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Connecting Galois Representations with Cohomology

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ABSTRACT

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In this paper, we examine the conjecture of Avner Ash, Darrin Doud, David Pollack, and Warren Sinnott relating Galois representations to the mod p cohomology of congruence subgroups of $\mathrm{GL}_n(\mathbb{Z})$. We present computational evidence for this conjecture (the ADPS Conjecture) for the case $n = 3$ by finding Galois representations which appear to correspond to cohomology eigenclasses predicted by the ADPS Conjecture for the prime $p = 5$, level $N \in \{3, 7, 11, 13, 17, 19\}$, and quadratic nebentype ϵ. The examples include representations which appear to be attached to cohomology eigenclasses which arise from D_8, S_3, A_5, and S_5 extensions. Other examples include representations which are reducible as sums of characters, representations which are symmetric squares, $Sym^2(\sigma)$, of two-dimensional representations, and representations which arise from modular forms, as predicted by Jean-Pierre Serre for $n = 2$.

Keywords: Arithmetic Cohomology, Galois Representations, Hecke Operators

Contents

Chapter 1. Introduction

In [20] Jean-Pierre Serre conjectured a relationship between continuous, odd, irreducible Galois representations $\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ and mod p reductions of modular forms. Serre's conjecture has since been proven true by the work of Chandrashekhar Khare and Jean-Pierre Wintenberger in [13], [14], and [15]. In [3], Avner Ash and Warren Sinnott conjectured a relationship between Galois representations which are odd, niveau 1, and of arbitrary dimension n, and certain cohomology groups of congruence subgroups of $GL_n(\mathbb{Z})$. For $n = 2$, this conjecture is closely related to Serre's conjecture. The work by Ash and Sinnott in [3] focused primarily on three-dimensional niveau 1 reducible representations.

In [2], Avner Ash, Darrin Doud, and David Pollack presented additional examples as evidence for the conjecture by Ash and Sinnott. Their work included examples of nontrivial weight, level, and nebentype as well as examples with higher niveau.

In Chapter 3 of this paper we describe a program which is used to construct tables of all eigenclasses defined over $\mathbb{F}_p$ for the prime $p = 5$, level N, where $N \in \{3, 7, 11, 13, 17, 19\}$, and quadratic nebentype ϵ_N. Following work of Ash and Sinnott in [3], Chapter 4 looks at the process of finding which eigenclasses from the tables appear to be attached to representations $\rho : G_{\mathbb{Q}} \to GL_3(\overline{\mathbb{F}}_p)$ which are reducible as sums of characters.

We next investigate representations which either reduce to a two-dimensional irreducible representation plus a character or are completely irreducible. In Chapter 5, we examine two two-dimensional representations with image isomorphic to D_8 and S_3 which can be added to a character to give a three-dimensional representation which appears to be attached to a cohomology eigenclass. We then look at a two-dimensional representation which arises from a modular form, to which we then add a character to make a three-dimensional representation. This modular form is easily understood using the recipes for weight, level, and nebentype found by Serre in [20]. The last representation looked at in Chapter 5 is a representation which is a symmetric square of a two-dimensional representation. Symmetric squares were studied in [2] and we describe the process of computing the level, weight, and nebentype, as well as computing traces and cotraces to show that the representation appears to be attached to the appropriate eigenclass in the table.

To understand irreducible representations, we digress in Chapter 6 to discuss finding orders of Frobenius elements for different eigenclasses. This information is helpful in determining the representation we expect to be attached to a system of eigenvalues. From there, we consider representations with image S_5 and A_5 and find that they appear to be attached to certain systems of eigenvalues. The examples use techniques from [9], [10], and [19]. We are able to explain which eigenclasses appear to have a representation attached in nearly all the entries of the tables in the appendix. However, using the methods in Chapter 6 we find there are eigenclasses attached to representations which are too big to compute with our current means.

It is important to note that throughout the paper, we often say that a representation appears to be attached to a system of eigenvalues, not that it actually is attached. For a representation to actually be attached to a system of eigenvalues, the trace and cotrace must match up for all primes $\ell \nmid pN$. Since our examples only look at primes $2 \leq \ell \leq 47$ with $\ell \nmid pN$, we can only say the representation appears to be attached. Hence, we only give computational evidence for the conjecture rather than proven cases.

Chapter 2. Conjecture of Ash, Doud, Pollack, Sinnott

2.1 Definitions

We begin with the basic definitions needed to understand the conjecture of Ash, Doud, Pollack, and Sinnott, which we call the ADPS Conjecture.

2.1.1 Basic Definitions. Let $p > 2$ be a prime number. Let $\mathbb{F}_p$ be a finite field with p elements, with algebraic closure $\overline{\mathbb{F}}_p$. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ and $G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Let $\rho : G_{\mathbb{Q}} \to GL_n(\overline{\mathbb{F}}_p)$ be a continuous, semisimple representation.

For a prime q, denote the decomposition group at q in $G_{\mathbb{Q}}$ by G_q. This decomposition group has a filtration by ramification groups $G_{q,i}$, with $G_{q,0} \supseteq G_{q,1} \supseteq \cdots$. The whole inertia group above q is equal to $G_{q,0}$ and we will let $G_{p,0}$ above p be denoted by I_p. We will fix a Frobenius element Frob_q for each prime q, as well as a complex conjugation Frob_∞.

Definition 2.1.1. For $n = 2$ and $n = 3$, ρ is **odd** if the image of complex conjugation under ρ is conjugate to a matrix which is nonscalar. For $n = 2$,

$$\rho(\mathrm{Frob}_\infty) \sim \pm \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

For $n = 3$,

$$\rho(\mathrm{Frob}_\infty) \sim \pm \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

Denote the fundamental characters of niveau n in characteristic p by $\Psi_{n,d}$, where $d = 1, \ldots, n$. Refer to [17] for the definition of a fundamental character. Since this paper focuses on $n = 3$, we define the following fundamental characters of niveau 1, 2, and 3:

$$\omega = \Psi_{1,1} \text{ (niveau 1)}$$

$$\Psi = \Psi_{2,1},\ \Psi' = \Psi_{2,2} \text{ (niveau 2)}$$

$$\Theta = \Psi_{3,1},\ \Theta' = \Psi_{3,2},\ \Theta'' = \Psi_{3,3} \text{ (niveau 3)}.$$

From [17], we note the following:

$$\Psi^p = \Psi',\ (\Psi')^p = \Psi,\ \Theta^p = \Theta',\ (\Theta')^p = \Theta'',\ (\Theta'')^p = \Theta$$

This tells us the fundamental characters $\Psi_{n,d}$ are all Galois conjugates of $\Psi_{n,1}$ over $\mathbb{F}_p$.

Definition 2.1.2. A representation ρ is niveau 1 if its restriction to I_p is the sum of only niveau 1 characters. ρ is niveau 2 if a niveau 2 character is included in its sum. It is niveau 3 if it has a niveau 3 character in its sum.

2.1.2 Hecke Operators. $\Gamma_0(N)$ is the subgroup of matrices in $SL_3(\mathbb{Z})$ whose first row is congruent to $(*, 0, 0)$ mod N. Let S_N be the semigroup of integral matrices in $GL_3(\mathbb{Q})$ satisfying the same congruence condition and having positive determinant relatively prime to N. $\mathscr{H}(N)$ is the $\overline{\mathbb{F}}_p$-algebra of double cosets $\Gamma_0(N)\backslash S_N/\Gamma_0(N)$. $\mathscr{H}(N)$ is a commutative algebra that acts on the cohomology and homology groups of $\Gamma_0(N)$ with coefficients in any $\overline{\mathbb{F}}_p[S_N]$-module. It contains double cosets of the form $\Gamma_0(N)D(\ell,k)\Gamma_0(N)$, where $\ell \nmid pN$, $0 \le k \le 3$, and

$$D(\ell,k) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \ell & & \\ & & & & \ddots & \\ & & & & & \ell \end{pmatrix}$$

is a diagonal matrix with $n-k$ 1's followed by k ℓ's. A double coset acting on a cohomology or homology group is a Hecke operator, and we define the double coset generated by $D(\ell,k)$ to be the Hecke operator $T(\ell,k)$. For more information, refer to [16].

Definition 2.1.3. [20] Let V be an $\mathscr{H}(pN)$-module and let $v \in V$ be a simultaneous eigenvector for all $T(\ell,k)$ such that $T(\ell,k)v = a(\ell,k)v$ with $a(\ell,k) \in \overline{\mathbb{F}}_p$ for all $\ell \nmid pN$ with ℓ

prime and $0 \leq k \leq n$. If the representation $\rho : G_{\mathbb{Q}} \to GL_n(\overline{\mathbb{F}}_p)$ is unramified outside pN and

$$\sum_{k=0}^{n}(-1)^k \ell^{k(k-1)/2} a(\ell, k)x^k = \det(I - \rho(\mathrm{Frob}_\ell)x)$$

for all $\ell \nmid pN$, then we say ρ is **attached** to v.

Definition 2.1.4. The **trace**, Tr, of a matrix M is the sum of the diagonal elements M. Normally, this will be the sum of eigenvalues M.

Definition 2.1.5. The **cotrace**, T_2, of a matrix M is the sum products of pairs of eigenvalues M.

In this paper, we deal with $n = 3$, so using these definitions we can say

$$\det(I - Mx) = 1 - Tr(M)x + T_2(M)x^2 - \det(M)x^3$$

In connection with definition 2.1.3, if the ADPS Conjecture is true and ρ is attached to an eigenvector v, we get the following:

$$a(\ell, 0) = 1$$
$$a(\ell, 1) = Tr(\rho(\mathrm{Frob}_\ell))$$
$$\ell a(\ell, 2) = T_2(\rho(\mathrm{Frob}_\ell))$$
$$\ell^3 a(\ell, 3) = \det(\rho(\mathrm{Frob}_\ell))$$

2.1.3 Level and Nebentype.

Let $\rho : G_{\mathbb{Q}} \to GL_3(\overline{\mathbb{F}}_p)$ be a continuous, semisimple representation as above. In [20] Serre defines the level and nebentype for representations in $GL_2(\overline{\mathbb{F}}_p)$, and we use a similar definition for representations into $GL_3(\overline{\mathbb{F}}_p)$.

First we look at the level of ρ. For a fixed prime $q \neq p$ and $i \geq 0$, let $g_i := \left|\rho(G_{q,i})\right|$. Note that the continuity of ρ forces the image of ρ to be finite, so $g_i < \infty$. Let $M = \overline{\mathbb{F}}_p^3$ be a vector space acted on by $GL_3(\overline{\mathbb{F}}_p)$ via ρ in the natural way. We define

$$n_q := \sum_{i=0}^{\infty} \frac{g_i}{g_0} \dim M/M^{G_{q,i}}$$

Because we are working mod p, the images $\rho(G_{q,i})$ are eventually trivial so for i large, $M^{G_{q,i}} = M$ and $\dim M/M^{G_{q,i}} = 0$. This means n_q is finite. By [19] we also know that n_q is a positive integer.

Definition 2.1.6. [20] The **level** of a representation ρ is $N(\rho) = \prod_{q \neq p} q^{n_q}$.

Remark. This is a finite product since ρ is ramified at only finitely many primes, and $n_q = 0 \iff \rho$ is unramified at q.

Next, we will define the nebentype of ρ. We do this by examining $\det(\rho)$. Note that $\det(\rho)$ is a map from $G_{\mathbb{Q}}$ to $GL_1(\overline{\mathbb{F}}_p) = \overline{\mathbb{F}}_p^{\times}$. Following [20], we factor $\det(\rho)$ into a power of the cyclotomic character mod p and a character ϵ which is unramified at p, namely $\det \rho = \omega^k \epsilon$. The character ϵ will have level dividing the level N of ρ. We call it the nebentype of ρ.

In the cases we consider, the nebentype of ρ will be a quadratic character ramified only at one prime $q \neq p$. For odd primes p, there is a unique quadratic character mod p ramified only at q which we will denote by ϵ_q.

By class field theory, we may consider ϵ as a Dirichlet character such that

$$\epsilon : (\mathbb{Z}/N(\rho)\mathbb{Z})^{\times} \to \overline{\mathbb{F}}_p^{\times}$$

We pull ϵ back to S_N by composing it with the map that takes an element of S_N to its $(1,1)$ entry. Define $\mathbb{F}_{\epsilon}$ to be the one-dimensional space $\overline{\mathbb{F}}_p$ with the action of S_N given by ϵ. For a $GL_3(\mathbb{F}_p)$-module V, we create an S_N-module $V(\epsilon)$ by defining $V(\epsilon) := V \otimes \mathbb{F}_{\epsilon}$, with S_N acting on V via reduction mod p and on $\mathbb{F}_{\epsilon}$ via ϵ as described above.

2.1.4 Actions of Common Characters on Frobenius Elements.

Throughout the paper, ϵ_q and ω are used extensively, so we will provide their definitions and actions on Frobenius elements for reference.

The map $\omega : G_{\mathbb{Q}} \to \mathbb{F}_p^{\times}$ is the mod p cyclotomic character and gives the action of $G_{\mathbb{Q}}$ on p^{th} roots of 1, namely $\zeta_p = e^{2\pi i/p}$. If $\eta \in G_{\mathbb{Q}}$ and $\eta(\zeta_p) = \zeta_p^k$ with $k \in \mathbb{Z}$, then we define

$\omega(\eta) = k$. Since ζ_p has order p, $\omega(\eta)$ is well defined mod p, so we may consider it as a map $\omega : G_{\mathbb{Q}} \to \mathbb{F}_p^{\times}$. From [1], we see that the Frobenius element Frob_ℓ acts as the ℓth power map on pth roots of 1, namely that $\text{Frob}_\ell(\zeta_p) = \zeta_p^\ell$. This allows us to say $\omega(\text{Frob}_\ell) \equiv \ell \pmod p$.

Next, consider the action of complex conjugation, Frob_∞, on ζ_p. Since ζ_p is mapped to its conjugate $\overline{\zeta_p} = \zeta_p^{-1}$ by Frob_∞, then we see that $\omega(\text{Frob}_\infty) \equiv -1 \pmod p$.

Now, consider the character ϵ_q which is the unique quadratic character ramified only at q. It cuts out a quadratic extension of $\mathbb{Q}$ ramified only at q. For q odd, this quadratic extension is $\mathbb{Q}(\sqrt{q^*})/\mathbb{Q}$, with $q^* = (-1)^{\frac{q-1}{2}} q$. We note $x^2 - q^*$ factors mod ℓ if and only if ℓ has inertial degree 1 in the extension. This results in $\epsilon_q(\text{Frob}_\ell) = 1$. Similarly, $x^2 - q^*$ does not factor mod $\ell \iff \ell$ has inertial degree 2 in the extension. This results in $\epsilon_q(\text{Frob}_\ell) = -1$. From this we see

$$\epsilon_q(\text{Frob}_\ell) = \left(\frac{q^*}{\ell}\right)$$

where $\left(\frac{q^*}{\ell}\right)$ is the Kronecker symbol.

If $q \equiv 1 \pmod 4$, then $\sqrt{q^*}$ is real and the extension above is real. Then Frob_∞ acts trivially, which gives $\epsilon_q(\text{Frob}_\infty) = -1$. If $q \equiv 3 \pmod 4$, then $\sqrt{q^*}$ is imaginary and the extension above is imaginary. Then Frob_∞ acts nontrivially and gives $\epsilon_q(\text{Frob}_\infty) = -1$.

2.1.5 Irreducible $GL_3(\mathbb{F}_p)$-modules. The formula to calculate the weight given by Serre in [20] works only for two-dimensional representations. In two dimensions, the Eichler-Shimura theorem [21] relates the space of modular forms of weight k to a cohomology group with coefficients in $Sym^{k-2}(\mathbb{C}^2)(\epsilon)$. An eigenform f of level N, nebentype ϵ, and weight k gives rise to a collection of Hecke eigenvalues which also occur in $H^1(\Gamma_0(N), Sym^{k-2}(\mathbb{F}_p^2)(\epsilon))$ when taken mod p. Here, $Sym^{k-2}(\mathbb{F}_p^2)$ is the space of two-variable homogeneous polynomials of degree $k-2$ over $\mathbb{F}_p$. In [4], Ash and Stevens showed that the system of eigenvalues which occur in the cohomology groups of $\Gamma_0(N)$ with coefficients in some $GL_n(\mathbb{F}_p)$-module also occurs in the cohomology groups of some irreducible $GL_n(\mathbb{F}_p)$ constituent of the original $GL_n(\mathbb{F}_p)$-module. We could ask which irreducible $GL_2(\mathbb{F}_p)$-module gives rise to the system of eigenvalues predicted by Serre's conjecture. This would be a refinement of the weight in Serre's conjecture. For n-dimensional Galois representations, rather than predicting a module of the form $Sym^{k-2}(\mathbb{F}_p^n)$, we will predict an irreducible $GL_n(\mathbb{F}_p)$-module. Thus the natural generalization of a weight in Serre's conjecture into $n = 3$ is an irreducible $GL_3(\mathbb{F}_p)$-module. We want to parametrize the set of irreducible $GL_3(\mathbb{F}_p)$-modules for any p.

Definition 2.1.7. [2] An integer triple $[b_1, b_2, b_3]$ is **p-restricted** if

$$0 \le b_i - b_{i+1} \le p - 1 \text{ for } 1 \le i \le 2 \text{ and } 0 \le b_3 \le p - 1.$$

The set of irreducible $GL_3(\mathbb{F}_p)$-modules is in $1-1$ correspondence with the collection of p-restricted triples [2]. We will let the module $F[b_1, b_2, b_3]$ correspond to the p-restricted triple $[b_1, b_2, b_3]$.

Let $[a_1, a_2, a_3]$ be any integer triple. We say $F[a_1, a_2, a_3]' = F[b_1, b_2, b_3]$ if $a_i \equiv b_i \pmod{p-1}$ for $1 \le i \le 3$. This means $F[a_1, a_2, a_3]'$ may not be well defined, namely when $a_i \equiv a_{i+1} \pmod{p-1}$ for $i = 1$ or $i = 2$.

Example 2.1.1. $F[2, 2, 0]'$=$F[2, 2, 0]$ and $F[6, 2, 0]$ for $p = 5$.

2.1.6 Strict Parity Condition.

Definition 2.1.8. [2] Let $V = \overline{\mathbb{F}}_p$ be a three-dimensional space with the standard action of $GL_3(\overline{\mathbb{F}}_p)$. A **Levi subgroup** L of $GL_3(\overline{\mathbb{F}}_p)$ is the simultaneous stabilizer of a collection of subspaces $D_1, \ldots, D_k$ such that $V = \oplus_i D_i$. If each D_i has a basis consisting of standard basis vectors of V, then L is called a **standard Levi subgroup**.

Example 2.1.2. The standard Levi subgroups of $GL_3(\overline{\mathbb{F}}_p)$ are the subgroups of diagonal matrices, the whole of $GL_3(\overline{\mathbb{F}}_p)$, and the following:

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}, \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

Definition 2.1.9. [2] Let $\rho : G_{\mathbb{Q}} \to GL_3(\overline{\mathbb{F}}_p)$ be a continuous representation. A standard Levi subgroup L of $GL_3(\overline{\mathbb{F}}_p)$ is **ρ-minimal** if L is minimal among all standard Levi subgroups that contain some conjugate of the image of ρ.

Definition 2.1.10. [2] A semisimple continuous representation ρ with image in a standard Levi subgroup L satisfies the **strict parity condition** with Levi subgroup L if it satisfies the following:

1) L is ρ-minimal.

2) The image of complex conjugation is conjugate inside L to a matrix of the form

$$\pm \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

Example 2.1.3. An odd, irreducible three-dimensional representation satisfies strict parity with Levi subgroup $L = GL_3(\overline{\mathbb{F}}_p)$.

Example 2.1.4. Let ρ be the direct sum of a two-dimensional odd irreducible representation and a one-dimensional representation with image contained inside

$$L = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \text{ or } L = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

Then ρ satisfies strict parity.

Example 2.1.5. Let ρ be the direct sum of a two-dimensional even irreducible representation and a one-dimensional representation with image inside

$$L = \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}$$

Then ρ satisfies strict parity if ρ is odd.

Remark. Any odd three-dimensional representation is conjugate to a representation that satisfies definition 2.1.10.

2.1.7 Weights. As with Serre's conjecture, we would expect the weights attached to a representation to be determined by the restriction of ρ to the decomposition group at p, so we want to study the representations of the decomposition group G_p. We will denote the inertia group $G_{p,0}$ by I_p and the wild ramification group $G_{p,1}$ by I_w.

Lemma 2.1.1. *[2] Let V be a simple n-dimensional $\overline{\mathbb{F}}_p[G_p]$-module, with the action of G_p given by a representation $\rho : G_p \to GL_n(V)$. We may choose a basis for V such that*

$$\rho|_{I_p} \sim \begin{pmatrix} \varphi_1 & & \\ & \ddots & \\ & & \varphi_n \end{pmatrix}$$

with the characters $\varphi_1, \ldots, \varphi_n$ equal to some permutation of the fundamental characters $\Psi_{n,1}^m, \ldots, \Psi_{n,n}^m$ for some $m \in \mathbb{Z}$.

Remark. There are n distinct values of $m \pmod{p^n - 1}$. If m_0 is one of these, then the others are congruent to $pm_0, p^2 m_0, \ldots, p^{n-1} m_0 \pmod{p^n - 1}$.

Definition 2.1.11. [2] Let V be a simple G_p-module, diagonalized as in the above lemma with some $m \in \mathbb{Z}$. If possible, write $m = a_1 + a_2 p + ... + a_n p^{n-1}$, with $0 \leq a_i - a_n \leq p - 1$ for all i. Suppose the n-tuple $[b_1, ..., b_n]$ satisfies $b_i \geq b_{i+1}$ for all $i < n$ and is obtained by permuting the entries of $[a_1, ..., a_n]$. Then $[b_1, ..., b_n]$ is derived from V. If the action of G_p on V is given by a representation ρ, we say the n-tuple is **derived** from ρ.

For a complete description of derived n-tuples when the representation ρ does not define a simple G_p-module, see [2, pg. 530]. In the case of a three-dimensional niveau 1 representation ρ, we obtain a derived n-tuple by upper-triangularizing $\rho|_{I_p}$. We then have

$$\rho|_{I_p} \sim \begin{pmatrix} \omega^a & * & * \\ 0 & \omega^b & * \\ 0 & 0 & \omega^c \end{pmatrix}$$

The derived triple for ρ is then $[a, b, c]$. If $\rho|_{I_p}$ is actually conjugate to a diagonal, we may permute the characters on the diagonal by conjugation. This does not happen if $\rho|_{I_p}$ is upper triangular but not diagonal.

2.2 The Conjecture

Conjecture 2.2.1. *[2] Let $\rho : G_{\mathbb{Q}} \to GL_n(\overline{\mathbb{F}}_p)$ be a continuous, semisimple Galois representation. Suppose ρ satisfies the strict parity condition with Levi group L. Let $[a_1, ..., a_n]$ be an n-tuple derived from ρ. Let $V = F[a_1 - (n-1), a_2 - (n-2), ..., a_n - 0]'$. Let $N = N(\epsilon)$ be the level and $\epsilon = \epsilon(\rho)$ be the nebentype. Then ρ is attached to a cohomology class in $H^*(\Gamma_0(N), V(\epsilon))$.*

Remark. Here, $*$ is between 0 and 3. For irreducible ρ we can let $* = 3$. In the calculations for this paper we use H^3 regardless of irreducibility.

Theorem 2.2.1. *[2] If Conjecture 2.2.1 above is true for a representation ρ, then it is true for a representation $\rho \otimes \omega^s$, where ω is the cyclotomic character mod p.*

Remark. A couple things to note.

- Our research focuses on the specific case $n = 3$.
- Twisting by ω^s does not affect the predicted level or nebentype.
- If ρ is niveau 1, then the conjecture is found in [3, prop. 2.6].
- For any niveau, twisting by ω^s adds s to each a_i.
- We consider the conjecture satisfied for ρ if we find a weight attached to some twist of ρ by ω^s for some $s \in \mathbb{Z}$.
- Our code only allows for calculating with $p \not\equiv 0 \pmod 3$. For $p \equiv 0 \pmod 3$, we would have to change the code to iterate through possible non-twisted weights differently.

In the conjecture, we do not predict all possible weights that yield an eigenclass with ρ attached. There are three types of examples in which additional weights do yield eigenclasses with ρ attached. These extra weights occur very infrequently, so refer to [2, pp. 532-534] for additional information.

Chapter 3. Constructing Tables of Eigenclasses

Let p be a prime. Let $N \in \mathbb{N}$. For this paper, N will be prime for ease of computation. As this paper focuses on finding numerous examples of representations attached to cohomology groups for various p and N, we will explain the construction of tables of eigenclasses for small p and N.

Our approach to creating the tables is to run through all possible weights $F[a, b, c]$, calculate the eigenclasses associated to them, and then show there is a representation ρ with the same level and nebentype for which the eigenclasses match up (through the determinant and cotrace of the representation of ρ) for all primes $2 \leq \ell \leq 47$ with $\ell \nmid pN$.

Given p and N above, we first form all triples $[a, b, c]$ which satisfy $a+b+c \equiv 0 \pmod{p-1}$ and $0 \leq a, b, c \leq p-1$. By Conjecture 2.2.1 we find all weights $F[a-2, b-1, c]'$. The prime notation allows us to add multiples of $p-1$ to $a-2$, $b-1$, and c to ensure they satisfy the following conditions:

$$0 \leq (a-2) - (b-1) \leq p-1$$
$$0 \leq (b-1) - c \leq p-1$$
$$0 \leq c \leq p-2$$

The set of weights resulting from this process will be put into our cohomology calculator as described later.

When $p \not\equiv 1 \pmod 3$, the remark after Theorem 2.2.1 allows twisting by ω^s to give $F[a', b', c'] \otimes \omega^s = F[a'+s, b'+s, c'+s]$. We only need the main conjecture to be true in one representative of the set of twisted eigenclasses. Since $\gcd(3, p-1) = 1$, then $a+b+c+3s \pmod{p-1}$ will run through all residue classes for all choices of s. This means we can focus on the weights arising from the triples $[a, b, c]$ with $a+b+c \equiv 0 \pmod{p-1}$ since all other weights will just be twists of such weights. Hence, the set of weights described above allows us to confirm conjugates for all weights mod p.

When $p \equiv 1 \pmod 3$, then $\gcd(3, p-1) = 3$ so $a+b+c+3s$ will not run through all residue classes. As such, to get all possible weights while ignoring twisting, we would run through all triples $[a, b, c]$ such that $a + b + c \equiv 0, 1, 2 \pmod{p-1}$ and at least one of a, b, and c is 0, 1, or 2. Our code is only designed for primes $p \not\equiv 1 \pmod 3$, so the tables in the appendix are restricted to those choices of p. In fact, all tables in the appendix have $p = 5$.

Using our program, we are able to calculate the action of the Hecke operators for each weight $F[a, b, c]$ in our set. The actions come from $D(\ell, k)$, where $\ell \nmid pN$ and $0 \leq k \leq 3$. By definition, for $k = 0$ and $k = 3$ respectively we get

$$D(\ell, 0) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \text{ and } D(\ell, 3) = \begin{pmatrix} \ell & & \\ & \ell & \\ & & \ell \end{pmatrix}$$

$D(\ell, 0)$ acts trivially on the cohomology group so its actions are ignored in our computations. Given a weight $F[a, b, c]$ predicted for p, then $D(\ell, 3)$ acts as

$$a(\ell, 3) = \ell^{a+b+c} \cdot \epsilon(\ell)$$

which is exactly what is needed to give $\ell^3 a(\ell, 3) = \det \rho$, thereby satisfying Definition 2.1.3. Since this case is automatically fulfilled, our computations also ignore this case. What we do compute is a set of matrices representing the actions of $T(\ell, k)$ for $k = 1, 2$ and primes $2 \leq \ell \leq 47$, where $\ell \nmid pN$, as well as the dimension of the homology group which is used later.

Referring back to Definition 2.1.3, we need to find the simultaneous eigenvectors v for these actions found above. We will form pairs $(a(\ell, 1), a(\ell, 2))$ where $(a(\ell, *))$ satisfies the equality $T(\ell, *)v = a(\ell, *)v$ and $T(\ell, *)$ is the Hecke operator associated to $D(\ell, *)$. To find these pairs $(a(\ell, 1), a(\ell, 2))$, our program does the following:

For each prime $2 \leq \ell \leq 47$, $\ell \nmid pN$, we find the roots of the characteristic polynomial $\det(Ix - A) \bmod \ell$, where I is the appropriately sized identity matrix determined by the dimension of the homology group and A is the matrix associated to the action of the Hecke operator. With these roots in hand, we find the kernel associated to each root. This gives us a set of eigenspaces for each ℓ as above. Given these sets, it is very straightforward to find the intersection of all the eigenspaces, which gives us our simultaneous eigenvectors v needed above. Given these v, for each $k = 1, 2$ we go back through the primes ℓ and find which root of the characteristic polynomial described above gives v. This gives the pair $(a(\ell, 1), a(\ell, 2))$ for each $\ell \nmid pN$, and each v found above.

In constructing the initial tables (without Galois representations attached), we first find all unique systems of eigenvalues and then list all weights $F[a, b, c]$ which give rise to each system, as well as the triple associated with the weight.

We first consider the niveau 1 representations. From the tables in Appendix B, we will identify all of the systems of eigenvalues arising in level N and character ϵ from sums of characters. We consider the representation $\rho = \omega^a \epsilon \oplus \omega^b \oplus \omega^c$. Start with

$$\rho = \begin{pmatrix} \chi_1 & & \\ & \chi_2 & \\ & & \chi_3 \end{pmatrix}$$

where χ_i is a character which factors as $\omega^{a_i}\epsilon_i$, with ϵ_i ramifying outside p.

To ensure that the level of the representation ρ is $N = q$ (a prime), consider

$$\rho = \begin{pmatrix} \omega^{a_1}\epsilon_1 & & \\ & \omega^{a_2}\epsilon_2 & \\ & & \omega^{a_3}\epsilon_3 \end{pmatrix}$$

Since ω is unramified at N, then $\omega|_{I_N} = 1$. This gives

$$\rho|_{I_N} \sim \begin{pmatrix} \epsilon_1 & & \\ & \epsilon_2 & \\ & & \epsilon_3 \end{pmatrix}$$

If any of the ϵ_i ramify outside N, then from Definition 2.1.6 we would get the level divisible by something other than pN. This is not what we want, so we will choose ϵ_i to ramify only at N. If more than one of the ϵ_i ramify at N, then when we let $\rho|_{I_N}$ act on a three-dimensional $\overline{\mathbb{F}}_p$ vector space matrix multiplication, the fixed space is of dimension 0 or 1, resulting in a level of N^2 or N^3 by Definition 2.1.6. Again, we don't want this result. Hence, we need only one of the ϵ_i to ramify at N and the others must be trivial.

Now we get χ_1, χ_2, and χ_3 to be some permutation of $\omega^{a_1}\epsilon$, ω^{a_2}, and ω^{a_3}. From this, we get

$$\det \rho = \omega^{a_1+a_2+a_3}\epsilon$$

with $\omega^{a_1+a_2+a_3}$ ramifying at p and ϵ ramifying outside p. This says that the nebentype of ρ is ϵ. Since we need the nebentype of ρ to be ϵ_N, we get $\epsilon = \epsilon_N$.

Recall from Definition 2.1.1 that for $n = 3$, the representation ρ is odd if

$$\rho(\mathrm{Frob}_\infty) \sim \pm \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

We know that $\omega(\mathrm{Frob}_\infty) = -1$ and $\epsilon_N(\mathrm{Frob}_\infty) = (-1)^{\frac{N-1}{2}}$ as in Definition 2.1.4. Then we see which combinations of characters with ϵ_N attached in the appropriate spot have alternating signs down the diagonal when ρ acts on Frob_∞.

Example 4.0.1. Let $p = 5$ and $N = 7$. Then $\epsilon_7(\mathrm{Frob}_\infty) = -1$. Consider

$$\begin{pmatrix} \omega^a & & \\ & \omega^b & \\ & & \omega^c \end{pmatrix}$$

If $a \equiv b \not\equiv c \pmod 2$ then we can add ϵ_7 to ω^a to make ρ odd. If $b \equiv c \not\equiv a \pmod 2$, then we can add ϵ_7 to ω^c to make ρ odd. This gives $\rho = \omega^a\epsilon_7 \oplus \omega^b \oplus \omega^c$ or $\rho = \omega^a \oplus \omega^b \oplus \omega^c\epsilon_7$. These are equivalent representations since we can switch the place of ω^a and ω^c. Recall from before that we want $a + b + c \equiv 0 \pmod{p-1}$, so we get the following triples:

$$[2,3,3],\ [3,3,2]$$
$$[0,3,1],\ [1,3,0]$$
$$[0,1,3],\ [3,1,0]$$
$$[1,1,2],\ [2,1,1]$$

If $a \equiv b \equiv c \pmod 2$, then we can add ϵ_7 to ω^b to make ρ odd. This gives $\rho = \omega^a \oplus \omega^b \epsilon_7 \oplus \omega^c$ and results in the triples:

$$[0, 0, 0]$$
$$[2, 2, 0],\ [0, 2, 2]$$
$$[2, 0, 2]$$

Once we have a possible representation ρ as a sum of characters, we need to check that $Tr(\rho(\text{Frob}_\ell))$ and $T_2(\rho(\text{Frob}_\ell))$ match up with $a(\ell, 1)$ and $a(\ell, 2)$ respectively for primes $2 \leq \ell \leq 47$ and $\ell \nmid pN$. From the previous example, it is easy to see that $Tr(\rho) = \omega^a \epsilon_7 + \omega^b + \omega^c$, $Tr(\rho) = \omega^a + \omega^b \epsilon_7 + \omega^c$, or $Tr(\rho) = \omega^a + \omega^b + \omega^c \epsilon_7$ depending on the triple $[a, b, c]$. Similarly, $T_2(\rho) = \omega^{a+b}\epsilon_7 + \omega^{a+c}\epsilon_7 + \omega^{b+c}$, $T_2(\rho) = \omega^{a+b}\epsilon_7 + \omega^{a+c} + \omega^{b+c}\epsilon_7$, or $T_2(\rho) = \omega^{a+b} + \omega^{a+c}\epsilon_7 + \omega^{b+c}\epsilon_7$ for the respective triple $[a, b, c]$.

Example 4.0.1 (continued)**.** Take the triples $[2, 3, 3]$ and $[3, 3, 2]$ from above. These correspond to the representation $\rho = \omega^2 \oplus \omega^3 \oplus \omega^3 \epsilon_7$, having

$$Tr(\rho) = \omega^2 + \omega^3 + \omega^3 \epsilon_7$$

$$T_2(\rho) = \omega + \omega \epsilon_7 + \omega^2 \epsilon_7$$

With $Tr(\rho)$ and $T_2(\rho)$ in hand, we run through Frob_ℓ and find the system of eigenvalues. To do this, recall from Definition 2.1.4 that

$$\omega(\text{Frob}_\ell) \equiv \ell \pmod p$$

and since we consider ϵ_N to be a quadratic character,

$$\epsilon_N(\text{Frob}_\ell) \equiv \left(\frac{N^*}{\ell}\right)$$

where $N^* = (-1)^{\frac{N-1}{2}} N$.

If the resulting system of eigenvalues matches up with a system of eigenvalues found from our program, we check to make sure the triple $[a, b, c]$ matches with the weight $F[a', b', c']$ by checking that $F[a-2, b-1, c] = F[a', b', c']$. If this matches, we write the representation ρ in line with the associated system of eigenvalues and weights, and say ρ appears to be attached to this system.

Example 4.0.1 (continued)**.** For the triples $[2, 3, 3]$ and $[3, 3, 2]$, we calculate $\omega(\text{Frob}_\ell)$, $\epsilon_7(\text{Frob}_\ell)$, $Tr(\rho(\text{Frob}_\ell))$ and $T_2(\rho(\text{Frob}_\ell))$. We get the following table.

ℓ	2	3	11	13	17	19	23	29	31	37	41	43	47
$\omega(\text{Frob}_\ell)$	2	3	1	3	2	4	3	4	1	2	1	3	2
$\epsilon_7(\text{Frob}_\ell)$	1	-1	1	-1	-1	-1	1	1	-1	1	-1	1	-1
$Tr(\rho(\text{Frob}_\ell)$	0	4	3	4	4	1	3	4	1	0	1	3	4
$T_2(\rho(\text{Frob}_\ell)$	3	1	3	1	1	4	0	4	4	3	4	0	1

Since we want to find out if $a(\ell, 1) = Tr(\rho(\text{Frob}_\ell))$ and $\ell a(\ell, 2) = T_2(\rho(\text{Frob}_\ell))$, we construct the following table.

ℓ	2	3	11	13	17	19	23	29	31	37	41	43	47
$Tr(\rho(\text{Frob}_\ell))$	0	4	3	4	4	1	3	4	1	0	1	3	4
$\ell^{-1}T_2(\rho(\text{Frob}_\ell))$	4	2	3	2	3	1	0	1	4	4	4	0	3

This system of eigenvalues matches up to the system corresponding to the weights $F[8, 6, 3]$ and $F[9, 6, 2]$. Note that the triple $[2, 3, 3]$ corresponds to the weight $F[8, 6, 3]$ and $[3, 3, 2]$ corresponds to $F[9, 6, 2]$. Then $\rho = \omega^2 \oplus \omega^3 \oplus \omega^3\epsilon_7$ appears to be attached to the system of eigenvalues corresponding to the weights $F[8, 6, 3]$ and $F[9, 6, 3]$.

Chapter 5. Other Reducible Representations

Once we have all representations that can be written as sums of characters, we move to representations involving an irreducible two-dimensional representation.

5.1 Irreducible Two-dimensional Representation plus a Character

The first representation we will look at is a representation $\rho = \sigma \oplus \omega^k$ where σ is an irreducible two-dimensional representation and ω is the cyclotomic character. Recall from Example 2.1.2 that we have three special forms of Levi subgroups constructed with a 2×2 block and a 1×1 block. They are the following:

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}, \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

The first and third subgroups here have the 2×2 block intact while the second subgroup has the 2×2 block in the corners. The two-dimensional representation will correspond to this 2×2 block and the character will correspond to the 1×1 block.

Let $\sigma : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ be a two-dimensional irreducible representation. Suppose σ is niveau 1 such that

$$\sigma|_{I_p} = \begin{pmatrix} \omega^a & \\ & \omega^b \end{pmatrix}$$

where σ is odd. Then we consider the representation $\rho = \sigma \oplus \omega^c$. This gives the predicted weights

$$F[a-2, b-1, c],\ F[b-2, a-1, c],\ F[c-2, a-1, b],\ F[c-2, b-1, a]$$

corresponding respectively to the triples

$$[a, b, c],\ [b, a, c],\ [c, a, b],\ [c, b, a]$$

Since these will correspond to the same eigenclass, we are interested in the following patterns of triples:

$$[a, b|c]$$
$$[b, a|c]$$
$$[c|a, b]$$
$$[c|b, a]$$

If this pattern shows up on the table, we predict there is a representation $\rho = \sigma \oplus \omega^c$ which is attached to eigenclasses in these weights.

Example 5.1.1. Examining the table of eigenvalues for $p = 5$ and $N = 11$, the eighth system of eigenvalues is the first in the table not to correspond to a sum of characters. We reproduce this portion of the table below.

2	3	7	13	17	19	23	29	31	37	41	43	47	Weights	Triples	Galois Rep
1,2	1,2	1,2	1,3	1,2	1,1	1,2	1,1	4,4	1,3	1,4	1,3	1,3	F[1, 0, 0] F[5, 4, 0] F[3, 2, 0] F[2, 2, 1] F[6, 2, 1] F[6, 4, 3]	[3, 1, 0] [1, 3, 0] [0, 3, 1] [0, 1, 3]	$(\sigma \otimes \omega) \oplus \omega^0$

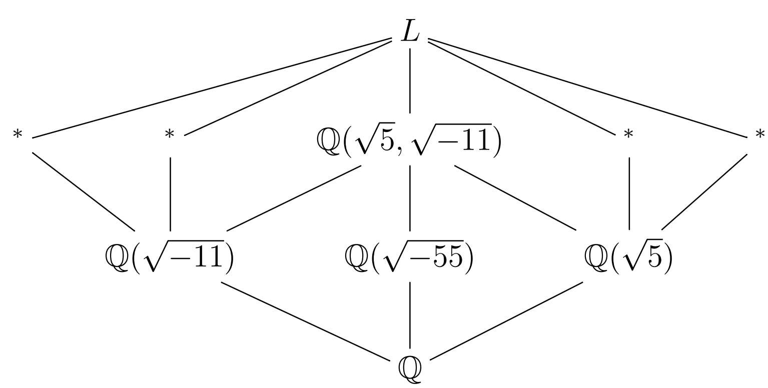
The triples corresponding to the weights in which the eigenvalues appear are

$$[3, 1, \mathbf{0}]$$
$$[1, 3, \mathbf{0}]$$
$$[\mathbf{0}, 3, 1]$$
$$[\mathbf{0}, 1, 3]$$

This is similar to the pattern we expect from above for a two-dimensional irreducible niveau 1 representation plus a character. We thus wish to search for a two-dimensional irreducible representation of niveau 1, level 11 and nebentype ϵ_{11}, to which we will add the character $\omega^0 = 1$.

Such a representation will cut out a Galois extension of $\mathbb{Q}$ ramified only at 5 and 11, with the ramification index of 5 dividing $5 - 1 = 4$. Examining the tables in [12], we discover that there is a D_8-extension L of $\mathbb{Q}$ ramified only at 5 and 11, defined as the splitting field of the polynomial $f(x) = x^4 - x^3 + 2x - 1$. In order to determine the ramification indices of 5 and 11, we need to write $L = \mathbb{Q}(\theta)$ for θ a root of some degree 8 polynomial $g(x)$. Using GP/Pari, we compute the compositum of f with itself to obtain such a polynomial $g(x) = x^8 - 3x^7 + 9x^6 - 13x^5 + 18x^4 - 11x^3 + 11x^2 - 4x + 1$. A quick check shows that g does have the correct Galois group, and that the discriminant of the number field $\mathbb{Q}(\theta)$ (where θ is a root of g) is divisible only by 5 and 11.

Again, by GP/Pari, we check that the ramification indices at 5 and 11 are both equal to 2. However, D_8 has many elements of order 2, some in its center, and others outside the center. The center of D_8 is generated by the element of order two which is a square in D_8. We wish to find whether the inertia groups at 5 and 11 are in the center or not.



Note that computations with GP/Pari show that the fixed field of an element of order 4 must be the quadratic field $\mathbb{Q}(\sqrt{-55})$, justifying its position in this diagram. Since the center is a normal subgroup, we know that the fixed field of the center of D_8 is the field $\mathbb{Q}(\sqrt{-11}, \sqrt{5})$. We are now ready to determine whether the inertia groups at 5 and 11 lie in the center of D_8. We know that both 5 and 11 ramify in the extension $\mathbb{Q}(\sqrt{-11}, \sqrt{5})$, which means that they cannot ramify in the extension $L/\mathbb{Q}(\sqrt{-11}, \sqrt{5})$. Hence, neither inertia group can lie in the center.

In order to find a two-dimensional representation, we examine the character table for D_8, which we copy from [11, pg. 161].

	e	a^2	a	b	ab
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Note here that $D_8 = \langle a, b | a^4 = b^2 = e, b^{-1}ab = a^{-1} \rangle$. The element a is a representative of the conjugacy class of elements of order 4, its square generates the center of D_8, and the elements b and ab (which label the last two columns) are representatives of the noncentral conjugacy classes of order 2. Since χ_5 is the only irreducible degree 2 character, there is a unique (up to equivalence) two-dimensional representation $\tilde{\sigma} : D_8 \to GL_2(\mathbb{F}_5)$, with $Tr \circ \tilde{\sigma} = \chi_5$.

Since b has order 2, its image under $\tilde{\sigma}$ also has order 2, and according to the character table must have trace 0. Since it has order 2, its eigenvalues are ± 1. Together with the fact that its trace is 0, we see that the two eigenvalues must be 1 and -1. Hence

$$\tilde{\sigma}(b) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

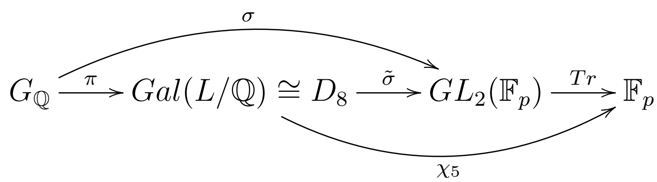
Similarly,

$$\tilde{\sigma}(ab) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that since the images of b and ab have determinant -1, we see that $\det \tilde{\sigma}$ must be χ_2.

To create a Galois representation $\sigma : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_p)$, we compose the projection π :

$G_{\mathbb{Q}} \to Gal(L/\mathbb{Q})$ with the representation $\tilde{\sigma}$. To see the relation between the various maps we have defined, see the diagram below.



We will often identify elements of $G_{\mathbb{Q}}$ with their images under π in $Gal(L/\mathbb{Q})$. Hence, we may make sense of saying $\sigma(a)$ or $\sigma(b)$.

From GP/Pari, we have already seen that the image of inertia at 11 has order 2 and is noncentral. Hence, this image must be generated by a matrix similar to

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This matrix (acting by multiplication on a two-dimensional vector space M) has a one-dimensional fixed space. Hence, by Definition 2.1.6, we want to find n_{11}. Since M^{I_q} is one-dimensional, then M/M^{I_q} is also one-dimensional. Hence, $n_q = 1$, and we see that σ has level $N = 11$.

For the nebentype, recall from section 2.1.3 that we factor $\det \sigma = \epsilon \cdot \omega^k$ where ω does not ramify at 11. Then $\det \sigma|_{I_{11}} = \epsilon$, since ω is trivial when restricted to I_{11}. Since the order of $\sigma|_{I_{11}}$ is 2, the image of $\sigma|_{I_{11}}$ is generated by

$$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

Then the image of $\det(\rho|_{I_{11}})$ is generated by -1. This says ϵ takes on the values 1 and -1, so is quadratic. Therefore $\epsilon = \epsilon_{11}$.

A similar study of I_5 shows that the image of I_5 under σ has order 2 and is noncentral.

Hence, the image of I_5 under σ is generated by a matrix similar to

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since $\sigma|_{I_5}$ must be similar to a representation of the form

$$\begin{pmatrix} \omega^a & \\ & \omega^b \end{pmatrix}$$

we see that $a = 0$ and $b = 2$ (since ω has order 4). Hence, we have

$$\sigma|_{I_5} \sim \begin{pmatrix} \omega^0 & \\ & \omega^2 \end{pmatrix}.$$

We can twist σ by ω to get

$$\sigma \otimes \omega = \begin{pmatrix} \omega^1 & \\ & \omega^3 \end{pmatrix}$$

Adding ω^0, we get the representation $\rho = (\sigma \otimes \omega^1) \oplus \omega^0$. This gives

$$\rho = \begin{pmatrix} \sigma \otimes \omega & \\ & \omega^0 \end{pmatrix}$$

Then

$$\rho|_{I_5} \sim \begin{pmatrix} \omega^1 & & \\ & \omega^3 & \\ & & \omega^0 \end{pmatrix}$$

At any point, we could switch the positions of ω^1 and ω^3 in σ. Similarly, ω^0 could have been positioned in the first or last entry of ρ. This gives us the triples $[1, 3, 0]$, $[3, 1, 0]$, $[0, 1, 3]$ and $[0, 3, 1]$ that we want.

Now that we have a representation, we want to check the traces and cotraces of the images of Frobenius for $2 \leq \ell \leq 47$, $\ell \nmid pN$. Let $\rho = (\sigma \otimes \omega^1) \oplus \omega^0$. Let $T = Tr(\sigma(\text{Frob}_\ell))$ and $D = \det(\sigma(\text{Frob}_\ell))$. Then $Tr(\rho(\text{Frob}_\ell)) = T\ell + 1$ and $T_2(\rho(\text{Frob}_\ell)) = \ell^2 D + \ell T$.

To find T, we need the order of Frobenius under σ (the same as the inertial degree of ℓ in L) for each ℓ and whether $\pi(\text{Frob}_\ell)$ is in the center of D_8 if it has order 2. From the character table for χ_5, we have the following for T and D mod 5:

$ord(\text{Frob}_\ell)$	$\text{Frob}_\ell \in Center$	T	D
1	Y	2	1
2	Y	3	1
2	N	0	4
4	–	0	1

To distinguish the order 2 elements, we need to find if Frob_ℓ lands in the center of D_8. To do this, we look at the inertia degree of ℓ in $L/\mathbb{Q}$ compared to the inertia degree of ℓ in $\mathbb{Q}(\sqrt{-11}, \sqrt{5})/\mathbb{Q}$. If the inertia occurs in the biquadratic extension, then ℓ cannot be inert in $L/\mathbb{Q}(\sqrt{-11}, \sqrt{5})$, so Frob_ℓ is not in the center and $T = 0$ and $D = -1$. Otherwise, $T = 3$ and $D = 1$.

Let f_1 be the inertial degree for ℓ in $L/\mathbb{Q}$ and f_2 be the inertial degree of ℓ in $\mathbb{Q}(\sqrt{-11}, \sqrt{5})/\mathbb{Q}$. We get the following for f_1, f_2, T, D, $Tr(\rho(\text{Frob}_\ell))$ and $\ell^{-1}T_2(\rho(\text{Frob}_\ell))$.

ℓ	2	3	7	13	17	19	23	29	31	37	41	43	47
f_1	4	2	4	4	4	2	2	2	2	2	2	4	2
f_2	–	2	–	–	–	2	2	2	1	2	2	–	2
T	0	0	0	0	0	0	0	0	3	0	0	0	0
D	1	-1	1	1	1	-1	-1	-1	1	-1	-1	1	-1
$Tr(\rho(\text{Frob}_\ell))$	1	1	1	1	1	1	1	1	4	1	1	1	1
$\ell^{-1}T_2(\rho(\text{Frob}_\ell))$	2	2	2	3	2	1	2	1	4	3	4	3	3

Here, note $a(\ell, 1) = Tr(\rho(\text{Frob}_\ell))$ and $a(\ell, 2) = \ell^{-1}T_2(\rho(\text{Frob}_\ell))$ as given on the table at the beginning of the example. Thus $\rho = (\sigma \otimes \omega) \oplus \omega^0$ appears to be attached to eigenclasses in the given weights.

Example 5.1.2. We refer to the same table, but this time look at the 9th system of eigenvalues. We see the triples $[2, 2, 0]$, $[2, 0, 2]$, and $[0, 2, 2]$. Consider $\rho = (\sigma \otimes \omega^k) \oplus \omega^2$, where σ is the representation in the previous example. This time let $k = 0$, so $\rho = \sigma \oplus \omega^2$. With D and T defined as in the previous example, we see that $Tr(\rho(\text{Frob}_\ell)) = T + \ell^2$ and $T_2(\rho(\text{Frob}_\ell)) = D + \ell^2 T$. Using GP/Pari, we run through $2 \leq \ell \leq 47$, $\ell \nmid pN$, and verify that $Tr(\rho(\text{Frob}_\ell))$ and $T_2(\rho(\text{Frob}_\ell))$ matches $a(\ell, 1)$ and $\ell a(\ell, 2)$ respectively. Thus, this representation appears to be attached to this system of eigenvalues.

Now let σ above be niveau 2. Then

$$\sigma|_{I_p} = \begin{pmatrix} (\Psi)^d & \\ & (\Psi')^d \end{pmatrix}$$

where σ is odd. Again, we consider the representation $\rho = \sigma \oplus \omega^c$ and this time we look for the following pattern:

$$[a, b|c]$$
$$[a', b'|c]$$
$$[c|a, b]$$
$$[c|a', b']$$

If this pattern shows up in one of the tables, we have reason to think there is a representation $\rho = \sigma \oplus \omega^c$ which is attached to the weights associated with the triples above.

Example 5.1.3. Consider the table for $p = 5$ and $N = 7$. The 8th system of eigenvalues does not correspond to a sum of characters. We reproduce the system below.

2	7	11	13	17	19	23	29	31	37	41	43	47	Weights	Triples	Galois Rep

0,0	1,3	0,0	1,3	1,2	1,1	0,0	0,0	1,4	0,0	1,4	0,0	1,2	F[1, 0, 0] F[5, 4, 0] F[4, 1, 0] F[2, 2, 1] F[6, 2, 1] F[6, 5, 2]	[3, 1, 0] [2, 2, 0] [0, 3, 1] [0, 2, 2]	$\sigma \oplus \omega^0$

This system corresponds to the triples

$$[3, 1, \mathbf{0}]$$

$$[2, 2, \mathbf{0}]$$

$$[\mathbf{0}, 3, 1]$$

$$[\mathbf{0}, 2, 2]$$

This is similar to the pattern we expect from above for a representation $\rho = \sigma \oplus \omega^0$, where σ is a two-dimensional irreducible niveau 2 representation. For this system of eigenvalues, we will restrict our search to finding a two-dimensional representation of niveau 2 and level 7, to which we will add the character $\omega^0 = 1$.

Similar to Example 5.1.1, we are looking for a representation which cuts out a Galois extension of $\mathbb{Q}$ ramified only at 5 and 7 with the ramification index of 5 dividing $5^2 - 1 = 24$ but not $5 - 1 = 4$. Referring to [12], we discover an S_3-extension L of $\mathbb{Q}$ ramified only at 5 and 7. This extension is defined as the splitting field of the polynomial $f(x) = x^3 - x^2 + 2x - 3$. A quick check in GP/Pari shows that the inertial degree at $q = 5$ is $e = 3$ and at $q = 7$ is $e = 2$

Referring to [11, pg. 121] we get the character table for S_3. Let $S_3 = \langle a, b : a^3 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$. Then

	e	a	a^2	b	ab	a^2b
χ_1	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	-1
χ_3	2	-1	-1	0	0	0

In this table, the elements a and a^2 are representatives of conjugacy classes of elements of order 2 while b, ab, and a^2b are representatives of conjugacy classes of elements of order 3.

Since χ_3 is the only irreducible degree 2 character, there is a unique (up to equivalence) two-dimensional representation $\tilde{\sigma} : S_3 \to GL_2(\mathbb{F}_5)$. Since a is order 3, its image under $\tilde{\sigma}$ must be order 3. According to the character table, its image must have trace -1. We can see that

$$\tilde{\sigma}(a) \sim \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

Note that b is of order 2, so its image under $\tilde{\sigma}$ must also have order 2. According to the character table it must have trace 0. Putting this all together, we can see that

$$\tilde{\sigma}(b) \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Notice the determinant of the image of a is 1 while the that of the image of b is -1. This shows that $\det \tilde{\sigma} = \chi_2$.

Similar to Example 5.1.1, to create the Galois representation $\sigma : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_p)$, we compose the projection $\pi : G_{\mathbb{Q}} \to Gal(L/\mathbb{Q})$ with the representation $\tilde{\sigma}$. Like last time, we identify elements of $G_{\mathbb{Q}}$ with their images under π in $Gal(L/\mathbb{Q})$ so we can make sense of saying $\sigma(a)$ or $\sigma(b)$.

For the level of the representation, we look at I_7. As noted earlier, for $q = 7$ we have $e = 2$. Then $\sigma|_{I_7}$ has image similar to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

By the same argument as in Example 5.1.1, we get $N = 7$ as expected. Similarly, we find that ϵ_7 must be quadratic.

For $q = 5$, $e = 3$ so the image of $\sigma|_{I_5}$ has order 3. This says the image of I_5 under σ must be generated by a matrix similar to

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

This is diagonalizable, so we might say

$$\sigma|_{I_5} \sim \begin{pmatrix} \omega^a & \\ & \omega^b \end{pmatrix}$$

But ω is order 4 and there is no $d \in \mathbb{F}_5^\times$ such that $(\omega^d)^3 = 1$. Therefore $\sigma|_{I_5}$ cannot be diagonalized in term of cyclotomic characters. Instead we use niveau 2 characters and get

$$\sigma|_{I_5} = \begin{pmatrix} (\Psi)^a & \\ & (\Psi')^a \end{pmatrix}$$

Here, Ψ and Ψ' have order 24. For Ψ^a to have order 3, we must have $\frac{24}{\gcd(a,24)} = 3$. This gives $a = 8$ or $a = 16$. If we take $a = 8$, then we have $8 = 3 + 1 * 5$ and $8 = -2 + 2 * 5$. These representations of 8 give triples $[3, 1, 0]$ and $[2, 2, 0]$ (mod 4) when we throw on the one-dimensional character ω^0. Therefore we take $\rho = \sigma \oplus \omega^0$ where

$$\sigma = \begin{pmatrix} (\Psi)^a & \\ & (\Psi')^a \end{pmatrix}$$

Now our representation has the right weight, level, and character as needed. All that remains is to check that $a(\ell, 1)$ and $a(\ell, 2)$ match up as necessary.

Using GP/Pari and the polynomial $f(x) = x^3 - x^2 + 2x - 3$, we find the inertial degrees of each prime $\ell \nmid pN$. Let $T = Tr(\sigma(\text{Frob}_\ell))$ and $D = \det(\sigma(\text{Frob}_\ell))$. Then $Tr(\rho(\text{Frob}_\ell)) =$

$T+1$ and $T_2(\rho(\mathrm{Frob}_\ell)) = (D+T)$. We get the following table.

ℓ	2	3	11	13	17	19	23	29	31	37	41	43	47
f	3	2	3	2	2	2	3	3	2	3	2	3	2
T	-1	0	-1	0	0	0	-1	-1	0	-1	0	-1	0
D	1	-1	1	-1	-1	-1	1	1	-1	1	-1	1	-1
$Tr(\rho(\mathrm{Frob}_\ell))$	0	1	0	1	1	1	0	0	1	0	1	0	1
$\ell^{-1}T_2(\rho(\mathrm{Frob}_\ell)$	0	3	0	3	2	1	0	0	4	0	4	0	2

As before, we see that $a(\ell,1) = Tr(\rho(\mathrm{Frob}_\ell))$ and $a(\ell,2) = \ell^{-1}T_2(\rho(\mathrm{Frob}_\ell))$ for the necessary ℓ. From this, $\rho = \sigma \oplus \omega^0$ appears to be attached to the weights associated to the before-mentioned triples.

5.2 σ Arising from a Modular Form

Let σ be a wildly ramified two-dimensional odd representation. Then

$$\sigma|_{I_p} \sim \begin{pmatrix} \omega^a & * \\ & \omega^b \end{pmatrix}$$

This time, the diagonal characters of $\sigma|_{I_p}$ cannot be interchanged as that would cause the conjugate to not be upper triangular. We can still add ω^c to get our representation ρ. In this case, we look for the following triples:

$$[a,b,c]$$
$$[c,a,b]$$

We predict the representation is $\rho = \sigma \oplus \omega^c$. Taking the pair (a,b), we will find a modular form of weight k, level N, and character ϵ_N associated to the two-dimensional representation σ, which when we add the one-dimensional character will give the correct representation ρ appearing to be attached to the right eigenclass.

Example 5.2.1. Refer to the table $p = 5$, $N = 13$. The first eigenclass not attached to a representation which is a sum of characters (row 8) has triples [3, 0, 1] and [0, 1, 3]. As stated, we predict the representation is $\rho = \sigma \oplus \omega^3$, where σ is a wildly ramified two-dimensional representation and

$$\sigma|_{I_5} = \begin{pmatrix} \omega^0 & * \\ & \omega^1 \end{pmatrix}$$

If $\sigma \oplus \omega^3$ really is attached to the system of eigenvalues that we are considering, then σ would have level 13 and quadratic nebentype ϵ_{13}. We would expect to be able to build several reducible representations from σ, with weights predicted in the table below.

Representation	*Triple*	*Weight*
$\sigma \oplus \omega^3$	$[3,0,1],[0,1,3]$	$F[5,3,1], F[6,4,3]$
$(\sigma \otimes \omega^2) \oplus \omega^3$	$[3,2,3],[2,3,3]$	$F[5,5,3], F[9,5,3], F[8,6,3]$
$(\sigma \otimes \omega^3) \oplus \omega$	$[1,3,0],[3,0,1]$	$F[3,2,0], F[5,3,1]$
$(\sigma \otimes \omega) \oplus \omega$	$[1,2,1],[1,1,2]$	$F[3,1,1], F[7,5,1], F[7,4,2]$

We note that we do find systems of eigenvalues in all these weights. In fact, it appears there are two complete sets of these eigenvalues.

For now, consider the eigenvalues $\{a(\ell, k)\}$, where $k \in \{1, 2\}$ and ℓ prime with $2 \leq \ell \leq 47$ and $\ell \nmid pN$, in the 8th row of the table, associated to the triples [0, 1, 3] and [0, 1, 3]. If this system is truly attached to a representation of the form $\rho = \sigma \oplus \omega^3$, we can extract the trace of $\sigma(\text{Frob}_\ell)$ very easily. Since $\rho = \sigma \oplus \omega^3$, then

$$Tr(\rho(\text{Frob}_\ell)) = Tr(\sigma(\text{Frob}_\ell)) + \omega^3(\text{Frob}_\ell)$$

Then by Definition 2.1.3, and rearranging the above equation, we get

$$Tr(\sigma(\text{Frob}_\ell)) = Tr(\rho(\text{Frob}_\ell)) - \omega^3(\text{Frob}_\ell) = a(\ell, 1) - \ell^3$$

Taking these values mod 5, we get the following table

ℓ	2	3	7	11	17
$a(\ell, 1)$	0	4	3	2	3
$Tr(\sigma(\text{Frob}_\ell))$	2	2	0	1	0

From this computation of the traces of $\sigma(\text{Frob}_\ell)$, we can also easily compute hypothetical values of the traces of the other three-dimensional representations listed above. They are:

ℓ	2	3	7	11	17
$Tr(\sigma(\text{Frob}_\ell))$	2	2	0	1	0
$Tr\big(((\sigma \otimes \omega^2) \oplus \omega^3)(\text{Frob}_\ell)\big)$	1	0	3	2	3
$Tr\big(((\sigma \otimes \omega^3) \oplus \omega)(\text{Frob}_\ell)\big)$	3	2	2	2	2
$Tr\big(((\sigma \otimes \omega) \oplus \omega)(\text{Frob}_\ell)\big)$	1	4	2	2	2

We note that these traces, up through $\ell = 17$, match the eigenvalues in the table. Using GP/Pari, these values actually match up through $\ell = 47$.

Similarly, assuming $\rho = \sigma \oplus \omega^3$ is attached to the 8th system of eigenvalues, we find

$$T_2(\rho(\text{Frob}_\ell)) = \ell a(\ell, 2) = \det(\sigma(\text{Frob}_\ell)) + \omega^3(\text{Frob}_\ell) Tr(\sigma(\text{Frob}_\ell))$$

and

$$\det(\rho(\text{Frob}_\ell)) = \det(\sigma(\text{Frob}_\ell))\omega^3(\text{Frob}_\ell)$$

We can easily find $\det(\sigma(\text{Frob}_\ell))$ and $\det(\rho(\text{Frob}_\ell))$, assuming $a(\ell, 2)$ really is an eigenvalue for this representation. This then allows us to find the cotraces for the other three-dimensional representations. We don't go through the work, but we get the following table:

ℓ	2	3	7	11	17
$\ell^{-1}T_2\big(\sigma \oplus \omega^3)(\text{Frob}_\ell)\big)$	2	4	4	0	1
$\ell^{-1}T_2\big(((\sigma \otimes \omega^2) \oplus \omega^3)(\text{Frob}_\ell)\big)$	1	3	4	0	1
$\ell^{-1}T_2\big(((\sigma \otimes \omega^3) \oplus \omega)(\text{Frob}_\ell)\big)$	2	3	1	0	4
$\ell^{-1}T_2\big(((\sigma \otimes \omega) \oplus \omega)(\text{Frob}_\ell)\big)$	0	0	1	0	4

Again, by GP/Pari we easily verify that the cotraces also match up through $\ell = 47$. Hence, if we can find an actual σ with the given traces, its existence would explain four rows of the table for $p = 5$ and $N = 13$.

Before we begin our search for σ, we will digress and consider the representation $\sigma \otimes \epsilon_{13} = \sigma'$. Restriction to I_{13} shows $\sigma|_{I_{13}} \sim (\sigma \otimes \epsilon_{13})|_{I_{13}}$, and we easily check that σ' is also level 13 and nebentype ϵ_{13}. It would give rise to the following four additional systems of eigenvalues, where the last five columns correspond to $Tr(\sigma(\text{Frob}_\ell))$:

ρ	$Triple$	$Weight$	2	3	7	11	17
$\sigma' \oplus \omega^3$	$[3,0,1],[0,1,3]$	$F[5,3,1], F[6,4,3]$	1	4	3	0	3
$(\sigma' \otimes \omega^2) \oplus \omega^3$	$[3,2,3],[2,3,3]$	$F[5,5,3], F[9,5,3], F[8,6,3]$	0	0	3	0	3
$(\sigma' \otimes \omega^3) \oplus \omega$	$[1,3,0],[3,0,1]$	$F[3,2,0], F[5,3,1]$	1	2	2	0	2
$(\sigma' \otimes \omega) \oplus \omega$	$[1,2,1],[1,1,2]$	$F[3,1,1], F[7,5,1], F[7,4,2]$	3	4	2	0	2

Note that these values match four more rows of the table up to $\ell = 17$. By GP/Pari, we find these values match up to $\ell = 47$. Therefore, finding a single two-dimensional representation σ with certain traces would explain eight rows of the table.

Now we return to a search for σ. If it were to exist, it would correspond to a modular form via Serre's conjecture. Recall at the beginning of this example we said

$$\sigma|_{I_5} \sim \begin{pmatrix} \omega^0 & * \\ & \omega^1 \end{pmatrix}$$

We will write this as

$$\sigma|_{I_5} \sim \begin{pmatrix} \omega^4 & * \\ & \omega^1 \end{pmatrix}$$

and twist by ω^{-1} to get

$$(\sigma \otimes \omega^{-1})|_{I_5} \sim \begin{pmatrix} \omega^3 & * \\ & \omega^0 \end{pmatrix}$$

To get the weight k of the modular form, we see by [20, pg. 185] that $a = 3$ and $b = 0$ so $k = 1 + 0 * 5 + 3 = 4$. We now use Magma, [7], to compute the mod 5 reductions of the eigenforms of weight 4, level 13, and nebentype ϵ_{13}. In particular, we find the form

$$f(x) = q + q^2 + 4q^3 + 4q^4 + 2q^5 + 4q^6 + 2q^8 + 4q^9 + 2q^{10} + q^{11} + q^{12} + 3q^{13} + \dots \pmod 5$$

Letting $f(x) = \sum a_\ell q^\ell$, we get the following for coefficients of prime powers of q:

ℓ	2	3	7	11	17
a_ℓ	1	4	0	1	0

By Deligne in [8], there must be a Galois representation $\sigma \otimes \omega^{-1}$ attached to $f(x)$ with

ℓ	2	3	7	11	17
$Tr((\sigma \otimes \omega^{-1})(\text{Frob}_\ell)$	1	4	0	1	0
$Tr(\sigma(\text{Frob}_\ell))$	2	2	0	1	0

This is exactly what is desired for σ, and we verify that the traces match up through $\ell = 47$. To verify the cotraces are correct as well, refer to [20] to see that given a representation ρ', $\det(\rho') = \omega^{k-1}\epsilon_N$. For our modular form, since $\rho' = \sigma \otimes \omega^{-1}$ and $k = 4$, $\det \rho' = \omega^3 \epsilon_{13}$. From this we easily see $\det \sigma = \det(\rho')\omega^2 = \omega\epsilon_{13}$. Although we don't show it, we compute $T_2(\rho)$ with this information and verify the cotraces match up with the $a(\ell, 2)$ as needed. This means that we have found Galois representations which appear to be attached to eigenclasses for eight rows of our table.

This process is used to understand a similar set of Galois representations for $p = 5$ and $N = 17$ as well, where σ arises from a modular form of weight 6, level 17, and character ϵ_{17}. We fill that table accordingly.

Chapter 6. Calculating Conjectured Orders of Frobenius

Many representations come from Galois groups which are far too big for us to calculate. Others are tricky as to how to begin searching for them in [12]. One thing that helps is to calculate the orders of various Frob_ℓ to get an idea of the divisors of the size of the group. This will help greatly to narrow down the representations we are interested in, or will let us know the representation is much too big.

To calculate the various orders of Frobenius, recall from definition 2.1.3 that ρ is attached to a simultaneous eigenvector $v \in V$, an $\mathscr{H}(pN)$-module, if

$$\sum_{k=0}^{n}(-1)^k \ell^{k(k-1)/2} a(\ell,k) x^k = det(I - \rho(\text{Frob}_\ell)x)$$

for all $\ell \nmid pN$. Since the order of Frobenius is dependent on the orders of its eigenvalues, we want to find the orders of the roots of $\det(I - \rho(\text{Frob}_\ell)x)$. To do this, we will look at the roots of $\sum_{k=0}^{3}(-1)^k \ell^{k(k-1)/2} a(\ell,k) x^k$ since we know $a(\ell,1)$ and $a(\ell,2)$ from the eigenclasses attached to the cohomology groups.

Theorem 6.0.1. *Given a representation ρ which is attached to an eigenvector with eigenvalues $a(\ell,k)$, the order of $Frob_\ell$ for primes $2 \le \ell \le 47$ with $\ell \nmid pN$ is $n5^k$, where n is the smallest, positive integer such that*

$$\gcd\left((x^n-1)^3, \sum_{k=0}^{3}(-1)^k \ell^{k(k-1)/2} a(\ell,k) x^k\right) = \sum_{k=0}^{3}(-1)^k \ell^{k(k-1)/2} a(\ell,k) x^k$$

Proof. From definition 2.1.3, we know

$$\sum_{k=0}^{n}(-1)^k \ell^{k(k-1)/2} a(\ell,k) x^k = det(I - \rho(\text{Frob}_\ell)x)$$

Because they are equal, $\sum_{k=0}^{n}(-1)^k \ell^{k(k-1)/2} a(\ell,k) x^k$ and $det(I - \rho(\text{Frob}_\ell)x)$ share the same roots. Call these roots a, b, c. Let their orders mod 5 be o_a, o_b, and o_c respectively. Because the orders are finite, a, b, and c must satisfy some equation $x^n - 1 = 0$. The first time this happens will occur at $lcm(o_a, o_b, o_c)$ since this is the smallest integer divisible by all of o_a, o_b, and o_c. Let $n = lcm(o_a, o_b, o_c)$. To account for algebraic multiplicity of roots, we consider $(x^n - 1)^3$. Let

$$\sum_{k=0}^{n} (-1)^k \ell^{k(k-1)/2} a(\ell, k) x^k = (x-a)(x-b)(x-c)$$

Then a, b, and c are roots of $x^n - 1$ so $(x-a)$, $(x-b)$, $(x-c)$ divide $x^n - 1$. Hence the product $(x-a)(x-b)(x-c)$ divides $(v^n - 1)^3$. Then for our choice of n,

$$\gcd\left((x^n - 1)^3, (x-a)(x-b)(x-c)\right) = (x-a)(x-b)(x-c)$$

This gives the result. □

Remark. For representations ρ which have image of order divisible by p, we lose the power of p attached to each Frobenius element because this power does not contribute to the eigenvalues which is all that the characteristic polynomial detects. Thus, our calculation of the order of Frob_ℓ will be missing any powers of p.

Chapter 7. Irreducible Representations

Now that we have exhausted the cases which are sums of characters and sums of irreducible two-dimensional representations plus a character, we move on to irreducible three-dimensional representations. We begin by looking at a representation ρ which arises as a symmetric square of a two-dimensional representation. Then we look at representations wildly ramified at $p = 5$ whose images are isomorphic to A_5 and S_5.

7.1 ρ as a Symmetric Square

By [2, Thm 4.1] Ash, Doud, and Pollack proved the ADPS conjecture for irreducible representations which are symmetric squares of two-dimensional representations having certain properties. Any two-dimensional representation σ has an associated three-dimensional representation $\rho = Sym^2\sigma$. We will discuss how to compute Tr and T_2 for $Sym^2\sigma$ as well as how to compute the predicted weight, level, and character. In cases where $Sym^2\sigma$ is irreducible, this allows us to find larger irreducible Galois representations whose invariants can be computed.

To understand $Sym^2\sigma$, consider an element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}_p)$$

Let M be the set of 2×2 matrices with trace 0. These matrices have the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - a\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

It is obvious that the set

$$\left\{ \mathbf{b_1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{b_2} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{b_3} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

is a basis for M.

Let $GL_2(\mathbb{F}_p)$ act on M via $g \circ M = gMg^{-1}$. This action is linear, so by examining what it does to the above basis, we get a homomorphism $Ad^0 : GL_2(\mathbb{F}p) \to GL_3(\mathbb{F}_p)$. Look at the action of g on each basis element of M and consider the result as a linear combination of the basis elements.

$$g(\mathbf{b_1}) = \tfrac{1}{\det g} \begin{pmatrix} -ac & a^2 \\ -c^2 & ac \end{pmatrix} = \tfrac{1}{\det g} \Big((a^2)\mathbf{b_1} + (ac)\mathbf{b_2} - (c^2)\mathbf{b_3} \Big)$$

$$g(\mathbf{b_2}) = \tfrac{1}{\det g} \begin{pmatrix} -ad - bc & 2ab \\ -2cd & bc + da \end{pmatrix} = \tfrac{1}{\det g} \Big((2ab)\mathbf{b_1} + (ad + bc)\mathbf{b_2} - (2cd)\mathbf{b_3} \Big)$$

$$g(\mathbf{b_3}) = \tfrac{1}{\det g} \begin{pmatrix} bd & -b^2 \\ d^2 & -bd \end{pmatrix} = \tfrac{1}{\det g} \Big((-b^2)\mathbf{b_1} + (-bd)\mathbf{b_2} + (d^2)\mathbf{b_3} \Big)$$

Then in terms of the basis $\{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$ we have

$$Ad^0(g) = \frac{1}{\det g} \begin{pmatrix} a^2 & 2ab & -b^2 \\ ac & ad + bc & -bd \\ -c^2 & -2cd & d^2 \end{pmatrix}$$

We define $\rho = Sym^2(g) = Ad^0(g) \cdot \det(g)$. By looking at the matrix above, we compute

$$Tr(Sym^2(g)) = Tr(g)^2 - \det(g)$$
$$T_2(Sym^2(g)) = \big(Tr(g)^2 - \det(g)\big) \cdot \det(g)$$
$$\det(Sym^2(g)) = \det(g)^3$$

For a two-dimensional representation $\sigma : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$, we will denote by $Sym^2\sigma$ the three-dimensional representation obtained by composing σ with

$$Sym^2 : GL_2(\overline{\mathbb{F}}_p) \to GL_3(\overline{\mathbb{F}}_p)$$

Example 7.1.1. From the table $p = 5$ and $N = 13$, let σ be as defined in example 5.2.1, where σ is irreducible, two-dimensional, and wildly ramified of level 13 and quadratic nebentype ϵ_{13}. Recall

$$\sigma|_{I_5} = \begin{pmatrix} \omega^0 & * \\ & \omega^1 \end{pmatrix}$$

Then with $a = \omega^0$, $b = *$, $c = 0$, and $d = \omega^1$, we see that

$$Sym^2(\sigma) = \begin{pmatrix} \omega^0 & * & * \\ 0 & \omega^1 & * \\ 0 & 0 & \omega^2 \end{pmatrix}$$

This gives the triple $[0, 1, 2]$. We will twist by ω^3 to give the triple $[3, 0, 1]$ which is what we want in order for our representation to correspond to the eigenclass in the table.

If we consider

$$\sigma|_{I_{13}} \sim \begin{pmatrix} \epsilon_{13} & \\ & 1 \end{pmatrix}$$

Then

$$(Sym^2(\sigma))|_{I_{13}} \sim \begin{pmatrix} \epsilon_{13}^2 = 1 & & \\ & \epsilon_{13} & \\ & & 1 \end{pmatrix}$$

which has a fixed space of dimension 2 and results in level $N = 13$ with quadratic character ϵ_{13}. Twisting by ω^3 won't change the level or nebentype, so we know the level, character, and weight are correct for $\rho = Sym^2\sigma \otimes \omega^3$. We want to check if the traces and cotraces of $Sym^2\sigma \oplus \omega^3$ match the computed values of $a(\ell, 1)$ and $a(\ell, 2)$ in the table.

From Example 5.2.1 we had the following table for $Tr(\sigma(\text{Frob}_\ell))$. Since we know $\sigma|_{I_5}$

and $\sigma|_{I_{13}}$, we see that $\det(\sigma) = \omega^{k-1}\epsilon_{13} = \omega\epsilon_{13}$. We will add $\det(\sigma(\text{Frob}_\ell))$ to the table:

ℓ	2	3	7	11	17	19	23	29	31	37	41	43	47
$Tr(\sigma(\text{Frob}_\ell))$	2	2	0	1	0	3	1	4	3	3	1	4	1
$\det(\sigma(\text{Frob}_\ell))$	3	3	3	4	2	1	3	4	4	3	4	3	3

Now

$$Tr(\rho(\text{Frob}_\ell)) = Tr\big((Sym^2(\sigma) \otimes \omega^3)(\text{Frob}_\ell)\big) = \big(Tr(\sigma(\text{Frob}_\ell))^2 - \det(\sigma(\text{Frob}_\ell))\big) * \ell^3$$

Similarly

$$\ell^{-1}T_2(\rho(\text{Frob}_\ell)) = \ell^{-1}T_2\big((Sym^2(\sigma) \otimes \omega^3)(\text{Frob}_\ell)\big) = \ell\big((Tr(\sigma)^2 - \det(\sigma))(\det(\sigma)\big)(\text{Frob}_\ell)$$

Using GP/Pari, we compute the following values:

ℓ	2	3	7	11	17	19	23	29	31	37	41	43	47
$Tr(\rho(\text{Frob}_\ell))$	3	2	1	2	4	2	1	3	0	3	2	1	4
$\ell^{-1}T_2(\rho(\text{Frob}_\ell))$	1	4	2	3	2	2	2	2	0	1	3	2	3

These values match the values of $a(\ell, 1)$ and $a(\ell, 2)$ computed for the eigenclass we are interested in, so $Sym^2\sigma \otimes \omega^3$ appears to be attached to this eigenclass.

7.2 Representations with Image Isomorphic to A_5 and S_5

The last two examples we look at do not break up as sums of characters, two-dimensional irreducible representations plus a character, or come from symmetric square representations. The examples we look at are representations with image isomorphic to A_5 and S_5. These extensions are wildly ramified at $p = 5$. The wild ramification prevents us from permuting the diagonal characters of $\rho|_{I_5}$, so only one weight will be predicted. There are, however, instances when an extra weight described in [2] reveals itself.

Example 7.2.1. For the table $p = 5$, $N = 17$, look at the 16th system of eigenvalues associated to the triples $[2, 0, 2]$ and $[3, 0, 1]$. We expect this eigenclass to be attached to a wildly ramified mod 5 representation with diagonal characters ω^2, ω^0, and ω^2. The additional weight coming from the triple $[3, 0, 1]$ is the extra weight described in [2].

Let $f(x)$ be an irreducible degree p polynomial such that for a root θ of $f(x)$, $\mathbb{Q}(\theta)$ is wildly ramified at p. In [9], Doud describes a relationship between the discriminant of $\mathbb{Q}(\theta)$, the order of the inertia group at p in the Galois closure of $f(x)$, and the depth of the filtration of ramification groups in this inertia group. If we let $n = v_p(disc(Q(\theta)))$, d be the depth of the filtration, and pt be the order of the inertia group, the relation is

$$n = (p - 1)(1 + d/t)$$

From the triple $[2, 0, 2]$ associated with the system of eigenvalues in which we are interested, we guess that the Galois representation that we want has

$$\rho|_{I_p} \sim \begin{pmatrix} \omega^2 & * & * \\ 0 & \omega^0 & * \\ 0 & 0 & \omega^2 \end{pmatrix}$$

One possibility for such a ρ would have the image of inertia at ρ of order 10 (hence, we would have $t = 2$). Since the GCD of t and d must be 1, we see that d is odd. We also have a bound from [19, pg. 72] stating that $n \leq 9$. We see that the only remaining possibility is $n = 6$, $d = 3$. Hence, we begin our search for a Galois representation by looking for a polynomial $f(x)$ of degree 5, defining a number field ramified only at 5 and 17, with discriminant exactly divisible by 5^6.

We return to [12] to find a polynomial which satisfies these conditions. We find the polynomial $f(x) = x^5 - 5x^2 + 5$ satisfies this, so we will find the associated system of eigenvalues. The Galois group associated with $f(x)$ is isomorphic to S_5, and we realize S_5 as a subgroup of $GL_3(\mathbb{F}_5)$ with generators

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 4 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{pmatrix}, C = \begin{pmatrix} 0 & 2 & 3 \\ 4 & 4 & 1 \\ 4 & 0 & 3 \end{pmatrix}$$

A has order 5, B has order 4, C has order 6, B^2 has order 2, C^2 has order 3, C^3 has order 2, and the identity I has order 1. Now we turn to GP/Pari and run through the prime decompositions for primes $2 \leq \ell \leq 47$ where $\ell \nmid pN$. This will give us the inertial degree f of each prime which corresponds to the order of Frob_ℓ via ρ. Since S_5 has only one conjugacy class of each order greater than two, with $\rho(\mathrm{Frob}_\ell) \sim A, B, C, B^2$, or C^2, we can quickly calculate the trace and cotrace of the images. We can also find the minimum polynomial $x^3 + c_1x^2 + c_2x + c_3$ of each generator. The trace is $-c_1$ mod 5 and the cotrace is c_2 mod 5. This gives the following table:

Generator	A	B	C	B^2	C^3	C^2	I
Order	5	4	6	2	2	3	1
Tr	3	1	2	4	4	0	3
T_2	3	1	2	4	4	0	3

Note that even though there are two conjugacy classes of elements of order 2 in S_5, under this map both classes have trace 4 (the images of both classes are in fact conjugate in $GL_3(\mathbb{F}_5)$ and so have the same trace and cotrace). These two conjugacy classes are represented by B^2 and C^3.

We next run through small primes $\ell \nmid pN$ to find the associated inertial degrees. We then get the following table:

ℓ	2	3	7	11	13
f	5	6	4	4	2
Tr	3	2	1	1	4
$\epsilon_{17}(\text{Frob}_\ell)$	1	-1	-1	-1	1

Comparing the traces on this table to $a(\ell, 1)$ in the system of eigenvalues, we note that the traces are off by a sign exactly when $\epsilon_{17}(\text{Frob}_\ell) = -1$. Instead of the representation ρ, we will consider the representation $\rho \otimes \epsilon_{17}$ to adjust for this. Twisting by ϵ_{17} will multiply each diagonal character of ρ by ϵ_{17} so the trace of the image of Frob_ℓ is multiplied by $\epsilon_{17}(\text{Frob}_\ell)$. This gives the following:

ℓ	2	3	7	11	13
f	5	6	4	4	2
$Tr((\rho \otimes \epsilon_{17}))(\text{Frob}_\ell))$	3	3	4	4	4

These traces are accurate, giving $a(\ell, 1) = Tr((\rho \otimes \epsilon_{17})(\text{Frob}_\ell))$ for primes $\ell \nmid pN$ with $2 \leq \ell \leq 47$. The weight of the representation remains the same since restricting ρ to I_5 causes ϵ_{17} to be trivial, thus having no effect on the resulting image of ρ.

Next we check the cotraces to see if they match $a(\ell, 2)$. Since the cotrace is the sum of products of pairs of diagonal elements and each diagonal has ϵ_{17} attached, then the cotrace is multiplied by $\epsilon_{17}^2 = 1$. This gives us $T_2(\rho \otimes \epsilon_{17}) = T_2(\rho) = Tr(\rho)$, where the last equality is an observation from the first table. Using GP/Pari, we find $\ell^{-1} Tr(\rho(\text{Frob}_\ell))$ for the appropriate ℓ and get

ℓ	2	3	7	11	13
$Tr(\rho \otimes \epsilon_{17})$	3	3	4	4	4
$T_2(\rho \otimes \epsilon_{17})$	4	4	3	1	3

This matches with our system of eigenvalues up through 13, so we finish checking the other $\ell \leq 47$ and verify those pairs of eigenvalues also match up.

To verify the level of $\rho \otimes \epsilon$ is 17, we see the ramification index for $q = 17$ is 2. This means the image of $\rho|_{I_{17}}$ is generated by B^2 or C^3, each having trace -1. The image of $\rho|_{I_{17}}$ has order 2 and trace -1, so we find the image of $\rho|_{I_{17}}$ is generated by a matrix similar to

$$\begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

Twisting by ϵ_{17} results in the image of $(\rho \otimes \epsilon_{17})|_{I_{17}}$ being generated by a matrix similar to

$$\begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

This matrix has a fixed space of dimension 2, resulting in level 17. It is easy to see that $\det(\rho \otimes \epsilon) = \epsilon_{17}$ must be quadratic.

Example 7.2.2. Refer to the table for $p = 5$ and $N = 13$, the eigenclass associated to the weight $F[4, 3, 2]$. Using GP/Pari, we run the program described in Chapter 6 to find the orders of Frobenius for primes $2 \leq \ell \leq 47$ with $\ell \nmid pN$. The resulting orders are 1, 2, or 3, with the attached power of $p = 5$ cut out as described above. Using this, the associated representation appears to be an A_5 representation. In fact, in [2, pg. 556] we are provided with a polynomial whose splitting field is cut out by an A_5 representation with the appropriate weight. Let $f(x) = x^5 + 5x^3 - 10x^2 - 45$. The number field discriminant is $5^6 13^2$. By [9], with $n = 6$ we have that $\frac{d}{t} = \frac{1}{2}$. Since $\gcd(d, t) = 1$, we must have $d = 1$ and $t = 2$. Then $\left|G_{0,5}\right| = pt = 10$.

There is a representation $\tilde{\rho} : A_5 \to GL_3(\overline{\mathbb{F}}_p)$ with image generated by the three matrices

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 4 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix}, C = \begin{pmatrix} 4 & 1 & 4 \\ 0 & 4 & 1 \\ 2 & 4 & 2 \end{pmatrix}$$

which have orders 5, 2, and 3 respectively. By letting $\pi : G_{\mathbb{Q}} \to Gal(f(x)) \cong A_5$ and setting $\rho = \tilde{\rho} \circ \pi$, we see as in the previous example that $\rho|_{I_5}$ must be conjugate to

$$\rho|_{I_5} = \begin{pmatrix} \omega^2 & * & * \\ & \omega^0 & * \\ & & \omega^2 \end{pmatrix}$$

This gives the triple $[2, 0, 2]$ which in turn gives the weight $F[4, 3, 2]$ as needed.

Using GP/Pari, we find that $q = 13$ has ramification index $e = 2$, so the image of $\rho|_{I_{13}}$ must have order 2. Since A_5 has only one conjugacy class of elements of order 2, the image of $\rho|_{I_{13}}$ must be conjugate to B and hence to

$$\begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

which has a fixed space of dimension 1. The level of ρ is computed to be 13^2 with trivial nebentype. In order to lower the level, we twist by ϵ_{13} in a manner similar to that in the last example and consider the representation $\rho \otimes \epsilon_{13}$. This representation has level $N = 13$ as desired. The nebentype is then easily seen to be ϵ_{13} as desired.

Now we need to compute $Tr\big((\rho \otimes \epsilon_{13})(\text{Frob}_\ell)\big)$ and $T_2\big((\rho \otimes \epsilon_{13})(\text{Frob}_\ell)\big)$ for $\ell \nmid pN$. We get the following table.

$Generator$	A	A^{-1}	B	C	I
$Order$	5	5	2	3	1
Tr	3	3	4	0	3
T_2	3	3	4	0	3

We note that A_5 has two conjugacy classes of order 5 (one represented by A and the other by A^{-1}) and one conjugacy class of elements of each other order.

With this information, we find the inertial degree f for each ℓ and find the appropriate trace and cotrace from the above table. Then

$$Tr\big((\rho \otimes \epsilon)(\mathrm{Frob}_\ell)\big) = Tr(\rho(\mathrm{Frob}_\ell))\epsilon(\mathrm{Frob}_\ell)$$
$$\ell^{-1}T_2\big((\rho \otimes \epsilon)(\mathrm{Frob}_\ell)\big) = \ell^{-1}T_2(\rho(\mathrm{Frob}_\ell))$$

We now construct the following table.

ℓ	2	3	7	11	17	19	23	29	31	37	41	43	47
f	5	3	3	5	3	3	3	5	5	2	3	5	5
$Tr(\rho(\mathrm{Frob}_\ell))$	3	0	0	3	0	0	0	3	3	4	0	3	3
$\left(\frac{\ell}{13}\right)$	-1	1	-1	-1	1	-1	1	1	-1	-1	-1	1	-1
$Tr\big((\rho \otimes \epsilon)(\mathrm{Frob}_\ell)\big)$	2	0	0	2	0	0	0	3	2	1	0	3	2
$\ell^{-1}T_2\big((\rho \otimes \epsilon)(\mathrm{Frob}_\ell)\big)$	4	0	0	3	0	0	0	2	3	2	0	1	4

The values in the last two rows match the system of eigenvalues that we are interested in, so $\rho \otimes \epsilon_{13}$ appears to be attached to the eigenclass corresponding to the weight $F[4, 3, 2]$. The other weight in which the system of eigenvalues appears is the extra weight described in [2].

Example 7.2.3. The last example we study in this paper comes from the 11th system of eigenvalues found in the table for $p = 5$ and $N = 11$. We reproduce the table here:

2	3	7	13	17	$\cdots$	$Weight$	$Triple$
$1, 3$	$3, 4$	$2, 0$	$0, 3$	$3, 0$	$\cdots$	$F[8, 6, 3]$	$[2, 3, 3]$

We run the program described in Chapter 6, and find that Frob_2 has order 62. This means the attached representation must be irreducible of dimension three. A theorem by Dedekind, in [6], tells us that the fixed field of the image of ρ must be the splitting field of a polynomial of degree at least 62. We have no hope of finding a polynomial this large, so we do not calculate it.

We find there are several other representations which would have huge order in the table for $p = 5$ and $N = 19$, so we likewise do not calculate them.

We have now found representations which appear to be attached to all eigenclasses in the tables for $p = 5$ and $N \in \{3, 7, 11, 13, 17, 19\}$, or discovered that their image would come from a polynomial of degree too big for us to calculate. These tables all appear in the appendix, filled in with the systems of eigenvalues and the representations which appear to be attached.

Appendix A. Computer Code

The following code runs through a certain choice of prime p and level N, and the output gives the tables contained at the end of the paper.

```
./ChoosePrime.txt $1 $2 >PrimeLevel.txt
gp2.5.1<GetLevels2.txt
chmod a+x ListOfLevels.txt
./ListOfLevels.txt
./PrepareTable.txt $2 $1 >GetTable.txt
gp2.5.1<GetTable.txt
chmod a+x DeleteFile.txt
./DeleteFile.txt
rm DeleteFile.txt
```

ChoosePrime.txt stores the choice of prime p and level N.

```
echo "{"
echo 'p='$1";"
echo 'N='$2";"
echo "}"
```

GetLevels2.txt finds all possible choices of triples that could correspond to Conjecture 2.2.1.

```
read("PrimeLevel.txt");
AllTriples=listcreate();
for(k=0,p-2,
    for(j=0,p-2,
        for(i=0,p-2,
            if(lift(Mod(i+j+k,p-1))==0,
                listput(AllTriples,[i,j,k]);
            )
        )
    )
);
```

```
AllTripleWeights=listcreate();
for(i=1,length(AllTriples),
    listput(AllTripleWeights,listcreate());
    x=AllTriples[i][1]-2;
    y=AllTriples[i][2]-1;
    z=AllTriples[i][3];
    TempWeight=[x,y,z];
    while(y<z,
        y=y+(p-1)
    );
    while(y-z<=p-1,
        while(x-y<=p-1,
            if(x>=y&&y>=z,
                listput(AllTripleWeights[i],[x,y,z]);
                File=Str("p",p,"l",N,"W",x,"_",y,"_",z,".txt");
                FileName=Str("./a.out ",p," ",N," ",x," ",y," ",z,"
                     -n -h -p 2 > ",File);
                write("ListOfLevels.txt",FileName);
                write("DeleteFile.txt","rm ",File);
                NextLine=Str("cp ",File," CurWeight.txt");
                write("ListOfLevels.txt",NextLine);
                WeightLine=Str("./Getabc.txt"," ",x," ",y," ",z," >Curabc.txt");
                write("ListOfLevels.txt",WeightLine);
                LastLine=Str("gp2.5.1<SimSingleton.txt");
                write("ListOfLevels.txt",LastLine)
            );
            x=x+(p-1)
        );
        y=y+(p-1);
        x=AllTriples[i][1]-2
    )
);
write("AllWeightsLevels.txt","{");
```

```
a1=Str("AllTriples=",AllTriples,";");
write("AllWeightsLevels.txt",a1);
a2=Str("AllTripleWeights=",AllTripleWeights,";");
write("AllWeightsLevels.txt",a2);
write("AllWeightsLevels.txt","}");
```

From Dr. Doud's program, we get the action from the Hecke operators. The following code spits out the systems of eigenpairs.

```
read("PrimeLevel.txt");
maxim=50;
read("Curabc.txt");
Curabc=Str("For F[",a,",",b,",",c,"]:");
m=matrix(maxim,2);
OpenFile=Str("EP_p",p,"l",N,"W",a,"_",b,"_",c,".txt");
write(OpenFile,"{");
write(OpenFile," ActualPairs=listcreate();");
trap(,
\\Do this if the matrix m is empty
write(OpenFile,"}")
,
\\Otherwise do this
    read("CurWeight.txt");
    /*p=11;
    N=5;*/
    s=matrix(maxim,2);
    n=length(m[2,1]);
    if(n==1,
        forprime(i=2,maxim,
            if(i!=p&&i!=N,
                m[i,1]=m[i,1][1]
            )
        );
        forprime(i=2,maxim,
            if(i!=p&&i!=N,
```

```
            m[i,2]=m[i,2][1]
        )
    );
);
Evalues=matrix(maxim,2);
Evectors=matrix(maxim,2*n);
forprime(i=1,maxim,
    roots=polrootsmod(charpoly(m[i,1]),p);
    Evalues[i,1]=roots;
    numev=length(roots);
    if(numev,
        for(j=1,numev,
            Evectors[i,j]=matker(m[i,1]-roots[j]*matid(n))
        )
    );
    roots=polrootsmod(charpoly(m[i,2]),p);
    numev=length(roots);
    Evalues[i,2]=roots;
    if(numev,
        for(j=1, numev,
            Evectors[i,j+n]=matker(m[i,2]-roots[j]*matid(n))
        )
    )
);
SimLenCur=length(Evalues[2,1]);
SimLenComp=length(Evalues[2,2]);
SimLenNew=SimLenCur*SimLenComp;
SimultaneousCur=vector(SimLenCur);
SimultaneousComp=vector(SimLenComp);
SimultaneousNew=vector(SimLenNew);
SimultaneousNew=vector(SimLenNew);
for(k=1,length(Evalues[2,1]),
    SimultaneousCur[k]=Evectors[2,k]
```

```
);
for(k=1,SimLenComp,
    SimultaneousComp[k]=Evectors[2,k+n]
);
r=1;
for(k=1,length(SimultaneousCur),
    for(l=1,length(SimultaneousComp),
        if(matintersect(SimultaneousCur[k],SimultaneousComp[l])!=[;],
            SimultaneousNew[r]=matintersect(SimultaneousCur[k],SimultaneousComp[l]);
            r=r+1
        )
    )
);
forprime(j=3,maxim,
    r=0;
    for(k=1,SimLenNew,
        if(SimultaneousNew[k]!=0,
            r=r+1
        )
    );
    SimultaneousCur=vector(r);
    for(k=1,r,
        SimultaneousCur[k]=SimultaneousNew[k]
    );
    SimLenCur=length(SimultaneousCur);
    SimLenComp=length(Evalues[j,1]);
    SimultaneousComp=vector(SimLenComp);
    SimLenNew=SimLenCur*SimLenComp;
    SimultaneousNew=vector(SimLenNew);
    for(i=1,SimLenComp,
        SimultaneousComp[i]=Evectors[j,i]
    );
    r=1;
```

```
    for(k=1,SimLenCur,
        for(l=1,SimLenComp,
            if(matintersect(SimultaneousCur[k],SimultaneousComp[l])!=[;],
                SimultaneousNew[r]=matintersect(SimultaneousCur[k],SimultaneousComp[l]);
                r=r+1
            )
        )
    )
);
forprime(j=3,maxim,
    r=0;
    for(k=1,SimLenNew,
        if(SimultaneousNew[k]!=0,
            r=r+1
        )
    );
    SimultaneousCur=vector(r);
    for(k=1,r,
        SimultaneousCur[k]=SimultaneousNew[k]
    );
    SimLenCur=length(SimultaneousCur);
    SimLenComp=length(Evalues[j,2]);
    SimultaneousComp=vector(SimLenComp);
    SimLenNew=SimLenCur*SimLenComp;
    SimultaneousNew=vector(SimLenNew);
    for(i=1,SimLenComp,
        SimultaneousComp[i]=Evectors[j,i+n]
    );
    r=1;
    for(k=1,SimLenCur,
        for(l=1,SimLenComp,
            if(matintersect(SimultaneousCur[k],SimultaneousComp[l])!=[;],
                SimultaneousNew[r]=matintersect(SimultaneousCur[k],SimultaneousComp[l]);
```

```
                r=r+1
            )
        )
    )
);
r=0;
for(k=1,SimLenNew,
    if(SimultaneousNew[k]!=0,
        r=r+1
)
);
SimultaneousCur=vector(r);
for(k=1,r,
    SimultaneousCur[k]=SimultaneousNew[k]
);
eigenvaluematrix=matrix(maxim,2);
DiffPairs=listcreate();
for(k=1,r,
    listput(DiffPairs,listcreate());
    forprime(i=1,maxim,
        for(j=1,length(Evalues[i,1]),
            if(matintersect(Evectors[i,j],SimultaneousCur[k])!=[;],
                eigenvaluematrix[i,1]=Evalues[i,1][j]
            )
        );
        for(j=1,length(Evalues[i,2]),
            if(matintersect(Evectors[i,j+n],SimultaneousCur[k])!=[;],
                eigenvaluematrix[i,2]=Evalues[i,2][j]
            )
        )
    );
    numberofprimes=0;
    forprime(i=1,50,
```

```
            if(i!=p&&i!=N,
                numberofprimes=numberofprimes+1
            )
        );
        eigenpairs=matrix(numberofprimes,2);
        t=1;
        forprime(i=2,maxim,
            if(i!=p&&i!=N,
                eigenpairs[t,1]=lift(eigenvaluematrix[i,1]);
                eigenpairs[t,2]=lift(eigenvaluematrix[i,2]);
                t=t+1
            )
        );
        ActualEigPair=Str(" listput(ActualPairs,",eigenpairs,");");
        write(OpenFile,ActualEigPair);
        \\print(eigenpairs)
    );
    write(OpenFile,"}");
)\\end the trap
```

Lastly, the following code puts everything together, including the tables following this code.

```
SumCharPos(a,b,c,i)= \\i represents the index corresponding to the triple it comes from
{
    listput(AllEigenPairs,listcreate());
    listput(CharSum[i],Str("w^",a,"\\epsilon+w^",b,"+w^",c));
    listput(CharSum[i],Str("w^",a,"+w^",b,"\\epsilon+w^",c));
    if(a!=c,
        listput(CharSum[i],Str("w^",a,"+w^",b,"+w^",c,"\\epsilon"))
    );
    TotSums=length(ModZeroTriple[i]);
    TotChar=length(CharSum[i]);
    CurWeight="";
    CurSums="";
```

```
    for(j=1,TotSums,
        CurWeight=Str(CurWeight," ",ModZeroTriple[i][j])
    );
    for(j=1,TotChar,
        CurSums=Str(CurSums," ",CharSum[i][j])
    );
    CurWeight=Str(CurWeight," corresponds to the character sum(s): ",CurSums);
    print(CurWeight);
    listput(Chars[i],[a,b,c;1,0,0]);
    listput(Chars[i],[a,b,c;0,1,0]);
    if(a!=c,
    listput(Chars[i],[a,b,c;0,0,1])
    );
    for(j=1,TotChar,
        print(CharSum[i][j]," corresponds to the E.Values: ");
        GetEigVal(Chars[i][j],N,p);
        listput(AllEigenPairs[i],listcreate());
        AllEigenPairs[i][j]=EigPair
    );
    e=1
}
SumCharNeg(a,b,c,i)=
{
    listput(AllEigenPairs,listcreate());
    w=-1;
    if(w^a==w^b&&w^a==w^c,
        listput(CharSum[i],Str("w^",a,"+w^",b,"\\epsilon+w^",c));
        listput(Chars[i],[a,b,c;0,1,0]),
        if(w^a==w^b,
            listput(CharSum[i],Str("w^",a,"\\epsilon+w^",b,"+w^",c));
            listput(Chars[i],[a,b,c;1,0,0]),
            listput(CharSum[i],Str("w^",a,"+w^",b,"+w^",c,"\\epsilon"));
            listput(Chars[i],[a,b,c;0,0,1])
```

```
        )
    );
    TotSums=length(ModZeroTriple[i]);
    CurWeight="";
    for(j=1,TotSums,
        CurWeight=Str(CurWeight," ",ModZeroTriple[i][j])
    );
    CurWeight=Str(CurWeight," corresponds to the character sum: ",CharSum[i][1]);
    print(CurWeight);
    print(CharSum[i][1]," corresponds to the E. Values: ");
    GetEigVal(Chars[i][1],N,p);
    listput(AllEigenPairs[i],listcreate());
    AllEigenPairs[i][1]=EigPair;
    e=-1
}
GetEigVal(M,N,p)=
{
    CharMat=M;
    a=CharMat[1,1];
    b=CharMat[1,2];
    c=CharMat[1,3];
    r=CharMat[2,1];
    s=CharMat[2,2];
    t=CharMat[2,3];
    Nstar=N*(-1)^((N-1)/2);
    if(N!=p,
        EigPair=matrix(13,3),
        EigPair=matrix(14,3)
    );
    count=1;
    forprime(l=1,50,
        if(l!=p&&l!=N,
            w=Mod(l,p);
```

```
            e=kronecker(Nstar,l);
            EigPair[count,1]=l;
            EigPair[count,2]=lift(Mod(w^a*e^r+w^b*e^s+w^c*e^t,p));
            EigPair[count,3]=lift(Mod(w^(a+b)*e^(r+s)+w^(a+c)*e^(r+t)+
                                                  w^(b+c)*e^(s+t),p)*l^(-1));
                count=count+1
        )
    );
    listput(SortedEigPairs,EigPair);
    print(EigPair)
}
CopyTable(N,p)=
{
    File=Str("P",p,".txt");
    NumPrimes=0;
    forprime(i=1,50,
        if(i!=N&&i!=p,
            TopRow=Str(TopRow,i,"& ");
            NumPrimes=NumPrimes+1;
        )
    );
    if(NumPrimes==13,
        Headline=Str("\\begin{longtable}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}"),
        Headline=Str("\\begin{longtable}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}")
    );
    TopRow=concat(TopRow,"Weights& Triple& Galois Rep\\cr");
    write(File,"");
    write(File,"For $p=",p,"$ and $N=",N,"$");
    write(File,"");
    write(File,Headline);
    write(File,"\\hline");
    write(File,TopRow);
    write(File,"\\hline");
```

```
read("AllWeightsLevels.txt");
ListedPairs=listcreate();
for(i=1,length(AllTriples),
    for(j=1,length(AllTripleWeights[i]),
        a=AllTripleWeights[i][j][1];
        b=AllTripleWeights[i][j][2];
        c=AllTripleWeights[i][j][3];
        ReadFile=Str("EP_p",p,"l",N,"W",a,"_",b,"_",c,".txt");
        read(ReadFile);
        for(k=1,length(ActualPairs),
            listput(ListedPairs,ActualPairs[k])
        );
        write("DeleteFile.txt","rm ",ReadFile);
    )
);
listsort(ListedPairs,1);
for(i=1,length(SimplifiedSortedEigPairs),
    Break=0;
    j=1;
    while(j<=length(ListedPairs),
        if(SimplifiedSortedEigPairs[i]==ListedPairs[j],
            listpop(ListedPairs,j);
            Break=1;
        );
        j=j+1;
        if(Break,break)
    );
);
for(i=1,NumEigPairs,
    FirstTriple=1;
    FirstWeight=1;
    for(j=1,NumWeights,
        for(k=1,length(AllEigenPairs[j]),
```

```
FirstWeight=1;
if(SortedEigPairs[i]==AllEigenPairs[j][k],
    for(l=1,length(ModZeroTriple[j]),
        ThisWeight="";
        if(FirstWeight,
            ThisChar=Str("$",CharSum[j][k],"$");
            FirstWeight=0,
            ThisChar=""
        );
        for(m=1,length(GroupedWeights[j][l]),
            a=GroupedWeights[j][l][m][1];
            b=GroupedWeights[j][l][m][2];
            c=GroupedWeights[j][l][m][3];
            OpenFile=Str("EP_p",p,"l",N,"W",a,"_",b,"_",c,".txt");
            read(OpenFile);
            PairSize=length(ActualPairs);
            NotAWeight=1;
            for(t=1,PairSize,
                if(ActualPairs[t]==SimplifiedSortedEigPairs[i],
                    NotAWeight=0
                )
            );
            if(!NotAWeight,
                ThisWeight=Str(ThisWeight,"F",GroupedWeights[j][l][m]," ")
            )
        );
        ThisTriple=ModZeroTriple[j][l];
        ThisLine="";
        for(m=1,NumPrimes,
            if(FirstTriple,
                ThisLine=Str(ThisLine,SortedEigPairs[i][m,2],",",
                                        SortedEigPairs[i][m,3],"&"),
                ThisLine=Str(ThisLine," &")
```

```
                    )
                );
                if(FirstTriple,
                    write(File,"\\hline")
                );
                FirstTriple=0;
                ThisLine=Str(ThisLine,ThisWeight,"& ",ThisTriple,"& ",ThisChar,
                                                "\\cr");
                write(File,ThisLine);
            )
        )
    )
  )
);
for(i=1,length(ListedPairs),
    FirstSpecial=1;
    FirstTriple=1;
    FirstWeight=1;
    GotOne=0;
    for(j=1,length(AllTriples),
        FirstWeight=1;
        ThisWeight="";
        if(FirstWeight,
            ThisChar=Str("?");
            FirstWeight=0;
            ThisChar="";
        );
        for(k=1,length(AllTripleWeights[j]),
            a=AllTripleWeights[j][k][1];
            b=AllTripleWeights[j][k][2];
            c=AllTripleWeights[j][k][3];
            ReadFile=Str("EP_p",p,"l",N,"W",a,"_",b,"_",c,".txt");
            read(ReadFile);
```

```
            for(l=1,length(ActualPairs),
                if(ListedPairs[i]==ActualPairs[l],
                    GotOne=1;
                    if(FirstSpecial,
                        write(File,"\\hline");
                        FirstSpecial=0
                    );
                    ThisWeight=Str(ThisWeight,"F",AllTripleWeights[j][k]," ");
                    ThisTriple=AllTriples[j]
                )
            )
        );
        if(GotOne,
            ThisLine="";
            for(m=1,NumPrimes,
                if(FirstTriple,
                    ThisLine=Str(ThisLine,ListedPairs[i][m,1],",",
                                             ListedPairs[i][m,2],"&"),
                    ThisLine=Str(ThisLine," &")
                )
            );
            if(FirstTriple,
                write(File,"\\hline")
            );
            FirstTriple=0;
            ThisLine=Str(ThisLine,ThisWeight,"& ",ThisTriple,"& ",ThisChar,"\\cr");
            write(File,ThisLine);
            GotOne=0
        )
    )
);
write(File,"\\hline");
FirstTriple=1;
```

```
FirstWeight=0;
for(i=1,length(AllTriples),
    ThisWeight="";
    ThisChar="";
    GotOne=0;
    for(j=1,length(AllTripleWeights[i]),
        a=AllTripleWeights[i][j][1];
        b=AllTripleWeights[i][j][2];
        c=AllTripleWeights[i][j][3];
        ReadFile=Str("EP_p",p,"l",N,"W",a,"_",b,"_",c,".txt");
        read(ReadFile);
        if(length(ActualPairs)==0,
            if(FirstTriple,
                write(File,"\\hline");
                FirstTriple=0;
                FirstWeight=1
            );
            GotOne=1;
            ThisWeight=Str(ThisWeight,"F",AllTripleWeights[i][j]," ");
            ThisTriple=AllTriples[j];
        )
    );
    if(GotOne,
        ThisLine="";
        for(m=1,NumPrimes,
            ThisLine=Str(ThisLine," &")
        );
        ThisLine=Str(ThisLine,ThisWeight,"& ",ThisTriple,"& ",ThisChar,"\\cr");
        write(File,ThisLine)
    )
);
if(FirstWeight,
    write(File,"\\hline")
```

```
    );
    write(File,"");
    write(File,"\\end{longtable}");
    write(File,"");
    write(File,"\\bigskip")
}
Characters(L,q)=
{
    N=L;
    p=q;
    print("For level N=",N," and for prime p=",p,", the following triples work:");
    w=-1;
    e=(-1)^((N-1)/2);
    Triple=listcreate();
    BadTriple=listcreate();
    t=0;
    if(e==1,
        for(i=0,p-2,
            for(j=0,p-2,
                for(k=0,p-2,
                    if(w^(i)!=w^(j)&&w^i==w^k,
                        t=t+1;
                        listinsert(Triple,[i,j,k],t),
                    \\otherwise
                        if(lift(Mod(i+j+k,p-1))==0,
                            listput(BadTriple,[i,j,k])
                        )
                    )
                )
            )
        )
    );
    if(e==-1,
```

```
    for(i=0,p-2,
        for(j=0,p-2,
            for(k=0,p-2,
                if(!(w^i!=w^j&&w^i==w^k),
                    t=t+1;
                    listinsert(Triple,[i,j,k],t),
                \\otherwise
                    if(lift(Mod(i+j+k,p-1))==0,
                        listput(BadTriple,[i,j,k])
                    )
                )
            )
        )
    )
);
print(Triple);
ModTriples=listcreate();
Weights=listcreate();
BadWeights=listcreate();
NumBadTriples=length(BadTriple);
for(i=1,NumBadTriples,
    maxweight=0;
    a=BadTriple[i][1]-2;
    b=BadTriple[i][2]-1;
    c=BadTriple[i][3];
    TempWeight=[a,b,c];
    listput(BadWeights,listcreate());
    while(b<c,
        b=b+(p-1)
    );
    while(b-c<=p-1,
        while(a-b<=p-1,
            if(a>=b&&b>=c,
```

```
                listput(BadWeights[i],[a,b,c])
            );
            a=a+(p-1)
        );
        b=b+(p-1);
        a=BadTriple[i][1]-2
    )
);
for(i=1,p-1,
    listput(ModTriples,listcreate())
);
for(i=1,t,
    Sum=lift(Mod(Triple[i][1]+Triple[i][2]+Triple[i][3],p-1));
    listput(ModTriples[Sum+1],Triple[i]);
);
NumWeights=length(ModTriples[1]);
for(i=1,NumWeights,
    maxweight=0;
    a=ModTriples[1][i][1]-2;
    b=ModTriples[1][i][2]-1;
    c=ModTriples[1][i][3];
    TempWeight=[a,b,c];
    listput(Weights,listcreate());
    while(b<c,
        b=b+(p-1)
    );
    while(b-c<=p-1,
        while(a-b<=p-1,
            if(a>=b&&b>=c,
            listput(Weights[i],[a,b,c])
            );
            a=a+(p-1)
        );
```

```
        b=b+(p-1);
        a=ModTriples[1][i][1]-2
    )
);
L=listcreate();
M=listcreate();
ModZeroTriple=listcreate();
GroupedWeights=listcreate();
if(e==1,
    for(i=1,NumWeights,
        Repeat=0;
        if(length(ModZeroTriple)==0,
            listput(ModZeroTriple,listcreate());
            listput(GroupedWeights,listcreate());
            listput(ModZeroTriple[1],ModTriples[1][1]);
            listput(GroupedWeights[1],Weights[1]),
        \\otherwise
            for(j=1,length(ModZeroTriple),
                for(z=1,3,
                    listput(L,ModTriples[1][i][z]);
                    listput(M,ModZeroTriple[j][1][z])
                );
                listsort(L);
                listsort(M);
                if(L==M,
                    listput(ModZeroTriple[j],ModTriples[1][i]);
                    listput(GroupedWeights[j],Weights[i]);
                    Repeat=1;
                );
                listkill(L);
                listkill(M)
            );
            if(!Repeat,
```

```
                listput(ModZeroTriple,listcreate());
                listput(GroupedWeights,listcreate());
                listput(ModZeroTriple[length(ModZeroTriple)],ModTriples[1][i]);
                listput(GroupedWeights[length(GroupedWeights)],Weights[i]);
            )
        )
    )
);
if(e==-1,
    for(i=1,NumWeights,
        Repeat=0;
        if(length(ModZeroTriple)==0,
            listput(ModZeroTriple,listcreate());
            listput(GroupedWeights,listcreate());
            listput(ModZeroTriple[1],ModTriples[1][1]);
            listput(GroupedWeights[1],Weights[1]),
        \\otherwise
            for(j=1,length(ModZeroTriple),
                if(ModZeroTriple[j][1][2]==ModTriples[1][i][2],
                    listput(L,ModTriples[1][i][1]);
                    listput(L,ModTriples[1][i][3]);
                    listput(M,ModZeroTriple[j][1][1]);
                    listput(M,ModZeroTriple[j][1][3]);
                    listsort(L);
                    listsort(M);
                    if(L==M,
                        listput(ModZeroTriple[j],ModTriples[1][i]);
                        listput(GroupedWeights[j],Weights[i]);
                        Repeat=1
                    );
                    listkill(L);
                    listkill(M)
                )
```

```
            );
            if(!Repeat,
                listput(ModZeroTriple,listcreate());
                listput(GroupedWeights,listcreate());
                listput(ModZeroTriple[length(ModZeroTriple)],ModTriples[1][i]);
                listput(GroupedWeights[length(GroupedWeights)],Weights[i])
            )
        )
    )
);
for(i=1,NumWeights,
    FollowingWeights="";
    for(j=1,length(Weights[i]),
        FollowingWeights=Str(FollowingWeights,"F",Weights[i][j],", ");
    );
);
NumWeights=length(ModZeroTriple);
Chars=listcreate();
CharSum=listcreate();
for(i=1,NumWeights,
    listput(Chars,listcreate());
    listput(CharSum,listcreate())
);
AllEigenPairs=listcreate();
SortedEigPairs=listcreate();
SimplifiedSortedEigPairs=listcreate();
for(i=1,NumWeights,
    a=ModZeroTriple[i][1][1];
    b=ModZeroTriple[i][1][2];
    c=ModZeroTriple[i][1][3];
    if(e==1,
        SumCharPos(a,b,c,i),
    \\or
```

```
            SumCharNeg(a,b,c,i)
        )
    );
    listsort(SortedEigPairs,1);
    NumEigPairs=length(SortedEigPairs);
    TopRow="";
    NumPrimes=0;
    forprime(i=1,50,
        if(i!=q&&i!=N,
            NumPrimes=NumPrimes+1
        )
    );
    for(i=1,NumEigPairs,
        tempPair=matrix(NumPrimes,2);
        t=1;
        forprime(j=1,50,
            if(j!=q&&j!=N,
                tempPair[t,1]=SortedEigPairs[i][t,2];
                tempPair[t,2]=SortedEigPairs[i][t,3];
                t=t+1
            )
        );
        listput(SimplifiedSortedEigPairs,tempPair)
    );
    CopyTable(N,p);
    write("DeleteFile.txt","rm ListOfLevels.txt");
    write("DeleteFile.txt","rm CurWeight.txt");
    write("DeleteFile.txt","rm Curabc.txt");
    write("DeleteFile.txt","rm AllWeightsLevels.txt");
}
```

The following code calculates the order of Frobenius as described in chapter 6.

```
f(al1,al2,l,N,a,b,c)=
{
```

```
    minpol=1-al1*kronecker(l,N)^0*Mod(1,5)*x+l*al2*Mod(1,5)*x^2-
                             kronecker(l,N)^1*l^(3+a+b+c)*Mod(1,5)*x^3;
    n=0;
    Done=1;
    while(Done,
        n=n+1;
        g=gcd((x^n-1)^3,minpol);
        if(g==minpol,
            Done=0
        )
    );
    print("Order of Frobenius= ",n)
}
```

Sample code to find $a(\ell, 1)$ and $a(\ell, 2)$ for the symmetric squares representations.

```
F(a)=
{
    j=1;
    forprime(i=2,47,
        if(i!=5&&i!=17,
            print([i,a[j]%5,((a[j]-i^1)*i^1)%5,((a[j]-i^1)*i^1*kronecker(i,17))%5]);
            j=j+1
        )
    )
}
```

Sample Magma code from [7] for finding modular forms attached to representations.

```
G<eps>:=DirichletGroup(13);
M:=ModularForms(eps,4);
S:=CuspidalSubspace(M);
f:=Newforms(S)[1];
Reductions(f[1],5);
qExpansion(Reductions(f[1],5)[1][1],50)
qExpansion(Reductions(f[1],5)[2][1],50)
```

Appendix B. Tables for $p = 5$ and small N

For $p = 5$, $N = 3$, $\epsilon = \epsilon_3$

2	7	11	13	17	19	23	29	31	37	41	43	47	Weights	Triple	Galois Rep
0,4	1,3	1,4	1,2	0,4	4,1	2,0	1,1	3,3	1,3	1,4	1,2	0,4	F[6, 4, 3] F[5, 4, 0]	[0, 1, 3] [3, 1, 0]	$\omega^0 \oplus \omega^1 \oplus \omega^3\epsilon$
1,2	3,4	1,4	3,1	1,2	3,2	1,3	1,1	3,3	3,4	1,4	3,1	1,2	F[6, 3, 0]	[0, 0, 0]	$\omega^0 \oplus \omega^0\epsilon \oplus \omega^0$
1,2	4,2	1,4	4,3	1,2	3,2	1,3	1,1	3,3	4,2	1,4	4,3	1,2	F[6, 5, 2] F[4, 1, 0]	[0, 2, 2] [2, 2, 0]	$\omega^0 \oplus \omega^2\epsilon \oplus \omega^2$
2,0	1,3	1,4	1,2	2,0	4,1	0,1	1,1	3,3	1,3	1,4	1,2	2,0	F[6, 2, 1] F[3, 2, 0]	[0, 3, 1] [1, 3, 0]	$\omega^0 \oplus \omega^3 \oplus \omega^1\epsilon$
2,4	4,2	1,4	4,3	2,4	3,2	2,1	1,1	3,3	4,2	1,4	4,3	2,4	F[4, 3, 2]	[2, 0, 2]	$\omega^2 \oplus \omega^0\epsilon \oplus \omega^2$
4,3	0,4	1,4	3,0	4,3	4,1	4,2	1,1	3,3	0,4	1,4	3,0	4,3	F[8, 6, 3] F[9, 6, 2]	[2, 3, 3] [3, 3, 2]	$\omega^2 \oplus \omega^3 \oplus \omega^3\epsilon$
4,3	3,0	1,4	0,1	4,3	4,1	4,2	1,1	3,3	3,0	1,4	0,1	4,3	F[7, 4, 2] F[8, 4, 1]	[1, 1, 2] [2, 1, 1]	$\omega^1\epsilon \oplus \omega^1 \oplus \omega^2$

For $p = 5$, $N = 7$, $\epsilon = \epsilon_7$

2	3	11	13	17	19	23	29	31	37	41	43	47	Weights	Triple	Galois Rep
0,4	4,2	3,3	4,2	4,3	1,1	3,0	4,1	1,4	0,4	1,4	3,0	4,3	F[8, 6, 3] F[9, 6, 2]	[2, 3, 3] [3, 3, 2]	$\omega^2 \oplus \omega^3 \oplus \omega^3\epsilon$
1,3	0,1	3,3	0,1	2,0	1,1	1,2	4,1	1,4	1,3	1,4	1,2	2,0	F[6, 2, 1] F[3, 2, 0]	[0, 3, 1] [1, 3, 0]	$\omega^0 \oplus \omega^3 \oplus \omega^1\epsilon$
1,3	2,0	3,3	2,0	0,4	1,1	1,2	4,1	1,4	1,3	1,4	1,2	0,4	F[6, 4, 3] F[5, 4, 0]	[0, 1, 3] [3, 1, 0]	$\omega^0 \oplus \omega^1 \oplus \omega^3\epsilon$
3,4	1,3	3,3	1,3	1,2	1,1	3,1	3,2	1,4	3,4	1,4	3,1	1,2	F[6, 3, 0]	[0, 0, 0]	$\omega^0 \oplus \omega^0\epsilon \oplus \omega^0$
3,0	4,2	3,3	4,2	4,3	1,1	0,1	4,1	1,4	3,0	1,4	0,1	4,3	F[7, 4, 2] F[8, 4, 1]	[1, 1, 2] [2, 1, 1]	$\omega^1\epsilon \oplus \omega^1 \oplus \omega^2$
4,2	1,3	3,3	1,3	1,2	1,1	4,3	3,2	1,4	4,2	1,4	4,3	1,2	F[6, 5, 2] F[4, 1, 0]	[0, 2, 2] [2, 2, 0]	$\omega^0 \oplus \omega^2\epsilon \oplus \omega^2$
4,2	2,1	3,3	2,1	2,4	1,1	4,3	3,2	1,4	4,2	1,4	4,3	2,4	F[4, 3, 2]	[2, 0, 2]	$\omega^2 \oplus \omega^0\epsilon \oplus \omega^2$
0,0	1,3	0,0	1,3	1,2	1,1	0,0	0,0	1,4	0,0	1,4	0,0	1,2	F[1, 0, 0] F[5, 4, 0] F[4, 1, 0] F[2, 2, 1] F[6, 2, 1] F[6, 5, 2]	[3, 1, 0] [2, 2, 0] [0, 3, 1] [0, 2, 2]	$\sigma \oplus \omega^0$
1,1	4,2	0,0	4,2	4,3	1,1	2,2	2,3	1,4	1,1	1,4	2,2	4,3	F[4, 1, 0] F[4, 4, 1] F[8, 4, 1] F[4, 3, 2] F[7, 4, 2]	[2, 2, 0] [2, 1, 1] [2, 0, 2] [1, 1, 2]	$(\sigma \otimes \omega^3) \oplus \omega^2$
2,1	1,3	0,0	1,3	1,2	1,1	2,4	0,0	1,4	2,1	1,4	2,4	1,2	F[6, 3, 0] F[3, 2, 0] F[6, 4, 3]	[0, 0, 0] [1, 3, 0] [0, 1, 3]	$(\sigma \otimes \omega^2) \oplus \omega^0$
2,3	4,2	0,0	4,2	4,3	1,1	1,4	2,3	1,4	2,3	1,4	1,4	4,3	F[4, 3, 2] F[6, 5, 2] F[5, 2, 2] F[9, 6, 2] F[8, 6, 3]	[2, 0, 2] [0, 2, 2] [3, 3, 2] [2, 3, 3]	$(\sigma \otimes \omega^1) \oplus \omega^2$
1,3	3,4	0,0	3,4	4,3	2,2	1,2	3,2	0,0	4,2	0,0	1,2	3,1	F[11, 7, 3]	[1, 0, 3]	$\rho \otimes \epsilon$

σ is the irreducible two-dimensional S_3 representation cutting out the splitting field of $f(x) = x^3 - x^2 + 2x - 3$.
ρ is the irreducible three-dimensional S_5 representation cutting out the splitting field of $f(x) = x^5 - 100x^2 - 100x - 55$.

For $p = 5$, $N = 11$, $\epsilon = \epsilon_{11}$

2	3	7	13	17	19	23	29	31	37	41	43	47	Weights	Triple	Galois Rep
0,4	1,2	0,4	2,0	0,4	1,1	1,2	1,1	3,3	1,3	1,4	2,0	1,3	F[6, 4, 3] F[5, 4, 0]	[0, 1, 3] [3, 1, 0]	$\omega^0 \oplus \omega^1 \oplus \omega^3\epsilon$
1,2	3,1	1,2	1,3	1,2	1,1	3,1	1,1	3,3	3,4	1,4	1,3	3,4	F[6, 3, 0]	[0, 0, 0]	$\omega^0 \oplus \omega^0\epsilon \oplus \omega^0$
1,2	4,3	1,2	1,3	1,2	1,1	4,3	1,1	3,3	4,2	1,4	1,3	4,2	F[6, 5, 2] F[4, 1, 0]	[0, 2, 2] [2, 2, 0]	$\omega^0 \oplus \omega^2\epsilon \oplus \omega^2$
2,0	1,2	2,0	0,1	2,0	1,1	1,2	1,1	3,3	1,3	1,4	0,1	1,3	F[6, 2, 1] F[3, 2, 0]	[0, 3, 1] [1, 3, 0]	$\omega^0 \oplus \omega^3 \oplus \omega^1\epsilon$
2,4	4,3	2,4	2,1	2,4	1,1	4,3	1,1	3,3	4,2	1,4	2,1	4,2	F[4, 3, 2]	[2, 0, 2]	$\omega^2 \oplus \omega^0\epsilon \oplus \omega^2$
4,3	0,1	4,3	4,2	4,3	1,1	0,1	1,1	3,3	3,0	1,4	4,2	3,0	F[7, 4, 2] F[4, 4, 1] F[8, 4, 1]	[1, 1, 2] [2, 1, 1]	$\omega^1\epsilon \oplus \omega^1 \oplus \omega^2$
4,3	3,0	4,3	4,2	4,3	1,1	3,0	1,1	3,3	0,4	1,4	4,2	0,4	F[8, 6, 3] F[5, 2, 2] F[9, 6, 2]	[2, 3, 3] [3, 3, 2]	$\omega^2 \oplus \omega^3 \oplus \omega^3\epsilon$
1,2	1,2	1,2	1,3	1,2	1,1	1,2	1,1	4,4	1,3	1,4	1,3	1,3	F[1, 0, 0] F[5, 4, 0] F[3, 2, 0] F[2, 2, 1] F[6, 2, 1] F[6, 4, 3]	[3, 1, 0] [1, 3, 0] [0, 3, 1] [0, 1, 3]	$(\sigma \otimes \omega) \oplus \omega^0$
4,3	4,3	4,3	4,2	4,3	1,1	4,3	1,1	4,4	4,2	1,4	4,2	4,2	F[4, 1, 0] F[4, 3, 2] F[6, 5, 2]	[2, 2, 0] [2, 0, 2] [0, 2, 2]	$\sigma \oplus \omega^2$
2,4	2,4	0,0	1,3	2,4	4,4	1,2	1,1	0,0	4,2	3,2	2,1	2,1	F[11, 7, 3]	[1, 0, 3]	$\rho \otimes \epsilon$
1,3	3,4	2,0	0,3	3,0	1,2	2,1	2,2	4,1	0,4	1,0	1,0	3,1	F[8, 6, 3]	[2, 3, 3]	Huge
4,2	2,1	0,4	1,0	0,1	2,1	3,4	2,2	1,4	3,0	0,4	0,3	2,4	F[7, 4, 2]	[1, 1, 2]	Huge

ρ is the irreducible three-dimensional S_5 representation cutting out the splitting field of $f(x) = x^5 - 25x^2 + 55$.
σ is the irreducible two-dimensional D_8 representation cutting out the splitting field of $f(x) = x^4 - x^3 + 2x - 2$.

For $p = 5$, $N = 13$, $\epsilon = \epsilon_{13}$

2	3	7	11	17	19	23	29	31	37	41	43	47	Weights	Triple	Galois Rep
0,4	0,1	0,4	1,4	3,0	2,2	0,1	4,1	1,4	0,4	1,4	0,1	0,4	F[3, 1, 1] F[7, 5, 1]	[1, 2, 1]	$\omega^1 \oplus \omega^2\epsilon \oplus \omega^1$
0,4	1,2	0,4	1,4	1,3	1,1	1,2	4,1	1,4	0,4	1,4	1,2	0,4	F[7, 3, 3] F[11, 7, 3] F[5, 3, 1]	[1, 0, 3] [3, 0, 1]	$\omega^1 \oplus \omega^0 \oplus \omega^3\epsilon$
2,0	1,2	2,0	1,4	1,3	1,1	1,2	4,1	1,4	2,0	1,4	1,2	2,0	F[7, 7, 3] F[11, 7, 3] F[5, 3, 1]	[1, 0, 3] [3, 0, 1]	$\omega^1\epsilon \oplus \omega^0 \oplus \omega^3$
2,0	3,0	2,0	1,4	0,4	2,2	3,0	4,1	1,4	2,0	1,4	3,0	2,0	F[5, 5, 3] F[9, 5, 3]	[3, 2, 3]	$\omega^3 \oplus \omega^2\epsilon \oplus \omega^3$
4,3	0,1	4,3	1,4	3,0	1,1	0,1	4,1	1,4	4,3	1,4	0,1	4,3	F[3, 1, 1] F[7, 5, 1]	[1, 2, 1]	$\omega^1\epsilon \oplus \omega^2 \oplus \omega^1$
4,3	1,2	4,3	1,4	1,3	2,2	1,2	4,1	1,4	4,3	1,4	1,2	4,3	F[3, 3, 3] F[7, 3, 3] F[7, 7, 3] F[11, 7, 3] F[5, 3, 1]	[1, 0, 3] [3, 0, 1]	$\omega^1 \oplus \omega^0\epsilon \oplus \omega^3$
4,3	3,0	4,3	1,4	0,4	1,1	3,0	4,1	1,4	4,3	1,4	3,0	4,3	F[5, 5, 3] F[9, 5, 3]	[3, 2, 3]	$\omega^3\epsilon \oplus \omega^2 \oplus \omega^3$
0,2	4,4	3,4	2,0	3,1	2,2	3,0	3,0	4,2	1,1	2,0	1,2	4,3	F[5, 3, 1] F[6, 4, 3]	[3, 0, 1] [0, 1, 3]	$\sigma \oplus \omega^3$
1,1	4,4	3,4	0,3	3,1	1,1	3,0	3,0	3,1	0,2	0,3	1,2	2,0	F[5, 3, 1] F[6, 4, 3]	[3, 0, 1] [0, 1, 3]	$\sigma\epsilon \oplus \omega^3$
1,1	0,3	3,4	2,0	3,1	2,2	1,2	3,0	4,2	0,2	2,0	3,0	2,0	F[5, 5, 3] F[9, 5, 3] F[8, 6, 3]	[3, 2, 3] [2, 3, 3]	$(\sigma \otimes \omega^2) \oplus \omega^3$
0,2	0,3	3,4	0,3	3,1	1,1	1,2	3,0	3,1	1,1	0,3	3,0	4,3	F[5, 5, 3] F[9, 5, 3] F[8, 6, 3]	[3, 2, 3] [2, 3, 3]	$(\sigma\epsilon \otimes \omega^2) \oplus \omega^3$
3,2	2,3	2,1	2,0	2,4	1,1	0,1	0,2	4,2	1,0	2,0	1,2	0,4	F[3, 2, 0] F[5, 3, 1]	[1, 3, 0] [3, 0, 1]	$(\sigma \otimes \omega^3) \oplus \omega$
1,0	2,3	2,1	0,3	2,4	2,2	0,1	0,2	3,1	3,2	0,3	1,2	4,3	F[3, 2, 0] F[5, 3, 1]	[1, 3, 0] [3, 0, 1]	$(\sigma\epsilon \otimes \omega^3) \oplus \omega$
1,0	4,0	2,1	2,0	2,4	1,1	1,2	0,2	4,2	3,2	2,0	0,1	4,3	F[3, 1, 1] F[7, 5, 1] F[7, 4, 2]	[1, 2, 1] [1, 1, 2]	$(\sigma \otimes \omega) \oplus \omega$
3,2	4,0	2,1	0,3	2,4	2,2	1,2	0,2	3,1	1,0	0,3	0,1	0,4	F[3, 1, 1] F[7, 5, 1] F[7, 4, 2]	[1, 2, 1] [1, 1, 2]	$(\sigma\epsilon \otimes \omega) \oplus \omega$
2,4	0,0	0,0	2,3	0,0	0,0	0,0	3,2	2,3	1,2	0,0	3,1	2,4	F[5, 3, 1] F[4, 3, 2]	[3, 0, 1] [2, 0, 2]	Extra $\rho \otimes \epsilon$
3,1	2,4	1,2	2,3	4,2	2,2	1,2	3,2	0,0	3,1	2,3	1,2	4,3	F[5, 3, 1]	[3, 0, 1]	$Sym^2(\sigma) \otimes \omega^3$

σ is the irreducible two-dimensional representation attached to the modular form found in the space of cuspidal newforms of weight 4, level 13, and quadratic nebentype ϵ_{13}.

$$f(x) = q + q^2 + 4q^3 + 4q^4 + 2q^5 + 4q^6 + 2q^8 + 4q^9 + 2q^{10} + q^{11} + q^{12} + 3q^{13} + \cdots \pmod 5$$

ρ is the irreducible three-dimensional A_5 representation cutting out the splitting field of $f(x) = x^5 + 5x^3 - 10x^2 - 45$

For $p = 5$, $N = 17$, $\epsilon = \epsilon_{17}$

2	3	7	11	13	19	23	29	31	37	41	43	47	Weights	Triple	Galois Rep
0,4	0,1	2,0	1,4	3,0	4,1	0,1	2,2	1,4	2,0	1,4	3,0	0,4	F[5, 5, 3] F[9, 5, 3]	[3, 2, 3]	$\omega^3 \oplus \omega^2\epsilon \oplus \omega^3$
0,4	4,2	4,3	1,4	3,0	4,1	4,2	1,1	1,4	4,3	1,4	3,0	0,4	F[5, 5, 3] F[9, 5, 3]	[3, 2, 3]	$\omega^3\epsilon \oplus \omega^2 \oplus \omega^3$
1,3	0,1	2,0	1,4	1,2	4,1	0,1	1,1	1,4	2,0	1,4	1,2	1,3	F[7, 7, 3] F[11, 7, 3] F[5, 3, 1]	[1, 0, 3] [3, 0, 1]	$\omega^1\epsilon \oplus \omega^0 \oplus \omega^3$
1,3	2,0	0,4	1,4	1,2	4,1	2,0	1,1	1,4	0,4	1,4	1,2	1,3	F[7, 3, 3] F[11, 7, 3] F[5, 3, 1]	[1, 0, 3] [3, 0, 1]	$\omega^1 \oplus \omega^0 \oplus \omega^3\epsilon$
1,3	4,2	4,3	1,4	1,2	4,1	4,2	2,2	1,4	4,3	1,4	1,2	1,3	F[3, 3, 3] F[7, 3, 3] F[7, 7, 3] F[11, 7, 3] F[5, 3, 1]	[1, 0, 3] [3, 0, 1]	$\omega^1 \oplus \omega^0\epsilon \oplus \omega^3$
3,0	2,0	0,4	1,4	0,1	4,1	2,0	2,2	1,4	0,4	1,4	0,1	3,0	F[3, 1, 1] F[7, 5, 1]	[1, 2, 1]	$\omega^1 \oplus \omega^2\epsilon \oplus \omega^1$
3,0	4,2	4,3	1,4	0,1	4,1	4,2	1,1	1,4	4,3	1,4	0,1	3,0	F[3, 1, 1] F[7, 5, 1]	[1, 2, 1]	$\omega^1\epsilon \oplus \omega^2 \oplus \omega^1$
0,4	0,1	3,4	4,2	2,1	2,4	3,3	1,1	0,3	3,4	0,3	3,0	0,4	F[5, 4, 0] F[7, 3, 3] F[11, 7, 3]	[3, 1, 0] [1, 0, 3]	$\sigma \oplus \omega^3$
0,4	4,2	3,4	3,1	2,1	2,4	1,0	2,2	2,0	3,4	2,0	3,0	0,4	F[5, 4, 0] F[7, 3, 3] F[11, 7, 3]	[3, 1, 0] [1, 0, 3]	$\sigma\epsilon \oplus \omega^3$
1,3	4,2	3,4	4,2	2,1	2,4	1,0	1,1	0,3	3,4	0,3	1,2	1,3	F[9, 6, 2] F[9, 5, 3]	[3, 3, 2] [3, 2, 3]	$(\sigma \otimes \omega^2) \oplus \omega^3$
1,3	0,1	3,4	3,1	2,1	2,4	3,3	2,2	2,0	3,4	2,0	1,2	1,3	F[9, 6, 2] F[9, 5, 3]	[3, 3, 2] [3, 2, 3]	$(\sigma\epsilon \otimes \omega^2) \oplus \omega^3$
1,3	2,0	2,1	4,2	3,4	1,3	1,4	2,2	0,3	2,1	0,3	1,2	1,3	F[8, 4, 1] F[7, 5, 1]	[2, 1, 1] [1, 2, 1]	$(\sigma \otimes \omega) \oplus \omega$
1,3	4,2	2,1	3,1	3,4	1,3	0,3	1,1	2,0	2,1	2,0	1,2	1,3	F[8, 4, 1] F[7, 5, 1]	[2, 1, 1] [1, 2, 1]	$(\sigma\epsilon \otimes \omega) \oplus \omega$
3,0	4,2	2,1	4,2	3,4	1,3	0,3	2,2	0,3	2,1	0,3	0,1	3,0	F[6, 2, 1] F[7, 7, 3] F[11, 7, 3]	[0, 3, 1] [1, 0, 3]	$(\sigma \otimes \omega^3) \oplus \omega$
3,0	2,0	2,1	3,1	3,4	1,3	1,4	1,1	2,0	2,1	2,0	0,1	3,0	F[6, 2, 1] F[7, 7, 3] F[11, 7, 3]	[0, 3, 1] [1, 0, 3]	$(\sigma\epsilon \otimes \omega^3) \oplus \omega$
3,4	3,4	4,3	4,1	4,3	3,2	4,2	4,4	3,2	3,1	3,2	0,0	0,0	F[5, 3, 1] F[4, 3, 2]	[3, 0, 1] [2, 0, 2]	Extra $\rho \otimes \epsilon$
1,3	4,2	1,2	0,0	4,3	0,0	3,4	2,2	2,3	1,2	2,3	1,2	1,3	F[11, 7, 3]	[1, 0, 3]	$Sym^2(\sigma) \otimes \omega^3$

σ is the irreducible two-dimensional representation attached to the modular form found in the space of cuspidal newforms of weight 6, level 13, and quadratic nebentype ϵ.

$$f(x) = q + 2q^2 + 3q^3 + 2q^4 + 3q^5 + q^6 + 2q^9 + q^{10} + 3q^{11} + q^{12} + 4q^{15} + \cdots \pmod 5$$

ρ is the irreducible three-dimensional S_5 representation cutting out the splitting field of $f(x) = x^5 - 5x^2 + 5$.

For $p = 5$, $N = 19$, $\epsilon = \epsilon_{19}$

2	3	7	11	13	17	23	29	31	37	41	43	47	Weights	Triple	Galois Rep
0,4	2,0	1,3	3,3	2,0	1,3	1,2	1,1	1,4	0,4	1,4	1,2	1,3	F[6, 4, 3] F[5, 4, 0]	[0, 1, 3] [3, 1, 0]	$\omega^0 \oplus \omega^1 \oplus \omega^3\epsilon$
1,2	1,3	3,4	3,3	1,3	3,4	3,1	1,1	1,4	1,2	1,4	3,1	3,4	F[6, 3, 0]	[0, 0, 0]	$\omega^0 \oplus \omega^0\epsilon \oplus \omega^0$
1,2	1,3	4,2	3,3	1,3	4,2	4,3	1,1	1,4	1,2	1,4	4,3	4,2	F[6, 5, 2] F[4, 1, 0]	[0, 2, 2] [2, 2, 0]	$\omega^0 \oplus \omega^2\epsilon \oplus \omega^2$
2,0	0,1	1,3	3,3	0,1	1,3	1,2	1,1	1,4	2,0	1,4	1,2	1,3	F[6, 2, 1] F[3, 2, 0]	[0, 3, 1] [1, 3, 0]	$\omega^0 \oplus \omega^3 \oplus \omega^1\epsilon$
2,4	2,1	4,2	3,3	2,1	4,2	4,3	1,1	1,4	2,4	1,4	4,3	4,2	F[4, 3, 2]	[2, 0, 2]	$\omega^2 \oplus \omega^0\epsilon \oplus \omega^2$
4,3	4,2	0,4	3,3	4,2	0,4	3,0	1,1	1,4	4,3	1,4	3,0	0,4	F[8, 6, 3] F[5, 2, 2] F[9, 6, 2]	[2, 3, 3] [3, 3, 2]	$\omega^2 \oplus \omega^3 \oplus \omega^3\epsilon$
4,3	4,2	3,0	3,3	4,2	3,0	0,1	1,1	1,4	4,3	1,4	0,1	3,0	F[7, 4, 2] F[4, 4, 1] F[8, 4, 1]	[1, 1, 2] [2, 1, 1]	$\omega^1\epsilon \oplus \omega^1 \oplus \omega^2$
4,3	4,2	4,2	4,4	4,2	4,2	4,3	1,1	1,4	4,3	1,4	4,3	4,2	F[4, 1, 0] F[4, 3, 2] F[6, 5, 2]	[2, 2, 0] [2, 0, 2] [0, 2, 2]	$\sigma \oplus \omega^2$
1,2	1,3	1,3	4,4	1,3	1,3	1,2	1,1	1,4	1,2	1,4	1,2	1,3	F[1, 0, 0] F[5, 4, 0] F[3, 2, 0] F[2, 2, 1] F[6, 2, 1] F[6, 4, 3] F[11, 7, 3]	[3, 1, 0] [1, 3, 0] [0, 3, 1] [0, 1, 3] [1, 0, 3]	$(\sigma \otimes \omega) \oplus \omega^0$ V
0,2	1,1	3,1	2,0	3,2	1,3	1,1	3,4	4,4	4,1	2,1	2,3	0,2	F[6, 2, 1]	[0, 3, 1]	Huge
0,1	2,2	0,4	0,3	3,3	1,2	0,3	4,4	3,1	3,1	2,2	2,2	3,2	F[7, 7, 3] F[11, 7, 3]	[1, 0, 3]	Huge
1,0	2,3	2,4	0,2	4,4	1,3	3,2	4,3	1,1	3,3	4,3	4,4	4,0	F[5, 4, 0]	[3, 1, 0]	Huge
3,0	4,1	3,0	3,0	1,4	4,3	4,0	4,4	4,2	3,1	3,3	1,4	4,4	F[7, 3, 3] F[11, 7, 3]	[1, 0, 3]	Huge

σ is the irreducible two-dimensional representation cutting out the splitting field of $f(x) = x^4 - 2x^3 + 2x^2 - x - 1$

V is a weight that we actually would not expect from the main conjecture. It possibly comes from a violation of strict parity.

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