\partial P_A$ that has endpoints disjoint from the parabolic locus $P$ and that intersects $P$ in exactly one transverse arc;

(iii) $E \cap S$ is not parallel in $S$ to an arc in $\partial S$ that contains at most one component of $S \cap P$.

Figure 3.10a shows a portion of the diagram of $P_A$. The dotted line represents the boundary of an EPD $D$. There is a parabolic compression disk $E$ for $D$. The thick black line shows the arc as in part (ii) of the above definition.

![Diagram](image)

(a) A parabolic compression arc  
(b) Two new EPDs

Figure 3.10: Parabolic compression

**Definition 39.** If $D$ is an EPD in $P_A$ with a parabolic compression disk $E$ for $D$, then tubing $D$ to the remnants of the knot along $E$ will produce a pair of new essential product disks, $D'$ and $D''$. We say that $D$ and $D' \cup D''$ are equivalent under parabolic compression.

Figure 3.10b shows the two new EPDs $D'$ and $D''$ obtained from the disk $D$ in Figure 3.10a. These new disks are equivalent to $D$ under parabolic compression.

**Definition 40.** An essential product disk $D$ in $P_A$ is called

1. simple if $D$ is the boundary of a regular neighborhood of a white bigon face of $P_A$. 

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2. semi-simple if $D$ is equivalent under parabolic compression to a union of simple disks (but $D$ is not simple).

3. complex if $D$ is neither simple nor semi-simple.

We see that the EPDs in Figure 3.10b are simple, and the EPD in Figure 3.10a is semi-simple. The following Lemma identifies semi-simple EPDs more generally.

**Lemma 41.** Let $D$ be an essential product disk in $P_A$. Then $D$ is semi-simple if it bounds a region in the projection plane that does not contain any non-bigon white faces of $P_A$.

**Proof.** Given an EPD $D$ in $P_A$, $\partial D$ runs through two colored shaded faces as in Figure 3.11a. Here the red line represents $\partial D$, and the black lines represent edges of $P_A$. By Lemma 32, $\partial D$ can only move between two colored faces at ideal vertices, since colored faces cannot share an edge. Suppose the “inside” of $\partial D$ is a region that does not contain any non-bigon white faces. This means that the white face to the right of the ideal vertex on the left of Figure 3.11a must be a bigon. This bigon is shown in Figure 3.11b. The dotted black line in Figure 3.11b shows a parabolic compression arc corresponding to a parabolic compression disk for $D$. Therefore $D$ is equivalent under parabolic compression to two new EPDs $D_1$ and $D_2$, whose boundaries are shown in Figure 3.11c. We can repeat this process on the disk to the right in Figure 3.11c, compressing off another bigon. There are finitely many bigons in this region, so we continue until there is only one left. Then $D$ is equivalent under parabolic compression to the union of the simple EPDs in Figure 3.11d. □

**Lemma 42** (Lemma 5.8 of [6]). There exists a set $E_s \cup E_c$ of essential product disks in $P_A$ such that

1. $E_s$ is the set of all simple disks in $P_A$.

2. $E_c$ consists of complex disks. Further, $E_c$ is minimal in the sense that no disk in $E_c$ is equivalent under parabolic compression to a subcollection of $E_s \cup E_c$.

3. $\|E_c\|$ is also maximal in the sense that if any complex disks were added to $E_c$, then $E_c$ would no longer be minimal.
We take Lemma 42 as a definition of the minimal set $E_c$. It is not easy to tell directly from definitions 38, 39, and Lemma 42 whether a set of complex disks $E_c$ is minimal. The following Lemma provides an easier way to characterize minimality.

**Lemma 43.** Let $E_s$ be the set of all simple EPDs in $P_A$, and let $E_1$ and $E_2$ be EPDs in $P_A$. Since $\partial E_1$ divides the projection plane into two components, it also partitions the white faces of $P_A$ into two sets. We call these sets $B_1$ and $B_2$. Let $C_1$ and $C_2$ be a partition of the white faces similarly defined for $\partial E_2$. Then $E_2$ is equivalent under parabolic compression to a subcollection of $E_s \cup E_1$ if one of the $B_i$ differs only by bigons from one of the $C_i$.

**Proof.** We may assume without loss of generality that $B_1$ differs from $C_1$ only by bigons. Define $G = (B_1 \cup C_1) \setminus (B_1 \cap C_1)$, so that $G$ consists of bigons.

Since $\partial E_1$ divides the projection plane into two connected components, let $S_1$ be the component that contains all the faces of the set $B_1$ and $S_2$ be the component that contains all the faces of the set $B_2$. Similarly for $\partial E_2$, let $T_1$ and $T_2$ be the connected components of the projection plane containing the faces of $C_1$ and $C_2$, respectively.

Corresponding to $G$, we define $H = (S_1 \cup T_1) \setminus (S_1 \cap T_1)$. Then $H$ is the part of the projection plane that contains the bigons of $G$, and in fact $H$ contains no non-bigon white faces.

We consider first the case that $\partial E_1$ and $\partial E_2$ do not intersect. This situation is depicted in
Figure 3.12. Without loss of generality we may say that $T_1$ is the region “inside” of $\partial E_2$. Then there are two possibilities. In Figure 3.12a, $S_1$ is the region on the outside of $\partial E_1$; in Figure 3.12b $S_1$ is on the inside. In each of these figures, $\mathcal{H}$ is the shaded region. The situation in Figure 3.12a is easy to analyze. Both $\partial E_1$ and $\partial E_2$ bound a region of the projection plane that doesn’t contain any non-bigon white faces. Then by Lemma 41, $E_1$ and $E_2$ are both semi-simple. In this case it is clear that $E_2$ is equivalent under parabolic compression to a subset of $E_s$, hence of $E_s \cup E_1$.

The situation in Figure 3.12b must be divided into further cases. First we will suppose that $\partial E_1$ and $\partial E_2$ both run through the same two colors, green and orange, as in Figure 3.13a. In this figure the black lines represent the edges of $P_A$. Recall from Lemma 32 that $P_A$ is checkerboard colored with 4-valent ideal vertices. Thus the boundary of an EPD can only move from one colored face to another colored face at an ideal vertex; four of these vertices are shown in the figure. We focus on the left-most vertex. Since $P_A$ is checkerboard colored, the face to the right of this vertex is a white face. Therefore this face must be a bigon, as in Figure 3.13b. We can find a parabolic compression disk whose parabolic compression arc is the dotted black line in the figure. Therefore, $\partial E_1$ is parabolically compressible to a union of two essential product disks, one of which is simple (Figure 3.13c).

Now we are again in the same situation we started in. We can compress off another bigon on the same side over and over again, creating a chain of bigons. Eventually this chain must connect up with one of the other three vertices in the figure. In Figure 3.14a, the chain connects up with the other vertex on $\partial E_1$. However this creates a contradiction, since the green and orange faces are both simply-connected. (In this figure for example the green face is not simply-connected.) Therefore the chain must hook up with one of the vertices on $\partial E_2$, as in Figure 3.14b. The same
process is repeated for the other two vertices, resulting in Figure 3.14c. In this figure, we see that $E_1$ is equivalent under parabolic compression to a subset if $E_s \cup E_2$.

This was all supposing that the two EPDs ran through the same two colors. Suppose now that they do not. In Figure 3.15, $\partial E_1$ runs through green and orange but $\partial E_2$ runs through blue and purple. We start building a chain of bigons on the left as before. However, this time the chain must connect up with the other vertex of $\partial E_1$, since the other vertices have the wrong colors. This results in Figure 3.15, contradicting the fact that the green and orange faces are simply-connected. Note that the same analysis will hold even if there are only three colors in the diagram (i.e. the
blue and orange faces are the same, for instance.)

This concludes our proof in the case that \( \partial E_1 \) and \( \partial E_2 \) are disjoint. Now we must prove our result in the case that \( \partial E_1 \) and \( \partial E_2 \) do intersect. In this case the region \( \mathcal{H} \) consists of one or more connected components. Take one of these connected components, say \( H \). Then \( \partial H \) consists of an arc of \( \partial E_1 \) and an arc of \( \partial E_2 \), with two intersections of \( \partial E_1 \) and \( \partial E_2 \). Suppose these intersections both take place in the green face, as in Figure 3.16a. Since the green face is simply-connected, we can push \( E_1 \) and \( E_2 \) to remove both of these intersections.

Thus we assume that the intersections of \( \partial E_1 \) and \( \partial E_2 \) occur in two different faces, green and orange as in Figure 3.16b. Here the black lines represent edges of \( P_A \). As before, by Lemma 32, the boundary of an EPD can only move from one colored face to another at an ideal vertex of \( P_A \). Suppose \( H \) is the region in the middle of this figure, so there are no non-bigon white faces in this region. The face just underneath the vertex at the top of the figure is a white face, which must be a bigon. This is shown in Figure 3.17a. The dotted black line shows the compression arc corresponding to a parabolic compression disk for \( E_1 \). Therefore \( E_1 \) is equivalent under parabolic compression to two new EPDs, one of which is simple (Figure 3.17b). Then we can repeat this process, compressing off simple EPDs in a chain until the chain connects to the other vertex of \( P_A \).
(Figure 3.17c). Thus in this region, $E_1$ is equivalent under parabolic compression to a subset of $E_s \cup E_2$.

To conclude the proof for the case that $\partial E_1$ and $\partial E_2$ intersect, we repeat the above arguments for each connected component of $\mathcal{H}$. We either push the intersections off, or compress bigons off of $E_1$ until $E_1$ is equivalent to a subset of $E_s \cup E_2$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3_17.png}
\caption{Figure 3.17}
\end{figure}

Using Lemma 43 we may more easily find a minimal set $E_c$ and therefore calculate $||E_c||$. For certain Montesinos links, $||E_c||$ is already known.

**Theorem 44** (Futer, Kalfagianni, and Purcell; Proposition 8.16 of [6]). Let $D(K)$ be a reduced, admissible, non-alternating Montesinos diagram with at least three positive tangles. Then $||E_c|| = 0$.

### 3.4 Finding complex EPDs

We now want to bound $||E_c||$ for other classes of Montesinos links. We will do so directly by finding a minimal spanning set of complex EPDs in $P_A$. The following results show how we might begin to look for complex EPDs.

**Lemma 45.** Let $D(K)$ be a reduced, admissible, $A$-adequate, non-alternating Montesinos diagram. Let $E$ be a complex EPD in $P_A$. Then either
1. $\partial E$ runs through a negative tangle $N_i$ of slope $-1 \leq q \leq -1/2$, along segments of $H_A$ that connect a state circle $I$ to the north and south sides of the tangle. See Figure 3.18.

2. There are exactly two positive tangles $P_1$ and $P_2$, and $\partial E$ runs along segments of $H_A$ that run through $P_1$ and $P_2$ from east to west.

Proof. By [6] Corollary 6.6, $\partial E$ runs over tentacles adjacent to segments of a 2-edge loop in $G'_A$. Further, [6] Lemma 8.14 gives three possible types of 2-edge loops in $G_A$. Type (1) loops correspond to crossings in a single twist region, in which the all-$A$ resolution is the short resolution. By [6] Lemma 5.17, we may remove all bigons in the short resolution of a twist region without changing $||E_i||$. Therefore we may ignore loops of type (1). For a 2-edge loop of type (2), Lemma 8.15 proves that $\partial E$ must run through $I$. (Notice that the first paragraph in the proof of Lemma 8.15 does not use the fact that $D(K)$ has at least three positive tangles; and the first paragraph proves the part of the Lemma we want to use.) For a 2-edge loop of type (3), the desired result is clear. □

![Figure 3.18](image)

Figure 3.18: A negative tangle with slope $-1 \leq q \leq -1/2$; from [6]

Notice Lemma 45 also applies to an EPD that has been pulled into a normal square. This is because we can pull an EPD $E$ into a normal square without changing which segments $\partial E$ runs along. (See Figure 3.9).

**Definition 46.** A type (1) EPD is an EPD in $P_A$ described by Lemma 45 (1); similarly for a type (2) EPD.

Note that by Lemma 45, every complex EPD must be either type (1) or type (2). This is not true for EPDs in general. Note also that an EPD may be both types (1) and (2).

The following lemma gives further information about type (1) EPDs.
Lemma 47. Let $K$ be $++-$ or $a+-+$ Montesinos link, and let $E$ be a type (1) complex EPD in $P_A$ as in Lemma 45, with $E$ pulled into a normal square. Then the white edges of $E$ may occur only in the following places:

1. A white edge may appear on the innermost disk $I$ (Figure 3.19).

![Figure 3.19: Type 1 white edges](image)

2. If $\partial E$ runs out of $I$ to the next adjacent positive tangle, a white edge may appear at the head of a tentacle running out of the negative block containing $I$ (Figure 3.20).

![Figure 3.20: Type 2 white edges](image)

3. If $\partial E$ runs downstream from $I$ to the next adjacent positive tangle, then across a segment spanning the positive tangle east to west, and then downstream across the outer state circle of the next negative block, a white edge may appear on the next adjacent negative block (Figure 3.21). This situation cannot occur for $++-$ links.

We will call these white edges of types (1N), (1S), (2N), (2S), (3N), and (3S).
Proof. This follows from the proof of Proposition 8.16 of [6]. In this reference there are 5 types of white edges; however, types (4) and (5) require non-prime arcs in $D(K)$, and we do not have any non-prime arcs in our diagrams. Notice that for a type (3) white edge, we must have a string of tangles $-+-$ moving either east or west, where the first negative tangle is the one containing $I$. This string of tangles cannot occur in a Montesinos link with two positive and one negative tangle. Thus $++-$ links may only have types (1) and (2) white edges.

Notice that two diagrams have been given for type (1S) white edges in Figure 3.19. This is because $\partial E$ may run south down any of the segments that connect $I$ to the bottom of the negative tangle. The resulting diagram will look slightly different depending on this choice. However for our purposes, it won’t matter which segment $\partial E$ runs down, since there are bigon white faces between these segments. (See Lemma 43.)

We will be able to obtain even more information with the following argument. Let $E$ be a complex EPD, and suppose $\partial E$ runs through a given face of $P_A$ and has a vertex at a given location. At this vertex, $\partial E$ runs into another colored face of $P_A$. We can tell which faces $\partial E$ may run into at the vertex by considering the most general form for $P_A$, as in Lemmas 33 and 34. We want to know firstly which state circles the new face may have originated in. Further, there may be multiple ways for the face to run in tentacles from a given state circle to the vertex; and we need to consider each of these possibilities. The following two lemmas provide this information for $++-$ links and $+-++$ links.

**Lemma 48.** Let $K$ be a $++-$ tangle with reduced, admissible diagram $D$. Suppose $E$ is a type (1) complex EPD in $P_A$ pulled into a normal square; then $\partial E$ runs through $T_e$ from north to south.
Lemma 47 enumerates the possible locations of white edges for $\partial E$. At each of these white edges, $\partial E$ runs into another colored face of $P_A$. This new face must have originated in:

(1N) • the state circle $G$, running across $T_b$ to reach the white edge

(1S) • the lower-left-most state circle of $T_a$, if $T_a$ has at least one state circle
  • $G$, if $T_a$ has no state circles

(2N) • the lower-left-most state circle of $T_a$, if the segment meeting the white edge is the unique segment on the southeast of $T_b$ and $T_a$ has at least one state circle
  • $G$, if the segment meeting the white edge is the unique segment on the southeast of $T_b$ and $T_a$ has no state circles
  • $G$, if the segment is not the unique segment at the southeast of $T_b$ and, after originating in $G$, the face runs in a tentacle through $T_b$
  • a state circle in $T_b$, if the segment is not the unique segment on the southeast of $T_b$, and is also not one of the segments at the north of $T_b$ that run all the way across $T_b$ from east to west

(2S) • a state circle in $T_a$
  • $G$, if the face reaches the white edge by running through a horizontal tentacle in $T_a$
  • $G$, if the face reaches the white edge by running through a horizontal tentacle across $T_b$ and around the top of the diagram, if the white edge is at the far north segment of $T_a$.

Proof. By Lemma 33, $P_A$ has the general form of Figure 3.7. We run through the (1N) white edge as an example. In Figure 3.7 we can see that at a type (1N) white edge, $\partial E$ will move from the state circle $I$ (colored blue in the figure) to the tentacle colored green in the figure. The green tentacle must originate in the state circle $G$, because $T_b$ has $b_{n-1}$ segments running across it on the north side. The other cases are similar.

Lemma 49. Let $K$ be a $++-+$ tangle with reduced, admissible diagram $D$. Suppose $E$ is a type (1) complex EPD in $P_A$ pulled into a normal square. Then $\partial E$ must run from north to south through a negative tangle; suppose $\partial E$ runs through $T_d$. Lemma 47 enumerates the possible locations of white
edges for $\partial E$. At each of these white edges, $\partial E$ runs into another colored face of $P_A$. This new face must have originated in:

(1N) or (3S)  
- the lower-right-most state circle in $T_b$

(1S)  
- the lower-left-most state circle in $T_a$
- the uppermost state circle in $T_b$, if $T_a$ has no state circles

(2N)  
- the lower-left-most state circle in $T_a$, if the white edge occurs at the unique segment at the southeast of $T_c$
- the uppermost state circle in $T_b$, if $T_a$ has no state circles and if the white edge occurs at the unique segment at the southeast of $T_c$. In both of these cases the tentacle reaches the (2N) white edge by running all the way across the bottom of the diagram.
- a state circle in $T_c$, if the white edge is not at the unique southeast segment of $T_c$ and the tentacle doesn’t run through a horizontal tentacle all the way across $T_c$
- the lower-right-most state circle in $T_b$, running along a horizontal tentacle all the way across $T_c$ to reach the white edge, otherwise.

(2S)  
- a state circle in $T_a$
- the uppermost state circle in $T_b$, if the tentacle runs along a horizontal segment all the way across $T_a$
- the lower-right-most state circle in $T_b$, if the tentacle runs from $T_b$ all the way across $T_c$ horizontally, and then over the north of $T_d$ and around the top of the diagram to $T_a$. In this case the white edge occurs on the unique northernmost segment of $T_a$.

(3N)  
- a state circle in $T_b$ that is connected by a segment to the state circle surrounding $T_b$. These are the uppermost state circle in $T_b$ and all the lower-most state circles in $T_b$.

Proof. This result follows from Lemma 34. The proof is left to the reader. □

We now have a lot of information about type (1) complex EPDs. The following argument summarizes how we will use the information to find all possible type (1) complex EPDs.
Lemma 50 (Two-Face Argument). Let $E$ be a normal square that meets shaded faces $F_1$ and $F_2$ at one white edge $W_1$, and meets shaded faces $F_1$ and $F_3$ at another white edge $W_2$. Then $F_2 = F_3$. Moreover, up to isotopy there is a unique embedded arc from $W_1$ to $W_2$ running through $F_2 = F_3$, and $\partial E$ must follow this arc.

Proof. By Definition 36, $\partial E$ meets exactly two shaded faces. Since one of these is $F_1$, then $F_2 = F_3$. Moreover, shaded faces are simply connected, so there is a unique embedded arc from $W_1$ to $W_2$ in $F_2 = F_3$. Then $\partial E$ follows this arc. \qed

Chapter 4. Bounds on $||E_c||$

Lemma 45 provides a useful method for finding complex EPDs in $P_A$. We look first for type (1) complex EPDs. Then we look for complex EPDs which are not type (1). These must necessarily be type (2). The following Lemmas use this method to find all possible complex EPDs in $P_A$ for $++-$ links. The results of the two Lemmas are combined in Proposition 53. The analysis is repeated for $+-+-$ links in Lemmas 54 and 55 and Proposition 56.

Lemma 51. Let $K$ be a $++-$ link with reduced, admissible diagram $D(K)$. Then the only possible type (1) complex EPDs in $P_A$ are those depicted in Figures 4.1b and 4.3d.

Proof. We will continually use Lemma 33. For $E$ a type (1) complex EPD in $P_A$, then by Lemma 45 $T_c$ must have slope $-1 \leq q \leq -1/2$, and $\partial E$ must run through $T_c$ from north to south as in Figure 3.18. We will pull $E$ into a normal square with two white edges (Lemma 37). Lemma 47 tells us all the possible locations for white edges of $\partial E$ with respect to the state circle $I$ in $T_c$. These possibilities are labeled (1N), (1S), (2N), and (2S). We must have one white edge north and one south of $I$. Therefore we must check only the following four combinations.

**White edges (1N), (1S).** By the notation (1N), (1S), we mean that our two white edges are described by type (1N) and type (1S) in Lemma 47. Using Figure 3.19, we see our EPD must have white edges as in Figure 4.1a. That is, $\partial E$ runs over white faces adjacent to $I$, meeting the green face in $I$, a red tentacle to the north of $I$, and a blue tentacle to the south of $I$. By the Two-Face Argument (Lemma 50), the red and blue faces must agree. We reference Lemmas 33 and Lemma 48 to see where the red and blue faces may originate. The red face originates in the state circle $G$. 41
Therefore the blue face must originate in $G$ also. This can only happen if $T_a$ has no state circles at all, by Lemma 48. Therefore we have the diagram of $\partial E$ as in Figure 4.1a. Here $\partial E$ may bound non-bigons on both sides, where indicated by $\star$. Thus we have found our first complex EPD.

Figure 4.1: An EPD with white edges type (1N) and (1S)

White edges (1N), (2S). The boundary of an EPD with white edges type (1N) and (2S) will look like 4.2a. In particular, to the north $\partial E$ runs into a red tentacle meeting $I$. To the south of $I$, $\partial E$ runs through a green tentacle extending across the bottom of $H_A$, and then meets a blue tentacle just inside $T_a$. The type (2S) white face requires that $T_c$ has only one state circle, so that there are no non-bigons in $T_c$. Lemma 48 tells us where the red and blue tentacles may originate. The blue tentacle may originate in $G$ or in a state circle in $T_a$; the red tentacle must originate in $G$. By Lemma 50 the red and blue faces must agree, so they must both originate in $G$. From $G$, the face reaches the type (1N) white edge by running across $T_b$; but it may reach the (2S) white edge by running across either $T_a$ or $T_b$. Therefore there are two possible EPDs, shown in Figures 4.2b and 4.2c. In each of these Figures the only places where non-bigons may occur in the diagram are indicated by $\star$. Each of the EPDs bounds only bigons to the outside, so they are both semi-simple.

Figure 4.2: Semi-simple EPDs with white edges type (1N), (2S)
**White edges (2N), (1S).** A normal square with white edges (2N) and (1S) will have a diagram as in Figure 4.3a. Here, $\partial E$ runs through a green tentacle from $I$ to the north of $T_c$, to meet a red tentacle just inside $T_b$. To the south of $I$, $\partial E$ runs into a blue tentacle meeting $I$. Lemma 48 tells us where the red and blue faces must have originated. Since we must pick a matching pair, there are two options: they either originated in the lower-left-most state circle of $T_a$, or in $G$ if $T_a$ has no state circles. We will color this face orange (the face that both green and blue agree with.) If the orange face originates in $T_a$ then the white face is on the unique southmost segment of $T_b$, and the orange face runs in a tentacle across the bottom of $H_A$. There exists an arc in the orange face running from the bottom of $I$ and then to the west, to meet the white face in $T_b$. By Lemma 50, $\partial E$ follows this arc. Thus we obtain the EPD shown in Figure 4.2b. This EPD may be simple as shown in this figure, or semi-simple if there are additional segments south from $I$ (see Figure 3.19). If the orange face originates in $G$, it may reach the white edge in $T_b$ either by wrapping around the bottom of $H_A$, or by traveling through a segment across $T_b$. These possibilities result in the two EPDs shown in Figures 4.3c and 4.3d respectively. The EPD shown in Figure 4.3d is complex, as it may bound nonbigon white faces on both sides where shown by $\star$. This is the second complex EPD we have found.

![Figure 4.3: EPDs with white edges type (2N), (1S)](image)

**White edges (2N), (2S).** The (2N) and (2S) white edges give a diagram as in Figure 4.4a. North of $I$, $\partial E$ runs through the green tentacle around the northwest of $T_c$ and then into a red
tentacle inside $T_b$. To the south, $\partial E$ runs through a green tentacle across the bottom of the diagram and then into a blue tentacle inside $T_a$. We use Lemma 48 to see where the red and blue faces may originate, and choose a matching pair. This case is somewhat more complicated than the previous three. By Lemma 48, the new face (which we color orange) may originate in either $G$ or the lower-left-most state circle of $T_a$. If the orange face originates in $T_a$, we obtain the EPD shown in Figure 4.4b. Here $\partial E$ runs through the orange tentacle across the bottom of $H_A$ from one white face to the other. Here the EPD is semi-simple, bounding bigons in $T_c$ and in $T_a$ on the inside.

If the orange face originates in $G$, it may reach the (2N) white edge by traveling across $T_a$ or $T_b$, and may also reach the (2S) white edge by traveling across $T_a$ or $T_b$. Therefore if the new face originates in $G$ there are four different possible diagrams, shown in Figures 4.4c - 4.4f. In particular, if the orange face travels across $T_a$ to reach both white faces, Figure 4.4c is the result. There is a unique way to connect $\partial E$ in the green and orange faces. Note that the interior of $\partial E$ consists only of bigons in $T_a$ and bigons in $T_c$. Hence this EPD is semi-simple. Figure 4.4d shows orange traveling across $T_a$ to reach the (2S) white face, and across $T_b$ to reach the (2N) white face. In this diagram $\partial E$ bounds only bigons on the outside. In Figure 4.4e the orange face travels across $T_a$ to reach the (2N) white face and across $T_b$ to reach the (2S) white face. Again there is a unique way for $\partial E$ to run through these orange tentacles from white face to white face. Notice that in this diagram non-bigon white faces occur only at the south side of $T_b$ (there can’t be any at the south of $T_a$ or at the east of $T_c$). Thus this EPD is semi-simple. Finally, Figure 4.4f shows the orange face reaching both white faces by traveling across $T_b$. Then on the outside, $\partial E$ bounds one or more bigons in $T_b$ and the large bigon on the outside of $H_a$.

All of the EPDs pictured are simple or semi-simple.

This concludes the search for type (1) complex EPDs in $P_A$. The complex EPDs we found are depicted in Figures 4.1b and 4.3d.

\begin{lemma}
Let $K$ be a ++−− link with reduced, admissible diagram $D(K)$. Then the only possible complex EPDs in $P_A$ which are not of type (1) are those shown in Figures 4.9 ($\alpha b$, $\beta b$) and 4.9 ($\alpha b$, $\beta c$)
\end{lemma}

\begin{proof}
Let $E$ be a complex EPD in $P_A$ not type (1); then $E$ must be type (2). Therefore, $\partial E$ runs along segments across both $T_a$ and $T_b$. For convenience, let’s call these two segments $\alpha$ (in $T_a$) and

\end{proof}
\( \beta \) (in \( T_b \)). Now, \( \partial E \) may run along either the north or south sides of these segments. Therefore we have four possibilities again, listed below. By Lemma 37, we may pull \( E \) into a normal square with white edges at tails of the green face, after we choose colors orange and green for faces of the EPD. We will make convenient choices in the cases below.

\( \alpha N, \beta N \). We interpret the notation \( \alpha N, \beta N \) to mean that \( \partial E \) runs along the north side of \( \alpha \) and the north side of \( \beta \). The tentacle running along \( \alpha N \) originates in the lower-right-most state circle of \( T_c \) (see Lemma 33); let’s color this face green. The tentacle running along \( \beta N \) originates in \( G \); we color this face orange. We must have \( \partial E \) running over a white face in the downstream direction from \( \beta \). That is, after running from \( G \) across \( \beta \), \( \partial E \) must meet a white edge. Also, a white face must occur at a head of an orange tentacle, by Lemma 37. Thus we look for possible heads of orange downstream from \( \beta \). From Lemma 33 we see there is only one place this can occur, at the unique segment on the north of \( T_c \) (Figure 4.5a). We must have another white face in the downstream direction from \( \alpha \). That is, \( \partial E \) must meet a white face after running across \( \alpha \) towards \( G \). Also, this white face must occur at a green tail. There are only two places for a green tail to occur downstream from \( \alpha \): on the east side of \( T_a \) (Figure 4.5b) or the northwest corner of \( T_b \) (Figure 4.5c). Note that in both cases, \( \partial E \) bounds the single bigon just north of \( G \), and possibly bigons in \( T_a \) for the EPD in Figure 4.5b. In both cases the EPD is simple or semi-simple, by Lemma 41.
\( \alpha N, \beta S \). The tentacle running along \( \alpha N \) originates in the lower-right-most state circle of \( T_c \). On the other hand, the tentacle running along \( \beta S \) originates in the upper-right-most state circle of \( T_c \). Suppose first that these state circles are the same; that is, \( T_c \) has only one state circle. Let’s color the corresponding face green. Then to find the white edges of \( \partial E \) we must look for tails of the green face, by Lemma 37. There must be a white edge downstream from \( \alpha N \); this can only occur on the state circle \( G \). Likewise there must be a white edge downstream from \( \beta S \), which can only occur on \( G \). This results in the EPD pictured in 4.6a. This EPD is of type (1). However, we are assuming \( E \) is not of type (1).

Therefore, we must assume that the upper-right-most and the lower-left-most state circles in \( T_c \) are not the same. Thus \( \partial E \) already runs through two distinct faces by running across \( \alpha N \) and \( \beta S \). If we color the face over \( \alpha N \) green, we must have a white edge at a tail of green downstream from \( \alpha \). At this white edge \( \partial E \) will jump into the face originating in \( G \), so \( \partial E \) runs through this new face. However \( \partial E \) already runs through two distinct faces which both originate in \( T_c \). Thus \( \partial E \) runs through three distinct colored faces. See Figure 4.6b. This contradicts Lemma 50.

\( \alpha S, \beta N \). The tentacles running along \( \alpha S \) and \( \beta N \) both originate in \( G \). We color this face green, as in Figure 4.7. We also include state circles in \( T_c \) because they will be relevant to the discussion.
The red state circle is the lower-right-most state circle in $T_c$; the blue is the uppermost in $T_c$. We must look for a white face downstream from $\alpha$, which must occur at a tail by Lemma 37. We have three possibilities for white faces downstream of $\alpha$:

\((\alpha a)\) The tentacle running across $\alpha S$ runs into a segment in $T_a$ and terminates. A white face occurs here in $T_a$, and meets a red tentacle.

\((\alpha b)\) The tentacle running over $\alpha$ continues across the bottom of $H_A$, until it hits a segment in $T_b$. A white edge may occur here, meeting the blue tentacle.

\((\alpha c)\) The tentacle over $\alpha S$ runs across the bottom of the diagram, and then $\partial E$ follows a new green tentacle into $T_c$. The white face is at the tip of this tentacle. At this white face $\partial E$ meets a tentacle that originates in $T_c$ (in any state circle of $T_c$ that is connected by a segment to the bottom of $T_c$).

Similarly, we must have a white edge downstream of $\beta$. This must occur at a tail of the green face. There are three possibilities:

\((\beta a)\) The tentacle running across $\beta N$ runs out of $T_b$ to the north without hitting any segments. It runs across the top of $H_A$, hits a segment in $T_a$, and terminates. A white edge may occur here, meeting a red tentacle.

\((\beta b)\) After running across $\beta N$ and heading north, the green tentacle hits a segment in $T_b$ and terminates. A white edge may occur here, meeting a blue tentacle.

\((\beta c)\) The tentacle running across $\beta N$ runs out of $T_b$ without hitting any segments. It runs over the top of $T_c$ and spawns a new tentacle on the unique segment at the north of $T_c$. It terminates when it hits a segment running south out of the blue state circle. A white edge may occur here, meeting a blue tentacle.
Notice that if there is only one state circle in $T_c$, then the red and blue faces are the same. Therefore all possible combinations of the above vertex types for $\alpha$ and $\beta$ produce EPDs. So there are nine ways these white faces could combine to produce an EPD. These nine are depicted in Figures 4.8 - 4.10. Since the red and blue faces must both agree as one new face, we color the new face orange in these figures.

Cases $(\alpha b, \beta b)$ and $(\alpha b, \beta c)$ both result in complex EPDs. Cases $(\alpha a, \beta a)$ and $(\alpha c, \beta a)$ result in semi-simple EPDs, bounding only bigons. In the other five cases, $\partial E$ runs through $T_c$ from north to south, contradicting our assumption that the $E$ is not type (1).

$\alpha S$, $\beta S$. We color the face running along $\alpha S$ green and the face running along $\beta S$ orange. We must have a white edge downstream from $\beta$, which must occur at an orange head. But the orange
tentacle terminates without producing any heads in that direction. Thus no EPD is possible in this case.

This concludes the search for complex EPDs not of type (1) in $P_A$. The complex EPDs we found are depicted in Figures 4.9 ($\alpha b, \beta b$) and 4.9 ($\alpha b, \beta c$).

Proposition 53. Let $K$ be a $++-$ link with reduced, admissible diagram $D(K)$. Then

$$||E_c|| \leq 1$$

where $||E_c||$ is the number of complex EPDs required to span $P_A$.

Proof. Recall the definition of $||E_c||$ in Lemma 42. Let $E_s$ be as in that Lemma. To obtain the desired result, we need to show that if $E_1$ and $E_2$ are two complex EPDs in $P_A$, then $E_1$ is equivalent under parabolic compression to a subset of $E_2 \cup E_s$.

In Lemmas 51 and 52, we have found all possible complex EPDs in $P_A$. They are shown in Figure 4.11 (originally from Figures 4.1b, 4.3d, 4.9 ($\alpha b, \beta b$), and 4.9 ($\alpha b, \beta c$)). Call these EPDs $E_{(a)}$, $E_{(b)}$, $E_{(c)}$, and $E_{(d)}$ respectively. We compare these EPDs pairwise.

Consider $E_{(a)}$ and $E_{(b)}$. In $T_a$, both EPDs must run across the southernmost railroad track. In $T_b$, both run across a railroad track. However, $E_{(a)}$ runs over the northernmost railroad track, and

Figure 4.11: Complex EPDs in $P_A$ for a $++-$ link
$E_{(b)}$ runs over any railroad track \textit{but} the northernmost. This doesn’t matter much; there are only bigons between the railroad tracks of $T_b$. In $T_c$, both $E_{(a)}$ and $E_{(b)}$ run from the north down the unique segment at the north of $T_c$. Then they may run down any of the segments which connect the green state circle $I$ to the south side of $T_c$ (although in these figures both EPDs are shown running down the westernmost of these segments. Recall the discussion of the type (1S) white edge after Lemma 47.) However, this choice will only change the EPDs by bigons, since there are only bigon white faces between these segments. Thus, in all of $P_A$, the sets of white faces bounded by $E_{(a)}$ and $E_{(b)}$ differ only by bigons. Thus by Lemma 43, $E_{(a)}$ is equivalent under parabolic compression to a subset of $E_{(b)} \cup E_s$.

The analysis of the other pairs is similar, and is left to the reader.

\hspace{1cm} \Box

**Lemma 54.** Let $K$ be a $a+-+-$ link with reduced, admissible diagram $D(K)$. Then the only possible type (1) complex EPDs in $P_A$ are those shown in Figures 4.12b, 4.15c, 4.20d, and 4.20e.

\textit{Proof.} We will continually use Lemma 34 to reference what the diagram of $P_A$ may look like. For $E$ a type (1) complex EPD in $P_A$, we pull $E$ into a normal square with two white edges (Lemma 37). Since $E$ is type (1), $\partial E$ runs through a negative tangle from north to south. We may assume, after possibly taking a cyclic permutation, that $\partial E$ runs through $T_d$. Lemma 47 tells us all the possible locations for white edges of $\partial E$. These possibilities are labeled (1N), (1S), (2N), (2S), (3N), and (3S). We must have one white edge north and one south. Therefore we must check nine combinations.

\textbf{White edges (1N), (1S).} By the notation (1N), (1S), we mean that our two white edges are described by type (1N) and type (1S) in Lemma 47. Using Figure 3.19, we see our EPD must have white edges as in Figure 4.12a. That is, $\partial E$ runs through the green face in the state circle $I$ in $T_d$. It runs through one white face adjacent to $I$ and then through an orange tentacle to the north; it runs through another white face adjacent to $I$ and then through a purple tentacle to the south. By the Two-Face Argument (Lemma 50), the orange and purple faces must agree. We reference Lemma 49, or Figure 3.8, to see where the orange and purple faces may originate. We reference Lemma 49, or Figure 3.8, to see where the orange and purple faces may originate. The purple tentacle must have originated in the lower-left-most state circle of $T_a$, or from the uppermost state circle of $T_b$ if $T_a$ has no state circles. The orange tentacle must have originated in the lower-right-
most state circle of \( T_b \). For these to agree, there must be only one state circle in \( T_b \), in which this face originated; and there must be no state circles in \( T_a \). This results in the diagram pictured in Figure 4.12b. This diagram may contain non-bigon white faces where indicated by \( \star \), so this EPD is complex.

![Diagram](image)

(a) (b) A complex EPD

Figure 4.12: White edges type (1N) and (1S)

**White edges (1N), (2S).** Using Figures 3.8, 3.19, and 3.20, we see that the diagram of \( \partial E \) must look like Figure 4.13a. In particular, there is one white edge adjacent to \( I \); here \( \partial E \) runs into the orange face and heads north. Also, \( \partial E \) runs in the green face around the bottom of \( H_A \) and spawns a new tentacle that runs into \( T_a \). A white edge occurs at the head of this new green tentacle, and \( \partial E \) runs into a purple tentacle here. Notice that since we have a (2S) vertex, we may only have one state circle in \( T_d \). The orange tentacle in this Figure must have originated in the lower-right-most state circle of \( T_b \), by Lemma 49. By Lemma 50, the purple tentacle must come from the same face. The purple tentacle can come from the lower-right-most state circle of \( T_b \) by traveling across \( T_c \), as in Figure 4.12b. Or it could come from the uppermost state circle of \( T_b \) by traveling across \( T_a \); in this case there must be only one state circle in \( T_b \) (by Lemma 50, the uppermost and the lower-left-most state circles of \( T_b \) must be the same). This results in Figure 4.13c. The EPD in Figure 4.13b bounds a single bigon on the outside, and so is simple. In Figure 4.13c, non-bigons can only occur where indicated by \( \star \). Notice there are only nonbigons on one side, so this EPD is semi-simple by Lemma 41.

**White edges (1N), (3S).** White edges type (1N) and (3S) produce a diagram as in Figure 4.14a. Here \( \partial E \) runs through a white face adjacent to \( I \) and into the purple face and then heads north, for the (1N) white edge. For the (3S) white edge, \( \partial E \) runs in the green face across the bottom of the diagram. On the west side of \( T_a \) a new green tentacle runs across \( T_a \) and over the top of \( T_b \). Another new tentacle spawns at the unique segment on the north of \( T_b \). A white face
occurs at the head of this tentacle, meeting an orange tentacle. Note that the (3S) white edge can only occur if $T_d$ has only one state circle. Here both the purple and orange tentacles must come from the lower-right-most state circle in $T_b$, by Lemma 49. Only one diagram is possible; see Figure 4.14b. Non-bigons may occur only where indicated by $\star$, so the EPD is semi-simple.

**White edges (2N), (1S).** A normal square with white edges type (2N) and (1S) will have a diagram as in Figure 4.15a. Here $\partial E$ runs through a white edge adjacent to $I$ into an orange tentacle running south of $I$. It runs north of $I$ through the green tentacle, over the top of $T_d$, and just into $T_c$ through a segment. A white edge is located on the east side of $T_c$, and $\partial E$ runs into a purple tentacle. By Lemma 50, the purple and orange faces must agree.

We reference Lemma 49 and Figure 3.8 to see where the orange and purple faces may have originated. The orange face may have originated in the lower-left-most state circle of $T_a$, or in the uppermost state circle of $T_b$ if $T_a$ has no state circles. However, in either case the orange tentacle reaches the (1S) white edge by running across the bottom of the diagram from $T_a$ to $T_d$. The purple tentacle may also reach the (2N) white edge by running across the bottom of the diagram. If both run across the bottom of the diagram, $\partial E$ is as in Figure 4.15b. This EPD bounds one or more bigons in $T_d$, so it is not complex. On the other hand, the purple tentacle may also originate in
the lower-right-most state circle of $T_b$ and reach the (2N) edge by running across $T_c$. In order for the purple face to agree with the orange face in this case, there must be only one state circle in $T_b$, in which the face originates. The resulting diagram is shown in Figure 4.15c. Non-bigons may be present where indicated by $\star$, so the EPD is complex.

White edges (2N), (2S). Using Figures 3.20 and 3.21, we obtain Figure 4.16a for the EPD with (2N) and (2S) white edges. Here $\partial E$ runs through $T_d$ from north to south through the green face. To the north, $\partial E$ runs around the northwest side of $T_d$ and into $T_c$ in the green face; where it enters $T_c$, there is a white edge and $\partial E$ runs into a orange tentacle. To the south, $\partial E$ runs around the bottom of $H_A$ and into $T_a$ in the green face; where it enters $T_a$, there is a white edge and $\partial E$ runs into a purple tentacle.

Notice that the (2S) white edge requires that $T_d$ has only one state circle. By Lemma 49, the purple tentacle meeting the type (2S) white edge originated either in $T_a$ or $T_b$. If $T_b$, it may originate in the uppermost state circle and run across $T_a$, or it may run across the lower-right-most state circle and run across $T_c$. We now look at the possibilities for the orange tentacle (2N). By Lemma 50, we must choose a face that matches one of the options for the purple tentacle. Thus we ignore the possibilities that are not in either $T_a$ or $T_b$. Thus we have that the orange tentacle may originate in the lower-left-most state circle of $T_a$; or in the lower-right-most state circle of $T_b$,
running across $T_c$; or in the uppermost state circle of $T_b$, running across $T_a$, if $T_a$ has no state circles. There are five possible diagrams. The first occurs if the orange face originates in $T_a$; this gives Figure 4.16b. This EPD bounds bigons in $T_a$ and $T_d$ on the inside.

There are four diagrams that arise if the orange face originates in $T_b$, based on whether the tentacles of that face run across $T_a$ or $T_c$ to reach the white edges. In Figure 4.17a, the orange tentacles reach both the (2N) and the (2S) edges by running across $T_a$. The EPD here bounds bigons in $T_a$ and $T_d$ on the inside. In Figure 4.17b, the orange tentacles run across $T_c$ to reach both white edges. Then to the outside, $\partial E$ bounds bigons in $T_c$ and the bigon on the outside of the diagram. In Figure 4.17c, one orange tentacle runs across $T_a$ to reach the (2N) edge, and one runs across $T_c$ to reach the (2S) edge. In this case the uppermost and lower-left-most state circles of $T_b$ are the same, so there is only one state circle in $T_b$. This EPD bounds bigons in $T_a$ and $T_d$ to the inside. Finally, in Figure 4.17d, one orange tentacle runs across $T_c$ to reach the (2N) edge, and one runs across $T_a$ to reach the (2S) edge. In this case there is also only one state circle in $T_b$.

Each of the five EPDs depicted in Figures 4.16 and 4.17 bound a region that contains no non-bigon white faces. By Lemma 41, they are all semi-simple.

\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{white_edges.png}
  \caption{Locations of white edges}
  \label{fig:white_edges}
\end{figure}

\textbf{White edges (2N), (3S).} White edges (2N) and (3S) give us the diagram in Figure 4.18a. In particular, $\partial E$ runs through $T_d$ from north to south in the face we color green. To the north, it runs around the top of $T_d$, to a white edge just inside $T_c$, and into a purple tentacle. To the south, it runs around the bottom of $H_A$, across $T_a$ over a segment, over the top of $T_b$, and down into $T_b$ via the unique segment at the top of $T_b$. There, it runs through a white edge and into an orange tentacle. Since we have a (3S) white edge, there can be only one state circle in $T_d$.

By Lemma 49, the orange tentacle must originate in the lower-right-most state circle of $T_b$. From this state circle in $T_b$, the (2N) white edge can be reached in two ways (see Lemma 49). The
orange face may travel across $T_c$, as in Figure 4.18b. This EPD bounds only bigon white faces to the outside. Or, if there is only one state circle in $T_b$ and no state circles in $T_a$, the orange tentacle may travel across $T_a$, as in Figure 4.18c. This EPD bounds only bigon white faces to the inside. In both figures, white non-bigons may occur where indicated by $\star$. Both EPDs are semi-simple.

Figure 4.17: The orange face originates in $T_b$

Figure 4.18: White edges type $(2N)$, $(3S)$
**White edges (3N), (1S).** The white edges appear as in Figure 4.19a. That is, there is a white edge on \( I \) in \( T_d \), where \( \partial E \) runs into an orange tentacle and heads south. To the north \( \partial E \) runs through the green face around the top of \( T_d \), across a segment of \( T_c \), around the bottom of \( T_b \) and into \( T_b \) along a segment. Just where \( \partial E \) runs into \( T_b \) there is a white edge, and \( \partial E \) runs into a purple tentacle. A (3N) white edge requires that \( T_c \) has no state circles. By Lemma 49, the purple tentacle must originate in \( T_b \). The orange tentacle must match; it may originate in the uppermost state circle of \( T_b \). For the orange tentacle, this means that \( T_a \) has no state circles. Thus we have the diagram shown Figure 4.19b. We may only have non-bigon white faces where indicated by \( \star \); so \( \partial E \) bounds only bigons to the inside. By Lemma 41, \( E \) is semi-simple.

![Diagram](a) Locations of white edges

![Diagram](b)

Figure 4.19: White edges type (3N), (1S)

**White edges (3N), (2S).** The EPD has white edges as in Figure 4.20a. In this figure \( \partial E \) runs through the green face, through \( T_d \) from north to south. To the north, \( \partial E \) runs in the green across \( T_c \) and into \( T_b \) from the south. As it enters \( T_b \) there is a white edge, and then \( \partial E \) runs into a purple tentacle. To the south, \( \partial E \) runs across the bottom of \( H_A \) and into \( T_a \) along a segment. Here it runs through a white edge and into an orange tentacle. Notice that the (3N) edge implies that \( T_c \) has no state circles, and the (2S) edges implies that \( T_d \) has only one state circle.

By Lemma 49, the purple tentacle must come from \( T_b \); either the uppermost state circle, or one of lowermost. The orange tentacle may come from the lower-right-most state circle of \( T_b \) and run across \( T_c \), or from the uppermost state circle of \( T_b \) and run across \( T_a \). There are four possible diagrams.

Figures 4.20b and 4.20c show the orange tentacle originating in the lower-right-most state circle of \( T_b \) and running across \( T_c \). In Figure 4.20b, the purple tentacle originates in the uppermost state circle of \( T_b \). Since this must agree with the orange face, there is only one state circle in \( T_b \). This EPD bounds only bigon white faces on the outside; bigons in \( T_b \) and \( T_c \) and the bigon on the outside.
of the diagram. In Figure 4.20c, the purple tentacle originates in one of the lowermost state circles of $T_b$; it must be the lower-right-most state circle, to agree with the orange face. On the outside, this EPD again bounds only bigon white faces. Both of these EPDs are semi-simple by Lemma 41.

In Figures 4.20d and 4.20e, the orange tentacle originates in the uppermost state circle of $T_b$ and running across $T_a$. Figure 4.20d shows the purple tentacle originating in the uppermost state circle of $T_b$ as well. There may be nonbigon white faces where indicated by ⋆ in this diagram. Figure 4.20d is similar and shows the purple tentacle originating in the uppermost state circle of $T_b$, and the orange tentacle originating in one of the lower-most state circles of $T_b$. There may be nonbigons again where indicated by ⋆. Since there are nonbigon white faces on both sides of the EPDs in each of these cases, they are both complex EPDs.

**White edges (3N), (3S).** The white edges of $\partial E$ are depicted in Figure 4.21a. In particular, $\partial E$ runs through the state circle $I$ in $T_d$, which we color green. To the north of $I$, $\partial E$ runs through the green face around the top of $T_d$, across $T_c$ along a segment, and into $T_b$ from the south up a segment. Just inside $T_b$ it runs through a white face and then runs into a blue tentacle. To the south of $I$, $\partial E$ runs through the green face around the bottom of the diagram, across $T_a$ along a segment, and into $T_b$ from the north down the unique segment at the north of $T_b$. Just inside $T_b$ it runs through a white face and then runs into a red tentacle. Notice the (3S) white edge requires that $T_d$ has only one state circle, and the (3N) white edge requires that $T_c$ has no state circles.
By Lemma 50, the blue and red faces must agree. We color this single new face orange in the
next two diagrams. The red tentacle must come from the lower-right-most state circle in $T_b$. The
blue tentacle may come from any of the lower-most state circles in $T_b$ or the uppermost state circle
in $T_b$. So there are two possible diagrams. In Figure 4.21b, the blue tentacle originates in the
lower-right-most state circle of $T_b$; to match the red tentacle, it must be the lower-right-most state
circle. This EPD bounds only bigon white faces on the outside. In Figure 4.21c the blue tentacle
originates in the uppermost state circle of $T_b$. In order for this to agree with the red tentacle, there
must be only one state circle in $T_b$. This EPD also bounds bigons on the outside. (In both Figures,
non-bigon white faces may occur only where indicated by $\ast$.) Both EPDs are semi-simple.

This concludes the search for type (1) complex EPDs in $P_A$. The complex EPDs we found are
depicted in Figures 4.12b, 4.15c, 4.20d, and 4.20e. 

\[\text{Lemma 55. Let } K \text{ be a } + - + - \text{ link with reduced, admissible diagram } D(K). \text{ Then there are no complex EPDs in } P_A \text{ which are not type (1).}\]

\[\text{Proof. Let } E \text{ be a complex EPD in } P_A \text{ not type (1); then } E \text{ must be type (2). Therefore, } \partial E \text{ runs along segments across both } T_a \text{ and } T_c. \text{ For convenience, let’s call these two segments } \alpha \text{ and } \beta. \text{ Now, } \partial E \text{ may run along either the north or south sides of these segments. Therefore we have four possibilities, listed below. By Lemma 37, we may pull } E \text{ into a normal square with white edges at tails of the green face. We will use Lemma 34 heavily, to show what the diagram of } P_A \text{ may look like.}\]

\[\alpha N, \beta N. \text{ We interpret the notation } \alpha N, \beta N \text{ to mean that } \partial E \text{ runs along the north side of } \alpha\]

and the north side of $\beta$. Using Lemma 34, we can see that the tentacle running along $\alpha N$ originates
in the lower-right-most state circle of $T_d$ (see Lemma 33). Let’s color this face orange. The tentacle
running along $\beta N$ originates in the lower-right-most state circle of $T_b$; we color this face green. See

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Figure 4.22a. $\partial E$ must run out of the $\alpha N$ segment to the east, and from there run to a white face. Since white faces occur at heads of orange, we look for a head of orange in that direction. Again by Lemma 34 we see that there is only one possible head of orange after running east out of $\alpha N$; if $\alpha$ is the uppermost segment in $T_a$, then the orange tentacle will run over $T_b$ and produce one head at the north of $T_b$. At this white edge, $\partial E$ will jump into the face originating in the lower-right-most-tentacle of $T_b$ (which happens to be our green face already.) See Figure 4.22b.

Figure 4.22: Setting up a $\alpha N$, $\beta N$ EPD

We now look for a second white edge. $\partial E$ must run out of the $\beta N$ tentacle to the east, and from there run to another white edge. Thus we look for green tails in that direction. There are three possibilities. If the green tentacle runs into another segment in $T_c$ north of $\beta$, it will terminate in a tail. At this white edge, $\partial E$ will run into the face originating in the uppermost state circle of $T_d$. In order for this to match the orange face (Lemma 50), there must be only one state circle in $T_d$. See Figure 4.23a. We see that $\partial E$ runs through $T_d$ from north to south; but we have assumed that $\partial E$ is not type (1). Contradiction.

Figure 4.23: The $\alpha N$, $\beta N$ EPDs

Now suppose the tentacle over $\beta N$ does not hit another segment in $T_c$ after running east over $\beta$. Then the green tentacle runs over the top of $T_d$, and there are possibilities for two other tails. First, the green tentacle will run down into $T_d$ from the north, terminating on the uppermost state
circle of $T_d$. If a white edge of $\partial E$ occurs here, then $\partial E$ will jump into the face originating in the uppermost state circle of $T_d$. For this face to match the orange, there must be only one state circle in $T_d$. This EPD is shown in Figure 4.23b, and runs through $T_d$; this contradicts our assumption that $E$ is not type (1). The second place a tail occurs is where the green tentacle runs across $T_d$, over the top of the diagram, and hits a segment in $T_a$. If a white edge occurs here, $\partial E$ runs into the orange face. The resulting diagram is shown in Figure 4.23c; the EPD is simple.

$\alpha N$, $\beta S$. Again, the tentacle running over $\alpha N$ originates in the lower-right-most state circle of $T_d$; color this face orange. The tentacle running over $\beta S$ originates in the uppermost state circle of $T_d$. Suppose first that these state circles of $T_d$ are not the same; so we color the face running over $\beta S$ green. Now, as in the previous case $\partial E$ must run east out of $\alpha N$ to a white edge, which must be at a head of orange. There is only one possible head of orange, at the top of $T_b$ as before. However, at this white edge $\partial E$ runs into the face originating in the lower-right-most state circle of $T_b$; we color this face purple. Then $\partial E$ runs through the green, purple, and orange faces, which are all distinct. See Figure 4.24a. This contradicts Lemma 50.

Therefore, it must be that the uppermost and the lower-left-most state circles of $T_d$ are the same; so $T_d$ has only one state circle. So $\partial E$ runs from $\alpha N$ to $\beta S$ through $T_d$; see Figure 4.24b. But then $E$ is a type (1) EPD; contradiction.

$\alpha S$, $\beta N$. Referencing Lemma 34 shows that the $\alpha S$ tentacle originates in the uppermost state circle of $T_b$; and the $\beta N$ tentacle originates in the lower-right-most state circle of $T_b$. Suppose first that these two state circles are the same; so there is only one state circle in $T_b$. This gives Figure 4.25a. But then $\partial E$ runs through $T_b$ from north to south, so $E$ is type (1). Contradiction.

Assume, therefore, that the uppermost and lower-left-most state circles of $T_b$ are different. Color these state circles as in Figure 4.25b. Then $\partial E$ must run out of the $\beta N$ tentacle to the east, and
then to a white edge of $\partial E$. This white edge occurs at a tail of the green face, so we look for tails of
green in that direction. The possibilities are the same as the three in Case $\alpha N, \beta N$ (Figure 4.23).
First, the green tentacle may run into another segment in $T_c$ and terminate with a tail in $T_c$. At
that white edge $\partial E$ runs into a face originating in the uppermost state circle of $T_d$. Second, if the
green tentacle doesn’t hit another segment in $T_c$, it may run across the top of $T_d$ and down into $T_d$.
At that white edge, $\partial E$ will jump into the uppermost state circle of $T_d$. Finally, if the green tentacle
doesn’t hit another segment in $T_c$, it may run across the top of $T_d$, all the way over the top of the
diagram, and terminate with a tail in $T_a$. At this white edge, $\partial E$ runs into a face originating in
the lower-right-most state circle of $T_d$.

In each of these three cases, $\partial E$ jumps into a face that originated in $T_d$. But by Lemma 50, $\partial E$
can only jump into the orange face, which originated in $T_b$. Contradiction.

$\alpha S$, $\beta S$. Lemma 34 tells us that the $\alpha S$ tentacle originates in the uppermost state circle of $T_b$
and the $\beta S$ tentacle originates in the uppermost state circle of $T_d$. Color these faces as in Figure
4.26a. $\partial E$ must run out of $\beta S$ to the west, and from there to a white edge. This must occur at
a head of the orange tentacle. If the orange face hits a segment before exiting $T_c$, it will produce
no heads in that direction. However, if it runs out of $T_c$, it will run along the bottom of $T_b$ and
produce a new head at every segment along the south of $T_b$. However we must choose an orange
head that is adjacent to a green tail. This may occur at the lower-left-most segment, as in Figure
4.26b; or at any segment which connects to the green state circle, as in Figure 4.26c. We may rule
out Figure 4.26c, since this would make $E$ a type (1) complex EPD.

Now we look for the second white edge of $\partial E$. After $\partial E$ runs out of the $\alpha S$ tentacle to the
west, it must run to a white edge. We look for tails of green in that direction. First, the green
tentacle may hit a segment and terminate before it leaves $T_a$. At this tail, by Lemma 34, it will
Figure 4.26: The first white edge for Case $\alpha S$, $\beta S$

meet a tentacle that originated in the lower-right-most state circle of $T_{d}$. This must agree with the orange face, so there must be only one state circle in $T_{d}$. This results in the EPD in Figure 4.27a. But this EPD is type (1); this contradicts our assumption that $E$ is not type (1). Next, suppose that the green tentacle runs out of $T_{a}$ without hitting a segment. Then it runs around the bottom of the diagram, to $T_{d}$. Here it may spawn new tentacles that run up a segment into $T_{d}$ and terminate. These green tails may meet the orange face if the orange face is connected by a segment to the bottom of $T_{d}$, as in Figure 4.27b. This EPD is also type (1). Finally, one green tail may occur in $T_{c}$, after the green tentacle runs over the south of $T_{d}$. This tail will meet the orange face, by Lemma 34. This results in Figure 4.27c. This EPD is simple, bounding a single bigon on the inside.

Figure 4.27: EPDs in Case $\alpha S$, $\beta S$

Proposition 56. Let $K$ be a $+--+$ link with reduced, admissible diagram $D(K)$. Then

$$||E_{c}|| \leq 1$$

where $||E_{c}||$ is the number of complex EPDs required to span $P_{A}$. 

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Proof. Recall the definition of $||E_c||$ in Lemma 42. Let $E_s$ be as in that Lemma. To obtain the desired result, we need to show that if $E_1$ and $E_2$ are two complex EPDs in $P_A$, then $E_1$ is equivalent under parabolic compression to a subset of $E_2 \cup E_s$.

In Lemmas 54 and 55, we have found all possible complex EPDs in $P_A$. These are shown again in Figure 4.28 (originally from Figures 4.12b, 4.15c, 4.20d, and 4.20e). We will call these EPDs $E_{(a)}$, $E_{(b)}$, $E_{(c)}$, and $E_{(d)}$ respectively.

Consider the pair of EPDs $E_{(a)}$ and $E_{(b)}$. These EPDs are complex only if there are nonbigon white faces at both of the places indicated by $\star$. In particular, there must be at least one nonbigon white face on the south side of $T_c$; this means $T_c$ has at least one state circle, by Lemma 34. And there must be at least one nonbigon white face on the east side of $T_d$; this means $T_d$ has more than one state circle by Lemma 34. Notice also that in these diagrams, $T_a$ may not have any state circles, since any state circles must occur beneath the segment running across $T_a$ and prevent the orange tentacle from running out of $T_a$. Likewise $T_b$ must have only one state circle.

Now consider the second pair of EPDs, $E_{(c)}$ and $E_{(d)}$. By the same arguments as in the previous paragraph, these diagrams are only complex if $T_a$ has at least one state circle and $T_b$ has more than one state circle. Also, $T_c$ may not have any state circles and $T_d$ may have only one state circle. This shows that if one of the EPDs $E_{(a)}$ or $E_{(b)}$ occur in $P_A$ for a given $+-+-$ link, then neither of $E_{(c)}$ or $E_{(d)}$ may occur. Likewise, if either of $E_{(c)}$ or $E_{(d)}$ occur, then neither of $E_{(a)}$ or $E_{(b)}$ may occur.
Look at the first pair of EPDs. In $T_a$ both EPDs must run across the southernmost railroad. In $T_b$ they run down the unique segment at the north of $T_b$ into the orange state circle, and then along the easternmost segment heading south. So in $T_a$ and $T_b$, the EPDs must be the same. However, they may differ in $T_c$ and $T_d$. In $T_c$, $\partial E(a)$ runs over the northernmost railroad track in $T_c$. On the other hand $\partial E(b)$ may run over any of the railroad tracks except the northernmost. However, there are only bigon white faces between these railroad tracks, so the two EPDs differ by bigons here. In $T_d$, there may be many segments running down from the green state circle $I$ to the south edge of $T_d$. $\partial E(a)$ and $\partial E(b)$ may run down any of these segments (recall the discussion of (1S) white edges after Lemma 47). However, there are only bigon white faces between these segments. Then, in the whole polyhedron, the sets of white faces these EPDs bound differ only by bigons. By Lemma 43, $E(a)$ is equivalent under parabolic compression to a subset of $E(b) \cup E_s$.

Next look at the second pair of EPDs, $E(c)$ and $E(d)$. Similarly to the above pair, these EPDs must follow the same path through $T_c$ and $T_d$. They differ in $T_b$ and possibly also in $T_a$. In $T_a$, either EPD may run across $T_a$ along any of the railroad tracks in $T_a$. However, the choice of which railroad track to run across will cause the EPDs to differ only by bigons, since there are only bigon white faces between these railroad tracks. In $T_b$, $E(c)$ does not contain any white faces. But $E(d)$ contains one or more white faces of $T_b$, depending on which segment it runs down from the orange state circle. However, this differs only by bigons from $E(c)$ here, since the west side of $T_b$ contains only bigon white faces. Then, in the whole polyhedron, the sets of white faces these EPDs bound differ only by bigons. By Lemma 43, $E(a)$ is equivalent under parabolic compression to a subset of $E(b) \cup E_s$.

\[ \square \]

**Chapter 5. Proofs of Main Results**

**Theorem 57.** Let $K$ be a hyperbolic Montesinos link. Then $K$ has a reduced admissible diagram $D$, and

\[
\text{vol}(S^3 \setminus K) \geq \nu_8(\chi_{\Lambda}(G'_{\Lambda}) - 1)
\]
where \( v_8 = 3.6638 \ldots \) is the hyperbolic volume of a regular ideal octahedron, \( \chi_{-}(\cdot) \) is the negative Euler characteristic, and \( G'_A \) is the reduced A-state graph.

Proof. Let \( K \) be a hyperbolic Montesinos link with reduced admissible diagram \( D \). We will use Theorem 3 to bound \( \text{vol}(S^3 \setminus K) \). The hypotheses of the theorem are that \( K \) is prime and that \( D \) is A-adequate. The diagram \( D \) is prime by Corollary 17. By Theorem 10, if \( D \) is not A-adequate then it must be B-adequate. But in this case the mirror image of \( D \) is A-adequate. Taking the mirror image does not change the volume of the knot complement, so we may safely assume that \( D \) is A-adequate.

Now we may use Theorem 3. Notice that we obtain the desired volume bound if we show \( ||E_c|| \leq 1 \). We divide into several simple cases.

**Case 1.** Suppose \( K \) has either all positive or all negative tangles. Then \( D \) is alternating, so \( ||E_c|| = 0 \) (see [9]).

**Case 2.** Suppose \( K \) has two tangles. Then by [3] \( K \) has a diagram with only one tangle, so Case 1 applies.

**Case 3.** Suppose \( K \) has three tangles, with slopes not all the same sign. If \( K \) is a \( + + - \) link, then by Proposition 53 we have \( ||E_c|| \leq 1 \), so the desired volume bound holds. Alternatively \( K \) may be a \( + - - \) tangle. Let \( J \) be the mirror image of \( K \); then \( J \) is a \( + + - \) tangle and the volume bound applies. But \( \text{vol}(S^3 \setminus K) = \text{vol}(S^3 \setminus J) \). Thus the volume bound applies to both \( + + - \) and \( + - - \) links. These are the only links in Case 3, up to cyclic permutation.

**Case 4.** Suppose \( K \) has four tangles, with slopes not all the same sign. Up to cyclic permutation, there are four ways the positive and negative tangles can be arranged. These are:

\( + - + - \). By Proposition 56, \( ||E_c|| \leq 1 \). Thus the desired volume bound holds.

\( + + - - \). By Lemma 19, a \( + + - - \) link is a mutation of a \( + - + - \) link. Ruberman proved that mutations do not change the volume of the knot complement [16]. Therefore the volume bound for a \( + - + - \) link also applies to a \( + + - - \) link.

\( + + + - \) or \( + - - - \). By Theorem 44, we know \( ||E_c|| = 0 \) for a \( + + + - \) link since it has three positive tangles. The mirror image of a \( + - - - \) link also has three positive tangles, so the same volume bound applies.

**Case 5.** Suppose \( K \) has five tangles. Then \( K \) must have at least three positive or three negative tangles. Thus Theorem 44 applies, taking the mirror image if \( K \) has three negative tangles.
Therefore $||E_c|| \leq 1$ in all cases.

Lemma 58. Let $D(K)$ be a reduced, admissible, hyperbolic Montesinos diagram with at least two positive and at least two negative tangles. Let $G'_A$ and $G'_B$ be the reduced all-$A$ and all-$B$ graphs associated to $D$. Then

$$-\chi(G'_A) - \chi(G'_B) \geq t(D) - Q_{1/2}(D) - 2$$

where $Q_{1/2}(D)$ is the number of rational tangles in $D$ whose slope has absolute value $|q| \in [1/2, 1)$.

Proof. We follow the proof of Lemma 9.10 in [6]. Since $D$ has at least two positive and at least two negative tangles, $D$ is still both $A$- and $B$- adequate. Let $v_A$ be the number of vertices in $G_A$, $e_A$ be the number of edges in $G_A$, and $e'_A$ be the number of edges in $G'_A$; and similarly for $v_B$, $e_B$, and $e'_B$. Then we have $-\chi(G_A) = v_A - e_A$ and $-\chi(G'_A) = v_A - e'_A$, and likewise for $-\chi(G_B)$ and $-\chi(G'_A)$. The first two paragraphs of the proof in [6] still hold, so we have

$$v_A - e_A + v_B = 0$$

Now consider the number of edges of $G_A$ that are discarded when we pass to $G'_A$. By Lemma 8.14 of [6], edges may be lost in three ways:

1. If $r$ is a twist region with $c(r) > 1$ crossings such that the $A$-resolution of $r$ is short, then $c(r) - 1$ of these edges will be discarded when we pass to $G'_A$.

2. If $N_i$ is a negative tangle with slope $q \in (-1,-1/2]$, then one edge of $G_A$ will be lost from the two-edge loop spanning $N_i$ from north to south.

3. If there are exactly two positive tangles $P_1$ and $P_2$, then one edge of $G_A$ will be lost from the two-edge loop that runs across $P_1$ and $P_2$ from east to west.

The same holds for $G_B$, with $A$ and $B$ switched and positive and negative tangles switched. Combining this information, we obtain

$$\sum\{c(r) - 1\} + \#\{i : |q_i| \in [1/2, 1)\} + 2 = c(D) - t(D) + Q_{1/2}(D) + 2$$
where the sum is over all twist regions $r$. Since the edges of $G_B$ are in one-to-one correspondence with the crossings of $D$,

\[-\chi(G'_A) - \chi(G'_B) \leq e'_A + e'_B - v_A - v_B \]
\[= (e'_A + e'_B - e_A - e_B) + e_B \]
\[+ (e_A - v_A - v_B) \]
\[\geq -c(D) + t(D) - Q_{1/2}(D) - 2 + c(D) \]
\[+ 0 \]
\[= t(D) - Q_{1/2}(D) - 2. \]

\[\square\]

**Theorem 59.** Let $K$ be a Montesinos link that is both $A$- and $B$-adequate, and let $D$ be a reduced diagram for $K$. Then

\[\text{vol}(S^3 \setminus K) \geq \frac{v_8}{4}(t(D) - \#K - 8)\]

where $v_8 = 3.6638\ldots$ is the hyperbolic volume of a regular ideal octahedron and $\#K$ is the number of link components of $K$.

**Proof.** First, notice that from the proof of Lemma 9.11 in [6], we still obtain

\[Q_{1/2} \leq \frac{t(D) + \#K}{2}. \]

Combining this fact with Lemma 58, we have

\[-\chi(G'_A) - \chi(G'_B) \geq \frac{t(D) - \#K - 4}{2}. \]

We now follow the proof of Theorem 9.12 in [6]. We can make the diagram $D(K)$ admissible by a sequence of flypes, without changing the twist number. Therefore we may assume $D(K)$ is admissible, so that Theorem 44 and Propositions 53 and 56 apply.

By Theorem 29, $S_A$ and $S_B$ are both essential in $S^3 \setminus A$. Then we can apply Theorem 1 to the
two surfaces $S_A$ and $S_B$, obtaining

$$\text{vol}(S^3 \setminus K) \geq \frac{v_8}{2} \left[ \chi_- \text{-guts}(S^3 \setminus S_A) + \chi_- \text{-guts}(S^3 \setminus S_B) \right]$$

$$= \frac{v_8}{2} \left[ \chi_-(G_A) + \chi_-(G_B) \right] - \frac{v_8}{2} \left( ||E_c||_A + ||E_c||_B \right), \text{ by Theorem 2}$$

$$= \frac{v_8}{2} (t(D) - \# K - 4) - \frac{v_8}{2} (2), \text{ by Propositions 53 and 56}$$

$$\geq \frac{v_8}{4} (t(D) - \# K - 8).$$

$\Box$
Bibliography


