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Representations Associated to the Group Matrix

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ABSTRACT

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For a finite group $G = \{g_0 = 1, g_1, \ldots, g_{n-1}\}$, we can associate independent variables $x_0, x_1, \ldots, x_{n-1}$ where $x_i = x_{g_i}$. There is a natural action of $\text{Aut}(G)$ on $\mathbb{C}[x_0, \ldots, x_{n-1}]$. Let $C_1, \ldots, C_r$ be the conjugacy classes of G. If $C = \{g_{i_1}, g_{i_2}, \ldots, g_{i_u}\}$ is a conjugacy class, then let $\overline{C} = x_{i_1} + x_{i_2} + \cdots + x_{i_u}$. Let ρ_G be the representation of $\text{Aut}(G)$ on $\mathbb{C}[x_0, \ldots, x_{n-1}]/\langle \overline{C}_1, \ldots, \overline{C}_r \rangle$ and let χ_G be the character afforded by ρ_G. If G is a dihedral group of the form D_{2p}, D_{4p} or D_{2p^2}, with p an odd prime, I show how χ_G splits into irreducible constituents. I also show how the module $\mathbb{C}[x_1, \ldots, x_n]/\langle \overline{C}_1, \ldots, \overline{C}_r \rangle$ decomposes into irreducible submodules. This problem is motivated by results of Humphries [2] relating to random walks on groups and the group determinant.

Keywords: Group matrix, finite group, dihedral group

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Chapter 1. Introduction

Representation theory is the branch of mathematics that involves looking at groups as groups of matrices. Since a lot is known about matrices, this can tell us a lot of information about groups. Unfortunately, matrices include almost too much information. The trace of the representation matrix, or the character, is a lot more compact and still carries a lot of information about the group. For a finite group $G = \{g_0 = 1, g_1, \ldots, g_{n-1}\}$, we can associate independent variables $x_0, x_1, \ldots, x_{n-1}$ to each element of G such that $x_i = x_{g_i}$. The *group matrix* M_G is the $n \times n$ matrix where the (i, j)-entry is defined to be x_k where $g_k = g_i g_j^{-1}$. The *group determinant* θ_G is just the determinant of the group matrix. From early in the history of representation theory, the factorization of the group determinant was sought. Let V be the vector space defined by $V = \mathbb{C}[x_0, \ldots, x_{n-1}]$. Since x_i corresponds to g_i, $\mathrm{Aut}(G)$ acts on V. Let $C_1, \ldots, C_r$ be the conjugacy classes of G and for any $C \subseteq G$ let $\overline{C} = \sum_{g_i \in C} x_i$. Each $\overline{C}_k$ determines a hyperplane (a subspace of dimension $n-1$) $H(C_k) \subset \mathbb{C}^n$. Let U_G be the intersection of all such hyperplanes. Let ρ_G be the representation of $\mathrm{Aut}(G)$ acting on $V/\langle \overline{C}_1, \ldots, \overline{C}_r \rangle$. Let χ_G be the character afforded by ρ_G. This is the same as the representation of $\mathrm{Aut}(G)$ at $0 \in U_G = \cap_{k=1}^{r} H(C_k)$. It has been shown that ρ_G is an irreducible representation if and only if $G = Q_8 \times A$, where A is a 2-group [2]. The question arises: how does ρ decompose when it is not irreducible? In this paper, I look at a few select cases. Let p be an odd prime. If G is a dihedral group of the form D_{2p}, I show that χ_G has $\frac{p+1}{2}$ distinct irreducible constituents, and more specifically $\frac{p-1}{2}$ linear constituents and one irreducible constituent of degree $p-1$. If G is a dihedral group of the form D_{4p}, I show that χ_G has $\frac{p+3}{2}$ distinct irreducible constituents, and more specifically $\frac{p-1}{2}$ linear constituents and two constituents of degree $p-1$. I also show that if $G = D_{2p^2}$, then χ_G breaks into $\frac{p^2+p}{2} + 2$ distinct irreducible constituents. I also prove that the trivial character cannot be a constituent of χ_G.

Chapter 2. Preliminaries

2.1 Representation and Character Theory

Let G be a group.

Definition 2.1. A *representation* of G over $\mathbb{C}$ is a homomorphism ρ from G to $GL(n, \mathbb{C})$, for some n. The *degree* of ρ is the integer n.

Definition 2.2. Let V be an $\mathbb{C}G$-module, and let $\mathscr{B}$ be a basis of V. For each $g \in G$, let $[g]_{\mathscr{B}}$ denote the matrix of the endomorphism $v \to vg$ of V, relative to the basis $\mathscr{B}$.

Theorem 2.3. *[1, pg. 40] (1) If $\rho : G \to GL(n, \mathbb{C})$ is a representation of G over $\mathbb{C}$ and $V = \mathbb{C}^n$, then V becomes an $\mathbb{C}G$-module if we define the multiplication vg by*

$$vg = v(g\rho) \;\; (v \in V, g \in G).$$

Moreover, there is a basis $\mathscr{B}$ of V such that

$$g\rho = [g]_{\mathscr{B}} \text{ for all } g \in G.$$

(2) Assume that V is an $\mathbb{C}G$-module and let $\mathscr{B}$ be a basis of V. Then the function

$$g \to [g]_{\mathscr{B}} \;\; (g \in G)$$

is a representation of G over $\mathbb{C}$.

Definition 2.4. Suppose that V is a finite-dimensional $\mathbb{C}G$-module with a basis $\mathscr{B}$. Then the *character* of V is the function $\chi : G \to \mathbb{C}$ defined by

$$\chi(g) = \operatorname{tr}[g]_{\mathscr{B}} \;\; (g \in G).$$

Definition 2.5. The character of a representation $\rho : G \to GL(n, \mathbb{C})$ is the character χ of the corresponding $\mathbb{C}G$-module $\mathbb{C}^n$ where

$$\chi(g) = \mathrm{tr}(g\rho) \quad (g \in G).$$

Definition 2.6. A $\mathbb{C}G$-module V is said to be irreducible if it is non-zero and it has no $\mathbb{C}G$-submodule apart from $\{0\}$ and V.

Definition 2.7. We say that χ is a *character* of G if χ is the character of some $\mathbb{C}G$-module. Further, χ is an *irreducible* character of G if χ is the character of an irreducible $\mathbb{C}G$-module.

Theorem 2.8. *[1, pg. 119] If x and y are conjugate elements of the group G, then*

$$\chi(x) = \chi(y)$$

for all characters χ of G.

Definition 2.9. Suppose that φ and θ are functions from G to $\mathbb{C}$. Define the inner product

$$\langle \theta, \varphi \rangle = \frac{1}{|G|} \sum_{g \in G} \theta(g)\overline{\varphi(g)}.$$

Definition 2.10. Suppose that ψ is a character of G, and that χ is an irreducible character of G. We say that χ is a *constituent* of ψ if $\langle \psi, \chi \rangle \neq 0$. Thus, the constituents of ψ are the irreducible characters χ_i of G for which the integer d_i in the expression $\psi = d_1\chi_1 + \ldots + d_k\chi_k$ is non-zero.

Theorem 2.11. *[1, pg. 143] Suppose that V and W are $\mathbb{C}G$-modules, with characters χ and ψ, respectively. Then V and W are isomorphic if and only if $\chi = \psi$.*

Theorem 2.12. *[1, pg. 145] Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G. Then $\chi_1, \ldots, \chi_k$ are linearly independent vectors in the vector space of all functions from G to*

$\mathbb{C}$. *Also, a character* $\chi_1, \ldots, \chi_k$ *form an orthonormal set; that is* $\langle \chi_i, \chi_j \rangle = \delta_{ij}$ *for all* i, j. *Also,* ψ *is irreducible if and only if* $\langle \psi, \psi \rangle = 1$.

Lemma 2.13. *[1, pg. 128,152] Let* $Irr(G)$ *denote the set of irreducible characters of a finite group* G. *Then* $|Irr(G)|$ *equals the number of conjugacy classes of* G *and*

$$\sum_{\chi \in Irr(G)} \chi(1)^2 = |G|.$$

Lemma 2.14. *[1, pg. 161] Row orthogonality relation: Let* $\chi_1, \ldots, \chi_k$ *be the irreducible characters of a finite group* G. *The following relations hold for any* $i, j \in \{1, \ldots, k\}$:

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g) \chi_j(g^{-1}) = \delta_{ij}.$$

Lemma 2.15. *[1, pg. 161] Column orthogonality relation: Let* G *be finite group and let* $g, h \in G$. *Then*

$$\sum_{\chi \in Irr(G)} \chi(g) \overline{\chi(h)} = 0,$$

if g *is not conjugate to* h *in* G. *Otherwise, the sum is equal to* $|C_G(g)|$, *where* $C_G(g)$ *is the centralizer subgroup.*

Definition 2.16. A *linear character* of a finite group is a character of degree one.

Definition 2.17. If $N \triangleleft G$ and $\tilde{\chi}$ is a character of G/N, then the character χ of G which is given by

$$\chi(g) = \tilde{\chi}(Ng) \ (g \in G)$$

is called the *lift* of $\tilde{\chi}$ to G.

Recall that the derived subgroup of a group G is the subgroup generated by all commutators $a^{-1}b^{-1}ab$ where $a, b \in G$.

Lemma 2.18. *[1, pg. 174] Let* G *be a finite group and let* G' *be the derived subgroup of* G. *Then the linear characters of* G *are precisely the lifts to* G *of the irreducible characters of*

G/G'. In particular, the number of distinct linear characters of G is equal to $|G/G'|$, and so divides $|G|$.

Definition 2.19. Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G and let $g_1, \ldots, g_k$ be representatives of the conjugacy classes of G. The $k \times k$ matrix whose ij-entry is $\chi_i(g_j)$ (for all i, j with $1 \le i \le k,\ 1 \le j \le k$) is called the *character table* of G.

Theorem 2.20. *[1, pg. 246] If χ is a character of G and $g \in G$, then $\chi(g)$ is an algebraic integer.*

2.2 Algebra and Dihedral Groups

Let $G = \{g_0 = 1, g_1, \ldots, g_{n-1}\}$ be a finite group of order n and let $R = \mathbb{C}[x_0, x_1, \ldots, x_{n-1}]$ be a polynomial ring. We think of x_i as a coordinate function on $\mathbb{C}^n$. Here x_i corresponds to $g_i, 0 \le i \le n-1$, and in general we write $x_i = x_{g_i}$.

Definition 2.21. The *group matrix* M_G of G is the matrix whose rows and columns are indexed by the group elements, where the (i, j) entry is x_k if $g_i g_j^{-1} = g_k$. The *group determinant* is defined to be $\Theta_G = \det M_G$.

Let $C_1 = \{1\}, C_2, \ldots, C_r$ be the conjugacy classes of G. Let V be the complex vector space spanned by $x_0, \ldots, x_{n-1}$. To any conjugacy class C of G we associate the element $\overline{C} = \sum_{g \in C} x_g$.

Let $\mathrm{Aut}(G)$ be the group of automorphisms of G. Since $\mathrm{Aut}(G)$ clearly acts on G and for each element g_i we have the associated variable x_i, then $\mathrm{Aut}(G)$ acts on V in the natural way: for $g \in G$, $\alpha \in \mathrm{Aut}(G)$ we have $(x_g)\alpha = x_{(g)\alpha}$. Note that we are acting on the x_i from the right.

If C is a conjugacy class, then so is $(C)\alpha$ for $\alpha \in \mathrm{Aut}(G)$. Thus $\mathrm{Aut}(G)$ acts on the set of conjugacy classes of G and so acts on $\{\overline{C}_1, \overline{C}_2, \ldots, \overline{C}_r\}$. We see that $\{\overline{C}_1, \overline{C}_2, \ldots, \overline{C}_r\}$ is

Aut(G)-invariant. Define W as the quotient

$$W = V/Span(\overline{C}_1, \overline{C}_2, \ldots, \overline{C}_r).$$

For each conjugacy class C of G, we can take one element $g \in C$ and let the associated x_g be a *non-basis element* of W. We can construct a basis of W by taking the remaining $n-r$ elements of G and their associated x variables. Then Aut(G) will also act on W. This determines a representation ρ_G of Aut(G) of degree $n - r$. Here,

$$\rho_G : \text{Aut}(G) \to GL(n - r, \mathbb{C}).$$

Let χ_G be the character afforded by the representation ρ_G.

Each row of the matrix $\rho_G(\alpha)$ is determined by where α sends a basis element of W. When calculating the matrix $\rho_G(\alpha)$ representing the automorphism action α, only the diagonal entries of the matrix $\rho_G(\alpha)$ matter in determining the character χ_G. For an element $x \in W$, $\alpha(x)$ can be x itself or the non-basis element of its conjugacy class, or another element. Consider the kth row of the matrix $\rho_G(\alpha)$ determined by $\alpha(x)$. If x is sent to x, there will be a 1 in the (k, k) entry, which is part of the diagonal of the matrix. If x is sent to the non-basis element b of its conjugacy class, we see that $\alpha(x) = b = -\sum_{c \in C, c \neq b} c$. Thus there will be a -1 in the (k, k) matrix entry. For all other elements that x is sent to, there will be a 0 in the (k, k) entry since no other elements are expressed using x. The sum of the diagonal entries results in $\chi_G(\alpha)$.

Example 2.22. Let $G = S_3$. Let $g_0 = 1, g_1 = (123), g_2 = (132), g_3 = (12), g_4 = (13), g_5 = (23)$. Note that $\text{Aut}(S_3) = S_3$. Let I_g be the action of conjugation on S_3 by g. The conjugacy classes of S_3 are $\{1\}, \{(123), (132)\}, \{(12), (13), (23)\}$. So $W = V/Span(x_0, x_1 + x_2, x_3 + x_4 + x_5)$ and generators for W can be taken to be x_2, x_4, x_5. Relative to this set of generators, the action of $I_{(13)}$ would be:

$$(x_2)I_{(13)} = -x_2; \qquad (x_4)I_{(13)} = x_4; \qquad (x_5)I_{(13)} = -x_4 - x_5.$$

So the matrix representing the action $I_{(13)}$ would be

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

and $\chi_{S_3}(I_{13}) = -1$.

Definition 2.23. The dihedral group D_{2n} is the finite group with the presentation

$$D_{2n} = \langle r, s | r^n = s^2 = 1, srs^{-1} = r^{-1} \rangle.$$

Lemma 2.24. *[1, pg. 182] Let $n \geq 3$ be an odd number. Let $G = D_{2n}$ be the dihedral group of order $2n$ presented as $\langle r, s | r^n = s^2 = 1, rs = sr^{-1} \rangle$. Then the conjugacy classes of D_{2n} are*

$$\{1\},\ \{r^1, r^{-1}\},\ \ldots,\ \{r^{\frac{n-1}{2}}, r^{\frac{1-n}{2}}\},\ \{sr^b : 0 \leq b \leq n-1\}.$$

There are $\frac{n+3}{2}$ conjugacy classes of D_{2n}.

Lemma 2.25. *[1, pg. 183] Let $n \geq 4$ be an even number with $n = 2m$. Let $G = D_{2n}$ be the dihedral group of order $2n$ presented as $\langle r, s | r^n = s^2 = 1, rs = sr^{-1} \rangle$. Then the conjugacy classes of D_{2n} are*

$$\{1\},\ \{r^m\},\ \{r^1, r^{-1}\},\ \ldots,\ \{r^{m-1}, r^{1-m}\},\ \{sr^b : b \text{ even}\},\ \{sr^b : b \text{ odd}\}.$$

There are $m + 3$ conjugacy classes of D_{2n}.

Lemma 2.26. *The automorphism group of D_{2n} consists of $n\phi(n)$ elements.*

Proof. An automorphism of D_{2n} is completely determined by where the generators r, s are sent. In D_{2n}, r has order n. In order to keep the structure of D_{2n} intact, any automorphism must send r to another element of order n. The only elements of D_{2n} that have order n are powers of r, r^k, such that $(k, n) = 1$. There are $\phi(n)$ of these. The element s has order 2. All elements of the form sr^m have order 2, since

$$(sr^m)(sr^m) = (sr^m s)r^m = r^{-m}r^m = 1.$$

The element s cannot be sent to a power of r by the automorphism because some power of r is already being sent there. Hence, s must be sent to sr^m with $0 \leq m < n$. We see that $(r^k)^n = 1$ and $(sr^m)^2 = 1$. Lastly

$$sr^m(r^k)r^{-m}s^{-1} = sr^k s^{-1} = (r^k)^{-1}.$$

Thus the relations in the presentation for D_{2n} are satisfied. Since $\langle r^k, sr^m \rangle = D_{2n}$, any such choices of k and m determine an automorphism. Thus there are $n\phi(n)$ different automorphisms. □

For ease of notation, the element of $\text{Aut}(D_{2n})$ that sends r to r^j and s to sr^k will be referred to from hereafter as (j, k).

Lemma 2.27. *Let $n > 2$ be an integer. Let $AGL(1, n)$ denote the subgroup of 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ where $a \in (\mathbb{Z}/n\mathbb{Z})^*$ and $b \in \mathbb{Z}/n\mathbb{Z}$. Then Aut($D_{2n}$) is isomorphic $AGL(1, n)$. The element (a, b) corresponds to $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$.*

Proof. Let $f : \text{Aut}(D_{2n}) \to AGL(1, n)$ be defined by $f((a, b)) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. Consider the composition of automorphisms $(a,b)\circ(c,d)$. We see that

$$(a,b)\circ(c,d)(r)=(a,b)(r^c)=r^{ac}$$

and

$$(a,b)\circ(c,d)(s)=(a,b)(sr^d)=sr^br^{ad}=sr^{b+ad}.$$

Thus $(a,b)\circ(c,d)=(ac,b+ad)$. Now

$$f((a,b)\circ(c,d))=f((ac,b+ad))=\begin{pmatrix}ac & b+ad\\ 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}c & d\\ 0 & 1\end{pmatrix}=f((a,b))f((c,d)).$$

Thus f is a homomorphism. It is clear that f is a bijection, so f is an isomorphism. □

2.3 Decomposing $\mathbb{C}G$-modules

Lemma 2.28. *[1, pg.146] If χ is an irreducible character of G, and V is any $\mathbb{C}G$-module, then*

$$V\left(\sum_{g\in G}\chi(g^{-1})g\right)$$

is equal to the sum of those $\mathbb{C}G$-submodules of V which have character χ (where for $r\in\mathbb{C}G$, we define $Vr=\{vr : v\in V\}$).

When the irreducible characters of G are known, we can find how V decomposes into irreducible $\mathbb{C}G$-modules. We have a procedure to do this given in [1, pg. 147].

1) Choose a basis $v_1,\ldots,v_n$ of V.

2) For each irreducible character χ of G, calculate the vectors $v_i(\sum_{g\in G}\chi(g^{-1})g)$ for $1\le i\le n$, and let C_χ be the subspace of V spanned by these vectors.

3) Then V is a direct sum of the $\mathbb{C}G$-modules V_χ as χ runs over the irreducible characters of G. The character of V_χ is an integer multiple of χ.

This gives us a way to decompose W into irreducible submodules. The number of irreducible constituents is the same number of irreducible submodules of W.

Chapter 3. Origins of the problem

This problem arises from a paper by Stephen Humphries, *Generalized cogrowth series, random walks, and the group determinant* [2].

The group matrix, which is linked to random walks, obviously determines the group G as it is essentially a multiplication table of G. From the group matrix M_G, the *group determinant* is $\Theta_G = \det M_G$. The group determinant determines G as well [4].

If G is finite, the group matrix M_G can be block diagonalized: there is a matrix Q such that

$$Q^{-1} M_G Q = \operatorname{diag}(d_1 \cdot D_1, d_2 \cdot D_2, \ldots, d_r \cdot D_r), \tag{3.1}$$

where r is the number of conjugacy classes, D_i is a square $d_i \times d_i$ matrix over $\mathbb{C}[x_0, x_1, \ldots, x_{n-1}]$, and $d_i \cdot D_i$ indicates that D_i occurs d_i times in this diagonal decomposition. Further $D_i = \sum_{j=1}^{n} x_j D_{i,j}$, where $\rho_i : g_j \mapsto D_{i,j}$ is an irreducible representation of G [2].

Let $\mathcal{I}_G$ denote the ideal of $C[x_0, x_1, \ldots, x_{n-1}]$ generated by $C_{1_G,k}, k \geq 1$ where$C_{1_G,k}$ is a polynomial in $x_0, \ldots, x_{n-1}$ of degree k. The polynomials $C_{1_G,k}$ help determine the random walk on G. It is shown in [2] that $\overline{C}$ is in the radical of $\mathcal{I}_G$ and the ideal $\langle \overline{C}_1, \ldots, \overline{C}_r \rangle$ is $\mathrm{Aut}(G)$-invariant. Let $\mathcal{V}_G$ denote the variety determined by the ideal $\mathcal{I}_G$. A *hyperplane* is a codimension-1 vector subspace. Each $\overline{C}_k$ determines a hyperplane $H(C_k) \subset \mathbb{C}^n$. Let the intersection of all r of these hyperplanes be

$$U_G = \cap_{k=1}^{r} H(C_k).$$

The variety $\mathcal{V}_G$ is contained in the U_G. The representation ρ_G, which is the representation of $\mathrm{Aut}(G)$ on W, is the same as the representation of $\mathrm{Aut}(G)$ at $0 \in U_G$ [2]. The decomposition of U_G into irreducible $\mathrm{Aut}(G)$-modules gives information about $\mathcal{V}_G$.

A condition is given for when ρ_G is irreducible:

Theorem 3.1. *[2] Let G be a finite non-abelian group. Then ρ_G is irreducible if and only if $G \cong Q_8 \times A$, where A is an elementary abelian 2-group.*

Some of the most basic non-abelian groups are the dihedral groups. We know that ρ_G will not be irreducible when G is a dihedral group. We will describe how ρ_G decomposes into irreducible representations.

In the following chapters p will be an odd prime. Recall the dihedral group

$$D_{2p} = \langle r, s | r^p = s^2 = 1, srs^{-1} = r^{-1} \rangle.$$

Lemma 4.1. *The automorphism group of D_{2p} has p conjugacy classes.*

Proof. Note that $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1}$ is $\begin{pmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{pmatrix}$. To find the conjugacy class of an element of $\mathrm{Aut}(D_{2p})$, we conjugate:

$$\begin{pmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & a^{-1}(bx + y - b) \\ 0 & 1 \end{pmatrix}.$$

If $x = 1$, we get $\begin{pmatrix} 1 & a^{-1}y \\ 0 & 1 \end{pmatrix}$. When $y = 0$, we see that the identity map is only conjugate to itself, as expected. When $y \neq 0$, $a^{-1}y$ can assume every value of $\mathbb{Z}_p$ except for 0. Thus $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ with $0 < y < p$ is a conjugacy class of $p - 1$ elements. If $x \neq 1$ and $y = 0$, we get $\begin{pmatrix} x & a^{-1}b(x-1) \\ 0 & 1 \end{pmatrix}$ with $a^{-1}b(x-1)$ nonzero and $a^{-1}b(x-1)$ can, in fact, assume every value of $\mathbb{Z}_p$. For each $x \neq 1$, there is a conjugacy class of p elements, the set of all automorphisms that send r to r^x: $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $0 \leq y < p$. There are $p - 2$ of these conjugacy classes, one for each x, $1 < x < p$. □

Lemma 4.2. *Let $n > 1$ be an odd positive integer. The derived subgroup of $Aut(D_{2n})$ is the set of automorphisms that send r to r.*

Proof. We will denote the derived subgroup of $G = \mathrm{Aut}(D_{2n})$ as G' for ease of notation. Then G' is generated by all elements of the form $x^{-1}y^{-1}xy$ with $x, y \in \mathrm{Aut}(D_{2n})$. Let

$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \in D_{2n}$. We have

$$\begin{pmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{-1} & -yx^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & x^{-1}y + ba^{-1}x^{-1} - a^{-1}x^{-1}y - a^{-1}b \\ 0 & 1 \end{pmatrix}.$$

If $a = 1$ and $x = 2^{-1}$, the matrix simplifies to $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Thus $G' = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z}_n \right\}$. □

From Lemma 4.2 and Lemma 2.18, $|\mathrm{Aut}(D_{2p}) : G'| = p - 1$ and so $\mathrm{Aut}(D_{2p})$ has $p - 1$ linear characters. Since $\mathrm{Aut}(D_{2p})$ has p conjugacy classes, there is only one irreducible nonlinear character. From Lemma 2.13, we have

$$(p-1)(1)^2 + x^2 = p(p-1),$$

where x is the degree of the non-linear character. This implies that $x = p - 1$ and the last irreducible character has degree $p - 1$.

Lemma 4.3. *$Aut(D_{2p})/G' \cong \mathbb{Z}/(p-1)\mathbb{Z}$.*

Proof. From Lemma 4.1 and Lemma 4.2, it follows that $\mathrm{Aut}(D_{2p})/G' \cong \mathbb{Z}_p^{\times}$, which is a cyclic group of order $p - 1$, since p is an odd prime. □

The character table of $\mathbb{Z}/(p-1)\mathbb{Z}$ is straightforward. All irreducible characters are linear, since this is an abelian group. Let a be the multiplicative generator of $\mathbb{Z}/(p-1)\mathbb{Z}$ and put $\omega = e^{2\pi i/(p-1)}$. The $p - 1$ irreducible representations of $\mathbb{Z}/(p-1)\mathbb{Z}$ over $\mathbb{C}$ are

ρ_{ω^j} $(0 \le j \le p-2)$, where

$$(a^k)\rho_{\omega_j} = (\omega^{jk}) \ (0 \le k \le p-2).$$

Note that if $k = \frac{p-1}{2}$, then $(a^{\frac{p-1}{2}})\rho_{\omega_j} = (\omega^{j\frac{p-1}{2}}) = (-1)^j$.

The character table of $\mathbb{Z}/(p-1)\mathbb{Z}$ looks like

	1	r^k $(1 < k < p-1)$
χ_1	1	1
χ_{j+1} $(0 < j \le p-2)$	1	ω^{jk}

By the column orthogonality relation,

$$0 = \sum_{\chi \in \mathrm{Irr}(G)} \chi(r^k)\overline{\chi(1)} = \sum_{\chi \in \mathrm{Irr}(G)} \chi(r^k) = \sum_{0 \le j \le p-2} \omega^{jk}.$$

We have defined the characters $\chi_i, 1 \le i \le p-1$, above. Let χ_p denote the unique non-linear character of $\mathrm{Aut}(D_{2p})$.

Since the identity automorphism and the automorphisms that send r to r are in G', they are also in the kernel of every linear irreducible character. Let a be a generator for $\mathbb{Z}/(p-1)\mathbb{Z} \cong \mathrm{Aut}(D_{2p})/G'$. Here $(p-1, 0)$ is in $a^{\frac{p-1}{2}}G'$, so $\chi^2((p-1,0)) = 1$ for linear irreducible χ. Thus $\chi((p-1,0)) = 1$ or $\chi((p-1,0)) = -1$.

Let $g = (1,1)$, $h = (1,0) \in \mathrm{Aut}(D_{2p})$. By the column orthogonality relation,

$$\sum_{\chi \in \mathrm{Irr}(G),} \chi((1,1))\overline{\chi((1,0))} = 0,$$

since $(1,1)$ and $(1,0)$ are not conjugate. Furthermore,

$$0 = \sum_{\chi \in \mathrm{Irr}(G)} \chi((1,1))\overline{\chi((1,0))}$$

$$= \sum_{1 \le i \le p-1} \chi_i((1,1))\overline{\chi_i(1)} + \chi_p((1,1))\chi_p((1,0))$$

$$= p - 1 + (p-1)\chi_p((1,1)).$$

This implies that $\chi_p((1,1)) = -1$. Employing the column orthogonality relation with $n \neq 1, 0$, we have

$$0 = \sum_{\chi \in \mathrm{Irr}(G)} \chi((n,0))\overline{\chi((1,0))} = \sum_{0 \le j \le p-2} \omega^{jn} + (p-1)\chi_p((n,0))$$

which implies $\chi_p((n,0)) = 0$, $n \neq 1$. Our character table for $\mathrm{Aut}(D_{2p})$ looks like:

	$(1,0)$	$(1,1)$	$(2,0)$	...	$(p-1,0)$
χ_{2i+1} $(0 \le i \le \frac{p-3}{2})$	1	1			1
χ_{2i} $(1 \le i \le \frac{p-1}{2})$	1	1			-1
χ_p	$p-1$	-1	0	0	0

For $G = D_{2p}$, let $g_n = r^n$ $(0 \le n \le p-1)$ and $g_{p+n} = sr^n$ $(0 \le n \le p-1)$. Let $R = \mathbb{C}[x_0, x_1, \ldots, x_{2p-1}]$ be a polynomial ring. The x_i act as coordinate functions on $\mathbb{C}^{2p}$. Then x_i corresponds to g_i. Let V be the vector space spanned by $x_0, \ldots, x_{2p-1}$. Let $C_1 = \{1\}, C_2, \ldots, C_{\frac{p+3}{2}}$ be the conjugacy classes of G. Let $W_{2p} = V/Span(\overline{C}_1, \ldots, \overline{C}_{\frac{p+3}{2}})$. We will choose a basis of W_{2p} to be the elements r^n $(1 \le n \le \frac{p-1}{2})$ and sr^n $(1 \le n \le p-1)$. Now $\mathrm{Aut}(D_{2p})$ acts on the set $\{\overline{C}_1, \overline{C}_2, \ldots, \overline{C}_{\frac{p+3}{2}}\}$ and so acts on W_{2p}. This determines the representation $\rho_{D_{2p}}$ of degree $2p - \frac{p+3}{2} = \frac{3p-3}{2}$. Let $\chi_{D_{2p}}$ be the character afforded by the representation $\rho_{D_{2p}}$.

Example 4.4. Let $G = D_{10}$.

The character table of $\mathrm{Aut}(D_{10})$ is given in Table 4.1.

We will number the elements of G as

$$g_0 = 1, g_1 = r, g_2 = r^2, g_3 = r^3, g_4 = r^4, g_5 = s, g_6 = sr, g_7 = sr^2, g_8 = sr^3, g_9 = sr^4.$$

	$(1,0)$	$(1,1)$	$(2,0)$	$(3,0)$	$(4,0)$
χ_1	1	1	1	1	1
χ_2	1	1	i	$-i$	-1
χ_3	1	1	-1	-1	1
χ_4	1	1	$-i$	i	-1
χ_5	4	-1	0	0	0

Table 4.1: Character table of $\mathrm{Aut}(D_{10})$

The conjugacy classes of D_{10} are $\{1\}, \{r, r^4\}, \{r^2, r^3\}, \{s, sr, sr^2, sr^3, sr^4\}$. The basis that we choose for W_{10} is $x_1, x_2, x_6, x_7, x_8, x_9$.

For the representation ρ_G, we have

$$\rho_G((1,0)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

since $(1,0)$ acts trivially on each basis element. Then the character $\chi_G((1,0)) = 6$.

Consider how $(1,1)$ acts on the basis elements of W_{10}:

$$(1,1)(x_1) = x_1, (1,1)(x_2) = x_2, (1,1)(x_6) = x_7, (1,1)(x_7) = x_8, (1,1)(x_8) = x_9, (1,1)(x_9) = -x_6 - x_7 - x_8 - x_9$$

Thus

$$\rho_G((1,1)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 & -1 \end{pmatrix}$$

and for the character we have $\chi_G((1,1)) = 1$.

Consider how $(2,0)$ acts on the basis elements of W_{10}:

$$(2,0)(x_1) = x_2, (2,0)(x_2) = -x_1, (2,0)(x_6) = x_7, (2,0)(x_7) = x_9, (2,0)(x_8) = x_6, (2,0)(x_9) = x_8.$$

For the representation we have

$$\rho_G((2,0)) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and the character value is $\chi_G((2,0)) = 0$.

Consider how $(3,0)$ acts on the basis elements of W_{10}:

$$(3,0)(x_1) = -x_2, (3,0)(x_2) = x_1, (3,0)(x_6) = x_8, (3,0)(x_7) = x_6, (3,0)(x_8) = x_9, (3,0)(x_9) = x_7.$$

Thus

$$\rho_G((3,0)) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and the character value is $\chi_G((3,0)) = 0$.

Consider how $(4,0)$ acts on the basis elements of W:

$$(4,0)(x_1) = -x_1, (4,0)(x_2) = -x_2, (3,0)(x_6) = x_9, (3,0)(x_7) = x_8, (3,0)(x_8) = x_7, (3,0)(x_9) = x_6.$$

Thus

$$\rho_G((4,0)) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and the character value is $\chi_G((4,0)) = -2$.

Taking the inner product with the irreducible characters of $\text{Aut}(D_{10})$, we see that

$$\chi_G = \chi_2 + \chi_4 + \chi_5.$$

The above is an example of the following general result:

Theorem 4.5. *For the dihedral group D_{2p} with p an odd prime, $Aut(D_{2p})$ has character degrees 1 (with multiplicity $p-1$) and $p-1$ (with multiplicity 1). The character $\chi_{D_{2p}}$ has $\frac{p+1}{2}$ distinct irreducible constituents. In particular, there are $\frac{p-1}{2}$ linear constituents and* 1 *non-linear constituent of degree $p-1$(this being the only non-linear irreducible character of $Aut(D_{2p})$).*

Proof. Clearly, $\rho_{D_{2p}}((1,0))$ is the identity matrix since $(1,0)$ acts trivially. There are $2p - \frac{p+3}{2} = \frac{3(p-1)}{2}$ basis elements of W, so $\rho_{D_{2p}}((1,0))$ is a $\frac{3(p-1)}{2} \times \frac{3(p-1)}{2}$ identity matrix and $\chi_{D_{2p}}((1,0)) = \frac{3(p-1)}{2}$.

The automorphism $(1,1)$ sends r^k to r^k. Since there are $\frac{p-1}{2}$ basis elements that are powers of r, this will result in $\frac{p-1}{2}$ ones in the diagonal. The automorphism $(1,1)$ will send sr^k to sr^{k+1}. There will be a 0 in the diagonal for each sr^k except for sr^{p-1}, which will be

	$(1,0)$	$(1,1)$	$(2,0)$	...	$(p-1,0)$
χ_{2i+1} $(0 \leq i \leq \frac{p-3}{2})$	1	1			1
χ_{2i} $(1 \leq i \leq \frac{p-1}{2})$	1	1			-1
χ_p	$p-1$	-1	0	0	0
$\chi_{D_{2p}}$	$\frac{3(p-1)}{2}$	$\frac{p-3}{2}$	0	0	$\frac{1-p}{2}$

Table 4.2: Character table of $\mathrm{Aut}(D_{2p})$

sent to s, a non-basis element, resulting in -1. We see that

$$\chi_{D_{2p}}((1,1)) = \frac{p-1}{2} - 1 = \frac{p-3}{2}.$$

Consider the automorphism $(j,0)$ with $j \neq 1, p-1$. We see that $(r^k)(j,0) = r^{jk}$. If $jk \equiv k \mod p$, then $p|k(j-1)$ so that either $k = p$ or $j-1 = p$. Since $k \neq 0, p$ and $j \neq 1$, this cannot happen. If $jk \equiv -k \mod p$, then $p|k(j+1)$. Since $j \neq p-1$, this also cannot happen. So $(j,0)$ will send r^k to another power of r that is not r^k or r^{-k}, resulting in $\frac{p-1}{2}$ zeroes in the diagonal. Similarly, $(sr^k)(j,0) = sr^{jk}$. If $jk \equiv k \mod p$ also implies that sr^k cannot be sent to sr^k. We see that $jk \equiv 0 \mod p$ is impossible, so sr^k cannot be sent s. This results in $p-1$ zeroes in the diagonal. Thus $\chi_{D_{2p}}((j,0)) = 0$ for $1 < j < p-1$.

The last automorphism is $(p-1,0)$. We see that

$$(r^k)(p-1,0) = r^{(p-1)k} = r^{-k}.$$

There will be $\frac{p-1}{2}$ negative ones in the diagonal. Now

$$(sr^k)(p-1,0) = sr^{(p-1)k} = sr^{-k}.$$

No basis element will be sent to s. If $sr^k = sr^{-k}$, then $k \equiv -k \mod p$ and $p|2k$. Thus no basis element will be sent to itself and $\chi_{D_{2p}}((p-1,0)) = \frac{1-p}{2}$.

The character table of $\mathrm{Aut}(D_{2p})$ with $\chi_{D_{2p}}$ added can be seen in Table 4.

To find the constituents of $\chi_{D_{2p}}$, we take the inner product of $\chi_{D_{2p}}$ with each irreducible character of $\mathrm{Aut}(D_{2p})$. If the inner product is zero, then the irreducible character is not a constituent. Otherwise, the irreducible character will be a constituent of $\chi_{D_{2p}}$.

The inner product

$$\langle \chi_{2i+1}, \chi_{D_{2p}} \rangle = \frac{\frac{3(p-1)}{2} + (p-1)\frac{p-3}{2} + \frac{p(1-p)}{2}}{p(p-1)} = 0.$$

For χ_{2i} $(1 \leq i \leq \frac{p-1}{2})$,

$$\langle \chi_{2i}, \chi_{D_{2p}} \rangle = \frac{\frac{3(p-1)}{2} + (p-1)\frac{p-3}{2} - \frac{p(1-p)}{2}}{p(p-1)} = 1.$$

This gives $\chi_{D_{2p}}$ $\frac{p-1}{2}$ irreducible constituents.

The inner product of $\chi_{D_{2p}}$ and the only non-linear irreducible character gives us

$$\langle \chi_p, \chi_{D_{2p}} \rangle = \frac{(p-1)\frac{3(p-1)}{2} + (p-1)(-1)\frac{p-3}{2}}{p(p-1)} = 1.$$

This results in another irreducible constituent.

Thus $\chi_{D_{2p}}$ has $\frac{p+1}{2}$ irreducible constituents, of which $\frac{p-1}{2}$ are linear and one constituent which is non-linear. □

Corollary 4.6. *Using the method from [1, pg. 147], the module W_{2p} decomposes as*

$$W_{2p} = \bigoplus_{i=1}^{\frac{p-1}{2}} Span(\sum_{n=1}^{\frac{p-1}{2}} \chi_{2i}((n,0))x_n) \bigoplus Span(x_{p+1}, \ldots, x_{2p-1}).$$

Example 4.7. Let $G = D_{10}$ and let W_{10} have the same basis as Example 4.4. From the same example, we know that $\chi_G = \chi_2 + \chi_4 + \chi_5$. So W_{10} should decompose into three irreducible submodules. From Theorem 4.5, we can decompose W_{10} as

$$W_{10} = Span(x_1 - ix_2) \bigoplus Span(x_1 + ix_2) \bigoplus Span(x_6, x_7, x_8, x_9).$$

Example 4.8. Let $G = D_{14}$. Let $g_i = r^i$ and $g_{i+7} = r^i s$ for $0 \leq i \leq 6$. Let x_i correspond to g_i and let ω be a primitive 3rd root of unity. We choose the basis elements of W_{14} to be $x_1, x_2, x_3, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}$. Then

$$W_{14} = Span(x_1 + x_2 - x_3) \bigoplus Span(x_1 + \omega x_2 - \omega^2 x_3)$$

$$\bigoplus Span(x_1 + \omega^2 x_2 - \omega x_3) \bigoplus Span(x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}).$$

CHAPTER 5. THE DIHEDRAL GROUP D_{4p}

We now consider the dihedral group D_{4p}. Much of what we do in this case will be similar to the D_{2p} case.

Lemma 5.1. *The derived subgroup of $Aut(D_{4p})$ is the set of automorphisms that send r to r and s to sr^{2k}.*

Proof. We will denote the derived subgroup of $G = \mathrm{Aut}(D_{4p})$ as G' for ease of notation. Then G' is generated by all elements of the form $x^{-1}y^{-1}xy$ with $x, y \in \mathrm{Aut}(D_{4p})$. Now

$$\begin{pmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{-1} & -yx^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & x^{-1}y + ba^{-1}x^{-1} - a^{-1}x^{-1}y - a^{-1}b \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & x^{-1}y(1-a^{-1}) - a^{-1}b(1-x^{-1}) \\ 0 & 1 \end{pmatrix}.$$

Note that the $(1, 2)$ entry can only be even, since a and x must be odd. If $a = 1$ and $x = 2p - 1$, the matrix simplifies to $\begin{pmatrix} 1 & -2b \\ 0 & 1 \end{pmatrix}$. Thus $G' = \left\{ \begin{pmatrix} 1 & 2b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z}_{2p} \right\}$. □

Lemma 5.2. *The automorphism group of D_{4p} has $2p$ conjugacy classes.*

Proof. To find the conjugacy class of an element of $\mathrm{Aut}(D_{4p})$, we conjugate:

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & ay + b(1-x) \\ 0 & 1 \end{pmatrix}.$$

If $x = 1$, we get $\begin{pmatrix} 1 & ay \\ 0 & 1 \end{pmatrix}$. When $y = 0$, we see that the identity map is only conjugate to itself, as expected. Note that since $\gcd(a, 2p) = 1$, a must be odd. When $y = p$, we see that $(1, p)$ is only conjugate to itself since a is odd and $ap \equiv p \mod 2p$. If $\gcd(y, 2p) = 1$, then $\gcd(ay, 2p) = 1$ as well. In fact, the elements $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ with $\gcd(y, 2p) = 1$ form their own conjugacy class with $p - 1$ elements. If $y = 2$, then $2a$ will be can be any even positive integer less than $2p$. Thus $\begin{pmatrix} 1 & 2y \\ 0 & 1 \end{pmatrix}$ with $0 < y < p$ compose another conjugacy class.

If $x \neq 1$ and $y = 0$, we get $\begin{pmatrix} x & b(1-x) \\ 0 & 1 \end{pmatrix}$. Since $1 - x$ will always be even, $b(1 - x)$ can assume any positive even integer less than $2p$. For each $x > 1$, there is a conjugacy class of p elements, the set of all automorphisms that send r to r^x and s to sr^{2k}: $\begin{pmatrix} x & 2y \\ 0 & 1 \end{pmatrix}$ with $0 \leq 2y < 2p$. There are $p-2$ of these conjugacy classes, one for each x, $1 < x < p$. Similarly, if $x \neq 1$ and $y = 1$, we get $\begin{pmatrix} x & b(1-x)+a \\ 0 & 1 \end{pmatrix}$. Since $b(1 - x)$ is always even and a is odd, $b(1 - x) + a$ is odd and can be congruent to any positive odd number less than $2p$. For each $x > 1$, there is a conjugacy class of p elements, the set of all automorphisms that send r to r^x and s to sr^{2k+1}: $\begin{pmatrix} x & 2y \\ 0 & 1 \end{pmatrix}$ with $0 \leq 2y < 2p$. There are $p - 2$ of these conjugacy classes, one for each x, $1 < x < p$. Thus there are $4 + (p - 2) + (p - 2) = 2p$ conjugacy classes. □

Lemma 5.3. *[1, pg. 249] If χ is an irreducible character of G, then $\chi(1)$ divides $|G|$.*

Lemma 5.4. *[1, pg. 173] If χ is a linear character of G, with G' being the derived subgroup, then $G' \leq Ker\ \chi$.*

In particular, the intersection of the kernels of all the irreducible linear characters of G is equal to the derived subgroup.

There are $|\text{Aut}(D_{4p})/G'| = \frac{2p(p-1)}{p} = 2(p - 1)$ irreducible linear characters for $\text{Aut}(D_{4p})$. Since there are $2p$ irreducible characters, there are only 2 non-linear irreducible characters.

Let x be the degree of the first and y be the degree of the second. Then we have

$$2p(p-1) = 2(p-1) + x^2 + y^2$$

so that $x^2 + y^2 = 2(p-1)^2$. Since x and y divide $2p(p-1)$ by Lemma 5.3, we let $x = \frac{2p(p-1)}{a}$ and $y = \frac{2p(p-1)}{b}$. Then $x^2 + y^2 = 2(p-1)^2$ and so

$$\left(\frac{2p(p-1)}{a}\right)^2 + \left(\frac{2p(p-1)}{b}\right)^2 = 2(p-1)^2.$$

Getting rid of the denominators, we get

$$(2p(p-1)b)^2 + (2p(p-1)a)^2 = 2(p-1)^2a^2b^2$$

which simplifies even further to

$$2p^2(a^2+b^2) = a^2b^2.$$

Since 2 divides the left side, 2 divides a or b. If 2 divides a then, 4 divides the right side and so 2 divides b as well. Similarly, p divides the left side, so p divides a or b. If p divides a then, p^2 divides the right side and so p divides b as well. If $a, b > 2p$, then $x, y < p-1$ and $x^2 + y^2 < 2(p-1)^2$. Also, if $a > 2p$ and $b = 2p$, then $x < p-1$, $y = p-1$ and $x^2 + y^2 < 2(p-1)^2$. So $a = 2p$ and $b = 2p$. Thus $x = p-1$ and $y = p-1$.

Since $(1,2) \in G'$, we have $\chi_{j+1}((1,2)) = 1$ for $0 < j < 2p-2$ by Lemma 5.4.

One can show by induction on $k \geq 0$ that

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} a^k & b\sum_{i=0}^{k-1} a^i \\ 0 & 1 \end{pmatrix}.$$

Thus if $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{Aut}(D_{4p})$, then

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{p-1} = \begin{pmatrix} a^{p-1} & b\sum_{i=0}^{p-2} a^i \\ 0 & 1 \end{pmatrix}.$$

If $x \equiv a^{p-1} \mod 2p$, then $x \equiv a^{p-1} \mod p$ and by Fermat's little theorem, $x \equiv 1 \mod 2p$, where again we use the fact that a is odd. The sum $\sum_{i=0}^{p-2} a^i$ has an even number of terms. Since $(a, 2p) = 1$, then each of the terms is odd. Thus $b\sum_{i=0}^{p-2} a^i$ is even. Hence, $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{p-1} \in G'$. This means that G/G' cannot be cyclic since it has no element of order $2(p-1)$.

We know that there exists an $(a, b) \in \mathrm{Aut}(D_{4p})$ such that $|a| = p - 1$ in $\mathbb{Z}/p\mathbb{Z}$. Then $(a, 0)$ has order $p - 1$ and $(a, 0)G' \in \mathrm{Aut}(D_{4p})/G'$ has order $p - 1$. Thus $\mathrm{Aut}(D_{4p})/G' \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}$.

Lemma 5.5. *[1, pg. 206] Let G and H be finite groups. Let $\chi_1, \ldots, \chi_a$ be the distinct irreducibles of G and let $\psi_1, \ldots, \psi_b$ be the distinct irreducibles of H. Then $G \times H$ has precisely ab distinct irreducible characters, and these are*

$$\chi_i \times \psi_j \ (1 \leq i \leq a,\ 1 \leq j \leq b).$$

Let ω be a $p-1$th root of unity. Let $(a, 0),\ (0, b) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}$ with $|a| = 2$ and $|b| = p - 1$ be generating elements of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}$. The character table of

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}$ looks like

	(1,1)	$(1, b^k)$ $(1 < k < p)$	(a,1)	(a, b^k) $(1 < k < p)$
χ_1	1	1	1	1
χ_{j+1} $(0 < j \leq p-2)$	1	ω^{jk}	1	ω^{jk}
χ_p	1	1	-1	-1
χ_{p+j+1} $(0 < j \leq p-2)$	1	ω^{jk}	-1	$-\omega^{jk}$

From Lemma 2.18, the linear characters of $\text{Aut}(D_{4p})$ are the lifts of the irreducible characters of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}$. The element $(1, b^{\frac{p-1}{2}})$ in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}$ corresponds to the element $(2p-1, 0) \in \text{Aut}(D_{4p})$. Similarly, the element $(a, b^{\frac{p-1}{2}})$ corresponds to $(2p-1, 1)$. Thus $\chi_i((2p-1, 0)$ equals 1 or -1 and $\chi_i((2p-1, 1)$ equals 1 or -1 as well. Let a_j be the power of -1 corresponding to χ_{j+1}. Clearly, a_j will be even $\frac{p-1}{2}$ times and will be odd $\frac{p-1}{2}$ times for $0 < j \leq p-2$.

Adding the lifts, we get the following incomplete character table for $\text{Aut}(D_{4p})$:

	(1,0)	$(1, p)$	(1,1)	$(1, 2)$	$(k, 0)$ $(1 < k < 2p-1, (k, 2p) = 1)$
χ_1	1	1	1	1	1
χ_{j+1} $(0 < j \leq p-2)$	1	1	1	1	
χ_p	1	-1	-1	1	
χ_{p+j+1} $(0 < j \leq p-2)$	1	-1	-1	1	
χ_{2p-1}	p-1				
χ_{2p}	p-1				

	$(k,1)$ $(1<k<2p-1,(k,2p)=1)$	$(2p-1,0)$	$(2p-1,1)$
χ_1	1	1	1
χ_{j+1} $(0<j\leq p-2)$		$(-1)^{a_j}$	$(-1)^{a_j}$
χ_p		1	-1
χ_{p+j+1} $(0<j\leq p-2)$		$(-1)^{a_{p+j}}$	$-(-1)^{a_{p+j}}$
χ_{2p-1}			
χ_{2p}			

Let $\chi_{2p-1}((1,p))=x$ and $\chi_{2p}((1,p))=y$. Using column orthogonality on the first two columns, we get

$$\sum_{\chi\in\mathrm{Irr}(\mathrm{Aut}(D_{4p}))}\chi((1,p))\overline{\chi((1,0))}=(p-1)(x+y)=0.$$

Thus $x=-y$ and $|x|=|y|$. Using column orthogonality on the $(1,p)$ column, we get

$$2p-2+|x|^2+|y|^2=2p(p-1).$$

Simplifying, we get

$$2(p-1)+2|x|^2=2p(p-1).$$

Thus $p-1$ divides $|x|$.

Lemma 5.6. *[1, pg. 285] Let $g\in G$ and χ be a character of G. Then $|\chi(g)|\leq\chi(1)$.*

Lemma 5.7. *[1, pg. 253] Let g be an element of order n in G. Suppose that g is conjugate to g^i for all i with $1\leq i\leq n$ and $\gcd(i,n)=1$. Then $\chi(g)$ is an integer for all characters χ of G.*

The automorphism $(1,p)$ is its own inverse. By the Lemma 5.7, $\chi_{2p-1}((1,p))$ and $\chi_{2p}((1,p))$ are integers. So $p-1$ divides x. Since $x\leq p-1$, $|x|=p-1$. Since we have not previously distinguished χ_{2p-1} and χ_{2p} from each other, let $\chi_{2p-1}((1,p)) = p-1$ and $\chi_{2p}((1,p)) = 1-p$.

Now let $x = \chi_{2p-1}((1,1))$ and $y = \chi_{2p}((1,1))$. Using column orthogonality on the first and third columns, we see that $(p-1)(x+y) = 0$ and so $x = -y$. Using column orthogonality on the second and third columns, we get $2p - 2 + (p-1)(x-y) = 0$. So $x - y = 2$. Thus $x = 1$ and $y = -1$ and we have $\chi_{2p-1}((1,1)) = 1$ and $\chi_{2p}((1,1)) = -1$.

Now let $x = \chi_{2p-1}((1,2))$ and $y = \chi_{2p}((1,2))$. Using column orthogonality on the $(1,2)$ column, we get

$$2p - 2 + |x|^2 + |y|^2 = 2p.$$

So $|x|^2 + |y|^2 = 2$. Using column orthogonality on the first and fourth columns, we obtain $2p-2+(p-1)(x+y) = 0$. So $x+y = -2$. Thus $x = -1$ and $y = -1$ and so $\chi_{2p-1}((1,2)) = -1$ and $\chi_{2p}((1,2)) = -1$.

Now let $x = \chi_{2p-1}((2p-1,0))$ and $y = \chi_{2p}((2p-1,0))$. Using column orthogonality on the $(2p-1,0)$ column, we see that

$$\sum_{\chi \in \mathrm{Irr}(\mathrm{Aut}(D_{4p}))} \chi((2p-1,0))\overline{\chi((2p-1,0))} = 2p - 2 + |x|^2 + |y|^2 = 2p - 2.$$

Thus $\chi_{2p-1}((2p-1,0)) = 0$ and $\chi_{2p}((2p-1,0)) = 0$. Similarly, we see $\chi_{2p-1}((2p-1,1)) = 0$ and $\chi_{2p}((2p-1,1)) = 0$.

We will not need to calculate the character values for $\chi_i((k,0))$ and $\chi_i((k,1))$. We will see in the proof of the next theorem that $\chi((n,0)) = 0$ and $\chi((n,1)) = 0$ and so these values will not affect the inner product $\langle \chi, \chi_{2p-1} \rangle$ or $\langle \chi, \chi_{2p-1} \rangle$. These columns will be omitted from the next character table. With these calculations, we have enough of the $\mathrm{Aut}(D_{4p})$ character table to proceed:

	(1,0)	$(1,p)$	(1,1)	$(1,2)$	$(2p-1,0)$	$(2p-1,1)$
χ_1	1	1	1	1	1	1
χ_{j+1} $(0 < j \leq p-2)$	1	1	1	1	$(-1)^{a_j}$	$(-1)^{a_j}$
χ_p	1	-1	-1	1	1	-1
χ_{p+j+1} $(0 < j \leq p-2)$	1	-1	-1	1	$(-1)^{a_j}$	$-(-1)^{a_j}$
χ_{2p-1}	$p-1$	$p-1$	-1	-1	0	0
χ_{2p}	$p-1$	$1-p$	1	-1	0	0

For $G = D_{4p}$, let $g_n = r^n$ $(0 \leq n \leq 2p-1)$ and $g_{2p+n} = sr^n$ $(0 \leq n \leq 2p-1)$. Let $R = \mathbb{C}[x_0, x_1, \ldots, x_{4p-1}]$ be a polynomial ring. The x_i act as coordinate functions on $\mathbb{C}^{4p}$. Then x_i corresponds to g_i. Let V be the vector space spanned by $x_0, \ldots, x_{4p-1}$. Let $C_1 = \{1\}, C_2, \ldots, C_{p+3}$ be the conjugacy classes of G. Let $W = V/Span(\overline{C}_1, \ldots, \overline{C}_{p+3})$. We will choose a basis of W to be the elements r^n $(1 \leq n \leq p-1)$ and sr^n $(2 \leq n \leq 2p-1)$. Now $\text{Aut}(D_{4p})$ acts on the set $\{\overline{C}_1, \overline{C}_2, \ldots, \overline{C}_{p+3}\}$ and so acts on W. This determines the representation $\rho_{D_{4p}}$ of degree $4p - (p+3) = 3p-3$. Let $\chi_{D_{4p}}$ be the character afforded by the representation $\rho_{D_{4p}}$.

Theorem 5.8. *For the dihedral group D_{4p} with p an odd prime, Aut(D_{4p}) has character degrees 1 (with multiplicity $2p-2$) and $p-1$ (with multiplicity 2). The character $\chi_{D_{4p}}$ has $\frac{p+3}{2}$ distinct irreducible constituents. In particular, there are $\frac{p-1}{2}$ linear constituents each with multiplicity 2 and two degree $p-1$ constituents with multiplicity 1(these being the non-linear irreducible characters of Aut(D_{4p})).*

Proof. Since W has $4p-(p+3)$ basis elements, $\rho_{D_{4p}}((1,0))$ is the $(3p-3) \times (3p-3)$ identity matrix. Thus $\chi_{D_{4p}}((1,0)) = 3p-3$.

Consider the matrix $\rho_{D_{4p}}((1,p))$. The automorphism $(1,p)$ fixes each r^k for $k = 1, \ldots, p-1$ and so contributes $p-1$ ones in the diagonal. We also have $(sr^k)(1,p) = sr^{k+p}$. If k is even, $k+p$ will be odd and cannot be congruent to $0 \mod 2p$. If k is odd, $k+p$ will be even and cannot be congruent to $1 \mod 2p$. It is also impossible for $k+p \equiv k \mod 2p$. Thus the rest of the diagonal will be zeroes and we have $\chi_{D_{4p}}((1,p)) = p-1$.

Consider the matrix $\rho_{D_{4p}}((1,1))$. The automorphism $(1,1)$ fixes each r^k for $k = 1,\ldots,p-1$ and so contributes $p-1$ ones in the diagonal. We also have $(sr^k)(1,1) = sr^{k+1}$. If k is even, $k+1$ will be odd and cannot be congruent to $0 \mod 2p$. If k is odd, $k+1$ will be even and cannot be congruent to $1 \mod 2p$. It is also impossible for $k+1 \equiv k \mod 2p$. Thus the rest of the diagonal will be zeroes and we have $\chi_{D_{4p}}((1,1)) = p-1$.

Consider the matrix $\rho_{D_{4p}}((1,2))$. The automorphism $(1,2)$ fixes each r^k for $k = 1,\ldots,p-1$ and so contributes $p-1$ ones in the diagonal. We also have $(sr^k)(1,2) = sr^{k+2}$. If $sr^{k+2} = s$, then $k = 2p-2$, contributing a -1 to the diagonal. If $sr^{k+2} = sr$, then $k = 2p-1$, contributing a -1 to the diagonal. We see that $sr^k = sr^{k+2}$ can never happen. Thus $\chi_{D_{4p}}((1,2)) = p-3$.

If $1 < n < 2p-1$, then consider the matrix for the representation $\rho_{D_{4p}}((n,0))$. We have $(r^k)(n,0) = r^{nk} \neq r^n, r^{-n}$. This will contribute $p-1$ zeros to the diagonal. If k is even, then $k = 2j$ from some j and $(sr^{2j})(n,0) = sr^{2nj}$. If $sr^{2nj} = s$, then $2nj \equiv 0 \mod 2p$, which is impossible. If $sr^{2nj} = sr^{2j}$, then $2nj \equiv 2j \mod 2p$. This implies that $2p|2j(n-1)$. This is not possible since $1 < n < 2p-1$ and $n \neq p+1$. Let k be odd. Then $(sr^k)(n,0) = sr^{kn}$. If $sr^{kn} = sr$, then $kn \equiv 1 \mod 2p$. Since n only has a unique inverse, this can only happen once, contributing one -1 to the diagonal. If $sr^{kn} = sr^k$, then $kn \equiv k \mod 2p$ and $p|k(n-1)$. This is only possible if $k = p$, contributing one 1 to the diagonal. Thus $\chi_{D_{4p}}((n,0)) = 0$.

If $1 < n < 2p-1$, then consider the matrix $\rho_{D_{4p}}((n,1))$. For $(n,1)$ we have $(r^k)(n,1) = r^{nk} \neq r^n, r^{-n}$. This will contribute $p-1$ zeros to the diagonal. Now $(sr^k)(n,1) = sr^{nk+1}$. If k is even, then sr^{nk+1} will have an odd power of r. If k is odd, then sr^{nk+1} will have an even power of r, since n is necessarily odd. So sr^k will never be sent to an element in its conjugacy class. This will result in all zeros in the diagonal and thus $\chi_{D_{4p}}((n,1)) = 0$.

Consider the matrix $\rho_{D_{4p}}((2p-1,0))$. Then for $(2p-1,0)$ we have $(r^k)(2p-1,0) = r^{k(2p-1)} = r^{-k}$ and so contributes $p-1$ negative ones to the diagonal. Then $(sr^k)(2p-1,0) =$

	(1,0)	$(1,p)$	(1,1)	$(1,2)$	$(2p-1,0)$	$(2p-1,1)$
χ_1	1	1	1	1	1	1
χ_{j+1} $(0 < j \leq p-2)$	1	1	1	1	$(-1)^{a_j}$	$(-1)^{a_j}$
χ_p	1	-1	-1	1	1	-1
χ_{p+j+1} $(0 < j \leq p-2)$	1	-1	-1	1	$(-1)^{a_j}$	$-(-1)^{a_j}$
χ_{2p-1}	$p-1$	$p-1$	-1	-1	0	0
χ_{2p}	$p-1$	$1-p$	1	-1	0	0
$\chi_{D_{4p}}$	$3p-3$	$p-1$	$p-1$	$p-3$	$1-p$	$1-p$

Table 5.1: Character Table of $\text{Aut}(D_{4p})$

sr^{-k}. Now $sr^{-k} \neq s$ if $k \neq 0$. If $sr^{-k} = sr$, then $-k \equiv 1 \mod 2p$ and so $2p|1+k$. This is only possible if $k = 2p-1$. This will contribute to one -1 to the diagonal. If $sr^{-k} = sr^k$, then $-k \equiv k \mod 2p$ which implies $2p|2k$ and $k = p$. This will contribute a single 1 to the diagonal. Thus $\chi_{D_{4p}}((2p-1,0)) = 1-p$.

Consider the matrix $\rho_{D_{4p}}((2p-1,0))$. Then $(r^k)(2p-1,1) = r^{-k}$ contributes $p-1$ negative ones to the diagonal. Then $(sr^k)(2p-1,1) = sr^{1-k}$. Since k and $1-k$ are of opposite parity, sr^k and sr^{1-k} are in different conjugacy classes. This will result in $2p-2$ zeros in the diagonal. Thus $\chi_{D_{4p}}((2p-1,0)) = 1-p$.

The character table of $\text{Aut}(D_{4p})$ including $\chi_{D_{4p}}$ is shown in Table 5.

In order to find the constituents of $\chi_{D_{4p}}$, we must take the inner product of $\chi_{D_{4p}}$ with each irreducible character.

The inner product

$$\langle \chi_1, \chi_{D_{4p}} \rangle = \frac{(3p-3)+(p-1)+(p-1)^2+(p-1)(p-3)+2p(1-p)}{2p(p-1)} = 0.$$

The inner product

$$\langle \chi_{j+1}, \chi_{D_{4p}} \rangle = \frac{(3p-3)+(p-1)+(p-1)^2+(p-1)(p-3)+2p(1-p)(-1)^{a_j}}{2p(p-1)}$$

simplifies to

$$\langle \chi_{j+1}, \chi_{D_{4p}} \rangle = 1-(-1)^{a_j}.$$

If a_j is even, $\langle \chi_{j+1}, \chi_{D_{4p}} \rangle = 0$. If a_j is odd, $\langle \chi_{j+1}, \chi_{D_{4p}} \rangle = 2$. This will result in $\frac{p-1}{2}$ linear constituents each having a multiplicity of 2.

The inner product

$$\langle \chi_{p+j+1}, \chi_{D_{4p}} \rangle = \frac{(p-1)(3-1-(p-1)+(p-3)-p((-1)^{a_i}-(-1)^{a_i})}{2p(p-1)} = 0.$$

The inner product

$$\langle \chi_{2p-1}, \chi_{D_{4p}} \rangle = \frac{(3p-3)(p-1)+(p-1)^2-(p-1)^2-(p-1)(p-3)}{2p(p-1)} = 1.$$

The inner product

$$\langle \chi_{2p}, \chi_{D_{4p}} \rangle = \frac{(3p-3)(p-1)-(p-1)^2+(p-1)^2-(p-1)(p-3)}{2p(p-1)} = 1.$$

Then χ_{2p-1} and χ_{2p} contribute two non-linear constituents to $\chi_{D_{4p}}$ each with multiplicity 1.

□

Corollary 5.9. *We see that W_{4p} can be decomposed into the following:*

$$W_{4p} = \bigoplus_{i=2}^{\frac{p+1}{2}} Span\left(\sum_{n=1}^{\frac{p-1}{2}} \chi_i((2n-1,0)^{-1})x_{2n-1}, \sum_{m=1}^{\frac{p-1}{2}} \chi_i((p-2m,0)^{-1})x_{2m}\right)$$

$$\bigoplus Span\left(x_{2p+2}+x_{3p+2}, \ldots, x_{3p-1}+x_{4p-1}, x_{3p} - \sum_{j=p+1}^{2p-1} x_{2j}\right)$$

$$\bigoplus Span\left(x_{2p+2}-x_{3p+2}, \ldots, x_{3p-1}-x_{4p-1}, x_{3p} + \sum_{k=p+1}^{2p-1} x_{2k}\right).$$

We now consider the dihedral group D_{2p^2}. Much of what we do in this case will be similar to the D_{2p} case.

By Lemma 3, there are $p^2\varphi(p^2) = p^2(p^2 - p) = p^3(p-1)$ elements of $\mathrm{Aut}(D_{2p^2})$.

Lemma 6.1. *The automorphism group Aut(D_{2p^2}) has p^2+1 conjugacy classes.*

Proof. To find the conjugacy class of an element of $\mathrm{Aut}(D_{2p^2})$, we conjugate:

$$\begin{pmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & a^{-1}(bx + y - b) \\ 0 & 1 \end{pmatrix}.$$

If $x = 1$, we get $\begin{pmatrix} 1 & a^{-1}y \\ 0 & 1 \end{pmatrix}$. When $y = 0$, we see that the identity map is only conjugate to itself, as expected. If $\gcd(y,p) = 1$, then $a^{-1}y$ can assume every value k of $\mathbb{Z}_{p^2}$ such that $\gcd(k,p) = 1$. Thus we have a conjugacy class of $\varphi(p^2) = p^2 - p$ elements, where the elements appear as $(1,k)$ with $\gcd(k,p) = 1$. If $y \neq 0$ and $\gcd(y,p) \neq 1$, then $a^{-1}y$ can assume $p-1$ different values. So there is a conjugacy class of $p-1$ elements whose elements look like $(1,pk)$ for $0 < k < p$. This accounts for 3 conjugacy classes.

Let $x \neq 1$, $x \equiv 1 \mod p$ and $x = pk+1$ for some positive integer k. Then the matrix

$$\begin{pmatrix} x & a^{-1}(bx + y - b) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & a^{-1}(b(pk+1) + y - b) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & a^{-1}(bkp + y) \\ 0 & 1 \end{pmatrix}.$$

If $\gcd(y,p) = 1$, then $a^{-1}(bkp + y)$ can be every value k of $\mathbb{Z}_{p^2}$ such that $\gcd(k,p) = 1$. This results in a conjugacy class of size $p^2 - p$ with elements of the form $(pk+1, y)$ with $\gcd(y,p) = 1$. There will be $p-1$ of these conjugacy classes since $0 < k < p$. If $\gcd(y,p) = p$, then $a^{-1}(bkp + y)$ can assume p different values. So there is a conjugacy class of p elements whose elements are of the form $(pk+1, pt)$ for $0 \leq t < p$. Again, there will be $p-1$ of these conjugacy classes since $0 < k < p$.

Let $x \not\equiv 1 \mod p$. Then we get

$$\begin{pmatrix} x & a^{-1}(b(x-1)+y) \\ 0 & 1 \end{pmatrix}.$$

If $y = 0$ and $b = 1$, the matrix simplifies to

$$\begin{pmatrix} x & a^{-1}(x-1) \\ 0 & 1 \end{pmatrix}.$$

We see that $a^{-1}(x-1)$ can assume any value k of $\mathbb{Z}_{p^2}$ such that $\gcd(k,p) = 1$. If $b = p$, the matrix simplifies to

$$\begin{pmatrix} x & a^{-1}p(x-1) \\ 0 & 1 \end{pmatrix}.$$

We see that $a^{-1}p(x-1)$ can assume any non-zero value k of $\mathbb{Z}_{p^2}$ such that $\gcd(k,p) = p$. Thus for each $x \not\equiv 1 \mod p$, the automorphisms of the form (x,k) where $x \not\equiv 1 \mod p$ form a conjugacy class of p^2 elements. There will be $p^2 - 2p$ of these conjugacy classes.

In total, $\mathrm{Aut}(D_{2p^2})$ has $3 + 2(p-1) + (p^2 - 2p) = p^2 + 1$ conjugacy classes. □

By Lemma 4.2, the derived subgroup of $\mathrm{Aut}(D_{2p^2})$ is the set of automorphisms that send r to r. Thus by Lemma 2.18, $\mathrm{Aut}(D_{2p^2})$ has $\frac{p^3(p-1)}{p^2} = p(p-1)$ linear irreducible characters. Thus there are also $p+1$ non-linear irreducible characters.

It follows that $\mathrm{Aut}(D_{2p^2})/G' \cong \mathbb{Z}_{p^2}^{\times}$, which is a cyclic group or order $p(p-1)$ since p is odd. So

$$\mathrm{Aut}(D_{2p^2})/G' \cong \mathbb{Z}/(p(p-1))\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}.$$

Let $\omega = e^{\frac{2\pi i}{p-1}}$ and let $x = e^{\frac{2\pi i}{p}}$. Let a be a generator of $\mathbb{Z}/(p-1)\mathbb{Z}$ and let $\chi_{j+1}(a^k) = \omega^{jk}$ with $0 \le j \le p-2$ be the irreducible characters of $\mathbb{Z}/(p-1)\mathbb{Z}$. Let b be a generator of $\mathbb{Z}/p\mathbb{Z}$ and let $\chi_{m+1}(b^h) = x^{mh}$ with $0 \le h \le p-1$ be the irreducible characters of $\mathbb{Z}/p\mathbb{Z}$. Then by Lemma 5.5, the character table of $\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ looks like:

	(a^k, b^h) $(0 \leq k \leq p-2; 0 \leq h \leq p-1)$
$\chi_{j+1} \times \psi_{m+1}$	$\omega^{jk} x^{hm}$

From Lemma 2.18, the linear characters of $\mathrm{Aut}(D_{2p^2})$ are the lifts of the irreducible characters of $\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. The elements $(0, b^h)$ in $\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ correspond to the elements $(n, 0)$ and $(n, 1)$ with $n \equiv 1 \mod p$ in $\mathrm{Aut}(D_{2p^2})$, since both sets of these elements have order p. The character $\chi_{j+1} \times \psi_{m+1}((0, b^h))$ simplifies to x^{hm}. The elements $(a^{\frac{p-1}{2}}, b^k)$ in $\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ correspond to the elements $(n, 0)$ with $n \equiv -1 \mod p$ in $\mathrm{Aut}(D_{2p^2})$. Note that all of these elements have order $2p$. The character $\chi_{j+1} \times \psi_{m+1}((a^{\frac{p-1}{2}}, b^h))$ simplifies to $(-1)^{c_j} x^{hm}$. We relabel the character $\chi_{j+1} \times \psi_{m+1}$ as $\chi_{m(p-1)+j+1}$. When $m = j = 0$, we see that χ_1 is the principal character. Note that $\chi_i((n, 0)) = \chi_i((n, 1)) = 1$ when $n \equiv 1 \mod p$ and $0 \leq i < p$. Also, $\chi_i((n, 0)) = 1$ or -1, when $n \equiv -1 \mod p$.

If $n \not\equiv 1, -1 \mod p$, then $\chi_i((n, 0))$ will be a $p(p-1)$th root of unity. We will show later that for these conjugacy classes, $\chi_{D_{2p^2}}(n, 0) = 0$, so no other information needs to be shown about their values.

	(1,0)	$(1,p)$	(1,1)	$(n,0)$ $n \not\equiv 1 \mod p$	$(n,0)$ $n \equiv 1 \mod p$	$(n,1)$ $n \equiv 1 \mod p$	$(p^2-1, 0)$
χ_1	1	1	1	1	1	1	1
χ_j $(1 < j \leq p^2 - p)$	1	1	1				$(-1)^a$
χ_{p^2-p+j} $(1 \leq j \leq p)$	$p-1$	$p-1$	-1	0			0
χ_{p^2+1}	$p(p-1)$	$-p$	0	0	0	0	0

There are $p+1$ non-linear irreducible characters. From Lemma 2.13, the squares of the degrees of these irreducibles add up to $p^3(p-1) - p(p-1) = p(p+1)(p-1)^2$. By Lemma 5.3, the degrees of these irreducibles divide $p^3(p-1)$.

Let H be a subgroup of G generated by the elements (n, k) with $n \equiv 1 \mod p$. Note that $|H| = p^3$ and H is normal in G. So $|G/H| = p-1$. Let B be a subgroup of H generated by (n,p) with $n \equiv 1 \mod p$. Then $|B| = p^2$ and note that B is normal in H. Let $Q = H/B$, so $Q \cong \mathbb{Z}/p\mathbb{Z}$. Thus the irreducible characters of Q are pth roots of 1. Let $\tilde{\phi}$ be a non-trivial irreducible character of Q. Then $\phi = \tilde{\phi}(Qh)$ $(h \in H)$, the lift of $\tilde{\phi}$, is a character of H.

Definition 6.2. Let $H \subseteq G$ and let φ be a class function of H. Then φ^G, the *induced class function* on G, is given by

$$\varphi^G(g) = \frac{1}{|H|}\sum_{x\in G}\varphi^\circ(xgx^{-1}),$$

where φ° is defined by $\varphi^\circ(h) = \varphi(h)$ if $h \in H$ and $\varphi^\circ(h) = 0$ if $h \notin H$.

So $\phi^G(g) = \frac{1}{|H|}\sum_{x\in G}\phi^\circ(xgx^{-1})$ is the induced character and $\phi^G(1) = \frac{|G|}{|H|}\phi(1) = p-1$. Since $H \triangleleft G$, if $g \notin H$, then $\phi^G(g) = 0$. We see that for $n \equiv 1 \mod p$,

$$\phi^G((n,p)) = \frac{1}{|p^3|}\sum_{x\in G}\phi(x(n,p)x^{-1}) = \frac{p^3(p-1)}{p^3}\phi((n,p)) = (p-1)\phi((n,p)) = p-1.$$

We also see that for $n \not\equiv 1 \mod p$, we have

$$\phi^G((n,1)) = \frac{1}{|p^3|}\sum_{x\in G}\phi(x(n,1)x^{-1}) = \frac{p^2}{p^3}\sum_{\substack{k=1\\(k,p)=1}}^{p^2-1}\phi((n,k)) = \sum_{k=1}^{p-1}\phi((n,k)) = -1.$$

Now consider the inner product $\langle\phi^G,\phi^G\rangle$. We see that

$$\langle\phi^G,\phi^G\rangle = \frac{1}{p^3(p-1)}\left((p-1)^2 + (p-1)^3 + (p^2-p)p(-1)^2 + p(p-1)(p-1)^2\right) = 1$$

and so ϕ^G is an irreducible character of G. We relabel ϕ^G as χ_{p^2-p+1}.

Lemma 6.3. *[1, pg. 176] Suppose that χ is a character of G and λ is a linear character of G. Then the product $\chi\lambda$, defined by*

$$\chi\lambda(g) = \chi(g)\lambda(g) \quad (g \in G)$$

is a character of G. Moreover, if χ is irreducible, then so is $\chi\lambda$.

From lemma 6.3, $\chi_i \chi_{p^2-p+1}$ for $1 \leq i \leq p^2 - p$ is an irreducible character. Since for $n \equiv 1 \mod p$, $\chi_i((n,0))$ assumes p different values (the pth roots of unity) and $\chi_{p^2-p+1}((n,0))$ is nonzero, we will have p irreducible characters $\chi_i \chi_{p^2-p+1}$ of degree $p-1$.

This means there is one more irreducible character for $\mathrm{Aut}(D_{2p^2})$ and say that is has degree x. Since $p^4 - p^3 = (p^2-p)(1)^2 + p(p-1)^2 + x^2$, the remaining irreducible has degree $p(p-1)$. Let χ_{p^2+1} be the unique irreducible character of degree $p(p-1)$. By Lemma 6.3, $\chi_{p^2+1}\chi_i$, for $1 \leq i \leq p^2 - p$, is an irreducible character. Since χ_{p^2+1} is the only irreducible character of its degree, the conjugacy classes in the derived subgroup are the only classes that can have non-zero values for χ_{p^2+1}. Thus $\chi_{p^2+1}(n,0) = 0$ and $\chi_{p^2+1}(n,1) = 0$ for $n \neq 1$. By column orthogonality, we have

$$0 = \sum_{\chi \in \mathrm{Irr}(G)} \chi((1,0))\chi((1,p)) = (p^2-p) + p(p-1)^2 + (p(p-1))\chi_{p^2+1}((1,p)).$$

This simplifies to $\chi_{p^2+1}((1,p)) = -p$. Since χ_{p^2+1} is irreducible, we have

$$1 = \langle \chi_{p^2+1}, \chi_{p^2+1} \rangle = \frac{1}{p^4-p^3}\left((p(p-1)^2 + (p-1)(-p)^2 + (p^2-p)(\chi_{p^2+1}((1,1)))^2\right).$$

This simplifies to $\chi_{p^2+1}((1,1)) = 0$.

Now with the character table, we can proceed to the main theorem of the chapter.

Theorem 6.4. *For the dihedral group D_{2p^2} with p an odd prime, $Aut(D_{2p^2})$ has character degrees 1 (with multiplicity p^2-p), $p-1$ (with multiplicity p) and a unique character of degree p^2-p. The character $\chi_{D_{2p^2}}$ has $\frac{p^2-p}{2}+2$ constituents. The character $\chi_{D_{2p^2}}$ has*

$\frac{p-1}{2}$ linear constituents (with multiplicity 2),

$\frac{(p-1)^2}{2}$ linear constituents (with multiplicity 1),

one constituent of degree $p-1$ (with multiplicity 1), and

one constituent of degree p^2-p (with multiplicity 1).

Proof. Since W has $2p^2 - \frac{p^2+3}{2} = \frac{3(p^2-1)}{2}$ basis elements, $\rho_{D_{2p^2}}((1,0))$ is the $(\frac{3(p^2-1)}{2}) \times (\frac{3(p^2-1)}{2})$ identity matrix.

Consider the matrix $\rho_{D_{2p^2}}((1,p))$. The automorphism $(r^k)(1,p) = r^k$ contributes $\frac{p^2-1}{2}$ ones to the diagonal. We also have $(sr^k)(1,p) = sr^{k+p}$. So $(sr^k)(1,p)$ cannot equal sr^k and $(sr^k)(1,p) = s$ only if $k = p^2 - p$. This will contribute -1 to the diagonal. Thus $\chi_{D_{2p^2}}((1,p)) = \frac{p^2-3}{2}$.

Consider the matrix $\rho_{D_{2p^2}}((1,1))$. The automorphism $(r^k)(1,1) = r^k$ contributes $\frac{p^2-1}{2}$ ones to the diagonal. We also have $(sr^k)(1,1) = sr^{k+1}$. So $(sr^k)(1,1)$ cannot equal sr^k and $(sr^k)(1,1) = s$ only if $k = p^2 - 1$. This will contribute -1 to the diagonal. Thus $\chi_{D_{2p^2}}((1,1)) = \frac{p^2-3}{2}$.

Consider the matrix $\rho_{D_{2p^2}}((n,0))$ where $n \not\equiv 1, -1 \mod p$. The automorphism $(r^k)(n,0) = r^{nk} \neq r^k, r^{-k}$ contributes only zeroes to the diagonal. We also have $(sr^k)(n,0) = sr^{nk}$ but $sr^{nk} \neq sr^k, s$. So there are only zeroes on the diagonal and $\chi_{D_{2p^2}}((n,0)) = 0$.

Consider the matrix $\rho_{D_{2p^2}}((n,0))$ where $n \equiv 1 \mod p$ and $n \neq 1$. If $\gcd(k,p) = 1$, then $(r^k)(n,0) = r^{nk} \neq r^k, r^{-k}$. Let $k = ap$ and $n = 1 + bp$ for positive integers a, b. Then $(r^k)(n,0) = r^{(1+bp)ap} = r^{ap+abp^2} = r^k$. This will contribute $\frac{p-1}{2}$ 1s to the diagonal of $\rho_{D_{2p^2}}((n,0))$. We also have $(sr^k)(n,0) = sr^{nk}$. If $\gcd(k,p) = 1$, then $sr^{nk} = sr^k$ only if $n \equiv 1 \mod p^2$ which gives a contradiction. Let $k = ap$ and $n = 1 + bp$ for positive integers a, b. Then

$$sr^{kn} = sr^{(ap)(1+bp)} = sr^{ap+abp^2} = sr^k.$$

This will result in $p - 1$ 1s in the diagonal. We see that $sr^{nk} = s$ cannot happen. Thus $\chi_{D_{2p^2}}((n,0)) = \frac{3(p-1)}{2}$.

Consider the matrix $\rho_{D_{2p^2}}((n,1))$ where $n \equiv 1 \mod p$ and $n \neq 1$. If $\gcd(k,p) = 1$, then $(r^k)(n,1) = r^{nk} \neq r^k, r^{-k}$. Let $k = ap$ and $n = 1 + bp$ for positive integers a, b. Then $(r^k)(n,1) = r^{(1+bp)ap} = r^{ap+abp^2} = r^k$. This will contribute $\frac{p-1}{2}$ 1s to the diagonal of $\rho_{D_{2p^2}}((n,1))$. We also have $(sr^k)(n,1) = sr^{nk+1}$. If $sr^{nk+1} = sr^k$, then $nk + 1 \equiv k \mod p^2$. Since $n \equiv 1 \mod p$, let $n = 1 + ap$ for some positive integer a. Then $(ap + 1)k + 1 \equiv k \mod p^2$ which simplifies to $akp \equiv -1 \mod p^2$. So $sr^{nk+1} = sr^k$ cannot occur. If $sr^{nk+1} = s$, then $k \equiv -n^{-1} \mod p^2$ which has a unique solution. This will result in a single -1 in the diagonal. Thus $\chi_{D_{2p^2}}((n,1)) = \frac{p-3}{2}$.

Consider the matrix $\rho_{D_{2p^2}}((n,0))$ where $n \equiv -1 \mod p$ and $n \neq p^2-1$. If $\gcd(k,p)=1$, then $(r^k)(n,0) = r^{nk} \neq r^k, r^{-k}$. Let $k = ap$ and $n = -1+bp$ for positive integers a, b. Then $(r^k)(n,0) = r^{(-1+bp)ap} = r^{-ap+abp^2} = r^{-k}$. This will contribute $\frac{p-1}{2}$ -1s to the diagonal. We also have $(sr^k)(n,0) = sr^{nk}$ but $sr^{nk} \neq sr^k, s$. Thus $\chi_{D_{2p^2}}((n,0)) = \frac{1-p}{2}$.

Consider the matrix $\rho_{D_{2p^2}}((p^2-1,0))$. Then $(r^k)(p^2-1,0) = r^{(p^2-1)k} = r^{-k}$. This contributes $\frac{p^2-1}{2}$ -1s to the diagonal. We also have $(sr^k)(p^2-1,0) = sr^{(p^2-1)k} = sr^{-k} \neq s, sr^k$ which contributes p^2-1 zeros to the diagonal. Thus $\chi_{D_{2p^2}}((p^2-1,0)) = \frac{1-p^2}{2}$.

The following is the character table of $\mathrm{Aut}(D_{2p^2})$ including the character $\chi_{D_{2p^2}}$.

	(1,0)	$(1,p)$	(1,1)	$(n,0)$ $n \not\equiv 1,-1 \mod p$	$(n,0)$ $n \equiv 1 \mod p$
χ_1	1	1	1	1	1
χ_j $(1 < j \leq p^2-p)$	1	1	1		
χ_{p^2-p+1}	$p-1$	$p-1$	-1	0	$p-1$
χ_{p^2+1}	$p(p-1)$	$-p$	0	0	0
$\chi_{D_{2p^2}}$	$\frac{3(p^2-1)}{2}$	$\frac{p^2-3}{2}$	$\frac{p^2-3}{2}$	0	$\frac{3(p-1)}{2}$

	$(n,1)$ $n \equiv 1 \mod p$	$(n,0)$ $n \equiv -1 \mod p$	$(p^2-1,0)$
χ_1	1	1	1
χ_j $(1 < j \leq p^2-p)$	1	1	1
χ_{p^2-p+1}	-1	0	0
χ_{p^2+1}	0	0	0
$\chi_{D_{2p^2}}$	$\frac{p-3}{2}$	$\frac{1-p}{2}$	$\frac{1-p^2}{2}$

We take the inner product of $\chi_{D_{2p^2}}$ with irreducible characters to find the constituents of $\chi_{D_{2p^2}}$.

For $1 \leq j \leq p-1$, $\chi_j((n,0)) = 1$ and $\chi_j((n,1)) = 1$ for $n \equiv 1 \mod p$. Also $\chi_j((n,0)) =$

-1 for $n \equiv -1 \mod p$ can occur $\frac{p-1}{2}$ times. The inner product

$$|\mathrm{Aut}(D_{2p^2})|\langle \chi_j, \chi_{D_{2p^2}} \rangle = \frac{3(p^2-1)}{2} + \frac{(p^2-3)(p-1)}{2} + \frac{(p^2-3)(p(p-1))}{2}$$

$$+\frac{3(p-1)p(p-1)}{2} + \frac{(p-3)p(p-1)}{2} + \frac{(-1)(1-p)p^2(p-1)}{2} + \frac{(-1)(1-p^2)}{2}$$

simplifies to $\langle \chi_j, \chi_{D_{2p^2}} \rangle = 2$. This results in $\frac{p-1}{2}$ linear constituents of multiplicity 2.

For $p-1 \leq j \leq p^2-p$, $\chi_j((n,0)) = -1$ for $n \equiv -1 \mod p$ occurs half of the time and we will choose those. Note that there are $\frac{(p-1)^2}{2}$ of these characters. Now $\chi_j((n,0)) = -x^a$, where $n \equiv -1 \mod p$, x is a pth root of unity and a is a some positive integer. Also, $\chi_j((n,0)) = \chi_j((n,1)) = x^a$, where $n \equiv 1 \mod p$, x is a pth root of unity and a is a some positive integer. Note that for $(n,0)$ and $(n,1)$ where $n \equiv 1 \mod p$, we have

$$\sum_{i=0}^{p-1} \chi_j((ip+1,0)) = \sum_{i=0}^{p-1} \chi_j((ip+1,1) = \sum_{l=0}^{p-1} x^l = 0.$$

For $(n,0)$ where $n \equiv -1 \mod p$, we have

$$\sum_{i=1}^{p} \chi_j((ip-1,0)) = \sum_{l=0}^{p-1} -x^l = 0.$$

The inner product

$$|\mathrm{Aut}(D_{2p^2})|\langle \chi_j, \chi_{D_{2p^2}} \rangle = \frac{3(p^2-1)}{2} + \frac{(p^2-3)(p-1)}{2} + \frac{(p^2-3)(p(p-1))}{2}$$

$$+\frac{3(p-1)}{2}(p)\sum_{i=1}^{p-1} x^i + \frac{p-3}{2}(p^2-p)\sum_{i=1}^{p-1} x^i + \frac{1-p}{2}(p^2)\sum_{i=1}^{p-1} -x^i + \frac{1-p^2}{2}(p^2)(-1)$$

simplifies to $\langle \chi_j, \chi_{D_{2p^2}} \rangle = 1$. This results in $\frac{(p-1)^2}{2}$ linear irreducible constituents of $\chi_{D_{2p^2}}$ all of which have multiplicity 1.

The inner product

$$|\mathrm{Aut}(D_{2p^2})|\langle \chi_{p^2-p+1}, \chi_{D_{2p^2}}\rangle = \frac{3(p^2-1)}{2}(p-1) + \frac{p^2-3}{2}(p-1)(p-1) + \frac{p^2-3}{2}(-1)(p^2-p)$$

$$+\frac{3(p-1)}{2}(p-1)(p)(p-1) + \frac{p-3}{2}(-1)(p^2-p)(p-1)$$

simplifies to $\langle \chi_{p^2-p+1}, \chi_{D_{2p^2}}\rangle = 1$.

So we see that χ has one irreducible constituent of degree $p-1$ which has multiplicity 1.

The inner product

$$\langle \chi_{p^2+1}, \chi_{D_{2p^2}}\rangle = \frac{\frac{3(p^2-1)p(p-1)}{2} + \frac{-p(p^2-3)(p-1)}{2}}{p^3(p-1)} = 1.$$

Thus the character $\chi_{D_{2p^2}}$ has a constituent of degree $p(p-1)$ and multiplicity 1.

Note that the characters shown to be constituents are all of the constituents of χ since $2(\frac{p-1}{2}) + \frac{(p-1)^2}{2} + (p-1) + p(p-1) = \frac{3(p^2-1)}{2}$, or rather

$$\chi(1) = \sum_{\langle \chi, \chi_i\rangle \neq 0} \chi_i(1).$$

□

Theorem 7.1. *For any finite group G, let $W_G = \mathbb{C}[x_0, \ldots, x_{n-1}]/\langle \overline{C}_1, \ldots, \overline{C}_r \rangle$. Then $W = W_G$ does not contain the trivial submodule.*

Proof. By Lemma 2.28,

$$W\left(\sum_{\alpha \in \mathrm{Aut}(G)} \alpha\right)$$

is equal to the sum of the trivial submodules of W. If x_g, with $g \in G$, is a basis element of W, then $x_g\left(\sum_{\alpha \in \mathrm{Aut}(G)} \alpha\right) = a \sum \overline{C}$. Since $\overline{C} = 0$ in W, $x_g\left(\sum_{\alpha \in \mathrm{Aut}(G)} \alpha\right) = 0$. Note that x_0 is not in W, neither is any x_g where $g \in Z(G)$. □

Of course this means that χ_0 can never be a constituent of ρ_G.

Although this paper has been focused on dihedral groups, here are some examples of the decomposition of W for other groups. All computations in this section have been done in Magma.

Example 7.2. Let $G = S_3$. The automorphism group of S_3 is itself. The character table of S_3 is the following:

	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Now $\rho_G = \chi_2 + \chi_3$. We can decompose W as $W = Span(x_1) \bigoplus Span(x_3, x_4)$ where x_1 corresponds to (123), x_3 corresponds to (12) and x_4 corresponds to (13). Since $S_3 = D_6$, this clearly follows Theorem 4.1.

Example 7.3. Let $G = S_4$. Note that $\mathrm{Aut}(S_4) = S_4$. The character table of S_4 is the following:

	1	(12)(34)	(1234)	(123)	(12)
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	2	2	0	-1	0
χ_4	3	-1	-1	0	1
χ_5	3	-1	1	0	-1

Then $\chi_G = \chi_2 + 3\chi_3 + 2\chi_4 + 2\chi_5$.

Example 7.4. Let $G = A_4$. We see that $\mathrm{Aut}(A_4) = S_4$. Referring to the character table above, $\chi_G = \chi_3 + \chi_4 + \chi_5$.

While studying the structure of the automorphism group of dihedral groups, I formulated the following conjecture about the number of conjugacy classes of $\mathrm{Aut}(D_{2p})$. While a little off topic, I wanted to include it as I thought it was interesting.

Conjecture 1. *Let $\sigma(n)$ be the sum of the divisors of n. The number of conjugacy classes of $Aut(D_{2n})$ is equal to*

$$\sum_{\substack{0<x<n \\ (x,n)=1}} \sigma(gcd(x-1, n)).$$

Let $n = p_1^{e_1} \dots p_k^{e_k}$ be the factorization of n. The number of conjugacy classes of $Aut(D_{2n})$ is also equal to

$$\prod_{i=1}^{k} \left(\left(\sum_{j=0}^{e_i} p_i^j \right) - p_i^{e_i - 1} \right).$$

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