2013-06-27

The Frobenius Manifold Structure of the Landau-Ginzburg A-model for Sums of An and Dn Singularities

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A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

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June 2013

ABSTRACT

The Frobenius Manifold Structure of the Landau-Ginzburg A-Model
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In this thesis we compute the Frobenius manifold of the Landau-Ginzburg A-model (FJRW theory) for certain polynomials. Specifically, our computations apply to polynomials that are sums of A_n and D_n singularities, paired with the corresponding maximal symmetry group. In particular this computation applies to several K3 surfaces. We compute the necessary correlators using reconstruction, the concavity axiom, and new techniques. We also compute the Frobenius manifold of the D_3 singularity.

Keywords: K3 surfaces, reconstruction lemma, concavity axiom, Frobenius algebra, Frobenius manifold, Landau-Ginzburg mirror symmetry, FJRW theory

Acknowledgments

This thesis began as an REU project at the University of Michigan, under the direction of Dr. Yongbin Ruan (NSF grant number DMS - 0918623). Many thanks for his guidance, and also to his graduate students for their help in explaining new concepts. Thanks is also due to my advisor Dr. Tyler Jarvis, especially for his patience in helping me understand reconstruction and the concavity axiom, and for thoughtful feedback on this paper. And again, thanks to the students in the BYU mirror symmetry research group, both past and present, on whose work this thesis builds. The support and feedback I have received from this group has been a great help.

I am also grateful for other individuals whose contributions to this thesis were less direct. Thanks to my parents, who taught me the work ethic and thinking skills that made completing a thesis possible. And a big thank-you to my husband Jared for always being willing to listen, either to my grouchings when research is stressful, or to endless yammerings when I am excited about it.

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Chapter 1. Introduction

First semester calculus teaches us that the zeros of a function's derivative signal interesting behavior of the function. Likewise, mirror symmetry studies functions $W : \mathbb{C}^n \to \mathbb{C}$ that have an isolated critical point at the origin. Mirror symmetry often includes a choice of a group of *symmetries* of W; i.e., invertible linear maps $g : \mathbb{C}^n \to \mathbb{C}^n$ satisfying $W \circ g = W$. The idea of mirror symmetry is to identify mirror pairs (W, G) and (W^T, G^T), and from these construct some sort of isomorphic A- and B-models, $\mathcal{A}_{W,G}$ and $\mathcal{B}_{W^T,G^T}$.

Landau-Ginzburg mirror symmetry is one example of this process that applies to a larger class of pairs (W, G) than other common types of mirror symmetry. It is also of interest for its relatively tractable computations. The Landau-Ginzburg A- and B-models have several layers of structure, the simplest being that of a graded algebra with a pairing satisfying the Frobenius property. Furthermore, the Landau-Ginzburg models are Frobenius manifolds, and they also exhibit higher genus structure. The Landau-Ginzburg A-model is given by the FJRW theory of the pair (W, G) (see [FJR13]), while the B-model is an older construction.

Some progress has been made in proving the Landau-Ginzburg isomorphism $\mathcal{A}_{W,G} \cong \mathcal{B}_{W^T,G^T}$ at various levels of structure. The paper [Kra10] proves that $\mathcal{A}_{W,G} \cong \mathcal{B}_{W^T,G^T}$ as graded vector spaces, and as Frobenius algebras when $G = G^{max}$. The Frobenius algebra isomorphism is proved for "two-thirds" of all pairs (W, G) in [FJJS12]. At the level of Frobenius manifolds, however, the isomorphism has been fully worked out for only 16 examples plus 3 infinite families of examples.

The reason for the sparsity of examples of Frobenius manifolds is twofold. First, the A-model structure of the manifold is described by certain *correlators*, which are integrals of cohomology classes on the moduli space of curves. Though there exist combinatorial axioms that can be used for their computation, the infinite number of necessary correlators makes computing all of them a difficult problem. Second, the B-model structure involves a choice of *primitive form*. For all but a few polynomials, the list of all possible primitive forms is not currently known, much less which one will yield a B-model isomorphic to the A-model.

The goal of this thesis is to extend our list of known constructions of Landau-Ginzburg A-model Frobenius manifolds to a much larger class of examples. My strategy for doing this is as follows. I use a property of the FJRW theory of a polynomial W, which states that if $W = W_1 + \ldots + W_r$, then as Frobenius algebras,

$$\mathcal{A}_{W, G_W^{max}} \cong \mathcal{A}_{W_1, G_{W_1}^{max}} \otimes \ldots \otimes \mathcal{A}_{W_r, G_{W_r}^{max}}. \tag{1.1}$$

This interpretation of $\mathcal{A}_{W, G_W^{max}}$ leads me to consider polynomials W that are sums of polynomials W_i whose corresponding algebras $\mathcal{A}_{W_i, G_{W_i}^{max}}$ are particularly straightforward. In particular, I consider polynomials W_i such that $\mathcal{A}_{W_i, G_{W_i}^{max}}$ has a unique primitive element with respect to its multiplication. That is, there is a unique non-scalar element $\alpha \in \mathcal{A}_{W_i, G_{W_i}^{max}}$, such that if $\alpha = \beta \star \gamma$, then either β or γ is an element of $\mathbb{C}$. It turns out that there are exactly two kinds of polynomials whose rings have this property: the A_{n-1}-polynomials, which look like x^n, and the D_{n+1} polynomials, which look like $x^2y + y^n$. Thus, the goal of this thesis is to prove the following:

Theorem 1.0.1. *Let W be a sum of A_{n-1} and D_{n+1} polynomials in distinct variables. Then the Frobenius manifold structure of $\mathcal{A}_{W, G_W^{max}}$ is completely determined by the pairing, the three-point correlators, and the four-point correlators*

$$\langle\, \mathbf{p}_{W_i},\, \mathbf{p}_{W_i},\, \mathbf{h}_{W_i},\, \mathbf{h}_W \,\rangle$$

for each polynomial summand W_i. The value of this correlator is $\frac{1}{n}$ when $W_i \neq D_3$. Here, $\mathbf{p}_{W_i}$ is the unique primitive basis element and $\mathbf{h}_{W_i}$ is the unique basis element of highest degree in the corresponding algebra. The element $\mathbf{h}_W$ is the unique basis element of highest degree in $\mathcal{A}_{W, G_W^{max}}$.

When W_i is a D_3 polynomial, the relevant correlator turns out to have a different value, which we also compute in this thesis.

This theorem applies to a wide range of examples, including several so-called K3 surfaces. These polynomials are important examples because they exhibit multiple kinds of mirror symmetry besides Landau-Ginzburg mirror symmetry. Table 1.1 contains a list of $K3$ surfaces whose Frobenius manifold structure is determined by Theorem 1.0.1 (there are 29 of them).

$x^2y + y^2 + z^{12} + w^6$	$x^2y + y^4 + z^{24} + w^3$	$x^2y + y^6 + z^4 + w^6$
$x^2y + y^2 + z^8 + w^8$	$x^2y + y^4 + z^4 + w^8$	$x^2y + y^9 + z^3 + w^9$
$x^2y + y^2 + z^{20} + w^5$	$x^2y + y^5 + z^{15} + w^3$	$x^2y + y^{10} + z^4 + w^5$
$x^2y + y^3 + z^{12} + w^4$	$x^2y + y^5 + z^5 + w^5$	$x^2y + y^{12} + z^3 + w^8$
$x^2y + y^3 + z^6 + w^6$	$x^2y + y^6 + z^{12} + w^3$	$x^2y + y^{21} + z^3 + w^7$
$x^4 + y^4 + z^4 + w^4$	$x^{10} + y^{15} + z^2 + w^3$	$x^{12} + y^2 + z^4 + w^6$
$x^{12} + y^{12} + z^2 + w^3$	$x^2 + y^6 + z^6 + w^6$	$x^2 + y^4 + z^8 + w^8$
$x^3 + y^4 + z^4 + w^6$	$x^3 + y^3 + z^6 + w^6$	$x^{20} + y^2 + z^4 + w^5$
$x^{42} + y^2 + z^3 + w^7$	$x^{10} + y^2 + z^5 + w^5$	$x^{18} + y^2 + z^3 + w^9$
$x^{24} + y^2 + z^3 + w^8$	$x^{12} + y^3 + z^3 + w^4$	

Table 1.1: The twenty-nine K3 surfaces whose Frobenius manifold structure is determined by Theorem 1.0.1.

In the course of proving Theorem 1.0.1, it came to my attention that while the Frobenius manifold structure of D_{n+1} is computed in [FJR13] for $n > 2$, the case $n = 2$ does work with the method used there. Thus, I also work out the Frobenius manifold structure of D_3 in this thesis.

In addition, the D_3 Frobenius manifold computation led me to discover that Equation (92) in [FJR13] is not quite right. This equation is particularly useful for computing Frobenius manifolds, and we use it in our D_3 computations. A correct version of this equation and its derivation is presented as Lemma 3.2.1 in this thesis.

Finally, the astute reader may observe that the isomorphism in Equation (1.1) also holds at the level of Frobenius manifolds. It turns out this is not helpful for a few reasons. First, there is no proven corresponding isomorphism on the B-side, and so knowing that the pieces $\mathcal{A}_{W_i, G_{W_i}^{max}}$ and $\mathcal{B}_{W_i, G_{W_i}^{max}}$ are isomorphic does not automatically lead to an isomorphism of the larger objects. The only way to prove an A to B isomorphism, then, is to use so-called *correlators*, or structure constants that determine the manifold. But it is not clear what Equation (1.1) says about correlators. One description of the relationship between the correlators of $\mathcal{A}_{W,G_W^{max}}$ and the correlators of $\mathcal{A}_{W_i,G_{W_i}^{max}}$ is worked out by Kauffmann in [Kau99], but this description is sufficiently general to make its application to a specific example quite a challenge. Kauffmann's approach would be an alternative (and probably doable) method to prove Theorem 1.0.1, but appears to be equally if not more difficult than the strategy we develop here.

In Chapter 2 of this thesis I will define a general Frobenius manifold over a vector space, and then describe the specific construction of the Frobenius manifold of $\mathcal{A}_{W,G}$. Chapter 3 will give an overview of several known strategies for computing correlators, including the correlator axioms and reconstruction techniques. Next, Chapter 4 contains the proof of Theorem 1.0.1, including the introduction of a new technique for computing correlators. Finally, Chapter 5 contains our computations of the D_3 Frobenius manifold, and Chapter 6 concludes this thesis with some ideas for future research.

Chapter 2. Frobenius manifolds

2.1 The general construction

The purpose of this section is to introduce the reader to the idea of a Frobenius manifold. We will begin by defining a Frobenius algebra, and then give a heuristic definition of a Frobenius manifold. We will conclude with the precise definition of a Frobenius manifold over a vector space, which will be sufficient for our needs.

If R is a ring, an *R-algebra* is an R-module equipped with an associative multiplication and identity element. Now suppose we have a finite rank R-module with such a multiplication. Suppose that our multiplication is also commutative, and that our R-algebra has a symmetric nondegenerate bilinear form, called a *pairing*. Denote the pairing by $\langle \cdot, \cdot \rangle$. We say this pairing is *Frobenius* if it satisfies the axiom $\langle ab, c \rangle = \langle a, bc \rangle$ for all elements a, b, and c in the algebra. A k-algebra equipped with a Frobenius pairing is a *Frobenius algebra*.

Example 2.1.1. *A familiar example of a Frobenius algebra is the cohomology ring of a closed orientable manifold, with the Frobenius pairing equal to the cup product evaluated on the fundamental class of the manifold. Equivalently, using de Rham cohomology, consider the ring of closed differential forms modulo exact forms. Then the Frobenius pairing of ω with σ is equal (up to a sign) to integration of ω over the Poincaré dual of σ.*

The idea of a Frobenius manifold is to define a local Frobenius algebra structure on the tangent space of a complex manifold M. Our ring of scalars will be local sections of $\mathcal{O}_M$. That is, we want to define an associative and commutative multiplication on our tangent vectors, as well as a nongenenerate pairing satisfying appropriate properties. When we have done this, for each point p in the manifold M, the space T_pM will have the structure of a Frobenius algebra over $\mathbb{C}$. Thus, a second intuitive definition of a Frobenius manifold is that of gluing together Frobenius $\mathbb{C}$-algebras in some smoothly varying way.

The precise definition of a general Frobenius manifold is quite involved (for an introduction see [Koc01] or [Loo09]). In practice, however, almost all concrete Frobenius manifolds of current interest have vector spaces for the base manifold. This simplification leads to a much more straightforward definition, which we now present.

Let $\{\alpha_i\}$ be a basis for M a $\mathbb{C}$-vector space. Then the dual vectors to the α_i, which we will denote t_i, form a natural set of coordinates for M. This allows us to write a function from M to $\mathbb{C}$ as a power series in the dual vectors t_i, so we identify $\mathcal{O}_M$ with $\mathbb{C}[[t_i]]$. Also, since T_pM is canonically isomorphic to M^{**}, we will frequently identify the coordinate tangent vectors $\frac{\partial}{\partial t_i}$ with the basis vectors α_i. Because M is a vector space, these form a (global) basis $B = \{\alpha_i\}$ for the sheaf of sections of TM as a free $\mathcal{O}_M$-module. This discussion shows that we can understand TM as $M \otimes \mathbb{C}[[t_i]]$, and in fact we will routinely think of TM in this way (see [Koc01]).

Definition 2.1.2. *Let M be a vector space with basis $\{\alpha_i\}$. Then using the notation defined above, a Frobenius manifold structure on M is given by*

1. *a constant symmetric nondegenerate $\mathcal{O}_M$-bilinear pairing $\eta : TM \times TM \to \mathbb{C}$, and*

2. a potential $T \in \mathcal{O}_M(U)$ *where* U *is a neighborhood of the origin of* M*:*

$$T = \sum_{k \geq 3} \sum_{(\alpha_1, \ldots, \alpha_k) \in B^k} \frac{\langle \alpha_1, \ldots, \alpha_k \rangle}{k!} t_1 \ldots t_k, \tag{2.1}$$

where the $\langle \alpha_1, \ldots, \alpha_k \rangle$ *are any complex numbers such that* T *satisfies the equations*

$$\sum_{k,l} T_{ijk} \eta^{kl} T_{lmn} = \sum_{k,l} T_{imk} \eta^{kl} T_{ljn} \tag{2.2}$$

$$T_{1ij} = \eta_{ij}. \tag{2.3}$$

Here, T_{ijk} *indicates the third partial derivative* $\frac{\partial^3 T}{\partial t_i \partial t_j \partial t_k}$*, and* η_{ij} *and* η^{ij} *represent the* ij^{th} *entry of the matrix for* η *(relative to* B*) and the inverse of the matrix for* η*, respectively.*

In practice, we usually define the potential T by writing down a power series that satisfies Equations (2.2) and (2.3). We then show that our power series converges in a neighborhood of the origin. Before we have shown convergence, we say that T defines a *formal* Frobenius manifold.

The equations labeled (2.2) are called the WDVV equations, and their purpose is to ensure that the potential T will define a multiplication on the tangent space TM that is associative. Equation (2.3) ensures that the multiplication has an identity. This is explained with the following theorem.

Theorem 2.1.3. *Given a pairing* η *and a potential* T *as in Definition 2.1.2, define a multiplication on* TM *by*

$$\alpha_i \star \alpha_j = \sum_{k,l} T_{ijk} \eta^{kl} \alpha_l. \tag{2.4}$$

This multiplication can be extended linearly to all of TM. The multiplication $\star$ *is commutative and associative with identity* α_1*, and with it* η *satisfies the Frobenius property, making* TM *into a Frobenius algebra over* $\mathcal{O}_M$*.*

Proof. That $\star$ is commutative follows from the fact that $T_{ijk} = T_{jik}$.

To check associativity, we proceed as follows:

$$
\begin{aligned}
(\alpha_i \star \alpha_j) \star \alpha_k &= \left(\sum_{l,m} T_{ijl}\eta^{lm}\,\alpha_m\right) \star \alpha_k \\
&= \sum_{l,m} T_{ijl}\eta^{lm}\,(\alpha_m \star \alpha_k) \\
&= \sum_{n,p}\left(\sum_{l,m} T_{ijl}\eta^{lm}T_{mkn}\right)\,\eta^{np}\,\alpha_p. \qquad (2.5)
\end{aligned}
$$

Alternatively, we compute

$$
\begin{aligned}
\alpha_i \star (\alpha_j \star \alpha_k) &= \alpha_i \star \left(\sum_{l,m} T_{jkl}\eta^{lm}\,\alpha_m\right) \\
&= \sum_{l,m} T_{jkl}\eta^{lm}\,(\alpha_i \star \alpha_m) \\
&= \sum_{n,p}\left(\sum_{l,m} T_{jkl}\eta^{lm}T_{imn}\right)\eta^{np}\,\alpha_p. \qquad (2.6)
\end{aligned}
$$

Equations (2.5) and (2.6) are equivalent via an application of Equation (2.2) to the terms in parentheses.

Next, we check that α_1 is the identity. The definition yields

$$\alpha_1 \star \alpha_i = \sum_{j,k} T_{1ij}\eta^{jk}\,\alpha_k.$$

Applying Equation (2.3) yields

$$\alpha_1 \star \alpha_i = \sum_{j,k} \eta_{ij}\eta^{jk}\,\alpha_k.$$

Note that $\sum_j \eta_{ij}\eta^{jk}$ is equal to the ik^{th} entry of $[\eta_{ij}][\eta^{ij}] = I$, where I is the identity matrix. Thus, $\sum_j \eta_{ij}\eta^{jk} = \delta_{ik}$, and we conclude

$$\alpha_1 \star \alpha_i = \sum_k \delta_{ik}\alpha_k = \alpha_i.$$

Finally, we show that η is Frobenius under $\star$.

$$\begin{aligned}
\eta(\, \alpha_i \star \alpha_j,\ \alpha_k\,) &= \eta\left(\sum_{l,m} T_{ijl}\eta^{lm}\, \alpha_m,\ \alpha_k \right) \\
&= \sum_{l,m} T_{ijl}\eta^{lm}\ \eta(\, \alpha_m,\ \alpha_k\,) \\
&= \sum_{l,m} T_{ijl}\eta^{lm}\eta_{mk} \\
&= \sum_{l} T_{ijl}\delta_{lk} = T_{ijk}.
\end{aligned}$$

On the other hand, we can compute

$$\begin{aligned}
\eta(\, \alpha_i,\ \alpha_j \star \alpha_k\,) &= \eta\left(\, \alpha_i,\ T_{jkl}\eta^{lm}\, \alpha_m\,\right) \\
&= \sum_{l,m} T_{jkl}\eta^{lm}\eta_{mi} \\
&= T_{ijk}.
\end{aligned}$$

For the second equality we used the fact that η is symmetric, so $\eta_{im} = \eta_{mi}$. Thus, η is Frobenius. □

Just as many important Frobenius algebras have a natural grading (for example the cohomology ring of Example 2.1.1), many important Frobenius manifolds also have some kind of "graded structure." We describe this structure with a vector field called an *Euler field.*

Definition 2.1.4. *Let M be a (vector space) Frobenius manifold with potential T and coor-*

dinates $\{t_i\}$. Then an Euler field *E on M is a vector field of the form*

$$E = \sum_i (d_i t_i + r_i) \frac{\partial}{\partial t_i}$$

satisfying

$$E(T) = d_T T$$

up to quadratic terms, for some complex numbers d_i, r_i and d_T.

Note that given a Frobenius manifold M, it is not always possible to define a corresponding Euler field. However, Frobenius manifolds of interest in this thesis will all have Euler fields. In fact, we will always use Euler fields with $r_i = 0$. In the case $r_i = 0$, existence of an Euler field is equivalent to a certain property of T which is very similar to quasihomogeneity (see Definition 2.2.1). This is the content of the following lemma.

Lemma 2.1.5. *Let M be a (vector space) Frobenius manifold with potential T and coordinates $\{t_i\}$. Then $E = \sum_i d_i t_i \frac{\partial}{\partial t_i}$ is an Euler field on M if and only if for every $c \in \mathbb{C}$,*

$$T(c^{d_1} t_1, \ldots, c^{d_\mu} t_\mu) = c^{d_T} T(t_1, \ldots, t_\mu). \tag{2.7}$$

Note the similarity of the property in Lemma 2.1.5 and the definition of quasihomogeneity (Definition 2.2.1). In fact, the only difference is that the d_i in the Lemma need only be complex numbers, whereas the q_i in Definition 2.2.1 must be positive rationals. We will call a function with Property (2.7) *Euler of degree d_T*. We will see that given the existence of an Euler field for an (Euler) function, the variable t_i often behaves as if it had "degree" d_i.

Proof. For simplicity, write the potential T as

$$T = \sum_{n_1, \ldots, n_\mu} b_{n_1 \ldots n_\mu} t_1^{n_1} \ldots t_\mu^{n_\mu}.$$

Then the equation $E(T) = d_T T$ becomes

$$\begin{aligned} E(T) &= \left(\sum_i d_i t_i \frac{\partial}{\partial t_i} \right) \left(\sum_{n_1, \ldots, n_\mu} b_{n_1 \ldots n_\mu} t_1^{n_1} \ldots t_\mu^{n_\mu} \right) \\ &= \sum_{n_1, \ldots, n_\mu} \left(\sum_i d_i n_i \right) b_{n_1 \ldots n_\mu} t_1^{n_1} \ldots t_\mu^{n_\mu} \\ &= \sum_{n_1, \ldots, n_\mu} d_T b_{n_1 \ldots n_\mu} t_1^{n_1} \ldots t_\mu^{n_\mu}. \end{aligned}$$

From equating coefficients, this is equivalent to the equations

$$\sum_i d_i n_i = d_T \tag{2.8}$$

for every choice of integers $n_1, \ldots, n_\mu$ such that $b_{n_1 \ldots n_\mu} \neq 0$.

On the other hand, the statement that T is an Euler function becomes

$$\begin{aligned} T(c^{d_1} t_1, \ldots, c^{d_\mu} t_\mu) &= \sum_{n_1, \ldots, n_\mu} b_{n_1 \ldots n_\mu} c^{d_1 n_1} t_1^{n_1} \ldots c^{d_\mu n_\mu} t_\mu^{n_\mu} \\ &= \sum_{n_1, \ldots, n_\mu} c^{\sum_i d_i n_i} b_{n_1 \ldots n_\mu} t_1^{n_1} \ldots t_\mu^{n_\mu} \\ &= \sum_{n_1, \ldots, n_\mu} c^{d_T} b_{n_1 \ldots n_\mu} t_1^{n_1} \ldots t_\mu^{n_\mu}. \end{aligned}$$

Again by equating coefficients, since this must hold for every $c \in \mathbb{C}$, this is equivalent to the equations

$$\sum_i d_i n_i = d_T$$

for every choice of integers $n_1, \ldots, n_\mu$ such that $b_{n_1 \ldots n_\mu} \neq 0$. $\square$

We have now defined a Frobenius manifold as a Frobenius algebra structure on the tangent space of a vector space. We are ready to investigate a particular kind of Frobenius manifold, the Landau-Ginzburg A-model.

2.2 The Frobenius manifold of the Landau-Ginzburg A-model

Landau-Ginzburg mirror symmetry is currently defined only for "invertible" polynomials with appropriate "groups of symmetries". We will first explain the definitions of these polynomials and groups, and then we will describe the construction of the corresponding Frobenius manifold.

2.2.1 Polynomials and symmetry groups.

We give the necessary definitions below, beginning with the polynomial.

Definition 2.2.1. *A function* $W : \mathbb{C}^N \to \mathbb{C}$ *is* quasihomogeneous *if there exist positive rational numbers* $q_1, q_2, \ldots, q_N$ *so that for every* $c \in \mathbb{C}$*,*

$$W(c^{q_1}x_1, c^{q_2}x_2, \ldots, c^{q_N}x_N) = cW(x_1, x_2, \ldots, x_N).$$

The numbers $q_1, \ldots, q_N$ *are called the* weights *of* W*.*

Definition 2.2.2. *A function* $W : \mathbb{C}^N \to \mathbb{C}$ *is* nondegenerate *if it has an isolated critical point at the origin of* $\mathbb{C}^N$*.*

Definition 2.2.3. *A nondegenerate, quasihomogeneous polynomial is* invertible *if (1) the weights are unique and (2) the polynomial has the same number of monomials as variables.*

In this thesis, we will focus on two families of invertible polynomials. These are the polynomials $A_{n-1} = x^n$ and $D_{n+1} = x^2y + y^n$. These A and D polynomials are two of the so-called simple singularities, and are of interest in many areas of mathematics. As discussed in the introduction, I chose to focus on these polynomials because their Frobenius algebras are particularly easy to work with. This will become clearer as we progress through this thesis.

Example 2.2.4. *The polynomial* $A_{n-1} = x^n$ *has the unique system of weights* $q_1 = \frac{1}{n}$*. It is nondegenerate since* $\frac{\partial}{\partial x}(x^n) = nx^{n-1}$ *vanishes only at the origin. Since* A_{n-1} *has one monomial and one variable,* x*, it is invertible.*

Similarly, the polynomial $D_{n+1} = x^2y + y^n$ has the unique system of weights $(q_1, q_2) = (\frac{n-1}{2n}, \frac{1}{n})$. The gradient $(\frac{\partial}{\partial x}D_{n+1}, \frac{\partial}{\partial y}D_{n+1}) = (2xy,\, x^2 + ny^{n-1})$ vanishes only at the origin, so it is nondegenerate. Since D_{n+1} has two monomials and two variables, it is invertible.

The next theorem will tell us how sums of invertible polynomials behave.

Theorem 2.2.5. *Let W be a sum of invertible polynomials in distinct variables, so $W = W_1 + W_2$ where $W_1 \in \mathbb{C}[x_1, \ldots, x_N]$ and $W_2 \in \mathbb{C}[y_1, \ldots, y_M]$, and each W_i is invertible. Then W is also invertible.*

Proof. It is clear that W is quasihomogeneous with a unique system of weights. Similarly, W is nondegenerate since the set of partial derivatives of W will equal the union of the sets of partials of W_1 and W_2. The partials of W_1 will force $x_1 = \ldots = x_N = 0$, and likewise the partials of W_2 will force $y_1 = \ldots = y_M = 0$. Finally, because each W_i has the same number of variables as monomials, W will also have equal numbers of variables and monomials. Thus, W is invertible. □

In this thesis, I will focus on sums of A and D polynomials in distinct variables (e.g., $x^3 + y^4$ is the sum of two A-polynomials).

Next, we define the "group of symmetries" of an invertible polynomial.

Definition 2.2.6. *Let $W : \mathbb{C}^N \to \mathbb{C}$ be an invertible polynomial. Then the* maximal symmetry group *of W, denoted G_W^{max}, is defined as follows:*

$$G_W^{max} = \{(\gamma_1, \gamma_2, \ldots, \gamma_N) \in (\mathbb{C}^\times)^N \;:\; W(\gamma_1 x_1, \gamma_2 x_2, \ldots, \gamma_N x_N) = W(x_1, x_2, \ldots, x_N)\}. \quad (2.9)$$

In other words, G_W^{max} is the set of diagonal linear maps g satisfying $W \circ g = W$. As in Definition 2.2.6, however, we will denote one of these maps by the elements $(\gamma_1, \ldots, \gamma_N)$ appearing on the diagonal of the matrix. It is a fact that the coordinates $\gamma_1, \ldots, \gamma_N$ of an element of G_W^{max} are always roots of unity; hence $\gamma_j = e^{2\pi i \Theta_j}$ for some unique $\Theta_j \in \mathbb{Q} \cap [0, 1)$. When working with explicit examples, we will write $(\Theta_1, \ldots, \Theta_N)$ for $(\gamma_1, \ldots, \gamma_N)$ to save space. Note that the group operation then becomes addition. From now on, we will use g to denote an element of G_W^{max}, $\gamma_1, \ldots, \gamma_N$ to denote its multiplicative coordinates, and $\Theta_1, \ldots, \Theta_N$ to denote its additive coordinates. Occasionally, we will use a superscript g on the coordinates to indicate what group element they came from; i.e., $g = (\Theta_1^g, \ldots, \Theta_N^g)$.

Example 2.2.7. *Let us compute $G_{A_{n-1}}^{max} = G_{x^n}^{max}$. From the definition, $\gamma \in \mathbb{C}^\times$ is in $G_{x^n}^{max}$ if and only if $(\gamma x)^n = x^n$, which is true if and only if $\gamma^n = 1$. So our symmetry group is exactly equal to the multiplicative group of n^{th} roots of unity. In additive notation, we can write any group element uniquely as $\frac{t}{n}$ for some $t \in \{0, \ldots, n-1\}$.*

Similarly, $(\gamma_1, \gamma_2) \in G_{D_{n+1}}^{max}$ if and only if $(\gamma_1 x)^2(\gamma_2)y + (\gamma_2 y)^n = x^2y + y^n$, which is true if and only if $\gamma_1^2\gamma_2 = 1$ and $\gamma_2^n = 1$. Then the second equation tells us that γ_2 is any n^{th} root of unity, and γ_1 is a square root of γ_2^{-1}. Equivalently, if γ_1 is any $2n^{th}$ root of unity, then γ_2^{-1} is the square of γ_1. Then our symmetry group consists of the multiplicative elements $(e^{2\pi ik/2n}, e^{-2\pi ik/n})$ for $k \in \{0, \ldots, 2n-1\}$. Additively, the group is generated by $(\frac{1}{2n}, \frac{-1}{n})$.

The following theorem tells us how G_W^{max} behaves when W is a sum of polynomials.

Theorem 2.2.8. *Let $W = W_1 + W_2$ be a sum of invertible polynomials in distinct variables $x_1 \ldots x_N$ and $y_1 \ldots y_M$. Then*

$$G_W^{max} = G_{W_1}^{max} \times G_{W_2}^{max}.$$

Proof. When we solve for $g = (\gamma_1, \ldots, \gamma_N, \gamma_{N+1}, \ldots, \gamma_{N+M}) \in G_W^{max}$, the resulting system of equations will partition naturally into two unrelated systems, one involving the variables $\gamma_1 \ldots \gamma_N$ and one in $\gamma_{N+1} \ldots \gamma_{N+M}$. The first system will define the first N coordinates of g as an element of $G_{W_1}^{max}$, and the second system will define the last M coordinates as an element of $G_{W_2}^{max}$. □

Because we always take W to be quasihomogeneous, G_W^{max} will always contain a certain non-identity element, which we now define.

Definition 2.2.9. *Let W be quasihomogeneous with weights $(q_1, q_2, \ldots, q_N)$. Then the* exponential grading operator $j_W = (e^{2\pi iq_1}, e^{2\pi iq_2}, \ldots, e^{2\pi iq_N})$ *is always in G_W^{max}.*

The fact that j_W is always in G_W^{max} follows directly from the definition of quasihomogeneous, taking $c = e^{2\pi i} = 1$. Note that in additive notation, $j_W = (q_1, q_2, \ldots, q_N)$.

We will need one more definition concerning elements of G_W^{max}.

Definition 2.2.10. *The* fixed locus *of a group element $g = (\gamma_1, \ldots, \gamma_N) \in G_W^{max}$ is the subspace of $\mathbb{C}^N$ where g restricts to the identity map.*

Thus, the dimension of the fixed locus of g is the number of the γ_j that are equal to 1. Equivalently, if we write $g = (\Theta_1, \ldots, \Theta_N)$ where $\gamma_j = e^{2\pi i \Theta_j}$, the dimension of the fixed locus is the number of the Θ_j that equal 0.

2.2.2 Frobenius manifold construction.

Recall that the Landau-Ginzburg A-model of a pair (W, G) is written $\mathcal{A}_{W,G}$. We will now define $\mathcal{A}_{W,G}$ as a Frobenius manifold, given an invertible polynomial W and corresponding group G.

We will begin by defining $\mathcal{A}_{W,G}$ as a Frobenius algebra over $\mathbb{C}$. We will then take this algebra to be the base manifold M in Definition 2.1.2. We will choose the Frobenius manifold pairing on TM to be the $\mathcal{O}_M$-linear extension of the pairing on M (by identifying M canonically with $M^{**} \cong TM$). However, the potential we will choose will define a multiplication on TM that is *not* just a linear extension of the multiplication on M. We will see that we can recover M including its multiplicative structure by looking at T_0M in the Frobenius manifold.

Let us begin, then, with a vector space basis for the Frobenius algebra $\mathcal{A}_{W,G}$. What follows will be a short summary of the construction; for more details, see [FJR13] for a theoretical or [Kra10] for a more computational approach. We will need the following definition.

Definition 2.2.11. *Let $W : \mathbb{C}^N \to \mathbb{C}$ be an invertible polynomial. Then the* Milnor ring *of W is defined to be*

$$\mathcal{Q}_W = \frac{\mathbb{C}[x_1, \ldots, x_N]}{\left(\frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_N}\right)}. \tag{2.10}$$

We will be particularly interested in the space $\mathcal{Q}_W \cdot \omega$, where $\omega = dx_1 \wedge \ldots \wedge dx_N$. If $G \leq G_W^{max}$, then this space has a natural G-action on it given by

$$(\gamma_1, \ldots, \gamma_N) \cdot (x_1^{a_1} \ldots x_N^{a_N}\, \omega) = \gamma_1^{a_1+1} \ldots \gamma_N^{a_N+1} x_1^{a_1} \ldots x_N^{a_N}\, \omega.$$

Here, $(\gamma_1, \ldots, \gamma_N) \in G_W^{max}$ is written multiplicatively. The $+1$'s in the exponents of the γ_i on the right hand side of Equation (2.10) come from the volume form ω.

We are now ready to describe the basis of the Frobenius algebra of $\mathcal{A}_{W,G}$.

Definition 2.2.12. *Let W be invertible, $G \leq G_W^{max}$. We require G to contain the exponential grading operator j_W. Then as a vector space, $\mathcal{A}_{W,G}$ is equal to*

$$\bigoplus_{g \in G} \left(\mathcal{Q}_{W|_{\mathrm{fix}(g)}} \cdot \omega_g \right)^G. \tag{2.11}$$

Here, $W|_{\mathrm{fix}(g)}$ indicates the restriction of W to the fixed locus of g, and $\omega_g = dx_{i_1} \wedge \ldots \wedge dx_{i_m}$ where $x_{i_1}, \ldots, x_{i_m}$ are the variables fixed by g. The exponent G indicates that we take only the G-invariant subspace of $\mathcal{Q}_{W|_{\mathrm{fix}(g)}} \cdot \omega_g$.

In this thesis, we will only be concerned with rings $\mathcal{A}_{W,G}$ where $G = G_W^{max}$.

We will usually write a basis element of $\mathcal{A}_{W,G}$ coming from the summand corresponding to g as me_g where m is a monomial in the appropriate Milnor ring (note that we omit the volume form). In this case we say that me_g "comes from the sector" g, or that "the sector of me_g is g."

We will now define a grading for the Frobenius algebra of $\mathcal{A}_{W,G}$ by defining the degree of our canonical basis vectors.

Definition 2.2.13. *Let α in $\mathcal{A}_{W,G}$ be a basis element coming from the sector $g = (\Theta_1, \Theta_2, \ldots, \Theta_N)$. Then we define the* W-degree *of α as*

$$\deg_W(\alpha) = D + 2\sum_{j=1}^{N} (\Theta_j - q_j),$$

where D is the dimension of the fixed locus of g.

Notice that the W-degree of an element depends only on the sector and not on the monomial.

Let us do an example of the computation of a basis for the Frobenius algebra of $\mathcal{A}_{W,G}$.

Example 2.2.14. *Let $W = x^3$ and $G = G_W^{max} = \langle\zeta\rangle$, where $\zeta = e^{2\pi i/3}$ (so G is the multiplicative group of 3^{rd} roots of unity). Then the sum in Equation (2.11) is taken over the three elements $g = 1$, $g = \zeta$, and $g = \zeta^2$.*

When $g = 1$, $\mathrm{fix}(g) = \mathbb{C}[x]$ because g is the identity map, so $\mathcal{Q}_{W|_{\mathrm{fix}(g)}} = \mathbb{C}[x]/(3x^2) = \mathrm{span}\{1, x\}$. We now compute the G-invariants of this space—note that we only need to check a generator of G. We find $\zeta\cdot(dx) = \zeta dx$ and $\zeta\cdot(x\,dx) = \zeta^2 x\,dx$, so the G-invariant subspace here is 0-dimensional. Thus, this summand contributes no basis elements.

If $g = \zeta$, then $\mathrm{fix}(g) = \{0\}$, so $\mathcal{Q}_{W|_{\mathrm{fix}(g)}} = \mathbb{C}/(0) = \mathrm{span}\{1\}$. Since $\zeta\cdot 1 = 1$, we have that any multiple of 1 is invariant, so that this summand contributes the basis element e_ζ, or $e_{1/3}$ in additive notation.

The case $g = \zeta^2$ is similar to $g = \zeta$. We conclude that a basis for $\mathcal{A}_{W,G}$ is $\{\, e_{1/3},\, e_{2/3}\,\}$. We can also compute the degrees of these basis elements using Definition 2.2.13 as follows:

$$\deg_W(e_{1/3}) = 0 + 2(1/3 - 1/3) = 0$$

$$\deg_W(e_{2/3}) = 0 + 2(2/3 - 1/3) = 2/3.$$

This completes the definition of the basis of the Frobenius algebra of $\mathcal{A}_{W,G}$; the second step is to define the pairing. To do this we will need the following definitions.

Definition 2.2.15. *Let W be an invertible polynomial with weights $(q_1, \ldots, q_N)$. The* weighted degree *of a monomial $x_1^{a_1}\ldots x_N^{a_N}$ in $\mathcal{Q}_W$ is defined to be $a_1q_1 + \ldots + a_Nq_N$.*

Note that this definition makes $\mathcal{Q}_W$ into a graded $\mathbb{C}$-algebra. The next two theorems tell us something about that graded structure.

Theorem 2.2.16 ([AGL98] p. 40, 4°). *As a graded $\mathbb{C}$-algebra, $\mathcal{Q}_W$ has a one-dimensional homogeneous subspace of highest degree. The degree of an element in this subspace is called the* central charge *of W, and is denoted $\hat{c}_W$. It is given by*

$$\hat{c}_W = \sum_{j=1}^{N} (1 - 2q_j).$$

Here, $(q_1, \ldots, q_N)$ are the weights of W.

As we will see later, $\hat{c}_W$ is in some sense the "dimension of the pairing."

Theorem 2.2.17. *The one-dimensional subspace of highest degree in $\mathcal{Q}_W$ is spanned by $Hess(W)$, or the* Hessian *of W, which is given by*

$$Hess(W) = \det \begin{pmatrix} \frac{\partial^2 W}{\partial x_1^2} & \frac{\partial^2 W}{\partial x_1 x_2} & \cdots & \frac{\partial^2 W}{\partial x_1 x_N} \\ \frac{\partial^2 W}{\partial x_2 x_1} & \frac{\partial^2 W}{\partial x_2^2} & \cdots & \frac{\partial^2 W}{\partial x_2 x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 W}{\partial x_N x_1} & \frac{\partial^2 W}{\partial x_N x_2} & \cdots & \frac{\partial^2 W}{\partial x_N^2} \end{pmatrix}.$$

Proof. A straightforward computation using the definition of quasihomogeneity shows that $Hess(W)$ has the degree given in Theorem 2.2.16. Showing that it is nonzero is harder. The idea is to compute the below-defined residue pairing of $Hess(W)$ with 1, using also the Transformation Law on p. 657 of [GH11]. This integral turns out to be nonzero. Since the residue pairing is bilinear, $Hess(W)$ must be nonzero. □

We are now ready to define a pairing on the Milnor ring, which we will then extend to a pairing on $\mathcal{A}_{W,G}$. Our pairing comes from the *Grothendiek residue pairing*, which is given by the residue of the integral

$$\langle f, g \rangle = \int_\Delta \frac{fg \, dx_1 \wedge \ldots \wedge dx_N}{\frac{\partial W}{\partial x_1} \cdots \frac{\partial W}{\partial x_N}},$$

where Δ is the boundary of a small polydisk about the origin in $\mathbb{C}^N$. It is a well-known fact (see [GH11] Chapter 5 Section 1, especially the Transformation Law on p. 657) that this residue can also be computed as

$$fg = \frac{\langle f, g\rangle}{\mu} Hess(W) + \text{ terms of lower degree,} \tag{2.12}$$

where μ is the dimension of the Milnor ring $\mathcal{Q}_W$. This justifies our earlier claim that $\hat{c}_W$ is "the dimension of the pairing," since the pairing of two monomials will be nonzero only if their degrees add to $\hat{c}_W$.

This pairing is extended to our basis for $\mathcal{A}_{W,G}$ in the following way.

Definition 2.2.18. *Let me_g and ne_h be two basis elements for $\mathcal{A}_{W,G}$. Then we define their pairing as follows:*

$$\langle me_g, ne_h\rangle = \begin{cases} \langle m, n\rangle & \text{if } g = -h \\ 0 & \text{otherwise} \end{cases},$$

where $\langle m, n\rangle$ refers to the pairing in $\mathcal{Q}_{W|_{\text{fix}(g)}}$.

Example 2.2.19. *Let us compute three pairings at the Frobenius algebra level which we will use later. First, in the ring $\mathcal{A}_{A_{n-1}, G^{max}_{A_{n-1}}}$, we will compute the pairing of $e_{\frac{1}{n}}$ with $e_{\frac{n-1}{n}}$. We will see later (Section 4.2) that this is actually the pairing of the identity element with the basis element of highest degree. Because $\frac{1}{n} + \frac{n-1}{n} = 0$ in $G^{max}_{A_{n-1}}$, the pairing is not automatically zero. Then, following Formula (2.12), we want to investigate the Milnor ring of x^{n-1} restricted to the fixed locus of $(\frac{1}{n})$. In fact, this group element fixes no variables, so the Milnor ring is just $\mathbb{C}$. Since $\mathbb{C}$ is one-dimensional as a vector space, $\mu = 1$. The integral definition of the pairing gives 1 in this case, which agrees with the convention that the determinant of an empty matrix is 1 (and hence $Hess(W) = 1$).*

Similarly, in the ring $\mathcal{A}_{D_{n+1}, G^{max}_{D_{n+1}}}$, the pairing of the identity with the basis element of highest degree is just the pairing of $e_{(\frac{n-1}{2n}, \frac{1}{n})}$ with $e_{(\frac{n+1}{2n}, \frac{n-1}{n})}$. Because the two group elements are inverses but have no fixed locus, by an argument identical to above, the pairing is 1.

Finally, in the ring $\mathcal{A}_{D_3, G^{max}_{D_3}}$, we will compute the pairing of $xe_{(0,0)}$ with itself. Clearly *the sector* $(0,0)$ *is the inverse of itself, so this pairing is not automatically zero. Because the identity fixes everything, restricting to the fixed locus just returns the original polynomial* $D_3 = x^2y + y^2$. *It can be shown that the Milnor ring for* D_3 *has dimension 3 and that* $Hess(D_3) = -6x^2$. *Then following the formula, we find that the pairing is equal to* $\frac{3}{-6} = -\frac{1}{2}$.

In order to complete our definition of $\mathcal{A}_{W,G}$ as a Frobenius algebra, we still have to define a multiplication on our basis vectors. We will temporarily postpone this definition. We will later define a potential for the Frobenius manifold of $\mathcal{A}_{W,G}$, which will in turn define a multiplication on the tangent space of this manifold. It turns out that under this multiplication, the tangent space at the origin is canonically isomorphic to the original Frobenius algebra of $\mathcal{A}_{W,G}$, and we will use this to define our multiplication.

We now completely understand $\mathcal{A}_{W,G}$ as a Frobenius algebra. Let us call this space M. We will turn M into a Frobenius manifold following Definition 2.1.2. According to that definition, we need to choose a constant pairing $\eta : TM \times TM \to \mathbb{C}$. The most natural choice is to extend the pairing on M bilinearly over $\mathcal{O}_M$ to TM (recall we identify TM with $M \otimes \mathbb{C}[[t_i]]$).

The important point about this pairing is that it is straightforward to compute. Moreover, since it is completely determined by pairings of basis elements, only a finite number of computations are needed to understand it. Thus, the pairing on an A-model Frobenius manifold is "easy," and in fact completely understood. We will not address it further in this thesis.

In fact, the difficulty in computing $\mathcal{A}_{W,G}$ as a Frobenius manifold comes from the one remaining part of our definition, the potential

$$T = \sum_{k\geq 3} \sum_{(\alpha_1 \ldots \alpha_k)\in B^k} \frac{\langle \alpha_1, \ldots, \alpha_k \rangle}{k!} t_1 \ldots t_k.$$

This is because the constants $\langle \alpha_1, \ldots, \alpha_k \rangle$ in the potential T are defined to be integrals of corresponding cohomology classes over the moduli space of curves:

$$\langle \alpha_1, \ldots, \alpha_k \rangle = \int_{\overline{\mathcal{M}}_{0,k}} \Lambda_{0,k}(\alpha_1, \ldots, \alpha_k).$$

Definition 2.2.20. *The constant $\langle \alpha_1, \ldots, \alpha_k \rangle$ is called a k-*point correlator *of the A-model $\mathcal{A}_{W,G}$, and the α_i's are called the* insertions *of the correlator.*

The Λ-classes are defined in [FJR13], and there it is proven that thus defined, T will satisfy Equations (2.2) and (2.3). However, we cannot be certain of the convergence of T without further investigation, so that in general we only understand $\mathcal{A}_{W,G}$ as a formal Frobenius manifold.

As promised, this definition of a potential yields the product for the Frobenius algebra of $\mathcal{A}_{W,G}$ by identifying the Frobenius algebra with T_0M. Then computing a product of basis vectors in the Frobenius algebra amounts to evaluating Equation (2.4) at $\vec{0}$. It is fairly straightforward to compute that this yields

$$\alpha_i \star \alpha_j = \sum_{k,l} \langle \alpha_i,\ \alpha_j,\ \alpha_k \rangle \eta^{kl}\, \alpha_l. \tag{2.13}$$

We have completely defined the Frobenius manifold of $\mathcal{A}_{W,G}$. In the next section, we will show (in Theorem 3.1.2) that this manifold has the Euler field

$$E = \sum_i \left(1 - \frac{1}{2}\deg_W(\alpha_i)\right) t_i \frac{\partial}{\partial t_i}.$$

Thus, given that the tangent vector α_i has degree $\deg_W(\alpha_i)$, we think of the corresponding coordinate t_i as having "degree" $1 - \frac{1}{2}\deg_W(\alpha_i)$. An intuitive reason for this is that the t_is are really the dual vectors of the α_is (see the discussion before Definition 2.1.2).

So now we have the complete definition of the Frobenius manifold of $\mathcal{A}_{W,G}$ and all its accessories. What, then, is so hard about writing down explicit examples of this object? The challenge is that the integral appearing in the definition of a correlator (Definition 2.2.20) is in general extremely difficult to compute. The cohomology classes $\Lambda_{0,k}(\alpha_1, \ldots, \alpha_k)$ are not at all easy to write down. Thus, the problem of computing $\mathcal{A}_{W,G}$ as a Frobenius manifold is really just the problem of computing these correlators. In the next section, we will review some known strategies for computing correlators.

Chapter 3. Known strategies for computing correlators

The first strategy for computing correlators is to not actually do any integrals over $\overline{\mathcal{M}}_{0,k}$, but instead to use some standard tricks to deduce the values of these integrals indirectly. These "standard tricks" are known as the correlator axioms, and are the subject of the first part of this section.

It should be noted that throughout this thesis, we will only work with correlators of basis elements of $\mathcal{A}_{W,G}$. We can then extend linearly to all of TM. Thus, when we write a correlator $\langle \alpha_1, \ldots, \alpha_k \rangle$, we can talk about "the" sector of α_i, for example.

3.1 The correlator axioms

We now present the correlator axioms, interspersed with the necessary definitions. These axioms come from the geometry of a space $\mathcal{W}_{0,k}$ which is associated with a correlator. This space is a Deligne-Mumford stack of algebraic curves with the addition of some special line bundles. The axioms have since been reduced to combinatorial rules, and this later version is what we will present in this thesis. We will cite most of these axioms from two sources: the first a higher-level paper explaining the geometric origins of the axiom, and the second a more computational presentation.

The first axiom tells us that the order of elements in a correlator doesn't matter. This drastically reduces the number of correlators we need to compute.

Axiom 1 (**Symmetry** [FJR13] Theorem 4.2.2 C1 and [KPA$^+$10] Axiom 2)**.** *Let* $\langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$

be a k-point correlator, and let σ be a permutation in S_k. Then

$$\langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle = \langle \alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(k)} \rangle.$$

Our next two axioms will give us strategies for identifying correlators that are zero. We begin by defining the degree of the cohomology class Λ corresponding to a correlator.

Definition 3.1.1 ([FJR13] Theorem 4.1.8 part (1)). *The degree of the cohomology class $\Lambda_{0,k}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is the following:*

$$\deg(\Lambda_{0,k}(\alpha_1, \alpha_2, \ldots, \alpha_k)) = -\hat{c}_W + \frac{1}{2}\sum_{i=1}^{k} \deg_W(\alpha_i).$$

This then allows us to prove the second correlator axiom.

Axiom 2 (**Dimension** [FJR13] Theorem 4.1.8 and [KPA+10] Axiom 1). *A genus-zero k-point correlator $\langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$ vanishes unless*

$$\frac{1}{2}\sum_{i=1}^{k} \deg_W(\alpha_i) = \hat{c}_W + k - 3.$$

Proof. Because the correlator $\langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$ is defined to be the integral $\int_{\overline{\mathcal{M}}_{0,k}} \Lambda_{0,k}(\alpha_1, \ldots, \alpha_k)$, the correlator will be zero unless

$$\deg(\Lambda_{0,k}(\alpha_1, \ldots, \alpha_k)) = \dim \overline{\mathcal{M}}_{0,k}.$$

Using the fact that the dimension of $\overline{\mathcal{M}}_{0,k}$ is $k-3$, this is equivalent to

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^{k} \deg_W(\alpha_i) - \hat{c}_W &= 0 - 3 + k \\ \iff \quad \frac{1}{2}\sum_{i=1}^{k} \deg_W(\alpha_i) &= \hat{c}_W + k - 3. \end{aligned}$$

□

We can now show that the Frobenius manifold of $\mathcal{A}_{W,G}$ has an Euler field.

Theorem 3.1.2. *The vector field* $E = \sum_i \left(1 - \frac{1}{2}\deg_W(\alpha_i)\right) t_i \frac{\partial}{\partial t_i}$ *is an Euler field on* $\mathcal{A}_{W,G}$.

Proof. Write the potential T as

$$T = \sum_{k\geq 3} \sum_{(\alpha_1,\ldots,\alpha_k)\in B^k} \frac{\langle \alpha_1, \ldots, \alpha_k\rangle}{k!} t_1 \ldots t_k = \sum_{n_1,\ldots,n_\mu} b_{n_1\ldots n_\mu} t_1^{n_1} \ldots t_\mu^{n_\mu}. \tag{3.1}$$

Now, recall that in the proof of Lemma 2.1.5 (see Equation (2.8)) we showed that E will be an Euler field on $\mathcal{A}_{W,G}$ if and only if there exists some constant d_T such that whenever $b_{n_1\ldots n_\mu} \neq 0$,

$$\sum_i (1 - \frac{1}{2}\deg_W(\alpha_i)) = d_T.$$

Now, from our definition of T in Equation (3.1), if $b_{n_1,\ldots n_\mu} \neq 0$, the corresponding correlator (which has n_i insertions equal to α_i) must not be zero. Thus, this correlator satisfies the Dimension Axiom. But this is equivalent to saying

$$\frac{1}{2}\sum_{i=1}^{\mu} n_i \deg_W(\alpha_i) = \hat{c}_W + \left(\sum_{i=1}^{\mu} n_i\right) - 3,$$

which we can rewrite as

$$\sum_i (1 - \frac{1}{2}\deg_W(\alpha_i)) n_i = 3 - \hat{c}_W.$$

Thus, if we take $d_T = 3 - \hat{c}_W$, then E is indeed the Euler field. $\square$

A second concept that assists us in determining nonzero correlators is the idea of *line bundle degrees.*

Definition 3.1.3. *The* j^{th} line bundle degree *of a correlator* $\langle \alpha_1, \alpha_2, \ldots, \alpha_k\rangle$ *is defined to be*

$$l_j = q_j(k-2) - \sum_{i=1}^{k} \Theta_j^{g_i}, \tag{3.2}$$

where α_i *comes from the sector corresponding to the element* $g_i = (\Theta_1^{g_i}, \Theta_2^{g_i}, \ldots, \Theta_N^{g_i})$.

Axiom 3 (**Line Bundle Degrees** [FJR13] Proposition 2.2.8 and [KPA$^+$10] Axiom 3)**.** *A correlator $\langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$ is zero unless all its line bundle degrees are integers.*

The remaining three axioms allow us to compute the values of nonzero correlators.

Axiom 4 (**Pairing** [FJR13] Theorem 4.2.2 C4b and [KPA$^+$10] Axiom 7)**.** *Suppose we have a correlator of the form $\langle \mathbf{1}, \alpha_1, \alpha_2 \rangle$. Then*

$$\langle \mathbf{1}, \alpha_1, \alpha_2 \rangle = \langle \alpha_1, \alpha_2 \rangle,$$

where $\langle \alpha_1, \alpha_2 \rangle$ is the pairing of α_1 and α_2.

Axiom 5 (**Forgetting Tails** [FJR13] Theorem 4.2.2 C4a)**.** *Let $\tau : \overline{\mathcal{M}}_{0,k+1} \to \overline{\mathcal{M}}_{0,k}$ be the map that forgets the $(k+1)^{st}$ marked point, given $k \geq 3$. Then*

$$\Lambda_{0,k+1}(\mathbf{1},\ \alpha_1,\ \alpha_2,\ \ldots,\ \alpha_k) = \tau^* \Lambda_{0,k}(\alpha_1,\ \ldots,\ \alpha_k).$$

A commonly used corollary to this axiom is as follows.

Corollary 3.1.4. *For $k \geq 3$,*

$$\langle \mathbf{1},\ \alpha_1,\ \alpha_2,\ \ldots,\ \alpha_k \rangle = 0.$$

Proof. The degree of $\Lambda_{0,k}(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_k)$ can be at most the dimension of the space $\overline{\mathcal{M}}_{0,k}$ where it lives, which is $k-3$. But then this is also the degree of $\Lambda_{0,k+1}(\mathbf{1},\ \alpha_1,\ \alpha_2,\ \ldots,\ \alpha_k)$, since $\mathbf{1}$ always has degree zero. Then our correlator cannot satisfy the Dimension Axiom, since doing so would require the degree of $\Lambda_{0,k+1}(\mathbf{1},\ \alpha_1,\ \alpha_2,\ \ldots,\ \alpha_k)$ to be $k+1-3 = k-2$. $\square$

Our final axiom, the Concavity Axiom, is our strongest tool for directly calculating the values of correlators. Recall that a correlator is defined as the integral of an appropriate Λ class over $\overline{\mathcal{M}}_{0,k}$. In general, we do not know how to compute this integral. The idea of the

Concavity Axiom is to rewrite Λ classes in terms of other cohomology classes whose integrals are known. To be precise, for correlators satisfying certain conditions, the Concavity Axiom allows us to rewrite the corresponding Λ class in terms of so-called tautological classes κ, ψ, and Δ on $\overline{\mathcal{M}}_{0,k}$. The integrals of polynomials in these classes have been extensively studied (see [Zvo11] for an introduction), and they can be computed using Carl Faber's Maple code [Fab] or Drew Johnson's Sage code [Joh].

Before we can state the Concavity Axiom, we need to take a moment to explain the concept of decorated graphs corresponding to a correlator on $\overline{\mathcal{M}}_{0,k}$.

Graphs on $\overline{\mathcal{M}}_{\mathbf{0,k}}$. As a set, we understand $\overline{\mathcal{M}}_{0,k}$ as the space of stable genus-zero (possibly nodal) complex curves with k marked points, labeled 1 through k. That is, each point in $\overline{\mathcal{M}}_{0,k}$ corresponds to a single such curve. We call the set of nodal curves the *boundary* of $\overline{\mathcal{M}}_{0,k}$ (see Figures 3.1 and 3.2). The stability condition is equivalent to requiring that each irreducible component of a curve (i.e., each ball) have at least three marked points or nodes.

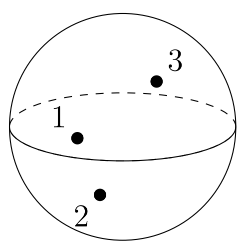

Figure 3.1: A smooth stable genus-zero curve with 3 marked points (an interior point in $\overline{\mathcal{M}}_{0,3}$)

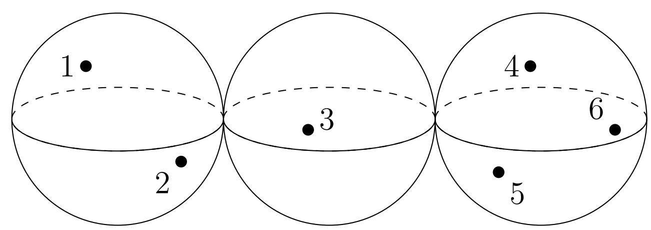

Figure 3.2: A stable nodal genus-zero curve with 6 marked points (a boundary point in $\overline{\mathcal{M}}_{0,6}$)

We define the dual graph to such a curve as follows. Begin with a vertex for each irreducible component of the curve. Connect vertices with an edge if the corresponding components are joined via a node. Then, attach half edges to the vertices corresponding to marked points. We will consider two such graphs identical if under some orientation, the sets of integers attached to half-edges for each node are identical. See Figures 3.3 and 3.4 for examples of dual graphs.

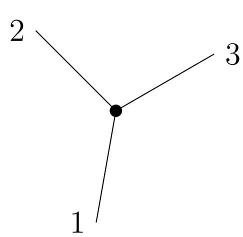

Figure 3.3: The dual graph to the curve in Figure 3.1 above

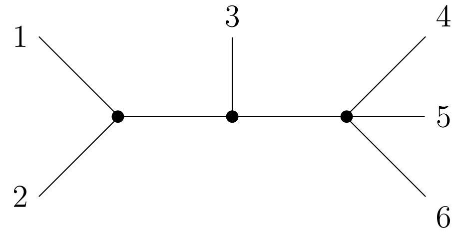

Figure 3.4: The dual graph to the curve in Figure 3.1 above. Swapping marks 4 and 6 yields an identical graph; swapping 1 and 4 does not.

Call the dual graph Γ. Recall that Γ is the dual graph to a curve in $\overline{\mathcal{M}}_{0,k}$, so it has k half-edges. If we number the insertions in a k-point correlator, then this induces a labeling of the half-edges of Γ by the group elements of the corresponding insertions. When Γ is labeled with group elements this way, we say that it is *decorated*. Note that different graphs may look the same once they are decorated. For example, if all the insertions in a correlator are equal, different numberings of the half-edges (which correspond to distinct graphs) will all look the same once the integers are replaced with the corresponding group elements.

A decorated graph Γ will induce decorations on the interior edges as follows. An edge is labeled with a group element g^+ one side and g^- on the other. Let v be the number of whole edges coming out of the vertex adjacent to g^+ (so $v \geq 1$, since g^+ labels such an edge). Let t be the total number of half and whole edges coming out of the vertex. If $n_1, \ldots, n_{t-v}$ are the marked points on the vertex adjacent to g^+, with corresponding sectors $g_{n_1}, \ldots, g_{n_{t-v}}$, then we define

$$g^+ = (t-2)j_W - \sum_i g_{n_i}, \tag{3.3}$$

with arithmetic occurring in the symmetry group (i.e, in $(\mathbb{Q}/\mathbb{Z})^N$). Define g^- similarly. We leave it to the reader to check that in fact, $g^- = -g^+$.

Example 3.1.5. *Let $W = D_4 = x^2y + y^3$ and $G = G_W^{max}$. Then $j_W = (1/3, 1/3)$. We will compute a basis for $\mathcal{A}_{W,G}$ in Section 4.2. Consider the correlator*

$\langle e_{(1/6,2/3)}, e_{(1/6,2/3)}, e_{(2/3,2/3)}, e_{(2/3,2/3)} \rangle$*. This correlator in fact satisfies the Dimension Axiom* *(so it is not trivially 0). Figure 3.5 shows an undecorated dual graph to a curve in* $\overline{\mathcal{M}}_{0,4}$*. Then if we number the insertions in our correlator 1 through 4, we get the decorated graph in Figure 3.6.*

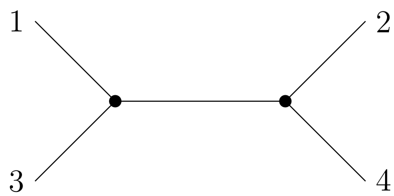

Figure 3.5: An undecorated dual graph.

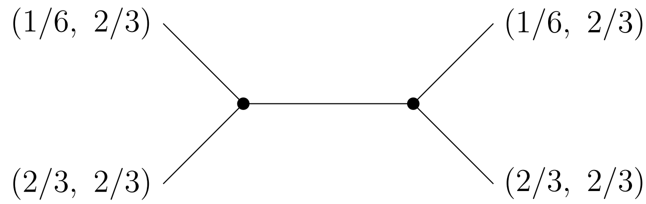

Figure 3.6: The graph in Figure 3.5 decorated with the insertions of the correlator $\langle e_{(1/6,2/3)}, e_{(1/6,2/3)}, e_{(2/3,2/3)}, e_{(2/3,2/3)}\rangle$.

Next, we compute g^+ *for the single edge. Note that it is adjacent to a vertex with two half edges, so in this case* $t = 3$*. Then using Equation (3.3), we have*

$$g^+ = (3 - 2)(1/3, 1/3) - (1/6, 2/3) - (2/3, 2/3) = (1/2, 0)$$

in G_W^{max}*. The decorations* g^+ *and* g^- *are diagrammed in Figure 3.7.*

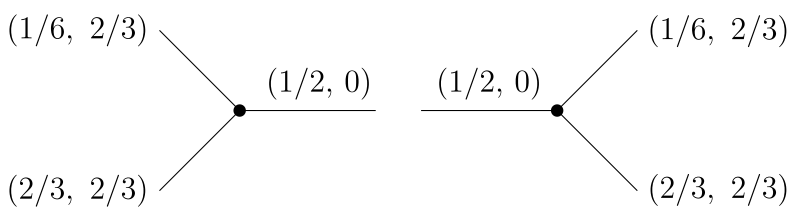

Figure 3.7: The graph in Figure 3.6 with g^+ and g^-.

We will use the idea of dual graphs of curves twice in our statement of the Concavity Axiom. The first reference will be to the line bundle degrees of such graphs. Let Γ be a fully decorated graph (so we have computed the elements g^+ as in the preceding paragraph). We will now define the j^{th} *line bundle degree* of a given vertex (as with correlators, we have a line bundle degree for each variable in W). We begin by collapsing any edges that are labeled with a group element that acts as the identity in the j^{th} coordinate. We want to think of half edges coming out of the bounding vertices of such an edge as living on the same vertex. Once we have done this, if a given vertex is adjacent to decorations $g_{n_1}, \ldots, g_{n_t}$ (including any decorations coming from edges), we define the j^{th} line bundle degree of the vertex to be

$$(t-2)q_j - \sum_i \Theta_j^{g_{n_i}}. \tag{3.4}$$

This time, we do our arithmetic in $\mathbb{Q}^N$. Note that this is well-defined since each $\Theta_j^{g_{n_i}}$ is a unique element in $\mathbb{Q} \cap [0, 1)$.

Example 3.1.6. *Let us compute the line bundle degrees for the fully decorated graph in Figure 3.7.*

We begin with the 1^{st} line bundle degrees. Because the first coordinate of g^+ is $1/2 \neq 0$, we do not collapse any edges. Instead, we compute two 1^{st} line bundle degrees, one for each vertex in the graph. Using Equation (3.4), we compute the 1^{st} line bundle degree of the left vertex to be

$$(3-2)(1/3) - 1/6 - 2/3 - 1/2 = -1.$$

The first line bundle degree on the right vertex is identical.

Now let us compute the 2^{nd} line bundle degrees. In fact, since the second coordinate of g^+ is 0, we contract the middle edge for this computation, leaving only one vertex (and hence only one line bundle degree to compute). This vertex is adjacent to 4 half-edges, and hence $t = 4$ in Equation (3.4). We compute the line bundle degree to be

$$(4-2)(1/3) - 2/3 - 2/3 - 2/3 - 2/3 = -2.$$

Note that because of the way we have defined g^+ and g^-, when we compute the j^{th} line bundle degree of a vertex adjacent to at least one of these elements, we will always get an integer. If none of the half edges coming out of our vertex are labeled with a g^+ or a g^-, non-integer degrees are possible. If we do get non-integer degrees, the corresponding correlator vanishes by the Line Bundle Degrees Axiom (Axiom 3).

The second time we will use the idea of these dual graphs in the statement of the Concavity Axiom is in the precise definition of the Δ classes. This Δ is in fact a whole family of classes indexed by (undecorated) graphs Γ with a single edge (i.e., graphs corresponding to curves with a single node). By convention, we always assume that the half-edge labeled with 1 is on the left in such a graph. Note that this defines a unique group element g^+, calculated as above. We will write the tautological class associated with such a graph as Δ_K, where $K \subset \{1, \dots, k\}$ is the set of integers labeling the half edges on the left of our edge (so $1 \in K$ always). In the Concavity Axiom, we will take a sum over all such classes.

Example 3.1.7. *In the context of our running example, let us consider all possible undecorated graphs Γ with a single edge corresponding to curves in $\overline{\mathcal{M}}_{0,4}$. Curves with the same marked points on each of the irreducible components correspond to the same point in $\overline{\mathcal{M}}_{0,4}$; hence the order of the numbers on the half edges coming out of a given vertex on Γ does not matter. With our convention that the half-edge labeled 1 is always on the left, we have exactly three possibilities, as diagrammed in Figure 3.8.*

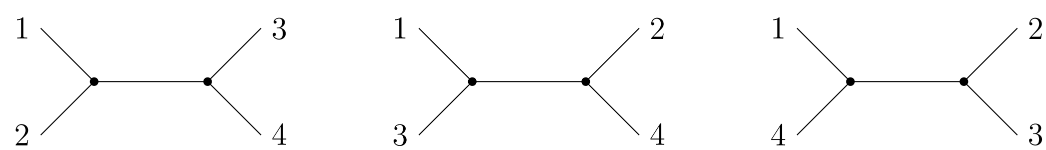

Figure 3.8: The three possibilities for a graph dual to a curve in $\overline{\mathcal{M}}_{0,4}$.

We can decorate each of these graphs with insertions from the correlator $\langle e_{(1/6,2/3)}, e_{(1/6,2/3)}, e_{(2/3,2/3)}, e_{(2/3,2/3)} \rangle$, as we did to produce Figure 3.6 from Figure 3.5. Note that when decorated, the second and third graphs will be identical (though distinct from the first graph). Because the second and third graphs came from distinct undecorated graphs, they will be counted separately in our Concavity Axiom computations. Each of these three graphs defines a unique g^+, as diagrammed in Figure 3.7 for the second and third graphs.

We are now ready to present the Concavity Axiom.

Axiom 6 (**Concavity** [FJR13] Theorem 4.1.8 5a and [Fra12] Section 4.1)**.** *Consider the correlator $\langle \alpha_1, \ldots, \alpha_k \rangle$. Assume that each α_j comes from a sector g_j whose fixed locus has dimension 0, and that all the line bundle degrees of all possible decorated dual graphs are negative integers (not zero). If the polynomial W has two variables with weights (q_1, q_2), then*

$$\Lambda_{0,k}(\alpha_1, \ldots, \alpha_k) = ch_2(\mathcal{E}_1) + ch_2(\mathcal{E}_2) + \frac{1}{2} \left(\, ch_1(\mathcal{E}_1) + ch_1(\mathcal{E}_2) \, \right)^2 \tag{3.5}$$

where we define the symbols $ch_d(\mathcal{E}_i)$ as follows:

$$ch_d(\mathcal{E}_i) = \frac{B_{d+1}(q_i)}{(d+1)!} \, \kappa_d - \sum_{j=1}^{k} \frac{B_{d+1}(\Theta_i^{g_j})}{(d+1)!} \, \psi_j^d + \sum_{\Gamma} \frac{B_{d+1}(\Theta_i^{g_-})}{(d+1)!} P_{d,\Gamma}(\Delta). \tag{3.6}$$

Here, B_d is the d^{th} Bernoulli polynomial, and we write a group element $g = (\Theta_1^g, \ldots, \Theta_N^g)$ in additive notation. The final sum is over all single-edged dual graphs Γ with the half-edge labeled 1 on the left. We define the polynomials $P_{d,\Gamma}(\Delta)$ below.

$$\begin{aligned} P_{1,\Gamma}(\Delta) &= \Delta_K \\ P_{2,\Gamma}(\Delta) &= \begin{cases} -\Delta_K \Delta_{1 \cup K^C} & |K| = 3 \\ \Delta_K \Delta_{K \cup \{j\}} & |K| = 2, \; j \notin K. \end{cases} \end{aligned}$$

The reader should be aware that what we are calling the Concavity Axiom is really a combination of that axiom (originally stated in Theorem 4.1.8 part 5a of [FJR13]) and Chiodo's formula (see [Chi08] Theorem 1.1.1 and [Fra12] Section 4.1) reduced to the two-variable case. I worked through a small amount of algebra to arrive at this simplified form. The strategy for the computation I used is outlined in Section 4.1 of [Fra12]. I omit the computation itself because it requires extensive vocabulary in areas with no bearing on the rest of this thesis. However, the reader should be aware that [Fra12] contains a few minor typos. First, Item 4 of Property 3 should read $c_t(\mathcal{E}) = \sum_{i=1}^{\infty} (1 - c_t(-\mathcal{E}))^i$. Second, the left hand side of the last two equalities in Equation (4.3) should be $(j_K)^*(\psi_-)$ instead of $(j_K)^*(\psi_+)$.

These six axioms are not all the known "correlator axioms," but just the ones we will use in this thesis. In general, not every correlator can be computed using these axioms, or even using the full set of correlator axioms. However, we will be able to compute every correlator needed in this thesis using the axioms we have presented. By no means does this make our task easy: there are still an infinite number of correlators appearing in a potential T, so computing them all (using just the axioms) cannot be done in a finite thesis.

Thankfully, there is another strategy that will significantly reduce the number of correlators we need to compute. The remainder of this chapter will explain that strategy, known as *reconstruction*. The reader should bear in mind that even though reconstruction will reduce the number of correlator computations we need to do, reconstruction will not reduce these computations to a finite number. Thus, there will still be a good bit of work to do after this chapter.

3.2 The reconstruction lemma

In this chapter we will prove the reconstruction lemma, an essential tool for computing correlators. The reader who is willing to take the proof on faith can skim this section.

The idea of the reconstruction lemma is to "reconstruct" all correlators in terms of a smaller, fixed set of correlators. This strategy was originally used in [FJR13], where the proof relied on a certain equation derived from WDVV. As it is formulated in [FJR13], this equation is incorrect, though the errors do not affect the proof of reconstruction itself. Because this is equation is useful in its own right for computing correlators and we will use it directly in Chapter 5, we will derive the correct version in the following lemma.

Lemma 3.2.1. *Let $K = (\xi_1, \ldots, \xi_{k-3})$ be a tuple of elements in $\mathcal{A}_{W,G}$. Let $\{\delta_l\}$ be a basis for $\mathcal{A}_{W,G}$ and let $\{\delta_l'\}$ be a dual basis with respect to the pairing. Then the following equality* *holds:*

$$\langle \xi \in K,\, \alpha,\, \beta,\, \epsilon \star \phi \rangle = \sum_{I \sqcup J = K} \sum_{l} c_{I,J} \langle \xi \in I,\, \alpha,\, \epsilon,\, \delta_l \rangle \langle \delta_l',\, \phi,\, \beta,\, \xi \in J \rangle$$
$$- \sum_{\substack{I \sqcup J = K \\ J \neq \emptyset}} \sum_{l} c_{I,J} \langle \xi \in I,\, \alpha,\, \beta,\, \delta_l \rangle \langle \delta_l',\, \phi,\, \epsilon,\, \xi \in J \rangle, \tag{3.7}$$

where

$$c_{I,J} = \frac{\prod n_K(\xi_k)!}{\prod n_I(\xi_i)! \prod n_J(\xi_j)!}$$

are integer coefficients. Here, $n_X(x)$ refers to the number of elements equal to x in the tuple X. Each product in $c_{I,J}$ is taken over all distinct elements $x \in X$.

Proof. Recall the WDVV equation

$$\sum_{k,l} T_{ijk} \eta^{kl} T_{lmn} = \sum_{k,l} T_{imk} \eta^{kl} T_{ljn}. \tag{3.8}$$

We wish to see what this equation says about correlators. If we let $C = \{\alpha_i\}$, then recall that the definition of T is

$$T = \sum_{\rho \geq 3} \sum_{(\alpha_1, \ldots, \alpha_\rho) \in C^\rho} \frac{\langle \alpha_1, \ldots, \alpha_\rho \rangle}{\rho!} t_1 \ldots t_\rho$$

where $(\alpha_1, \ldots, \alpha_\rho) \in C^\rho$ is an *ordered* tuple. For compactness, denote such a tuple by $X \in C^\rho$ and rewrite T as follows:

$$T = \sum_{\rho \geq 3} \sum_{X \in C^\rho} \frac{\langle X \rangle}{|X|!} \prod_{x \in X} t_x.$$

Equating coefficients of an arbitrary term $t_1 \ldots t_\rho$ in Equation (3.8) should tell us something about the correlators. To this end, let us compute a third partial derivative of T.

$$T_{ijk} = \sum_{\rho \geq 3} \sum_{X \in C^\rho} \frac{\langle X \rangle}{|X|!} \frac{\prod_{x \in X} t_x}{t_{\alpha_i} t_{\alpha_j} t_{\alpha_k}} \, n_X(\alpha_i) \, n_{X \setminus \alpha_i}(\alpha_j) \, n_{X \setminus \{\alpha_i, \alpha_j\}}(\alpha_k).$$

Now let $A \in C^\rho$. We will compute the coefficient of $\prod_{a\in A} t_a$ in T_{ijk}. We find this equals

$$\sum_{\substack{X\in C^{\rho+3} \\ X=A\cup\{\alpha_i,\alpha_j,\alpha_k\}}} \frac{\langle X\rangle}{|X|!}\, n_X(\alpha_i)\, n_{X\setminus\alpha_i}(\alpha_j)\, n_{X\setminus\{\alpha_i,\alpha_j\}}(\alpha_k).$$

Now, each of the terms in this sum is the same. The number of terms is equal to the number of ways you can arrange the elements of $X = A \cup \{\alpha_i, \alpha_j, \alpha_k\}$. So, our coefficient equals

$$\frac{|X|!}{\prod_{x\in X} n_X(x)!} \frac{\langle X\rangle}{|X|!}\, n_X(\alpha_i)\, n_{X\setminus\alpha_i}(\alpha_j)\, n_{X\setminus\{\alpha_i,\alpha_j\}}(\alpha_k).$$

The $|X|!$ terms cancel. Also, I claim that in fact we can write our coefficient in terms of A as follows:

$$\frac{\langle A, \alpha_i, \alpha_j, \alpha_k\rangle}{\prod_{a\in A} n_A(a)!}.$$

This is proved separately for the case where $\alpha_i = \alpha_j = \alpha_k$, the case where $\alpha_i = \alpha_j \neq \alpha_k$, and the case where α_i, α_j, and α_k are all distinct. We will prove the second case; the other two are similar. If $\alpha_i = \alpha_j \neq \alpha_k$, then $n_X(\alpha_i) = n_A(\alpha_i) + 2$, $n_{X\setminus\alpha_i}(\alpha_j) = n_A(\alpha_i) + 1$, and $n_{X\setminus\{\alpha_i,\alpha_j\}}(\alpha_k) = n_A(\alpha_k) + 1$. So our coefficient can be written

$$\begin{aligned}
&\frac{\langle X\rangle\ (n_A(\alpha_i)+2)\ (n_A(\alpha_i)+1)\ (n_A(\alpha_k)+1)}{(n_A(\alpha_i)+2)!\,(n_A(\alpha_k)+1)!\,\prod_{x\in A\setminus\{\alpha_i,\alpha_k\}} n_X(a)!} \\
&= \frac{\langle X\rangle}{n_A(\alpha_i)!n_A(\alpha_k)!\prod_{a\in A\setminus\{\alpha_i,\alpha_k\}} n_A(a)!} \\
&= \frac{\langle A, \alpha_i, \alpha_j, \alpha_k\rangle}{\prod_{a\in A} n_A(a)!}.
\end{aligned}$$

This was the desired equality.

The next step is to use our formula for the coefficients of T_{ijk} to compare coefficients of monomials in Equation (3.8). So let $X \in B^k$. We will look at the coefficient of $\prod_{x\in X} t_x$ on each side of Equation (3.8); these must be equal. We find

$$\sum_{k,l} \eta^{kl} \sum_{A\sqcup B=X} \frac{\langle A, \alpha_i, \alpha_j, \alpha_k\rangle}{\prod_{a\in A} n_A(a)!} \frac{\langle B, \alpha_l, \alpha_m, \alpha_n\rangle}{\prod_{b\in B} n_B(b)!} = \sum_{k,l} \eta^{kl} \sum_{A\sqcup B=X} \frac{\langle A, \alpha_i, \alpha_m, \alpha_k\rangle}{\prod_{a\in A} n_A(a)!} \frac{\langle B, \alpha_l, \alpha_j, \alpha_n\rangle}{\prod_{b\in B} n_B(b)!}.$$

The final step is to manipulate this a bit and use the definition of the product. We break up the sum on the left side to obtain

$$\sum_{k,l}\eta^{k,l}\frac{\langle X,\alpha_i,\alpha_j,\alpha_k\rangle\langle\alpha_l,\alpha_m,\alpha_n\rangle}{\prod_{x\in X}n_X(x)!} = \sum_{k,l}\eta^{kl}\sum_{A\sqcup B=X}\frac{\langle A,\alpha_i,\alpha_m,\alpha_k\rangle}{\prod_{a\in A}n_A(a)!}\frac{\langle B,\alpha_l,\alpha_j,\alpha_n\rangle}{\prod_{b\in B}n_B(b)!}$$
$$-\sum_{k,l}\eta^{kl}\sum_{A\sqcup B=X,B\neq\emptyset}\frac{\langle A,\alpha_i,\alpha_j,\alpha_k\rangle}{\prod_{a\in A}n_A(a)!}\frac{\langle B,\alpha_l,\alpha_m,\alpha_n\rangle}{\prod_{b\in B}n_B(b)!}.$$

We multiply by the denominator of the left side and use linearity of correlators to obtain

$$\langle X,\ \alpha_i,\ \alpha_j,\ \sum_{k,l}\langle\alpha_m,\alpha_n,\alpha_l\rangle\eta^{lk}\alpha_k\rangle = \sum_{A\sqcup B=X}c_{A,B}\sum_l\langle A,\alpha_i,\alpha_m,\sum_k\eta^{kl}\alpha_k\rangle\langle B,\alpha_l,\alpha_j,\alpha_n\rangle$$
$$-\sum_{\substack{A\sqcup B=X\\ B\neq\emptyset}}c_{A,B}\sum_l\langle A,\alpha_i,\alpha_j,\sum_k\eta^{kl}\alpha_k\rangle\langle B,\alpha_l,\alpha_m,\alpha_n\rangle$$

where

$$c_{A,B}=\frac{\prod n_X(x)!}{\prod n_A(a)!\prod n_B(b)!}.$$

Now, $\sum_{k,l}\langle\alpha_m,\alpha_n,\alpha_l\rangle\eta^{l,k}\alpha_k$ is just the definition of the product $\alpha_m\star\alpha_n$ (see Equation 2.13). Moreover, we calculate the pairing of $\sum_k\eta^{kl}\alpha_k$ and α_l to be

$$\begin{aligned}\langle\sum_k\eta^{kl}\alpha_k,\ \alpha_l\rangle &= \sum_k\eta^{kl}\langle\alpha_k,\ \alpha_l\rangle\\ &= \sum_k\eta^{lk}\eta_{kl}=\sum_k\delta_{lk}\\ &= 1.\end{aligned}$$

Then

$$\langle X,\alpha_i,\alpha_j,\alpha_m\star\alpha_n\rangle = \sum_{A\sqcup B=X}c_{A,B}\sum_l\langle A,\alpha_i,\alpha_m,\alpha_l'\rangle\langle B,\alpha_l,\alpha_j,\alpha_n\rangle$$
$$-\sum_{A\sqcup B=X,B\neq\emptyset}c_{A,B}\sum_l\langle A,\alpha_i,\alpha_j,\alpha_l'\rangle\langle B,\alpha_l,\alpha_m,\alpha_n\rangle$$

where α_l' is the dual basis element to α_l. $\square$

Using Lemma 3.2.1, the reconstruction lemma will argue that all correlators can be written as sums involving *basic* correlators. We define these below.

Definition 3.2.2. *A correlator* $\langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$ *is* basic *if* α_i *is primitive for* $i \leq k-2$.

An element $\alpha \in \mathcal{A}_{W,G}$ which is not a scalar multiple of the identity is *primitive* if whenever $\alpha = \phi \star \epsilon$, either ϕ or ϵ is in $\mathbb{C}$.

Theorem 3.2.3 (Reconstruction Lemma [FJR13] Lemma 6.2.6)**.** *A k-point correlator* $\langle \xi_1, \ldots, \xi_{k-3}, \alpha, \beta, \epsilon \star \phi \rangle$ *satisfies*

$$\begin{aligned} \langle \xi_1, \ldots, \xi_{k-3}, \alpha, \beta, \epsilon \star \phi \rangle = S + &\langle \xi_1, \ldots, \xi_{k-3}, \alpha, \epsilon, \beta \star \phi \rangle \\ &+ \langle \xi_1, \ldots, \xi_{k-3}, \alpha \star \epsilon, \beta, \phi \rangle \\ &- \langle \xi_1, \ldots, \xi_{k-3}, \alpha \star \beta, \epsilon, \phi \rangle, \end{aligned} \tag{3.9}$$

where S is a linear combination of correlators with fewer than k insertions. In addition, the k-point correlators are uniquely determined by the pairing, the three-point correlators, and by basic t-point correlators for $t \leq k$.

Proof. Consider Equation (3.7). We can pull out the terms where either I or J is the empty set (there are two of these in the first sum, but only one in the second). Remaining terms will be a sum of products of correlators with strictly fewer than k insertions. We call this sum S. This shows

$$\begin{aligned} \langle \xi_1, \ldots, \xi_{k-3}, \alpha, \beta, \epsilon \star \phi \rangle = S + &\sum_l \langle \xi_1, \ldots, \xi_{k-3}, \alpha, \epsilon, \delta_l \rangle \langle \delta_l', \phi, \beta \rangle \\ &+ \sum_l \langle \alpha, \epsilon, \delta_l \rangle \langle \delta_l', \phi, \beta, \xi_1, \ldots, \xi_{k-3} \rangle \\ &- \sum_l \langle \alpha, \beta, \delta_l \rangle \langle \delta_l', \phi, \epsilon, \xi_1, \ldots, \xi_{k-3} \rangle. \end{aligned}$$

Note that in each of the three summands explicitly written out, the coefficients $c_{I,J}$ are equal to 1. From the definition of the multiplication, we know $\sum_l \langle \delta_l', \phi, \beta \rangle \delta_l$ equals $\phi \star \beta$. Similar computations prove Equation (3.9).

Now, for the second claim, we will repeatedly apply Equation (3.9) to a k-point correlator with $m < k-2$ primitive elements. Such a correlator can be written as

$$\langle \xi_1, \ldots, \xi_{k-3}, \alpha, \beta, \nu \rangle$$

where α, β, and ν are all nonprimitive. Then we can write $\nu = \epsilon \star \phi$ where ϵ is primitive. Then Equation (3.9) allows us to write this correlator in terms of correlators with fewer than k insertions, and in terms of the sum

$$\langle \xi_1, \ldots, \xi_{k-3}, \alpha, \epsilon, \beta \star \phi \rangle + \langle \xi_1, \ldots, \xi_{k-3}, \alpha \star \epsilon, \beta, \phi \rangle - \langle \xi_1, \ldots, \xi_{k-3}, \alpha \star \beta, \epsilon, \phi \rangle.$$

Note that two of these correlators (the ones with ϵ in them) have $m+1$ primitive elements. This is not the case for the correlator $\langle \xi_1, \ldots, \xi_{k-3}, \alpha \star \epsilon, \beta, \phi \rangle$, so we apply Equation (3.9) to this correlator with the factorization $\epsilon' \star \phi'$ of ϕ, where ϵ' is again primitive. Since ϕ is an element of a graded ring, and the degree of ϕ' decreases with each iteration of our algorithm, this process must terminate at some point. That is, at some point we will see that all three special correlators in Equation (3.9) have more than m primitive elements.

We can apply this process inductively to write our original k-point correlator solely in terms of basic t-point correlators with $t \leq k$, the three-point correlators, and the pairing. $\square$

Thus, the reconstruction lemma reduces our problem to the computation of only the *basic* correlators of $\mathcal{A}_{W,G}$. The challenge is that there is still an infinite number of basic correlators (in particular, the number of insertions in a correlator can be arbitrarily large). All known examples of Landau-Ginzburg A-model Frobenius manifolds have been computed by finding a bound, using the Dimension Axiom (Axiom 2), on the number of insertions in a nonzero basic correlator. The same strategy fails in our case because our class of examples is “too general.” Thus, we need to develop a new strategy to complete our infinite number of correlator computations. This strategy is presented in the next section, along with its application to our problem.

Chapter 4. Computation of Frobenius manifolds

4.1 General strategy for computing correlators

The canonical basis for the graded algebra $\mathcal{A}_{W,G}$ is the one described in Section 2.2.2. It is a property of such algebras that there will be a unique element of highest degree in this basis. If we consider the entire algebra $\mathcal{A}_{W,G}$, however, this element will lose its uniqueness, as it will span a 1-dimensional subspace of elements of the same (highest) degree. The reader should keep this in mind as we present the following notation.

Notation 4.1.1. *Let W be an invertible polynomial, and let $A = \mathcal{A}_{W,G}$. Then we will use $\mathbf{1}$ to denote the identity in A and $\mathbf{h}_W$ to denote the unique basis element of highest W-degree in our canonical basis for A. We will also use $\mathbf{p}_W$ to denote a primitive element in our canonical basis for A.*

Note that because $\mathcal{A}_{W,G}$ is an algebra, $\mathbf{h}_W$ will span a 1-dimensional subspace of elements of $\mathcal{A}_{W,G}$ that all have the same (highest) degree. We choose the canonical element $\mathbf{h}_W$ to be the element in this subspace that is given by our standard computation of a vector space basis of $\mathcal{A}_{W,G}$, as described in Section 2.2.2.

In the remainder of this thesis, we will consider polynomials $W = \sum W_\ell$ where each W_ℓ is an invertible polynomial in distinct variables. (Note that by Theorem 2.2.5, W is also invertible.) In such a situation, we will call the variables appearing in W_ℓ (and in no other summand of W) the variables *subtended* by W_ℓ. Now we present a key theorem which will allow us to exploit the structure of $W = \sum W_\ell$ in computations of correlators.

Theorem 4.1.2 ([FJJS12], Axiom 4 and [FJR13], Theorem 4.2.2)**.** *Let W be an invertible* *polynomial and let G_W^{max} be its maximal symmetry group. Suppose W can be written as the sum of two invertible polynomials $W = Y + Z$ with distinct variables. Then as Frobenius algebras,*

$$\mathcal{A}_{Y,G_Y^{max}} \otimes \mathcal{A}_{Z,G_Z^{max}} \cong \mathcal{A}_{W,G_W^{max}}$$

via the isomorphism $(me_g, ne_h) \mapsto mne_{g+h}$. When we write the sum $g+h$, we are thinking of g and h as elements of G_W^{max} via inclusion (see Theorem 2.2.8). Moreover,

$$\begin{aligned}\Lambda_{0,k}(m_1e_{g_1},\, m_2e_{g_2},\, \ldots,\, m_ke_{g_k}) &\otimes \Lambda_{0,k}(n_1e_{h_1},\, n_2e_{h_2},\, \ldots,\, n_ke_{h_k}) \\ &= \Lambda_{0,k}(m_1n_1e_{g_1+h_1},\; m_2n_2e_{g_2+h_2},\; \ldots,\; m_kn_ke_{g_k+h_k}).\end{aligned}$$

If $W = W_1 + W_2 + \ldots + W_r$, we will routinely identify $\mathcal{A}_{W,G_W^{max}}$ with $\mathcal{A}_{W_1,G_{W_1}^{max}} \otimes \ldots \otimes \mathcal{A}_{W_r,G_{W_r}^{max}}$ via this isomorphism. Moreover, we will identify elements $\alpha \in \mathcal{A}_{W_\ell,G_{W_\ell}^{max}}$ with their images $(\mathbf{1}, \ldots, \alpha, \ldots, \mathbf{1}) \in \mathcal{A}_{W,G_W^{max}}$. One consequence of this theorem is that to calculate a basis for W, we only need calculate bases for $W_1, \ldots, W_r$. Another consequence is that primitive elements in $\mathcal{A}_{W,G_W^{max}}$ will be of the form $(\mathbf{1}, \ldots, \mathbf{1}, \mathbf{p}_{W_i}, \mathbf{1}, \ldots, \mathbf{1})$, where $\mathbf{p}_{W_i}$ is a primitive element in $\mathcal{A}_{W_i,G_{W_i}^{max}}$. Also, from the definition of the W-degree of an element $\alpha \in \mathcal{A}_{W,G_W^{max}}$, it is clear that if $(\alpha_1, \ldots, \alpha_r) \mapsto \alpha$ in the isomorphism of Theorem 4.1.2, then $deg_W(\alpha) = \sum_{i=1}^r deg_{W_i}(\alpha_i)$. Thus, $\mathbf{h}_W = (\mathbf{h}_{W_1}, \mathbf{h}_{W_2}, \ldots, \mathbf{h}_{W_r})$ up to a constant.

The idea behind our new strategy for computing correlators is to use some new notation for correlators that takes advantage of the tensor product structure described in the above theorem. Once we have this notation and some associated vocabulary, it will be simple enough to compute specific examples of Frobenius manifolds for polynomials that are sums of A_{n-1} and D_{n+1} polynomials. The difficulty, again, will be to find a general argument that works for all examples. Our new notation, however, will still be used in the general proof.

Notation 4.1.3. *Because it is helpful to work with tensor products of our algebras $\mathcal{A}_{W,G}$, we will find it useful to think of correlators as grids, with rows corresponding to insertions in the correlator and columns corresponding to polynomial summands in a decomposition* *of W. Thus, let $W = W_1 + W_2 + \ldots + W_r$ be a decomposition of W. If $(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ir})$ maps to α_i in the isomorphism of Theorem 4.1.2, we will represent the genus-zero correlator $\langle \alpha_1, \alpha_2, \ldots \alpha_k \rangle$ with the grid*

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1r} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{k1} & \alpha_{k2} & \ldots & \alpha_{kr} \end{bmatrix}.$$

Definition 4.1.4. *A* basic column *is a column in such a grid as above, given that the grid corresponds to a nonzero basic correlator. The* basic entries *of a basic column are the first $k-2$ entries of the column.*

Note that the basic entries of a basic column must be either $\mathbf{1}$ or primitive. Thus, with this notation, it is easy to identify a basic correlator. Such a correlator will have at most one primitive element $\mathbf{p}_{W_i}$ in each of the top $k-2$ rows, with the remaining entries in these top rows equal to $\mathbf{1}$.

Example 4.1.5. *Figure 4.1 contains three examples of correlator grids for 5-point correlators of a polynomial with 3 summands. Grid (1) corresponds to the basic correlator*

$$\underset{(1)}{\begin{bmatrix} \mathbf{p}_{W_1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{p}_{W_2} & \mathbf{1} \\ \mathbf{1} & \mathbf{p}_{W_2} & \mathbf{1} \\ * & * & * \\ * & * & * \end{bmatrix}} \quad \underset{(2)}{\begin{bmatrix} \mathbf{p}_{W_1} & \mathbf{p}_{W_2} & \mathbf{1} \\ \mathbf{1} & \mathbf{p}_{W_2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{p}_{W_3} \\ * & * & * \\ * & * & * \end{bmatrix}} \quad \underset{(3)}{\begin{bmatrix} \mathbf{p}_{W_1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{p}_{W_1} & \mathbf{1} & \mathbf{1} \\ * & * & * \\ * & * & * \end{bmatrix}}$$

Figure 4.1: Examples of grids corresponding to basic and non-basic correlators.

$\langle \mathbf{p}_{W_1}, \mathbf{p}_{W_2}, \mathbf{p}_{W_2}, *, * \rangle$. Grid (2) does not correspond to a basic correlator because the first row has 2 primitive elements. Equivalently, the corresponding correlator is $\langle \mathbf{p}_{W_1}\mathbf{p}_{W_2}, \mathbf{p}_{W_2}, \mathbf{p}_{W_3}, *, * \rangle$, and $\mathbf{p}_{W_1}\mathbf{p}_{W_2}$ is not primitive. Grid (3) corresponds to the basic correlator $\langle \mathbf{p}_{W_1}, \mathbf{1}, \mathbf{p}_{W_1}, *, * \rangle$. In each grid, the $*$s indicate that this entry of the grid could be any element of the appropriate

ring without changing the example.

The next step is to combine Axioms 2 and 3 into a form that is particularly useful with our new grid notation.

Lemma 4.1.6. *A non-vanishing genus-zero k-point correlator corresponding to a polynomial W in N variables must satisfy*

$$\frac{1}{2}\sum_{i=1}^{k} D_i - \sum_{j=1}^{N} l_j = N + k - 3 \tag{4.1}$$

where l_j is the j^{th} line bundle degree, and D_i is the complex dimension of the fixed locus of the group element corresponding to the i^{th} insertion in the correlator.

Proof. We can combine Axiom 2 and Definition 2.2.13 to get the following, which must be satisfied by any nonvanishing k-point correlator:

$$\frac{1}{2}\sum_{i=1}^{k}\left(D_i + 2\sum_{j=1}^{N}(\Theta_j^{g_i} - q_j)\right) = \hat{c}_W + k - 3. \tag{4.2}$$

Also, we can rewrite Equation (3.2), which defines the line bundle degrees, as follows:

$$\sum_{i=1}^{k} \Theta_j^{g_i} = (k-2)q_j - l_j. \tag{4.3}$$

Hence, when we combine Equations (4.2) and (4.3), we find that a nonvanishing correlator must satisfy

$$\frac{1}{2}\sum_{i=1}^{k} D_i + \sum_{j=1}^{N}\left(\sum_{i=1}^{k}\Theta_j^{g_i} - \sum_{i=1}^{k} q_j\right) = \hat{c}_W + k - 3$$
$$\frac{1}{2}\sum_{i=1}^{k} D_i + \sum_{j=1}^{N}((k-2)q_j - l_j - kq_j) = \hat{c}_W + k - 3$$
$$\frac{1}{2}\sum_{i=1}^{k} D_i + \sum_{j=1}^{N}(-2q_j - l_j) = \sum_{j=1}^{N}(1-2q_j) + k - 3$$
$$\frac{1}{2}\sum_{i=1}^{k} D_i - \sum_{j=1}^{N} l_j = N + k - 3.$$

□

To see how this lemma relates to our grid notation, let $W = \sum_{\ell=1}^{r} W_\ell$ be a sum of invertible polynomials. Suppose we have a nonvanishing correlator $\langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$, and suppose $(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ir})$ maps to α_i in the isomorphism of Theorem 4.1.2. Note that we can break up the left hand side of Equation (4.1) as a sum over the columns of our correlator (equivalently: polynomial summands) in the following way: if W_ℓ corresponds to coordinates a_ℓ through b_ℓ, then

$$\frac{1}{2}\sum_{i=1}^{k} D_i - \sum_{j=1}^{N} l_j = \sum_{\ell=1}^{r}\left(\frac{1}{2}\sum_{i=1}^{k} D_i^\ell - \sum_{j=a_\ell}^{b_\ell} l_j\right)$$

where D_i^ℓ is the complex dimension of the fixed locus of the element $\alpha_{i\ell}$.

This leads to the following definition.

Definition 4.1.7. *Let W_ℓ be a summand (in distinct variables) of the invertible polynomial W. Then the quantity*

$$C_\ell = \frac{1}{2}\sum_{i=1}^{k} D_i^\ell - \sum_{j=a_\ell}^{b_\ell} l_j \tag{4.4}$$

is the contribution *of the ℓ^{th} column of a correlator grid.*

In this thesis we will be concerned with polynomials $W = W_1 + \ldots + W_r$ where each summand W_ℓ is an A_{n-1} or D_{n+1} polynomial. We will see in Section 4.2 that in these two cases, $D_i^\ell \in 2\mathbb{Z}$ for any $\alpha_{i\ell} \in W_\ell$. Since a nonvanishing correlator must have integer line bundle degrees, in this case the contribution C_ℓ must be an integer. Then Lemma 4.1.6 says that a nonvanishing basic correlator consists of r basic columns whose integer contributions sum to $n+k-3$. (Note that for a general polynomial W, C_ℓ need only be a half-integer.)

Hence, the remainder of our analysis will be to identify the possible contributions of a basic column, and from that describe all nonzero basic correlators. For convenience we will rewrite a column contribution C_ℓ using Equation (3.2) as follows:

$$\begin{aligned}
C_\ell &= \frac{1}{2}\sum_{i=1}^{k} D_i^\ell - \sum_{j=a_\ell}^{b_\ell} l_j \\
&= \frac{1}{2}\sum_{i=1}^{k} D_i^\ell - \sum_{j=a_\ell}^{b_\ell}\left(q_j(k-2) - \sum_{i=1}^{k}\Theta_j^{g_i}\right) \\
&= -(k-2)\sum_{j=a_\ell}^{b_\ell} q_j + \sum_{i=1}^{k}\left(\frac{1}{2}D_i^\ell + \sum_{j=a_\ell}^{b_\ell}\Theta_j^{g_i}\right).
\end{aligned}$$

This leads to the notion of the contribution of an *entry* in a correlator grid.

Definition 4.1.8. *Let W be an invertible polynomial, and let $\alpha_{i\ell}$ be the entry in the i^{th} row and ℓ^{th} column of a nonzero basic correlator grid. Then we define the* contribution *of $\alpha_{i\ell}$ to be the quantity*

$$c(\alpha_{i\ell}) = \frac{1}{2}D_i^\ell + \sum_{j=a_\ell}^{b_\ell}\Theta_j^{g_i}.$$

(Recall that since we restrict our attention to correlators of basis elements only, there is indeed a unique sector for $\alpha_{i\ell}$ equal to the appropriate restriction of g_i.)

Our new notation and definitions allow us to describe a nonzero basic correlator in a computationally straightforward way. That is, a nonzero basic correlator is defined by a grid consisting of r basic columns, whose contributions are integers C_ℓ (in the cases considered in this thesis) with

$$C_\ell = -(k-2)\sum_{j=a_\ell}^{b_\ell} q_j + \sum_{i=1}^{k} c(\alpha_{i\ell})$$

that also satisfy $\sum_{\ell=1}^{r} C_\ell = N+k-3$.

4.2 Frobenius algebra structure of A_{n-1} and D_{n+1}

For the remainder of this thesis, let $W = \sum W_i$ where each W_i is an A_{n-1} or D_{n+1} polynomial in distinct variables. According to Theorem 4.1.2, understanding the Frobenius algebra structure of $\mathcal{A}_{W,G_W^{max}}$ amounts to understanding the Frobenius algebra structure of $\mathcal{A}_{A_{n-1},G^{max}_{A_{n-1}}}$ and $\mathcal{A}_{D_{n+1},G^{max}_{D_{n+1}}}$. In this section we will analyze the structure of these two rings. For each, we will do the following: describe the canonical vector space basis for the algebra; compute the contribution c of each element; identify $\mathbf{1}$, $\mathbf{p}$, and $\mathbf{h}$ in this basis and show that $\mathbf{p}$ is unique relative to the basis, and characterize when the product of two elements will equal $\mathbf{h}$.

This section is mostly a summary of computations which will be cited from elsewhere in the literature. All we add is the computation of the contributions of the basis elements.

4.2.1 Frobenius algebra of A_{n-1}.

The Frobenius algebra structure of $A_{n-1} = x^n$ can be found in Proposition 6.1 of [JKV01] (note that this paper uses different notation than we do). Note that the weights vector of A_{n-1} is $(1/n)$. The computations in [JKV01] tell us that the ring $\mathcal{A}_{A_{n-1},G^{max}_{A_{n-1}}}$ has a unique primitive basis element. Also, when two basis elements come from inverse sectors, their product is a scalar mulitple of $\mathbf{h}_{A_{n-1}}$.

As explained in [JKV01], the multiplicative identity is given by $\mathbf{1} = e_{\frac{1}{n}}$, and if $n > 2$ the unique primitive basis element is given by $\mathbf{p}_W = e_{\frac{2}{n}}$. If $n = 2$, the ring $\mathcal{A}_{A_{n-1},G^{max}_{A_{n-1}}}$ contains only the identity. The basis element of highest W-degree is $\mathbf{h}_W = e_{\frac{n-1}{n}}$. We summarize the contribution of each element in Table 4.1 below.

Element	$e_a,\ 0 < a < n$
Expanded form	$e_{(a/n)}$
Contribution $c(e_a)$	a/n

Table 4.1: Vector space basis elements for $\mathcal{A}_{A_{n-1},G^{max}_{A_{n-1}}}$ and their contributions.

4.2.2 Frobenius algebra of D_{n+1}.

The Frobenius algebra structure of $D_{n+1} = x^2y + y^n$ is described in Section 5.3.1 of [FJR13], which gives an explicit isomorphism from the

Milnor ring $\mathcal{Q}_{x^2+xy^n}$ to $\mathcal{A}_{D_{n+1},G^{max}_{D_{n+1}}}$. This isomorphism shows that $\mathcal{A}_{D_{n+1},G^{max}_{D_{n+1}}}$ has a unique primitive element. The computations in this paper also tell us that when two elements come from inverse sectors, including the identity sector, their product is a scalar multiple of $\mathbf{h}_{D_{n+1}}$ (the unique element of highest weight in our usual basis). Note that the weights of D_{n+1} are $\left(\frac{n-1}{2n}, \frac{1}{n}\right)$.

Because the appearance of the primitive element in D_{n+1} depends on whether $n > 2$ or $n = 2$, a priori we may have to do these cases separately. For $n > 2$, the vector space basis for D_{n+1} is summarized in Table 4.2 below.

Element	xe_0	e_{n-a} , $0 < a < n$	e_{n+a} $0 < a < n$
Expanded form	$xe_{(0,0)}$	$e_{(\frac{n+a}{2n}, \frac{n-a}{n})}$	$e_{(\frac{n-a}{2n}, \frac{a}{n})}$
Contribution c	1	$\frac{3n-a}{2n}$	$\frac{n+a}{2n}$

Table 4.2: Vector space basis elements for $\mathcal{H}_{D_{n+1},G^{max}_{D_{n+1}}}$ and their contributions.

As explained in [FJR13], we know that $\mathbf{1} = e_{n+1}$ with contribution $\frac{n+1}{2n}$, $\mathbf{p}_{D_{n+1}} = e_{n+2}$ with contribution $\frac{n+2}{2n}$, and $\mathbf{h}_{D_{n+1}} = e_{n-1}$ with contribution $\frac{3n-1}{2n}$. Note that $\mathbf{h}_{D_{n+1}}$ has the largest contribution of any element in this basis as well as the largest W-degree.

For $n = 2$, a basis for the state space is summarized in Table 4.3 below.

Element	$\mathbf{1}$	$\mathbf{p}$	$\mathbf{h}$
Expanded form	$e_{(1/4,1/2)}$	$xe_{(0,0)}$	$e_{(3/4,1/2)}$
Contribution c	3/4	1	5/4

Table 4.3: Vector space basis elements for $\mathcal{H}_{D_3,G^{max}_{D_3}}$ and their contributions.

Because the elements $\mathbf{1}$, $\mathbf{p}_{D_3}$, and $\mathbf{h}_{D_3}$ all have contributions in agreement with the formulas listed in Table 4.2, we can consider the cases $n > 2$ and $n = 2$ together.

4.3 Basic four-point correlators

In this section, as in the remainder of this thesis, we take $W = \sum W_\ell$ be be a sum of A_{n-1} and D_{n+1} polynomials in distinct variables. We call the A_{n-1} and D_{n+1} polynomials the *atomic summands* of W.

Recall that the goal of this thesis is to compute the Frobenius manifold structure of $\mathcal{A}_{W,G_W^{max}}$ for these polynomials. We have seen that to do so, all we really need to do is compute all possible correlators. Moreover, thanks to the Reconstruction Lemma, we can focus our attention on 3-point correlators and basic k-point correlators for $k \geq 4$. Now, let $W = W_1 + W_2$, let $\alpha_i \in \mathcal{A}_{W_1,G_{W_1}^{max}}$, and let $\beta_i \in \mathcal{A}_{W_2,G_{W_2}^{max}}$. Then it follows from Theorem 4.1.2 and the definition of a correlator as an integral that

$$\langle \alpha_1\beta_1,\, \alpha_2\beta_2,\, \alpha_3\beta_3 \rangle = \langle \alpha_1,\, \alpha_2,\, \alpha_3 \rangle \langle \beta_1,\, \beta_2,\, \beta_3 \rangle,$$

where $(\alpha_i, \beta_i) \mapsto \alpha_i\beta_i$ in the isomorphism of Theorem 4.1.2. Thus, the 3-point correlators of $\mathcal{A}_{W,G_W^{max}}$ are determined by the 3-point correlators of the summands. Because the 3-point correlators for A_{n-1} are all computed in [JKV01] and for D_{n+1} in [FJR13], it is easy to compute 3-point correlators for the polynomials we are interested in. Thus, for the remainder of this thesis we will focus on basic k-point correlators with $k \geq 4$.

The goal of this section is to use the Reconstruction Lemma 3.2.3 to identify a small collection of nonzero basic four-point correlators that, with the three-point correlators and the pairing, completely determine all four-point correlators. We begin with some new notation and a general lemma about the possible values for the contribution of a basic column.

Notation 4.3.1. *Let W_ℓ be an atomic summand of an invertible polynomial $W : \mathbb{C}^N \to \mathbb{C}$. Then denote by v_ℓ the number of variables subtended by W_ℓ. In particular, $v_\ell = 1$ when $W_\ell = A_{n-1}$ and $v_\ell = 2$ when $W_\ell = D_{n+1}$. Note that $\sum_\ell v_\ell = N$.*

Lemma 4.3.2. *Let W_ℓ be an atomic summand of the invertible polynomial W. Then we have the following bound on the value of the contribution of a basic column in a k-point correlator:*

$$v_\ell \leq C_\ell \leq 1 + v_\ell + \frac{k-4}{v_\ell\, n}.$$

The maximum contribution is realized by the column $(\mathbf{p}_{W_\ell}, \ldots, \mathbf{p}_{W_\ell}, \mathbf{h}_{W_\ell}, \mathbf{h}_{W_\ell})^T$, except in the case $W_\ell = A_1$. If $W_\ell = A_1$, then $C_\ell = 1$.

Proof. Denote the ℓ^{th} column of a basic k-point correlator by $(a_{1\ell}, a_{2\ell}, \ldots, a_{k\ell})^T$. Tables 4.1 and 4.2 and the surrounding discussion show that the identity $\mathbf{1}$ has the smallest contribution of any element in $\mathcal{A}_{W_\ell, G^{max}_{W_\ell}}$. Since the contribution of a column includes the sum of the contributions of the elements in that column, we have that C_ℓ is minimized when $a_{i\ell} = \mathbf{1}$. We compute

$$
\begin{aligned}
C_\ell &\geq -(k-2)\sum_j q_j + \sum_{i=1}^{k} c(\mathbf{1}) \\
&= k\, c(\mathbf{1}) - (k-2)\, c(\mathbf{1}) \\
&= 2\, c(\mathbf{1}).
\end{aligned}
$$

Here we used the fact that for any FJRW ring, $c(\mathbf{1})$ is exactly equal to $\sum_j q_j$. Then $C_\ell \geq \frac{2}{n}$ in the A_{n-1} case and $\frac{n+1}{n}$ in the D_{n+1} case. Because C_ℓ must be an integer, we have $C_\ell \geq v_\ell$.

Similarly, an analysis of Tables 4.1 and 4.2 and the surrounding discussion shows that the largest element contributions come from $\mathbf{p}_{W_\ell}$ (if we require a basic element) and $\mathbf{h}_{W_\ell}$ (otherwise). Thus, C_ℓ is maximized when

$$
(a_{1\ell}, a_{2\ell}, \ldots, a_{k\ell})^T = (\mathbf{p}_{W_\ell}, \ldots, \mathbf{p}_{W_\ell}, \mathbf{h}_{W_\ell}, \mathbf{h}_{W_\ell})^T.
$$

In the A_{n-1} case,

$$
\begin{aligned}
C_\ell &\leq -(k-2)(q_1) + (k-2)\, c(\mathbf{p}_{A_{n-1}}) + 2\, c(\mathbf{h}_{A_{n-1}}) \\
&= -(k-2)\frac{1}{n} + (k-2)\frac{2}{n} + 2\frac{n-1}{n} \\
&= 2 + \frac{k-4}{n}.
\end{aligned}
$$

In the D_{n+1} case,

$$
\begin{aligned}
C_\ell &\leq -(k-2)\left(\frac{n-1}{2n} + \frac{1}{n}\right) + (k-2)\frac{n+2}{2n} + 2\frac{3n-1}{2n} \\
&= \frac{k-2}{2n} + \frac{6n-2}{2n} = 3 + \frac{k-4}{2n}.
\end{aligned}
$$

This shows $C_\ell \leq 1 + v_\ell + \frac{k-4}{v_\ell\, n}$.

Finally, if $W_\ell = A_1$, the ring $\mathcal{A}_{W_\ell, G^{max}_{W_\ell}}$ has only the identity element $\mathbf{1}$. Then the contribution of this column is equal to

$$
\begin{aligned}
C_\ell &= -(k-2)q_1 + \sum_{i=1}^{k} c(\mathbf{1}) \\
&= 2\, c(\mathbf{1}) = \frac{2}{2} = 1.
\end{aligned}
$$

$\square$

We can now prove that for our polynomial $W = W_1 + \ldots + W_r$, all four-point correlators are completely determined by the pairing, the three-point correlators, and at most r basic four-point correlators.

Theorem 4.3.3. *Let W be an invertible polynomial with $W = W_1 + \ldots + W_r$, where each atomic summand W_ℓ is either an A_{n-1} or D_{n+1} polynomial. Then the four-point correlators of $\mathcal{A}_{W,G}$ are uniquely determined by the pairing, the three-point correlators, and the correlators*

$$\langle \mathbf{p}_{W_\ell},\ \mathbf{p}_{W_\ell},\ \mathbf{h}_{W_\ell},\ \mathbf{h}_W \rangle$$

where $W_\ell \neq A_1$.

Proof. Figure 4.2 documents the following argument using our grid notation. The reader may find it helpful to refer to the grids frequently to help visualize the proof.

From the Reconstruction Lemma 3.2.3 we know that we need only consider basic four-point correlators. Let us consider the form of such a correlator. Now, from Lemma 4.3.2, we know that $C_\ell \geq v_\ell$. Also, from Lemma 4.1.6, we know that $\sum C_\ell = N+1$. Then since $\sum v_\ell = N$, we must have some integer $m \in 1, \ldots, r$ such that $C_m = v_m + 1$, but $C_\ell = v_\ell$ for $\ell \neq m$. However, Lemma 4.3.2 also tells us that if $W_\ell = A_1$, $C_\ell = 1$. Thus, W_m is not an A_1-type polynomial.

Because $v_\ell + 1$ is the maximum value for C_ℓ when $k = 4$, we must have the m^{th} column equal to $(\mathbf{p}_{W_m}, \mathbf{p}_{W_m}, \mathbf{h}_{W_m}, \mathbf{h}_{W_m})^T$. From the definition of a basic correlator, the remaining columns in our grid must have all their basic elements equal to 1. See grids (1) and (2) in Figure 4.2.

$$\begin{bmatrix} \mathbf{p}_{W_1} & * & * & \ldots & * \\ \mathbf{p}_{W_1} & * & * & \ldots & * \\ \mathbf{h}_{W_1} & * & * & \ldots & * \\ \mathbf{h}_{W_1} & * & * & \ldots & * \end{bmatrix} \qquad \begin{bmatrix} \mathbf{p}_{W_1} & 1 & 1 & \ldots & 1 \\ \mathbf{p}_{W_1} & 1 & 1 & \ldots & 1 \\ \mathbf{h}_{W_1} & * & * & \ldots & * \\ \mathbf{h}_{W_1} & * & * & \ldots & * \end{bmatrix}$$

(1) (2)

$$\begin{bmatrix} \mathbf{p}_{W_1} & 1 & 1 & \ldots & 1 \\ \mathbf{p}_{W_1} & 1 & 1 & \ldots & 1 \\ \mathbf{h}_{W_1} & \xi_{12} & \xi_{13} & \ldots & \xi_{1r} \\ \mathbf{h}_{W_1} & \xi_{22} & \xi_{23} & \ldots & \xi_{2r} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{p}_{W_1} & 1 & 1 & \ldots & 1 \\ \mathbf{p}_{W_1} & 1 & 1 & \ldots & 1 \\ \mathbf{h}_{W_1} & 1 & 1 & \ldots & 1 \\ \mathbf{h}_{W_1} & \mathbf{h}_{W_2} & \mathbf{h}_{W_3} & \ldots & \mathbf{h}_{W_r} \end{bmatrix}$$

(3) (4)

Figure 4.2: Filling in the grid for a nonzero basic four-point correlator. Step 1: one column must have the form $(\mathbf{p}_{W_m}, \mathbf{p}_{W_m}, \mathbf{h}_{W_m}, \mathbf{h}_{W_m})^T$. For simplicity we take $m = 1$. Step 2: this forces the remaining columns to have $\mathbf{1}$s for their basic elements (top 2 elements). Step 3: we label the remaining non primitive elements. Step 4: using the Reconstruction Lemma we "slide" the ξ_{1i}s into the bottom row.

For $\ell \neq m$, call the two non-basic elements in the ℓ^{th} column $\xi_{1\ell}$ and $\xi_{2\ell}$. Recall that in this case $C_\ell = v_\ell$. Then

$$\begin{aligned} C_\ell &= -(k-2)\sum_j q_j + (k-2)\, c(\mathbf{1}) + c(\xi_{1\ell}) + c(\xi_{2\ell}) \\ &= c(\xi_{1\ell}) + c(\xi_{2\ell}) \\ &= v_\ell. \end{aligned}$$

Here again we are using the fact that in any FJRW ring, $c(\mathbf{1}) = \sum_j q_j$. Then in the case

$W_\ell = A_{n-1}$, since $c(\xi_{1\ell}) = \frac{a}{n}$ where a is an integer and $0 < a < n$, we have $c(\xi_{2\ell}) = \frac{n-a}{n}$. Then since in this case $v_\ell = 1$ we must have $\xi_{1\ell} = e_a$ and $\xi_{2\ell} = e_{n-a}$, and in particular $\xi_{1\ell}$ and $\xi_{2\ell}$ come from inverse sectors. Thus, their product is a scalar multiple of $\mathbf{h}_{A_{n-1}}$. Similarly, if $W_\ell = D_{n+1}$, an identical argument shows that $\xi_{1\ell}$ and $\xi_{2\ell}$ come from inverse sectors, so that again $\xi_{1\ell} \star \xi_{2\ell}$ is a scalar multiple of $\mathbf{h}_{D_{n+1}}$.

Now, we have deduced that our nonzero basic 4-point correlator must have the form

$$\langle \mathbf{p}_{W_m},\ \mathbf{p}_{W_m},\ \mathbf{h}_{W_m}\xi_{11}\dots\xi_{1r},\ \mathbf{h}_{W_m}\xi_{21}\dots\xi_{2r}\rangle.$$

This correlator corresponds to multiplying across the rows of correlator grid (3) in Figure 4.2. We will use the Reconstruction Lemma 3.2.3 to show that after fixing m, all of these can be determined by a single correlator. To do this, set $\alpha = \mathbf{p}_{W_m}$, $\beta = \mathbf{h}_{W_m}\xi_{11}\dots\xi_{1r}$, $\epsilon = \mathbf{h}_{W_m}$, and $\phi = \xi_{21}\dots\xi_{2r}$. Then $\epsilon \star \phi = \mathbf{h}_{W_m}\xi_{21}\dots\xi_{2r}$, as required. Also $\alpha \star \epsilon = \alpha \star \beta = 0$, since both of these products involve multiplying $\mathbf{h}_{W_m}$ by some element of smaller (but nonzero) degree in $\mathcal{A}_{W_m, G_{W_m}^{max}}$. Finally, $\beta \star \phi = \mathbf{h}_{W_m}\xi_{11}\xi_{21}\dots\xi_{1r}\xi_{2r} = d_1\mathbf{h}_{W_1}\dots d_r\mathbf{h}_{W_r} = d\mathbf{h}_W$. Here, d and each d_i are complex numbers. Because correlators are linear, we can pull the constant d out to the front. We conclude that all four-point correlators are determined by the pairing, the three-point correlators, and the correlators

$$\langle \mathbf{p}_{W_\ell},\ \mathbf{p}_{W_\ell},\ \mathbf{h}_{W_\ell},\ \mathbf{h}_W\rangle,$$

for $\ell = 1, \dots r$, omitting ℓ where $W_\ell = A_1$. □

We note a fact which follows directly from the proof above that we will return to later.

Corollary 4.3.4. *If a basic column has only* $\mathbf{1}$*s for its basic elements, then* $C_\ell = v_\ell$*.*

Proof. Let the ℓ^{th} column equal $(\mathbf{1}, \dots, \mathbf{1}, \xi_1, \xi_2)^T$. Then in the above proof we showed $c(\xi_1) + c(\xi_2) = C_\ell$. It follows from a quick check of Tables 4.1 and 4.2 that it is impossible to choose ξ_1 and ξ_2 so that $c(\xi_1) + c(\xi_2)$ is any integer greater than v_ℓ. □

It remains to calculate the values of these correlators. The following theorem takes us most of the way.

Theorem 4.3.5. *For any correlator described in Theorem 4.3.3, the following equality holds.*

$$\langle \mathbf{p}_{W_\ell},\ \mathbf{p}_{W_\ell},\ \mathbf{h}_{W_\ell},\ \mathbf{h}_{W} \rangle = \langle \mathbf{p}_{W_\ell},\ \mathbf{p}_{W_\ell},\ \mathbf{h}_{W_\ell},\ \mathbf{h}_{W_\ell} \rangle.$$

This theorem is helpful because the correlator $\langle \mathbf{p}_{W_\ell},\ \mathbf{p}_{W_\ell},\ \mathbf{h}_{W_\ell},\ \mathbf{h}_{W_\ell} \rangle$ comes from the simpler ring $\mathcal{A}_{D, G_D^{max}}$ or $\mathcal{A}_{A, G_A^{max}}$, and its value is known in almost all cases.

Proof. The idea of this proof is to use the second part of Theorem 4.1.2 to write

$$\Lambda_{0,4}(\mathbf{p}_{W_\ell},\ \mathbf{p}_{W_\ell},\ \mathbf{h}_{W_\ell},\ \mathbf{h}_{W}) = \Lambda_{0,4}(\mathbf{p}_{W_\ell},\ \mathbf{p}_{W_\ell},\ \mathbf{h}_{W_\ell},\ \mathbf{h}_{W_\ell}) \bigotimes_{m \neq \ell} \Lambda_{0,4}(\mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \mathbf{h}_{W_m}).$$

We will show that the degree of each class $\Lambda_{0,4}(\mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \mathbf{h}_{W_m})$ is zero, so that each of these terms is just a complex number, and hence the degree of $\Lambda_{0,4}(\mathbf{p}_{W_\ell},\ \mathbf{p}_{W_\ell},\ \mathbf{h}_{W_\ell},\ \mathbf{h}_{W})$ is equal to the degree of $\Lambda_{0,4}(\mathbf{p}_{W_\ell},\ \mathbf{p}_{W_\ell},\ \mathbf{h}_{W_\ell},\ \mathbf{h}_{W_\ell})$. Using the definition

$$\langle \mathbf{p}_{W_\ell},\ \mathbf{p}_{W_\ell},\ \mathbf{h}_{W_\ell},\ \mathbf{h}_{W} \rangle = \int_{\overline{\mathcal{M}}_{0,4}} \Lambda_{0,4}(\mathbf{p}_{W_\ell},\ \mathbf{p}_{W_\ell},\ \mathbf{h}_{W_\ell},\ \mathbf{h}_{W}),$$

we can then conclude that $\langle \mathbf{p}_{W_\ell},\ \mathbf{p}_{W_\ell},\ \mathbf{h}_{W_\ell},\ \mathbf{h}_{W} \rangle$ is equal to $\langle \mathbf{p}_{W_\ell},\ \mathbf{p}_{W_\ell},\ \mathbf{h}_{W_\ell},\ \mathbf{h}_{W_\ell} \rangle$, multiplied by the constants $\langle \mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \mathbf{h}_{W_m} \rangle$ for $m \neq \ell$.

First we show that the classes $\Lambda_{0,4}(\mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \mathbf{h}_{W_m})$ have degree zero. We compute the degree using Definition 3.1.1:

$$\begin{aligned}
\deg(\ \Lambda_{0,4}(\mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \mathbf{h}_{W_m})) &= \frac{1}{2}(\deg_W(\mathbf{1}) + \deg_W(\mathbf{1}) + \deg_W(\mathbf{1}) + \deg_W(\mathbf{h}_{W_m})) - \hat{c}_W \\
&= \frac{1}{2}(0 + 0 + 0 + 2\hat{c}_W) - \hat{c}_W \\
&= 0.
\end{aligned}$$

Second, to compute the value of these (constant) classes we use the Forgetting Tails

Axiom (Axiom 5). We find

$$\Lambda_{0,4}(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{h}_{W_m}) = \tau^*\Lambda_{0,3}(\mathbf{1},\mathbf{1},\mathbf{h}_{W_m}).$$

Any cohomology class on $\overline{\mathcal{M}}_{0,3} = \{\mathrm{pt}\}$ is a constant, so $\tau^*\Lambda_{0,3}(\mathbf{1},\mathbf{1},\mathbf{h}_{W_m}) = \Lambda_{0,3}(\mathbf{1},\mathbf{1},\mathbf{h}_{W_m})$. Moreover, $\Lambda_{0,3}(\mathbf{1},\mathbf{1},\mathbf{h}_{W_m}) = \langle \mathbf{1},\mathbf{1},\mathbf{h}_{W_m}\rangle$, since the integral of a constant over a point is equal to the constant. Using the Pairing Axiom (Axiom 4),

$$\langle \mathbf{1},\ \mathbf{1},\ \mathbf{h}_{W_m}\rangle = \langle \mathbf{1},\ \mathbf{h}_{W_m}\rangle = 1.$$

Recall that we computed this pairing in Example 2.2.19. Thus,

$$\Lambda_{0,4}(\mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \mathbf{h}_{W_m}) = 1.$$

We have shown that

$$\langle \mathbf{p}_{W_\ell},\ \mathbf{p}_{W_\ell},\ \mathbf{h}_{W_\ell},\ \mathbf{h}_{W}\rangle = \langle \mathbf{p}_{W_\ell},\ \mathbf{p}_{W_\ell},\ \mathbf{h}_{W_\ell},\ \mathbf{h}_{W_\ell}\rangle.$$

□

Corollary 4.3.6. *If W_ℓ is a D_{n+1} type polynomial with $n > 2$ or an A_{n-1} type polynomial, then*

$$\langle \mathbf{p}_{W_\ell},\ \mathbf{p}_{W_\ell},\ \mathbf{h}_{W_\ell},\ \mathbf{h}_{W}\rangle = \frac{1}{n}.$$

If $W_\ell = D_3$, then

$$\langle \mathbf{p}_{W_\ell},\ \mathbf{p}_{W_\ell},\ \mathbf{h}_{W_\ell},\ \mathbf{h}_{W}\rangle = \pm\frac{1}{8}.$$

Proof. The value of $\langle \mathbf{p}_{W_\ell},\ \mathbf{p}_{W_\ell},\ \mathbf{h}_{W_\ell},\ \mathbf{h}_{W_\ell}\rangle$ is calculated in the D_{n+1} case for $n > 2$ in Section 6.3.6 of [FJR13] and in the A_{n-1} case for all values of n in Proposition 6.1 of [JKV01]. Because the idea used in [FJR13] for the D_{n+1} polynomials does not work when $n = 2$, we calculate this correlator in the D_3 case in Chapter 5 of this thesis. □

4.4 Basic higher-point correlators

The goal of this section is to prove the following theorem.

Theorem 4.4.1. *Let W be an invertible polynomial with $W = W_1 + \ldots + W_r$, where each summand W_ℓ is either an A_{n-1} or D_{n+1} polynomial. Then every genus-zero basic k-point correlator with $k > 4$ vanishes.*

Note that this will completely determine the Frobenius manifold structure for W in terms of the Frobenius manifold structures of the summands W_ℓ.

The idea of the proof is to look at the grid corresponding to such a correlator, supposing (for contradiction) that the correlator is nonzero. In Lemma 4.4.3 we will show that the requirement that the correlator be basic puts a lower bound on the number of $\mathbf{1}$s that can appear in this grid. However, in Lemma 4.4.4 we will show that the contribution rule $\sum C_\ell = N + k - 3$ puts an upper bound on the number of $\mathbf{1}$s. We will show that these bounds are incompatible, and thus any basic k-point correlator vanishes.

Before beginning the proof of Theorem 4.4.1, we do an example which contains the flavor of the general proof.

Example 4.4.2. *Let $W = x^3 + y^3 + z^3 + w^3$. For this example, let us consider 7-point correlators. In Section 2.2.2 we computed the basis for $\mathcal{A}_{x^3, G_{x^3}^{max}}$ to be the two elements $\mathbf{1} = e_{1/3}$ and $\mathbf{p}_{x^3} = \mathbf{h}_{x^3} = e_{2/3}$ with contributions $1/3$ and $2/3$ respectively.*

Now, we wish to fill in the entries of our correlator grid with these two elements. We are subject to the constraint $\sum C_\ell = N + k - 3 = 4 + 7 - 3 = 8$, as well as the bound in Lemma 4.3.2, which says $C_\ell \leq 1 + v_\ell + \frac{k-4}{v_\ell n} = 1 + 1 + \frac{7-4}{1 \cdot 3} = 3$. Thus, our contributions partition 8 into four pieces, which can occur in one of three ways: $1+1+3+3$, $1+2+2+3$, or $2+2+2+2$.

Now, the grid of a nonzero basic correlator needs a lot of $\mathbf{1}$ entries. Thus, given a value *for C_ℓ, let us see how many* **1***s the ℓ^{th} column can hold. We have the formula*

$$C_\ell = -(k-2)\sum_{j=a_\ell}^{b_\ell} q_j + \sum_{i=1}^{k} c(\alpha_{i\ell}) = -5/3 + \sum_{i=1}^{7} c(\alpha_{i\ell}).$$

A quick computation leads to Table 4.4:

C_ℓ	Maximum number of **1**s	Computation of C_ℓ
1	6 **1**s, 1 **p**	$-5/3 + 6(1/3) + 1(2/3) = 1$
2	3 **1**s, 4 **p**s	$-5/3 + 3(1/3) + 4(2/3) = 2$
3	7 **p**s	$-5/3 + 0(1/3) + 7(2/3) = 3$

Table 4.4: Possible column contributions for a nonzero basic 7-point correlator for $W = x^3 + y^3 + z^3 + w^3$ and the corresponding maximal number of **1**s. Note that in this case, the number of **1**s is actually uniquely determined by C_ℓ; this is not true for more complicated examples.

Now, we simply cycle through our three cases for the partition of 8 into values for the C_ℓ, and see if anything works. The three possibilities are diagrammed in Figure 4.3. In each case, the required column contributions make it so that we "don't have enough **1***s" to fill out the grid.*

$$\begin{array}{r|ccc}
\text{Correlator grid} & \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{p} & * \\ \mathbf{1} & \mathbf{1} & \mathbf{p} & * \\ \mathbf{1} & \mathbf{1} & \mathbf{p} & * \\ \mathbf{1} & \mathbf{1} & \mathbf{p} & * \\ \mathbf{1} & \mathbf{1} & \mathbf{p} & * \\ \mathbf{1} & \mathbf{1} & \mathbf{p} & \mathbf{p} \\ \mathbf{p} & \mathbf{p} & \mathbf{p} & \mathbf{p} \end{bmatrix} & \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{p} & * \\ \mathbf{1} & \mathbf{1} & \mathbf{p} & * \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{p} \\ \mathbf{1} & \mathbf{p} & \mathbf{1} & * \\ \mathbf{1} & \mathbf{p} & \mathbf{1} & * \\ \mathbf{1} & \mathbf{p} & \mathbf{p} & \mathbf{p} \\ \mathbf{p} & \mathbf{p} & \mathbf{p} & \mathbf{p} \end{bmatrix} & \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{p} & \mathbf{1} \\ \mathbf{1} & \mathbf{p} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{p} & \mathbf{1} & \mathbf{1} \\ \mathbf{p} & \mathbf{1} & \mathbf{1} & * \\ \mathbf{p} & \mathbf{1} & * & * \\ \mathbf{p} & \mathbf{p} & \mathbf{p} & \mathbf{p} \\ \mathbf{p} & \mathbf{p} & \mathbf{p} & \mathbf{p} \end{bmatrix} \\
\sum C_\ell = 8 & 1{+}1{+}3{+}3 & 1{+}2{+}2{+}3 & 2{+}2{+}2{+}2
\end{array}$$

Figure 4.3: Grids for potentially nonzero basic 7-point correlators. In each case we have left $*$s where the the definition of a basic correlator requires that we put a **1**, but the column contributions stipulate that we have already used up all our **1**s.

In theory, the method of the previous example could be used to prove Theorem 4.4.1 for any fixed W and k. Of course, we want to do all cases of W and k at once, which requires insight beyond simple manipulation of correlator grids. We begin our general proof of Theorem 4.4.1 by establishing a lower bound on the number of $\mathbf{1}$s in any p columns of a basic correlator.

Lemma 4.4.3. *Any p columns of a basic correlator will have at least (p-1)(k-2) entries equal to $\mathbf{1}$.*

Proof. The basic elements in a correlator grid must all be either $\mathbf{1}$ or $\mathbf{p}$, and exactly one $\mathbf{p}$ can appear in each row of the grid. The number of $\mathbf{1}$s in our p columns is minimized when every row's $\mathbf{p}$ appears in one of our p columns. Likewise, the number of ones is minimized when the bottom two rows of the correlator grid (the non-basic elements) have no $\mathbf{1}$s in them. This means we have at least $p-1$ columns times $k-2$ rows of $\mathbf{1}$s. □

Now, let us step back and consider what the contribution rule $\sum C_\ell = N+k-3$ requires. We are to partition $N+k-3$ into r pieces, called $C_1, \ldots, C_r$. From Lemma 4.3.2, we know that $C_\ell \geq v_\ell$. Then since $\sum v_\ell = N$, we have $k-3$ "extra" units to partition among the C_ℓ.

Now, let $P = \{\ell \in 1, 2, \ldots, r \mid C_\ell > v_\ell\}$, so that P is the set of columns with some of the "extra" $k-3$ units. Let $p = |P|$. The following lemma states an upper bound on the number of $\mathbf{1}$s in a basic column W_ℓ for $\ell \in P$.

Lemma 4.4.4. *Let W_ℓ be a A_{n-1} or D_{n+1}-type atomic summand of W. If $C_\ell = v_\ell + t$ with $t \geq 1$, then the number of ones in the ℓ^{th} column is at most $k-3t-1$.*

Proof. Write the ℓ^{th} column of the correlator grid as $(\mathbf{1}, \ldots, \mathbf{1}, \mathbf{p}_{W_\ell}, \ldots, \mathbf{p}_{W_\ell}, \xi_1, \xi_2)^T$. By Corollary 4.3.4, since $C_\ell > v_\ell$, we know we have at least one basic element equal to $\mathbf{p}_{W_\ell}$. Now, suppose this column has the maximal number of $\mathbf{1}$s yielding the contribution C_ℓ. We claim that $\xi_1 = \xi_2 = \mathbf{h}_{W_\ell}$. If $\xi_i \neq \mathbf{h}_{W_\ell}$, then we could replace ξ_i by some element whose contribution is $\frac{1}{n}$ higher (in the A_{n-1} case) or $\frac{1}{2n}$ higher (in the D_{n+1} case). This would allow us to replace some $\mathbf{p}_{W_\ell}$ (since we know there is at least one) by another $\mathbf{1}$, since

$c(\mathbf{p}_{W_\ell}) - c(\mathbf{1}) = \frac{1}{n}$ or $\frac{1}{2n}$ in the A_{n-1} or D_{n+1} case, respectively. Thus, we have increased the number of $\mathbf{1}$s, which is a contradiction.

Let T be a basic column satisfying the above hypotheses that also contains the maximum number of $\mathbf{1}$s. Then we have shown that

$$T = (\mathbf{1}, \ldots, \mathbf{1}, \mathbf{p}_{W_\ell}, \ldots, \mathbf{p}_{W_\ell}, \mathbf{h}_{W_\ell}, \mathbf{h}_{W_\ell})^T,$$

where the exact number of $\mathbf{1}$s and $\mathbf{p}_{W_\ell}$s is still unknown. The remainder of the proof is done in two cases.

$\mathbf{A_{n-1}}$ case: Let $\mathcal{O}$ be the number of $\mathbf{1}$s in T. Then T also has two $\mathbf{h}_{W_\ell}$s and $k - \mathcal{O} - 2$ elements equal to $\mathbf{p}_{W_\ell}$. Thus,

$$\begin{aligned}
C_\ell &= -(k-2)\sum_j q_j + \mathcal{O}\, c(\mathbf{1}) + (k - \mathcal{O} - 2)\, c(\mathbf{p}_{W_\ell}) + 2\, c(\mathbf{h}_{W_\ell}) \\
C_\ell &= -(k-2)\left(\frac{1}{n}\right) + \mathcal{O}\left(\frac{1}{n}\right) + (k - \mathcal{O} - 2)\left(\frac{2}{n}\right) + 2\left(\frac{n-1}{n}\right) \\
&= \frac{k - 2 - \mathcal{O}}{n} + 2\left(\frac{n-1}{n}\right).
\end{aligned}$$

Since we have $C_\ell = 1 + t$, we can set these two equations equal and solve for $\mathcal{O}$ in terms of t. We find

$$\mathcal{O} = k - 4 - n(t-1).$$

Now, note that the requirement $t \geq 1$, coupled with Lemma 4.3.2, implies that $n \geq 3$. Moreover, since $t \geq 1$, the quantity $t - 1$ is nonnegative. We conclude

$$\begin{aligned}
\mathcal{O} &\leq k - 4 - 3(t-1) \\
&= k - 3t - 1.
\end{aligned}$$

$\mathbf{D_{n+1}}$ case: Let $\mathcal{O}$ be the number of $\mathbf{1}$s in T. Then T also has two $\mathbf{h}_{W_\ell}$s and $k - \mathcal{O} - 2$

elements equal to $\mathbf{p}_{W_\ell}$. Thus,

$$\begin{aligned} C_\ell &= -(k-2)\left(\frac{n+1}{2n}\right) + \mathcal{O}\left(\frac{n+1}{2n}\right) + (k-\mathcal{O}-2)\left(\frac{n+2}{2n}\right) + 2\left(\frac{3n-1}{2n}\right) \\ &= \frac{k-2-\mathcal{O}}{2n} + 3 - \frac{1}{n}. \end{aligned}$$

Since we have $C_\ell = 2 + t$, we can set these two equations equal and solve for $\mathcal{O}$ in terms of t. We find

$$\mathcal{O} = k - 4 - 2n(t-1).$$

Now, since $t \geq 1$, the quantity $t-1$ is nonnegative. This allows us to use the fact $2 \leq n$ to conclude

$$\begin{aligned} \mathcal{O} &\leq k - 4 - 2(2)(t-1) \\ &= k - 4t = k - 3t - t \\ &\leq k - 3t - 1, \end{aligned}$$

where the last inequality follows again since $t \geq 1$. □

We are now ready to prove that all basic k-point correlators with $k > 4$ vanish. The idea is that the upper bound on the number of $\mathbf{1}$s found in Lemma 4.4.4 contradicts the lower bound found in Lemma 4.4.3.

Proof of Theorem 4.4.1. As before, let $P = \{\ell \in 1, 2, \ldots, r \mid C_\ell > v_\ell\}$. Let $p = |P|$. Now, for $\ell \in P$, $C_\ell = v_\ell + t_\ell$ where $\sum_{\ell \in P} t_\ell = k - 3$. In particular, note that $p \leq k - 3$. From Lemma 4.4.4, the columns of P have at most $\sum_{\ell \in P}(k - 3t_\ell - 1) = pk - 3(k-3) - p$ elements equal to $\mathbf{1}$. Conversely, by Lemma 4.4.3 these columns must have at least $(p-1)(k-2)$ elements equal to $\mathbf{1}$. This yields the inequality

$$(p-1)(k-2) \leq pk - 3(k-3) - p.$$

We solve for k to obtain

$$k \leq \frac{p+7}{2}.$$

When we substitute $p \leq k-3$, we conclude

$$k \leq 4.$$

This shows that no nonzero basic correlator can have $k > 4$. $\square$

Chapter 5. The D_3 Frobenius manifold

The goal of this section is to compute the correlator $\langle \mathbf{p}_{D_3},\ \mathbf{p}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3} \rangle$. The analogous correlator in the D_{n+1} case with $n > 2$ was computed directly in [FJR13] using the Concavity Axiom. Unfortunately, we cannot do the same for the D_3 correlator because it does not satisfy the hypotheses of that axiom. In fact, the insertion $\mathbf{p}_{D_3}$ comes from the sector $(0,0)$, which has a fixed locus of dimension 2 (dimension 0 is required for the Concavity Axiom).

Instead, we will use the Reconstruction Lemma 3.2.3 to relate our desired correlator to the 5-point correlator $\langle \mathbf{h}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3} \rangle$. Because $\mathbf{h}_{D_3}$ comes from the sector $(3/4, 1/2)$ which has a fixed locus of dimension 0, we will see that we can use the Concavity Axiom to compute this 5-point correlator. This allows us to compute the desired 4-point correlator up to a sign. Unfortunately, we cannot do better than this with any of the known tools.

Our first step, then, is to relate the correlators $\langle \mathbf{p}_{D_3},\ \mathbf{p}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3} \rangle$ and $\langle \mathbf{h}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3} \rangle$ using the reconstruction.

Lemma 5.0.5. *Let* $\alpha = \langle \mathbf{p}_{D_3},\ \mathbf{p}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3}\rangle$ *and* $\beta = \langle \mathbf{h}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3}\rangle$. *Then*

$$\alpha = \sqrt{\frac{\beta}{8}}.$$

Proof. Let $X = \mathbf{p}_{D_3}$, so X^2 will be a scalar multiple of $\mathbf{h}_{D_3}$. Let us apply Lemma 3.2.1 to the correlator $\langle X^2, X^2, X^2, X^2, X^2\rangle$ setting $\alpha = \beta = X^2$ and $\epsilon = \phi = X$. We find

$$\begin{aligned}\langle X^2, X^2, X^2, X^2, X^2\rangle = &\sum_l \langle X^2, X^2, X^2, X, \delta_l\rangle\langle \delta_l', X, X^2\rangle + 2\sum_l \langle X^2, X^2, X, \delta_l\rangle\langle \delta_l', X, X^2, X^2\rangle \\ &+ \sum_l \langle X^2, X, \delta_l\rangle\langle \delta_l', X, X^2, X^2, X^2\rangle - 2\sum_l \langle X^2, X^2, X^2, \delta_l\rangle\langle \delta_l', X, X, X^2\rangle \\ &- \sum_l \langle X^2, X^2, \delta_l\rangle\langle \delta_l', X, X, X^2, X^2\rangle.\end{aligned}$$

Now, the correlator $\langle X^2, X, \delta_l\rangle$ will vanish for every value of δ_l by the Dimension Axiom (Axiom 2), as will $\langle X^2, X^2, \delta_l\rangle$. Moreover, by the Dimension Axiom the correlator $\langle X^2, X^2, X^2, \delta_l\rangle$ can be nonzero only if δ_l is a scalar multiple of $\mathbf{1}$. But in this case, by Corollary 3.1.4, the correlator is zero anyway. Thus, we have

$$\langle X^2, X^2, X^2, X^2, X^2\rangle = 2\sum_l \langle X^2, X^2, X, \delta_l\rangle\langle \delta_l', X, X^2, X^2\rangle.$$

By the Dimension Axiom, the correlator $\langle X^2, X^2, X, \delta_l\rangle$ can be nonzero only if $\delta_l = X$. Now, in Example 2.2.19 we computed that $\langle X, X\rangle = -\frac{1}{2}$. This implies that if $\delta_l = X$, then $\delta_l' = -2X$. Then

$$\langle X^2, X^2, X^2, X^2, X^2\rangle = -4\langle X^2, X^2, X, X\rangle^2.$$

We will now cite the fact that $\mathbf{p}_{D_3} \star \mathbf{p}_{D_3} = -\frac{1}{2}\mathbf{h}_{D_3}$. The reader can in fact check this from the definition of the Frobenius algebra multiplication (Equation 2.13), the definition of the pairing (Equation 2.12), and the correlator axioms. It is a straightforward but somewhat lengthy computation. Then we have the relation $X^2 = -\frac{1}{2}\mathbf{h}_{D_3}$. Let $\alpha = \langle \mathbf{h}_{D_3}, \mathbf{h}_{D_3}, X, X\rangle =$

$\langle -2X^2, -2X^2, X, X\rangle$. Thus $\langle X^2, X^2, X, X\rangle = \frac{\alpha}{4}$. Similarly if

$$\beta = \langle \mathbf{h}_{D_3}, \mathbf{h}_{D_3}, \mathbf{h}_{D_3}, \mathbf{h}_{D_3}, \mathbf{h}_{D_3}\rangle = \langle -2X^2, -2X^2, -2X^2, -2X^2, -2X^2\rangle,$$

then $\langle X^2, X^2, X^2, X^2, X^2\rangle = \frac{\beta}{(-2)^5}$. We have the equality

$$\frac{\beta}{(-2)^5} = -4\left(\frac{\alpha}{4}\right)^2.$$

Solving the equation for α, we find

$$\alpha = \sqrt{\frac{\beta}{8}}.$$

□

It remains to calculate the value of β using the Concavity Axiom (Axiom 6). The first step is to check that we can in fact use this axiom; that is, we need to check that our correlator satisfies all the hypotheses. This is the purpose of the following lemma.

Lemma 5.0.6. *All the line bundle degrees of the correlator $\langle \mathbf{h}_{D_3}, \mathbf{h}_{D_3}, \mathbf{h}_{D_3}, \mathbf{h}_{D_3}, \mathbf{h}_{D_3}\rangle$ are negative integers.*

Proof. We begin by listing all possible graphs Γ dual to stable nodal curves with five marked points. Since all the insertions in the correlator are the same, the numbering of the marks doesn't matter in this case. Up to symmetry, then, we have only three possibilities, diagrammed in Figures 5.1-5.3. To arrive at these possibilities, I considered the possible distributions of marks on a stable curve given the number of smooth components.

Since $\mathbf{h}_{D_3} = e_{(3/4,1/2)}$, all external half edges are marked with the group element $g = (3/4, 1/2)$. First, we check that the line bundle degrees coming from Figure 5.1 are negative integers. In this case, Equation 3.4 is exactly the same as the definition of the line bundle degrees of our 5-point correlator. We compute the line bundle degrees (l_1, l_2) as follows:

$$(5-2)(1/4, 1/2) - 5(3/4, 1/2) = (-3, -1).$$

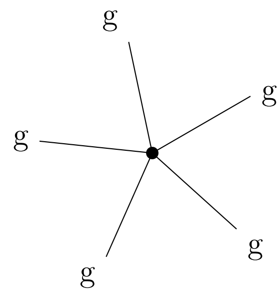

Figure 5.1: Dual graph for a 5-point correlator with 1 vertex; here $g = (3/4, 1/2)$.

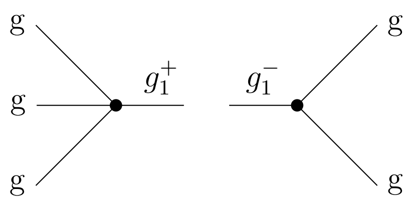

Figure 5.2: Dual graph for a 5-point correlator with 2 vertices; here $g = (3/4, 1/2)$.

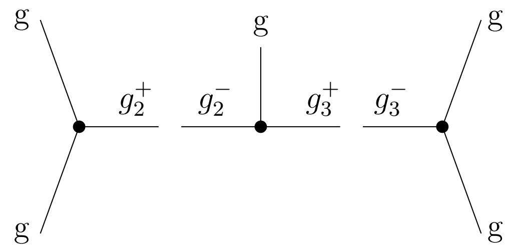

Figure 5.3: Dual graph for a 5-point correlator with 3 vertices; here $g = (3/4, 1/2)$.

Recall that by definition, we use arithmetic in $\mathbb{Q}^2$ to do this computation (instead of arithmetic in the group G_W^{max}). This is well-defined because the fractions Θ are chosen specifically to lie in $\mathbb{Q} \cap [0, 1)$.

The next step is to use Equation 3.3 to calculate the values of the group elements g_1^+, etc. We find

$$g_1^+ = (4 - 2)(1/4, 1/2) - 3(3/4, 1/2) = (1/4, 1/2).$$

Recall that this is a group element equation, so we are working in $(\mathbb{Q}/\mathbb{Z})^2$. Then

$$g_1^- = -g_1^+ = -(1/4, 1/2) = (3/4, 1/2).$$

(Note that this is equivalent to g^+ for the same graph reflected about a vertical axis). Because no coordinates of g^+ are zero, we do not have to contract any edges in the computation of the line bundle degrees for the vertices of this graph. Then using Equation 3.4, the line bundle degrees (l_1, l_2) for the g_1^+ node are

$$(4-2)(1/4, 1/2) - 3(3/4, 1/2) - (1/4, 1/2) = (-2, -1).$$

Recall that here we are using arithmetic in $(\mathbb{Q})^2$. These line bundle degrees are negative; checking the other components is similar. We find that the line bundle degrees for the right hand node in Figure 5.2 are also $(-2, -1)$, while from left to right the line bundle degrees for the nodes in Figure 5.3 are $(-2, -1)$, $(-1, -1)$, and $(-2, -1)$. $\square$

The final step is to apply the concavity axiom.

Lemma 5.0.7. *The following equality holds.*

$$\langle \mathbf{h}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3} \rangle = \frac{1}{8}.$$

Proof. All of the sectors in this correlator are $(3/4, 1/2)$ which has a 0-dimensional fixed locus. Since we know from Lemma 5.0.6 that this correlator is also concave, we just have to plug our numbers into the formulas in the Concavity Axiom (Axiom 6). In order to do so, we will need to know what all possible one-edged graphs Γ are. This time, graphs with different labelings of the half-edges will be considered distinct, even if their decorations by group elements end up being the same. Let $K \subset \{1, \ldots, 5\}$ be the set of marks appearing to the left of the edge. Recall that by convention, $1 \in K$. Before labeling, all graphs will look like the underlying graph in Figure 5.2 (if $|K| = 3$) or they will look like its reflection about a vertical axis (if $|K| = 2$). Thus we have two cases.

If $|K| = 3$, since we know $1 \in K$, there are $\binom{4}{2} = 6$ ways of picking the remaining elements of K. Each of these six graphs will have $g^- = (3/4, 1/2)$ as computed in Lemma 5.0.6 above.

If $|K| = 2$, since we know $1 \in K$, there are just 4 ways of picking the remaining element of K. The underlying graph here is just the reflection of the underlying graph in the case above, so that g^- here is equal to $-(3/4, 1/2) = (1/4, 1/2)$ as explained in Lemma 5.0.6

above. Note that this gives a total of 10 graphs that we have to sum over in the final term of Equation 3.6.

Then using the fact that $q_1 = 1/4$, $q_2 = 1/2$, and $B_2(x) = x^2 - x + 1/6$, we compute

$$\begin{aligned}
ch_1(\mathcal{E}_1) &= \frac{B_2(1/4)}{2}\,\kappa_1 - \sum_{j=1}^{5} \frac{B_2(3/4)}{2}\,\psi_j + \sum_{\Gamma,\,|K|=2} \frac{B_2(1/4)}{2}\Delta_K + \sum_{\Gamma,\,|K|=3} \frac{B_2(3/4)}{2}\Delta_K \\
&= \frac{1}{96}(-\kappa_1 + \sum_{j=1}^{5}\psi_j - \sum_{\Gamma}\Delta_K), \\
ch_1(\mathcal{E}_2) &= \frac{B_2(1/2)}{2}\,\kappa_1 - \sum_{j=1}^{5} \frac{B_2(1/2)}{2}\,\psi_j + \sum_{\Gamma,\,|K|=2} \frac{B_2(1/2)}{2}\Delta_K + \sum_{\Gamma,\,|K|=3} \frac{B_2(1/2)}{2}\Delta_K \\
&= \frac{1}{24}(-\kappa_1 + \sum_{j=1}^{5}\psi_j - \sum_{\Gamma}\Delta_K).
\end{aligned}$$

Now, $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$. Then $B_3(1/2) = 0$, so $ch_2(\mathcal{E}_2) = 0$. We further compute

$$\begin{aligned}
ch_2(\mathcal{E}_1) &= \frac{B_3(1/4)}{6}\,\kappa_2 - \sum_{j=1}^{5} \frac{B_3(3/4)}{6}\,\psi_j^2 + \sum_{\Gamma,\,|K|=2} \frac{B_3(1/4)}{6}P_{2,\Gamma}(\Delta) + \sum_{\Gamma,\,|K|=3} \frac{B_3(3/4)}{6}P_{2,\Gamma}(\Delta) \\
&= \frac{1}{128}(\kappa_2 + \sum_{j=1}^{5}\psi_j + \sum_{\Gamma,\,|K|=2} \Delta_K\Delta_{K\cup j} + \sum_{\Gamma,\,|K|=3} \Delta_K\Delta_{1\cup K^C}).
\end{aligned}$$

When we plug this information into Equation (3.5) and integrate the resulting polynomial in κ's, ψ's, and Δ's using Drew Johnson's Sage package [Joh], we get the value $\frac{1}{8}$. As a check on computation, the same integration was performed with Carl Faber's Maple package [Fab]. The result was the same.

□

We have proved the following theorem.

Theorem 5.0.8. *The following equality holds:*

$$\langle \mathbf{p}_{D_3},\ \mathbf{p}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3}\rangle = \pm\frac{1}{8}.$$

Proof. Combining Lemmas 5.0.5 and 5.0.7, we compute

$$\begin{aligned}\langle \mathbf{p}_{D_3},\ \mathbf{p}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3}\rangle &= \sqrt{\left(\frac{1}{8}\right)\left(\frac{1}{8}\right)} \\ &= \pm\frac{1}{8}.\end{aligned}$$

□

The conclusion of Theorem 5.0.8 leads us to hope that the Frobenius manifolds resulting from the two possible values of $\langle \mathbf{p}_{D_3},\ \mathbf{p}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3}\rangle$ should at least be isomorphic. This, unfortunately, is not the case. This is the content of the final definition and theorem in this thesis.

Definition 5.0.9. *Let M_1 and M_2 be Frobenius manifolds over vector spaces with potentials T_1 and T_2 and Euler fields E_1 and E_2, respectively. Let U_j be a neighborhood of the origin in M_j where the potential T_j is known to converge. Then an* isomorphism *of M_1 and M_2 is a biholomorphic map $\phi : U_1 \to U_2$ such that*

1. $\phi(0) = 0$
2. $T_1 = T_2 \circ \phi$
3. $\phi_*(id_1) = id_2$ *where id_j is the identity on M_j*
4. $\phi_*(E_1) = E_2$.

Theorem 5.0.10. *The D_3 Frobenius manifolds defined by the two possible values of the correlator*

$$\langle \mathbf{p}_{D_3},\ \mathbf{p}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3}\rangle = \pm\frac{1}{8}$$

are not isomorphic.

Proof. Let $\alpha = \langle \mathbf{p}_{D_3},\ \mathbf{p}_{D_3},\ \mathbf{h}_{D_3},\ \mathbf{h}_{D_3}\rangle$. Then the potential for the D_3 Frobenius manifold M

is given by

$$\begin{aligned} T &= 3\frac{\langle \mathbf{1}, \mathbf{1}, \mathbf{h} \rangle}{3!} t_1^2 t_h + 3\frac{\langle \mathbf{1}, \mathbf{p}, \mathbf{p} \rangle}{3!} t_1 t_p^2 + 6\frac{\langle \mathbf{p}, \mathbf{p}, \mathbf{h}, \mathbf{h} \rangle}{4!} t_p^2 t_h^2 + \frac{\langle \mathbf{h}, \mathbf{h}, \mathbf{h}, \mathbf{h}, \mathbf{h} \rangle}{5!} t_h^5 \\ &= \frac{1}{2} t_1^2 t_h - \frac{1}{4} t_1 t_p^2 + \frac{\alpha}{4} t_p^2 t_h^2 + \frac{8\alpha^2}{5!} t_h^5. \end{aligned} \tag{5.1}$$

Here, t_1, t_p, and t_h correspond to $\mathbf{1}$, $\mathbf{p}$, and $\mathbf{h}$, respectively. This formula was computed directly from the definition, Equation 2.1. The coefficients in front of each term came from the fact that the sum in Equation 2.1 is over all *unordered* tuples of basis elements, so that if correlator insertions are not all distinct, a given term will appear multiple times. For example, $\langle \mathbf{1}, \mathbf{1}, \mathbf{h} \rangle$ will appear once for each possible position of the $\mathbf{h}$ insertion, so we multiply the corresponding term by 3.

The computation of the 3-point correlators is a straightforward application of the Pairing Axiom (Axiom 4), since we already computed the pairings of the corresponding elements in Example 2.2.19. The value of the 5-point correlator was computed in terms of α in Lemma 5.0.5.

Let M_1 be the D_3 Frobenius manifold with potential T_1 equal to T in Equation (5.1) with $\alpha = \frac{1}{8}$. Label the coordinates of M_1 by $(t_1,\, t_p,\, t_h)$. Similarly, let N be the D_3 Frobenius manifold with potential T_2 equal to T in Equation (5.1) with $\alpha = -\frac{1}{8}$. Use coordinates $(r_1,\, r_p,\, r_t)$ on N.

Now we desire to find a map $\phi : \mathbb{C}^3_{t_1,t_p,t_h} \to \mathbb{C}^3_{r_1,r_p,r_h}$ satisfying the conditions of Definition 5.0.9. Write

$$\phi = (\, \phi_1(t_1, t_p, t_h),\ \phi_p(t_1, t_p, t_h),\ \phi_h(t_1, t_p, t_h)\,),$$

so that $r_i = \phi_i$. From condition (1) of Definition 5.0.9, we know that $\phi(0) = 0$. Let us next explore what condition (4) means for ϕ.

From the definition of W-degree, we easily compute

$$\deg_W(\mathbf{1}) = 0 \qquad \deg_W(\mathbf{p}) = \frac{1}{2} \qquad \deg_W(\mathbf{h}) = 1.$$

Because the W-degrees are the same for M_1 and M_2, using Theorem 3.1.2 we have Euler fields

$$E_1 = t_1 \frac{\partial}{\partial t_1} + \frac{3}{4} t_p \frac{\partial}{\partial t_p} + \frac{1}{2} t_h \frac{\partial}{\partial t_h}$$

and

$$E_2 = r_1 \frac{\partial}{\partial r_1} + \frac{3}{4} r_p \frac{\partial}{\partial r_p} + \frac{1}{2} r_h \frac{\partial}{\partial r_h}.$$

Then using the formula for the pushforward in coordinates, we require

$$\begin{bmatrix} \frac{\partial r_1}{\partial t_1} & \frac{\partial r_1}{\partial t_p} & \frac{\partial r_1}{\partial t_h} \\ \frac{\partial r_p}{\partial t_1} & \frac{\partial r_p}{\partial t_p} & \frac{\partial r_p}{\partial t_h} \\ \frac{\partial r_h}{\partial t_1} & \frac{\partial r_h}{\partial t_p} & \frac{\partial r_h}{\partial t_h} \end{bmatrix} \begin{bmatrix} t_1 \\ \frac{3}{4} t_p \\ \frac{1}{2} t_h \end{bmatrix} = \begin{bmatrix} r_1 \\ \frac{3}{4} r_p \\ \frac{1}{2} r_h \end{bmatrix}.$$

This is equivalent to the system of equations

$$t_1 \frac{\partial \phi_i}{\partial t_1} + \frac{3}{4} t_p \frac{\partial \phi_i}{\partial t_p} + \frac{1}{2} t_h \frac{\partial \phi_i}{\partial t_h} = d_i \phi_i$$

for $i = 1, p, h$, where $d_1 = 1$, $d_p = \frac{3}{4}$, and $d_h = \frac{1}{2}$. It is easy to see that this is equivalent to requiring $E(\phi_i) = d_i \phi_i$. Using the vocabulary of Lemma 2.1.5, the coordinate functions ϕ_i of ϕ must be Euler of degree d_i. Thus, we know

$$\phi = (\, c_1 t_1 + c_2 t_h^2,\ c_p t_p,\ c_h t_h \,),$$

where the constants c_1, c_2, c_p, and c_h are complex numbers still to be solved for.

Next, condition (3) says that $\phi_*(\frac{\partial}{\partial t_1}) = \frac{\partial}{\partial r_1}$. Again from the coordinate pushforward matrix, we find this implies $\frac{\partial \phi_1}{\partial t_1} = 1$, so we know $c_1 = 1$.

Finally, we consider condition (2). This condition says that we want $T_1 = T_2 \circ \phi$. This is an equality of two polynomials which we can solve by equating like coefficients. First, when we expand the term on the right, we get a term $c_h c_2 t_1 t_h^3$, but no $t_1 t_h^3$-term appears on the left. Thus $c_h = 0$ or $c_2 = 0$; it is easy to see that if $c_2 = 0$ the map ϕ will not be bijective.

So $c_h = 0$. Thus, ϕ must have the form

$$\phi = (t_1, c_p t_p, c_h t_h).$$

Next, equating coefficients of $t_1^2 t_h$-terms yields $\frac{1}{2} = \frac{c_h}{2}$, so $c_h = 1$. Similarly, equating coefficients of the $t_1 t_p^2$-terms yields $-\frac{1}{4} = -\frac{c_p^2}{4}$, which means $c_p^2 = 1$. But then fixing $\alpha = \frac{1}{8}$ and equating coefficients of the $t_p^2 t_h^2$-term (which has opposite sign in the two potentials), we have $\frac{\alpha}{4} = -\frac{\alpha c_p^2 c_h^2}{4} = -\frac{\alpha}{4}$, which is a contradiction.

$\square$

We close by noting the reason for the bad news behind Theorem 5.0.10: simply put, the WDVV equations (or the reconstruction algorithm) were not strong enough to uniquely determine the Frobenius manifold structure of D_3. Though this is disappointing, the reader should keep in mind that this Frobenius manifold is still uniquely determined by the definition of the correlator (Definition 2.2.20). And, in fact, reconstruction narrowed the possibilities down to only two.

Chapter 6. Conclusion

We have completely determined the A-model Frobenius manifold structure for a pair (W, G_W^{max}) when W is a sum of A_{n-1} and D_{n+1}-type polynomials (though only up to the sign of a 4-point correlator for polynomials with D_3 summands). To do this, we had to develop a new strategy for computing correlators specifically designed for sums of polynomials. We discovered that the Frobenius manifold structure of W is in fact determined by the structure of its summands in a very clean way. In the course of this proof, we computed the Frobenius manifold structure of D_3, as it had been overlooked in earlier literature.

We restricted our investigations to the polynomials A_{n-1} and D_{n+1} because they are exactly those polynomials with a unique primitive element in the usual basis for their Frobenius algebra structure. A natural question for subsequent investigation is, can the techniques in this thesis be easily extended to polynomials whose summands have arbitrarily many primitive elements? This would allow for a complete understanding of the Frobenius manifold structure of the pair (W, G_W^{max}) based on our understanding of its component pieces. Such an understanding, derived from the methods used in this thesis, might be easier to compute with than the ideas already worked out in [Kau99].

A second question of great interest is, now that we understand the A-model Frobenius manifold for this large class of polynomials, can we compute the corresponding B-model Frobenius manifold and show that they are isomorphic? As explained in the introduction, this is a hard question because the pairing on a B-model Frobenius manifold is determined by a *primitive form*. The primitive form is just a differential form in a straightforward space, but we are required to choose it so that the induced pairing is flat (i.e., constant in some coordinate system). It is not at all clear what differential forms will do the trick. An understanding of the primitive forms for the B-models of our class of polynomials would be substantial progress in this area.

This leads us to a third question, by far the most difficult. The murky nature of primitive forms is a huge obstacle to our understanding of the Landau-Ginzburg B-model and hence of the mirror symmetry relationship. A useful theorem would be to characterize the primitive form of the B-model of a sum of polynomials in terms of the (easier to understand) primitive forms of the pieces. Such a characterization might lead to an analog of Theorem 4.1.2 on the B-side. This would be a huge leap forward in our understanding of Landau-Ginzburg mirror symmetry.

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