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The Minimum Rank of Schemes on Graphs

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ABSTRACT

The Minimum Rank of Schemes on Graphs

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Let G be an undirected graph on n vertices and let $\mathcal{S}(G)$ be the class of all real-valued symmetric $n \times n$ matrices whose nonzero off-diagonal entries occur in exactly the positions corresponding to the edges of G. Let $V = \{1, 2, \ldots, n\}$ be the vertex set of G. A scheme on G is a function $f : V \to \{0, 1\}$. Given a scheme f on G, there is an associated class of matrices $\mathcal{S}_f(G) = \{A \in \mathcal{S}(G) | a_{ii} = 0$ if and only if $f(i) = 0\}$. A scheme f is said to be constructible if there exists a matrix $A \in \mathcal{S}_f(G)$ with $\operatorname{rank} A = \min\{\operatorname{rank} M | M \in \mathcal{S}(G)\}$. We explore properties of constructible schemes and give a complete classification of which schemes are constructible for paths and cycles. We also consider schemes on complete graphs and show the existence of a graph for which every possible scheme is constructible.

Keywords: Combinatorial Matrix Theory, Diagonal Entry Restrictions, Graph, Minimum Rank, Scheme, Symmetric, Zero forcing

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Contents

Chapter 1. Introduction

The *minimum rank problem* for graphs asks what is the minimum rank among all real symmetric matrices whose off-diagonal zero/nonzero pattern is given by a simple graph. This is equivalent to asking what is the maximum nullity among such matrices. Nylen published the first paper on the minimum rank of a graph in 1996 [1].

We consider a modification of this problem which extends the specified zero/nonzero pattern to include the diagonal entries. This modified minimum rank problem for graphs asks what is the minimum rank among all symmetric matrices whose off-diagonal zero/nonzero pattern is given by a simple graph and whose diagonal zero/nonzero pattern is given by a function on the vertex set. This problem is a generalization of the work done in [2] on diagonal entry restrictions. This problem is also equivalent to a modification to the minimum rank problem explored in [3] and [4] for graphs that allow single loops at vertices. It is also closely related to the study of sign-patterns and sign-solvable linear systems, a topic that has applications in economics (see [5]).

In Chapter 1 we provide preliminary results and some background information necessary to develop the topics addressed throughout this thesis. In Chapter 2 we present the concept of a scheme on a graph, compute the minimum rank of all schemes on graphs with four or fewer vertices, and compute the minimum rank of all schemes on complete graphs. In Chapter 3 we examine the topic of constructible schemes on paths and cycles. In Chapter 4 we consider the question of counting constructible schemes and determine the existence of a graph for which all schemes are constructible.

1.1 Preliminaries

This section presents definitions, examples, and previous results.

A graph G is an ordered pair (V, E) where V is a set of vertices and E is a set of edges, defined by two element subsets of V. If i and j are vertices of a graph G, we use the convention ij to denote the edge $\{i, j\}$. In this paper, all graphs are simple and undirected unless otherwise indicated.

Definition 1.1. Given a graph G on n vertices, let $\mathcal{S}(G)$ be the set of all real symmetric $n \times n$ matrices $A = [a_{ij}]$ such that $a_{ij} \in \mathbb{R}$ and $a_{ij} \neq 0$, $i \neq j$, if and only if ij is an edge of G.

Throughout the paper G will always be a graph on n vertices.

Definition 1.2. The *minimum rank* of G is

$$\operatorname{mr}(G) = \min\{\operatorname{rank} A \mid A \in \mathcal{S}(G)\}.$$

The *maximum nullity* of G is

$$\operatorname{M}(G) = \max\{\operatorname{nullity} A \mid A \in \mathcal{S}(G)\}.$$

We note that the minimum rank of G is labeling invariant in the sense that a relabeling of the vertices of G does not affect the minimum rank achievable by matrices in $\mathcal{S}(G)$. This is because relabeling the vertices of G is equivalent to a conjugation of matrices in $\mathcal{S}(G)$ and matrix rank is invariant under matrix conjugation.

Finding the minimum rank of a graph and finding the maximum nullity of a graph are equivalent problems since the rank-nullity theorem implies $\operatorname{mr}(G) + \operatorname{M}(G) = n$.

We list here some standard graph terminology and define some common graphs.

Definition 1.3.

- The *degree* of a vertex is the number of edges incident to the vertex.
- A *pendant* vertex is a vertex of degree 1.
- A *dominating* vertex in a graph with n vertices is a vertex of degree $n-1$.
- The *complete graph* on n vertices, K_n, is the graph in which every vertex is a dominating vertex.
- The *path* on n vertices, P_n, is the connected graph with 2 vertices of degree 1 and $n-2$ vertices of degree 2.
- The *cycle* on n vertices, C_n, is the connected graph in which every vertex has degree 2.

The following is a well know result in matrix theory (see p. 13 in [6]).

Proposition 1.4. *Let A and B be $m \times n$ matrices. Then*

$$\operatorname{rank}(A+B) \leq \operatorname{rank} A + \operatorname{rank} B.$$

Example 1.5. We show that $\operatorname{mr}(K_n) = 1, n \geq 2$ and $\operatorname{mr}(P_n) = n-1, n \geq 2$.

First consider K_n where $n \geq 2$. The all ones matrix, denoted J_n, is in $\mathcal{S}(K_n)$ and has rank 1. Thus $\operatorname{mr}(K_n) \leq 1$. Since $n \geq 2$, the all zero matrix is not in $\mathcal{S}(K_n)$. The only matrix with rank 0 is the all zero matrix so every matrix in $\mathcal{S}(K_n)$ has rank at least one. Thus $\operatorname{mr}(K_n) = 1$.

Now consider P_n where $n \geq 2$. Under the standard labeling

$$P_n = (\{1, 2, \ldots, n\}, \{12, 23, 34, \ldots, n-1n\}),$$

$\mathcal{S}(P_n)$ is the set of $n \times n$ tridiagonal real symmetric matrices

$$M = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{12} & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & a_{n-1\,n} \\ 0 & \cdots & 0 & a_{n-1\,n} & a_{n\,n} \end{bmatrix}$$

where $a_{i\,i+1} \neq 0$ and a_{ii} may be zero or nonzero. The first $n-1$ rows of any such matrix are linearly independent because in each of these rows there is a nonzero entry in a column where every previous row has a zero. Thus every matrix in $\mathcal{S}(P_n)$ has rank at least $n-1$. Observe that

$$A = \begin{bmatrix} 1 & 1 & 0 & \cdots & & 0 & 0 \\ 1 & 1 & 0 & \ddots & & 0 & 0 \\ 0 & 0 & 0 & & & \vdots & \vdots \\ \vdots & \ddots & & & 0 & 0 & 0 \\ 0 & 0 & & \cdots & 0 & 0 & 0 \\ 0 & 0 & & \cdots & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \cdots & & 0 & 0 \\ 0 & 1 & 1 & \ddots & & 0 & 0 \\ 0 & 1 & 1 & & & \vdots & \vdots \\ \vdots & \ddots & & & 0 & 0 & 0 \\ 0 & 0 & & \cdots & 0 & 0 & 0 \\ 0 & 0 & & \cdots & 0 & 0 & 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 & 0 & 0 & \cdots & & 0 & 0 \\ 0 & 0 & 0 & \ddots & & 0 & 0 \\ 0 & 0 & 0 & & & \vdots & \vdots \\ \vdots & \ddots & & & 0 & 0 & 0 \\ 0 & 0 & & \cdots & 0 & 1 & 1 \\ 0 & 0 & & \cdots & 0 & 1 & 1 \end{bmatrix}$$

is in $\mathcal{S}(G)$ and by Proposition 1.4, rank $A \leq \sum_{i=1}^{n-1} \operatorname{rank} J_2 = \sum_{i=1}^{n-1} 1 = n-1$.

Definition 1.6. Let G be a graph with a vertex labeled v. The graph $G - v$ is the graph obtained from G by removing vertex v and all edges incident to v.

Remark. For brevity, we use the phrase *deleting vertex* v to refer to removing the vertex and all its incident edges.

Definition 1.7. Let G be a graph and let H be obtained by deleting a sequence of vertices from G. The graph H is called an *induced subgraph* of G.

Definition 1.8. Let A be an $n \times n$ matrix and let $\{i_1, i_2, \ldots, i_k\}$ be a subset of $\{1, 2, \ldots, n\}$. The matrix $A[i_1, i_2, \ldots, i_k]$ is the matrix obtained by deleting the rows and columns of A corresponding to the indices *not in* $\{i_1, i_2, \ldots, i_k\}$ and the matrix $A(i_1, i_2, \ldots, i_k)$ is the matrix obtained by deleting the rows and columns of A corresponding to the indices *in* $\{i_1, i_2, \ldots, i_k\}$.

The following is a proposition from [1].

Proposition 1.9 (Nylen's Lemma)**.** *Let G be a graph and let $A \in \mathcal{S}(G)$ with* $\operatorname{rank} A = \operatorname{mr}(G)$*. Let $p \in \{1, 2, ..., n\}$. Then* $\operatorname{rank} A(p) = \operatorname{rank} A$ *or* $\operatorname{rank} A(p) = \operatorname{rank} A - 2$.

Chapter 2. Schemes of graphs

Definition 2.1. Given a graph $G = (V, E)$, a *scheme* of G is a function $f : V \to \{0, 1\}$.

Definition 2.2. Given a graph G and a scheme f,

$$\mathcal{S}_f(G) = \{M \in \mathcal{S}(G) | M_{vv} = 0 \text{ iff } f(v) = 0\}.$$

Remark. A matrix $M \in \mathcal{S}(G)$ is said to satisfy a scheme f if $M \in \mathcal{S}_f(G)$.

Definition 2.3. Given a graph G and a scheme f, f is called *constructible* if there exists a matrix $M \in \mathcal{S}_f(G)$ such that $\operatorname{rank} M = \operatorname{mr}(G)$.

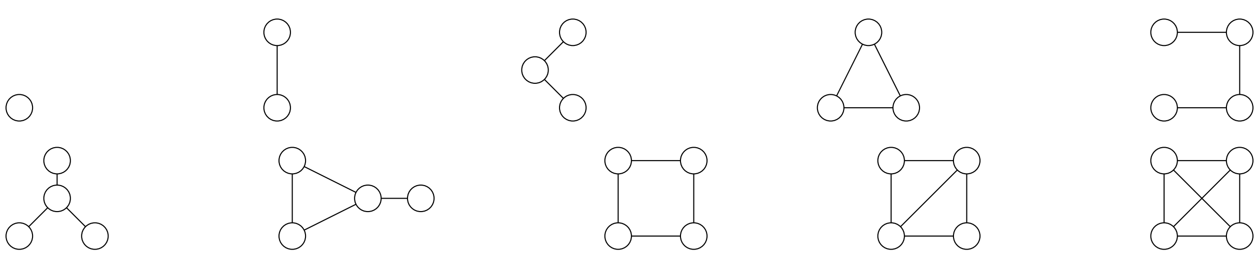
Definition 2.4. The minimum rank of a scheme f on a graph G is

$$\operatorname{mr}_f(G) = \min\{\operatorname{rank} A \mid A \in \mathcal{S}_f(G)\}.$$

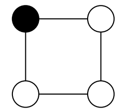
Observation 2.5. *Given a graph G and a scheme f, $\mathcal{S}_f(G) \subset \mathcal{S}(G)$ so $\operatorname{mr}_f(G) \geq \operatorname{mr}(G)$ with equality if and only if f is constructible.*

2.1 The Minimum Rank of Schemes on Small Connected Graphs

In this section we will compute the minimum rank of all schemes on connected graphs with four or fewer vertices. The connected graphs with four or fewer vertices are



from left to right, starting with the top row, $K_1, K_2, P_3, K_3, P_4, S_4$, paw, C_4, diamond, K_4. For this section we will pictorially display schemes by coloring vertices black if they are mapped to one and coloring vertices white if they are mapped to zero. Thus



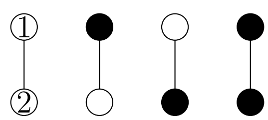
depicts the graph C_4 together with the scheme mapping the upper left vertex to one and mapping the other three vertices to zero.

For each of the following examples f_1 refers to the first depicted scheme, f_2 to the second scheme, and so forth. We will indicate the vertex labeling in the graph depicting the scheme that maps all vertices to zero and will omit the vertex labeling from the other graphs.

Example 2.6 (Schemes on K_1)**.** There are two schemes on K_1.

The first scheme f_1 restricts $\mathcal{S}(K_1)$ to include only the 1×1 zero matrix, thus $\mathrm{mr}_{f_1}(K_1) = 0$. Then $\mathcal{S}_{f_2}(G)$ is the set of 1×1 nonzero matrices so $\mathrm{mr}_{f_2}(K_1) = 1$.

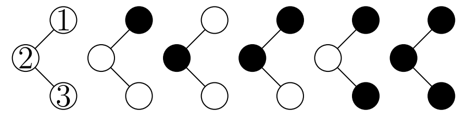
Example 2.7 (Schemes on K_2)**.** There are four schemes on K_2.



All matrices in $\mathcal{S}_{f_1}(K_2)$ have the form $\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$ where a is nonzero. All matrices in $\mathcal{S}_{f_2}(K_2)$ have the form $\begin{bmatrix} b & a \\ a & 0 \end{bmatrix}$ where a and b are nonzero. All matrices in $\mathcal{S}_{f_3}(K_2)$ have the form $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix}$ where a and b are nonzero. Hence, all matrices satisfying either f_1, f_2, or f_3 have nonzero determinant and are thus invertible so $\mathrm{mr}_{f_1}(K_2) = \mathrm{mr}_{f_2}(K_2) = \mathrm{mr}_{f_3}(K_2) = 2$. All matrices in $\mathcal{S}_{f_4}(K_2)$ have the form $\begin{bmatrix} b & a \\ a & c \end{bmatrix}$ where $abc \neq 0$. Since $J_2 \in \mathcal{S}_{f_4}(K_2)$ and the zero matrix is not in $\mathcal{S}_{f_4}(K_2)$, $\mathrm{mr}_{f_4}(K_2) = 1$.

We observe that the vertex coloring in the second and third graphs of K_2 are isomorphic so there are only three unique schemes on K_2. In the subsequent examples we will only consider the nonisomorphic schemes.

Example 2.8 (Schemes on P_3)**.** There are 6 unique schemes.

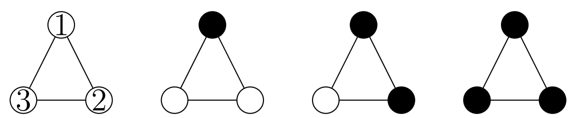


Every matrix satisfying either f_2 or f_4 has nonzero determinant and is therefore invertible. Thus $\operatorname{mr}_{f_2}(P_3) = \operatorname{mr}_{f_4}(P_3) = 3$. The matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

each have rank 2 and satisfy the schemes f_1, f_3, f_5, f_6 respectively. Since $\operatorname{mr}(P_3) = 2$, we conclude $\operatorname{mr}_{f_1}(P_3) = \operatorname{mr}_{f_3}(P_3) = \operatorname{mr}_{f_5}(P_3) = \operatorname{mr}_{f_6}(P_3) = \operatorname{mr}(P_3) = 2$.

Example 2.9 (Schemes on K_3)**.** There are 4 schemes.

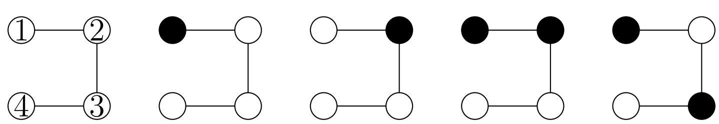


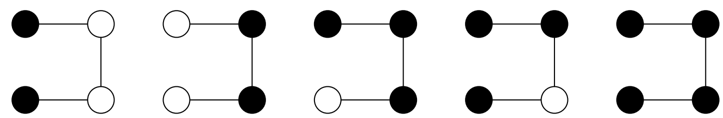
Every matrix satisfying f_1 is invertible so $\operatorname{mr}_{f_1}(K_3) = 3$. Every matrix satisfying either f_2 or f_3 has a 2×2 invertible submatrix so $\operatorname{mr}_{f_2}(K_3), \operatorname{mr}_{f_3}(K_3) \geq 2$. Since

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

are rank 2 matrices and satisfy f_2 and f_3 respectively, $\operatorname{mr}_{f_2}(K_3) = \operatorname{mr}_{f_3}(K_3) = 2$. Then $J_3 \in \mathcal{S}_{f_4}(K_3)$ so $\operatorname{mr}_{f_4}(K_3) = 1$.

Example 2.10 (Schemes on P_4)**.** There are 10 schemes.



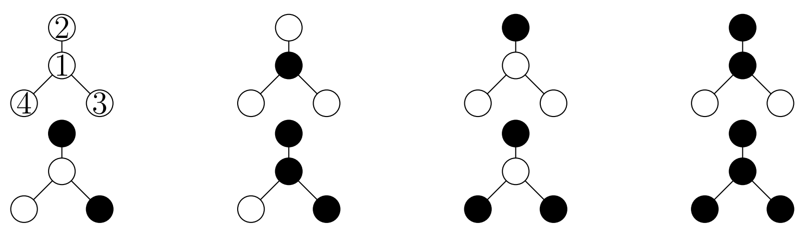


Every matrix satisfying one of f_1, f_2, f_3, f_5 or f_7 is invertible so $\mathrm{mr}_{f_1}(P_4) = \mathrm{mr}_{f_2}(P_4) = \mathrm{mr}_{f_3}(P_4) = \mathrm{mr}_{f_5}(P_4) = \mathrm{mr}_{f_7} = 4$. Each of

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

satisfy one of the remaining schemes and has rank 3. Since $\mathrm{mr}(P_4) = 3$, $\mathrm{mr}_{f_4}(P_4) = \mathrm{mr}_{f_6}(P_4) = \mathrm{mr}_{f_8}(P_4) = \mathrm{mr}_{f_9}(P_4) = \mathrm{mr}_{f_{10}}(P_4) = 3$.

Example 2.11 (Schemes on S_4)**.** There are 8 schemes.



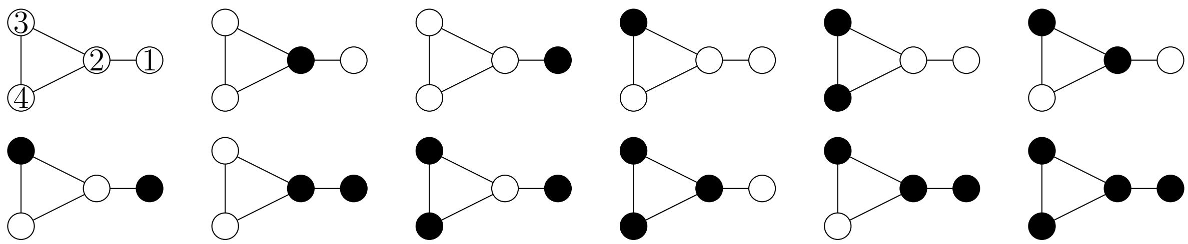
Every matrix in $\mathcal{S}_{f_5}(S_4)$ and $\mathcal{S}_{f_6}(S_4)$ is invertible so $\mathrm{mr}_{f_5}(S_4) = \mathrm{mr}_{f_6}(S_4) = 4$. Every matrix satisfying one of f_3, f_4, f_7, or f_8 has a full rank 3×3 submatrix so $\mathrm{mr}_{f_3}(S_4) = \mathrm{mr}_{f_4}(S_4) = \mathrm{mr}_{f_7}(S_4) = \mathrm{mr}_{f_8}(S_4) \geq 3$. Each of

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 & 1 & 1 \\ -2 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

has rank 3 so $\mathrm{mr}_{f_3}(S_4) = \mathrm{mr}_{f_4}(S_4) = \mathrm{mr}_{f_7}(S_4) = \mathrm{mr}_{f_8}(S_4) = 3$. All matrices satisfying f_1

and f_2 have a full rank 2×2 submatrix and $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ are rank 2 matrices so $\mathrm{mr}_{f_1}(S_4) = \mathrm{mr}_{f_2}(S_4) = 2$.

Example 2.12 (Schemes on the paw)**.** There are 12 schemes.



Every matrix satisfying one of f_1, f_2, f_4 or f_6 is invertible so $\mathrm{mr}_{f_1}(\text{paw}) = \mathrm{mr}_{f_2}(\text{paw}) = \mathrm{mr}_{f_4}(\text{paw}) = \mathrm{mr}_{f_6}(\text{paw}) = 4$. Every matrix satisfying one of $f_3, f_5, f_7, f_8, f_{10}$, or f_{11} has a full rank 3×3 submatrix and

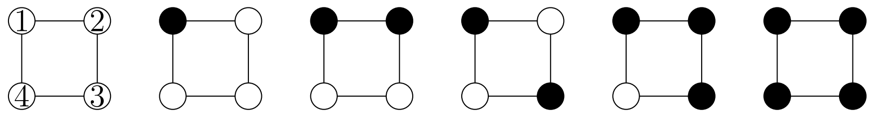
$$\begin{bmatrix} -2 & 2 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

are rank 3 matrices so $\mathrm{mr}_{f_3}(\text{paw}) = \mathrm{mr}_{f_5}(\text{paw}) = \mathrm{mr}_{f_7}(\text{paw}) = \mathrm{mr}_{f_8}(\text{paw}) = \mathrm{mr}_{f_{10}}(\text{paw}) = \mathrm{mr}_{f_{11}}(\text{paw}) = 3$. All matrices satisfying f_9 and f_{12} have a rank 2 submatrix and

$$\begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

are rank 2 matrices so $\mathrm{mr}_{f_9}(\text{paw}) = \mathrm{mr}_{f_{12}}(\text{paw}) = 2$.

Example 2.13 (Schemes on C_4)**.** There are 6 schemes.



All matrices satisfying one of f_2, f_3 and f_5 have a full rank 3×3 submatrix and

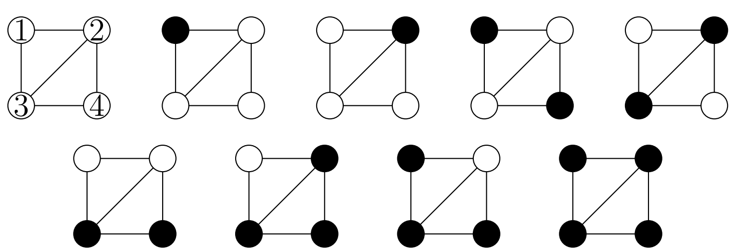
$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

are rank 3 matrices so $\mathrm{mr}_{f_2}(C_4) = \mathrm{mr}_{f_3}(C_4) = \mathrm{mr}_{f_5}(C_4) = 3$. All matrices satisfying one of f_1, f_4 and f_6 have a rank 2 submatrix and

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 2 \end{bmatrix}$$

are rank 2 matrices so $\mathrm{mr}_{f_1}(C_4) = \mathrm{mr}_{f_4}(C_4) = \mathrm{mr}_{f_6}(C_4) = 2$.

Example 2.14 (Schemes on the diamond)**.** There are 9 schemes.



All matrices satisfying one of f_1, f_2, f_6, or f_7 have a full rank 3×3 submatrix and

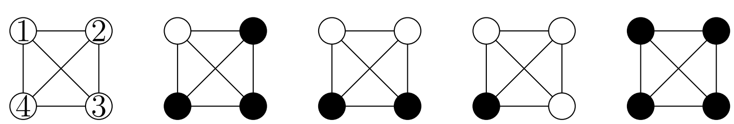
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 0 & -2 & 1 \\ 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 2 & 0 \\ 2 & 0 & -1 & 2 \\ 2 & -1 & 2 & -2 \\ 0 & 2 & -2 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

are rank 3 matrices satisfying the four respective schemes so $\mathrm{mr}_{f_1}(\text{diamond}) = \mathrm{mr}_{f_2}(\text{diamond}) = \mathrm{mr}_{f_6}(\text{diamond}) = \mathrm{mr}_{f_7}(\text{diamond}) = 3$. It is straightforward to tell that the remaining schemes must have minimum rank at least 2 and

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & -1 & 1 & 0 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

are rank 2 matrices satisfying f_3, f_4, f_5, f_8, f_9 respectively so each of these schemes has minimum rank 2.

Example 2.15 (Schemes on K_4)**.** There are 5 schemes.



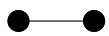
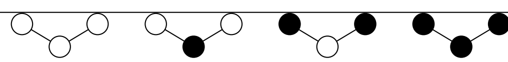
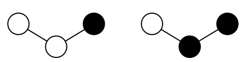
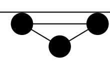
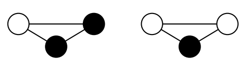
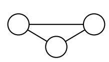
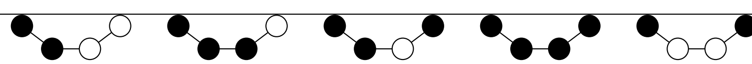
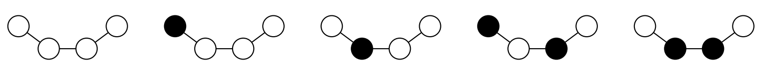
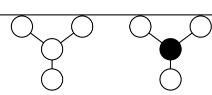
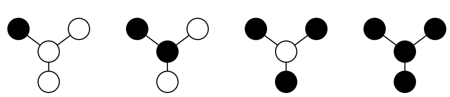
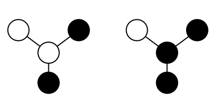
Since $J_4 \in \mathcal{S}_{f_5}(K_4)$, $\mathrm{mr}_{f_5}(K_4) = 1$. All matrices satisfying either f_1 or f_4 have a full rank 3×3 submatrix and

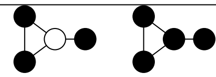
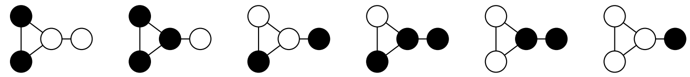
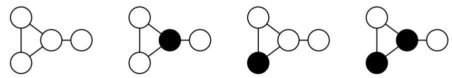
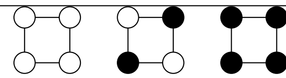
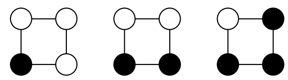
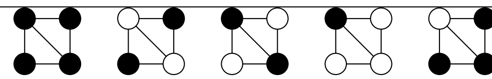
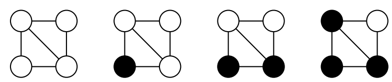
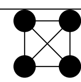
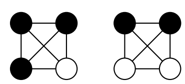
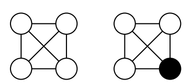
$$\begin{bmatrix} 0 & 1 & 4 & 9 \\ 1 & 0 & 1 & 4 \\ 4 & 1 & 0 & 1 \\ 9 & 4 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 4 & 5 \\ 1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 5 \\ 5 & 2 & 5 & 12 \end{bmatrix}$$

are rank 3 matrices satisfying the respective schemes so $\mathrm{mr}_{f_1}(K_4) = \mathrm{mr}_{f_4}(K_4) = 3$. All matrices satisfying either f_2 or f_3 have a full rank 2×2 submatrix and

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix}$$

are rank 2 matrices satisfying the respective schemes so $\mathrm{mr}_{f_2}(K_4) = \mathrm{mr}_{f_3}(K_4) = 2$.

The Minimum Rank of Schemes on Small Connected Graphs		
Graph	Minimum Rank	Scheme
K_1	0	
	1	
K_2	1	
	2	
P_3	2	
	3	
K_3	1	
	2	
	3	
P_4	3	
	4	
S_4	2	
	3	
	4	

The Minimum Rank of Schemes on Small Connected Graphs (cont.)		
Graph	Minimum Rank	Scheme
paw	2	
	3	
	4	
C_4	2	
	3	
diamond	2	
	3	
K_4	1	
	2	
	3	

2.2 The Minimum Rank of Schemes on Complete Graphs

We classify the minimum rank of schemes on K_n where K_n has its vertices labeled $1, 2, \ldots, n$. Schemes on K_n are uniquely determined up to isomorphism by the number of vertices mapped to zero. Let f be a scheme on K_n that maps k vertices to zero. Without loss of generality, f maps vertices $1, 2, \ldots, k$ to zero. Note $\mathrm{mr}_f(K_n) \geq \mathrm{mr}(K_n) = 1, n > 1$. For $k = 0$, Example 2.6 shows $\mathrm{mr}_f(K_1) = 1$ and since $J_n \in \mathcal{S}_f(K_n)$ we have $\mathrm{mr}_f(K_n) = 1$ for all n.

Next we consider the other extreme, $k = n$. If $n = 1$, $\mathcal{S}_f(K_1) = \{[0]\}$ so $\mathrm{mr}_f(K_1) = 0$. If $n = 2$, $\mathcal{S}_f(K_2) = \left\{ \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \middle| a \neq 0 \right\}$ so $\mathrm{mr}_f(K_2) = 2$. If $n \geq 3$, we define a Toeplitz matrix $A = [a_{ij}]$ where $a_{ij} = (i - j)^2$. Note that A is symmetric since $(i - j)^2 = (j - i)^2$. Also, the diagonal entries of A are each zero and the off-diagonal entries are nonzero so $A \in \mathcal{S}_f(K_n)$.

Let $v^{(i)}$ be the ith column of A. For $i > 3$, we show $v^{(i)}$ is a linear combination of $v^{(1)}, v^{(2)}$, and $v^{(3)}$. Observe that

$$\begin{aligned}
&\frac{1}{2}[(i-3)(i-2)v_j^{(1)} - 2(i-3)(i-1)v_j^{(2)} + (i-2)(i-1)v_j^{(3)}] \\
&= \frac{1}{2}[(i-3)(i-2)(1-j)^2 - 2(i-3)(i-1)(2-j)^2 + (i-2)(i-1)(3-j)^2] \\
&= \frac{1}{2}[(i^2-5i+6)(1-2j+j^2) - 2(i^2-4i+3)(4-4j+j^2) + (i^2-3i+2)(9-6j+j^2)] \\
&= \frac{1}{2}[2i^2 - 4ij + 2j^2] = (i-j)^2 = v_j^{(i)}.
\end{aligned}$$

Hence, the jth entry of $v^{(i)}$ is a combination of the jth entries of $v^{(1)}, v^{(2)}$, and $v^{(3)}$ and we conclude $v^{(i)} = \frac{1}{2}[(i-3)(i-2)v^{(1)} - 2(i-3)(i-1)v^{(2)} + (i-2)(i-1)v^{(3)}]$. Thus $\operatorname{rank} A \leq 3$ and $\operatorname{mr}_f(K_n) \leq 3$. For any $M \in \mathcal{S}_f(K_n)$, $M[1,2,3]$ has the form $\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$ with $a, b, c \neq 0$. Thus $\det M[1,2,3] = 2abc \neq 0$ so $\operatorname{rank} M \geq 3$ and we conclude $\operatorname{mr}_f(K_n) = 3$.

If $k = 1, n \geq 2$, then for any $M \in \mathcal{S}_f(K_n)$, $M[1,2] = \begin{bmatrix} 0 & a \\ a & b \end{bmatrix}$ with $a, b \neq 0$. Thus, $\operatorname{mr}_f(K_n) \geq 2$. For $n \geq 2$, let $C = \begin{bmatrix} 0 & e^T \\ e & J_{n-1} \end{bmatrix}$ where e is the all ones vector. Then C is a rank 2 matrix so $\operatorname{mr}_f(K_n) = 2$.

If $k = 2, n \geq 3$, then for any $M \in \mathcal{S}_f(K_n)$, $M[1,2] = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$ with $a \neq 0$ so $\operatorname{rank} M \geq 2$. Let $C = \begin{bmatrix} 0 & 1 & e^T \\ 1 & 0 & e^T \\ e & e & B \end{bmatrix}$ where $B = 2J_{n-1}$ and e is the all ones vector. Since $\operatorname{rank} C = 2$, $\operatorname{mr}_f(K_n) \leq 2$ so $\operatorname{mr}_f(K_n) = 2$.

The final case requires $k \geq 3, n > k$. For every $D \in \mathcal{S}_f(K_n)$, $\det D[1,2,3] \neq 0$ so $\operatorname{mr}_f(K_n) \geq 3$. Define a block matrix $M = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix}$ with columns $w^{(i)}$ where the blocks are defined as follows: $C = 12J_{n-k}$, A is the previously defined $k \times k$ Toeplitz matrix, and B is an $(n-k) \times k$ matrix whose ith column is $[(i-1)^2 + (i-2)^2 + (i-3)^2]e$. For $3 < i \leq k$,

$$w^{(i)} = \frac{1}{2}[(i-3)(i-2)w^{(1)} - 2(i-3)(i-1)w^{(2)} + (i-2)(i-1)w^{(3)}].$$

We have already established

$$w_j^{(i)} = \frac{1}{2}[(i-3)(i-2)w_j^{(1)} - 2(i-3)(i-1)w_j^{(2)} + (i-2)(i-1)w_j^{(3)}] \text{ for } j \leq k.$$

If $j > k$,

$$\begin{aligned}
&\frac{1}{2}[(i-3)(i-2)w_j^{(1)} - 2(i-3)(i-1)w_j^{(2)} + (i-2)(i-1)w_j^{(3)}] \\
&= \frac{1}{2}[(i-3)(i-2)(5) - 2(i-3)(i-1)(2) + (i-2)(i-1)(5)] \\
&= \frac{1}{2}[5(i^2-5i+6) - 4(i^2-4i+3) + 5(i^2-3i+2)] \\
&= \frac{1}{2}[6i^2 - 24i + 28] \\
&= 3i^2 - 12i + 14 \\
&= (i-1)^2 + (i-2)^2 + (i-3)^2 = w_j^{(i)}.
\end{aligned}$$

For $k < i \leq n$, we claim $w^{(i)} = w^{(1)} + w^{(2)} + w^{(3)}$. For $j \leq k$, $w_j^{(1)} + w_j^{(2)} + w_j^{(3)} = (1-j)^2 + (2-j)^2 + (3-j)^2 = M_{ij} = M_{ji} = w_j^{(i)}$. For $j > k$, note $w_j^{(i)} = 12$ and

$$\begin{aligned}
&w_j^{(1)} + w_j^{(2)} + w_j^{(3)} \\
&= [0^2 + 1^2 + 2^2] + [1^2 + 0^2 + 1^2] + [2^2 + 1^2 + 0^2] \\
&= 5 + 2 + 5 = 12 = w_j^{(i)}.
\end{aligned}$$

Hence, $\operatorname{rank} M \leq 3$ and $\operatorname{mr}_f(K_n) = 3$.

Theorem 2.16. *Given any schemes h, g, f on K_1, K_2, or K_n, $n \geq 3$ respectively, $0 \leq \operatorname{mr}_h(K_1) \leq 1 \leq \operatorname{mr}_g(K_2) \leq 2$, $1 \leq \operatorname{mr}_f(K_n) \leq 3$ where $\operatorname{mr}_h(K_1), \operatorname{mr}_g(K_2), \operatorname{mr}_f(K_n) = 1$ if and only if the schemes h, g, f map all vertices to one and $\operatorname{mr}_f(K_n) = 3$ if and only if f maps at least three vertices to zero.*

Chapter 3. Paths and Cycles

We will use a modification of a graph parameter, the zero forcing number Z, as a tool to develop a classification of constructible schemes on paths. Then we will use the classification of constructible schemes on paths to classify constructible schemes on cycles. The parameter Z is used to give an upper bound on the maximum nullity of a graph and it first appeared in [7]. Physicists studying quantum systems have also employed the zero forcing process, under the name "graph infection" ([8]). A modification of Z that appears in [9] may be used to put an upper bound on the maximum nullity of a loop graph. Since loop graphs impose restrictions on diagonal entries, there is a very natural connection between loop graphs and schemes. The following definitions from [7] and [9] define Z and its modification to loop graphs.

Research done in [2] and [10] study some properties of the structure of minimum rank matrices for paths and cycles. A classification of constructible schemes is intended to enhance the understanding of the structure of matrices which achieve the minimum rank for paths and cycles.

Definition 3.1.

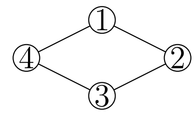
- *Color-change rule for a simple graph*: If G is a graph with each vertex colored either white or black, u is a black vertex of G, and exactly one neighbor v of u is white, then change the color of v to black.

- Given a coloring of G, the *derived coloring* is the result of applying the color-change rule for a simple graph until no more changes are possible.

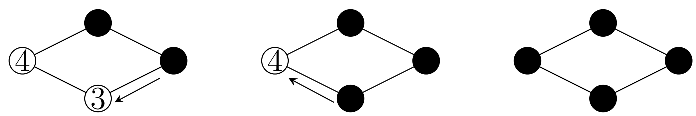
- A *zero forcing set* for a graph G is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, the derived coloring of G is all black.

- The zero forcing number of a graph G, $\mathrm{Z}(G)$ is the minimum of $|\,\mathrm{Z}\,|$ over all zero forcing sets $\mathrm{Z} \subset V(G)$.

Example 3.2. Consider the graph C_4.



Vertices 1 and 2 comprise a zero forcing set. Begin by coloring vertices 1 and 2 black (see the illustration below). Since 2 has exactly one white neighbor, 3, it can force 3 black by the color-change rule for a simple graph. Since 3 has exactly one white neighbor, 4, it can force 4 black.



Note in the above example the zero forcing set $1, 2$ is not unique nor is the order of the vertex forcing. Since there is no single vertex in C_4 that constitutes a zero forcing set, $\mathrm{Z}(C_4) = 2$.

Definition 3.3. A *loop graph* is a graph that allows single loops at vertices, i.e., $\widehat{G} = (V, E)$ where V is the set of vertices of $\widehat{G}$ and the set of edges E is a set of two-element multisets. Vertex u is a *neighbor* of vertex v in $\widehat{G}$ if $uv \in E$; note that u is a neighbor of itself if and only if the loop uu is an edge. The *underlying simple graph* of a loop graph $\widehat{G}$ is the graph G obtained from $\widehat{G}$ by deleting all loops.

Remark. In a loop graph, every vertex is specified as being looped or unlooped.

Definition 3.4. The *set of real symmetric matrices described by a loop graph* $\widehat{G}$ is

$$\mathcal{S}(\widehat{G}) = \{A = [a_{ij}] \mid A \text{ is symmetric and } a_{ij} \neq 0 \text{ if and only if } ij \in E\}.$$

and the *maximum nullity of* $\widehat{G}$ is

$$\mathrm{M}(\widehat{G}) = \max\{\text{nullity } A \mid A \in \mathcal{S}(\widehat{G})\}$$

Definition 3.5 (Color-change rule for a loop graph)**.** Let $\widehat{G}$ be a loop graph with each vertex colored white or black. If exactly one neighbor u of v is white, then change the color of u to black.

The color-change rule for a loop graph and the color-change rule for simple graphs are almost identical, the only differences being that when using a loop graph, two additional coloring forces are valid. First, a looped white vertex that has no other white neighbors may be colored black. Second, if an unlooped white vertex has only one white neighbor u, u may be colored black. By $\mathrm{Z}(\widehat{G})$, we mean the same thing as in Definition 3.1, except we use the color-change rule for a loop graph. (We distinguish the two cases by whether or not the graph is a loop graph.)

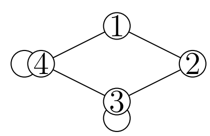
The following result is from [9].

Theorem 3.6. *For any loop graph $\widehat{G}$, $\mathrm{M}(\widehat{G}) \leq \mathrm{Z}(\widehat{G})$.*

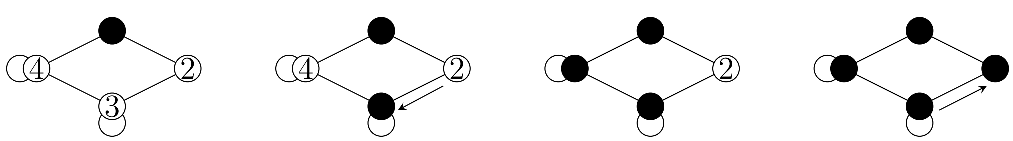
The following example illustrates the color-change rules for a loop graph.

Example 3.7. Consider the loop graph $\widehat{C}_4$ whose underlying simple graph is C_4.

$\widehat{C}_4$:



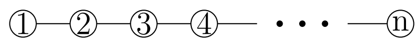
Color vertex 1 black (see illustration below). Since 2 is an unlooped vertex and only has one white neighbor 3, 3 can be colored black. Since 4 is looped and has no white neighbors, 4 can be colored black. Then 3 forces 2. Thus $\mathrm{Z}(\widehat{C}_4) \leq 1$. It is clear that $\mathrm{Z}(\widehat{C}_4) \geq 1$. Thus $\mathrm{Z}(\widehat{C}_4) = 1$.



3.1 Schemes on Paths

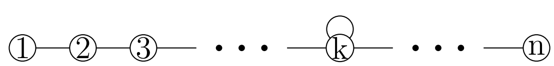
Unless otherwise stated, we label the vertices of P_n such that

$$P_n = (\{1, \ldots, n\}, \{12, 23, \ldots, n-1n\}).$$

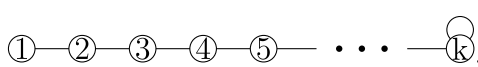


Theorem 3.8. *Let P_n be a path on n vertices and let f be a scheme on P_n. If n is odd, then f is not constructible if and only if there exists a unique odd vertex k such that $f(k) = 1$. The scheme f may take on either value, 0 or 1, on even vertices.*

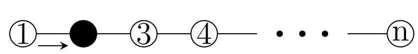
Proof. We make use of the zero forcing parameter for loop graphs to prove the reverse direction. Suppose there exists a unique odd vertex k such that $f(k) = 1$. Then either $f(1) = 0$ or $f(n) = 0$ or both. We may assume $f(1) = 0$. Let $\widehat{P}_n$ be the loop graph where a vertex i is looped if and only if $f(i) = 1$. Thus k is looped and all other odd vertices are unlooped. Even vertices may be looped or unlooped and thus remain unspecified throughout the proof. We show that the empty set is a zero forcing set for this graph. We consider two cases: either $1 < k < n$



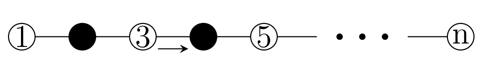
or $k = n$



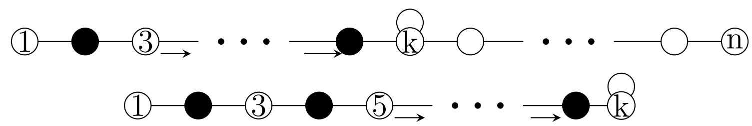
In either case, since 1 is unlooped and its only neighbor is 2, we may color 2 black.



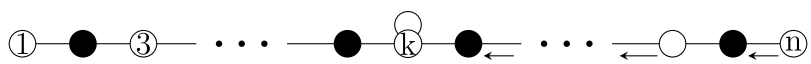
Now 3 has exactly one white neighbor, vertex 4. Thus if $f(3) = 0$, 3 is unlooped and we may color 4 black.



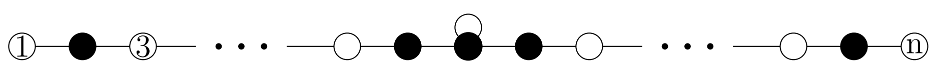
Since k is the only odd vertex not mapped to zero by f, this process will continue until all even vertices less than k are colored black. We have one of the following depending on whether $k < n$ or $k = n$.



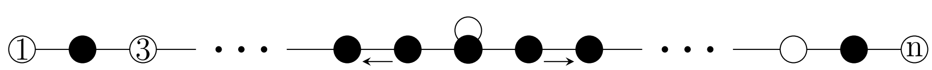
I. If $k < n$, then the same procedure starting with n results in all even vertices greater than k being colored black.



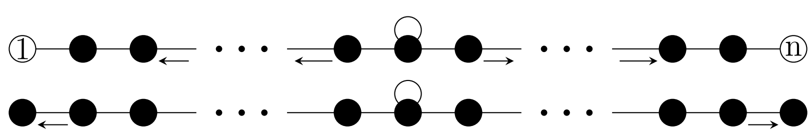
Then $k-1$ and $k+1$ are both even vertices and therefore black. Since $f(k) = 1$, k is looped. Also both its neighbors are black so k may be colored black.



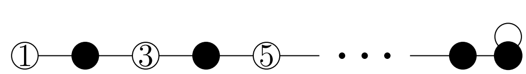
Since vertices $k-1$ and $k+1$ are black and each have exactly one white neighbor, vertices $k-2$ and $k+2$ respectively, we may color them black.



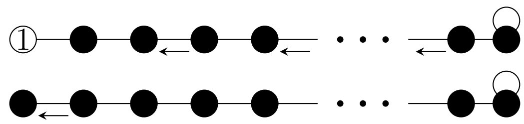
Since $k-3$ is even, it is black and has exactly one white neighbor, $k-4$. Thus we may color $k-4$ black. Similarly $k+3$ forces $k+4$ to be colored black. This process continues until all odd vertices are black and hence all vertices are black.



II. If $k = n$, $k-1$ is an even vertex less than k and therefore black. Since k has no other neighbors and is looped, we may color it black.



Then as above all odd vertices less than k may be forced black.



Thus all vertices are colored black. Therefore, in both cases, we conclude the empty set is a forcing set for $\widehat{P}_n$. Thus $Z(\widehat{P}_n) = 0$. Since $M(\widehat{P}_n) \leq Z(\widehat{P}_n) = 0$, $M(\widehat{P}_n) = 0$ and $\text{mr}(\widehat{P}_n) = n$. We conclude f is not constructible, or equivalently, all matrices in $\mathcal{S}_f(P_n)$ are invertible.

We prove the contrapositive of the forward direction. Thus we must show that schemes which either map all odd vertices to zero or map at least two odd vertices to one are constructible. We proceed by induction on the number of vertices in the path. When $n = 1$ there is only one scheme that satisfies the hypothesis. The scheme f which maps vertex 1 to zero. Then $S_f(P_1) = \{[0]\}$ and $[0]$ achieves the minimum rank. When $n = 3$, there are exactly two odd vertices. Thus a valid scheme must map both odd vertices to zero or map both to one. Hence, there are four schemes that satisfy the hypothesis. We showed in Example 2.8 these schemes have rank 2, or, equivalently, are constructible.

By way of induction, we assume schemes on a path with $2k+1$ vertices which either map all odd vertices to zero or map at least two odd vertices to one are constructible. We then show such schemes on paths with $2k+3$ vertices are constructible. Let f be a scheme on P_n where $n = 2k+3$. First suppose f maps either vertex 1 or vertex n to zero. Without loss of generality, let $f(1) = 0$. Let G_1 be the subgraph of P_n induced by the vertex set $\{1, 2, 3\}$ and let G_2 be the subgraph induced by the vertex set $\{3, \ldots, n\}$.

$$P_n := 1 - 2 - 3 - 4 - \cdots - n$$

$$G_1 := 1 - 2 - 3 \qquad\qquad G_2 := 3 - 4 - 5 - \cdots - n$$

We define schemes g_1 and g_2 on G_1 and G_2 respectively by letting $g_1(1) = f(1), g_1(2) = f(2), g_1(3) = 0$ and $g_2(i) = f(i)$ for $3 \leq i \leq n$. Since $f(1) = 0$, the scheme f must send all odd vertices to zero or map at least two odd vertices in the set $\{3, \ldots, n\}$ to one. In either case, G_2 has $2k+1$ vertices and the scheme g_2 is constructible by the inductive hypothesis.

Thus there exists a matrix

$$B = \begin{bmatrix} b_{33} & b_{34} & 0 & \cdots & 0 \\ b_{43} & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & b_{n-1\,n} \\ 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} \end{bmatrix} \in S_{g_2}(G_2)$$

with $\operatorname{rank} B = 2k$. Also from the $n = 3$ case we know there exists a matrix

$$A = \begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix} \in S_{g_1}(G_1)$$

with $\operatorname{rank} A = 2$. We define a matrix from A and B by letting

$$C = \begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & \cdots & 0 \\ 0 & a_{32} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & b_{11} & b_{12} & 0 & \cdots & 0 \\ 0 & 0 & b_{21} & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & b_{n-1\,n} \\ 0 & 0 & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 & \cdots & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 & \cdots & 0 \\ 0 & a_{32} & b_{11} & b_{12} & 0 & \cdots & 0 \\ 0 & 0 & b_{21} & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & b_{n-1\,n} \\ 0 & 0 & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} \end{bmatrix} \in \mathcal{S}_f(P_{2k+3}).$$

Clearly, $C \in \mathcal{S}(P_{2k+3})$. Hence, we have $2k + 2 = \operatorname{mr}(P_{2k+3}) \leq \operatorname{rank} C \leq \operatorname{rank} A + \operatorname{rank} B = 2k + 2$. Hence, f is constructible.

Next we must consider the case $f(1) = f(n) = 1$. Let G_1 be the subgraph of P_{2k+3} induced by the vertex set $\{1, 2, 3\}$ and let G_2 be the subgraph induced by the vertex set $\{3, \ldots, n\}$. We define schemes g_1 and g_2 on G_1 and G_2 respectively by letting $g_1(1) = f(1), g_1(2) = f(2), g_1(3) = 1$ and $g_2(3) = 1, g_2(i) = f(i)$ for $4 \leq i \leq n$. Note that G_2 has $2k+1$ vertices and $g_2(3) = g_2(n) = 1$. Thus by induction we know there exists a matrix

$$B = \begin{bmatrix} b_{33} & b_{34} & 0 & \cdots & 0 \\ b_{43} & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & b_{n-1\,n} \\ 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} \end{bmatrix} \in S_{g_2}(G_2)$$

with rank $B = 2k$. Then since $g_1(1) = g_1(3) = 1$ we know from the $n = 3$ case there exists

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \in S_{g_1}(G_1)$$

with rank $A = 2$. We note in particular that $b_{33}, a_{33} \neq 0$. We define a matrix from A and B by letting

$$C = s \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & \cdots & 0 \\ 0 & a_{32} & a_{33} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & b_{33} & b_{34} & 0 & \cdots & 0 \\ 0 & 0 & b_{43} & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & b_{n-1\,n} \\ 0 & 0 & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} \end{bmatrix}$$

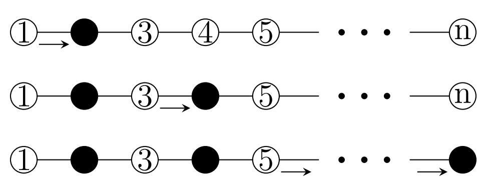
$$
= \begin{bmatrix}
sa_{11} & sa_{12} & 0 & 0 & 0 & \cdots & 0 \\
sa_{12} & sa_{22} & sa_{23} & 0 & 0 & \cdots & 0 \\
0 & sa_{32} & sa_{33}+b_{33} & b_{34} & 0 & \cdots & 0 \\
0 & 0 & b_{43} & \ddots & \ddots & & \vdots \\
0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & \vdots & & \ddots & \ddots & b_{n-1\,n} \\
0 & 0 & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn}
\end{bmatrix}.
$$

where s is a nonzero scalar chosen such that $sa_{33} + b_{33} = 0$ if $f(3) = 0$ and $sa_{33} + b_{33} \neq 0$ if $f(3) = 1$. Then $C \in S_f(P_{2k+3})$ and we have $2k + 2 = \mathrm{mr}(P_{2k+3}) \leq \mathrm{rank}(C) \leq \mathrm{rank}\, sA + \mathrm{rank}\, B = 2k + 2$. Therefore f is constructible.

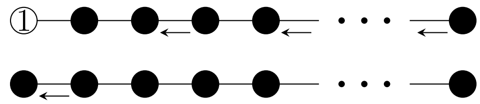
$\square$

Theorem 3.9. *Let P_n be a path on n vertices and let f be a scheme on P_n. If n is even, then f is constructible if and only if there exists an odd vertex i and an even vertex j such that $i < j$ and $f(i) = f(j) = 1$.*

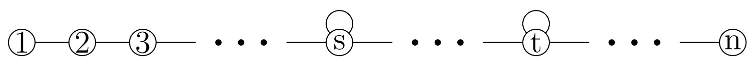
Proof. We make use of the zero forcing parameter for loop graphs to prove the contrapositive of the forward direction. Let $\widehat{P}_n$ be the loop graph where a vertex v is looped if and only if $f(v) = 1$. Assuming there do not exist vertices i, j with $i < j$, i odd, j even, and $f(i) = f(j) = 1$, we show the empty set is a zero forcing set for $\widehat{P}_n$. First consider the case where all odd vertices are mapped to zero by the scheme f. Thus all odd vertices are unlooped in $\widehat{P}_n$. Therefore, 1 is unlooped so 2 may be colored black, then 3 is unlooped so 4 may be colored black. The process will continue until all even vertices have been colored black.



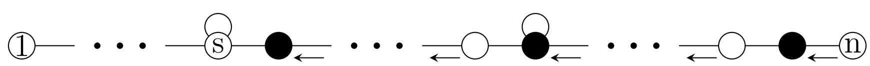
Then since n is even, it is black. Also, since n has exactly one white neighbor, $n - 1$, we may color $n - 1$ black. Since $n - 2$ is even, hence black, and has one white neighbor, $n - 3$, we may color $n - 3$ black. This process continues until all odd vertices are black.



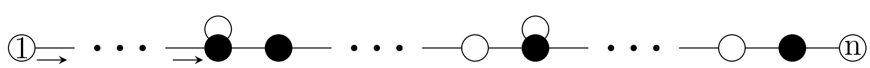
Thus the empty set is a forcing set. The case where all even vertices are mapped to zero by the scheme is similar. Next consider the case where at least one odd vertex and at least one even vertex is mapped to one. Let $t = \min\{k|k$ is odd and $f(k) = 1\}$ and $s = \max\{k|k$ is even and $f(k) = 1\}$. Then to satisfies the negation of the conclusion we must have $1 < s < t < n$. Hence $f(1) = f(n) = 0$.



Since n is unlooped and has only one neighbor, we may color $n-1$ black. Then $n-2$ is unlooped so we may color $n-3$ black. This will continue until all odd vertices greater than s are black.



Similarly all even vertices less than t may be colored black. In particular s is even and $s < t$ so s is black.



Also $s+1$ is black since it is odd and greater than s. We therefore have two adjacent vertices colored black which is sufficient to force all remaining vertices in the path to be black. Therefore, the empty set is a forcing set. We conclude $Z(\widehat{P}_n) = 0$. Since $M(\widehat{P}_n) \leq Z(\widehat{P}_n) = 0$, $M(\widehat{P}_n) = 0$ and we conclude f is not constructible.

We prove the reverse direction. We proceed by induction on size of the path. When $n = 2$ there is only one scheme that satisfies the hypothesis. The scheme f defined by $f(1) = f(2) = 1$. Then $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ achieves the minimum rank. By way of induction, we assume schemes on paths of size $n = 2k$ which satisfy the hypothesis of the reverse implication are constructible. We then show such schemes on paths of size $2k+2$ are constructible. Let f be a scheme on P_n where $n = 2k+2$. First suppose f maps either vertex 1 or vertex n to zero. Without loss of generality, let $f(1) = 0$. Let G_1 be the subgraph of P_n induced by the vertex set $\{1, 2, 3\}$ and let G_2 be the subgraph induced by the vertex set $\{3, \ldots, n\}$. We define schemes g_1 and g_2 on G_1 and G_2 respectively by letting $g_1(1) = f(1), g_1(2) = f(2), g_1(3) = 0$ and $g_2(i) = f(i)$ for $3 \leq i \leq n$. Since $f(1) = 0$, the scheme f must send vertices $i, j \in \{3, \ldots, n\}$ to one, where i is odd, j is even, and $i < j$. Since G_2 has $2k$ vertices the scheme g_2 is constructible by the inductive hypothesis. Thus there exists a matrix

$$B = \begin{bmatrix} b_{33} & b_{34} & 0 & \cdots & 0 \\ b_{43} & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & b_{n-1\,n} \\ 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} \end{bmatrix} \in \mathcal{S}_{g_2}(G_2)$$

with $\operatorname{rank} B = 2k - 1$. Also from the $n = 3$ case we know there exists a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \in S_{g_1}(G_1)$$

with $\operatorname{rank} A = 2$. Since A satisfies g_1, we know $a_{11} = a_{33} = 0$. We define a matrix from A and B by letting

$$C = \begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & \cdots & 0 \\ 0 & a_{32} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & b_{33} & b_{34} & 0 & \cdots & 0 \\ 0 & 0 & b_{43} & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & b_{n-1\,n} \\ 0 & 0 & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 & \cdots & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 & \cdots & 0 \\ 0 & a_{32} & b_{33} & b_{34} & 0 & \cdots & 0 \\ 0 & 0 & b_{43} & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & b_{n-1\,n} \\ 0 & 0 & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} \end{bmatrix}.$$

Clearly, $C \in \mathcal{S}(P_{2k+2})$. Hence, we have $2k+1 = \mathrm{mr}(P_{2k+2}) \leq \operatorname{rank} C \leq \operatorname{rank} A + \operatorname{rank} B = 2 + 2k - 1 = 2k+1$. Also C satisfies f. Hence, f is constructible.

Next we must consider the case $f(1) = f(n) = 1$. Let G_1 be the subgraph of P_n induced by the vertex set $\{1, 2, 3\}$ and let G_2 be the subgraph induced by the vertex set $\{3, \ldots, n\}$. We define schemes g_1 and g_2 on G_1 and G_2 respectively by letting $g_1(1) = f(1), g_1(2) = f(2), g_1(3) = 1$ and $g_2(3) = 1, g_2(i) = f(i)$ for $4 \leq i \leq n$. Note that G_2 has $2k$ vertices and $g_2(3) = g_2(n) = 1$. Thus by induction we know there exists a matrix

$$B = \begin{bmatrix} b_{33} & b_{34} & 0 & \cdots & 0 \\ b_{43} & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & b_{n-1\,n} \\ 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} \end{bmatrix} \in S_{g_2}(G_2)$$

with $\operatorname{rank} B = 2k-1$. Then since $g_1(1) = g_1(3) = 1$ we know from the $n = 3$ case there exists

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \in S_{g_1}(G_1)$$

with $\operatorname{rank} A = 2$. We note in particular that $b_{33}, a_{33} \neq 0$. We define a matrix from A and B

by letting

$$C = s\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & \cdots & 0 \\ 0 & a_{32} & a_{33} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & b_{33} & b_{34} & 0 & \cdots & 0 \\ 0 & 0 & b_{43} & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & b_{n-1\,n} \\ 0 & 0 & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} sa_{11} & sa_{12} & 0 & 0 & 0 & \cdots & 0 \\ sa_{12} & sa_{22} & sa_{23} & 0 & 0 & \cdots & 0 \\ 0 & sa_{32} & sa_{33}+b_{33} & b_{34} & 0 & \cdots & 0 \\ 0 & 0 & b_{43} & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & b_{n-1\,n} \\ 0 & 0 & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} \end{bmatrix}$$

where s is a nonzero scalar chosen such that $sa_{33} + b_{33} = 0$ if $f(3) = 0$ and $sa_{33} + b_{33} \neq 0$ if $f(3) = 1$. Then $C \in S_f(P_{2k+2})$ and we have $2k + 1 = \mathrm{mr}(P_{2k+2}) \leq \operatorname{rank} C \leq \operatorname{rank} sA + \operatorname{rank} B = 2 + 2k - 1 = 2k + 1$. Hence, f is constructible. $\square$

Corollary 3.10. *Any scheme g on a path P_n which maps both degree one vertices to one is constructible.*

Theorem 3.11. *Let f be a constructible scheme on P_n that sends two consecutive vertices to zero. We asssume vertices k and $k+1$ are mapped to zero. Let G be the graph obtained by deleting k and $k+1$ from P_n and if $1 < k < n-1$ adding an edge between $k-1$ and $k+2$. Let g be the scheme on G defined by $g(i) = f(i)$ for $i \in \{1, 2, \ldots, k-1, k+2, k+3 \ldots, n\}$. Then g is constructible.*

Proof. First note from Example 2.7, P_2 has no constructible schemes with consecutive vertices mapped to zero so $n \geq 3$. Hence, G is not the empty graph. Also, note G is a path on $n-2$ vertices with a nonstandard vertex labeling. Let h be a scheme defined on P_{n-2}, where P_{n-2} has the standard labeling, by $h(i) = f(i)$ for $i < k$ and $h(i) = f(i+2)$ for $i \geq k$. Then showing g is constructible and showing h is constructible are equivalent problems. We opt to show h is constructible.

First suppose n is odd. Since f is constructible Theorem 3.8 implies f either maps each vertex in $T = \{1, 3, 5, \dots, n\}$ to zero or it maps at least two of them to one. If f maps every vertex in T to zero, then h maps every vertex in $T' = \{1, 3, 5, \dots, n-2\}$ to zero so h is constructible by Theorem 3.8. Suppose f maps vertex $i, j \in T$ with $i < j$ to one. Since $f(k) = f(k+1) = 0$, $i, j \neq k, k+1$ and we must have $n \geq 5$. If $j < k$, $h(i) = f(i) = 1, h(j) = f(j) = 1$ so h maps at least two vertices in T' to one. If $i > k$, $h(i-2) = f(i) = 1, h(j-2) = f(j) = 1$ and since $i-2, j-2$ are odd h maps at least two vertices in T' to one. If $i < k < j$, $h(i) = f(i) = 1, h(j-2) = f(j) = 1$. Since i, j are odd and either k or $k+1$ is odd, we have $i < k, k+1 < j$ so $i \neq j-2$ and h maps at least two vertices in T' to one. Hence, h is constructible by Theorem 3.8.

Suppose n is even so $n \geq 4$. Since f is constructible Theorem 3.9 implies there exists vertices i, j such that $i < j$, i is odd, j is even, and $f(i) = f(j) = 1$. Note $i, j \neq k, k+1$. If $j < k$, $h(i) = f(i) = 1, h(j) = f(j) = 1$ so h is constructible. If $i > k$, $h(i-2) = f(i) = 1, h(j-2) = f(j) = 1$. Since $i-2 < j-2$ and $i-2$ is odd and $j-2$ is even, Theorem 3.9 implies h is constructible. If $i < k < j$, $h(i) = f(i) = 1, h(j-2) = f(j) = 1$. Since i is odd, j is even, $i < k$, and $j > k+1$, we note $i < j-2$. Also $j-2$ is even. Thus h is constructible by Theorem 3.9. □

Theorem 3.12. *Let P_n be a path with n even and let f be a scheme on P_n. If f sends more than half the vertices of P_n to one, then f is constructible.*

Proof. If f sends more than half the vertices of P_n to one, then by the pigeonhole principle f sends at least two consecutive vertices to one. Hence there exists a vertex i such that

$f(i) = f(i+1) = 1$. If i is odd, $i+1$ is even and we're done since the result clearly follows from Theorem 3.9. Similarly if i is even and there exists an odd vertex $j < i$ with $f(j) = 1$, we're done. If i is even and there exists an even vertex $j > i+1$ with $f(j) = 1$, then $f(i+1) = f(j) = 1$ and we are also done. If i is even and there does not exist a vertex j satisfying one of the previous two cases, then f maps every odd vertex less than i to zero and every even vertex greater than $i+1$ to zero so f maps at least half its vertices to zero contradicting the hypothesis. $\square$

We may restate Theorems 3.8 and 3.9 in terms of the following two nonstandard labelings of P_n: for the first, we allow the vertices of P_n to be labeled by any set of distinct positive integers but with alternating parity so a vertex may not be adjacent to a vertex of the same parity. We will refer to this as an alternating parity labeling of P_n.

$$6 - 7 - 2 - 5 - 4$$

$$3 - 8 - 5 - 4 - 1 - 2$$

Under this labeling, note that the parity of the degree one vertices must be the same if P_n has an odd number of vertices and the degree one vertices must have opposite parity if P_n has an even number of vertices.

For the second labeling, we allow the vertices of P_n to be labeled by any set of distinct positive integers for which exactly two mutually adjacent vertices have the same parity and all remaining pairs of adjacent vertices have opposite parity. We will refer to this as a split alternating parity labeling of P_n. In this paper, the split alternating parity labeling will only be used in connection with paths on an even number of vertices.

$$3 - 8 - 2 - 1 - 6 - 5$$

$$3 - 8 - 5 - 1 - 4 - 9$$

Under both labellings, let one of the degree one vertices be referred to as the start vertex and the other as the finish vertex. We define an ordering on the vertices such that $i \preceq j$ if

$\mathrm{dist}(w, i) \leq \mathrm{dist}(w, j)$ where w is the start vertex and equality occurs if and only if $i = j$. If $\mathrm{dist}(w, i) < \mathrm{dist}(w, j)$, we say $i \prec j$.

For the split alternating parity labelling, traversing the path from start vertex to finish vertex the first of the two adjacent vertices with the same parity is called the split vertex.

For the following theorem statements the choice of start vertex and finish vertex does not affect the statement of the theorem.

Theorem 3.13. *Let P_n be a path on an odd number of vertices with an alternating parity labeling and let f be a scheme on P_n. Then f is constructible if and only if either f maps every vertex with the same parity as the start vertex to zero or f maps at least two vertices with the same parity as the start vertex to one.*

Theorem 3.14. *Let P_n be a path on an even number of vertices with an alternating parity labeling and let f be a scheme on P_n. Then f is constructible if and only if traversing the path from start vertex to finish vertex there exist a vertex i with the same parity as the start vertex and a vertex j with the same parity as the finish vertex such that $i \prec j$ and $f(i) = f(j) = 1$.*

Theorem 3.15. *Let P_n be a path on an even number of vertices with a split alternating parity labeling and split vertex s. Let f be a scheme on P_n. Then f is constructible if and only if traversing the path from start vertex to finish vertex one of the following occurs:*

(i) there exists a vertex $i \prec s$ with the same parity as the start vertex and a vertex $j \preceq s$ with the opposite parity as the start vertex such that $i \prec j$ and $f(i) = f(j) = 1$.

(ii) there exists a vertex i with the opposite parity as the start vertex and a vertex j with the same parity as the start vertex such that $s \prec i \prec j$ and $f(i) = f(j) = 1$.

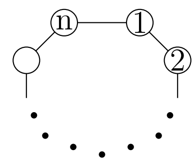
(iii) there exists a vertex i with the same parity as the start vertex and a vertex j with the same parity as the start vertex such that $i \preceq s \prec j$ and $f(i) = f(j) = 1$.

3.2 Schemes on Cycles

Unless otherwise stated, we label the vertices of C_n such that

$$C_n = (\{1, \ldots, n\}, \{12, 23, \ldots, n-1\,n, n1\}).$$

Also, recall $\mathrm{mr}(C_n) = n - 2$.



Definition 3.16. Given a graph $G = (V, E)$, a *partial scheme* is a function $f' : U \to \{0, 1\}$ where $U \subset V$.

Definition 3.17. Given a partial scheme $f' : U \to \{0, 1\}$ of G,

$$\mathcal{S}_{f'}(G) = \{M \in \mathcal{S}(G) \,|\, \forall v \in U, M_{vv} = 0 \text{ if and only if } f'(v) = 0\}.$$

Note matrices in $\mathcal{S}_{f'}(G)$ have no restrictions placed on diagonal entries corresponding to vertices not defined by the partial scheme. A matrix $M \in \mathcal{S}(G)$ is said to satisfy a partial scheme if $M \in \mathcal{S}_{f'}(G)$.

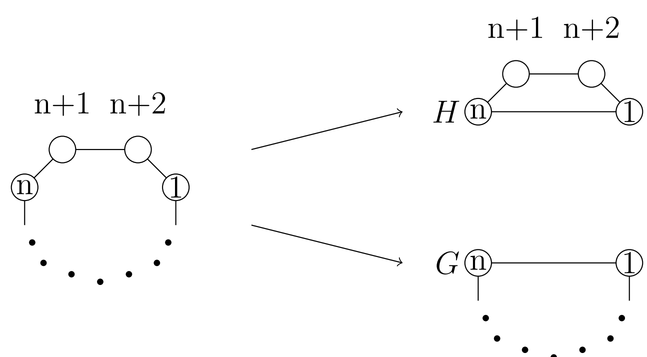
Lemma 3.18. *Let C_n be a cycle on an even number of vertices n. Let f' be a partial scheme on C_n defined for $n - 2$ consecutive vertices of C_n. Let v, w be the two vertices for which f' is undefined. It is possible to find a matrix $M \in \mathcal{S}(C_n)$ such that M satisfies the partial scheme f', $\operatorname{rank} M = \mathrm{mr}(C_n)$, and $\operatorname{rank} M[v, w] = 2$.*

Proof. Without loss of generality, we assume the partial scheme is defined on the vertex set $\{1, 2, \ldots, n-2\}$. We proceed by induction on n. If $n = 4$, there are four possible partial schemes defined by: $f'(1) = f'(2) = 0$; $g'(1) = g'(2) = 1$; $h'(1) = 0, h'(2) = 1$; and

$j'(1) = 1, j'(2) = 0$. Observe that the matrices

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \in \mathcal{S}(C_4)$$

satisfy the partial schemes f', g', h', j' respectively. Also $\operatorname{mr}(C_4) = 2$ and each of these matrices has rank 2. Lastly, in each matrix the 2×2 submatrix obtained by deleting rows and columns 1 and 2 has rank 2. Thus for any partial scheme defined on two consecutive vertices of C_4 it is possible to find a matrix with the desired properties. Let $n \geq 4$ be even. We assume that for any partial scheme defined on $n-2$ consecutive vertices of C_n there exists a matrix in $\mathcal{S}(C_n)$ that has the three desired properties. Let f' be a scheme defined on the subset $\{1, 2, \ldots, n\}$ of vertices in C_{n+2}. We must show there exists $M \in C_{n+2}$ such that M satisfies f', $\operatorname{rank} M = \operatorname{mr}(C_{n+2})$, and $\operatorname{rank} M[n+1, n+2] = 2$. We construct a graph G from C_{n+2} by deleting vertices $n+1$ and $n+2$ from C_{n+2} and adding an edge between vertices 1 and n. We also construct a graph H by deleting vertices $2, 3, \ldots, n-1$ from C_{n+2} and adding an edge between vertices 1 and n.



The graph G is a cycle on n vertices and H is a cycle on 4 vertices. The cycles G and H inherit the vertex labels of C_{n+2} resulting in G having the standard labeling of C_n but $H = (\{1, n, n+1, n+2\}, \{1\,n, n\,n+1, n+1\,n+2, 1\,n+2\})$. Define a partial scheme g' on G by setting $g'(i) = f'(i)$ for $i \in \{2, 3, \ldots, n-1\}$. Since $G = C_n$ and g' is defined on $n-2$

consecutive vertices of G, we may apply the inductive hypothesis to obtain a matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & 0 & \cdots & 0 & b_{1n} \\ b_{21} & \ddots & \ddots & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & \ddots & \ddots & b_{n-1\,n} \\ b_{n1} & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} \end{bmatrix} \in \mathcal{S}(G)$$

such that B satisfies g', $\operatorname{rank} B = \operatorname{mr}(G) = n-2$, and $\operatorname{rank} B[1,n] = 2$. We must consider four cases based on the diagonal entries of $B[1,n]$.

Case 1 ($b_{11} = b_{nn} = 0$): We define a partial scheme h' on H by setting $h'(1) = f'(1)$ and $h'(n) = f'(n)$. From the base case, we know there exists

$$A = \begin{bmatrix} a_{11} & a_{1n} & 0 & a_{1\,n+2} \\ a_{n1} & a_{nn} & a_{n\,n+1} & 0 \\ 0 & a_{n+1\,n} & a_{n+1\,n+1} & a_{n+1\,n+2} \\ a_{n+2\,1} & 0 & a_{n+2\,n+1} & a_{n+2\,n+2} \end{bmatrix} \in \mathcal{S}_{h'}(H)$$

with $\operatorname{rank} A = 2$ and $\operatorname{rank} A[n+1, n+2] = 2$. We construct a matrix from A and B by letting C equal

$$\begin{bmatrix} 0 & b_{12} & 0 & \cdots & 0 & b_{1n} & 0 & 0 \\ b_{21} & b_{22} & \ddots & & & 0 & & \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & & \\ 0 & & & \ddots & b_{n-1\,n-1} & b_{n-1\,n} & 0 & 0 \\ b_{n1} & 0 & \cdots & 0 & b_{n\,n-1} & 0 & 0 & 0 \\ 0 & & \cdots & & 0 & 0 & 0 & 0 \\ 0 & & \cdots & & 0 & 0 & 0 & 0 \end{bmatrix} + s \begin{bmatrix} a_{11} & 0 & \cdots & \cdots & 0 & a_{1n} & 0 & a_{1\,n+2} \\ 0 & 0 & \ddots & & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & & \ddots & 0 & 0 & 0 & 0 \\ a_{n1} & 0 & & & 0 & a_{nn} & a_{n\,n+1} & 0 \\ 0 & 0 & \cdots & \cdots & 0 & a_{n+1\,n} & a_{n+1\,n+1} & a_{n+1\,n+2} \\ a_{n+2\,1} & 0 & \cdots & \cdots & 0 & 0 & a_{n+2\,n+1} & a_{n+2\,n+2} \end{bmatrix}$$

$$
= \begin{bmatrix}
sa_{11} & b_{12} & 0 & \cdots & 0 & b_{1n}+sa_{1n} & 0 & sa_{1\,n+2} \\
b_{21} & b_{22} & \ddots & & & 0 & 0 & 0 \\
0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 & & \\
0 & & & \ddots & b_{n-1\,n-1} & b_{n-1\,n} & 0 & 0 \\
b_{n1}+sa_{n1} & 0 & \cdots & 0 & b_{n\,n-1} & sa_{nn} & sa_{n\,n+1} & 0 \\
0 & 0 & \cdots & & 0 & sa_{n+1\,n} & sa_{n+1\,n+1} & sa_{n+1\,n+2} \\
sa_{n+2\,1} & 0 & \cdots & & 0 & 0 & sa_{n+2\,n+1} & sa_{n+2\,n+2}
\end{bmatrix}
$$

where s is a nonzero number chosen such that $b_{1n} + sa_{1n} = 0$. Thus $C \in \mathcal{S}_{f'}(C_{n+2})$ and $\operatorname{rank} C[n+1, n+2] = 2$. Also, $n = \operatorname{mr}(C_{n+2}) \leq \operatorname{rank} C \leq \operatorname{rank} A + \operatorname{rank} B = 2 + n - 2 = n$ so $\operatorname{rank} C = \operatorname{mr}(C_{n+2})$.

Case 2 ($b_{11}, b_{nn} \neq 0$): We make the following constructions depending on the values of $f'(1)$ and $f'(n)$.

Subcase 1 ($f'(1) = f'(n) = 1$): Define

$$
P = \begin{bmatrix}
b_{11} & b_{12} & 0 & \cdots & 0 & b_{1n} & 0 & 0 \\
b_{21} & \ddots & \ddots & & & 0 & & \\
0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 & & \\
0 & & & \ddots & \ddots & b_{n-1\,n} & 0 & 0 \\
b_{n1} & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} & 0 & 0 \\
0 & & \cdots & & 0 & 0 & 0 & 0 \\
0 & & \cdots & & 0 & 0 & 0 & 0
\end{bmatrix}
- b_{1n}
\begin{bmatrix}
0 & 0 & & \cdots & 0 & 1 & 0 & 1 \\
0 & \ddots & \ddots & & & 0 & & 0 \\
& \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & & & \\
0 & & & \ddots & 0 & 0 & 0 & 0 \\
1 & 0 & \cdots & & 0 & 0 & 1 & 0 \\
0 & & \cdots & & 0 & 1 & 0 & 1 \\
1 & 0 & \cdots & & 0 & 0 & 1 & 0
\end{bmatrix}
$$

$$
= \begin{bmatrix}
b_{11} & b_{12} & 0 & \cdots & 0 & 0 & 0 & -b_{1n} \\
b_{21} & \ddots & \ddots & & & 0 & & 0 \\
0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 & & \\
0 & & & \ddots & \ddots & b_{n-1\,n} & 0 & 0 \\
0 & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} & -b_{1n} & 0 \\
0 & & \cdots & & 0 & -b_{1n} & 0 & -b_{1n} \\
-b_{1n} & 0 & \cdots & & 0 & 0 & -b_{1n} & 0
\end{bmatrix}.
$$

Subcase 2 ($f'(1) = 1, f'(n) = 0$): Define

$$Q = \begin{bmatrix} b_{11} & b_{12} & 0 & \cdots & 0 & b_{1n} & 0 & 0 \\ b_{21} & \ddots & \ddots & & & 0 & & \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & & \\ 0 & & & \ddots & b_{n-1\,n-1} & b_{n-1\,n} & 0 & 0 \\ b_{n1} & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} & 0 & 0 \\ 0 & & \cdots & & 0 & 0 & 0 & 0 \\ 0 & & \cdots & & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & & \cdots & 0 & -b_{1n} & 0 & -b_{1n} \\ 0 & \ddots & \ddots & & & 0 & & 0 \\ & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & & \\ 0 & & & \ddots & 0 & 0 & 0 & 0 \\ -b_{1n} & 0 & \cdots & & 0 & -b_{nn} & -b_{1n} & 0 \\ 0 & & \cdots & & 0 & -b_{1n} & 0 & -b_{1n} \\ -b_{1n} & 0 & \cdots & & 0 & 0 & -b_{1n} & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & b_{12} & 0 & \cdots & 0 & 0 & 0 & -b_{1n} \\ b_{21} & \ddots & \ddots & & & 0 & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & & \\ 0 & & & \ddots & b_{n-1\,n-1} & b_{n-1\,n} & 0 & 0 \\ 0 & 0 & \cdots & 0 & b_{n\,n-1} & 0 & -b_{1n} & 0 \\ 0 & & \cdots & & 0 & -b_{1n} & 0 & -b_{1n} \\ -b_{1n} & 0 & \cdots & & 0 & 0 & -b_{1n} & b_{nn} \end{bmatrix}.$$

Subcase 3 ($f'(1) = 0, f'(n) = 1$: Define

$$R = \begin{bmatrix} b_{11} & b_{12} & 0 & \cdots & 0 & b_{1n} & 0 & 0 \\ b_{21} & b_{22} & \ddots & & & 0 & & \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & & \\ 0 & & & \ddots & \ddots & b_{n-1\,n} & 0 & 0 \\ b_{n1} & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} & 0 & 0 \\ 0 & & \cdots & & 0 & 0 & 0 & 0 \\ 0 & & \cdots & & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -b_{11} & 0 & & \cdots & 0 & -b_{1n} & 0 & -b_{1n} \\ 0 & 0 & \ddots & & & 0 & & 0 \\ & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & & \\ 0 & & & \ddots & 0 & 0 & 0 & 0 \\ -b_{1n} & 0 & \cdots & & 0 & 0 & -b_{1n} & 0 \\ 0 & & \cdots & & 0 & -b_{1n} & b_{11} & -b_{1n} \\ -b_{1n} & 0 & \cdots & & 0 & 0 & -b_{1n} & 0 \end{bmatrix}$$

$$
= \begin{bmatrix}
0 & b_{12} & 0 & \cdots & 0 & 0 & 0 & -b_{1n} \\
b_{21} & b_{22} & \ddots & & & 0 & & 0 \\
0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 & & \\
0 & & & \ddots & \ddots & b_{n-1\,n} & 0 & 0 \\
0 & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} & -b_{1n} & 0 \\
0 & & \cdots & & 0 & -b_{1n} & b_{11} & -b_{1n} \\
-b_{1n} & 0 & \cdots & & 0 & 0 & -b_{1n} & 0
\end{bmatrix}.
$$

Subcase 4 ($f'(1) = f'(n) = 0$): Define S to be

$$
\begin{bmatrix}
b_{11} & b_{12} & 0 & \cdots & 0 & b_{1n} & 0 & 0 \\
b_{21} & \ddots & \ddots & & & 0 & & \\
0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 & & \\
0 & & & \ddots & \ddots & b_{n-1\,n} & 0 & 0 \\
b_{n1} & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} & 0 & 0 \\
0 & & \cdots & & 0 & 0 & 0 & 0 \\
0 & & \cdots & & 0 & 0 & 0 & 0
\end{bmatrix}
+
\begin{bmatrix}
-b_{11} & 0 & & \cdots & 0 & -b_{1n} & 0 & x \\
0 & 0 & \ddots & & & 0 & & 0 \\
 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & & & \\
0 & & & \ddots & 0 & 0 & 0 & 0 \\
-b_{1n} & 0 & \cdots & & 0 & -b_{nn} & y & 0 \\
0 & & \cdots & & 0 & y & y & y \\
x & 0 & \cdots & & 0 & 0 & y & z
\end{bmatrix}
$$

$$
= \begin{bmatrix}
0 & b_{12} & 0 & \cdots & 0 & 0 & 0 & \frac{b_{11}b_{nn}}{b_{1n}} - b_{1n} \\
b_{21} & \ddots & \ddots & & & 0 & & 0 \\
0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 & & \\
0 & & & \ddots & b_{n-1\,n-1} & b_{n-1\,n} & 0 & 0 \\
0 & 0 & \cdots & 0 & b_{n\,n-1} & 0 & \frac{b_{1n}^2}{b_{11}} - b_{nn} & 0 \\
0 & & \cdots & & 0 & \frac{b_{1n}^2}{b_{11}} - b_{nn} & \frac{b_{1n}^2}{b_{11}} - b_{nn} & \frac{b_{1n}^2}{b_{11}} - b_{nn} \\
\frac{b_{11}b_{nn}}{b_{1n}} - b_{1n} & 0 & \cdots & & 0 & 0 & \frac{b_{1n}^2}{b_{11}} - b_{nn} & b_{nn} - \frac{b_{11}b_{nn}^2}{b_{1n}^2}
\end{bmatrix}
$$

where $x = \frac{b_{11}b_{nn}}{b_{1n}} - b_{1n}$, $y = \frac{b_{1n}^2}{b_{11}} - b_{nn}$, and $z = b_{nn} - \frac{b_{11}b_{nn}^2}{b_{1n}^2}$.

It is clear $P, Q, R \in \mathcal{S}_f(G)$ and $\operatorname{rank} P[n+1, n+2] = \operatorname{rank} Q[n+1, n+2] = \operatorname{rank} R[n+1, n+2] = 2$. To compute the rank of P, Q, R the same argument from case 1 may be used to show $\operatorname{rank} P = \operatorname{rank} Q = \operatorname{rank} R = \operatorname{mr}(C_{n+2})$. To show S has the same properties requires the following observations. First, by assumption $\operatorname{rank} B[1, n] = 2$ so

$\det B[1,n] = b_{11}b_{nn} - b_{1n}^2 \neq 0$. Therefore, $\frac{b_{1n}^2}{b_{11}} - b_{nn} \neq 0$ and $\frac{b_{11}b_{nn}}{b_{1n}} - b_{1n} \neq 0$ so $S \in \mathcal{S}_f(C_{n+2})$. Then

$$\begin{aligned}
\det S[n+1, n+2] &= \left(\frac{b_{1n}^2}{b_{11}} - b_{nn}\right)\left(b_{nn} - \frac{b_{11}b_{nn}^2}{b_{1n}^2}\right) - \left(\frac{b_{1n}^2}{b_{11}} - b_{nn}\right)^2 \\
&= \left(\frac{b_{1n}^2}{b_{11}} - b_{nn}\right)\left[\left(b_{nn} - \frac{b_{11}b_{nn}^2}{b_{1n}^2}\right) - \left(\frac{b_{1n}^2}{b_{11}} - b_{nn}\right)\right] \\
&= \left(\frac{b_{1n}^2}{b_{11}} - b_{nn}\right)\left(2b_{nn} - \frac{b_{11}b_{nn}^2}{b_{1n}^2} - \frac{b_{1n}^2}{b_{11}}\right) \\
&= \frac{1}{b_{11}}\left(b_{nn} - \frac{b_{1n}^2}{b_{11}}\right)\left[\left(\frac{b_{11}b_{nn}}{b_{1n}}\right)^2 - 2b_{11}b_{nn} + b_{1n}^2\right] \\
&= \frac{1}{b_{11}}\left(b_{nn} - \frac{b_{1n}^2}{b_{11}}\right)\left(\frac{b_{11}b_{nn}}{b_{1n}} - b_{1n}\right)^2 \\
&= \frac{\det B[1,n]}{b_{11}^2}\left(\frac{\det B[1,n]}{b_{1n}}\right)^2 \neq 0.
\end{aligned}$$

Therefore, $\operatorname{rank} S[n+1, n+2] = 2$. That $\operatorname{rank} S = \operatorname{mr}(C_{n+2})$ follows from the same reasoning as with the matrix in case 1.

Case 3 ($b_{11} = 0, b_{nn} \neq 0$): We construct matrices as follows:

Subcase 1 ($f'(1) = f'(n) = 0$): Define P to be

$$\begin{bmatrix}
0 & b_{12} & 0 & \cdots & 0 & b_{1n} & 0 & 0 \\
b_{21} & b_{22} & \ddots & & & 0 & & \\
0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 & & \\
0 & & & \ddots & \ddots & b_{n-1\,n} & 0 & 0 \\
b_{n1} & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} & 0 & 0 \\
0 & & \cdots & & 0 & 0 & 0 & 0 \\
0 & & \cdots & & 0 & 0 & 0 & 0
\end{bmatrix}
+
\begin{bmatrix}
0 & 0 & & \cdots & 0 & -b_{1n} & 0 & -b_{1n} \\
0 & \ddots & \ddots & & & 0 & & 0 \\
& \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & & & \\
0 & & & \ddots & 0 & 0 & 0 & 0 \\
-b_{1n} & 0 & \cdots & & 0 & -b_{nn} & -b_{1n} & 0 \\
0 & & \cdots & & 0 & -b_{1n} & 0 & -b_{1n} \\
-b_{1n} & 0 & \cdots & & 0 & 0 & -b_{1n} & b_{nn}
\end{bmatrix}$$

$$
= \begin{bmatrix}
0 & b_{12} & 0 & \cdots & 0 & 0 & 0 & -b_{1n} \\
b_{21} & \ddots & \ddots & & & 0 & & 0 \\
0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 & & \\
0 & & & \ddots & b_{n-1\,n-1} & b_{n-1\,n} & 0 & 0 \\
0 & 0 & \cdots & 0 & b_{n\,n-1} & 0 & -b_{1n} & 0 \\
0 & & \cdots & & 0 & -b_{1n} & 0 & -b_{1n} \\
-b_{1n} & 0 & \cdots & & 0 & 0 & -b_{1n} & b_{nn}
\end{bmatrix}.
$$

Subcase 2 ($f'(1) = f'(n) = 1$): Define Q to be

$$
\begin{bmatrix}
0 & b_{12} & 0 & \cdots & 0 & b_{1n} & 0 & 0 \\
b_{21} & b_{22} & \ddots & & & 0 & & \\
0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 & & \\
0 & & & \ddots & \ddots & b_{n-1\,n} & 0 & 0 \\
b_{n1} & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} & 0 & 0 \\
0 & & \cdots & & 0 & 0 & 0 & 0 \\
0 & & \cdots & & 0 & 0 & 0 & 0
\end{bmatrix}
+
\begin{bmatrix}
-b_{11} & 0 & & \cdots & 0 & -b_{1n} & 0 & -b_{1n} \\
0 & 0 & \ddots & & & 0 & & 0 \\
 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & & & \\
0 & & & \ddots & 0 & 0 & 0 & 0 \\
-b_{1n} & 0 & \cdots & & 0 & 0 & -b_{1n} & 0 \\
0 & & \cdots & & 0 & -b_{1n} & b_{11} & -b_{1n} \\
-b_{1n} & 0 & \cdots & & 0 & 0 & -b_{1n} & 0
\end{bmatrix}
$$

$$
= \begin{bmatrix}
-b_{11} & b_{12} & 0 & \cdots & 0 & 0 & 0 & -b_{1n} \\
b_{21} & b_{22} & \ddots & & & 0 & & 0 \\
0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 & & \\
0 & & & \ddots & \ddots & b_{n-1\,n} & 0 & 0 \\
0 & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} & -b_{1n} & 0 \\
0 & & \cdots & & 0 & -b_{1n} & b_{11} & -b_{1n} \\
-b_{1n} & 0 & \cdots & & 0 & 0 & -b_{1n} & 0
\end{bmatrix}.
$$

Subcase 3 ($f'(1)=0, f'(n)=1$): Define

$$R = \begin{bmatrix} 0 & b_{12} & 0 & \cdots & 0 & b_{1n} & 0 & 0 \\ b_{21} & b_{22} & \ddots & & & 0 & & \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & & \\ 0 & & & \ddots & \ddots & b_{n-1\,n} & 0 & 0 \\ b_{n1} & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} & 0 & 0 \\ 0 & & \cdots & & 0 & 0 & 0 & 0 \\ 0 & & \cdots & & 0 & 0 & 0 & 0 \end{bmatrix} - b_{1n} \begin{bmatrix} 0 & 0 & & \cdots & 0 & 1 & 0 & 1 \\ 0 & \ddots & \ddots & & & 0 & & 0 \\ & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & & \\ 0 & & & \ddots & 0 & 0 & 0 & 0 \\ 1 & 0 & \cdots & & 0 & 0 & 1 & 0 \\ 0 & \cdots & & & 0 & 1 & 0 & 1 \\ 1 & 0 & \cdots & & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & b_{12} & 0 & \cdots & 0 & 0 & 0 & -b_{1n} \\ b_{21} & b_{22} & \ddots & & & 0 & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & & \\ 0 & & & \ddots & \ddots & b_{n-1\,n} & 0 & 0 \\ 0 & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} & -b_{1n} & 0 \\ 0 & & \cdots & & 0 & -b_{1n} & 0 & -b_{1n} \\ -b_{1n} & 0 & \cdots & & 0 & 0 & -b_{1n} & 0 \end{bmatrix}.$$

Subcase 4 ($f'(1)=1, f'(n)=0$): Define

$$S = \begin{bmatrix} 0 & b_{12} & 0 & \cdots & 0 & b_{1n} & 0 & 0 \\ b_{21} & b_{22} & \ddots & & & 0 & & \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & & \\ 0 & & & \ddots & \ddots & b_{n-1\,n} & 0 & 0 \\ b_{n1} & 0 & \cdots & 0 & b_{n\,n-1} & b_{nn} & 0 & 0 \\ 0 & & \cdots & & 0 & 0 & 0 & 0 \\ 0 & & \cdots & & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{b_{1n}^2}{b_{nn}} & 0 & & & 0 & -b_{1n} & 0 & -2b_{1n} \\ 0 & 0 & \ddots & & & 0 & & 0 \\ & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & & \\ 0 & & & \ddots & 0 & 0 & 0 & 0 \\ -b_{1n} & 0 & \cdots & & 0 & -b_{nn} & -2b_{nn} & 0 \\ 0 & & \cdots & & 0 & -2b_{nn} & -2b_{nn} & -2b_{nn} \\ -2b_{1n} & 0 & \cdots & & 0 & 0 & -2b_{nn} & 2b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{b_{1n}^2}{b_{nn}} & b_{12} & 0 & \cdots & 0 & 0 & 0 & -2b_{1n} \\ b_{21} & \ddots & \ddots & & & 0 & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & & \\ 0 & & & \ddots & b_{n-1\,n-1} & b_{n-1\,n} & 0 & 0 \\ 0 & 0 & \cdots & 0 & b_{n\,n-1} & 0 & -2b_{nn} & 0 \\ 0 & & \cdots & & 0 & -2b_{nn} & -2b_{nn} & -2b_{nn} \\ -2b_{1n} & 0 & \cdots & & 0 & 0 & -2b_{nn} & 2b_{nn} \end{bmatrix}.$$

Then $P, Q, R, S \in \mathcal{S}_f(C_{n+1})$, $\operatorname{rank} P[n+1, n+2] = \operatorname{rank} Q[n+1, n+2] = \operatorname{rank} R[n+1, n+2] = \operatorname{rank} S[n+1, n+2] = 2$, and $\operatorname{rank} P = \operatorname{rank} Q = \operatorname{rank} R = \operatorname{rank} S = \operatorname{mr}(C_{n+2})$.

Case 4 ($b_{11} \neq 0, b_{nn} = 0$): This case does not vary significantly from case 3. $\square$

Theorem 3.19. *Let f be a scheme on C_n, with n even. Suppose f alternates sending vertices to zero and one so adjacent vertices are not mapped to the same value. Then f is constructible.*

Proof. Assume $f(\{1, 3, 5, \ldots, n-1\}) = 1$ and $f(\{2, 4, \ldots, n\}) = 0$. Recall $\operatorname{mr}(C_n) = n - 2$. For C_4, $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ is rank 2 so f is constructible for C_4. Assume f is constructible for $C_k, k < n$. We show f is constructible for C_n. Let g be the scheme on C_{n-2} defined by $g(\{1, 3, 5, \ldots, n-3\}) = 1$ and $g(\{2, 4, \ldots, n-2\}) = 0$. By assumption g is constructible so there exists

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1\,n-2} \\ a_{12} & 0 & a_{23} & & & 0 \\ 0 & a_{23} & a_{33} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & & 0 \\ 0 & & & & a_{n-3\,n-3} & a_{n-3\,n-2} \\ a_{1\,n-2} & 0 & \cdots & 0 & a_{n-3\,n-2} & 0 \end{bmatrix} \in \mathcal{S}_g(C_{n-2})$$

with rank $A = \operatorname{mr}(C_{n-2}) = n-4$. Define $B = \begin{bmatrix} -a_{11} & a_{1\,n-2} & 0 & a_{1\,n-2} \\ a_{1\,n-2} & 0 & a & 0 \\ 0 & a & a & a \\ a_{1\,n-2} & 0 & a & 0 \end{bmatrix}$ where $a = \frac{a_{1\,n-2}^2}{a_{11}}$

so rank $B = 2$. Define C to be

$$\begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1\,n-2} & 0 & 0 \\ a_{12} & 0 & a_{23} & & & 0 & \vdots & \vdots \\ 0 & a_{23} & a_{33} & \ddots & & \vdots & & \\ \vdots & & \ddots & \ddots & & 0 & \vdots & \vdots \\ 0 & & & & a_{n-3\,n-3} & a_{n-3\,n-2} & 0 & 0 \\ a_{1\,n-2} & 0 & \cdots & 0 & a_{n-3\,n-2} & 0 & 0 & 0 \\ 0 & \cdots & & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -a_{11} & 0 & \cdots & \cdots & 0 & a_{1\,n-2} & 0 & a_{1\,n-2} \\ 0 & 0 & \ddots & & & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & & & & & \\ \vdots & & & & & & & \\ 0 & & & & & & & \\ a_{1\,n-2} & & & & & 0 & a & 0 \\ 0 & & & & & a & a & a \\ a_{1\,n-2} & & & & & 0 & a & 0 \end{bmatrix}$$

$$
= \begin{bmatrix}
2a_{11} & a_{12} & 0 & \cdots & 0 & 0 & 0 & -a_{1\,n-2} \\
a_{12} & 0 & a_{23} & & & 0 & \vdots & \vdots \\
0 & a_{23} & & \ddots & & \vdots & & \\
\vdots & & \ddots & \ddots & & 0 & \vdots & \vdots \\
0 & & & & a_{n-3\,n-3} & a_{n-3\,n-2} & 0 & 0 \\
0 & 0 & \cdots & 0 & a_{n-3\,n-2} & 0 & a & 0 \\
0 & \cdots & & \cdots & 0 & a & a & a \\
-a_{1\,n-2} & \cdots & & \cdots & 0 & 0 & a & 0
\end{bmatrix}.
$$

Note that $C \in \mathcal{S}_f(C_n)$ and $n-2 = \operatorname{mr}(C_n) \leq \operatorname{rank} C \leq \operatorname{rank} A + \operatorname{rank} B = n-4+2 = n-2$ so $\operatorname{rank} C = n-2$. Therefore, f is constructible. $\square$

We now aim to provide a classification of the constructible schemes on cycles. Since the matrices in $\mathcal{S}(C_n)$ have the property that any $n-1 \times n-1$ principal submatrix is a matrix in $\mathcal{S}(P_{n-1})$, it is natural to use the classification of constructible schemes of paths given in Theorems 3.8 and 3.9 to provide necessary and sufficient conditions for the constructible schemes on cycles.

Definition 3.20. Let f be a scheme on a graph G. We say the scheme f has the *vertex deletion property* if for every vertex v of G the scheme induced by f on $G-v$ is constructible.

Theorem 3.21. *Let C_n be a cycle on n vertices and let f be a scheme on C_n. Then f is a constructible scheme on C_n if and only if f has the vertex deletion property.*

Proof. First, suppose f is a constructible scheme on C_n and v is a vertex of C_n. Since f is constructible, there exists a matrix $M \in \mathcal{S}_f(C_n)$ such that $\operatorname{rank} M = \operatorname{mr}(C_n) = n-2$. Let g be the scheme on $C_n - v$ induced by f. Note that $C_n - v$ is a path on $n-1$ vertices so $M(v) \in \mathcal{S}(P_n)$. Also $M(v)$ satisfies g. We must show g is constructible. By Nylen's lemma, either $\operatorname{rank} M(v) = \operatorname{rank} M - 2$ or $\operatorname{rank} M(v) = \operatorname{rank} M$. In the first case, $\operatorname{rank} M(v) = \operatorname{rank} M - 2 = n-2-2 = n-4$. However, $M(v) \in \mathcal{S}(P_{n-1})$ and $\operatorname{mr}(P_{n-1}) = n-2$ so $\operatorname{rank} M(v) = n-4$ is not possible. Therefore, we have $\operatorname{rank} M(v) = \operatorname{rank} M = n-2 =$

$\mathrm{mr}(P_{n-1}) = \mathrm{mr}(C_n - v)$. Therefore g is constructible. Hence, f has the vertex deletion property.

To prove the reverse implication we proceed by induction on the number of vertices n. Let f be a scheme on C_3 and suppose f has the property that for each vertex v, the scheme induced by f on $C_3 - v$ is constructible. For each vertex v, $C_3 - v$ is a complete graph on 2 vertices. Recall a scheme on a complete graph is only constructible if it sends each vertex to one. Thus since the scheme induced by f after deleting vertex 1 must be constructible, we have $f(2) = f(3) = 1$. Also the scheme induced by f after deleting vertex 2 must be constructible so we must have $f(1) = f(3) = 1$. Hence, the scheme defined by $f(1) = f(2) = f(3) = 1$ is the only scheme on C_3 satisfying the necessary property. Since $J_3 \in \mathcal{S}_f(C_3)$ and $\operatorname{rank} J_3 = \mathrm{mr}(C_3) = 1$, f is constructible. The computations of the minimum ranks of schemes on C_4 in Example 2.13 show the implication holds for C_4. By way of induction, suppose C is a cycle on fewer than $n \geq 5$ vertices and suppose any scheme g of C is constructible if it has the vertex deletion property. Let f be a scheme on $C_n, n \geq 5$ with the vertex deletion property. We must show f is constructible.

Case 1. We first suppose f sends at least two consecutive vertices to zero. Without loss of generality, $f(n-1) = f(n) = 0$. Let G be the graph obtained by deleting vertices $n-1$ and n from C_n and adding an edge between vertices 1 and $n-2$. Let g be the scheme on G induced by f. Note G is a cycle on $n-2$ vertices. We claim g has the vertex deletion property. Let v be a vertex of G. Note $C_n - v$ is a path. The scheme that f induces on $C_n - v$ is constructible since f has the vertex deletion property and the induced scheme maps two consecutive vertices $n-1, n$ to zero. It follows from Theorem 3.11 that the scheme induced by g on the path $G - v$ is constructable. Therefore, g has the vertex deletion property so g

is constructible by induction. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1\,n-2} \\ a_{21} & \ddots & \ddots & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & \ddots & \ddots & a_{n-3\,n-2} \\ a_{1\,n-2} & 0 & \cdots & 0 & a_{n-3\,n-2} & a_{n-2\,n-2} \end{bmatrix} \in \mathcal{S}_g(G)$$

with $\operatorname{rank} A = \operatorname{mr}(G) = \operatorname{mr}(C_{n-2}) = n-4$. Define $B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$. Note $\operatorname{rank} B = 2$.

Define M to be

$$\begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1\,n-2} & 0 & 0 \\ a_{21} & \ddots & \ddots & & & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & 0 & 0 \\ 0 & & & \ddots & \ddots & a_{n-3\,n-2} & 0 & 0 \\ a_{1\,n-2} & 0 & \cdots & 0 & a_{n-3\,n-2} & a_{n-2\,n-2} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - a_{1\,n-2} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\ 0 & \ddots & \ddots & & & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & 0 & 0 \\ 0 & & & \ddots & \ddots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$
= \begin{bmatrix}
a_{11} & a_{12} & 0 & \cdots & 0 & 0 & 0 & -a_{1\,n-2} \\
a_{21} & \ddots & \ddots & & & 0 & 0 & 0 \\
0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 & 0 & 0 \\
0 & & & \ddots & \ddots & a_{n-3\,n-2} & 0 & 0 \\
0 & 0 & \cdots & 0 & a_{n-3\,n-2} & a_{n-2\,n-2} & -a_{1\,n-2} & 0 \\
0 & 0 & \cdots & 0 & 0 & -a_{1\,n-2} & 0 & -a_{1\,n-2} \\
-a_{1\,n-2} & 0 & \cdots & 0 & 0 & 0 & -a_{1\,n-2} & 0
\end{bmatrix}.
$$

Observe that $M \in \mathcal{S}_f(C_n)$ and $n-2 = \operatorname{mr}(C_n) \le \operatorname{rank} M \le \operatorname{rank} A + \operatorname{rank} B = n-4+2 = n-2$ so $\operatorname{rank} M = n-2$. Therefore, f is constructible.

Case 2. We suppose f sends at least three consecutive vertices to one and f doesn't send any consecutive vertices to zero. Without loss of generality, $f(n-1) = f(n) = f(1) = 1$. Let G be the graph obtained by deleting vertex n from C_n and adding an edge between vertices 1 and $n-1$. We must separately consider the cases where n is even and where n is odd.

We start with the case where n is even. Since n is even, $C_n - (n-1) = P_{n-1}$ is a path on an odd number of vertices. Since f has the vertex deletion property the scheme induced on $C_n - (n-1)$ must be constructible. Since $f(n) = 1$, Theorem 3.13 implies f must map at least one vertex in $D = \{2, 4, \ldots, n-2\}$ to one, say $w \in D$ with $f(w) = 1$.

Let g be a scheme on G defined by $g(i) = f(i)$ for $1 \le i \le n-1$ so g agrees with f on all vertices. We claim g has the vertex deletion property. Let u be a vertex of G. Let h be the scheme on $G-u$ induced by g. If $u = n-1$, $G-u$ is an even path with the standard labeling and $h(1) = h(w) = 1$ so h is constructible by Theorem 3.9. If $u = 1$, $G-u$ is an even path with the alternating parity labeling with vertex 2 as the start vertex. Then $w \prec n-1$, w has the same parity as the start vertex, $n-1$ has the opposite parity, and $h(w) = h(n-1) = 1$ so h is constructible by Theorem 3.14. If $u \neq 1, n-1$, $G-u$ is an even path with the split alternating parity labeling and the split vertex is either 1 or $n-1$ depending on the selection of the start vertex. Suppose u is even. Then the even path has a start vertex with odd parity and 1 and $n-1$ satisfy condition (iii) of Theorem 3.15 so h is constructible. Suppose u is odd so $u \neq w$. Then the even path $G-u$ has a start vertex/finish vertex with even parity. Choose the start vertex such that $w \prec n-1$ and $w \prec 1$. Without loss of generality we assume $n-1 \prec 1$ so w and $n-1$ satisfy condition (i) of Theorem 3.14 and therefore h is constructible.

Hence, g has the vertex deletion property. Since G is a cycle on $n-1$ vertices, g is constructible by induction. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & & 0 & a_{1\,n-1} \\ a_{12} & \ddots & \ddots & & & & 0 \\ 0 & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & & \ddots & 0 \\ 0 & & & \ddots & & \ddots & a_{n-2\,n-1} \\ a_{1\,n-1} & 0 & \cdots & 0 & & a_{n-2\,n-1} & a_{n-1\,n-1} \end{bmatrix} \in \mathcal{S}_g(G)$$

with $\operatorname{rank} A = n-3$. Let $B = \begin{bmatrix} t & a_{1\,n-1} & t \\ a_{1\,n-1} & \frac{a_{1\,n-1}^2}{t} & a_{1\,n-1} \\ t & a_{1\,n-1} & t \end{bmatrix}$ where $t \neq 0$ is selected so that $a_{11}-t \neq 0$ and $a_{n-1\,n-1} - \frac{a_{1\,n-1}^2}{t} \neq 0$. Note $\operatorname{rank} B = 1$. Define M to be

$$\begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1\,n-1} & 0 \\ a_{21} & \ddots & \ddots & & & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & & & \ddots & \ddots & a_{n-2\,n-1} & 0 \\ a_{1\,n-1} & 0 & \cdots & 0 & a_{n-2\,n-1} & a_{n-1\,n-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} t & 0 & 0 & \cdots & 0 & a_{1\,n-1} & t \\ 0 & \ddots & \ddots & & & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & & & \ddots & \ddots & 0 & 0 \\ a_{1\,n-1} & 0 & \cdots & 0 & 0 & \frac{a_{1\,n-1}^2}{t} & a_{1\,n-1} \\ t & 0 & \cdots & 0 & 0 & a_{1\,n-1} & t \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}-t & a_{12} & 0 & \cdots & 0 & 0 & -t \\ a_{21} & \ddots & \ddots & & & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & & & \ddots & \ddots & a_{n-2\,n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_{n-2\,n-1} & a_{n-1\,n-1}-\frac{a_{1\,n-1}^2}{t} & -a_{1\,n-1} \\ -t & 0 & \cdots & 0 & 0 & -a_{1\,n-1} & -t \end{bmatrix}.$$

Observe that $M \in \mathcal{S}_f(C_n)$ and $n-2 = \operatorname{mr}(C_n) \leq \operatorname{rank} M \leq \operatorname{rank} A + \operatorname{rank} B = n-3+1 = n-2$ so $\operatorname{rank} M = n-2$. Therefore, f is constructible.

Suppose n is odd so G is a cycle on an even number of vertices. Define a partial scheme g' on G by g'(i)=f(i) for $2 \leq i \leq n-2$ then by Lemma 3.18 there exists

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1\,n-1} \\ a_{12} & \ddots & \ddots & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & \ddots & \ddots & a_{n-2\,n-1} \\ a_{1\,n-1} & 0 & \cdots & 0 & a_{n-2\,n-1} & a_{n-1\,n-1} \end{bmatrix} \in \mathcal{S}_{g'}(G)$$

with $\operatorname{rank} A = n-3$ and $\operatorname{rank} A[1, n-1] = 2$. Let $B = \begin{bmatrix} t & a_{1\,n-1} & t \\ a_{1\,n-1} & \frac{a_{1\,n-1}^2}{t} & a_{1\,n-1} \\ t & a_{1\,n-1} & t \end{bmatrix}$ where $t \neq 0$ is selected so that $a_{11} - t \neq 0$ and $a_{n-1\,n-1} - \frac{a_{1\,n-1}^2}{t} \neq 0$. Note it is possible to choose t

regardless of whether a_{11} and $a_{n-1\,n-1}$ are zero or nonzero. Also rank $B = 1$. Define M as

$$\begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1\,n-1} & 0 \\ a_{21} & \ddots & \ddots & & & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & & & \ddots & \ddots & a_{n-2\,n-1} & 0 \\ a_{1\,n-1} & 0 & \cdots & 0 & a_{n-2\,n-1} & a_{n-1\,n-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} t & 0 & 0 & \cdots & 0 & a_{1\,n-1} & t \\ 0 & \ddots & \ddots & & & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & & & \ddots & \ddots & 0 & 0 \\ a_{1\,n-1} & 0 & \cdots & 0 & 0 & \frac{a_{1\,n-1}^2}{t} & a_{1\,n-1} \\ t & 0 & \cdots & 0 & 0 & a_{1\,n-1} & t \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}-t & a_{12} & 0 & \cdots & 0 & 0 & -t \\ a_{21} & \ddots & \ddots & & & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & & & \ddots & \ddots & a_{n-2\,n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_{n-2\,n-1} & a_{n-1\,n-1} - \frac{a_{1\,n-1}^2}{t} & -a_{1\,n-1} \\ -t & 0 & \cdots & 0 & 0 & -a_{1\,n-1} & -t \end{bmatrix}.$$

Observe that $M \in \mathcal{S}_f(C_n)$ and $n-2 = \mathrm{mr}(C_n) \leq \operatorname{rank} M \leq \operatorname{rank} A + \operatorname{rank} B = n-3+1 = n-2$ so $\operatorname{rank} M = n-2$. Therefore, f is constructible.

Case 3. Suppose f sends at most two consecutive vertices to one and does not send any consecutive vertices to zero.

If f sends no consecutive vertex pairs to one then since f sends no consecutive vertex pairs to zero, n must be even and f must alternate sending vertices to zero and one so Theorem 3.19 implies f is constructible.

Therefore we may assume f sends at least one pair of consecutive vertices to one. Without loss of generality, $f(n-1) = f(n) = 1$. Then $f(n-2) = f(1) = 0$ because no more than two consecutive vertices are sent to one. Then $f(n-3) = f(2) = 1$ since consecutive vertices cannot be sent to zero.

We now consider the case where n is even. Let g be a scheme on G defined by $g(i) = f(i)$ for $2 \leq i \leq n-2$, $g(1) = 1, g(n-1) = 0$ so g agrees with f on all vertices except vertices 1 and $n-1$. We claim g has the vertex deletion property. Let u be a vertex of G. Let h be the scheme on $G - u$ induced by g. If $u = n-1$, $G - u$ is an even path with the standard labeling and $h(1) = h(2) = 1$ so by Theorem 3.9 h is constructible. If $u = 1$, $G - u$ is an even path with the alternating parity labeling with vertex 2 as the start vertex. Then $2 \prec n-3$ and $n-3$ has the opposite parity as the start vertex. Also $h(2) = h(n-3) = 1$ so h is constructible by Theorem 3.14. If $u \neq 1, n-1$, $G - u$ is an even path with the split alternating parity labeling and the split vertex is either 1 or $n-1$ depending on the selection of the start vertex. Suppose u is odd. Then the even path has a start vertex with even parity. Choose the start vertex so that $2 \prec 1$. Then 2 and 1 satisfy condition (i) of Theorem 3.15 so h is constructible. Suppose u is even so $u \neq n-3$. Then the even path $G - u$ has a start vertex/finish vertex with odd parity. Choose the start vertex such that $n-3 \prec 1$ and $n-3 \prec 2$. Without loss of generality we assume $1 \prec 2$ so $n-3$ and 1 satisfy condition (iii) of Theorem 3.14 and therefore h is constructible. Hence, g has the vertex deletion property so g is constructible.

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1\,n-1} \\ a_{12} & \ddots & \ddots & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & \ddots & a_{n-2\,n-2} & a_{n-2\,n-1} \\ a_{1\,n-1} & 0 & \cdots & 0 & a_{n-2\,n-1} & 0 \end{bmatrix} \in \mathcal{S}_g(G)$$

with rank $A = n-3$. Let $B = \begin{bmatrix} a_{11} & a_{1\,n-1} & a_{11} \\ a_{1\,n-1} & \frac{a_{1\,n-1}^2}{a_{11}} & a_{1\,n-1} \\ a_{11} & a_{1\,n-1} & a_{11} \end{bmatrix}$. Note rank $B = 1$. Define M as

$$\begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1\,n-1} & 0 \\ a_{21} & \ddots & \ddots & & & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & & & \ddots & a_{n-2\,n-2} & a_{n-2\,n-1} & 0 \\ a_{1\,n-1} & 0 & \cdots & 0 & a_{n-2\,n-1} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 & a_{1\,n-1} & a_{11} \\ 0 & \ddots & \ddots & & & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & & & \ddots & \ddots & 0 & 0 \\ a_{1\,n-1} & 0 & \cdots & 0 & 0 & \frac{a_{1\,n-1}^2}{a_{11}} & a_{1\,n-1} \\ a_{11} & 0 & \cdots & 0 & 0 & a_{1\,n-1} & a_{11} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a_{12} & 0 & \cdots & 0 & 0 & -a_{11} \\ a_{21} & a_{22} & \ddots & & & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & & & \ddots & \ddots & a_{n-2\,n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_{n-2\,n-1} & -\frac{a_{1\,n-1}^2}{a_{11}} & -a_{1\,n-1} \\ -a_{11} & 0 & \cdots & 0 & 0 & -a_{1\,n-1} & -a_{11} \end{bmatrix}.$$

Observe that $M \in \mathcal{S}_f(C_n)$ and $n-2 = \operatorname{mr}(C_n) \leq \operatorname{rank} M \leq \operatorname{rank} A + \operatorname{rank} B = n-3+1 = n-2$ so rank $M = n-2$. Therefore, f is constructible.

Suppose n is odd so G is a cycle on an even number of vertices. Also f must induce a constructible scheme on $C_n - n$. Since $C_n - n$ is an even path and $f(1) = f(n-2) = 0$, there is $w \in \{3, 5, n-4\}$ such that $f(w) = 1$. Define a scheme g on G as follows: $g(i) = f(i)$ for $2 \leq i \leq n-2$, $g(n-1) = 0$, $g(1) = 1$. Let u be a vertex of G and let h be the scheme on $G - u$ induced by g. Note $G - u$ has the alternating parity labeling. If u is odd, then $u \neq 2, n-3$, $G - u$ has a start vertex with even parity, and 2 and $n-3$ have the same parity. Since $h(2) = h(n-3) = 1$, Theorem 3.13 implies h is constructible. If u is even, then $u \neq 1, w$, $G - u$ has a start vertex with odd parity, and 1 and w have the same parity. Since $h(1) = h(w) = 1$, Theorem 3.13 implies h is constructible. Hence, g has the vertex deletion property so g is constructible. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1\,n-1} \\ a_{12} & \ddots & \ddots & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & \ddots & \ddots & a_{n-2\,n-1} \\ a_{1\,n-1} & 0 & \cdots & 0 & a_{n-2\,n-1} & 0 \end{bmatrix} \in \mathcal{S}_g(G)$$

with $\operatorname{rank} A = n-3$. Let $B = \begin{bmatrix} -a_{11} & -a_{1\,n-1} & -a_{11} \\ -a_{1\,n-1} & -\frac{a_{1\,n-1}^2}{a_{11}} & -a_{1\,n-1} \\ -a_{11} & -a_{1\,n-1} & -a_{11} \end{bmatrix}$. Note $\operatorname{rank} B = 1$. Define M as

$$\begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1\,n-1} & 0 \\ a_{21} & \ddots & \ddots & & & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & & & \ddots & \ddots & a_{n-2\,n-1} & 0 \\ a_{1\,n-1} & 0 & \cdots & 0 & a_{n-2\,n-1} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -a_{11} & 0 & 0 & \cdots & 0 & -a_{1\,n-1} & -a_{11} \\ 0 & \ddots & \ddots & & & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & & & \ddots & \ddots & 0 & 0 \\ -a_{1\,n-1} & 0 & \cdots & 0 & 0 & -\frac{a_{1\,n-1}^2}{a_{11}} & -a_{1\,n-1} \\ -a_{11} & 0 & \cdots & 0 & 0 & -a_{1\,n-1} & -a_{11} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a_{12} & 0 & \cdots & 0 & 0 & -a_{11} \\ a_{21} & a_{22} & \ddots & & & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & & & \ddots & \ddots & a_{n-2\,n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_{n-2\,n-1} & -\frac{a_{1\,n-1}^2}{a_{11}} & -a_{1\,n-1} \\ -a_{11} & 0 & \cdots & 0 & 0 & -a_{1\,n-1} & -a_{11} \end{bmatrix}.$$

Observe that $M \in \mathcal{S}_f(C_n)$ and $n-2 = \operatorname{mr}(C_n) \leq \operatorname{rank} M \leq \operatorname{rank} A + \operatorname{rank} B = n-3+1 = n-2$ so $\operatorname{rank} M = n-2$. Therefore, f is constructible.

□

It is a simple matter of counting to see that for a graph on n vertices the total number of schemes is 2^n. For a given graph G, we ask how many of the schemes of G are constructible. We compute the total number of construcible schemes of paths.

Proposition 4.1. *Given a path P_n on an odd number of vertices $n = 2k+1$, the number of constructructible schemes is $2^{2k+1} - (k+1)2^k$.*

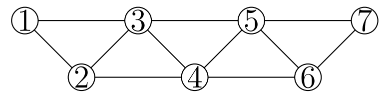
Proof. By Theorem 3.8 the schemes which are not constructible send exactly one odd labeled vertex to one. We count the nonconstructible schemes. There are $k+1$ odd vertices. Given a chosen odd vertex, all odd vertices are fixed. Then each of the k even vertices may be mapped to zero or one. Hence, the total number of noncostructible schemes is $(k+1)2^k$ and the result follows by deducting from the 2^{2k+1} total possible schemes on P_{2k+1}. □

Proposition 4.2. *Given a path P_n on an even number of vertices $n = 2k$, the number of constructible schemes is $2^{2k} - (k+2)2^{k-1}$.*

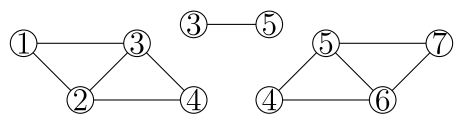
Proof. By Theorem 3.9 the constructible schemes must have vertices $i < j$ such that i is odd, j is even, and $f(i) = f(j) = 1$. We count nonconstructible schemes. First, let all odd vertices be mapped to zero. Then each even vertex is free to be mapped to zero or one so there are 2^k such schemes. Next pick an odd vertex s to be the $\min\{t|t \text{ is odd and } f(t) = 1\}$. There are k choices for s. For each choice the mapping of the odd vertices less than s are fixed and the even vertices greater than s are fixed. Hence, there are $k-1$ free vertices resulting in $k2^{k-1}$ nonconstructible schemes where at least one odd vertex is mapped to one. Combining with the total number of schemes where every odd vertex is mapped to zero gives $k2^{k-1} + 2^k = (k+2)2^{k-1}$ nonconstructible schemes. Deducting from the 2^{2k} total possible schemes gives the desired result. □

It can be verified that there are nonconstructible schemes of all graphs on six or fewer vertices. In fact, from Theorem 2.16 we see K_n has exactly one constructible scheme. Therefore, the gap between the number of constructible schemes and the total number of schemes may be arbitrarily large. Finding a graph for which all schemes are constructible is possible on seven vertices.

Example 4.3. Let G be a 2-path on 7 vertices. It is known that $\mathrm{mr}(G) = 5$.



We will demonstrate that all 2^7 schemes on G are constructible. We will use specific decompositions of the graph into subgraphs to build matrices for each scheme. For instance, there are 2^4 schemes which map vertices $1, 4$, and 7 to zero. To construct these we consider the following decomposition of G into subgraphs:



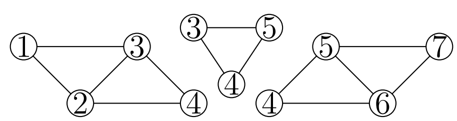
Then we combine minimum rank matrices from each subgraph to construct a matrix in $\mathcal{S}(G)$ satisfying a scheme f that maps vertices $1, 4$, and 7 to zero.

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & x & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & y & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & \sqrt{ab} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{ab} & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & w & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & z & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & x & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & y-a & 1 & -\sqrt{ab} & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -\sqrt{ab} & 1 & w-b & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & z & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

where $x = \begin{cases} 0, & \text{if } f(2) = 0 \\ 1, & \text{if } f(2) = 1 \end{cases}$, $y = \begin{cases} 1, & \text{if } f(2) = 1 \\ 2, & \text{if } f(2) = 0 \end{cases}$, $z = \begin{cases} 0, & \text{if } f(6) = 0 \\ 1, & \text{if } f(6) = 1 \end{cases}$,
$w = \begin{cases} 1, & \text{if } f(6) = 1 \\ 2, & \text{if } f(6) = 0 \end{cases}$, $a = \begin{cases} y, & \text{if } f(3) = 0 \\ 2y, & \text{if } f(3) = 1 \end{cases}$, $b = \begin{cases} w, & \text{if } f(5) = 0 \\ 2w, & \text{if } f(5) = 1 \end{cases}$. Note $a, b > 0$ so $\sqrt{ab}$ is a nonzero real number. Then $M \in \mathcal{S}_f(G)$ and $5 = \operatorname{mr}(G) \leq \operatorname{rank} M \leq 2 + 1 + 2 = 5$.

Let f be a scheme mapping vertices 1 and 7 to zero and vertex 4 to one. We build minimum rank matrices by considering the following decomposition.



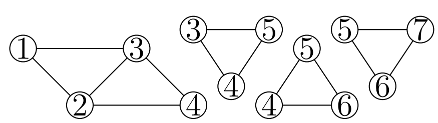
Then let

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & x & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & y & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 3 & \sqrt{ab} & 0 & 0 \\ 0 & 0 & 3 & \frac{9}{a} & 3\sqrt{\frac{b}{a}} & 0 & 0 \\ 0 & 0 & \sqrt{ab} & 3\sqrt{\frac{b}{a}} & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & w & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & z & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & x & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & y-a & -2 & -\sqrt{ab} & 0 & 0 \\ 0 & 1 & -2 & -\frac{9}{a} & 1-3\sqrt{\frac{b}{a}} & 1 & 0 \\ 0 & 0 & -\sqrt{ab} & 1-3\sqrt{\frac{b}{a}} & w-b & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & z & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

where $x = \begin{cases} 0, & \text{if } f(2) = 0 \\ 1, & \text{if } f(2) = 1 \end{cases}$, $y = \begin{cases} 1, & \text{if } f(2) = 1 \\ 2, & \text{if } f(2) = 0 \end{cases}$, $z = \begin{cases} 0, & \text{if } f(6) = 0 \\ 1, & \text{if } f(6) = 1 \end{cases}$, $w = \begin{cases} 1, & \text{if } f(6) = 1 \\ 2, & \text{if } f(6) = 0 \end{cases}$, $a = \begin{cases} y, & \text{if } f(3) = 0 \\ 2y, & \text{if } f(3) = 1 \end{cases}$, $b = \begin{cases} w, & \text{if } f(5) = 0 \\ 2w, & \text{if } f(5) = 1 \end{cases}$. Note a, b may take on the values $1, 2$, or 4 so $\sqrt{ab}$, $\sqrt{\frac{b}{a}}$ are real and $1 - 3\sqrt{\frac{b}{a}} \neq 0$. Then $M \in \mathcal{S}_f(G)$ and $5 = \mathrm{mr}(G) \leq \operatorname{rank} M \leq 2+1+2 = 5$. This shows another 2^4 schemes on G are constructible.

Next let f be a scheme mapping vertex 1 to zero and vertex 7 to one. Consider the decomposition



together with the matrix construction

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & x & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & y & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 2a & 4a & 0 & 0 \\ 0 & 0 & 2a & 4a & 8a & 0 & 0 \\ 0 & 0 & 4a & 8a & 16a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & b & b & 0 \\ 0 & 0 & 0 & b & b & b & 0 \\ 0 & 0 & 0 & b & b & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & A & \\ 0 & 0 & 0 & 0 & & & \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & x & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & y-a & 1-2a & -4a & 0 & 0 \\ 0 & 1 & 1-2a & b-4a & b-8a & b & 0 \\ 0 & 0 & -4a & b-8a & b-16a & b & 0 \\ 0 & 0 & 0 & b & b & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & A & \\ 0 & 0 & 0 & 0 & & & \end{bmatrix}.$$

where $x = \begin{cases} 0, & \text{if } f(2) = 0 \\ 1, & \text{if } f(2) = 1 \end{cases}$, $y = \begin{cases} 1, & \text{if } f(2) = 1 \\ 2, & \text{if } f(2) = 0 \end{cases}$, $a = \begin{cases} y, & \text{if } f(3) = 0 \\ 2y, & \text{if } f(3) = 1 \end{cases}$,

$b = \begin{cases} 4a, & \text{if } f(4) = 0 \\ 6a, & \text{if } f(4) = 1 \end{cases}$ and

$$A = \begin{cases} (16a-b)J_3, & \text{if } f(5)=0, f(6)=1 \\ \begin{bmatrix} -4b & -2b & -8b \\ -2b & -b & -4b \\ -8b & -4b & -16b \end{bmatrix}, & \text{if } f(5)=1, f(6)=0 \\ \begin{bmatrix} -b & -2b & -4b \\ -2b & -4b & -8b \\ -4b & -8b & -16b \end{bmatrix}, & \text{if } f(5)=f(6)=1 \end{cases}$$

Then $M \in \mathcal{S}_f(G)$ and $5 = \mathrm{mr}(G) \leq \operatorname{rank} M \leq 2+1+1+1 = 5$. This constructs $3(2^3)$ of the 2^5 schemes mapping vertex 1 to zero and vertex 7 to one. The remaining schemes are those for which $f(5) = f(6) = 0$. Minimum rank matrices satisfying these eight schemes are given by:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2\sqrt{2}-2 & 2 & 0 \\ 0 & 0 & -2 & 2\sqrt{2}-2 & 0 & 2\sqrt{2}-2 & -2 \\ 0 & 0 & 0 & 2 & 2\sqrt{2}-2 & 0 & -2 \\ 0 & 0 & 0 & 0 & -2 & -2 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2\sqrt{2}-2 & 2 & 0 \\ 0 & 0 & -2 & 2\sqrt{2}-2 & 0 & 2\sqrt{2}-2 & -2 \\ 0 & 0 & 0 & 2 & 2\sqrt{2}-2 & 0 & -2 \\ 0 & 0 & 0 & 0 & -2 & -2 & -2 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & -2 & -3 & -4 & 0 & 0 \\ 0 & 1 & -3 & 0 & 4\sqrt{2}-4 & 4 & 0 \\ 0 & 0 & -4 & 4\sqrt{2}-4 & 0 & 4\sqrt{2}-4 & -4 \\ 0 & 0 & 0 & 4 & 4\sqrt{2}-4 & 0 & -4 \\ 0 & 0 & 0 & 0 & -4 & -4 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -2 & -3 & -4 & 0 & 0 \\ 0 & 1 & -3 & 0 & 4\sqrt{2}-4 & 4 & 0 \\ 0 & 0 & -4 & 4\sqrt{2}-4 & 0 & 4\sqrt{2}-4 & -4 \\ 0 & 0 & 0 & 4 & 4\sqrt{2}-4 & 0 & -4 \\ 0 & 0 & 0 & 0 & -4 & -4 & -4 \end{bmatrix},$$

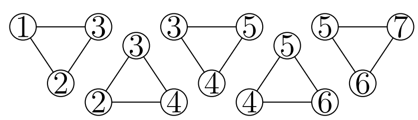
$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -2 & 0 & 0 \\ 0 & 1 & -1 & 2 & 2 & 2\sqrt{2} & 0 \\ 0 & 0 & -2 & 2 & 0 & 2\sqrt{2}-2 & -2 \\ 0 & 0 & 0 & 2\sqrt{2} & 2\sqrt{2}-2 & 0 & -2 \\ 0 & 0 & 0 & 0 & -2 & -2 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & -2 & 0 & 0 \\ 0 & 1 & -1 & 2 & 2 & 2\sqrt{2} & 0 \\ 0 & 0 & -2 & 2 & 0 & 2\sqrt{2}-2 & -2 \\ 0 & 0 & 0 & 2\sqrt{2} & 2\sqrt{2}-2 & 0 & -2 \\ 0 & 0 & 0 & 0 & -2 & -2 & -2 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & -2 & -3 & -4 & 0 & 0 \\ 0 & 1 & -3 & 4 & 4 & 4\sqrt{2} & 0 \\ 0 & 0 & -4 & 4 & 0 & 4\sqrt{2}-4 & -4 \\ 0 & 0 & 0 & 4\sqrt{2} & 4\sqrt{2}-4 & 0 & -4 \\ 0 & 0 & 0 & 0 & -4 & -4 & -4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -2 & -3 & -4 & 0 & 0 \\ 0 & 1 & -3 & 4 & 4 & 4\sqrt{2} & 0 \\ 0 & 0 & -4 & 4 & 0 & 4\sqrt{2}-4 & -4 \\ 0 & 0 & 0 & 4\sqrt{2} & 4\sqrt{2}-4 & 0 & -4 \\ 0 & 0 & 0 & 0 & -4 & -4 & -4 \end{bmatrix}.$$

A computer algebra system can easily verify these are rank 5 matrices.

The minimum rank matrices for the 2^5 schemes mapping vertex 1 to one and vertex 7 to zero are produced similarly.

Finally suppose f is a scheme that maps vertices 1 and 7 to one. Consider



and let

$$M = \begin{bmatrix} 1 & 2 & 4 & 0 & 0 & 0 & 0 \\ 2 & 4 & 8 & 0 & 0 & 0 & 0 \\ 4 & 8 & 16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & a & a & 0 & 0 & 0 \\ 0 & a & a & a & 0 & 0 & 0 \\ 0 & a & a & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & b & b & 0 & 0 \\ 0 & 0 & b & b & b & 0 & 0 \\ 0 & 0 & b & b & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c & c & c & 0 \\
0 & 0 & 0 & c & c & c & 0 \\
0 & 0 & 0 & c & c & c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
+
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & & A & \\
0 & 0 & 0 & 0 & & &
\end{bmatrix}
$$

$$
=
\begin{bmatrix}
1 & 2 & 4 & 0 & 0 & 0 & 0 \\
2 & 4+a & 8+a & a & 0 & 0 & 0 \\
4 & 8+a & 16+a+b & a+b & b & 0 & 0 \\
0 & a & a+b & a+b+c & b+c & c & 0 \\
0 & 0 & b & b+c & b+c & c & 0 \\
0 & 0 & 0 & c & c & c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
+
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & & A & \\
0 & 0 & 0 & 0 & & &
\end{bmatrix}
$$

where $a = \begin{cases} -4, & \text{if } f(2) = 0 \\ 4, & \text{if } f(2) = 1 \end{cases}$, $b = \begin{cases} -(16+a), & \text{if } f(3) = 0 \\ 16+a, & \text{if } f(3) = 1 \end{cases}$, $c = \begin{cases} -(a+b), & \text{if } f(4) = 0 \\ a+b, & \text{if } f(4) = 1 \end{cases}$, and

$$A = \begin{cases} -(b+c)J_3, & \text{if } f(5)=0, f(6)=1 \\ \begin{bmatrix} -4c & -2c & -8c \\ -2c & -c & -4c \\ -8c & -4c & -16c \end{bmatrix}, & \text{if } f(5)=1, f(6)=0 \\ \begin{bmatrix} -c & -2c & -4c \\ -2c & -4c & -8c \\ -4c & -8c & -16c \end{bmatrix}, & \text{if } f(5)=f(6)=1 \end{cases}$$

Then $M \in \mathcal{S}_f(G)$ and $5 = \operatorname{mr}(G) \leq \operatorname{rank} M \leq 1+1+1+1+1 = 5$. This gives $3(2^3)$ of the 2^5 schemes. Of the remaining 8 schemes, 6 are isomorphic to schemes constructed above. Minimum rank matrices satisfying the remaining two schemes are

$$\begin{bmatrix} 1 & 2 & 4 & 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & -4 & 0 & 0 & 0 \\ 4 & -4 & 0 & -16 & -12 & 0 & 0 \\ 0 & -4 & -16 & -20 & -16 & -4 & 0 \\ 0 & 0 & -12 & -16 & 0 & 4 & 4 \\ 0 & 0 & 0 & -4 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 4 & 0 & 0 & 0 & 0 \\ 2 & 0 & 8+8\sqrt{2} & -4 & 0 & 0 & 0 \\ 4 & 8+8\sqrt{2} & 0 & 16\sqrt{2} & 16 & 0 & 0 \\ 0 & -4 & 16\sqrt{2} & 0 & 16\sqrt{2} & -4 & 0 \\ 0 & 0 & 16 & 16\sqrt{2} & 0 & 8+8\sqrt{2} & 4 \\ 0 & 0 & 0 & -4 & 8+8\sqrt{2} & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 & 2 & 1 \end{bmatrix}.$$

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