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Representations Associated to the Group Matrix

Joseph Aaron Keller
*Brigham Young University - Provo*

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Representations Associated to the Group Matrix

Joseph Keller

A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

Stephen Humphries, Chair
Darrin Doud
Pace Nielsen

Department of Mathematics
Brigham Young University
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ABSTRACT

Representations Associated to the Group Matrix

Joseph Keller
Department of Mathematics, BYU
Master of Science

For a finite group $G = \{g_0 = 1, g_1, \ldots, g_{n-1}\}$, we can associate independent variables $x_0, x_1, \ldots, x_{n-1}$ where $x_i = x_{g_i}$. There is a natural action of $\text{Aut}(G)$ on $\mathbb{C}[x_0, \ldots, x_{n-1}]$. Let $C_1, \ldots, C_r$ be the conjugacy classes of $G$. If $C = \{g_{i_1}, g_{i_2}, \ldots, g_{i_u}\}$ is a conjugacy class, then let $\overline{C} = x_{i_1} + x_{i_2} + \cdots + x_{i_u}$. Let $\rho_G$ be the representation of $\text{Aut}(G)$ on $\mathbb{C}[x_0, \ldots, x_{n-1}]/\langle C_1, \ldots, \overline{C}_r \rangle$ and let $\chi_G$ be the character afforded by $\rho_G$. If $G$ is a dihedral group of the form $D_{2p}$, $D_{4p}$ or $D_{2p^2}$, with $p$ an odd prime, I show how $\chi_G$ splits into irreducible constituents. I also show how the module $\mathbb{C}[x_1, \ldots, x_n]/\langle \overline{C}_1, \ldots, \overline{C}_r \rangle$ decomposes into irreducible submodules. This problem is motivated by results of Humphries [2] relating to random walks on groups and the group determinant.

Keywords: Group matrix, finite group, dihedral group
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Chapter 1. Introduction

Representation theory is the branch of mathematics that involves looking at groups as groups of matrices. Since a lot is known about matrices, this can tell us a lot of information about groups. Unfortunately, matrices include almost too much information. The trace of the representation matrix, or the character, is a lot more compact and still carries a lot of information about the group. For a finite group \( G = \{ g_0 = 1, g_1, \ldots, g_{n-1} \} \), we can associate independent variables \( x_0, x_1, \ldots, x_{n-1} \) to each element of \( G \) such that \( x_i = x_{g_i} \). The group matrix \( M_G \) is the \( n \times n \) matrix where the \((i,j)\)-entry is defined to be \( x_k \) where \( g_k = g_i g_j^{-1} \). The group determinant \( \theta_G \) is just the determinant of the group matrix. From early in the history of representation theory, the factorization of the group determinant was sought. Let \( V \) be the vector space defined by \( V = \mathbb{C}[x_0, \ldots, x_{n-1}] \). Since \( x_i \) corresponds to \( g_i \), \( \text{Aut}(G) \) acts on \( V \). Let \( C_1, \ldots, C_r \) be the conjugacy classes of \( G \) and for any \( C \subseteq G \) let \( \overline{C} = \sum_{g_i \in C} x_i \). Each \( \overline{C}_k \) determines a hyperplane (a subspace of dimension \( n - 1 \)) \( H(C_k) \subseteq \mathbb{C}^n \). Let \( U_G \) be the intersection of all such hyperplanes. Let \( \rho_G \) be the representation of \( \text{Aut}(G) \) acting on \( V/\langle \overline{C}_1, \ldots, \overline{C}_r \rangle \). Let \( \chi_G \) be the character afforded by \( \rho_G \). This is the same as the representation of \( \text{Aut}(G) \) at \( 0 \in U_G = \cap_{k=1}^r H(C_k) \). It has been shown that \( \rho_G \) is an irreducible representation if and only if \( G = Q_8 \times A \), where \( A \) is a 2-group \([2]\). The question arises: how does \( \rho \) decompose when it is not irreducible? In this paper, I look at a few select cases. Let \( p \) be an odd prime. If \( G \) is a dihedral group of the form \( D_{2p} \), I show that \( \chi_G \) has \( \frac{p+1}{2} \) distinct irreducible constituents, and more specifically \( \frac{p-1}{2} \) linear constituents and one irreducible constituent of degree \( p - 1 \). If \( G \) is a dihedral group of the form \( D_{4p} \), I show that \( \chi_G \) has \( \frac{p+3}{2} \) distinct irreducible constituents, and more specifically \( \frac{p-1}{2} \) linear constituents and two constituents of degree \( p - 1 \). I also show that if \( G = D_{2p^2} \), then \( \chi_G \) breaks into \( \frac{p^2+p}{2} + 2 \) distinct irreducible constituents. I also prove that the trivial character cannot be a constituent of \( \chi_G \).
Chapter 2. Preliminaries

2.1 Representation and Character Theory

Let $G$ be a group.

**Definition 2.1.** A representation of $G$ over $\mathbb{C}$ is a homomorphism $\rho$ from $G$ to $GL(n, \mathbb{C})$, for some $n$. The degree of $\rho$ is the integer $n$.

**Definition 2.2.** Let $V$ be an $\mathbb{C}G$-module, and let $\mathcal{B}$ be a basis of $V$. For each $g \in G$, let $[g]_{\mathcal{B}}$ denote the matrix of the endomorphism $v \mapsto vg$ of $V$, relative to the basis $\mathcal{B}$.

**Theorem 2.3.** [1, pg. 40] (1) If $\rho : G \rightarrow GL(n, \mathbb{C})$ is a representation of $G$ over $\mathbb{C}$ and $V = \mathbb{C}^n$, then $V$ becomes an $\mathbb{C}G$-module if we define the multiplication $vg$ by

$$vg = v(g\rho) \text{ (} v \in V, g \in G\text{).}$$

Moreover, there is a basis $\mathcal{B}$ of $V$ such that

$$g\rho = [g]_{\mathcal{B}} \text{ for all } g \in G.$$  

(2) Assume that $V$ is an $\mathbb{C}G$-module and let $\mathcal{B}$ be a basis of $V$. Then the function

$$g \rightarrow [g]_{\mathcal{B}} \text{ (} g \in G\text{)}$$

is a representation of $G$ over $\mathbb{C}$.

**Definition 2.4.** Suppose that $V$ is a finite-dimensional $\mathbb{C}G$-module with a basis $\mathcal{B}$. Then the character of $V$ is the function $\chi : G \rightarrow \mathbb{C}$ defined by

$$\chi(g) = \text{tr}[g]_{\mathcal{B}} \text{ (} g \in G\text{).}$$
**Definition 2.5.** The character of a representation $\rho : G \to GL(n, \mathbb{C})$ is the character $\chi$ of the corresponding $\mathbb{C}G$-module $\mathbb{C}^n$ where

$$\chi(g) = \text{tr}(g\rho) \ (g \in G).$$

**Definition 2.6.** A $\mathbb{C}G$-module $V$ is said to be irreducible if it is non-zero and it has no $\mathbb{C}G$-submodule apart from $\{0\}$ and $V$.

**Definition 2.7.** We say that $\chi$ is a character of $G$ if $\chi$ is the character of some $\mathbb{C}G$-module. Further, $\chi$ is an irreducible character of $G$ if $\chi$ is the character of an irreducible $\mathbb{C}G$-module.

**Theorem 2.8.** [1, pg. 119] If $x$ and $y$ are conjugate elements of the group $G$, then

$$\chi(x) = \chi(y)$$

for all characters $\chi$ of $G$.

**Definition 2.9.** Suppose that $\varphi$ and $\theta$ are functions from $G$ to $\mathbb{C}$. Define the inner product

$$\langle \theta, \varphi \rangle = \frac{1}{|G|} \sum_{g \in G} \theta(g)\overline{\varphi(g)}.$$

**Definition 2.10.** Suppose that $\psi$ is a character of $G$, and that $\chi$ is an irreducible character of $G$. We say that $\chi$ is a constituent of $\psi$ if $\langle \psi, \chi \rangle \neq 0$. Thus, the constituents of $\psi$ are the irreducible characters $\chi_i$ of $G$ for which the integer $d_i$ in the expression $\psi = d_1\chi_1 + \ldots + d_k\chi_k$ is non-zero.

**Theorem 2.11.** [1, pg. 143] Suppose that $V$ and $W$ are $\mathbb{C}G$-modules, with characters $\chi$ and $\psi$, respectively. Then $V$ and $W$ are isomorphic if and only if $\chi = \psi$.

**Theorem 2.12.** [1, pg. 145] Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of $G$. Then $\chi_1, \ldots, \chi_k$ are linearly independent vectors in the vector space of all functions from $G$ to
Also, a character $\chi_1, \ldots, \chi_k$ form an orthonormal set; that is $\langle \chi_i, \chi_j \rangle = \delta_{ij}$ for all $i, j$. Also, $\psi$ is irreducible if and only if $\langle \psi, \psi \rangle = 1$.

**Lemma 2.13.** [1, pg. 128,152] Let $\text{Irr}(G)$ denote the set of irreducible characters of a finite group $G$. Then $|\text{Irr}(G)|$ equals the number of conjugacy classes of $G$ and

$$\sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = |G|.$$ 

**Lemma 2.14.** [1, pg. 161] Row orthogonality relation: Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of a finite group $G$. The following relations hold for any $i, j \in \{1, \ldots, k\}$ :

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g)\chi_j(g^{-1}) = \delta_{ij}.$$ 

**Lemma 2.15.** [1, pg. 161] Column orthogonality relation: Let $G$ be finite group and let $g, h \in G$. Then

$$\sum_{\chi \in \text{Irr}(G)} \chi(g)\overline{\chi(h)} = 0,$$

if $g$ is not conjugate to $h$ in $G$. Otherwise, the sum is equal to $|C_G(g)|$, where $C_G(g)$ is the centralizer subgroup.

**Definition 2.16.** A linear character of a finite group is a character of degree one.

**Definition 2.17.** If $N \triangleleft G$ and $\bar{\chi}$ is a character of $G/N$, then the character $\chi$ of $G$ which is given by

$$\chi(g) = \bar{\chi}(Ng) \ (g \in G)$$

is called the lift of $\bar{\chi}$ to $G$.

Recall that the derived subgroup of a group $G$ is the subgroup generated by all commutators $a^{-1}b^{-1}ab$ where $a, b \in G$.

**Lemma 2.18.** [1, pg. 174] Let $G$ be a finite group and let $G'$ be the derived subgroup of $G$. Then the linear characters of $G$ are precisely the lifts to $G$ of the irreducible characters of
In particular, the number of distinct linear characters of $G$ is equal to $|G/G'|$, and so divides $|G|$. 

**Definition 2.19.** Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of $G$ and let $g_1, \ldots, g_k$ be representatives of the conjugacy classes of $G$. The $k \times k$ matrix whose $ij$-entry is $\chi_i(g_j)$ (for all $i, j$ with $1 \leq i \leq k$, $1 \leq j \leq k$) is called the character table of $G$.

**Theorem 2.20.** [1, pg. 246] If $\chi$ is a character of $G$ and $g \in G$, then $\chi(g)$ is an algebraic integer.

## 2.2 Algebra and Dihedral Groups

Let $G = \{g_0 = 1, g_1, \ldots, g_{n-1}\}$ be a finite group of order $n$ and let $R = \mathbb{C}[x_0, x_1, \ldots, x_{n-1}]$ be a polynomial ring. We think of $x_i$ as a coordinate function on $\mathbb{C}^n$. Here $x_i$ corresponds to $g_i, 0 \leq i \leq n - 1$, and in general we write $x_i = x_{g_i}$.

**Definition 2.21.** The group matrix $M_G$ of $G$ is the matrix whose rows and columns are indexed by the group elements, where the $(i,j)$ entry is $x_k$ if $g_i g_j^{-1} = g_k$. The group determinant is defined to be $\Theta_G = \det M_G$.

Let $C_1 = \{1\}, C_2, \ldots, C_r$ be the conjugacy classes of $G$. Let $V$ be the complex vector space spanned by $x_0, \ldots, x_{n-1}$. To any conjugacy class $C$ of $G$ we associate the element $\overline{C} = \sum_{g \in C} x_g$.

Let $\text{Aut}(G)$ be the group of automorphisms of $G$. Since $\text{Aut}(G)$ clearly acts on $G$ and for each element $g_i$ we have the associated variable $x_i$, then $\text{Aut}(G)$ acts on $V$ in the natural way: for $g \in G, \alpha \in \text{Aut}(G)$ we have $(x_g)\alpha = x_{(g)\alpha}$. Note that we are acting on the $x_i$ from the right.

If $C$ is a conjugacy class, then so is $(C)\alpha$ for $\alpha \in \text{Aut}(G)$. Thus $\text{Aut}(G)$ acts on the set of conjugacy classes of $G$ and so acts on $\{\overline{C_1}, \overline{C_2}, \ldots, \overline{C_r}\}$. We see that $\{\overline{C_1}, \overline{C_2}, \ldots, \overline{C_r}\}$ is
Aut($G$)-invariant. Define $W$ as the quotient

$$W = V/\text{Span}(\overline{C_1}, \overline{C_2}, \ldots, \overline{C_r}).$$

For each conjugacy class $C$ of $G$, we can take one element $g \in C$ and let the associated $x_g$ be a \textit{non-basis element} of $W$. We can construct a basis of $W$ by taking the remaining $n - r$ elements of $G$ and their associated $x$ variables. Then Aut($G$) will also act on $W$. This determines a representation $\rho_G$ of Aut($G$) of degree $n - r$. Here,

$$\rho_G : \text{Aut}(G) \to GL(n - r, \mathbb{C}).$$

Let $\chi_G$ be the character afforded by the representation $\rho_G$.

Each row of the matrix $\rho_G(\alpha)$ is determined by where $\alpha$ sends a basis element of $W$. When calculating the matrix $\rho_G(\alpha)$ representing the automorphism action $\alpha$, only the diagonal entries of the matrix $\rho_G(\alpha)$ matter in determining the character $\chi_G$. For an element $x \in W$, $\alpha(x)$ can be $x$ itself or the non-basis element of its conjugacy class, or another element. Consider the $k$th row of the matrix $\rho_G(\alpha)$ determined by $\alpha(x)$. If $x$ is sent to $x$, there will be a 1 in the $(k, k)$ entry, which is part of the diagonal of the matrix. If $x$ is sent to the non-basis element $b$ of its conjugacy class, we see that $\alpha(x) = b = -\sum_{c \in C, c \neq b} c$. Thus there will be a $-1$ in the $(k, k)$ matrix entry. For all other elements that $x$ is sent to, there will be a 0 in the $(k, k)$ entry since no other elements are expressed using $x$. The sum of the diagonal entries results in $\chi_G(\alpha)$.

\textbf{Example 2.22.} Let $G = S_3$. Let $g_0 = 1, g_1 = (123), g_2 = (132), g_3 = (12), g_4 = (13), g_5 = (23)$. Note that Aut($S_3$) $= S_3$. Let $I_g$ be the action of conjugation on $S_3$ by $g$. The conjugacy classes of $S_3$ are \{1\}, \{(123), (132)\}, \{(12), (13), (23)\}. So $W = V/\text{Span}(x_0, x_1 + x_2, x_3 + x_4 + x_5)$ and generators for $W$ can be taken to be $x_2, x_4, x_5$. Relative to this set of generators,
the action of $I_{(13)}$ would be:

$$(x_2)I_{(13)} = -x_2; \quad (x_4)I_{(13)} = x_4; \quad (x_5)I_{(13)} = -x_4 - x_5.$$ 

So the matrix representing the action $I_{(13)}$ would be

$$
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & -1
\end{pmatrix}
$$

and $\chi_{S_3}(I_{13}) = -1$.

**Definition 2.23.** The dihedral group $D_{2n}$ is the finite group with the presentation

$$D_{2n} = \langle r, s \mid r^n = s^2 = 1, srs^{-1} = r^{-1} \rangle.$$ 

**Lemma 2.24.** [1, pg. 182] Let $n \geq 3$ be an odd number. Let $G = D_{2n}$ be the dihedral group of order $2n$ presented as $\langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$. Then the conjugacy classes of $D_{2n}$ are

$$\{1\}, \{r^1, r^{-1}\}, \ldots, \{r^{n-1}, r^{1-n}\}, \{sr^b : 0 \leq b \leq n - 1\}.$$ 

There are $\frac{n+3}{2}$ conjugacy classes of $D_{2n}$.

**Lemma 2.25.** [1, pg. 183] Let $n \geq 4$ be an even number with $n = 2m$. Let $G = D_{2n}$ be the dihedral group of order $2n$ presented as $\langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$. Then the conjugacy classes of $D_{2n}$ are

$$\{1\}, \{r^m\}, \{r^1, r^{-1}\}, \ldots, \{r^{m-1}, r^{1-m}\}, \{sr^b : b \text{ even}\}, \{sr^b : b \text{ odd}\}.$$ 

There are $m + 3$ conjugacy classes of $D_{2n}$.

**Lemma 2.26.** The automorphism group of $D_{2n}$ consists of $n\phi(n)$ elements.
Proof. An automorphism of $D_{2n}$ is completely determined by where the generators $r, s$ are sent. In $D_{2n}$, $r$ has order $n$. In order to keep the structure of $D_{2n}$ intact, any automorphism must send $r$ to another element of order $n$. The only elements of $D_{2n}$ that have order $n$ are powers of $r$, $r^k$, such that $(k, n) = 1$. There are $\phi(n)$ of these. The element $s$ has order 2. All elements of the form $sr^m$ have order 2, since
\[(sr^m)(sr^m) = (sr^m s)r^m = r^{-m}r^m = 1.\]
The element $s$ cannot be sent to a power of $r$ by the automorphism because some power of $r$ is already being sent there. Hence, $s$ must be sent to $sr^m$ with $0 \leq m < n$. We see that $(r^k)^n = 1$ and $(sr^m)^2 = 1$. Lastly
\[sr^m(r^k)r^{-m}s^{-1} = sr^ks^{-1} = (r^k)^{-1}.\]
Thus the relations in the presentation for $D_{2n}$ are satisfied. Since $\langle r^k, sr^m \rangle = D_{2n}$, any such choices of $k$ and $m$ determine an automorphism. Thus there are $n\phi(n)$ different automorphisms. \hfill \Box

For ease of notation, the element of $\text{Aut}(D_{2n})$ that sends $r$ to $r^j$ and $s$ to $sr^k$ will be referred to from hereafter as $(j, k)$.

Lemma 2.27. Let $n > 2$ be an integer. Let $\text{AGL}(1, n)$ denote the subgroup of $2 \times 2$ matrices of the form
\[
\begin{pmatrix}
a & b \\
0 & 1
\end{pmatrix}
\]
where $a \in (\mathbb{Z}/n\mathbb{Z})^*$ and $b \in \mathbb{Z}/n\mathbb{Z}$. Then $\text{Aut}(D_{2n})$ is isomorphic $\text{AGL}(1, n)$. The element $(a, b)$ corresponds to
\[
\begin{pmatrix}
a & b \\
0 & 1
\end{pmatrix}.
\]

Proof. Let $f : \text{Aut}(D_{2n}) \to \text{AGL}(1, n)$ be defined by $f((a, b)) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. Consider the
composition of automorphisms \((a, b) \circ (c, d)\). We see that

\[(a, b) \circ (c, d)(r) = (a, b)(r^c) = r^{ac}\]

and

\[(a, b) \circ (c, d)(s) = (a, b)(sr^d) = sr^{b+ad}.\]

Thus \((a, b) \circ (c, d) = (ac, b + ad)\). Now

\[f((a, b) \circ (c, d)) = f((ac, b + ad)) = \begin{pmatrix} ac & b + ad \\ 0 & 1 \end{pmatrix}\]

\[= \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} = f((a, b))f((c, d)).\]

Thus \(f\) is a homomorphism. It is clear that \(f\) is a bijection, so \(f\) is an isomorphism.

\[\square\]

2.3 Decomposing \(\mathbb{C}G\)-modules

**Lemma 2.28.** [1, pg.146] If \(\chi\) is an irreducible character of \(G\), and \(V\) is any \(\mathbb{C}G\)-module, then

\[V \left( \sum_{g \in G} \chi(g^{-1})g \right)\]

is equal to the sum of those \(\mathbb{C}G\)-submodules of \(V\) which have character \(\chi\) (where for \(r \in \mathbb{C}G\), we define \(Vr = \{vr : v \in V\}\)).

When the irreducible characters of \(G\) are known, we can find how \(V\) decomposes into irreducible \(\mathbb{C}G\)-modules. We have a procedure to do this given in [1, pg. 147].

1) Choose a basis \(v_1, \ldots, v_n\) of \(V\).

2) For each irreducible character \(\chi\) of \(G\), calculate the vectors \(v_i(\sum_{g \in G} \chi(g^{-1})g)\) for \(1 \leq i \leq n\), and let \(C_\chi\) be the subspace of \(V\) spanned by these vectors.
3) Then $V$ is a direct sum of the $\mathbb{C}G$-modules $V_\chi$ as $\chi$ runs over the irreducible characters of $G$. The character of $V_\chi$ is an integer multiple of $\chi$.

This gives us a way to decompose $W$ into irreducible submodules. The number of irreducible constituents is the same number of irreducible submodules of $W$. 
Chapter 3. Origins of the Problem

This problem arises from a paper by Stephen Humphries, *Generalized cogrowth series, random walks, and the group determinant* [2].

The group matrix, which is linked to random walks, obviously determines the group $G$ as it is essentially a multiplication table of $G$. From the group matrix $M_G$, the *group determinant* is $\Theta_G = \det M_G$. The group determinant determines $G$ as well [4].

If $G$ is finite, the group matrix $M_G$ can be block diagonalized: there is a matrix $Q$ such that

$$Q^{-1}M_GQ = \text{diag}(d_1 \cdot D_1, d_2 \cdot D_2, \ldots, d_r \cdot D_r),$$

(3.1)

where $r$ is the number of conjugacy classes, $D_i$ is a square $d_i \times d_i$ matrix over $\mathbb{C}[x_0, x_1, \ldots, x_{n-1}]$, and $d_i \cdot D_i$ indicates that $D_i$ occurs $d_i$ times in this diagonal decomposition. Further $D_i = \sum_{j=1}^n x_j D_{i,j}$, where $\rho_i : g_j \mapsto D_{i,j}$ is an irreducible representation of $G$ [2].

Let $I_G$ denote the ideal of $\mathbb{C}[x_0, x_1, \ldots, x_{n-1}]$ generated by $C_{1G,k}, k \geq 1$ where $C_{1G,k}$ is a polynomial in $x_0, \ldots, x_{n-1}$ of degree $k$. The polynomials $C_{1G,k}$ help determine the random walk on $G$. It is shown in [2] that $\mathcal{C}$ is in the radical of $I_G$ and the ideal $\langle \mathcal{C}_1, \ldots, \mathcal{C}_r \rangle$ is $\text{Aut}(G)$-invariant. Let $V_G$ denote the variety determined by the ideal $I_G$. A *hyperplane* is a codimension-1 vector subspace. Each $\mathcal{C}_k$ determines a hyperplane $H(C_k) \subset \mathbb{C}^n$. Let the intersection of all $r$ of these hyperplanes be

$$U_G = \cap_{k=1}^r H(C_k).$$

The variety $V_G$ is contained in the $U_G$. The representation $\rho_G$, which is the representation of $\text{Aut}(G)$ on $W$, is the same as the representation of $\text{Aut}(G)$ at $0 \in U_G$ [2]. The decomposition of $U_G$ into irreducible $\text{Aut}(G)$-modules gives information about $V_G$.

A condition is given for when $\rho_G$ is irreducible:

**Theorem 3.1.** [2] Let $G$ be a finite non-abelian group. Then $\rho_G$ is irreducible if and only if $G \cong Q_8 \times A$, where $A$ is an elementary abelian 2-group.
Some of the most basic non-abelian groups are the dihedral groups. We know that $\rho_G$ will not be irreducible when $G$ is a dihedral group. We will describe how $\rho_G$ decomposes into irreducible representations.
In the following chapters $p$ will be an odd prime. Recall the dihedral group

$$D_{2p} = \langle r, s | r^p = s^2 = 1, sr s^{-1} = r^{-1} \rangle.$$ 

**Lemma 4.1.** The automorphism group of $D_{2p}$ has $p$ conjugacy classes.

**Proof.** Note that $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1}$ is $\begin{pmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{pmatrix}$. To find the conjugacy class of an element of $\text{Aut}(D_{2p})$, we conjugate:

$$\begin{pmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & a^{-1}(bx + y - b) \\ 0 & 1 \end{pmatrix}.$$ 

If $x = 1$, we get $\begin{pmatrix} 1 & a^{-1}y \\ 0 & 1 \end{pmatrix}$. When $y = 0$, we see that the identity map is only conjugate to itself, as expected. When $y \neq 0$, $a^{-1}y$ can assume every value of $\mathbb{Z}_p$ except for 0. Thus $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ with $0 < y < p$ is a conjugacy class of $p - 1$ elements. If $x \neq 1$ and $y = 0$, we get $\begin{pmatrix} x & a^{-1}b(x - 1) \\ 0 & 1 \end{pmatrix}$ with $a^{-1}b(x - 1)$ nonzero and $a^{-1}b(x - 1)$ can, in fact, assume every value of $\mathbb{Z}_p$. For each $x \neq 1$, there is a conjugacy class of $p$ elements, the set of all automorphisms that send $r$ to $r^x$: $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $0 \leq y < p$. There are $p - 2$ of these conjugacy classes, one for each $x$, $1 < x < p$. 

**Lemma 4.2.** Let $n > 1$ be an odd positive integer. The derived subgroup of $\text{Aut}(D_{2n})$ is the set of automorphisms that send $r$ to $r$.

**Proof.** We will denote the derived subgroup of $G = \text{Aut}(D_{2n})$ as $G'$ for ease of notation. Then $G'$ is generated by all elements of the form $x^{-1}y^{-1}xy$ with $x, y \in \text{Aut}(D_{2n})$. Let
\[
\begin{pmatrix}
  a & b \\
  0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
  x & y \\
  0 & 1
\end{pmatrix} \in D_{2n}. \quad \text{We have}
\]

\[
\begin{pmatrix}
  a^{-1} & -ba^{-1} \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  x^{-1} & -y^{-1} \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  a & b \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  x & y \\
  0 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  1 & x^{-1}y + ba^{-1}x^{-1} - a^{-1}x^{-1}y - a^{-1}b \\
  0 & 1
\end{pmatrix}.
\]

If \(a = 1\) and \(x = 2^{-1}\), the matrix simplifies to \(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\). Thus \(G' = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z}_n \right\} \).

From Lemma 4.2 and Lemma 2.18, \(|\text{Aut}(D_{2p}) : G'| = p - 1\) and so \(\text{Aut}(D_{2p})\) has \(p - 1\) linear characters. Since \(\text{Aut}(D_{2p})\) has \(p\) conjugacy classes, there is only one irreducible nonlinear character. From Lemma 2.13, we have

\[
(p - 1)(1)^2 + x^2 = p(p - 1),
\]

where \(x\) is the degree of the non-linear character. This implies that \(x = p - 1\) and the last irreducible character has degree \(p - 1\).

**Lemma 4.3.** \(\text{Aut}(D_{2p})/G' \cong \mathbb{Z}/(p - 1)\mathbb{Z}\).

**Proof.** From Lemma 4.1 and Lemma 4.2, it follows that \(\text{Aut}(D_{2p})/G' \cong \mathbb{Z}^\times_p\), which is a cyclic group of order \(p - 1\), since \(p\) is an odd prime. \(\square\)

The character table of \(\mathbb{Z}/(p - 1)\mathbb{Z}\) is straightforward. All irreducible characters are linear, since this is an abelian group. Let \(a\) be the multiplicative generator of \(\mathbb{Z}/(p - 1)\mathbb{Z}\) and put \(\omega = e^{2\pi i/(p - 1)}\). The \(p - 1\) irreducible representations of \(\mathbb{Z}/(p - 1)\mathbb{Z}\) over \(\mathbb{C}\) are
\[ \rho_{\omega} \quad (0 \leq j \leq p - 2), \] where

\[ (a^k)\rho_{\omega_j} = (\omega^{jk}) \quad (0 \leq k \leq p - 2). \]

Note that if \( k = \frac{p-1}{2} \), then \( (a^{p-1})\rho_{\omega_j} = (\omega^{j\frac{p-1}{2}}) = (-1)^j \).

The character table of \( \mathbb{Z}/(p - 1)\mathbb{Z} \) looks like

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>( r^k ) (1 &lt; k &lt; p - 1)</th>
<th>( \chi_{j+1} ) (0 &lt; j &lt; p - 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi_1 )</td>
<td>1</td>
<td>1</td>
<td>( \omega^{jk} )</td>
</tr>
</tbody>
</table>

By the column orthogonality relation,

\[ 0 = \sum_{\chi \in \text{Irr}(G)} \chi(r^k)\overline{\chi(1)} = \sum_{\chi \in \text{Irr}(G)} \chi(r^k) = \sum_{0 \leq j \leq p - 2} \omega^{jk}. \]

We have defined the characters \( \chi_i, 1 \leq i \leq p - 1 \), above. Let \( \chi_p \) denote the unique non-linear character of \( \text{Aut}(D_{2p}) \).

Since the identity automorphism and the automorphisms that send \( r \) to \( r \) are in \( G' \), they are also in the kernel of every linear irreducible character. Let \( a \) be a generator for \( \mathbb{Z}/(p - 1)\mathbb{Z} \cong \text{Aut}(D_{2p})/G' \). Here \((p - 1,0)\) is in \( a^{\frac{p-1}{2}}G' \), so \( \chi^2((p - 1,0)) = 1 \) for linear irreducible \( \chi \). Thus \( \chi((p - 1,0)) = 1 \) or \( \chi((p - 1,0)) = -1 \).

Let \( g = (1,1), h = (1,0) \in \text{Aut}(D_{2p}) \). By the column orthogonality relation,

\[ \sum_{\chi \in \text{Irr}(G)} \chi((1,1))\overline{\chi((1,0))} = 0, \]

since \((1,1)\) and \((1,0)\) are not conjugate. Furthermore,

\[ 0 = \sum_{\chi \in \text{Irr}(G)} \chi((1,1))\overline{\chi((1,0))} \]
\[
\begin{align*}
&= \sum_{1 \leq i \leq p-1} \chi_i((1,1))\overline{\chi_i(1)} + \chi_p((1,1))\chi_p((1,0)) \\
&= p - 1 + (p - 1)\chi_p((1,1)).
\end{align*}
\]

This implies that \(\chi_p((1,1)) = -1\). Employing the column orthogonality relation with \(n \neq 1, 0\), we have

\[
0 = \sum_{\chi \in \text{Irr}(G)} \chi((n,0))\overline{\chi((1,0))} = \sum_{0 \leq j \leq p-2} \omega^{jn} + (p - 1)\chi_p((n,0))
\]

which implies \(\chi_p((n,0)) = 0, n \neq 1\). Our character table for \(\text{Aut}(D_{2p})\) looks like:

<table>
<thead>
<tr>
<th>(\chi_{2i+1}) ((0 \leq i \leq \frac{p-3}{2}))</th>
<th>(1,0)</th>
<th>(1,1)</th>
<th>(2,0)</th>
<th>...</th>
<th>(p-1,0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\chi_{2i}) ((1 \leq i \leq \frac{p-1}{2}))</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\chi_p)</td>
<td>(p-1)</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

For \(G = D_{2p}\), let \(g_n = r^n\) (\(0 \leq n \leq p-1\)) and \(g_{p+n} = sr^n\) (\(0 \leq n \leq p-1\)). Let \(R = \mathbb{C}[x_0, x_1, \ldots, x_{2p-1}]\) be a polynomial ring. The \(x_i\) act as coordinate functions on \(\mathbb{C}^{2p}\). Then \(x_i\) corresponds to \(g_i\). Let \(V\) be the vector space spanned by \(x_0, \ldots, x_{2p-1}\). Let \(C_1 = \{1\}, C_2, \ldots, C_{\frac{p+3}{2}}\) be the conjugacy classes of \(G\). Let \(W_{2p} = V/\text{Span}(\overline{C}_1, \ldots, \overline{C}_{\frac{p+3}{2}})\). We will choose a basis of \(W_{2p}\) to be the elements \(r^n\) (\(1 \leq n \leq \frac{p-1}{2}\)) and \(sr^n\) (\(1 \leq n \leq p-1\)). Now \(\text{Aut}(D_{2p})\) acts on the set \(\{\overline{C}_1, \overline{C}_2, \ldots, \overline{C}_{\frac{p+3}{2}}\}\) and so acts on \(W_{2p}\). This determines the representation \(\rho_{D_{2p}}\) of degree \(2p - \frac{p+3}{2} = \frac{3p-3}{2}\). Let \(\chi_{D_{2p}}\) be the character afforded by the representation \(\rho_{D_{2p}}\).

**Example 4.4.** Let \(G = D_{10}\).

The character table of \(\text{Aut}(D_{10})\) is given in Table 4.1.

We will number the elements of \(G\) as

\[
g_0 = 1, g_1 = r, g_2 = r^2, g_3 = r^3, g_4 = r^4, g_5 = s, g_6 = sr, g_7 = sr^2, g_8 = sr^3, g_9 = sr^4.
\]
Table 4.1: Character table of \( \text{Aut}(D_{10}) \)

<table>
<thead>
<tr>
<th></th>
<th>(1, 0)</th>
<th>(1, 1)</th>
<th>(2, 0)</th>
<th>(3, 0)</th>
<th>(4, 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_2 )</td>
<td>1</td>
<td>1</td>
<td>( i )</td>
<td>( -i )</td>
<td>(-1)</td>
</tr>
<tr>
<td>( \chi_3 )</td>
<td>1</td>
<td>1</td>
<td>(-1)</td>
<td>(-1)</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_4 )</td>
<td>1</td>
<td>1</td>
<td>(-i)</td>
<td>( i )</td>
<td>(-1)</td>
</tr>
<tr>
<td>( \chi_5 )</td>
<td>4</td>
<td>(-1)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The conjugacy classes of \( D_{10} \) are \{1\}, \{r, r^4\}, \{r^2, r^3\}, \{s, sr, sr^2, sr^3, sr^4\}. The basis that we choose for \( W_{10} \) is \( x_1, x_2, x_6, x_7, x_8, x_9 \).

For the representation \( \rho_G \), we have

\[
\rho_G((1, 0)) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

since \((1, 0)\) acts trivially on each basis element. Then the character \( \chi_G((1, 0)) = 6 \).

Consider how \((1, 1)\) acts on the basis elements of \( W_{10} \):

\[
(1, 1)(x_1) = x_1, (1, 1)(x_2) = x_2, (1, 1)(x_6) = x_7, (1, 1)(x_7) = x_8, (1, 1)(x_8) = x_9, (1, 1)(x_9) = -x_6 - x_7 - x_8 - x_9
\]

Thus

\[
\rho_G((1, 1)) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & -1 & -1 & -1 \\
\end{pmatrix}
\]
and for the character we have $\chi_G((1,1)) = 1$.

Consider how $(2,0)$ acts on the basis elements of $W_{10}$:

$$(2,0)(x_1) = x_2, (2,0)(x_2) = -x_1, (2,0)(x_6) = x_7, (2,0)(x_7) = x_9, (2,0)(x_8) = x_6, (2,0)(x_9) = x_8.$$ 

For the representation we have

$$\rho_G((2,0)) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}$$

and the character value is $\chi_G((2,0)) = 0$.

Consider how $(3,0)$ acts on the basis elements of $W_{10}$:

$$(3,0)(x_1) = -x_2, (3,0)(x_2) = x_1, (3,0)(x_6) = x_8, (3,0)(x_7) = x_6, (3,0)(x_8) = x_9, (3,0)(x_9) = x_7.$$ 

Thus

$$\rho_G((3,0)) = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}$$

and the character value is $\chi_G((3,0)) = 0$. 

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Consider how \((4, 0)\) acts on the basis elements of \(W\):

\[
(4, 0)(x_1) = -x_1, \ (4, 0)(x_2) = -x_2, \ (3, 0)(x_6) = x_9, \ (3, 0)(x_7) = x_8, \ (3, 0)(x_8) = x_7, \ (3, 0)(x_9) = x_6.
\]

Thus

\[
\rho_G((4, 0)) = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

and the character value is \(\chi_G((4, 0)) = -2\).

Taking the inner product with the irreducible characters of \(\text{Aut}(D_{10})\), we see that

\[
\chi_G = \chi_2 + \chi_4 + \chi_5.
\]

The above is an example of the following general result:

**Theorem 4.5.** For the dihedral group \(D_{2p}\) with \(p\) an odd prime, \(\text{Aut}(D_{2p})\) has character degrees \(1\) (with multiplicity \(p - 1\)) and \(p - 1\) (with multiplicity \(1\)). The character \(\chi_{D_{2p}}\) has \(\frac{p+1}{2}\) distinct irreducible constituents. In particular, there are \(\frac{p-1}{2}\) linear constituents and 1 non-linear constituent of degree \(p - 1\) (this being the only non-linear irreducible character of \(\text{Aut}(D_{2p})\)).

**Proof.** Clearly, \(\rho_{D_{2p}}((1, 0))\) is the identity matrix since \((1, 0)\) acts trivially. There are \(2p - \frac{p+3}{2} = \frac{3(p-1)}{2}\) basis elements of \(W\), so \(\rho_{D_{2p}}((1, 0))\) is a \(\frac{3(p-1)}{2} \times \frac{3(p-1)}{2}\) identity matrix and \(\chi_{D_{2p}}((1, 0)) = \frac{3(p-1)}{2}\).

The automorphism \((1, 1)\) sends \(r^k\) to \(r^k\). Since there are \(\frac{p-1}{2}\) basis elements that are powers of \(r\), this will result in \(\frac{p-1}{2}\) ones in the diagonal. The automorphism \((1, 1)\) will send \(sr^k\) to \(sr^{k+1}\). There will be a 0 in the diagonal for each \(sr^k\) except for \(sr^{p-1}\), which will be
sent to $s$, a non-basis element, resulting in -1. We see that

$$\chi_{D_{2p}}((1, 1)) = \frac{p - 1}{2} - 1 = \frac{p - 3}{2}. $$

Consider the automorphism $(j, 0)$ with $j \neq 1, p - 1$. We see that $(r^k)(j, 0) = r^{jk}$. If $jk \equiv k \mod p$, then $p|k(j - 1)$ so that either $k = p$ or $j - 1 = p$. Since $k \neq 0, p$ and $j \neq 1$, this cannot happen. If $jk \equiv -k \mod p$, then $p|k(j + 1)$. Since $j \neq p - 1$, this also cannot happen. So $(j, 0)$ will send $r^k$ to another power of $r$ that is not $r^k$ or $r^{-k}$, resulting in $\frac{p - 1}{2}$ zeroes in the diagonal. Similarly, $(sr^k)(j, 0) = sr^{jk}$. If $jk \equiv k \mod p$ also implies that $sr^k$ cannot be sent to $sr^k$. We see that $jk \equiv 0 \mod p$ is impossible, so $sr^k$ cannot be sent $s$. This results in $p - 1$ zeroes in the diagonal. Thus $\chi_{D_{2p}}((j, 0)) = 0$ for $1 < j < p - 1$.

The last automorphism is $(p - 1, 0)$. We see that

$$(r^k)(p - 1, 0) = r^{(p-1)k} = r^{-k}. $$

There will be $\frac{p - 1}{2}$ negative ones in the diagonal. Now

$$(sr^k)(p - 1, 0) = sr^{(p-1)k} = sr^{-k}. $$

No basis element will be sent to $s$. If $sr^k = sr^{-k}$, then $k \equiv -k \mod p$ and $p|2k$. Thus no basis element will be sent to itself and $\chi_{D_{2p}}((p - 1, 0)) = \frac{1-p}{2}$.

The character table of $\text{Aut}(D_{2p})$ with $\chi_{D_{2p}}$ added can be seen in Table 4.

To find the constituents of $\chi_{D_{2p}}$, we take the inner product of $\chi_{D_{2p}}$ with each irreducible

<table>
<thead>
<tr>
<th>$\chi_{2i+1}$ ($0 \leq i \leq \frac{p-3}{2}$)</th>
<th>(1, 0)</th>
<th>(1, 1)</th>
<th>(2, 0)</th>
<th>...</th>
<th>$(p-1, 0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_{2i}$ ($1 \leq i \leq \frac{p-1}{2}$)</td>
<td>1</td>
<td>1</td>
<td></td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>$\chi_p$</td>
<td>$p-1$</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_{D_{2p}}$</td>
<td>$\frac{3(p-1)}{2}$</td>
<td>$\frac{p-3}{2}$</td>
<td>0</td>
<td>0</td>
<td>$\frac{1-p}{2}$</td>
</tr>
</tbody>
</table>

Table 4.2: Character table of $\text{Aut}(D_{2p})$
character of \(\text{Aut}(D_{2p})\). If the inner product is zero, then the irreducible character is not a constituent. Otherwise, the irreducible character will be a constituent of \(\chi_{D_{2p}}\).

The inner product
\[
\langle \chi_{2i+1}, \chi_{D_{2p}} \rangle = \frac{3(p-1)}{2} + (p-1)\frac{p-3}{2} + \frac{p(1-p)}{2} = 0.
\]

For \(\chi_{2i}\) (\(1 \leq i \leq \frac{p-1}{2}\)),
\[
\langle \chi_{2i}, \chi_{D_{2p}} \rangle = \frac{3(p-1)}{2} + (p-1)\frac{p-3}{2} - \frac{p(1-p)}{2} = 1.
\]

This gives \(\chi_{D_{2p}}\) \(\frac{p-1}{2}\) irreducible constituents.

The inner product of \(\chi_{D_{2p}}\) and the only non-linear irreducible character gives us
\[
\langle \chi_p, \chi_{D_{2p}} \rangle = \frac{(p-1)3(p-1)}{2} + (p-1)(-1)\frac{p-3}{2} = 1.
\]

This results in another irreducible constituent.

Thus \(\chi_{D_{2p}}\) has \(\frac{p+1}{2}\) irreducible constituents, of which \(\frac{p-1}{2}\) are linear and one constituent which is non-linear.

\begin{corollary}
Using the method from [1, pg. 147], the module \(W_{2p}\) decomposes as
\[
W_{2p} = \bigoplus_{i=1}^{\frac{p-1}{2}} \text{Span}(\sum_{n=1}^{\frac{p-1}{2}} \chi_{2i}((n,0))x_n) \bigoplus \text{Span}(x_{p+1}, \ldots, x_{2p-1}).
\]
\end{corollary}

\begin{example}
Let \(G = D_{10}\) and \(W_{10}\) have the same basis as Example 4.4. From the same example, we know that \(\chi_G = \chi_2 + \chi_4 + \chi_5\). So \(W_{10}\) should decompose into three irreducible submodules. From Theorem 4.5, we can decompose \(W_{10}\) as
\[
W_{10} = \text{Span}(x_1 - ix_2) \bigoplus \text{Span}(x_1 + ix_2) \bigoplus \text{Span}(x_6, x_7, x_8, x_9).
\]
\end{example}

\begin{example}
Let \(G = D_{14}\). Let \(g_i = r^i\) and \(g_{i+7} = r^i s\) for \(0 \leq i \leq 6\). Let \(x_i\) correspond
to $g_i$ and let $\omega$ be a primitive 3rd root of unity. We choose the basis elements of $W_{14}$ to be $x_1, x_2, x_3, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}$. Then

$$W_{14} = \text{Span}(x_1 + x_2 - x_3) \bigoplus \text{Span}(x_1 + \omega x_2 - \omega^2 x_3) \bigoplus \text{Span}(x_1 + \omega^2 x_2 - \omega x_3) \bigoplus \text{Span}(x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}).$$
Chapter 5. The Dihedral Group $D_{4p}$

We now consider the dihedral group $D_{4p}$. Much of what we do in this case will be similar to the $D_{2p}$ case.

**Lemma 5.1.** The derived subgroup of $\text{Aut}(D_{4p})$ is the set of automorphisms that send $r$ to $r$ and $s$ to $sr^{2k}$.

**Proof.** We will denote the derived subgroup of $G = \text{Aut}(D_{4p})$ as $G'$ for ease of notation. Then $G'$ is generated by all elements of the form $x^{-1}y^{-1}xy$ with $x, y \in \text{Aut}(D_{4p})$. Now

$$
\begin{pmatrix}
a^{-1} & -ba^{-1} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x^{-1} & -yx^{-1} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
a & b \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x & y
\end{pmatrix}
$$

$$
= \begin{pmatrix}
1 & x^{-1}y + ba^{-1}x^{-1} - a^{-1}x^{-1}y - a^{-1}b \\
0 & 1
\end{pmatrix}
$$

$$
= \begin{pmatrix}
1 & x^{-1}y(1 - a^{-1}) - a^{-1}b(1 - x^{-1}) \\
0 & 1
\end{pmatrix}.
$$

Note that the $(1, 2)$ entry can only be even, since $a$ and $x$ must be odd. If $a = 1$ and $x = 2p - 1$, the matrix simplifies to

$$
\begin{pmatrix}
1 & -2b \\
0 & 1
\end{pmatrix}.
$$

Thus $G' = \left\{ \begin{pmatrix}
1 & 2b \\
0 & 1
\end{pmatrix} : b \in \mathbb{Z}_{2p} \right\}$. □

**Lemma 5.2.** The automorphism group of $D_{4p}$ has $2p$ conjugacy classes.

**Proof.** To find the conjugacy class of an element of $\text{Aut}(D_{4p})$, we conjugate:

$$
\begin{pmatrix}
a & b \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x & y \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
a^{-1} & -ba^{-1} \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
x & ay + b(1 - x) \\
0 & 1
\end{pmatrix}.
$$

If $x = 1$, we get $\begin{pmatrix}
1 & ay \\
0 & 1
\end{pmatrix}$. When $y = 0$, we see that the identity map is only conjugate
to itself, as expected. Note that since \( \gcd(a, 2p) = 1 \), \( a \) must be odd. When \( y = p \), we see that \((1, p)\) is only conjugate to itself since \( a \) is odd and \( ap \equiv p \mod 2p \). If \( \gcd(y, 2p) = 1 \), then \( \gcd(ay, 2p) = 1 \) as well. In fact, the elements \( \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \) with \( \gcd(y, 2p) = 1 \) form their own conjugacy class with \( p - 1 \) elements. If \( y = 2 \), then \( 2a \) will be can be any even positive integer less than \( 2p \). Thus \( \begin{pmatrix} 1 & 2y \\ 0 & 1 \end{pmatrix} \) with \( 0 < y < p \) compose another conjugacy class.

If \( x \neq 1 \) and \( y = 0 \), we get \( \begin{pmatrix} x & b(1-x) \\ 0 & 1 \end{pmatrix} \). Since \( 1 - x \) will always be even, \( b(1-x) \) can assume any positive even integer less than \( 2p \). For each \( x > 1 \), there is a conjugacy class of \( p \) elements, the set of all automorphisms that send \( r \) to \( r^x \) and \( s \) to \( sr^{2k} \): \( \begin{pmatrix} x & 2y \\ 0 & 1 \end{pmatrix} \) with \( 0 \leq 2y < 2p \). There are \( p - 2 \) of these conjugacy classes, one for each \( x, 1 < x < p \). Similarly, if \( x \neq 1 \) and \( y = 1 \), we get \( \begin{pmatrix} x & b(1-x) + a \\ 0 & 1 \end{pmatrix} \). Since \( b(1-x) \) is always even and \( a \) is odd, \( b(1-x) + a \) is odd and can be congruent to any positive odd number less than \( 2p \). For each \( x > 1 \), there is a conjugacy class of \( p \) elements, the set of all automorphisms that send \( r \) to \( r^x \) and \( s \) to \( sr^{2k+1} \): \( \begin{pmatrix} x & 2y \\ 0 & 1 \end{pmatrix} \) with \( 0 \leq 2y < 2p \). There are \( p - 2 \) of these conjugacy classes, one for each \( x, 1 < x < p \). Thus there are \( 4 + (p - 2) + (p - 2) = 2p \) conjugacy classes. \( \Box \)

**Lemma 5.3.** \([1, \text{pg. } 249]\) If \( \chi \) is an irreducible character of \( G \), then \( \chi(1) \) divides \( |G| \).

**Lemma 5.4.** \([1, \text{pg. } 173]\) If \( \chi \) is a linear character of \( G \), with \( G' \) being the derived subgroup, then \( G' \leq \ker \chi \).

In particular, the intersection of the kernels of all the irreducible linear characters of \( G \) is equal to the derived subgroup.

There are \( |\text{Aut}(D_{4p})/G'| = \frac{2p(p-1)}{p} = 2(p - 1) \) irreducible linear characters for \( \text{Aut}(D_{4p}) \). Since there are \( 2p \) irreducible characters, there are only \( 2 \) non-linear irreducible characters.
Let $x$ be the degree of the first and $y$ be the degree of the second. Then we have

$$2p(p - 1) = 2(p - 1) + x^2 + y^2$$

so that $x^2 + y^2 = 2(p - 1)^2$. Since $x$ and $y$ divide $2p(p - 1)$ by Lemma 5.3, we let $x = \frac{2p(p-1)}{a}$ and $y = \frac{2p(p-1)}{b}$. Then $x^2 + y^2 = 2(p - 1)^2$ and so

$$\left( \frac{2p(p-1)}{a} \right)^2 + \left( \frac{2p(p-1)}{b} \right)^2 = 2(p - 1)^2.$$ 

Getting rid of the denominators, we get

$$(2p(p - 1)b)^2 + (2p(p - 1)a)^2 = 2(p - 1)^2a^2b^2$$

which simplifies even further to

$$2p^2(a^2 + b^2) = a^2b^2.$$ 

Since 2 divides the left side, 2 divides $a$ or $b$. If 2 divides $a$ then, 4 divides the right side and so 2 divides $b$ as well. Similarly, $p$ divides the left side, so $p$ divides $a$ or $b$. If $p$ divides $a$ then, $p^2$ divides the right side and so $p$ divides $b$ as well. If $a, b > 2p$, then $x, y < p - 1$ and $x^2 + y^2 < 2(p - 1)^2$. Also, if $a > 2p$ and $b = 2p$, then $x < p - 1, y = p - 1$ and $x^2 + y^2 < 2(p - 1)^2$. So $a = 2p$ and $b = 2p$. Thus $x = p - 1$ and $y = p - 1$.

Since $(1, 2) \in G'$, we have $\chi_{j+1}((1, 2)) = 1$ for $0 < j < 2p - 2$ by Lemma 5.4.

One can show by induction on $k \geq 0$ that

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} a^k & b \sum_{i=0}^{k-1} a^i \\ 0 & 1 \end{pmatrix}.$$
Thus if \( \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \text{Aut}(D_{4p}) \), then
\[
\left( \begin{array}{cc} a & b \\ 0 & 1 \end{array} \right)^{p-1} = \left( \begin{array}{cc} a^{p-1} & b \sum_{i=0}^{p-2} a^i \\ 0 & 1 \end{array} \right).
\]

If \( x \equiv a^{p-1} \mod 2p \), then \( x \equiv a^{p-1} \mod p \) and by Fermat’s little theorem, \( x \equiv 1 \mod 2p \), where again we use the fact that \( a \) is odd. The sum \( \sum_{i=0}^{p-2} a^i \) has an even number of terms. Since \( (a, 2p) = 1 \), then each of the terms is odd. Thus \( b \sum_{i=0}^{p-2} a^i \) is even. Hence, \( \left( \begin{array}{cc} a & b \\ 0 & 1 \end{array} \right)^{p-1} \in G' \). This means that \( G/G' \) cannot be cyclic since it has no element of order \( 2(p-1) \).

We know that there exists an \( (a, b) \in \text{Aut}(D_{4p}) \) such that \( |a| = p-1 \) in \( \mathbb{Z}/p\mathbb{Z} \). Then \( (a, 0) \) has order \( p-1 \) and \( (a, 0)G' \in \text{Aut}(D_{4p})/G' \) has order \( p-1 \). Thus \( \text{Aut}(D_{4p})/G' \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z} \).

**Lemma 5.5.** [1, pg. 206] Let \( G \) and \( H \) be finite groups. Let \( \chi_1, \ldots, \chi_a \) be the distinct irreducibles of \( G \) and let \( \psi_1, \ldots, \psi_b \) be the distinct irreducibles of \( H \). Then \( G \times H \) has precisely \( ab \) distinct irreducible characters, and these are
\[
\chi_i \times \psi_j \ (1 \leq i \leq a, \ 1 \leq j \leq b).
\]

Let \( \omega \) be a \( p - 1 \)th root of unity. Let \( (a, 0), \ (0, b) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z} \) with \( |a| = 2 \) and \( |b| = p - 1 \) be generating elements of \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z} \). The character table of
\[ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z} \] looks like

<table>
<thead>
<tr>
<th></th>
<th>(1,1)</th>
<th>(1, b^k) (1 &lt; k &lt; p)</th>
<th>(a,1)</th>
<th>(a, b^k) (1 &lt; k &lt; p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\chi_1)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(\chi_{j+1}) ((0 &lt; j \leq p-2))</td>
<td>1</td>
<td>(\omega^{jk})</td>
<td>1</td>
<td>(\omega^{jk})</td>
</tr>
<tr>
<td>(\chi_p)</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>(\chi_{p+j+1}) ((0 &lt; j \leq p-2))</td>
<td>1</td>
<td>(\omega^{jk})</td>
<td>-1</td>
<td>(-\omega^{jk})</td>
</tr>
</tbody>
</table>

From Lemma 2.18, the linear characters of \(\text{Aut}(D_{4p})\) are the lifts of the irreducible characters of \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}\). The element \((1, b^{\frac{p-1}{2}})\) in \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}\) corresponds to the element \((2p - 1, 0)\) in \(\text{Aut}(D_{4p})\). Similarly, the element \((a, b^{\frac{p-1}{2}})\) corresponds to \((2p - 1, 1)\). Thus \(\chi_i((2p - 1, 0)\) equals 1 or -1 and \(\chi_i((2p - 1, 1)\) equals 1 or -1 as well. Let \(a_j\) be the power of \(-1\) corresponding to \(\chi_{j+1}\). Clearly, \(a_j\) will be even \(\frac{p-1}{2}\) times and will be odd \(\frac{p-1}{2}\) times for \(0 < j \leq p-2\).

Adding the lifts, we get the following incomplete character table for \(\text{Aut}(D_{4p})\):

<table>
<thead>
<tr>
<th></th>
<th>(1,0)</th>
<th>(1, p)</th>
<th>(1,1)</th>
<th>(1,2)</th>
<th>(k,0) (1 &lt; k &lt; 2p - 1, (k, 2p) = 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\chi_1)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(\chi_{j+1}) ((0 &lt; j \leq p-2))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>(\chi_p)</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>(\chi_{p+j+1}) ((0 &lt; j \leq p-2))</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>(\chi_{2p-1})</td>
<td>p-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\chi_{2p})</td>
<td>p-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Let $\chi_{2p-1}((1,p)) = x$ and $\chi_{2p}((1,p)) = y$. Using column orthogonality on the first two columns, we get

$$\sum_{\chi \in \text{Irr}(\text{Aut}(D_{4p}))} \chi((1,p)) \overline{\chi((1,0)}) = (p-1)(x+y) = 0.$$ 

Thus $x = -y$ and $|x| = |y|$. Using column orthogonality on the $(1,p)$ column, we get

$$2p - 2 + |x|^2 + |y|^2 = 2p(p-1).$$

Simplifying, we get

$$2(p-1) + 2|x|^2 = 2p(p-1).$$

Thus $p - 1$ divides $|x|$.

**Lemma 5.6.** [1, pg. 285] Let $g \in G$ and $\chi$ be a character of $G$. Then $|\chi(g)| \leq \chi(1)$.

**Lemma 5.7.** [1, pg. 253] Let $g$ be an element of order $n$ in $G$. Suppose that $g$ is conjugate to $g^i$ for all $i$ with $1 \leq i \leq n$ and $\gcd(i,n) = 1$. Then $\chi(g)$ is an integer for all characters $\chi$ of $G$.

The automorphism $(1,p)$ is its own inverse. By the Lemma 5.7, $\chi_{2p-1}((1,p))$ and $\chi_{2p}((1,p))$ are integers. So $p - 1$ divides $x$. Since $x \leq p - 1$, $|x| = p - 1$. Since we have not previously
distinguished \( \chi_{2p-1} \) and \( \chi_{2p} \) from each other, let \( \chi_{2p-1}(1, p) = p - 1 \) and \( \chi_{2p}(1, p) = 1 - p \).

Now let \( x = \chi_{2p-1}(1, 1) \) and \( y = \chi_{2p}(1, 1) \). Using column orthogonality on the first and third columns, we see that \( (p - 1)(x + y) = 0 \) and so \( x = -y \). Using column orthogonality on the second and third columns, we get \( 2p - 2 + (p - 1)(x - y) = 0 \). So \( x - y = 2 \). Thus \( x = 1 \) and \( y = -1 \) and we have \( \chi_{2p-1}(1, 1) = 1 \) and \( \chi_{2p}(1, 1) = -1 \).

Now let \( x = \chi_{2p-1}(1, 2) \) and \( y = \chi_{2p}(1, 2) \). Using column orthogonality on the \( (1, 2) \) column, we get

\[
2p - 2 + |x|^2 + |y|^2 = 2p.
\]

So \( |x|^2 + |y|^2 = 2 \). Using column orthogonality on the first and fourth columns, we obtain \( 2p - 2 + (p - 1)(x + y) = 0 \). So \( x + y = -2 \). Thus \( x = -1 \) and \( y = -1 \) and so \( \chi_{2p-1}(1, 2) = -1 \) and \( \chi_{2p}(1, 2) = -1 \).

Now let \( x = \chi_{2p-1}(2p - 1, 0) \) and \( y = \chi_{2p}(2p - 1, 0) \). Using column orthogonality on the \( (2p - 1, 0) \) column, we see that

\[
\sum_{\chi \in \text{Irr}(\text{Aut}(D_{4p}))} \chi((2p - 1, 0))\overline{\chi((2p - 1, 0))} = 2p - 2 + |x|^2 + |y|^2 = 2p - 2.
\]

Thus \( \chi_{2p-1}(2p - 1, 0) = 0 \) and \( \chi_{2p}(2p - 1, 0) = 0 \). Similarly, we see \( \chi_{2p-1}(2p - 1, 1) = 0 \) and \( \chi_{2p}(2p - 1, 1) = 0 \).

We will not need to calculate the character values for \( \chi_i((k, 0)) \) and \( \chi_i((k, 1)) \). We will see in the proof of the next theorem that \( \chi((n, 0)) = 0 \) and \( \chi((n, 1)) = 0 \) and so these values will not affect the inner product \( \langle \chi, \chi_{2p-1} \rangle \) or \( \langle \chi, \chi_{2p} \rangle \). These columns will be omitted from the next character table. With these calculations, we have enough of the \( \text{Aut}(D_{4p}) \) character table to proceed:
For $G = D_{4p}$, let $g_n = r^n (0 \leq n \leq 2p - 1)$ and $g_{2p+n} = sr^n (0 \leq n \leq 2p - 1)$. Let $R = \mathbb{C}[x_0, x_1, \ldots, x_{4p-1}]$ be a polynomial ring. The $x_i$ act as coordinate functions on $\mathbb{C}^{4p}$. Then $x_i$ corresponds to $g_i$. Let $V$ be the vector space spanned by $x_0, \ldots, x_{4p-1}$. Let $C_1 = \{1\}, C_2, \ldots, C_{p+3}$ be the conjugacy classes of $G$. Let $W = V/Span(\overline{C}_1, \ldots, \overline{C}_{p+3})$. We will choose a basis of $W$ to be the elements $r^n (1 \leq n \leq p - 1)$ and $sr^n (2 \leq n \leq 2p - 1)$. Now $\text{Aut}(D_{4p})$ acts on the set $\{\overline{C}_1, \overline{C}_2, \ldots, \overline{C}_{p+3}\}$ and so acts on $W$. This determines the representation $\rho_{D_{4p}}$ of degree $4p - (p + 3) = 3p - 3$. Let $\chi_{D_{4p}}$ be the character afforded by the representation $\rho_{D_{4p}}$.

**Theorem 5.8.** For the dihedral group $D_{4p}$ with $p$ an odd prime, $\text{Aut}(D_{4p})$ has character degrees $1$ (with multiplicity $2p - 2$) and $p - 1$ (with multiplicity 2). The character $\chi_{D_{4p}}$ has $\frac{p+3}{2}$ distinct irreducible constituents. In particular, there are $\frac{p-1}{2}$ linear constituents each with multiplicity 2 and two degree $p - 1$ constituents with multiplicity 1 (these being the non-linear irreducible characters of $\text{Aut}(D_{4p})$).

**Proof.** Since $W$ has $4p - (p + 3)$ basis elements, $\rho_{D_{4p}}((1, 0))$ is the $(3p - 3) \times (3p - 3)$ identity matrix. Thus $\chi_{D_{4p}}((1, 0)) = 3p - 3$.

Consider the matrix $\rho_{D_{4p}}((1, p))$. The automorphism $(1, p)$ fixes each $r^k$ for $k = 1, \ldots, p - 1$ and so contributes $p - 1$ ones in the diagonal. We also have $(sr^k)(1, p) = sr^{k+p}$. If $k$ is even, $k + p$ will be odd and cannot be congruent to 0 mod 2p. If $k$ is odd, $k + p$ will be even.
and cannot be congruent to 1 mod 2p. It is also impossible for \( k + p \equiv k \mod 2p \). Thus the rest of the diagonal will be zeroes and we have \( \chi_{D_{4p}}((1, p)) = p - 1 \).

Consider the matrix \( \rho_{D_{4p}}((1, 1)) \). The automorphism \((1, 1)\) fixes each \( r^k \) for \( k = 1, \ldots, p-1 \) and so contributes \( p - 1 \) ones in the diagonal. We also have \((sr^k)(1, 1) = sr^{k+1}\). If \( k \) is even, \( k + 1 \) will be odd and cannot be congruent to 0 mod 2p. If \( k \) is odd, \( k + 1 \) will be even and cannot be congruent to 1 mod 2p. It is also impossible for \( k + 1 \equiv k \mod 2p \). Thus the rest of the diagonal will be zeroes and we have \( \chi_{D_{4p}}((1, 1)) = p - 1 \).

Consider the matrix \( \rho_{D_{4p}}((1, 2)) \). The automorphism \((1, 2)\) fixes each \( r^k \) for \( k = 1, \ldots, p-1 \) and so contributes \( p - 1 \) ones in the diagonal. We also have \((sr^k)(1, 2) = sr^{k+2}\). If \( sr^{k+2} = s \), then \( k = 2p - 2 \), contributing a \(-1\) to the diagonal. If \( sr^{k+2} = sr \), then \( k = 2p - 1 \), contributing a \(-1\) to the diagonal. We see that \( sr^k = sr^{k+2} \) can never happen. Thus \( \chi_{D_{4p}}((1, 2)) = p - 3 \).

If \( 1 < n < 2p - 1 \), then consider the matrix for the representation \( \rho_{D_{4p}}((n, 0)) \). We have \((r^k)(n, 0) = r^{nk} \neq r^n, r^{-n}\). This will contribute \( p - 1 \) zeros to the diagonal. If \( k \) is even, then \( k = 2j \) from some \( j \) and \((sr^{2j})(n, 0) = sr^{-2nj}\). If \( sr^{2nj} = s \), then \( 2nj \equiv 0 \mod 2p \), which is impossible. If \( sr^{2nj} = sr^{2j} \), then \( 2nj \equiv 2j \mod 2p \). This implies that \( 2p|2j(n-1) \). This is not possible since \( 1 < n < 2p - 1 \) and \( n \neq p + 1 \). Let \( k \) be odd. Then \((sr^k)(n, 0) = sr^{kn}\). If \( sr^{kn} = sr \), then \( kn \equiv 1 \mod 2p \). Since \( n \) only has a unique inverse, this can only happen once, contributing one \(-1\) to the diagonal. If \( sr^{kn} = sr^k \), then \( kn \equiv k \mod 2p \) and \( p|k(n-1) \). This is only possible if \( k = p \), contributing one \(-1\) to the diagonal. Thus \( \chi_{D_{4p}}((n, 0)) = 0 \).

If \( 1 < n < 2p - 1 \), then consider the matrix \( \rho_{D_{4p}}((n, 1)) \). For \((n, 1)\) we have \((r^k)(n, 1) = r^{nk} \neq r^n, r^{-n}\). This will contribute \( p - 1 \) zeros to the diagonal. Now \((sr^k)(n, 1) = sr^{nk+1}\). If \( k \) is even, then \( sr^{nk+1} \) will have an odd power of \( r \). If \( k \) is odd, then \( sr^{nk+1} \) will have an even power of \( r \), since \( n \) is necessarily odd. So \( sr^k \) will never be sent to an element in its conjugacy class. This will result in all zeros in the diagonal and thus \( \chi_{D_{4p}}((n, 1)) = 0 \).

Consider the matrix \( \rho_{D_{4p}}((2p - 1, 0)) \). Then for \((2p - 1, 0)\) we have \((r^k)(2p - 1, 0) = r^{k(2p-1)} = r^{-k} \) and so contributes \( p - 1 \) negative ones to the diagonal. Then \((sr^k)(2p - 1, 0) =
\[\chi_{j+1} (0 < j \leq p - 2) \quad 1 \quad 1 \quad 1 \quad 1 \quad (-1)^{a_j} \quad (-1)^{a_j}\]

\[\chi_{p+j+1} (0 < j \leq p - 2) \quad 1 \quad -1 \quad -1 \quad 1 \quad (-1)^{a_j} \quad -(1)^{a_j}\]

\[\chi_{2p-1} \quad p - 1 \quad p - 1 \quad -1 \quad -1 \quad 0 \quad 0\]

\[\chi_{2p} \quad p - 1 \quad 1 - p \quad 1 \quad -1 \quad 0 \quad 0\]

\[\chi_{\text{D}_4p} \quad 3p - 3 \quad p - 1 \quad p - 1 \quad p - 3 \quad 1 - p \quad 1 - p\]

**Table 5.1: Character Table of `Aut(D_{4p})`**

\[sr^{-k}. \text{ Now } sr^{-k} \neq s \text{ if } k \neq 0. \text{ If } sr^{-k} = sr, \text{ then } -k \equiv 1 \mod 2p \text{ and so } 2p|1+k. \text{ This is only possible if } k = 2p-1. \text{ This will contribute to one } -1 \text{ to the diagonal. If } sr^{-k} = sr^k, \text{ then } -k \equiv k \mod 2p \text{ which implies } 2p|2k \text{ and } k = p. \text{ This will contribute a single } 1 \text{ to the diagonal. Thus } \chi_{\text{D}_4p}((2p - 1, 0)) = 1 - p.\]

Consider the matrix \(\rho_{\text{D}_4p}(2p - 1, 0)\). Then \((r^k)(2p - 1, 1) = r^{-k}\) contributes \(p - 1\) negative ones to the diagonal. Then \((sr^{-k})(2p - 1, 1) = sr^{-1-k}\). Since \(k\) and \(1 - k\) are of opposite parity, \(sr^k\) and \(sr^{-1-k}\) are in different conjugacy classes. This will result in \(2p - 2\) zeros in the diagonal. Thus \(\chi_{\text{D}_4p}((2p - 1, 0)) = 1 - p.\)

The character table of `Aut(D_{4p})` including \(\chi_{\text{D}_4p}\) is shown in Table 5.

In order to find the constituents of \(\chi_{\text{D}_4p}\), we must take the inner product of \(\chi_{\text{D}_4p}\) with each irreducible character.

The inner product

\[\langle \chi_1, \chi_{\text{D}_4p} \rangle = \frac{(3p - 3) + (p - 1) + (p - 1)^2 + (p - 1)(p - 3) + 2p(1 - p)}{2p(p - 1)} = 0.\]

The inner product

\[\langle \chi_{j+1}, \chi_{\text{D}_4p} \rangle = \frac{(3p - 3) + (p - 1) + (p - 1)^2 + (p - 1)(p - 3) + 2p(1 - p)(-1)^{a_j}}{2p(p - 1)}\]

simplifies to

\[\langle \chi_{j+1}, \chi_{\text{D}_4p} \rangle = 1 - (-1)^{a_j}.\]

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If \( a_j \) is even, \( \langle \chi_{j+1}, \chi_{D_4p} \rangle = 0 \). If \( a_j \) is odd, \( \langle \chi_{j+1}, \chi_{D_4p} \rangle = 2 \). This will result in \( \frac{p-1}{2} \) linear constituents each having a multiplicity of 2.

The inner product

\[
\langle \chi_{p+j+1}, \chi_{D_4p} \rangle = \frac{(p - 1)(3 - 1 - (p - 1) + (p - 3) - p((-1)^{a_i} - (-1)^{a_i})}{2p(p - 1)} = 0.
\]

The inner product

\[
\langle \chi_{2p-1}, \chi_{D_4p} \rangle = \frac{(3p - 3)(p - 1) + (p - 1)^2 - (p - 1)^2 - (p - 1)(p - 3)}{2p(p - 1)} = 1.
\]

The inner product

\[
\langle \chi_{2p}, \chi_{D_4p} \rangle = \frac{(3p - 3)(p - 1) - (p - 1)^2 + (p - 1)^2 - (p - 1)(p - 3)}{2p(p - 1)} = 1.
\]

Then \( \chi_{2p-1} \) and \( \chi_{2p} \) contribute two non-linear constituents to \( \chi_{D_4p} \) each with multiplicity 1.

\[ \square \]

**Corollary 5.9.** We see that \( W_{4p} \) can be decomposed into the following:

\[
W_{4p} = \bigoplus_{i=2}^{p+1} \text{Span} \left( \sum_{n=1}^{p-1} \chi_i(\(2n - 1, 0\))^{-1}x_{2n-1}, \sum_{m=1}^{p-1} \chi_i((p - 2m, 0)^{-1})x_{2m} \right)
\]

\[
\bigoplus \text{Span} \left( x_{2p+2} + x_{3p+2}, \ldots, x_{3p-1} + x_{4p-1}, x_{3p} - \sum_{j=1+p}^{2p-1} x_{2j} \right)
\]

\[
\bigoplus \text{Span} \left( x_{2p+2} - x_{3p+2}, \ldots, x_{3p-1} - x_{4p-1}, x_{3p} + \sum_{k=1+p}^{2p-1} x_{2k} \right).
\]

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Chapter 6. The Dihedral Group $D_{2p^2}$

We now consider the dihedral group $D_{2p^2}$. Much of what we do in this case will be similar to the $D_{2p}$ case.

By Lemma 3, there are $p^2 \varphi(p^2) = p^2(p^2 - p) = p^3(p - 1)$ elements of $\text{Aut}(D_{2p^2})$.

**Lemma 6.1.** The automorphism group $\text{Aut}(D_{2p^2})$ has $p^2 + 1$ conjugacy classes.

**Proof.** To find the conjugacy class of an element of $\text{Aut}(D_{2p^2})$, we conjugate:

$$
\begin{pmatrix}
a^{-1} & -ba^{-1} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x & y \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
a & b \\
0 & 1
\end{pmatrix} =
\begin{pmatrix}
x & a^{-1}(bx + y - b) \\
0 & 1
\end{pmatrix}.
$$

If $x = 1$, we get

$$
\begin{pmatrix}
1 & a^{-1}y \\
0 & 1
\end{pmatrix}.
$$

When $y = 0$, we see that the identity map is only conjugate to itself, as expected. If $\gcd(y, p) = 1$, then $a^{-1}y$ can assume every value $k$ of $\mathbb{Z}_{p^2}$ such that $\gcd(k, p) = 1$. Thus we have a conjugacy class of $\varphi(p^2) = p^2 - p$ elements, where the elements appear as $(1, k)$ with $\gcd(k, p) = 1$. If $y \neq 0$ and $\gcd(y, p) \neq 1$, then $a^{-1}y$ can assume $p - 1$ different values. So there is a conjugacy class of $p - 1$ elements whose elements look like $(1, pk)$ for $0 < k < p$. This accounts for 3 conjugacy classes.

Let $x \neq 1$, $x \equiv 1 \mod p$ and $x = pk + 1$ for some positive integer $k$. Then the matrix

$$
\begin{pmatrix}
x & a^{-1}(bx + y - b) \\
0 & 1
\end{pmatrix} =
\begin{pmatrix}
x & a^{-1}(b(pk + 1) + y - b) \\
0 & 1
\end{pmatrix} =
\begin{pmatrix}
x & a^{-1}(bkp + y) \\
0 & 1
\end{pmatrix}.
$$

If $\gcd(y, p) = 1$, then $a^{-1}(bkp + y)$ can be every value $k$ of $\mathbb{Z}_{p^2}$ such that $\gcd(k, p) = 1$. This results in a conjugacy class of size $p^2 - p$ with elements of the form $(pk + 1, y)$ with $\gcd(y, p) = 1$. There will be $p - 1$ of these conjugacy classes since $0 < k < p$. If $\gcd(y, p) = p$, then $a^{-1}(bkp + y)$ can assume $p$ different values. So there is a conjugacy class of $p$ elements whose elements are of the form $(pk + 1, pt)$ for $0 \leq t < p$. Again, there will be $p - 1$ of these conjugacy classes since $0 < k < p$. 

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Let $x \not\equiv 1 \mod p$. Then we get
\[
\begin{pmatrix}
x & a^{-1}(b(x-1)+y) \\
0 & 1
\end{pmatrix}
\]
If $y = 0$ and $b = 1$, the matrix simplifies to
\[
\begin{pmatrix}
x & a^{-1}(x-1) \\
0 & 1
\end{pmatrix}
\]
We see that $a^{-1}(x-1)$ can assume any value $k$ of $\mathbb{Z}_p$ such that $\gcd(k,p) = 1$. If $b = p$, the matrix simplifies to
\[
\begin{pmatrix}
x & a^{-1}p(x-1) \\
0 & 1
\end{pmatrix}
\]
We see that $a^{-1}p(x-1)$ can assume any non-zero value $k$ of $\mathbb{Z}_p$ such that $\gcd(k,p) = p$.
Thus for each $x \not\equiv 1 \mod p$, the automorphisms of the form $(x, k)$ where $x \not\equiv 1 \mod p$ form a conjugacy class of $p^2$ elements. There will be $p^2 - 2p$ of these conjugacy classes.

In total, $\text{Aut}(D_{2p^2})$ has $3 + 2(p-1) + (p^2 - 2p) = p^2 + 1$ conjugacy classes.

By Lemma 4.2, the derived subgroup of $\text{Aut}(D_{2p^2})$ is the set of automorphisms that send $r$ to $r$. Thus by Lemma 2.18, $\text{Aut}(D_{2p^2})$ has $\frac{p^3(p-1)}{p^2} = p(p-1)$ linear irreducible characters. Thus there are also $p+1$ non-linear irreducible characters.

It follows that $\text{Aut}(D_{2p^2})/G' \cong \mathbb{Z}_{p^2}$, which is a cyclic group of order $p(p-1)$ since $p$ is odd. So
\[
\text{Aut}(D_{2p^2})/G' \cong \mathbb{Z}/(p(p-1))\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}.
\]
Let $\omega = e^{2\pi i/p^2}$ and let $x = e^{2\pi i/p}$.
Let $a$ be a generator of $\mathbb{Z}/(p-1)\mathbb{Z}$ and let $\chi_{j+1}(a^k) = \omega^{jk}$ with $0 \leq j \leq p-2$ be the irreducible characters of $\mathbb{Z}/(p-1)\mathbb{Z}$. Let $b$ be a generator of $\mathbb{Z}/p\mathbb{Z}$ and let $\chi_{m+1}(b^h) = x^{mh}$ with $0 \leq h \leq p-1$ be the irreducible characters of $\mathbb{Z}/p\mathbb{Z}$. Then by Lemma 5.5, the character table of $\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ looks like:
From Lemma 2.18, the linear characters of $\text{Aut}(D_{2p^2})$ are the lifts of the irreducible characters of $\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. The elements $(0,b^h)$ in $\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ correspond to the elements $(n,0)$ and $(n,1)$ with $n \equiv 1 \mod p$ in $\text{Aut}(D_{2p^2})$, since both sets of these elements have order $p$. The character $\chi_{j+1} \times \psi_{m+1}((0,b^h))$ simplifies to $x^{hm}$. The elements $(a^{\frac{p-1}{2}},b^h)$ in $\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ correspond to the elements $(n,0)$ with $n \equiv -1 \mod p$ in $\text{Aut}(D_{2p^2})$. Note that all of these elements have order $2p$. The character $\chi_{j+1} \times \psi_{m+1}((a^{\frac{p-1}{2}},b^h))$ simplifies to $(-1)^{\epsilon_i}x^{hm}$. We relabel the character $\chi_{j+1} \times \psi_{m+1}$ as $\chi_{m(p-1)+j+1}$. When $m = j = 0$, we see that $\chi_1$ is the principal character. Note that $\chi_i((n,0)) = \chi_i((n,1)) = 1$ when $n \equiv 1 \mod p$ and $0 \leq i < p$. Also, $\chi_i((n,0)) = 1$ or $-1$, when $n \equiv -1 \mod p$.

If $n \not\equiv 1, -1 \mod p$, then $\chi_i((n,0))$ will be a $p(p-1)$th root of unity. We will show later that for these conjugacy classes, $\chi_{D_{2p^2}}(n,0) = 0$, so no other information needs to be shown about their values.

<table>
<thead>
<tr>
<th></th>
<th>(1,0)</th>
<th>(1,p)</th>
<th>(1,1)</th>
<th>(n,0) $\mod p$</th>
<th>(n,0) $\mod p$</th>
<th>(n,1) $\mod p$</th>
<th>(p$^2 - 1$, 0)</th>
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<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>$(-1)^{a}$</td>
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<td>(1&lt;j≤p$^2$-p)</td>
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<td></td>
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<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(1≤j≤p)</td>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\chi_{p^2+1}$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
</tr>
</tbody>
</table>

There are $p+1$ non-linear irreducible characters. From Lemma 2.13, the squares of the degrees of these irreducibles add up to $p^2(p-1) - p(p-1) = p(p+1)(p-1)^2$. By Lemma 5.3, the degrees of these irreducibles divide $p^2(p-1)$.

Let $H$ be a subgroup of $G$ generated by the elements $(n,k)$ with $n \equiv 1 \mod p$. Note that $|H| = p^3$ and $H$ is normal in $G$. So $|G/H| = p-1$. Let $B$ be a subgroup of $H$ generated
by \((n, p)\) with \(n \equiv 1 \mod p\). Then \(|B| = p^2\) and note that \(B\) is normal in \(H\). Let \(Q = H/B\), so \(Q \cong \mathbb{Z}/p\mathbb{Z}\). Thus the irreducible characters of \(Q\) are \(p\)th roots of \(1\). Let \(\tilde{\phi}\) be a non-trivial irreducible character of \(Q\). Then \(\phi = \tilde{\phi}(Qh) \ (h \in H)\), the lift of \(\tilde{\phi}\), is a character of \(H\).

**Definition 6.2.** Let \(H \subseteq G\) and let \(\varphi\) be a class function of \(H\). Then \(\varphi^G\), the induced class function on \(G\), is given by

\[
\varphi^G(g) = \frac{1}{|H|} \sum_{x \in G} \varphi(xgx^{-1}),
\]

where \(\varphi^\circ\) is defined by \(\varphi^\circ(h) = \varphi(h)\) if \(h \in H\) and \(\varphi^\circ(h) = 0\) if \(h \notin H\).

So \(\phi^G(g) = \frac{1}{|H|} \sum_{x \in G} \varphi^\circ(xgx^{-1})\) is the induced character and \(\phi^G(1) = \frac{|G|}{|H|} \phi(1) = p - 1\). Since \(H \trianglelefteq G\), if \(g \notin H\), then \(\phi^G(g) = 0\). We see that for \(n \equiv 1 \mod p\),

\[
\phi^G((n, p)) = \frac{1}{|p^3|} \sum_{x \in G} \phi(x(n, p)x^{-1}) = \frac{p^3(p - 1)}{p^3} \phi((n, p)) = (p - 1)\phi((n, p)) = p - 1.
\]

We also see that for \(n \not\equiv 1 \mod p\), we have

\[
\phi^G((n, 1)) = \frac{1}{|p^3|} \sum_{x \in G} \phi(x(n, 1)x^{-1}) = \frac{p^2}{p^3} \sum_{k=1}^{p^2-1} \phi((n, k)) = \sum_{k=1}^{p-1} \phi((n, k)) = -1.
\]

Now consider the inner product \(\langle \phi^G, \phi^G \rangle\). We see that

\[
\langle \phi^G, \phi^G \rangle = \frac{1}{p^3(p - 1)} \left( (p - 1)^2 + (p - 1)^3 + (p^2 - p)p(-1)^2 + p(p - 1)(p - 1)^2 \right) = 1
\]

and so \(\phi^G\) is an irreducible character of \(G\). We relabel \(\phi^G\) as \(\chi_{p^2-p+1}\).

**Lemma 6.3.** \([1, \text{pg. 176}]\) Suppose that \(\chi\) is a character of \(G\) and \(\lambda\) is a linear character of \(G\). Then the product \(\chi \lambda\), defined by

\[
\chi \lambda(g) = \chi(g)\lambda(g) \ (g \in G)
\]

is a character of \(G\). Moreover, if \(\chi\) is irreducible, then so is \(\chi \lambda\).
From lemma 6.3, \( \chi_i \chi_{p^2-p+1} \) for \( 1 \leq i \leq p^2 - p \) is an irreducible character. Since for \( n \equiv 1 \mod p \), \( \chi_i((n,0)) \) assumes \( p \) different values (the \( p \)th roots of unity) and \( \chi_{p^2-p+1}((n,0)) \) is nonzero, we will have \( p \) irreducible characters \( \chi_i \chi_{p^2-p+1} \) of degree \( p - 1 \).

This means there is one more irreducible character for \( \text{Aut}(D_{2p^2}) \) and say that is has degree \( x \). Since \( p^4 - p^3 = (p^2 - p)(1)^2 + p(p-1)^2 + x^2 \), the remaining irreducible has degree \( p(p-1) \). Let \( \chi_{p^2+1} \) be the unique irreducible character of degree \( p(p-1) \). By Lemma 6.3, \( \chi_{p^2+1} \chi_i \), for \( 1 \leq i \leq p^2 - p \), is an irreducible character. Since \( \chi_{p^2+1} \) is the only irreducible character of its degree, the conjugacy classes in the derived subgroup are the only classes that can have non-zero values for \( \chi_{p^2+1} \). Thus \( \chi_{p^2+1}(n,0) = 0 \) and \( \chi_{p^2+1}(n,1) = 0 \) for \( n \neq 1 \).

By column orthogonality, we have

\[
0 = \sum_{\chi \in \text{Irr}(G)} \chi((1,0)) \chi((1,p)) = (p^2 - p) + p(p-1)^2 + (p(p-1))\chi_{p^2+1}((1,p)).
\]

This simplifies to \( \chi_{p^2+1}((1,p)) = -p \). Since \( \chi_{p^2+1} \) is irreducible, we have

\[
1 = \langle \chi_{p^2+1}, \chi_{p^2+1} \rangle = \frac{1}{p^4 - p^3} \left((p(p-1)^2 + (p-1)(-p)^2 + (p^2 - p)(\chi_{p^2+1}((1,1)))^2\right).
\]

This simplifies to \( \chi_{p^2+1}((1,1)) = 0 \).

Now with the character table, we can proceed to the main theorem of the chapter.

**Theorem 6.4.** For the dihedral group \( D_{2p^2} \) with \( p \) an odd prime, \( \text{Aut}(D_{2p^2}) \) has character degrees \( 1 \) (with multiplicity \( p^2 - p \)), \( p-1 \) (with multiplicity \( p \)) and a unique character of degree \( p^2 - p \). The character \( \chi_{D_{2p^2}} \) has \( \frac{p^2 - p}{2} + 2 \) constituents. The character \( \chi_{D_{2p^2}} \) has

- \( \frac{p-1}{2} \) linear constituents (with multiplicity 2),
- \( \frac{(p-1)^2}{2} \) linear constituents (with multiplicity 1),
- one constituent of degree \( p-1 \) (with multiplicity 1), and
- one constituent of degree \( p^2 - p \) (with multiplicity 1).

**Proof.** Since \( W \) has \( 2p^2 - \frac{p^2 + 3}{2} = \frac{3(p^2-1)}{2} \) basis elements, \( \rho_{D_{2p^2}}((1,0)) \) is the \( \left(\frac{3(p^2-1)}{2}\right) \times \left(\frac{3(p^2-1)}{2}\right) \) identity matrix.
Consider the matrix $\rho_{D_{2p^2}}((1,p))$. The automorphism $(r^k)(1,p) = r^k$ contributes $\frac{p^2 - 1}{2}$ ones to the diagonal. We also have $(sr^k)(1,p) = sr^{k+p}$. So $(sr^k)(1,p)$ cannot equal $sr^k$ and $(sr^k)(1,p) = s$ only if $k = p^2 - p$. This will contribute $-1$ to the diagonal. Thus $\chi_{D_{2p^2}}((1,p)) = \frac{p^2 - 3}{2}$.

Consider the matrix $\rho_{D_{2p^2}}((1,1))$. The automorphism $(r^k)(1,1) = r^k$ contributes $\frac{p^2 - 1}{2}$ ones to the diagonal. We also have $(sr^k)(1,1) = sr^{k+1}$. So $(sr^k)(1,1)$ cannot equal $sr^k$ and $(sr^k)(1,1) = s$ only if $k = p^2 - 1$. This will contribute $-1$ to the diagonal. Thus $\chi_{D_{2p^2}}((1,1)) = \frac{p^2 - 3}{2}$.

Consider the matrix $\rho_{D_{2p^2}}((n,0))$ where $n \not\equiv 1, -1 \mod p$. The automorphism $(r^k)(n,0) = r^{nk} \neq r^k, r^{-k}$ contributes only zeroes to the diagonal. We also have $(sr^k)(n,0) = sr^{nk}$ but $sr^{nk} \neq sr^k, s$. So there are only zeroes on the diagonal and $\chi_{D_{2p^2}}((n,0)) = 0$.

Consider the matrix $\rho_{D_{2p^2}}((n,0))$ where $n \equiv 1 \mod p$ and $n \neq 1$. If gcd($k,p$) = 1, then $(r^k)(n,0) = r^{nk} \neq r^k, r^{-k}$. Let $k = ap$ and $n = 1 + bp$ for positive integers $a,b$. Then $(r^k)(n,0) = r^{(1+bp)ap} = r^{ap+abp^2} = r^k$. This will contribute $\frac{p-1}{2}$ 1s to the diagonal of $\rho_{D_{2p^2}}((n,0))$. We also have $(sr^k)(n,0) = sr^{nk}$. If gcd($k,p$) = 1, then $sr^{nk} = sr^k$ only if $n \equiv 1 \mod p^2$ which gives a contradiction. Let $k = ap$ and $n = 1 + bp$ for positive integers $a,b$. Then

$$sr^{kn} = sr^{(ap)(1+bp)} = sr^{ap+abp^2} = sr^k.$$ This will result in $p - 1$ 1s in the diagonal. We see that $sr^{nk} = s$ cannot happen. Thus $\chi_{D_{2p^2}}((n,0)) = \frac{3(p-1)}{2}$.

Consider the matrix $\rho_{D_{2p^2}}((n,1))$ where $n \equiv 1 \mod p$ and $n \neq 1$. If gcd($k,p$) = 1, then $(r^k)(n,1) = r^{nk} \neq r^k, r^{-k}$. Let $k = ap$ and $n = 1 + bp$ for positive integers $a,b$. Then $(r^k)(n,1) = r^{(1+bp)ap} = r^{ap+abp^2} = r^k$. This will contribute $\frac{p-1}{2}$ 1s to the diagonal of $\rho_{D_{2p^2}}((n,1))$. We also have $(sr^k)(n,1) = sr^{nk+1}$. If $sr^{nk+1} = sr^k$, then $nk + 1 \equiv k \mod p^2$. Since $n \equiv 1 \mod p$, let $n = 1 + ap$ for some positive integer $a$. Then $(ap + 1)k + 1 \equiv k \mod p^2$ which simplifies to $akp \equiv -1 \mod p^2$. So $sr^{nk+1} = sr^k$ cannot occur. If $sr^{nk+1} = s$, then $k \equiv -n^{-1} \mod p^2$ which has a unique solution. This will result in a single $-1$ in the
diagonal. Thus \( \chi_{D_{2p^2}}((n, 1)) = \frac{p-3}{2} \).

Consider the matrix \( \rho_{D_{2p^2}}((n, 0)) \) where \( n \equiv -1 \mod p \) and \( n \neq p^2 - 1 \). If \( \gcd(k, p) = 1 \), then \((r^k)(0, 0) = r^{nk} \neq r^k, r^{-k} \). Let \( k = ap \) and \( n = -1 + bp \) for positive integers \( a, b \). Then \((r^k)(n, 0) = r^{(-1+bp)ap} = r^{-ap+abp^2} = r^{-k} \). This will contribute \( \frac{p-1}{2} \) -1s to the diagonal. We also have \((sr^k)(n, 0) = sr^{nk} \) but \( sr^{nk} \neq sr^k, s \). Thus \( \chi_{D_{2p^2}}((n, 0)) = \frac{1-p}{2} \).

Consider the matrix \( \rho_{D_{2p^2}}((p^2 - 1, 0)) \). Then \((r^k)(p^2 - 1, 0) = r^{(p^2-1)k} = r^{-k} \). This contributes \( \frac{p^2-1}{2} \) -1s to the diagonal. We also have \((sr^k)(p^2 - 1, 0) = sr^{(p^2-1)k} = sr^{-k} \neq s, sr^k \) which contributes \( p^2 - 1 \) zeros to the diagonal. Thus \( \chi_{D_{2p^2}}((p^2 - 1, 0)) = \frac{1-p^2}{2} \).

The following is the character table of \( \text{Aut}(D_{2p^2}) \) including the character \( \chi_{D_{2p^2}} \).

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<thead>
<tr>
<th></th>
<th>(1,0)</th>
<th>(1,p)</th>
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<td>1</td>
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<tr>
<td>\chi_j 1 &lt; j \leq p^2 - p</td>
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<tr>
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</tr>
<tr>
<td>\chi_{p^2+1}</td>
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<td>-p</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>\chi_{D_{2p^2}}</td>
<td>\frac{3(p^2-1)}{2}</td>
<td>\frac{p^2-3}{2}</td>
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<tr>
<td>\chi_{p^2+1}</td>
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We take the inner product of \( \chi_{D_{2p^2}} \) with irreducible characters to find the constituents of \( \chi_{D_{2p^2}} \).

For \( 1 \leq j \leq p - 1 \), \( \chi_j((n, 0)) = 1 \) and \( \chi_j((n, 1)) = 1 \) for \( n \equiv 1 \mod p \). Also \( \chi_j((n, 0)) = \)
−1 for \( n \equiv -1 \mod p \) can occur \( \frac{p-1}{2} \) times. The inner product

\[
|\text{Aut}(D_{2p^2})| \langle \chi_j, \chi_{D_{2p^2}} \rangle = \frac{3(p^2 - 1)}{2} + \frac{(p^2 - 3)(p - 1)}{2} + \frac{(p^2 - 3)(p(p - 1))}{2}
\]

\[
+ \frac{3(p - 1)p(p - 1)}{2} + \frac{(p - 3)p(p - 1)}{2} + \frac{(-1)(1 - p)p^2(p - 1)}{2} + \frac{(-1)(1 - p^2)}{2}
\]

simplifies to \( \langle \chi_j, \chi_{D_{2p^2}} \rangle = 2 \). This results in \( \frac{p-1}{2} \) linear constituents of multiplicity 2.

For \( p - 1 \leq j \leq p^2 - p \), \( \chi_j((n, 0)) = -1 \) for \( n \equiv -1 \mod p \) occurs half of the time and we will choose those. Note that there are \( \frac{(p-1)^2}{2} \) of these characters. Now \( \chi_j((n, 0)) = -x^a \), where \( n \equiv -1 \mod p \), \( x \) is a \( p \)th root of unity and \( a \) is a some positive integer. Also, \( \chi_j((n, 0)) = \chi_j((n, 1)) = x^a \), where \( n \equiv 1 \mod p \), \( x \) is a \( p \)th root of unity and \( a \) is a some positive integer. Note that for \( (n, 0) \) and \( (n, 1) \) where \( n \equiv 1 \mod p \), we have

\[
\sum_{i=0}^{p-1} \chi_j(ip + 1, 0) = \sum_{i=0}^{p-1} \chi_j(ip + 1, 1) = \sum_{l=0}^{p-1} x^l = 0.
\]

For \( (n, 0) \) where \( n \equiv -1 \mod p \), we have

\[
\sum_{i=1}^{p} \chi_j((ip - 1, 0)) = \sum_{l=0}^{p-1} -x^l = 0.
\]

The inner product

\[
|\text{Aut}(D_{2p^2})| \langle \chi_j, \chi_{D_{2p^2}} \rangle = \frac{3(p^2 - 1)}{2} + \frac{(p^2 - 3)(p - 1)}{2} + \frac{(p^2 - 3)(p(p - 1))}{2}
\]

\[
+ \frac{3(p - 1)}{2} \sum_{i=1}^{p} x^i + \frac{p - 3}{2} \sum_{i=1}^{p} x^i + \frac{1 - p}{2} \sum_{i=1}^{p} x^i + \frac{1 - p^2}{2} \sum_{i=1}^{p} x^i
\]

simplifies to \( \langle \chi_j, \chi_{D_{2p^2}} \rangle = 1 \). This results in \( \frac{(p-1)^2}{2} \) linear irreducible constituents of \( \chi_{D_{2p^2}} \) all of which have multiplicity 1.
The inner product

\[ |\text{Aut}(D_{2p^2})| \langle \chi_{p^2-p+1}, \chi_{D_{2p^2}} \rangle = \frac{3(p^2 - 1)}{2}(p - 1) + \frac{p^2 - 3}{2}(p - 1)(p - 1) + \frac{p^2 - 3}{2}(-1)(p^2 - p) \]

\[ + \frac{3(p - 1)}{2}(p - 1)(p - 1) + \frac{p - 3}{2}(-1)(p^2 - p)(p - 1) \]

simplifies to \( \langle \chi_{p^2-p+1}, \chi_{D_{2p^2}} \rangle = 1 \).

So we see that \( \chi \) has one irreducible constituent of degree \( p - 1 \) which has multiplicity 1.

The inner product

\[ \langle \chi_{p^2+1}, \chi_{D_{2p^2}} \rangle = \frac{3(p^2-1)p(p-1)}{2} + \frac{-p(p^2-3)(p-1)}{2} = 1. \]

Thus the character \( \chi_{D_{2p^2}} \) has a constituent of degree \( p(p - 1) \) and multiplicity 1.

Note that the characters shown to be constituents are all of the constituents of \( \chi \) since

\[ 2\left(\frac{p-1}{2}\right) + \frac{(p-1)^2}{2} + (p - 1) + p(p - 1) = \frac{3(p^2-1)}{2}, \]

or rather

\[ \chi(1) = \sum_{\langle \chi, \chi_i \rangle \neq 0} \chi_i(1). \]
Theorem 7.1. For any finite group $G$, let $W_G = \mathbb{C}[x_0, \ldots, x_{n-1}] / \langle C_1, \ldots, C_r \rangle$. Then $W = W_G$ does not contain the trivial submodule.

Proof. By Lemma 2.28,

$$W \left( \sum_{\alpha \in \text{Aut}(G)} \alpha \right)$$

is equal to the sum of the trivial submodules of $W$. If $x_g$, with $g \in G$, is a basis element of $W$, then $x_g \left( \sum_{\alpha \in \text{Aut}(G)} \alpha \right) = a \sum C$. Since $C = 0$ in $W$, $x_g \left( \sum_{\alpha \in \text{Aut}(G)} \alpha \right) = 0$. Note that $x_0$ is not in $W$, neither is any $x_g$ where $g \in Z(G)$. 

Of course this means that $\chi_0$ can never be a constituent of $\rho_G$.

Although this paper has been focused on dihedral groups, here are some examples of the decomposition of $W$ for other groups. All computations in this section have been done in Magma.

Example 7.2. Let $G = S_3$. The automorphism group of $S_3$ is itself. The character table of $S_3$ is the following:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>(12)</th>
<th>(123)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

Now $\rho_G = \chi_2 + \chi_3$. We can decompose $W$ as $W = \text{Span}(x_1) \oplus \text{Span}(x_3, x_4)$ where $x_1$ corresponds to (123), $x_3$ corresponds to (12) and $x_4$ corresponds to (13). Since $S_3 = D_6$, this clearly follows Theorem 4.1.

Example 7.3. Let $G = S_4$. Note that $\text{Aut}(S_4) = S_4$. The character table of $S_4$ is the
The following:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>(12)(34)</th>
<th>(1234)</th>
<th>(123)</th>
<th>(12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>3</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>3</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

Then $\chi_G = \chi_2 + 3\chi_3 + 2\chi_4 + 2\chi_5$.

**Example 7.4.** Let $G = A_4$. We see that $\text{Aut}(A_4) = S_4$. Referring to the character table above, $\chi_G = \chi_3 + \chi_4 + \chi_5$.

While studying the structure of the automorphism group of dihedral groups, I formulated the following conjecture about the number of conjugacy classes of $\text{Aut}(D_{2p})$. While a little off topic, I wanted to include it as I thought it was interesting.

**Conjecture 1.** Let $\sigma(n)$ be the sum of the divisors of $n$. The number of conjugacy classes of $\text{Aut}(D_{2n})$ is equal to

$$\sum_{0 < x < n \atop (x,n) = 1} \sigma(\gcd(x - 1, n)).$$

Let $n = p_1^{e_1} \cdots p_k^{e_k}$ be the factorization of $n$. The number of conjugacy classes of $\text{Aut}(D_{2n})$ is also equal to

$$\prod_{i=1}^k \left( \sum_{j=0}^{e_i} p_i^j \right) - p_i^{e_i - 1}.$$
Bibliography


