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k-S-Rings

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ABSTRACT

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For a finite group G we study certain rings $\mathfrak{S}_G^{(k)}$ called k-S-rings, one for each $k \geq 1$, where $\mathfrak{S}_G^{(1)}$ is the centralizer ring $Z(\mathbb{C}G)$ of G. These have the property that $\mathfrak{S}_G^{(k+1)}$ determines $\mathfrak{S}_G^{(k)}$ for all $k \geq 1$. We show that $\mathfrak{S}^{(4)}$ determines G when G is any group with finite classes. We show that $\mathfrak{S}_G^{(3)}$ determines G for any finite group G, thus giving an answer to a question of Brauer. We show the 2-characters defined by Frobenius and extended 2-characters of Ken Johnson are characters of representations of $\mathfrak{S}_G^{(2)}$. We find the character table for the 2-S-ring of the dihedral groups of order $2n$, n odd, and classify groups with commutative 3-S-ring.

Keywords: S-ring, character, k-character, group algebra, finite group, Frobenius, FC group

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Contents

Chapter 1. Introduction

In 1963, Richard Brauer collected several known results and unsolved problems from the field of Group Theory into an expository article [3]. In that article, Brauer asked what else needs to be known, in addition to the character table of a finite group, to determine the group. That question, and some of the results it inspired, motivate and direct our work. In particular, Brauer's question rekindled interest in the group determinant and the k-characters of a finite group, objects which played a fundamental role in the development of character theory.

In 1896, while studying the group determinant, Ferdinand Georg Frobenius defined functions he called k-characters of the group (where k is any positive integer) which allowed him to factor the group determinant. Frobenius' 1-characters are now called the irreducible characters of a finite group.

Soon after, in 1897, Frobenius discovered that the 1-characters were related in a natural way to the representations of groups. The characters of a finite group G are class functions on the conjugacy classes of G. The class sums form a basis for $Z(\mathbb{C}G)$, the center of the group algebra. From the irreducible characters of G it is possible to determine $Z(\mathbb{C}G)$ up to isomorphism, and from $Z(\mathbb{C}G)$, the irreducible characters can be determined.

The k-characters are also class function on special subsets of G^k called the k-classes of G. The k-class sums form a a basis for a subring of $\mathbb{C}G^k$ which we call the k-S-ring of G, and the k-S-rings of groups are the primary object of study in this paper.

It is straightforward to show that D_8 and Q_8, the dihedral group of order 8 and the quaternions respectively, have the same 2-S-rings. Thus, the 2-S-ring does not determine the group, but in answer to Brauer's question we show that the 3-S-ring determines the group, when G is a group for which all conjugacy classes have finite size. We also prove two stronger results for finite groups. The first is that finite groups with the same character table are determined by the products of 3-classes of elements of type (x, x, x), $(x, x, 1)$, and $(x, 1, 1)$ for $x \in G$ with 1 the identity in G. The second, and stronger, condition is that a finite group

G is determined by a group invariant we call the non-determined incomplete Cayley table (NDICT) of G.

In 1991, motivated by Brauer's question, Edward Sibley and David Formanek showed that the group determinant determines the group [7]. In 1992, Ken Johnson and Surinder K. Sehgal showed that the 1- and 2-characters do NOT determine a group [14], and in the same year Ken Johnson working with Hans-Jürgen Hoehnke showed that 1-, 2-, and 3- characters do determine a group [8].

Ken Johnson also defined a 2- and 3-character table for a finite group G in [12]. In the definition of the 2-character table of a group the notion of the 2-S-ring is used [12]. See also [25] for a discussion of k-S-rings and k-characters.

The character table of an S-ring (and hence k-S-ring) has also been defined and studied by Olaf Tamaschke [24]. Unlike the character tables of groups, the character tables of S-rings do not have to be square. For example, the 2-S-ring of the Frobenius group of order 20 has a character table which is not square. We calculate the character table for the 2-S-ring of D_{2n}, n odd, which does have a square 2-character table. We explore the relationship between the irreducible characters of G^2, the characters of the 2-S-ring of G, and the 2-characters of G. We show that the 2-characters that Frobenius defined correspond to representations of (and hence characters of) the 2-S-ring in a natural way.

In partial answer to the question of when a k-S-ring has a square character table, we show that the only groups with commutative 3-S-rings are the generalized dihedral groups of order $2n$, n odd. A result of Frobenius guarantees a square character table for commutative S-rings.

1.1 Definitions and Notations

In this paper we are primarily concerned with finite groups and FC groups.

Definition 1.1. [21] An *FC group* is a group for which the sizes of all conjugacy classes are finite.

We use e or 1 to denote the identity element of a group G. For $x \in G$, let $o(x)$ denote the order of x. Also, we let $x^y = y^{-1}xy, [x, y] = x^{-1}y^{-1}xy$, and for $K \subseteq G$ we let $h^K = \{g^{-1}hg | g \in K\}$. We let $C_G(a)$ denote the centralizer in G of the element $a \in G$, and we let $C_G(A)$ denote the centralizer of the set $A \subseteq G$.

A group G is *generalized dihedral* if $G = N \rtimes C_2$ where N is a finite abelian group and C_2 is a cyclic group of order 2 which acts on N by inversion.

Definition 1.2. Let G be an FC group. The (complex)*group algebra of* G, denoted $\mathbb{C}G$, consists of formal sums of type $\sum\limits_{g \in G} \alpha_g g$, where only finitely many of the $\alpha_g \in \mathbb{C}$ are non-zero and $(\alpha g)(\beta h) = (\alpha\beta)gh$ for $\alpha, \beta \in \mathbb{C}$.

For a finite set $S \subseteq G$, we define $\overline{S} \in \mathbb{C}G$ to be $\overline{S} = \sum\limits_{g \in S} g$.

If the sets $\{C_i\}$ are the conjugacy classes of an FC group G, then the corresponding elements $\overline{C}_i \in \mathbb{C}G$ are called the *class sums* of G and generate $Z(\mathbb{C}G)$, the center of the group algebra. The subring $Z(\mathbb{C}G)$ of $\mathbb{C}G$ is an example of a special type of subring of a group algebra called an S-ring.

The concept of an S-ring was introduced by Schur, but it was Wielandt who first called them S-rings [20, 27, 4]:

Definition 1.3. An *S-ring* over an FC group G is a subring (or subalgebra) of the group algebra $\mathbb{C}G$ spanned by elements of the form $\overline{\Gamma}_i$ where $\mathbb{S} =\{\Gamma_1, \Gamma_2, \dots\}$ is a partition of G:

$$G = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_m \cup \dots,$$

with $\Gamma_1 = \{e\}$ and $|\Gamma_i| < \infty$ which satisfies:

(1) If $i \geq 1$ and $\Gamma_i = \{g_1, \dots, g_s\}$ then there is some $j \geq 1$ such that $\Gamma_i^{-1} := \{g_1^{-1}, \dots, g_s^{-1}\}$ is equal to Γ_j;

(2) If $i, j \geq 1$, then $\overline{\Gamma}_i\overline{\Gamma}_j = \sum_k \lambda_{ijk}\overline{\Gamma}_k$ where λ_{ijk} is a non-negative integer for all i, j, k.

The Γ_i are called the *principal sets* of the S-ring. The $\overline{\Gamma_i}$ form a basis for the S-ring and are called the *principal elements*. The λ_{ijk} are called the *structure constants* of the ring.

It is also common to define the *S-ring* of a finite group as a subring of the integral group ring $\mathbb{Z}G$ which has the principal elements $\{\overline{\Gamma}_i\}$ as the basis. We will occasionally think of the k-S-ring as a subring of the integral group ring, and when we do, this should be clear from the context.

Let G be a finite group. Fix $k \geq 1$ and let the symmetric group S_k act on G^k by permuting entries:

$$(g_1, g_2, \ldots, g_k)^\sigma = (g_{(1)\sigma}, g_{(2)\sigma}, \ldots, g_{(k)\sigma}),$$

and let G act on the right on G^k by diagonal conjugation:

$$(g_1, g_2, \ldots, g_k)^g = (g_1^g, g_2^g, \ldots, g_k^g).$$

Let $\tilde{G}_k$ denote the permutation group generated by these actions of S_k and G on G^k.

Definition 1.4. The $\tilde{G}_k$-orbits of the action are called *k-classes* of G. The $\tilde{G}_k$ -orbit of $g = (g_1, g_2, \ldots, g_k) \in G^k$ is denoted $K_G^{(k)}(g)$ or $K_G(g_1, g_2, \ldots, g_k)$. We write $K^{(k)}(g)$ or $K(g_1, g_2, \ldots, g_k)$ if G is understood.

Because $\tilde{G}_k$ acts via automorphisms on G^k, the k-classes determine an S-ring over G^k which we call the *k-S-ring of* G and which we denote by $\mathfrak{S}_G^{(k)}$.

In the case $k = 1$, we see that $\mathfrak{S}_G^{(1)}$ is just the centralizer ring $Z(\mathbb{C}G)$. We think of the k-S-rings of G as generalized centralizer rings.

The k-classes as defined here are the same k-classes used by Frobenius in his study of k-characters. The k-characters are defined recursively.

Definition 1.5. Let ψ be a character of G, and define $\psi^{(1)} = \psi$. For $(g, h) \in G^2$ define

$$\psi^{(2)}(g, h) = \psi(g)\psi(h) - \psi(gh).$$

Recursively define

$$\begin{array}{rl}\psi^{(k)}(g_1, g_2, \ldots, g_k) = & \psi(g_1)\psi^{(k-1)}(g_2, g_3, \ldots, g_k) - \psi^{(k-1)}(g_1g_2, g_3, \ldots, g_k) - \\ & \psi^{(k-1)}(g_2, g_1g_3, \ldots, g_k) - \cdots - \psi^{(k-1)}(g_2, g_3 \ldots, g_1g_k)\end{array}.$$

Let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G. If $\mathrm{Irr}(G) = \{\psi_1, \psi_2, \ldots, \psi_r\}$ is a complete set of irreducible characters of G, then the maps $\{\psi_i^{(k)} : G \to \mathbb{C}\}$ are called *the k-characters of G*. We will also call them the *Frobenius k-characters*.

Example 1.6. Consider $S_3 = \{e, (123), (132), (12), (13), (23)\}$, which has character table

S_3	e	(123)	(12)
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

Using the formula $\psi^{(2)}(g, h) = \psi(g)\psi(h) - \psi(gh)$, one can calculate some of the 2-character values for S_3.

$$\begin{aligned}
\chi_3^{(2)}(e, e) &= \chi(e)\chi(e) - \chi(e) = 4 - 2 = 2 \\
\chi_3^{(2)}((123), (123)) &= \chi((123))\chi((123)) - \chi((132)) = 1 - (-1) = 2 \\
\chi_3^{(2)}((12), (12)) &= \chi((12))\chi((12)) - \chi(e) = 0 - 2 = -2 \\
\chi_3^{(2)}((12), (23)) &= \chi((12))\chi((23)) - \chi((123)) = 0 - (-1) = 1
\end{aligned}$$

We write $GL_n(\mathbb{C})$ for the set of invertible $n \times n$ matrices with complex entries and $M_n(\mathbb{C})$ for the $n \times n$ matrices with complex entries. For $A \in M_n(\mathbb{C})$, $\mathrm{tr}(A)$ denotes the trace of the matrix A. If $A \in M_n(\mathbb{C})$ is a diagonal matrix with diagonal entries $a_1, a_2, \ldots, a_n$, then we write $A = \mathrm{D}(a_1, a_2, \ldots, a_n)$.

Definition 1.7. A *representation* of a group G is a group homomorphism $\rho : G \to GL_n(\mathbb{C})$. A representation of an S-ring T is an algebra (or ring) homomorphism $\rho : T \to M_n(\mathbb{C})$.

If $\{\rho_i\}$ is a complete set of pairwise non-isomorphic irreducible representations of a finite group G, then the functions $\chi_i : G \to \mathbb{C} : g \mapsto \mathrm{tr}(\rho_i(g))$ are the irreducible characters of

G. The value of $\chi_i(1)$ is the *degree* of χ_i. Characters for S-rings are defined analogously, as follows:

Definition 1.8 (Tamaschke)**.** Let T be an S-ring of a finite group $G = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_m$, generated by the principal sets Γ_i. Let $\tau_j = \dfrac{\overline{\Gamma_j}}{|\Gamma_j|}$ for $j = 1, \ldots, m$. For any representation $F : T \to M_n(\mathbb{C})$ the complex valued function $\phi : G \to \mathbb{C} : g \mapsto \text{trace} F(\tau_i)$ where $g \in \Gamma_i$, is called the *T-character of G related to F*. The character is *irreducible* if the representation is irreducible.

Every Schur-ring (as a subalgebra of $\mathbb{C}G$) is a semisimple algebra [26, p. 386, footnote]. Thus, every representation of T is completely reducible, and hence each character of T is a linear combination of the irreducible characters of T with non-negative integral coefficients [24, p. 342].

We also have the notion of a T-character table.

Definition 1.9. If T is an S-ring over a finite group G, the *character table of T* or *T-character table of G* has rows indexed by the irreducible characters and columns indexed by an element of the principal set and entries the character values. We let $CT(T)$ denote the character table of the S-ring T.

If G is finite, we also define an inner product on characters of S-rings.

Definition 1.10. Let χ, ψ be T-characters of G. Then we define

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}$$

where $\overline{\psi(g)}$ denotes the complex conjugate.

When $T = Z(\mathbb{C}G)$, this definition agrees with the usual definition of inner product on characters of a group.

If $\rho : G \to GL_n(\mathbb{C})$ is a representation of G, and T is an S ring of G, then ρ extends naturally to a representation $\mathbb{C}G \to M_n(\mathbb{C})$ and we let $\widehat{\rho}$ denote the restriction of this map to the S-ring T.

If $\phi : G \to H$ is any map between groups G, H, then we we define $\phi^{(k)} : G^k \to H^k : (g_1, \dots g_k) \to (\phi(g_1), \dots, \phi(g_k))$.

Definition 1.11. We say FC groups G, H *have the same k-S-ring* if there exists a bijection $\phi : G \to H$ for which the map $\hat{\phi}^{(k)}$ is a ring isomorphism of the k-S-rings of G and H.

Example 1.12. We have defined the concept of having the same k-S-ring. There is a weaker concept: Let A, B be S-rings over groups G, H respectively with principal sets $\{C_1, \dots, C_r, \dots\}$, $\{D_1, \dots, D_r, \dots\}$. Then an *algebraic isomorphism of S-rings* is a map $\phi : \{C_1, \dots, C_r, \dots\} \to \{D_1, \dots, D_r, \dots\}$ which induces an algebra isomorphism

$$\phi : \langle \bar{C}_1, \dots, \bar{C}_r, \dots \rangle \to \langle \bar{D}_1, \dots, \bar{D}_r, \dots \rangle.$$

A principal element C of $\mathfrak{S}_G^{(k)}$ will be called *diagonal* if every element of C has the form $(g, g, \dots, g)$ for some $g \in G$. Note that every element $g \in G$ determines a diagonal k-class.

As a generalization of diagonal classes we have the following:

Definition 1.13. The *uniform* 3*-classes* of an FC group G are the classes $K(x, x, x)$, $K(x, x, 1)$, $K(x, 1, 1)$ for $x \in G$.

We are also interested in the following maps, which allow us to answer Brauer's question regarding what, in addition to the character table, determines a finite group G.

Definition 1.14. A bijection $\psi : G \to H$ which induces an isomorphism of centralizer algebras and which also has the property that $\psi^{(3)}(AB) = \psi^{(3)}(A)\psi^{(3)}(B)$ for all uniform 3-classes A, B of G^3 is called a *UTCCI map* or UTCCI bijection, (where UTC stands for uniform 3-class and CI stands for centralizer isomorphism).

All computations made in the preparation of this paper were done using Magma [2].

1.2 Background Results and related results

Julius Wilhelm Richard Dedekind began studying the group matrix and the group determinant in the 1880s. He was interested in factoring the group determinant, as part of his work looking for a nice basis for normal fields [4, p. 51].

Definition 1.15. If G is a finite group with elements $\{g_1, g_2, \ldots g_n\}$, then the matrix $X_G = [x_{g_i g_j^{-1}}]$ is called the *group matrix of* G. Here $x_{g_1}, x_{g_2}, \ldots, x_{g_n}$ are commuting indeterminates, and $\Theta_G = det(X_G)$ is the *group determinant* of G.

For example, for $C_4 = \{e = g_1, t = g_2, t^2 = g_3, t^3 = g_4\}$, we have

$$\begin{aligned}\Theta_{C_4} &= \begin{vmatrix} x_1 & x_4 & x_3 & x_2 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & x_2 & x_1 & x_4 \\ x_4 & x_3 & x_2 & x_1 \end{vmatrix} \\ &= x_1^4 - x_2^4 + x_3^4 - x_4^4 - 2x_1^2x_3^2 + 2x_2^2x_4^2 - 4x_1^2x_2x_4 + 4x_1x_2^2x_3 + 4x_1x_3x_4^2 - 4x_2x_3^2x_4\end{aligned}$$

In this case the group determinant of C_4 can be written as a product of linear factors:

$$\Theta_{C_4} = (x_1 + x_2 + x_3 + x_4)(x_1 - x_2 + x_3 - x_4)(x_1 + ix_2 - x_3 - ix_4)(x_1 - ix_2 - x_3 + ix_4)$$

For abelian groups, Dedekind proved:

Theorem 1.16 (Dedekind)**.** *If $G = \{g_1, g_2, \ldots, g_n\}$ is an abelian group of order n and $\psi', \psi'', \ldots \psi^{(n)}$ are the characters corresponding to it, then the determinant Θ_G is decomposable, namely as the product of n linear factors*

$$\psi^{(s)}(g_1)x_1 + \cdots + \psi^{(s)}(g_n)x_n$$

that correspond to the n values of s.

Non-abelian groups have linear factors and factors of higher degree in their group determinants, when factored completely over $\mathbb{C}$. For example $\Theta_{S_3} = Q_1Q_2Q_3^2$ where

$$
\begin{aligned}
Q_1 =&\quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6, \\
Q_2 =&\quad x_1 + x_2 + x_3 - x_4 - x_5 - x_6, \text{ and} \\
Q_3 =&\quad x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 - x_6^2 - x_1x_2 - x_1x_3 - x_2x_3 + x_4x_5 + x_4x_6 + x_5x_6,
\end{aligned}
$$

and the x_i are commuting indeterminates.

In 1896 Frobenius proved a generalization of Dedekind's result for non-abelian groups [4]. His work involved the definition of characters and k-characters, and relies on the following result:

Theorem 1.17. *([Frobenius]) If there are n^3 variables a_{ijk} satisfying*

- $a_{ijk} = a_{jik}$;
- $\sum_j a_{ijk}a_{jpq} = \sum_j a_{ijq}a_{jpk}$;
- $\sum_{i,j} a_{ijk}a_{jil}$ *are all non-zero.*

Then the equations $r_jr_k = \sum_i a_{ijk}r_i$ have exactly n different complex solutions, each of which is an n-tuple $(r_1, r_2, \ldots, r_n)$ and they are linearly independent.

The first two requirements guarantee that we have a commutative, associative algebra, and the third guarantees that the algebra be semisimple. For Frobenius, the a_{ijk} were the structure constants of $Z(\mathbb{C}G)$, the center of the group algebra. Wielandt [26, p. 386, footnote] showed that all S-rings are semisimple, and so in particular k-S-rings are also, so this theorem also applies to the structure constants of commutative k-S-rings, giving a set of n linearly independent complex solutions.

When the a_{ijk} are the structure constants of $Z(\mathbb{C}G)$, then the associated solutions $(r_1, r_2, \ldots, r_n)$ are 'almost' the irreducible characters of G. Frobenius scaled the solutions so they would satisfy certain conditions called 'orthogonality conditions' known to hold for characters of abelian groups, and he called these n-tuples the *characters* of G.

For example, consider $S_3 = \{e, (123), (132), (12), (13), (23)\}$. Then S_3 has three conjugacy classes and so $Z(\mathbb{C}S_3)$ is generated by the class sums

$$\overline{C}_1 = e, \overline{C}_2 = (123) + (132), \overline{C}_3 = (12) + (13) + (23).$$

To find the structure constants, we calculate

$$\begin{aligned}
\overline{C}_1\overline{C}_1 &= \overline{C}_1 \\
\overline{C}_1\overline{C}_2 &= \overline{C}_2 \\
\overline{C}_1\overline{C}_3 &= \overline{C}_3 \\
\overline{C}_2\overline{C}_2 &= 2\overline{C}_1 + \overline{C}_2 \\
\overline{C}_2\overline{C}_3 &= 2\overline{C}_3 \\
\overline{C}_3\overline{C}_3 &= 3\overline{C}_1 + 3\overline{C}_2
\end{aligned}$$

Theorem 1.17 states that the system of equations in variables x_1, x_2, x_3 with the same structure constants will have three complex solutions.

$$\begin{aligned}
x_1x_1 &= x_1 \\
x_1x_2 &= x_2 \\
x_1x_3 &= x_3 \\
x_2x_2 &= 2x_1 + x_2 \\
x_2x_3 &= 2x_3 \\
x_3x_3 &= 3x_1 + 3x_2
\end{aligned}$$

The solutions to this system are (1,2,3), (1,2,-3), (1,-1,0), or, in tabular from we have:

S_3	C_1	C_2	C_3
	1	2	3
	1	2	-3
	1	-1	0

Dividing all entries of the second column by $|C_2| = 2$ and all entries of the third column by $|C_3| = 3$, and then multiplying entries of the third row by 2 produces the character table of S_3:

S_3	e	(123)	(12)
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

.

To get the character table for the 1-S-ring of S_3 we only divide the columns by the class sizes, but do not multiply the third row by 2.

$\mathfrak{S}_{S_3}^{(1)}$	e	(123)	(12)
ψ_1	1	1	1
ψ_2	1	1	-1
ψ_3	1	$-1/2$	0

.

The maps $\psi_i : G \to \mathbb{C}$ induce representations $\hat{\psi}_i : \mathfrak{S}_{S_3}^{(1)} \to \mathbb{C} : \tau_j \to \psi_i(g_j)$, where $\tau_j = \frac{\overline{K(g_j)}}{|K(g_j)|}$. When we evaluate $\langle \psi_3, \psi_3 \rangle = \frac{1}{6}(1^2 + 2(-1/2)^2 + 3(0^2)) = \frac{1}{4}$, we see that the characters defined this way are not 'ortho-normal.' However, the inner product is meaningful, as discussed in Chapter 5.

Theorem 1.18 (Frobenius)**.** *If ψ is a character of degree k, then the corresponding irreducible factor P_ψ of Θ_G is given by*

$$P_\psi = \frac{1}{k!} \sum \psi^{(k)}(g_1, g_2, \ldots, g_k) x_{g_1} x_{g_2} \cdots x_{g_k},$$

where the summation is over all elements of G^k.

The characters of a finite group do *not* determine the group. For example D_8 and Q_8 are non-isomorphic and have the same character table.

D_8, Q_8	C_1	C_2	C_3	C_4	C_5
ψ_1	1	1	1	1	1
ψ_2	1	1	1	-1	-1
ψ_3	1	1	-1	1	-1
ψ_4	1	1	-1	-1	1
ψ_5	2	-2	0	0	0

However, D_8 and Q_8 are determined by their 2-characters [12, pp. 303-305].

A renewed interest in Frobenius's approach to representation theory was initiated by K. W. Johnson in [12] and has led (among other things) to the following results:

(1) the group determinant determines the group [7, 15];

(2) the $1-, 2-$ and $3-$ characters determine the group [8].

Also, in [12] the notion of the 2-character table was defined. It was shown that there are non-isomorphic groups with the same 2-character table [14]. Another relevant concept is that of the *weak Cayley table* of a group $G:$ this is the matrix with rows and columns indexed by the elements of $G = \{g_1 = 1, g_2 \ldots, g_n\}$, such that the g_i, g_j entry is the conjugacy class of the element $g_i g_j$ [13]. One says that groups G, H *have the same weak Cayley table* if there is a bijection $\alpha : G \to H$ which takes classes to classes and satisfies $\alpha(g_1 g_2) \sim \alpha(g_1)\alpha(g_2)$, where $\sim$ denotes conjugacy in H [13]. The map α is called a *weak Cayley table isomorphism.* We note that the concept of groups having the same weak Cayley table makes sense even if the groups are infinite, so that the weak Cayley table is defined for any group.

It is known that there are non-isomorphic groups with the same weak Cayley table [13]. It is also known that the information in each of

(i) the weak Cayley table;

(ii) the $1-$ and $2-$characters;

(iii) the 2-character table;

of a finite group G is the same [13].

There are also the related questions about what properties of the group are determined by the character table, 2-character table, and so forth. Mattarei [17, 18, 16] answered one question of Brauer [3] by showing that there are non-isomorphic solvable groups with the same character table, but with different derived lengths. Of course the character table of G determines G/G'. But there are non-isomorphic groups G, H with the same character table but with $G'/G'' \ncong H'/H''$ [16]. Johnson, Mattarei and Sehgal [13, Corollary 3.5,3.6] showed further that there are non-isomorphic groups which have the same weak Cayley table, but different derived lengths.

We show in Lemma 1.29 that the diagonal k-classes generate a subring of the k-S-ring of G which is isomorphic to $Z(\mathbb{C}G)$. Thus, both the weak Cayley table of G and the 2-S-ring of G determine the character table of G, but the properties of having the same weak Cayley table and having the same 2-S-ring are independent:

Groups with the same weak Cayley table have the same number of involutions [13]. In general, the 2-S-ring does not determine the inverse map of a group. However, in Lemma 1.31 we show that the inverse map is determined by $\mathfrak{S}_G^{(2)}$ in the case where G has odd order.

Theorem 1.19. *[9] There are groups which have the same weak Cayley table, but not the same* 2*-S-rings, and groups which have the same* 2*-S-rings but not the same weak Cayley table:*

(a) the dihedral and quaternion groups of order 8*,* D_8 *and* Q_8*, have the same* 2*-S-rings, but do not have the same weak Cayley table;*

(b) the two non-abelian groups of order p^3*, where* $p > 3$ *is a prime, have the same weak Cayley table, but do not have the same* 2*-S-rings.*

In particular, the 2*-S-ring of a group does not determine the group.*

Proof. For part (a) we use the presentations for the quaternion and dihedral groups of order eight given below:

$$Q_8 = \langle x, y, z | x^2 = z, y^2 = z, z^2, (xy)^2 = z, (x, z), (y, z) \rangle,$$
$$D_8 = \langle a, b, c | a^2, b^2, c^2, (a, c), (b, c), (ab)^2 = c \rangle.$$

Groups with the same weak Cayley table have the same number of involutions [13], but D_8 and Q_8 have different numbers of involutions. Thus D_8 and Q_8 do not have the same weak Cayley tables. The bijection $\phi : D_8 \to Q_8$ below induces an isomorphism of $\mathfrak{S}^{(2)}_{D_8}$ and $\mathfrak{S}^{(2)}_{Q_8}$:

$$\phi(e_{D_8}) = e_{Q_8}, \quad \phi(a) = x, \quad \phi(b) = xy, \quad \phi(c) = z,$$
$$\phi(ab) = y, \quad \phi(bc) = xyz, \quad \phi(ac) = xz, \quad \phi(abc) = yz.$$

One can check by hand that this gives an isomorphism of 2-S-rings. Thus D_8 and Q_8 have the same 2-S-rings, but not the same weak Cayley tables.

For a proof of part (b) see [9]. $\square$

Example 1.20. The computer algebra system MAGMA [2] has a database of finite groups. We use the notation $G_{n,k}$ to denote the kth group among all groups of order n in the MAGMA database. Each of the following pairs of groups

$$(G_{192,1023}, G_{192,1025}), \qquad (G_{768,1085030}, G_{768,1085037}),$$
$$(G_{768,1083600}, G_{768,1083604}), (G_{1280,1116310}, G_{1280,1116312}),$$

have the same character tables, but have different 2-S-rings, since they have a different number of 2-classes. One checks that each of these eight groups is a Frobenius group and that in each case the groups in each pair have different numbers of involutions and so do not have the same weak Cayley table. We note that in [13, p. 408] the authors give a necessary and sufficient condition for two Frobenius groups to have the same weak Cayley table.

Let

$$G^{(0)} = G, G^{(1)} = [G, G], \ldots, G^{(i)} = [G^{(i-1)}, G^{(i-1)}], \ldots$$

denote the *derived series* of the group G. Another result in [9] contrasts to the results of [13, 17, 18, 16] showing that the character table and the weak Cayley table do not determine the derived length of a group:

Theorem 1.21. *[9] Let G be a finite group. The weak Cayley table of G and $\mathfrak{S}_G^{(2)}$ together determine the classes of G that lie in each element $G^{(i)}$ of the derived series of G. In particular, they determine the size of each $G^{(i)}$ and the length of the derived series of G.*

Thus the weak Cayley Table and the 2-S-ring together give us more information about the group than does the weak Cayley table alone.

We say groups G, H *have the same WCT$\mathfrak{S}^{(2)}$* if there is a weak Cayley table isomorphism $\phi : G \to H$ such that $\phi^{(2)}$ determines an isomorphism $\mathfrak{S}_G^{(2)} \to \mathfrak{S}_H^{(2)}$.

Theorem 1.22. *[9] If G and H have the same WCT$\mathfrak{S}^{(2)}$ determined by $\phi : G \to H$, then $\phi(G^{(i)}) = H^{(i)}$ for all $i \geq 1$; in particular, the sizes of the derived factors of G and H are the same.*

Example 1.23. Let $G_{27,3} = \langle a_1, a_2, a_3 | a_1^3, a_2^3, a_3^3, a_2^{a_1} = a_2a_3,$ and a_3 is central$\rangle$ and $G_{27,4} = \langle b_1, b_2, b_3 | b_1^3 = b_3, b_2^3, b_3^3, b_2^{b_1} = b_2b_3,$ and b_3 is central$\rangle$.

One can check that the map $\phi : G_{27,3} = \langle a_1, a_2, a_3\rangle \to G_{27,4} = \langle b_1, b_2, b_2\rangle$ defined by

$$\begin{aligned} \phi(a_1^{\varepsilon_1} a_2^{\varepsilon_2} a_3^{\varepsilon_3}) &= b_1^{\varepsilon_1} b_2^{\varepsilon_2} b_3^{\varepsilon_3} \quad \text{if } \varepsilon_1 = 0, 1; \\ \phi(a_1^{\varepsilon_1} a_2^{\varepsilon_2} a_3^{\varepsilon_3}) &= b_1^{\varepsilon_1} b_2^{\varepsilon_2} b_3^{\varepsilon_3+2} \text{ if } \varepsilon_1 = 2, \end{aligned}$$

is a weak Cayley table isomorphism and that $\phi : G_{27,3} \to G_{27,4}$ induces an isomorphism of the 2-S-rings of these groups. This shows that $G_{27,3}$ and $G_{27,4}$ have the same WCT$\mathfrak{S}^{(2)}$.

Recall that a pair (G, H) of non-isomorphic groups form a *Brauer pair* if there is a bijection $\phi : G \to H$ that maps classes to classes, that determines an isomorphism of centralizer algebras, and which also respects the power maps: the H-class of $\phi(g^k)$ is the same as the class of $\phi(g)^k$ for all $g \in G, k \in \mathbb{Z}$.

With respect to Brauer's problem of which properties together with the character table determine a group, it can be shown that having the same weak Cayley table and the same 2-S-ring does not determine a group up to isomorphism:

Theorem 1.24. *[9] There are non-isomorphic groups of order 2^9 which have the same weak Cayley table and the same 2-S-rings. They form a Brauer pair.*

Example 1.25. We have defined the concept of an algebraic isomorphism of S-rings. One can check that the 2-S-rings for the non-abelian groups $G_{5^3,3}, G_{5^3,4}$ of order 5^3 are algebraically isomorphic but that these groups do not have isomorphic 2-S-rings [9].

One can also check that the 2-S-rings for $G_{32,30}$ and $G_{32,31}$ are not algebraically isomorphic, although they have the same character table. Thus, groups with the same character tables do not necessarily have algebraically isomorphic 2-S-rings, and in this case certainly do not have the same 2-S-rings.

Let $\phi^{(2)} : \mathfrak{S}_G^{(2)} \to \mathfrak{S}_H^{(2)}$ be an isomorphism induced from $\phi : G \to H$. We define $\Delta : G \to G^2$ by the rule $g \mapsto (g, g)$. Let C_i be the principal elements of $\mathfrak{S}_G^{(2)}$ and let $E_i = \phi^{(2)}(C_i)$. Let N be a normal subgroup of G and write $N = A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_t}$ as a union of classes of G. Then $\overline{\Delta(A_{i_j})} \in \mathfrak{S}_G^{(2)}$ is diagonal and so $\phi^{(2)}(\overline{\Delta(A_{i_j})})$ is also diagonal. It follows that $\Delta(N)$ is a normal subgroup of $G \times G$ which is isomorphic to N. Thus $\phi^{(2)}(\Delta(N))$ is a normal subgroup of $H \times H$ which projects isomorphically under the first projection map to a normal subgroup M of H. We say that M *corresponds to* N in this circumstance.

Example 1.26. Here we show that there are groups G, H with the same 2-S-rings, which also have the property that G has a normal subgroup N corresponding to $M \triangleleft H$ such that $\mathfrak{S}_{G/N}^{(2)}$ and $\mathfrak{S}_{H/M}^{(2)}$ are not algebraically isomorphic and so certainly do not have the same 2-S-rings. Thus the property of having the same 2-S-ring does not behave well under quotients. This is in contrast to the case of the weak Cayley table, where one can show that if $\phi : G \to H$ is a the weak Cayley table isomorphism and $N \lhd G, M = \phi(N)$, then G/N and H/M do have the same weak Cayley table.

To describe our example, take $G = G_{64,13}, H = G_{64,14}$, using the Magma notation. One can check directly that G, H have the same 2-S-rings (one can use [9, Theorem 5.5] to give this result also). Further, one finds that G and H each have three normal subgroups of order 2. The quotients of G by these three normal subgroups are groups having isomorphism types

$$G_{32,8},\ G_{32,9},\ G_{32,10},\ (*)$$

while the quotients of H by its three normal subgroups of order two are of isomorphism types

$$G_{32,7},\ G_{32,10},\ G_{32,10}.(**)$$

Thus if the above statement is not true, then we should be able to pair up the groups in (*) with the groups in (**) so that corresponding pairs have algebraically isomorphic 2-S-rings. One can check that $G_{32,7}$ and $G_{32,8}$ have the same 2-S-rings, but that $G_{32,9}$ and $G_{32,10}$ do not have algebraically isomorphic 2-S-rings. Thus there can be no such pairing and the result follows. $\square$

Here is another a property of a group which is not determined by the character table, but which is determined by the 2-S-ring.

Theorem 1.27. *[9] (i) If G is a group and $x, y \in G, x \not\sim y$, then*

$$|K_G(x,y)| = 2 \times [G : C_G(\langle x, y\rangle)].$$

(ii) If G is a group of odd order and $x, y \in G, x \neq y$, then

$$|K_G(x,y)| = 2 \times [G : C_G(\langle x, y\rangle)].$$

In particular, for a group G of odd order and any $x, y \in G$, we can determine $|C_G(\langle x, y\rangle)|$

from $\mathfrak{S}_G^{(2)}$.

(iii) $\mathfrak{S}_G^{(2)}$ determines the multiset of sizes $|C_{C_G(a)}(b)| = |C_G(\langle a,b\rangle)|$ where $a, b \in G$ are chosen to be non-conjugate. In particular, there are groups with the same character table, but with different such multisets.

For a proof see [9].

1.3 Fundamental Results on k-S-rings

The k-class sums are the basis for the k-S-ring. We will use the following basic properties of 2-classes, in particular.

Lemma 1.28. *Fix $g, h \in G$. We have the following:*

(i) $K(e,g) = \big(\{e\} \times g^G\big) \cup \big(g^G \times \{e\}\big)$.

(ii) $K(g,h) \in (g^G \times h^G) \cup (h^G \times g^G)$.

(iii) If $g \not\sim h$, then $K(g,h) = (g^G \times h^G) \cup (h^G \times g^G)$ if and only if $C_G(g)C_G(h) = G$.

(iv) When g is not central, $g^G \times g^G$ is a union of two or more 2-classes, one of which is always the diagonal class $K(g,g)$.

Proof. Statements (i),(ii), and (iv) are direct consequences of the action of $\tilde{G}_2$ on G^2. And (iii) follows from a homework problem in [10, p.7]. □

Lemma 1.29. *The set of all $\bar{C}$, where C is a diagonal principal element, generates a subring of $\mathfrak{S}_G^{(k)}$ which is isomorphic to $Z(\mathbb{C}G)$, the centralizer S-ring of G.*

Proof. Let $A_1, \ldots, A_u$ be the diagonal principal elements of $\mathfrak{S}_G^{(k)}$ and let $\pi_1 : G^k \to G$ be the first projection function $\pi_1(g_1, g_2, \ldots, g_k) = g_1$.

Let $\Delta(G) = \{(g, g, \ldots, g) : g \in G\} \subset G^k$ and note that the restriction of π_1 to $\Delta(G)$ gives a group isomorphism $\pi_1 : \Delta(G) \to G$ which in turn determines an isomorphism of algebras $\mathbb{C}\Delta(G) \cong \mathbb{C}G$. Since each $\pi_1(A_i)$ is a class of G the result follows. □

One can easily show that $\Delta(G)$ is contained in the center of $\mathfrak{S}_G^{(k)}$.

Corollary 1.30. *Let $\phi^{(k)} : \mathfrak{S}_G^{(k)} \to \mathfrak{S}_H^{(k)}$ be an isomorphism of k-S-rings that is induced from the bijection $\phi : G \to H$. Then G and H have the same character tables. In particular, the map ϕ induces an isomorphism of centraliser algebras $\phi : Z(\mathbb{C}G) \to Z(\mathbb{C}H)$.*

Proof. The isomorphism $\phi^{(k)} : \mathfrak{S}_G^{(k)} \to \mathfrak{S}_H^{(k)}$ maps diagonal classes to diagonal classes bijectively and induces an isomorphism between the subrings that they generate. These subrings are isomorphic to the respective centralizer rings and so the result follows from Lemma 1.29, upon recalling that the character table is determined by the structure constants of the centralizer ring. See Theorem 1.17. □

If $\phi : G \to H$ is a bijection which induces a k-S-ring isomorphism $\hat{\phi}^{(k)}$, and the k-classes of G and H are $\{C_i : i \in I\}, \{E_j : j \in J\}$ (respectively), then

(i) for all $i \in I$ there is $j \in J$ with $\phi^{(k)}(C_i) = E_j$; and

(ii) if we relabel the E_j so that $\phi^{(k)}(C_i) = E_i$, then we have $\bar{C}_i\bar{C}_j = \sum_k \lambda_{ijk}\bar{C}_k$ if and only if $\bar{E}_i\bar{E}_j = \sum_k \lambda_{ijk}\bar{E}_k$.

Here (i) follows from the fact that principal sets must go to principal sets, and (ii) follows from (i) and the fact that the rings are isomorphic.

The weak Cayley table determines inverses of elements. In general, the 2-S-ring does not. However, we do have the following:

Lemma 1.31. *Let G be a group of odd order, $x \in G$. Then $\mathfrak{S}_G^{(2)}$ enables us to find x^{-1}.*

Proof. We may assume that $x \neq 1$. Then there is certainly some 2-class, D say, that contains (x, x^{-1}); that class satisfies $(1, 1) = (x, x^{-1})(x^{-1}, x) \in D^2$.

On the other hand, if D is a class that contains (x, y) and satisfies $(1, 1) \in D^2$, then we will first show that $(x^{-1}, y^{-1}) \in D$ and then we will show that $y = x^{-1}$.

Since $(1, 1) \in D^2$ there are $s, t \in G$ such that either

(a) $(x^s, y^s)(x^t, y^t) = (1, 1)$; or

(b) $(x^s, y^s)(y^t, x^t) = (1, 1)$.

Conjugating (a) or (b) by s^{-1} gives $(x, y)(u, v) = (1, 1)$, where $(u, v) \in D$ and so $(x^{-1}, y^{-1}) = (u, v) \in D$.

Since $(x^{-1}, y^{-1}) \in D = K(x, y)$ we must have either

(i) $(x^{-1}, y^{-1}) = (x^r, y^r)$, for some $r \in G$, or

(ii) $(x^{-1}, y^{-1}) = (y^r, x^r)$, for some $r \in G$.

Note that (i) is forbidden since $|G|$ is odd. Thus we have (ii): $x^{-1} = y^r, y^{-1} = x^r$, showing that

$$x = (y^{-1})^r = x^{r^2},$$

so that $r^2 \in C_G(x)$. But $|G|$ is odd and so $r \in C_G(x)$. This gives $y^{-1} = x^r = x$, so that $(x, y) = (x, x^{-1})$.

Thus, given x, to find x^{-1} we look for a 2-class D such that $(x, y) \in D$ for some $y \in G$, and $\bar{D}^2 = \lambda(1, 1) + \ldots, \lambda \neq 0$. We conclude, as in the above, that $y = x^{-1}$. □

Theorem 1.32. *Let $k \geq 1$. Then for any FC group G there is an epimorphism $\pi^{(k)}$: $\mathfrak{S}_G^{(k+1)} \to \mathfrak{S}_G^{(k)}$. Also, FC groups having the same k-S-ring have the same r-S-ring for all $1 \leq r \leq k$. In particular, finite groups having the same k-S-ring (for some $k \geq 1$) have the same character table.*

Proof. Fix $k \geq 1$ and let $\pi = \pi^{(k)} : G^{k+1} \to G^k$ be the projection so that $\pi(g_1, \ldots, g_k, g_{k+1}) = (g_1, \ldots, g_k)$. Then π induces a ring homomorphism $\mathbb{Z}G^{k+1} \to \mathbb{Z}G^k$ that we will also denote by π. We will show that

(1) if C is a principal set of $\mathfrak{S}_G^{(k+1)}$, then $\pi(\overline{C}) \in \mathfrak{S}_G^{(k)}$; this will show that π restricts to a ring homomorphism $\pi : \mathfrak{S}_G^{(k+1)} \to \mathfrak{S}_G^{(k)}$.

(2) the ring homomorphism $\pi : \mathfrak{S}_G^{(k+1)} \to \mathfrak{S}_G^{(k)}$ is onto.

Showing (1) and (2) will prove that $\mathfrak{S}_G^{(k+1)}$ determines $\mathfrak{S}_G^{(k)}$ for all $k \geq 1$.

To show (1), let C be a principal set of $\mathfrak{S}_G^{(k+1)}$. Since C is invariant under the action of diagonal conjugation and under the S_{k+1}-permutation action, it is easy to see that $\pi(C)$ is invariant under the action of diagonal conjugation and under the S_k-permutation action. It remains to show that if $\alpha = (g_1, \ldots, g_k), \beta = (h_1, \ldots, h_k) \in \pi(C)$ are in the same k-class, then the cardinalities of $\pi^{-1}(\alpha) \cap C$ and $\pi^{-1}(\beta) \cap C$ are the same.

Now, since $\alpha = (g_1, \ldots, g_k), \beta = (h_1, \ldots, h_k) \in \pi(C)$ are in the same k-class, then there is some permutation $\phi \in S_k$ and some $g \in G$ such that $\beta = (\alpha^g)\phi$. Thus it suffices to prove the result in the two cases where either (i) $\beta = \alpha^g$; or (ii) $\beta = (\alpha)\phi$.

So suppose that we have (i). Then for $\gamma \in \pi^{-1}(\alpha) \cap C$ we have $\gamma^g \in \pi^{-1}(\beta) \cap C$, because C is closed under action by diagonal conjugation. This shows that $(\pi^{-1}(\alpha) \cap C)^g \subseteq \pi^{-1}(\beta) \cap C$; similarly we have $(\pi^{-1}(\beta) \cap C)^{g^{-1}} \subseteq \pi^{-1}(\alpha) \cap C$, since $\alpha = \beta^{g^{-1}}$. Thus $\pi^{-1}(\beta) \cap C \subseteq (\pi^{-1}(\alpha)) \cap C)^g$ and so $\pi^{-1}(\beta) \cap C = (\pi^{-1}(\alpha) \cap C)^g$. This proves case (i).

Now suppose that we have (ii). We may think of $\phi \in S_k$ as an element of S_{k+1} in the obvious way. Then for $\gamma \in \pi^{-1}(\alpha) \cap C$ we have $(\gamma)\phi \in \pi^{-1}(\beta) \cap C$, because C is closed under the S_{k+1} action. This shows that $(\pi^{-1}(\alpha) \cap C)\phi \subseteq \pi^{-1}(\beta) \cap C$, and, as in case (i), one easily shows that $(\pi^{-1}(\alpha) \cap C)\phi = \pi^{-1}(\beta) \cap C$. This proves case (ii) and concludes the proof of (1).

Now for (2) we let C be a principal set of $\mathfrak{S}_G^{(k)}$, and let $\alpha = (g_1, \ldots, g_k) \in C$. Let $\varepsilon(\alpha)$ denote the number of entries g_i that are equal to the identity $1 \in G$. Clearly $\varepsilon(\alpha)$ is a function on C. We prove the result by (descending) induction on $0 \leq \varepsilon(C) \leq k$ for fixed k. Thus we assume that the image of π contains all k-classes C with $\varepsilon(C) = r$ and then show that it contains all k-classes C with $\varepsilon(C) = r - 1$. (Clearly the image contains all k-classes C with $\varepsilon(C) = k$.)

So assume that the image of π contains all k-classes C with $\varepsilon(C) = r$, and let D be a k-class with $\varepsilon(D) = r - 1$. Let $\alpha = (g_1, \ldots, g_k) \in D$ and let D' be the $(k+1)$-class that contains $\alpha' = (g_1, \ldots, g_k, 1)$. Then the elements in D' have either the form (a) $(a, 1)$, where $a \in G^k$; or (b) $(a, 1, a')$, where $(a, a') \in G^k$ and the last entry of a' is not 1.

The image under π of an element of type (a) has ε value equal to $r - 1$, and is an element of C; while the image of an element of type (b) has ε value equal to r. By induction we know that these latter classes are in the image of π, and so D is in the image of π. $\square$

Chapter 2. The 4-S-Ring Determines the Group

In this chapter, we show that the 4-S-ring of an FC group G determines the group G. It is true that, for finite groups, this result is subsumed in the result that the 3-S-ring of a finite group determines the group, since the 4-S-ring determines the 3-S-ring. But this result is cleaner and motivates a theorem that we use in the 3-S-ring case.

Definition 2.1. The *conjugacy incomplete Cayley Table* or *CICT* of a group G is the map $G^2 \to G : (g, h) \mapsto gh$ when $g \not\sim h$.

Thus, when we are given the CICT of G, we do not know the entire Cayley Table, because on conjugacy classes our information is incomplete. Related to the idea of a CICT we introduce the idea of a CICT map:

Definition 2.2. Let G, H be groups. A bijection $\psi : G \to H$ which satisfies $\psi(gh) = \psi(g)\psi(h)$ for all $g, h \in G$ with $g \not\sim h$ is called a CICT map. If there is a CICT map from G to H, we say that G and H *have the same CICT*.

In the first section of this chapter, we show that if two FC groups G and H have the same 4-S-ring, then they have the same CICT. In the second section we show that two FC groups with the same CICT are isomorphic.

Our main result for this chapter is the following:

Theorem 2.3. *Let G and H be FC groups which have the same 4-S-ring. Then G and H are isomorphic.*

Throughout this chapter, unless otherwise mentioned, G and H are FC groups.

2.1 Determining xy from the 4-S-ring

We know that we can determine the identity element 1 of G from the 4-S-ring of G, as it is the unique element $x \in G$ for which $K(x, x, x, x)K(x, x, x, x) = K(x, x, x, x) \in \mathfrak{S}_G^{(4)}$. We begin this section by proving the following lemma:

Lemma 2.4. *Let $x, y \in G \setminus \{1\}$. If $x \not\sim y$, then $(xy, xy, x, y) \in K(x, x, x, 1)K(y, y, y, 1)$, and if $(g, g, x, y) \in K(x, x, x, 1)K(y, y, y, 1)$, then $g = xy$.*

Proof. Let $x, y \in G \setminus \{1\}$, and assume $x \not\sim y$. Let $A = K(x, x, x, 1)$, $B = K(y, y, y, 1)$. By the definition of 4-classes, we have $A = \{(x, x, x, 1)^s, (x, x, 1, x)^s, (x, 1, x, x)^s, (1, x, x, x)^s\}_{s\in G}$ and $B = \{(y, y, y, 1)^t, (y, y, 1, y)^t, (y, 1, y, y)^t, (1, y, y, y)^t\}_{t\in G}$. In particular, $(x, x, x, 1) \in A$ and $(y, y, 1, y) \in B$, so $(x, x, x, 1)(y, y, 1, y) = (xy, xy, x, y) \in AB$, and it has form (g, g, x, y).

Suppose for some $g \in G$ we have $(g, g, x, y) \in AB$. Then we have $(g, g, x, y) = \alpha\beta$ for some $\alpha \in A$ and some $\beta \in B$, and looking at the types of elements in A and B, we see that $\alpha\beta$ could be one of sixteen possible product types. We check each of these sixteen cases and show that if $(g, g, x, y) \in AB$, then $g = xy$.

Case 1: $\alpha = (x, x, x, 1)^s$, $\beta = (y, y, y, 1)^t$, $\alpha\beta = (x^s y^t, x^s y^t, x^s y^t, 1) = (g, g, x, y)$. Here we have $x^s y^t = g, x^s y^t = g, x^s y^t = x, 1 = y$, and the last equation gives a contradiction, because by assumption $y \neq 1$.

Case 2: $\alpha = (x, x, x, 1)^s$, $\beta = (y, y, 1, y)^t$, $\alpha\beta = (x^s y^t, x^s y^t, x^s, y^t) = (g, g, x, y)$. Here we have $x^s y^t = g, x^s y^t = g, x^s = x, y^t = y$ so we have $x^s = x$, $y^t = y$, and $g = x^s y^t = xy$.

Case 3: $\alpha = (x, x, x, 1)^s$, $\beta = (y, 1, y, y)^t$, $\alpha\beta = (x^s y^t, x^s, x^s y^t, y^t) = (g, g, x, y)$. Here we have $x^s y^t = g, x^s = g, x^s y^t = x, y^t = y$ so that $g = x^s y^t = x^s$. Solving $x^s y^t = x^s$ for y, we get $y = 1$. This gives a contradiction, because we assume $y \neq 1$.

Case 4: $\alpha = (x, x, x, 1)^s$, $\beta = (1, y, y, y)^t$, $\alpha\beta = (x^s, x^s y^t, x^s y^t, y^t) = (g, g, x, y)$. Here we have $x^s = g, x^s y^t = g, x^s y^t = x, y^t = y$ so again we have $y = 1$, a contradiction.

Case 5: $\alpha = (x, x, 1, x)^s$, $\beta = (y, y, y, 1)^t$, $\alpha\beta = (x^s y^t, x^s y^t, y^t, x^s) = (g, g, x, y)$. Here we have $x^s y^t = g, x^s y^t = g, y^t = x, x^s = y$ so we have a contradiction, since, by assumption $x \not\sim y$.

Case 6: $\alpha = (x, x, 1, x)^s$, $\beta = (y, y, 1, y)^t$, $\alpha\beta = (x^s y^t, x^s y^t, 1, x^s y^t) = (g, g, x, y)$. Here, if $\alpha\beta = (g, g, x, y)$ then we get $x^s y^t = g, x^s y^t = g, 1 = x, x^s y^t = y$ so we have $x = 1$, a contradiction.

Case 7: $\alpha = (x, x, 1, x)^s$, $\beta = (y, 1, y, y)^t$, $\alpha\beta = (x^s y^t, x^s, y^t, x^s y^t) = (g, g, x, y)$.

Here we have $x^s y^t = g, x^s = g, y^t = x, x^s y^t = y$ so we have another contradiction in $y^t = x$.

Case 8: $\alpha = (x, x, 1, x)^s$, $\beta = (1, y, y, y)^t$, $\alpha\beta = (x^s, x^s y^t, y^t, x^s y^t) = (g, g, x, y)$.

Here we have $x^s = g, x^s y^t = g, y^t = x, x^s y^t = y$ so we have $x \sim y$, a contradiction again.

Case 9: $\alpha = (x, 1, x, x)^s$, $\beta = (y, y, y, 1)^t$, $\alpha\beta = (x^s y^t, y^t, x^s y^t, x^s) = (g, g, x, y)$.

Here we have $x^s y^t = g, y^t = g, x^s y^t = x, x^s = y$ so we again have the contradiction $x \sim y$.

Case 10: $\alpha = (x, 1, x, x)^s$, $\beta = (y, y, 1, y)^t$, $\alpha\beta = (x^s y^t, y^t, x^s, x^s y^t) = (g, g, x, y)$.

Here we have $x^s y^t = g, y^t = g, x^s = x, x^s y^t = y$ so we have $g = x^s y^t = y^t$, so $x = 1$, giving another contradiction.

Case 11: $\alpha = (x, 1, x, x)^s$, $\beta = (y, 1, y, y)^t$, $\alpha\beta = (x^s y^t, 1, x^s y^t, x^s y^t) = (g, g, x, y)$.

Here we have $x^s y^t = g, 1 = g, x^s y^t = x, x^s y^t = y$ so we have $x^s y^t = x = y = g = 1$ and so this case also gives us a contradiction, since $x \neq 1$ by assumption.

Case 12: $\alpha = (x, 1, x, x)^s$, $\beta = (1, y, y, y)^t$, $\alpha\beta = (x^s, y^t, x^s y^t, x^s y^t) = (g, g, x, y)$.

Here, if $\alpha\beta = (g, g, x, y)$ then we get $x^s = g, y^t = g, x^s y^t = x, x^s y^t = y$ so we have $x \sim g \sim y$, a contradiction.

Case 13: $\alpha = (1, x, x, x)^s$, $\beta = (y, y, y, 1)^t$, $\alpha\beta = (y^t, x^s y^t, x^s y^t, x^s) = (g, g, x, y)$.

Here we have $y^t = g, x^s y^t = g, x^s y^t = x, x^s = y$ so we have $x \sim y$, a contradiction.

Case 14: $\alpha = (1, x, x, x)^s$,$\beta = (y, y, 1, y)^t$, $\alpha\beta = (y^t, x^s y^t, x^s, x^s y^t) = (g, g, x, y)$.

Here we have $y^t = g, x^s y^t = g, x^s = x, x^s y^t = y$ so we have $y^t = g = x^s y^t$, so $x = 1$, a contradiction.

Case 15: $\alpha = (1, x, x, x)^s$, $\beta = (y, 1, y, y)^t$, $\alpha\beta = (y^t, x^s, x^s y^t, x^s y^t) = (g, g, x, y)$.

Here, we have $y^t = g, x^s = g, x^s y^t = x, x^s y^t = y$ so we have $x = x^s y^t = y$, a contradiction.

Case 16: $\alpha = (1, x, x, x)^s$, $\beta = (1, y, y, y)^t$, $\alpha\beta = (1, x^s y^t, x^s y^t, x^s y^t) = (g, g, x, y)$.

Here we have $1 = g, x^s y^t = g, x^s y^t = x, x^s y^t = y$ so we have $x^s y^t = x = y = g = 1$, a contradiction.

This shows if $x, y \in G \setminus \{1\}$ and $(g, g, x, y) \in AB$, then we must have $g = xy$. $\square$

We want to apply this result to FC groups G, H which have the same 4-S-rings. By definition there is a bijection $\psi : G \to H$ which induces an isomorphism $\psi^{(4)} : \mathfrak{S}_G^{(4)} \to \mathfrak{S}_H^{(4)}$.

Recall that $\psi : G \to H$ induces an isomorphism $\psi : Z(\mathbb{C}G) \to Z(\mathbb{C}H)$ and takes classes of G to classes of H, by Lemma 1.32.

Now, fix $x, y \in G \setminus \{1\}$, with $x \not\sim y$. Because ψ takes classes to classes, we know that $\psi(x) \not\sim \psi(y)$. Also, because $\psi^{(4)}$ is a ring isomorphism, we have

$$\begin{aligned}\psi^{(4)}\left(K(x,x,x,1)K(y,y,y,1)\right) &= \psi^{(4)}\left(K(x,x,x,1)\right)\psi^{(4)}\left(K(y,y,y,1)\right)\\ &= K(\psi(x),\psi(x),\psi(x),1)K(\psi(y),\psi(y),\psi(y),1).\end{aligned}$$

And $(xy, xy, x, y) \in K(x,x,x,1)K(y,y,y,1)$, so

$$\psi^{(4)}(xy,xy,x,y) \in K(\psi(x),\psi(x),\psi(x),1)K(\psi(y),\psi(y),\psi(y),1)$$

But $\psi^{(4)}(xy,xy,x,y) = (\psi(xy),\psi(xy),\psi(x),\psi(y))$ is of type $(h,h,\psi(x),\psi(y))$. As shown in the lemma above, $K(\psi(x),\psi(x),\psi(x),1)K(\psi(y),\psi(y),\psi(y),1)$ contains only one term of type $(h,h,\psi(x),\psi(y))$, and this occurs when $h = \psi(x)\psi(y)$. So we must have $\psi(xy) = \psi(x)\psi(y)$ in the case where $x \not\sim y$. We have just shown the following:

Theorem 2.5. *Let G, H be FC groups with 4-S-ring isomorphism induced by $\psi : G \to H$. Then $\psi(xy) = \psi(x)\psi(y)$ for any $x, y \in G$ with $x \not\sim y$, i.e. ψ is a CICT map.* □

For a finite group G, we can picture the CICT of G as being simply the Cayley table of G with no entries in products of elements which are conjugate. Here is the CICT of S_3, where we are intentionally not giving traditional labels to the elements of the group for purposes of this example.

S_3	e	s	t	u	v	w
e	e	s	t	u	v	w
s	s			v	w	u
t	t			w	u	v
u	u	w	v			
v	v	u	w			
w	w	v	u			

In the next section, we want to show that given the CICT of an FC group, there is a unique way to complete it as the Cayley table of a group. Or equivalently, that an FC group G is determined by its products $gh, g \not\sim h$.

Before proceeding with the general proof, we consider a specific case using the CICT of S_3. We know if G is a group with the same CICT as S_3, then G is non-abelian of order 6, and because S_3 is the only non-abelian group of order 6, this CICT must be the CICT of S_3. But what we want to demonstrate is how to use the information of the CICT to fill in the undetermined products. For this example, we show how the product uv is determined by the CICT of S_3

First, note that u and v are in a class of size $\frac{|G|}{2}$, so u and v must be involutions. One way to determine the product uv is to look in the u row of the CICT. We see that v occurs in this row, and more particularly that v occurs in the t column, so $ut = v$. We know that multiplying both sides by u, we get $t = uv$, so that we have determined from the CICT that uv must be t. This method can be used to determine any of the values uv, vu, uw, wu, vw, wv, and relies on the fact that u, v and w are involutions. It also relies on the fact that v 'happened' to occur in the u row of the CICT.

This example is just for a specific type of class, but it highlights the main technique that we have available. For a fixed element h in G, we look among elements of $\{hg | g \in G\}$, i.e. elements of the h row of the CICT, for an element a with a desirable characteristic, because

$h^{-1}a$ is determined by the CICT. The proof relies heavily on the associative property of groups.

We introduce two sets that allow us to talk about these ideas in general.

Let G be an FC group, and assume that for any $g, h \in G$ with $h \not\sim g$, the product gh is known or, equivalently, that we have the CICT of G. Then, given $h \in G$, for any $g \in G \setminus h^G$, the product hg is known and we define

$$P_h = \{hg | g \not\sim h\}.$$

For a finite group, if we fill in the CICT as we did in the S_3 example, P_h is the set of entries in the h-row of the CICT of G.

If $b \in P_h$, i.e. "b occurs in the h row of the conjugate incomplete Cayley Table", then $b = ha$ for some $a \in G$, and we can determine $a = h^{-1}b$ from the CICT. (If we have a filled out CICT for a finite group G, this corresponds to finding the entry b in the h row and identify the column a in which b occurs.) So, the set P_h has the following two properties:

(P1) For $a \in G$, $ha \in P_h$ if and only if $a \not\sim h$;

(P2) If $b \in P_h$, then $h^{-1}b$ can be determined from the CICT of G.

In the next section, we will be interested in those $h \in G$ for which we can calculate hx and also $(hx)y$. We know hx can be determined from the CICT exactly when $h \not\sim x$, and, when hx can be calculated, we want to be able to calculate $(hx)y$. This can be determined exactly when $hx \not\sim y$, or equivalently when $hx \not\sim x$, because $y \sim s$. We define

$$Q_x = G \setminus \left(\{h \in G | hx \sim x\} \cup x^G\right),$$

so that Q_x consists exactly of those $h \in G$ for which hx and $(hx)y$ can be determined for any $y \sim x$.

We note that $1 \in \{g \in G | gx \sim x\}$, since $1x \sim x$, so $1 \notin Q_x$. Also, we know that

$|\{g \in G | gx \sim x\}| = |x^G|$ because this set is exactly $x^G x^{-1}$. From this it follows that $|Q_x| \geq |G| - 2|x^G|$. Also, the notation for Q_x is independent of y, because the set itself is independent of y. For any $h \in Q_x, z \in x^G$, we can determine $(hx)z$.

From the CICT, we can only calculate gx for $g \not\sim x$, so we cannot necessarily determine $\{g \in G | gx \sim x\}$ directly. But when $g \not\sim x$ we can determine gx and we know from the CICT whether $gx \sim x$. So we can determine the set $\{h \in G | hx \sim x\} \cup x^G$, and from that we can determine Q_x, using only the information in the CICT.

Summarizing, we know that the set Q_x can be determined from the CICT and has the following properties:

(Q1) $1 \notin Q_x$;

(Q2) $|Q_x| \geq |G| - 2|x^G|$;

(Q3) For any $h \in Q_x$, we can compute $(hx)y$.

For example, in the CICT of S_3, we have $P_u = \{u, v, w\}$ and $Q_u = S_3 \setminus [\{1, s, t\} \cup \{u, v, w\}] = \emptyset$, while $Q_s = G \setminus [\{s, t\} \cup \emptyset] = \{e, u, v, w\}$.

2.2 The CICT of G determines G

In this section our goal is to show that FC groups with the same CICT are isomorphic. First we prove the following:

Theorem 2.6. *Let G be an FC group and suppose that we know each product gh where g and h are not conjugate. Then we can determine the Cayley table of G algorithmically.*

Proof. Let G be an FC group and fix $x, y \in G$ with $x \sim y$. We want to show that xy is determined by the CICT of G. We will do this proof by cases based on the size of $x^G = y^G$. The first case, $|x^G| = \frac{|G|}{2}$ generalizes the method of our S^3 example.

Case 1: G is a finite group and $|x^G| = \frac{|G|}{2}$. Then we have $|C_G(x)| = 2$, so that $C_G(x) = \{1, x\}$ and $o(x) = 2$. Because $y \sim x$ we know $o(y) = 2$ and $C_G(y) = \{1, y\}$ as well. If $y = x$, then $xy = x^2 = 1$ is known, so assume $y \neq x$. Because $(xy)^2 = 1$ implies $xy = (xy)^{-1} = y^{-1}x^{-1} = yx$, but we know that y does not centralize x, it follows that xy is not an involution. In particular, $xy \not\sim x$, so $y = x(xy)$ is in P_x by property (P1). And we know from property (P2) that when $y \in P_x$, we can solve for $x^{-1}y = xy$. So when $|x^G| = \frac{|G|}{2}$, we can always calculate xy.

Case 2a: G is a finite group and $|x^G| < \frac{|G|}{4}$. In this case, from property (Q2), we have

$$|Q_x| \geq |G| - 2|x^G| > |G| - 2\frac{|G|}{4} = \frac{|G|}{2},$$

so that more than half the elements of G are in Q_x. Fix an element $h \in Q_x$. By (Q3), we can determine $(hx)y$, and by (P2), if $(hx)y$ is in P_h, then we can also compute $h^{-1}(hx)y = xy$.

In the case where G is finite and we have a filled out CICT in front of us, we could fix $h \in Q_x$, compute $(hx)y$, and then look for $(hx)y$ in P_h, the entries of the h row of the CICT. If we find $(hx)y$ in P_h, we see what column it occurred in, i.e. we compute $h^{-1}(hx)y$ (using property (P2)), and we have determined xy. If we do not find $h(xy)$ in P_h, we can fix another element $h' \in Q_x$ and try again. The question is whether we are guaranteed to eventually find an element $g \in Q_x$ for which $(gx)y \in P_g$, so that we can use property (P2) to determine $g^{-1}(gx)y = xy$. We show that there is always such a g.

By property (P1), we have $(hx)y = h(xy) \in P_h$ exactly when $h \not\sim xy$. Because $|(xy)^G| \leq \frac{|G|}{2}$ and $|Q_x| > \frac{|G|}{2}$, we know $|Q_x \setminus (xy)^G| > 0$, so $Q_x \setminus (xy)^G \neq \emptyset$. Fix $g \in Q_x \setminus (xy)^G$. Because $g \in Q_x$, it follows from (Q3) that we can determine $(gx)y$. And because $g \not\sim xy$, it follows from (P1) that $g(xy) \in P_g$. So we know $g(xy)$ and by (P2) can solve for $g^{-1}(gx)y = xy$. This means that if we try the method described above for enough elements, we must eventually find an element $g \in Q_x$ for which $(gx)y \in P_g$, so that we can calculate $g^{-1}(gx)y = xy$. This shows that the product xy is determined in the case where $|x^G| < \frac{|G|}{4}$.

Case 2b: G is a non-finite FC group. Let $s = |x^G|$. In this case, from property (Q2), we have

$$|Q_x| \geq |G| - 2s$$

so that Q_x is clearly nonempty.

When $|x^G| \not< \frac{|G|}{4}$, then G is a finite group and $|x^G| \in \left\{\frac{|G|}{2}, \frac{|G|}{3}, \frac{|G|}{4}\right\}$. We have already shown that when $|x^G| = \frac{|G|}{2}$, we can calculate xy. So there are only the cases $|x^G| = \frac{|G|}{3}$ and $|x^G| = \frac{|G|}{4}$ remaining.

When $|x^G| \not< \frac{|G|}{4}$, then $|x^G| \in \left\{\frac{|G|}{2}, \frac{|G|}{3}, \frac{|G|}{4}\right\}$. We have already shown that when $|x^G| = \frac{|G|}{2}$, we can calculate xy. So there are only the cases $|x^G| = \frac{|G|}{3}$ and $|x^G| = \frac{|G|}{4}$ remaining.

Case 3: G is finite and $|x^G| = \frac{|G|}{3}$. In this case, by property (Q2), we have $|Q_x| \geq |G| - 2|x^G| = \frac{|G|}{3}$, so in particular Q_x is nonempty. Also for this case $|C_G(x)| = 3$, and so $C_G(x)$ must be the cyclic group generated by x, where $o(x) = 3$. If $y = x$, we first look for $1 = x(x^2) \in P_x$. If $1 \in P_x$, then by (P2) we can calculate $x^{-1}1 = x^2 = xy$. It follows from property (P1) that we have $1 = xx^2 \in P_x$ exactly when $x \not\sim x^2$. So if $1 \notin P_x$, then $x \sim x^2$. In this case, fix $h \in Q_x$. Because $x \sim x^2$ and $h \not\sim x$, we know $h \not\sim x^2$. Thus $(hx)x$ can be calculated by (Q3), and by (P1), $(hx)x = hx^2 \in P_h$. So by property(P2) we can find $h^{-1}(hx)x = x^2$. This shows that we can determine x^2 whenever $|x^G| = \frac{|G|}{3}$.

To finish the $|x^G| = \frac{|G|}{3}$ case, we assume $x \neq y$. We know we can determine x^2, so first we can look for y in P_{x^2}. If $y \in P_{x^2}$, then by (P2) we can calculate $(x^2)^{-1}y = xy$. If $y \notin P_{x^2}$, then it is because $xy \sim x^2$. As was the case when we determined x^2, our method will vary depending on whether or not x and x^2 are conjugate.

We first consider the subcase where $x \sim x^2$ and $xy \sim x^2$. In this case, fix $g \in Q_x$. Then by (Q3), $(gx)y$ can be calculated, and $g \not\sim x$, so $g \not\sim xy$, so $g(xy) \in P_g$, by (P1). Thus, by property (P2) $g^{-1}(gx)y = xy$ is determined by the CICT.

Next, we consider the subcase where $x \not\sim x^2$ and $xy \sim x^2$. Then $x \not\sim xy$ and $y \not\sim x^2$, because $x \sim y$. But $y \not\sim x^2$ implies that x^2y is known. And because $x \not\sim xy$, from (P1) it follows that $x(xy) \in P_x$. So by (P2) we can determine $x^{-1}x^2y = xy$.

Because this was the final subcase, we have shown that we can always determine the product xy from the CICT when $|x^G| = \frac{|G|}{3}$.

Case 4: G is finite and $|x^G| = \frac{|G|}{4}$. In this, our final case, by (Q2) we know that

$|Q_x| \geq |G| - 2\frac{|G|}{4} = \frac{|G|}{2}$. Also, $|C_G(x)| = 4$, so we know that x has order 2 or order 4. We work by cases, but in all cases we will work to show that $Q_x \setminus (xy)^G \neq \emptyset$. As we found earlier, when we have $g \in Q_x \setminus (xy)^G \neq \emptyset$, then we know that $(gx)y$ can be calculated by (Q3) and also that $g \not\sim xy$, so $g(xy) \in P_g$. It then follows by (P2) that $g^{-1}(gx)y = xy$ can be determined.

First, we consider the case $x \not\sim x^{-1}$. We know we are in this case when $1 \in P_x$. When this occurs, we are able to determine x^{-1}, but more importantly for what we are doing, when $x \not\sim x^{-1}$, we know that G has two classes of size $\frac{|G|}{4}$, and so by size considerations cannot have a class of size $\frac{|G|}{2}$. In particular, $|(xy)^G| < \frac{|G|}{2}$. Since $|Q_x| \geq \frac{|G|}{2}$, we must have $Q_x \setminus (xy)^G \neq \emptyset$.

Now suppose $x = x^{-1}$. When x is an involution, then $(yx)x = y$ so $(yx)x \sim x$. Thus, by definition we have $yx \notin Q_x$. But $yx \in (xy)^G$, so we have $Q_x \setminus (xy)^G = Q_x \setminus \big((xy)^G \setminus \{yx\}\big)$. Since Q_x contains at least half the elements of G, and $(xy)^G \setminus \{yx\}$ contains at most $\frac{|G|}{2} - 1$ elements, by size consideration we must have $Q_x \setminus (xy)^G \neq \emptyset$.

The final subcase of the $|x^G| = \frac{|G|}{4}$ is the case $x \sim x^{-1}$, $x \neq x^{-1}$. As mentioned earlier, this implies that $o(x) = 4$. We know that $C_G(x^2)$ contains $C_G(x)$, so $|C_G(x^2)| \geq 4$ and $|(x^2)^G| \leq \frac{|G|}{4}$. This ensures that $Q_x \setminus (x^2)^G \neq \emptyset$, so that x^2 can be determined from the CICT, or equivalently when $x = y$ we can determine xy.

Suppose $x \neq y$. We have $o(x) = 4$, so $o(x^2) = 2$ and $x^2 \not\sim y$. Thus x^2y is determined by the CICT. If, in addition, we have $x^{-1} \not\sim x^2y$, then by $(P2)$ we know $x^{-1}x^2y$ can be determined from the CICT.

If $x^{-1} \sim x^2y$, then $xy = x^{-1}(x^2y) \in x^{-1}x^G = x^Gx^{-1}$. As part of our earlier discussion, we showed that $Q_x = G \setminus (x^G \cup x^Gx^{-1})$. From this it follows that $xy \notin Q_x$ so that $Q_x \setminus (xy)^G = Q_x \setminus \{(xy)^G \setminus \{yx\}\}$. Recall that Q_x contains at least half the elements of G, and $(xy)^G \setminus \{yx\}$ contains at most $\frac{|G|}{2} - 1$ elements, so we have $Q_x \setminus (xy)^G \neq \emptyset$. This is the final subcase when we have $|x^G| = \frac{|G|}{4}$, we can always determine xy from the CICT when $|x^G| = |G|/4$.

In all cases, we showed that we can determine xy from the CICT of G. This shows that there is a unique way to complete the CICT of G to get the Cayley Table of an FC group. $\square$

We note that this theorem is not true for the class of finitely-generated groups, a counter example being given by Osin's construction of an uncountable set of finitely generated infinite groups with only two conjugacy classes [19, Cor 1.3]. In fact Osin constructs infinite finitely-generated groups with exactly n conjugacy classes for every $n \geq 2$.

We conclude this section with the following theorem:

Theorem 2.7. *Let G, H be FC groups and let $\psi : G \to H$ be a CICT map. Then ψ is an isomorphism.*

Proof. Let G, H be FC groups with $\psi : G \to H$ a CICT isomorphism. Then ψ is a bijection, and for each $g \in G$ we write $\tilde{g}$ for the corresponding element of H. We relabel elements of H, so that we think of H as $H = \{\tilde{g}\}_{g \in G}$. Thus elements of G and H differ in labeling only by the inclusion of a tilde.

If we have $x \sim y$, $x, y \in G$, and consider the corresponding elements $\tilde{x}, \tilde{y} \in H$, when we go through the steps of the proof of Theorem 2.6 above, using such elements as g, h, to determine the product xy, we can simultaneously construct $\tilde{x}\tilde{y}$ using the elements $\tilde{g}, \tilde{h}$. We get a unique element $z = xy$ in G and we get a unique element of H, which must be $\tilde{z}$. So we have $\psi(xy) = \psi(z) = \tilde{z} = \tilde{x}\tilde{y} = \psi(x)\psi(y)$ when $x \sim y$. This shows that $\psi(gh) = \psi(g)\psi(h)$ for any $g, h \in G$, so that ψ is an isomorphism. $\square$

Theorem 2.3 follows from Theorems 2.7 and 2.5. So when G and H have the same 4-S-ring then G and H are isomorphic.

Chapter 3. UTCCI maps and NDICT maps

Because the 2-S-ring does not determine the group, the next question is whether the 3-S-ring of an FC group determines the group. This will motivate the work of the next two chapters.

The method is similar to the method of the 4-S-ring case. First we extract information about possible products of certain pairs of elements $x, y \in G$ using information about products of principal elements of the 3-S-ring of G and collect this information into what we will call the 'non-determined' incomplete Cayley Table (NDICT). We then show that two groups with the same NDICT are isomorphic.

In the $\mathfrak{S}_G^{(4)}$ case we looked for $(g, g, x, y) \in K(x, x, x, 1)K(y, y, y, 1)$. We found that there was a unique possibility for g, namely $g = xy$. We did not use the full structure of the 4-S-ring, just products of diagonal 4-classes. We do a similar thing in the 3-S-ring case, considering class products of the classes $K(x, x, x)$, $K(x, x, 1)$, $K(x, 1, 1)$, $x \in G$, which we call the the *uniform* 3*-classes* of G.

Because we are motivated by Brauer's question, we are interested in the following maps:

Definition 3.1. A *uniform* 3*-class centralizer algebra isomorphism* (UTCCI) map is a bijection $\psi : G \to H$ which (1) takes classes of G to classes of H, (2) induces an isomorphism of the centralizer algebras $Z(\mathbb{C}G)$ and $Z(\mathbb{C}H)$, and (3) also has the property that $\psi^{(3)}(AB) = \psi^{(3)}(A)\psi^{(3)}(B)$ for all uniform 3-classes A, B of G^3.

We are going to collect information from looking at products of uniform 3-classes, where the information we extract depends on the relationship of x and y for $x, y \in G$. In addition to paying attention to whether $x \sim y$, we also need to know whether the pair (x, y) satisfies a condition that we call 'non-determined'.

Definition 3.2. A pair (u, v) with $u, v \in H$, $u \not\sim v$, is called an *ND pair* (not determined) if u, v satisfy the following list of conditions:

(i) $uv \neq vu$, which implies that u is not a power of v and v is not a power of u.)

(ii) $v^2 \neq 1$.

(iii) $v^u = v^{-1}$, which implies $u^v = uv^2$.

(iv) $u \sim uv \sim uv^{-1} \sim uv^2 \sim u^{-1}$.

Note that (*i*) and (*iii*) imply (*ii*). As the name implies, these are exactly the pairs x, y of elements, $x \not\sim y$ in G for which we cannot determine exactly the product xy by taking products of uniform 3-classes.

Definition 3.3. Let G be a finite group. We define the *'non-determined' incomplete Cayley Table* or NDICT of G be the table indexed by $g \in G$, which has three types of entries.

(i) If $x \not\sim y$ and (x, y) and (y, x) are not ND pairs, then the entry is xy. Also, if $x \sim y$ and x is a power of y or y is a power of x, we have the entry xy.

(ii) If $x \not\sim y$ and either (x, y) or (y, x) is an ND pair, then the entry of the NDICT is the set $\{xy, yx\}$.

(iii) If $x \sim y$ and x is not a power or y and y is not a power of x, then the entry of the NDICT is $\emptyset$.

Definition 3.4. Let G, H be FC groups. A bijection $\psi : G \to H$ is called an NDICT map if ψ maps classes to classes and has the following characteristics:

(1) for $x \not\sim y$, $x, y \in G$, if neither $(\psi(x), \psi(y))$ nor $(\psi(y), \psi(x))$ is an ND pair, then $\psi(xy) = \psi(x)\psi(y)$.

(2) for $x \not\sim y$, if $(\psi(x), \psi(y))$ or $(\psi(y), \psi(x))$ is an ND pair, then $\psi(xy) \in \{\psi(x)\psi(y), \psi(y)\psi(x)\}$.

(3) $\psi(x^k) = (\psi(x))^k$ for any $x \in G$, $k \in \mathbb{Z}$.

If such a map exists, we say that FC groups G and H *have the same NDICT*.

In this chapter, we prove the following:

Theorem 3.5. *Let G, H be FC groups with $\psi : G \to H$ a UTCCI map. Then ψ is an NDICT map.*

We also show that finite groups G, H with the same NDICT form a Brauer pair. Using this fact and known information about Brauer pairs, we will show that groups G for which

$|G| < 256$ is determined by the NDICT. We also discuss the NDICT of generalized dihedral groups of order $2n$, n odd.

In chapter 4 we show for finite groups of order 256 or greater which are not generalized dihedral of order $2n$, n odd, that an NDICT map is a CICT map. Application of Theorem 2.7 will then show that G and H are in fact isomorphic.

We show in the next section (Lemma 3.6) that a bijection $\psi : G \to H$ which induces a 3-S-ring isomorphism is a UTCCI map. Thus, as a corollary, it will also follow that the 3-S-ring of a finite group determines the group.

3.1 Some Definitions and preliminary results

Proposition 3.6. *If $\psi : G \to H$ induces a 3-S-ring isomorphism for G, H FC groups, then ψ induces an isomorphism of centralizer algebras, takes classes to classes, and has the property that $\psi(AB) = \psi(A)\psi(B)$ for any uniform 3-classes A, B. Or, in other words, ψ is a UTCCI map.*

Proof. Let G, H be FC groups. Let A, B be uniform 3-classes in G. It follows from Theorem 1.32 that ψ induces an isomorphism of centralizer algebras. Because $\hat{\psi}^{(3)}$ is a 3-S-ring isomorphism, and $\overline{A}, \overline{B}$ are elements of $\mathfrak{S}_G^{(3)}$, we know that $\psi^{(3)}(\overline{A}\,\overline{B}) = \psi^{(3)}(\overline{A})\psi^{(3)}(\overline{B})$. The result for set products follows. □

Lemma 3.7. *Let $\psi : G \to H$ be a UTCCI bijection. Then*

(a) $\psi(AB) = \psi(A)\psi(B)$ *for any classes A, B of G.*

(b) $\psi(1_G) = 1_H$.

(c) $K(\psi(x^{-1})) = K(\psi(x)^{-1})$.

(d) Let A, B be uniform 3-classes of G. Then $(x, y, z) \in AB$ if and only if $(\psi(x), \psi(y), \psi(z)) \in \psi(A)\psi(B)$.

Proof. (a) Fix $x, y \in G$ and let $A = K(x)$, $B = K(y)$. Because ψ is a UTCCI map, $\psi : Z(\mathbb{C}G) \to Z(\mathbb{C}H)$ is an isomorphism, and takes classes of G to classes of H.

(b) The class $K(1)$ is the unique class for which $K(x)K(x) = K(x)$.

(c) The class $K(x^{-1})$ is the unique class for which $1 \in K(x)K(x^{-1})$. Because $1 \in K(x)K(x^{-1})$, we have $1 \in \psi(K(x)K(x^{-1})) = \psi(K(x))\psi(K(x^{-1}) = K(\psi(x))K(\psi(x^{-1}))$, so that in fact $K(\psi(x^{-1})) = K(\psi(x)^{-1})$.

(d) When A and B are uniform 3-classes, then $\psi(\overline{AB}) = \psi(\overline{A})\psi(\overline{B})$ so $\psi(A)\psi(B) = \psi(AB)$. And ψ is a bijection, so we know $(x, y, z) \in AB$ if and only if $\psi^{(3)}(x, y, z) \in \psi(AB)$ if and only if $(\psi(x), \psi(y), \psi(z)) \in \psi(A)\psi(B)$. $\square$

Throughout the remainder of this section, G, H are FC groups and $\psi : G \to H$ is a UTCCI map. In order to show that $\psi(g^{-1}) = \psi(g)^{-1}$ for any $g \in G$, we first need a technical lemma. We define

$$I_x = \{g \in G | (1, x, g) \in K(x, x, 1)K(x^{-1}, x^{-1}, 1)\}.$$

Then we have the following result about I_x:

Lemma 3.8. *Let $x \in G \setminus \{1\}$, for G an FC group. Then*

(a) Both $|I_x| = 2$ and $I_x = \{x, x^{-1}\}$ hold if and only if both $x \neq x^{-1}$ and $x \in x^G(x^{-1})^G$ hold;

(b) $I_x = \{x^{-1}\}$ otherwise.

Proof. Let G be an FC group and $x \in G \setminus \{1\}$.

Let $A = K(x, x, 1)$, $B = K(x^{-1}, x^{-1}, 1)$. From the definition of 3-classes we have $A = \{(x, x, 1)^s, (x, 1, x)^s, (1, x, x)^s\}_{s \in G}$, $B = \{(x^{-1}, x^{-1}, 1)^t, (x^{-1}, 1, x^{-1})^t, (1, x^{-1}, x^{-1})^t\}_{t \in G}$, so that $(x, x, 1)(x^{-1}, 1, x^{-1}) = (1, x, x^{-1}) \in AB$. If $(1, x, g) \in AB$, then $(1, x, g) = \alpha\beta$ for some $\alpha \in A$, $\beta \in B$. So $(1, x, g)$ could occur as one of nine possible types of products.

Case 1: $\alpha = (x, x, 1)^s, \beta = (x^{-1}, x^{-1}, 1)^t, \alpha\beta = (x^s(x^{-1})^t, x^s(x^{-1})^t, 1) = (1, x, g)$. Here we have $x^s(x^{-1})^t = 1, x^s(x^{-1})^t = x, 1 = g$, so we get $x = x^s(x^{-1})^t = 1$, a contradiction because by assumption $x \neq 1$.

Case 2: $\alpha = (x, x, 1)^s, \beta = (x^{-1}, 1, x^{-1})^t, \alpha\beta = (x^s(x^{-1})^t, x^s, (x^{-1})^t) = (1, x, g)$. Here we have $x^s(x^{-1})^t = 1, x^s = x, (x^{-1})^t = g$, so that $x^s = x$ and $x(x^{-1})^t = 1$. Solving we get $(x^{-1})^t = x^{-1}$ and so $g = (x^{-1})^t = x^{-1}$.

Case 3: $\alpha = (x, x, 1)^s, \beta = (1, x^{-1}, x^{-1})^t, \alpha\beta = (x^s, x^s(x^{-1})^t, (x^{-1})^t) = (1, x, g)$. Here we have $x^s = 1$, giving $x = 1$, a contradiction.

Case 4: $\alpha = (x, 1, x)^s, \beta = (x^{-1}, x^{-1}, 1)^t, \alpha\beta = (x^s(x^{-1})^t, (x^{-1})^t, x^s) = (1, x, g)$. Here we have $x^s(x^{-1})^t = 1, (x^{-1})^t = x, x^s = g$, and so we get $x^s = x^t$ from the first equation, $x^t = x^{-1}$ from the second, so that $g = x^s = x^t = x^{-1}$.

Case 5: $\alpha = (x, 1, x)^s, \beta = (x^{-1}, 1, x^{-1})^t, \alpha\beta = (x^s(x^{-1})^t, 1, x^s(x^{-1})^t) = (1, x, g)$. Again we have $1 = x$, which gives a contradiction, because by assumption $x \neq 1$.

Case 6: $\alpha = (x, 1, x)^s, \beta = (1, x^{-1}, x^{-1})^t, \alpha\beta = (x^s, (x^{-1})^t, x^s(x^{-1})^t) = (1, x, g)$. Here we have $x^s = 1$, a contradiction, because $x \neq 1$.

Case 7: $\alpha = (1, x, x)^s, \beta = (x^{-1}, x^{-1}, 1)^t, \alpha\beta = ((x^{-1})^t, x^s(x^{-1})^t, x^s) = (1, x, g)$. Here we have $(x^{-1})^t = 1$, again a contradiction.

Case 8: $\alpha = (1, x, x)^s, \beta = (x^{-1}, 1, x^{-1})^t, \alpha\beta = ((x^{-1})^t, x^s, x^s(x^{-1})^t) = (1, x, g)$. Again we have $(x^{-1})^t = 1$, and so we get $x = 1$, a contradiction.

Case 9: $\alpha = (1, x, x)^s, \beta = (1, x^{-1}, x^{-1})^t, \alpha\beta = (1, x^s(x^{-1})^t, x^s(x^{-1})^t) = (1, x, g)$. Here we have $1 = 1, x^s(x^{-1})^t = x, x^s(x^{-1})^t = g$, and so we get $g = x^s(x^{-1})^t = x$.

This shows that it is only possible to get x and x^{-1} in I_x. We showed earlier that x^{-1} is always an element of I_x. Case 9 is the only case where we pick up x as a possible element of I_x for $x \neq x^{-1}$, and this occurs only when $x = x^s(x^{-1})^t$. So $|I_x| = 2$ only when x is not an involution or the identity and $x \in x^G(x^{-1})^G$. □

Just as an aside, when $G = S_3$ we have $(123) = (132)(132) \in (123)^G((123)^{-1})^G$, so it is possible to have $|I_x| = 2$.

Now we want to apply this result to FC groups G, H for which there is a UTCCI bijection $\psi : G \to H$. By Lemma 3.7(d), we know that that

$$(1, \psi(x), \psi(x^{-1})) \in K(\psi(x), \psi(x), 1)K(\psi(x)^{-1}, \psi(x)^{-1}, 1).$$

But $(1, \psi(x), \psi(x^{-1}))$ is of type $(1, \psi(x), h) \in K(\psi(x), \psi(x), 1)K(\psi(x)^{-1}, \psi(x)^{-1}, 1)$, and so is a term of type $(1, \psi(x), h)$ in $K(\psi(x), \psi(x), 1)K(\psi(x)^{-1}, \psi(x)^{-1}, 1)$. So by Lemma 3.8, $\psi(x^{-1}) \in I_{\psi(x)}$.

Also, $(1, \psi(x), \psi(x)^{-1}) \in K(\psi(x), \psi(x), \psi(x)K(\psi(x)^{-1}, \psi(x)^{-1}, 1)$ implies

$$(1, x, \psi^{-1}(\psi(x)^{-1})) \in K(x, x, 1)K(x^{-1}, x^{-1}, 1)$$

by Lemma 3.7, so that $\psi^{-1}(\psi(x)^{-1}) \in I_x$.

So, if $|I_x| = 1$ then $x^{-1} = \psi^{-1}(\psi(x)^{-1})$, so that $\psi(x^{-1}) = \psi(x)^{-1}$. Similarly, if $|I_{\psi(x)}| = 1$, then $\psi(x^{-1}) = \psi(x)^{-1}$. If $|I_x| = 2$, then by way of contradiction assume $\psi^{-1}(\psi(x)^{-1}) = x$. Then we have $\psi(x) = \psi(x)^{-1}$, so that $|I_{\psi(x)}| = 1$, a contradiction, because $x \neq x^{-1}$.

We have shown the following:

Theorem 3.9. *Let G, H be FC groups and $\psi : G \to H$ a UTCCI map. Then $\psi(x^{-1}) = \psi(x)^{-1}$.* □

We now attempt to mimic what we did for the 4-S-ring, where we considered $(g, g, x, y) \in K(x, x, x, 1)K(y, y, y, 1)$. We found that there was a unique possibility for g, namely $g = xy$. In the 3-S-ring case, we consider, for $x, y \in G$, the set

$$A_{x,y} = \{g \in G | (g, x, y) \in K(x, x, 1)K(y, y, 1)\}.$$

We will see that we always get $xy \in A_{x,y}$, but $A_{x,y}$ may also include other elements of G. Even so, it turns out to be a very useful set.

The identity element 1 of a group is determined by its 3-S-ring, Lemma 3.7, and for any

$x \in G$, the product $1x = x$ is known, so in the statement of the following lemma nothing is lost by assuming that that $x \neq 1, y \neq 1$.

Lemma 3.10. *Let G be an FC group and let $x, y \in G \setminus \{1\}$. Then*

(a) If $x \not\sim y$, then $xy \in A_{x,y} \subseteq \{xy, x^{-1}y, xy^{-1}\}$.

(b) If $x \sim y$, then $\{xy, yx\} \subseteq A_{x,y} \subseteq \{xy, yx, xy^{-1}, x^{-1}y, y^{-1}x, yx^{-1}\}$.

Proof. Let G be a finite group. Let $x, y \in G \setminus \{1\}$. Let $A = K(x, x, 1)$, $B = K(y, y, 1)$. Then by the definition of 3-class we know that $A = \{(x, x, 1)^s, (x, 1, x)^s, (1, x, x)^s\}_{s \in G}$ and $B = \{(y, y, 1)^t, (y, 1, y)^t, (1, y, y)^t\}_{t \in G}$. We have $(x, x, 1)(y, 1, y) = (xy, x, y)$, so $(xy, x, y) \in AB$ and $xy \in A_{x,y}$. If $x \sim y$, then $(y, 1, y) \in K(x, x, 1)$ and $(x, x, 1) \in K(y, y, 1)$, so $(y, 1, y)(x, x, 1) = (yx, x, y) \in AB$. So $yx \in A_{x,y}$ when $x \sim y$.

Suppose $(g, x, y) \in AB$. Then $(g, x, y) = \alpha\beta$ where $\alpha \in A$, $\beta \in B$. So (g, x, y) can occur as one of nine possible types of products, corresponding to the types of elements in A and B. We check each possibility to determine when it is possible to get $(g, x, y) \in AB$.

Case 1: $\alpha = (x, x, 1)^s, \beta = (y, y, 1)^t, \alpha\beta = (x^s y^t, x^s y^t, 1) = (g, x, y)$.

Here we have $x^s y^t = g, x^s y^t = x, 1 = y$, and the last equation gives a contradiction, because by assumption $y \neq 1$. So we don't get any elements of $A_{x,y}$ from this case.

Case 2: $\alpha = (x, x, 1)^s, \beta = (y, 1, y)^t, \alpha\beta = (x^s y^t, x^s, y^t) = (g, x, y)$.

Here we have $x^s y^t = g, x^s = x, y^t = y$, and so that $g = x^s y^t = xy$. This is what we already showed, namely that $xy \in A_{x,y}$.

Case 3: $\alpha = (x, x, 1)^s, \beta = (1, y, y)^t, \alpha\beta = (x^s, x^s y^t, y^t) = (g, x, y)$.

Here we have $x^s = g, x^s y^t = x, y^t = y$, and so we get $x = x^s y^t = x^s y$, and solving for x^s we get $x^s = xy^{-1}$, so $g = x^s = xy^{-1}$. So we can get $xy^{-1} \in A_{x,y}$.

Case 4: $\alpha = (x, 1, x)^s, \beta = (y, y, 1)^t, \alpha\beta = (x^s y^t, y^t, x^s) = (g, x, y)$.

Here we have $x^s y^t = g, y^t = x, x^s = y$, and so this case can only occur when $x \sim y$. We get $g = x^s y^t = yx$. So we can get $yx \in A_{x,y}$ if $x \sim y$.

Case 5: $\alpha = (x, 1, x)^s, \beta = (y, 1, y)^t, \alpha\beta = (x^s y^t, 1, x^s y^t) = (g, x, y)$.

Here we have $x^s y^t = g, 1 = x, x^s y^t = y$, and the second equation gives a contradiction, because by assumption $x \neq 1$.

Case 6: $\alpha = (x, 1, x)^s, \beta = (1, y, y)^t, \alpha\beta = (x^s, y^t, x^s y^t) = (g, x, y)$.

Here we have $x^s = g, y^t = x, x^s y^t = y$, and so this case occurs only when $x \sim y$. We get $y = x^s y^t = x^s x$. Solving for x^s we get $x^s = yx^{-1}$. Since $g = x^s$, it is possible to get $yx^{-1} \in A_{x,y}$.

Case 7: $\alpha = (1, x, x)^s, \beta = (y, y, 1)^t, \alpha\beta = (y^t, x^s y^t, x^s) = (g, x, y)$.

Here we have $y^t = g, x^s y^t = x, x^s = y$, and so this case occurs only when $x \sim y$. We get $x = x^s y^t = yy^t$. Solving for y^t we get $y^t = y^{-1}x$. So it is possible to get $y^{-1}x \in A_{x,y}$ if $x \sim y$.

Case 8: $\alpha = (1, x, x)^s, \beta = (y, 1, y)^t, \alpha\beta = (y^t, x^s, x^s y^t) = (g, x, y)$.

Here we have $y^t = g, x^s = x, x^s y^t = y$, and so we get $y = x^s y^t = xy^t$ and solving for y^t we get $y^t = x^{-1}y$, so we can get $g = y^t = x^{-1}y$ in $A_{x,y}$.

Case 9: $\alpha = (1, x, x)^s, \beta = (1, y, y)^t, \alpha\beta = (1, x^s y^t, x^s y^t) = (g, x, y)$.

Here we have $1 = g, x^s y^t = x, x^s y^t = y$, and so we get $g = 1$, and $y = x$ and $x \in (x^G)^2$. Note that in this case we have $g = xy^{-1}$.

So, if $x \not\sim y$, then $A_{x,y} \subset \{xy, xy^{-1}, x^{-1}y\}$ and $xy \in A_{x,y}$.

If $x \sim y$, then $A_{x,y} \subseteq \{xy, xy^{-1}, x^{-1}y, yx, yx^{-1}, y^{-1}x\}$ and $\{xy, yx\} \subseteq A_{x,y}$. □

We will use $A_{x,y}$ to show that a UTCCI map $\psi : G \to H$ maps powers of $g \in G$ to powers of $\psi(g)$. In the next section we will use $A_{x,y}$ to show that when groups of odd order have the same 3-S-ring, then the groups are isomorphic.

Theorem 3.11. *Let G, H be FC groups. If $\psi : G \to H$ is a UTCCI map, then $\psi(g^k) = (\psi(g))^k$ for all integers k.*

To simplify notation, we let $\psi(x) = \tilde{x}$, $\psi(y) = \tilde{y}$, etc. We will write $\tilde{x}^k$ for $\psi(x)^k$ and note that $\tilde{x}^{-1}$ is unambiguous because of the result of Theorem 3.9.

Proof. Let G,H be FC groups, with $\psi : G \to H$ a UTCCI map. Fix $x \in G$. We know $\psi(1) = 1$, and it follows from Theorem 3.9 that when $x^2 = 1$, then $\psi(x)^2 = 1$. So we assume

$x^2 \neq 1$. We have $(x,x,1)(x,1,x) = (x^2,x,x) \in K(x,x,1)K(x,x,1)$, and so we also have $(\psi(x^2),\tilde{x},\tilde{x}) \in K(\tilde{x},\tilde{x},1)K(\tilde{x},\tilde{x},1)$. But from the definition of $A_{\tilde{x},\tilde{x}}$, this means we have

$$\psi(x^2) \in A_{\tilde{x},\tilde{x}} \subseteq \{\tilde{x}\tilde{x}, \tilde{x}\tilde{x}^{-1}, \tilde{x}^{-1}\tilde{x}\} = \{1, \tilde{x}^2\},$$

and $\psi(x^2) \neq 1$, because ψ is a bijection, so we must have $\psi(x^2) = \tilde{x}^2$. Thus, when $o(x) \leq 3$, $\psi(x^k) = \tilde{x}^k$ for any k.

Without loss of generality, we assume $o(x) \geq 4$.

The proof is by (strong) induction on $k, 1 \leq k \leq o(x)$. Inductively, we assume $\psi(x^j) = \tilde{x}^j$, $1 \leq j \leq k$ for some k. We have shown that $\psi(g^{-1}) = (\psi(g))^{-1}$ for all $g \in G$, so it follows that

$$\psi(x^{-j}) = \psi((x^j)^{-1}) = (\psi(x^j))^{-1} = ((\tilde{x})^j)^{-1} = \tilde{x}^{-j}$$

for $1 \leq j \leq k$.

Because $(x^{k+1}, x^k, x) \in K(x,x,1)K(x^k,x^k,1)$ and, by induction $\psi(x^k) = \tilde{x}^k$, it follows from Lemma 3.7 that we have

$$\psi^{(3)}(x^{k+1}, x, x^k) = (\psi(x^{k+1}), \tilde{x}, \psi(x^k)) \in K(\tilde{x},\tilde{x},1)K(\tilde{x}^k,\tilde{x}^k,1).$$

But $(\psi(x^{k+1}), \tilde{x}, \psi(x^k))$ is of type $(g, \tilde{x}, \tilde{x}^k)$. From the definition of $A_{\tilde{x},\tilde{x}^k}$ and Lemma 3.10, we know $(g,\tilde{x},\tilde{x}^k) \in K(\tilde{x},\tilde{x},1)K(\tilde{x}^k,\tilde{x}^k,1)$ implies $g \in A_{\tilde{x},\tilde{x}^k}$. So that

$$\psi(x^{k+1}) \in A_{\tilde{x},\tilde{x}^k} \subseteq \{\tilde{x}\tilde{x}^k, \tilde{x}^k\tilde{x}, \tilde{x}^{-1}\tilde{x}^k, \tilde{x}\tilde{x}^{-k}, \tilde{x}^k\tilde{x}^{-1}, \tilde{x}^{-k}\tilde{x}\} = \{\tilde{x}^{k+1}, \tilde{x}^{k-1}, \tilde{x}^{1-k}\}.$$

By induction, we have $\psi(x^{k-1}) = \tilde{x}^{k-1}$ and $\psi(x^{1-k}) = \tilde{x}^{1-k}$. Because ψ is a bijection, we must have $\psi(x^{k+1}) = \tilde{x}^{k+1}$. $\square$

Finally, as a corollary to this result, recall that a pair (G,H) of non-isomorphic groups form a *Brauer pair* if there is a bijection $\phi : G \to H$ that maps classes to classes, that determines an isomorphism of centralizer algebras, and which also respects the power maps:

the H-class of $\phi(g^k)$ is the same as the class of $\phi(g)^k$ for all $g \in G, k \in \mathbb{Z}$.

From the definition of Brauer pair and Theorem 3.11 we get the following result:

Corollary 3.12. *Let G, H be FC groups. If there exists a UTCCI map $\psi : G \to H$, then G, H are a Brauer pair.* $\square$

3.2 Image of xy under ψ when $x \not\sim y$, $x \not\sim x^{-1}$, and $y \not\sim y^{-1}$

In this section, G, H are still FC groups and $\psi : G \to H$ is a UTCCI map. Also, $x, y \in G$ and $x \not\sim y$.

In the last section, we found the set $A_{x,y}$ by inspecting the possibilities for entries in products of certain uniform classes. We did a similar thing to determine I_x. Then we looked at $A_{\tilde{x},\tilde{y}}$ and $I_{\tilde{x}}$ to determine possible images of x^{-1} and xy under ψ (respectively).

We continue that method in this section, finding sets in H which contain $\tilde{x}\tilde{y}$ and intersecting these sets to further restrict the possible image of xy. For $x \not\sim y$, $x \not\sim x^{-1}$, and $y \not\sim y^{-1}$ we will show that $\widetilde{xy} = \tilde{x}\tilde{y}$.

For $x, y \in G$, define

$$R_{x,y} = \{g \in G | (g, x, x) \in K(x, x, x)K(y, 1, 1)\}, \text{ and}$$

$$L_{x,y} = \{h \in G | (h, y, y) \in K(x, 1, 1)K(y, y, y)\}.$$

Lemma 3.13. *Let G be a finite group. Let $x, y \in G \setminus \{1\}$ with $x \not\sim y$. Then $R_{x,y} = \{xy^t\}_{t\in G}$ and $L_{x,y} = \{x^s y\}_{s\in G}$.*

Proof. Let G be a finite group. Let x, y be non-conjugate, non-identity elements of G. Let $A = K(x, x, x)$ and $B = K(y, 1, 1)$. Then by the definition of 3-classes, we know $A = \{(x, x, x)^s\}_{s\in G}$ and $B = \{(1, 1, y)^t, (1, y, 1)^t, (y, 1, 1)^t\}_{t\in G}$. Since $(x, x, x) \in A$ and $(y^t, 1, 1) \in B$ for any $t \in G$, we have $(x, x, x)(y^t, 1, 1) = (xy^t, x, x) \in AB$ for any $t \in G$. These elements (xy^t, x, x) are of type $(g, x, x) \in AB$, $g \in G$. This shows that $\{xy^t\}_{t\in G} \subseteq R_{x,y}$. To show equality, we show that if $(g, x, x) \in AB$, then $g \in \{xy^t\}_{t \in G}$. We know that an element $(g, x, x) \in AB$ can occur as one of three possible types of products.

Case 1: $(x, x, x)^s(1, 1, y)^t = (x^s, x^s, x^s y^t) = (g, x, x)$.

Here we get $x = x^s y^t = x^s$ and solving for y gives $y = 1$, a contradiction.

Case 2: $(x, x, x)^s(1, y, 1)^t = (x^s, x^s y^t, x^s) = (g, x, x)$.

Again we get $x = x^s = x^s y^t$, and solving for y gives $y = 1$, a contradiction.

Case 3: $(x, x, x)^s(y, 1, 1)^t = (x^s y^t, x^s, x^s) = (g, x, x)$.

Here we have $x^s = x$, $g = x^s y^t = xy^t$.

So, in all cases, we have $g \in \{xy^t\}_{t \in G}$, so that, in fact $\{xy^t\}_{t \in G} = \{g \in G | (x, x, g) \in K(x, x, x)K(y, 1, 1)\}$.

To see that $L_{x,y} = \{x^s y\}_{s \in G}$, let $C = K(x, 1, 1)$ and $D = K(y, y, y)$. By the definition of 3-class we have $C = \{(x, 1, 1)^s, (1, x, 1)^s, (1, 1, x)^s\}_{s \in G}$, $D = \{(y, y, y)^t\}_{t \in G}$. Because $(x^s, 1, 1) \in C$ for any $s \in G$ and $(y, y, y) \in D$, we have $(x^s y, y, y) \in CD$ for any $s \in G$, so $\{x^s y\}_{s \in G} \subseteq \{h \in G | (h, y, y) \in K(x, 1, 1)K(y, y, y)\}$. To show equality, we show that if $(h, y, y) \in CD$, then $h \in \{x^s y\}_{s \in G}$. We know that an element $(h, y, y) \in CD$ can occur as one of three types of products.

Case 1: $(x, 1, 1)^s(y, y, y)^t = (x^s y^t, y^t, y^t) = (h, y, y)$.

Here we have $y^t = y$, $h = x^s y^t = x^s y$.

Case 2: $(1, x, 1)^s(y, y, y)^t = (y^t, x^s y^t, y^t) = (h, y, y)$.

Here we get $y = x^s y^t = y^t$, and solving for x gives $x = 1$, a contradiction.

Case 3: $(1, 1, x)^s(y, y, y)^t = (y^t, y^t, x^s y^t) = (h, y, y)$.

Here we get $y = y^t = x^s y^t$ and solving for x gives $x = 1$, a contradiction.

So, in all cases, we have $h \in \{x^s y\}_{s \in G}$. This concludes the proof that $\{x^s y\}_{s \in G} = \{h \in G | (h, y, y) \in K(x, 1, 1)K(y, y, y)\}$. $\square$

Suppose $x \not\sim y$, with $x, y \in G$. Then $(xy, x, x) \in K(x, x, x)K(y, 1, 1)$, and because ψ is a UTCCI map it follows from Lemma 3.7 that $(\widetilde{xy}, \tilde{x}, \tilde{x}) \in K(\tilde{x}, \tilde{x}, \tilde{x})K(\tilde{y}, 1, 1)$. So by the definition of $R_{\tilde{x},\tilde{y}}$ we have $\widetilde{xy} \in R_{\tilde{x},\tilde{y}}$. Similarly $(xy, y, y) \in K(x, 1, 1)K(y, y, y)$ implies

$(\widetilde{xy}, \tilde{y}, \tilde{y}) \in K(\tilde{x}, 1, 1)K(\tilde{y}, \tilde{y}, \tilde{y})$ so that $\widetilde{xy} \in L_{\tilde{x},\tilde{y}}$. Also, $(xy, x, y) \in K(x, x, 1)K(y, 1, y)$ implies $(\widetilde{xy}, \tilde{x}, \tilde{y}) \in K(\tilde{x}, \tilde{x}, 1)K(\tilde{y}, 1, \tilde{y})$ so that $\widetilde{xy} \in A_{\tilde{x},\tilde{y}}$. So we must have $\widetilde{xy} \in A_{\tilde{x},\tilde{y}} \cap R_{\tilde{x},\tilde{y}} \cap L_{\tilde{x},\tilde{y}}$.

Our next goal is to understand $A_{\tilde{x},\tilde{y}} \cap R_{\tilde{x},\tilde{y}} \cap L_{\tilde{x},\tilde{y}}$ when $x \not\sim x^{-1}$, $y \not\sim y^{-1}$. Because ψ maps classes to classes and inverses to inverses, this implies $\tilde{x} \not\sim \tilde{y}$, $\tilde{x} \not\sim \tilde{x}^{-1}$, and $\tilde{y} \not\sim \tilde{y}^{-1}$.

First, because $\tilde{x}\tilde{y} \in A_{\tilde{x},\tilde{y}}$, $\tilde{x}\tilde{y} \in R_{\tilde{x},\tilde{y}}$ and $\tilde{x}\tilde{y} \in L_{\tilde{x},\tilde{y}}$, we have $\tilde{x}\tilde{y} \in A_{\tilde{x},\tilde{y}} \cap R_{\tilde{x},\tilde{y}} \cap L_{\tilde{x},\tilde{y}}$.

Also, we know that $A_{\tilde{x},\tilde{y}} \cap R_{\tilde{x},\tilde{y}} \cap L_{\tilde{x},\tilde{y}} \subseteq A_{x,y} \subseteq \{xy, xy^{-1}, x^{-1}y\}$. Because $y \not\sim y^{-1}$, we have $\tilde{y}^{-1} \neq (\tilde{y})^t$ for any $t \in H$, so that $\tilde{x}\tilde{y}^{-1} \neq \tilde{x}\tilde{y}^t$ for any $t \in H$. So $\tilde{x}\tilde{y}^{-1} \notin R_{\tilde{x},\tilde{y}}$. Similarly, because $\tilde{x} \not\sim \tilde{x}^{-1}$, then $\tilde{x}^{-1}\tilde{y} \neq (\tilde{x})^s\tilde{y}$ for any $s \in H$, so that $\tilde{x}^{-1}\tilde{y} \notin L_{\tilde{x},\tilde{y}}$. Thus we have $\tilde{x}\tilde{y}^{-1}, \tilde{x}^{-1}\tilde{y} \notin A_{\tilde{x},\tilde{y}} \cap R_{\tilde{x},\tilde{y}} \cap L_{\tilde{x},\tilde{y}}$, so that in fact $A_{\tilde{x},\tilde{y}} \cap R_{\tilde{x},\tilde{y}} \cap L_{\tilde{x},\tilde{y}} = \{\tilde{x}\tilde{y}\}$ when $x \not\sim y$, $x \not\sim x^{-1}$, $y \not\sim y^{-1}$. Because $\widetilde{xy} \in A_{\tilde{y},\tilde{y}}$, it follows that $\tilde{x}\tilde{y} = \widetilde{xy}$. We have shown the following:

Lemma 3.14. *Let G, H be FC groups and $\psi : G \to H$ a UTCCI bijection. Let $x, y \in G$. If $x \not\sim y$, $x \not\sim x^{-1}$, $y \not\sim y^{-1}$, then $\psi(xy) = \psi(x)\psi(y)$.* □

If particular, if G is a group of odd order, and $\psi : G \to H$ is a UTCCI map, then no element of $G \setminus \{1\}$ is conjugate to its inverse, so it follows from Lemma 3.14 that $\psi(gh) = \psi(g)\psi(h)$ for any $g, h \in G$ with $g \not\sim h$. So, ψ is a CICT isomorphism, and hence G and H are isomorphic by Theorem 2.7.

Corollary 3.15. *Let G, H be finite groups of odd order and $\psi : G \to H$ a UTCCI map. Then ψ is an isomorphism. In particular, if G, H have the same 3-S-ring, then G and H are isomorphic.* □

3.3 Determining $\psi(xy)$ when $x \not\sim y$ and either $x \sim x^{-1}$ or $y \sim y^{-1}$

With Lemma 3.14 in hand we are ready to focus our efforts on pairs of elements $x, y \in G$ for which $x \not\sim y$ and either $x \sim x^{-1}$ or $y \sim y^{-1}$. We obtained Lemma 3.14 by considering sets in H containing $\widetilde{xy}$ and then intersecting those sets until we restricted to a single element $\tilde{x}\tilde{y}$ that was the only possible image for $\widetilde{xy}$. If $x \sim x^{-1}$ or $y \sim y^{-1}$ we may not always be so lucky. But we can do better than what we've got so far.

In this section, G, H are still FC groups and $\psi : G \to H$ a UTCCI map. Also, $x, y \in G$ and $x \not\sim y$.

We are going to need a couple more sets. For $x, y \in G$ with $x \not\sim y$, we let

$$M_{x,y} = \{(g \in G | (x, g, y^{-1}) \in K(g, g, 1)K(y^{-1}, y^{-1}, 1)\}, \text{ and}$$

$$N_{x,y} = \{h \in G | (x^{-1}, g, y) \in K(x^{-1}, x^{-1}, 1)K(g, g, 1)\}.$$

Then $M_{x,y}$ and $N_{x,y}$ have the following characteristics.

Lemma 3.16. *Let G be a finite group. Let $x, y \in G \setminus \{1\}$ with $x \not\sim y$, $x \not\sim y^{-1}$. Then $xy \in M_{x,y} \subseteq \{xy, yx, xy^{-1}\}$, $xy \in N_{x,y} \subseteq \{xy, yx, x^{-1}y\}$ and*

(a) If $yx \neq xy$ and $yx \neq x^{-1}y$, then $yx \in M_{x,y}$ if and only if $yx \sim y^{-1}$. If $xy^{-1} \neq xy$ and $xy^{-1} \neq yx$, then $xy^{-1} \in M_{x,y}$ if and only if $xy^{-1} \sim x$.

(b) If $yx \neq xy$ and $yx \neq x^{-1}y$, then $yx \in N_{x,y}$ if and only if $yx \sim x^{-1}$. If $x^{-1}y \neq xy$ and $x^{-1}y \neq yx$, then $x^{-1}y \in N_{x,y}$ if and only if $x^{-1}y \sim y$.

Before beginning the proof, we mention that if, for example $xy = yx$, then we have $yx \in M_{x,y}$, regardless of whether or not $yx \sim y^{-1}$. Also, the restriction that $x \not\sim y^{-1}$ holds is not much of a restriction because we are now primarily interested in the case where $x \not\sim y$ and either $x \sim x^{-1}$ or $y \sim y^{-1}$. And when we have either of these cases, it is straightforward to verify that $x \not\sim y^{-1}$ and, equivalently, that $x^{-1} \not\sim y$.

Proof. Let G be a finite group. Fix $x, y \in G \setminus \{1\}$ with $x \not\sim y$, $x \not\sim y^{-1}$.

We have $(x, xy, y^{-1}) = (xy, xy, 1)(y^{-1}, 1, y^{-1}) \in K(xy, xy, 1)K(y^{-1}, y^{-1}, 1)$ so $xy \in M_{x,y}$.

If $x \sim xy^{-1}$, then $(x, xy^{-1}, y^{-1}) = (x, x, 1)(1, y^{-1}, y^{-1}) \in K(xy^{-1}, xy^{-1}, 1)K(y^{-1}, y^{-1}, 1)$, so that $xy^{-1} \in M_{x,y}$.

If $y^{-1} \sim yx$, then $(x, yx, y^{-1}) = (y^{-1}, 1, y^{-1})(yx, yx, 1) \in K(yx, yx, 1)K(y^{-1}, y^{-1}, 1)$, so that $yx \in M_{x,y}$.

Also, we have $(x^{-1}, xy, y) = (x^{-1}, 1, x^{-1})(1, xy, xy) \in K(x^{-1}, x^{-1}, 1)K(xy, xy, 1)$, so $xy \in N_{x,y}$.

If $y \sim x^{-1}y$, then $(x^{-1}, x^{-1}y, y) = (x^{-1}, x^{-1}, 1)(1, y, y) \in K(x^{-1}, x^{-1}, 1)K(x^{-1}y, x^{-1}y, 1)$, so $x^{-1}y \in N_{x,y}$.

If $x^{-1} \sim yx$ then $(x^{-1}, yx, y) = (1, yx, yx)(x^{-1}, 1, x^{-1}) \in K(x^{-1}, x^{-1}, 1)K(yx, yx, 1)$, so $yx \in N_{x,y}$

To see that $M_{x,y} \subseteq \{xy, yx, xy^{-1}\}$, fix $g \in G$. We let $A = K(y^{-1}, y^{-1}, 1)$, and $C = K(1, g, g)$. By the definition, we have $A = \{(y^{-1}, y^{-1}, 1)^t, (y^{-1}, 1, y^{-1})^t, (1, y^{-1}, y^{-1})^t\}_{t \in G}$ and $C = \{(g, g, 1)^s, (g, 1, g)^s, (1, g, g)^s\}_{s \in G}$. If $g \in M_{x,y}$, then $(x, g, y^{-1}) \in CA$ and so (x, g, y^{-1}) occurs as one of nine possible types of products, which we check:

Case 1: $(g^s, g^s, 1)((y^{-1})^t, (y^{-1})^t, 1) = (g^s(y^{-1})^t, g^s(y^{-1})^t, 1) = (x, g, y^{-1})$.
Here we get $y^{-1} = 1$, a contradiction.

Case 2: $(g^s, g^s, 1)((y^{-1})^t, 1, (y^{-1})^t) = (g^s(y^{-1})^t, g^s, (y^{-1})^t) = (x, g, y^{-1})$.
Here we get $(y^{-1})^t = y^{-1}$ and $g^s = g$, substituting in the equations from the first component we get $x = g^s(y^{-1})^t = gy^{-1}$. Solving for g, we get $g = xy$.

Case 3: $(g^s, g^s, 1)(1, (y^{-1})^t, (y^{-1})^t) = (g^s, g^s(y^{-1})^t, (y^{-1})^t) = (x, g, y^{-1})$.
Here we get $(y^{-1})^t = y^{-1}$ and $x = g^s$, substituting into the equations from the second component, we get $g = g^s(y^{-1})^t = xy^{-1}$, $g \sim x$.

Case 4: $(g^s, 1, g^s)((y^{-1})^t, (y^{-1})^t, 1) = (g^s(y^{-1})^t, (y^{-1})^t, g^s) = (x, g, y^{-1})$.
Here we get $g^s = y^{-1}$ and $(y^{-1})^t = g$. Substituting into the equations from the first component, we get $x = g^s(y^{-1})^t = y^{-1}g$. Solving for g, we get $g = yx$, $g \sim y^{-1}$.

Case 5: $(g^s, 1, g^s)((y^{-1})^t, 1, (y^{-1})^t) = (g^s(y^{-1})^t, 1, g^s(y^{-1})^t) = (x, g, y^{-1})$.
Here we get $y^{-1} = g^s(y^{-1})^t = x$, a contradiction. By hypothesis $x \not\sim y^{-1}$.

Case 6: $(g^s, 1, g^s)(1, (y^{-1})^t, (y^{-1})^t) = (g^s, (y^{-1})^t, g^s(y^{-1})^t) = (x, g, y^{-1})$.
Here we get $x = g^s$ and $g = (y^{-1})^t$, which implies $x \sim y^{-1}$, a contradiction.

Case 7: $(1, g^s, g^s)((y^{-1})^t, (y^{-1})^t, 1) = ((y^{-1})^t, g^s(y^{-1})^t, g^s) = (x, g, y^{-1})$.
Here we get $x = (y^{-1})^t$, a contradiction.

Case 8: $(1, g^s, g^s)((y^{-1})^t, 1, (y^{-1})^t) = ((y^{-1})^t, g^s, g^s(y^{-1})^t) = (x, g, y^{-1})$.

Here we get $x = (y^{-1})^t$, a contradiction.

Case 9: $(1, g^s, g^s)(1, (y^{-1})^t, (y^{-1})^t) = (1, g^s(y^{-1})^t, g^s(y^{-1})^t) = (x, g, y^{-1})$.

Here we get $x = 1$, a contradiction.

This shows that $M_{x,y} \subseteq \{xy, yx, xy^{-1}\}$, and also gives the restrictions for when yx, xy^{-1} can occur in $M_{x,y}$.

Similarly, to show that $N_{x,y} \subseteq \{xy, yx, x^{-1}y\}$, we let $B = K(x^{-1}, x^{-1}, 1)$. Then from the definition of 3-class we have $B = \{(x^{-1}, x^{-1}, 1)^t, (x^{-1}, 1, x^{-1})^t, (1, x^{-1}, x^{-1})^t\}_{t \in G}$. Again let $C = K(1, g, g)$. Consider $(x^{-1}, g, y) \in BC$. Out of nine possible products there are only three that fail to give a contradiction:

Case A: $((x^{-1})^t, 1, (x^{-1})^t)(1, g^s, g^s) = ((x^{-1})^t, g^s, (x^{-1})^t g^s) = (x^{-1}, g, y)$.

Here we get $(x^{-1})^t = x^{-1}$, $g^s = g$, and $y = (x^{-1})^t g^s = x^{-1}g$ so $g = xy$.

Case B: $((x^{-1})^t, (x^{-1})^t, 1)(1, g^s, g^s) = ((x^{-1})^t, (x^{-1})^t g^s, g^s) = (x^{-1}, g, y)$.

Here we get $(x^{-1})^t = x^{-1}$, $g^s = y$, and $g = (x^{-1})^t g^s = x^{-1}y$, $g \sim y$.

Case C: $(1, (x^{-1})^t, (x^{-1})^t)(g^s, 1, g^s) = (g^s, (x^{-1})^t, (x^{-1})^t) = (x^{-1}, g, y)$.

Here we get $g^s = x^{-1}$, $(x^{-1})^t = g$, and $y = (x^{-1})^t g^s = gx^{-1}$ so $g = yx$, $g \sim x^{-1}$.

This gives us the inclusion $N_{x,y} \in \{xy, yx, x^{-1}y\}$ and the restrictions for when we can have $yx, x^{-1}y \in N_{x,y}$. $\square$

Now we consider the consequences of this with regards to a UTCCI map $\psi : G \to H$.

Fix $x, y \in G \setminus \{1\}$ with $x \not\sim y$ and with either $x \sim x^{-1}$ or $y \sim y^{-1}$. As mentioned earlier, it follows that $x \not\sim y^{-1}$ and $y \not\sim x^{-1}$. And, because ψ takes classes of G to classes of H, we also have that $\tilde{x} \not\sim \tilde{y}$, and either $\tilde{x} \sim \tilde{x}^{-1}$ or $\tilde{y} \sim \tilde{y}^{-1}$.

We have $(x, xy, y^{-1}) = (xy, xy, 1)(y^{-1}, 1, y^{-1}) \in K(xy, xy, 1)K(y^{-1}, y^{-1}, 1)$ and ψ a UTCCI map, so $(\tilde{x}, \widetilde{xy}, \tilde{y}^{-1}) \in K(\widetilde{xy}, \widetilde{xy}, 1)K(\tilde{y}^{-1}, \tilde{y}^{-1}, 1)$. By definition of $M_{\tilde{x},\tilde{y}}$, we have $\widetilde{xy} \in M_{\tilde{x},\tilde{y}}$.

Similarly, we have $(x^{-1}, xy, y) \in K(1, x^{-1}, x^{-1})K(xy, xy, 1)$ from which it follows that $(\tilde{x}^{-1}, \widetilde{xy}, \tilde{y}) \in K(\tilde{x}^{-1}, \tilde{x}^{-1}, 1)K(\widetilde{xy}, \widetilde{xy}, 1)$. So we also have $\widetilde{xy} \in N_{\tilde{x},\tilde{y}}$. We have shown the following:

Lemma 3.17. *Let G, H be finite groups. If $\psi : G \to H$ is a UTCCI map, then for $x, y \in G$ satisfying $x \not\sim y$, and either $x \not\sim x^{-1}$ or $y \not\sim y^{-1}$, we have $\widetilde{xy} \in M_{\tilde{x},\tilde{y}} \cap N_{\tilde{x},\tilde{y}}$.* □

Thus elements of $M_{\tilde{x},\tilde{y}} \cap N_{\tilde{x},\tilde{y}}$ are the possible images of xy under ψ.

We let $M = M_{\tilde{x},\tilde{y}}$, $N = N_{\tilde{x},\tilde{y}}$. If $|M \cap N| = 1$, then $M \cap N = \{\tilde{x}\tilde{y}\}$, so $\widetilde{xy} = \tilde{x}\tilde{y}$.

We consider what happens when $|M| = 3$. By Lemma 3.16, this can occur only if $\tilde{x}\tilde{y} \neq \tilde{y}\tilde{x} \neq \tilde{x}\tilde{y}^{-1}$, $\tilde{y}\tilde{x} \sim \tilde{y}^{-1}$ and $\tilde{x}\tilde{y}^{-1} \sim \tilde{x}$, and in this case we have $M = \{\tilde{x}\tilde{y}, \tilde{y}\tilde{x}, \tilde{x}\tilde{y}^{-1}\}$. Because $\tilde{y}\tilde{x} \sim \tilde{y}^{-1}$ we know $\tilde{y}\tilde{x} \not\sim \tilde{x}^{-1}$. Also, taking $\tilde{x}\tilde{y}^{-1} \sim \tilde{x}$ and inverting both sides, we get $\tilde{y}\tilde{x}^{-1} \sim \tilde{x}^{-1}$, so that $\tilde{x}^{-1}\tilde{y} = (\tilde{y}\tilde{x}^{-1})^{\tilde{x}} \sim \tilde{x}^{-1}$, and from this it follows that $\tilde{x}\tilde{y}^{-1} \not\sim y$. Because we have $\tilde{x}^{-1}\tilde{y} \sim \tilde{x}^{-1}$ and $\tilde{x}^{-1}\tilde{y} \not\sim \tilde{y}$, it follows from Lemma 3.16 that $N = \{\tilde{x}\tilde{y}\}$. This proves:

Lemma 3.18. *Let $x, y \in G$ a finite group with $x \not\sim y$, $x \not\sim y^{-1}$. If $|M| = 3$, then $|N| = 1$ and $M \cap N = \{\tilde{x}\tilde{y}\}$.* □

As our first consequence of this, we note that it is impossible to have $|M \cap N| = 3$. As a second consequence, we note that if $|M \cap N| = 2$, then $|N| \geq 2$, so that $|M| \neq 3$, so that when $|M \cap N| = 2$, we are forced to have $|M| = 2$.

So all that remains for us to consider is the possibility that $|M \cap N| = 2$. Because this implies $|M| = 2$, there are two possibilities for M: either (A) $M = \{\tilde{x}\tilde{y}, \tilde{y}\tilde{x}\}$ with $\tilde{x}\tilde{y} \neq \tilde{y}\tilde{x}$ and $\tilde{y}\tilde{x} \sim \tilde{y}^{-1}$, or (B) we have $M = \{\tilde{x}\tilde{y}, \tilde{x}\tilde{y}^{-1}\}$ with $\tilde{x}\tilde{y}^{-1} \sim \tilde{x}$ and $\tilde{x}\tilde{y}^{-1} \neq \tilde{x}\tilde{y}$.

Case A: First, suppose $M = \{\tilde{x}\tilde{y}, \tilde{y}\tilde{x}\}$ with $\tilde{y}\tilde{x} \neq \tilde{x}\tilde{y}$ and $\tilde{y}\tilde{x} \sim x^{-1}$, then $\tilde{y}\tilde{x} \not\sim y^{-1}$, as shown above. Because $\tilde{y}\tilde{x} \neq \tilde{x}\tilde{y}$, we can get $\tilde{y}\tilde{x} \in N$ only if $\tilde{y}\tilde{x} = \tilde{x}^{-1}\tilde{y}$.

Case B: Next, suppose $|M \cap N| = 2$ and $M = \{\tilde{x}\tilde{y}, \tilde{x}\tilde{y}^{-1}\}$ with $\tilde{x}\tilde{y}^{-1} \sim \tilde{x}$ and $\tilde{x}\tilde{y}^{-1} \neq \tilde{x}\tilde{y}$. Again, we must have $\tilde{x}\tilde{y}^{-1} \in N$. And because $\tilde{x}\tilde{y}^{-1} \neq \tilde{x}\tilde{y}$, this can occur because $\tilde{x}\tilde{y}^{-1} = \tilde{y}\tilde{x}$ (where $\tilde{x}\tilde{y} \neq \tilde{y}\tilde{x}$), or because $\tilde{x}\tilde{y}^{-1} = \tilde{x}^{-1}\tilde{y}$. In this case, because $\tilde{x}\tilde{y}^{-1} \neq \tilde{x}\tilde{y}$ we get as a consequence that $\tilde{x}^{-1}\tilde{y} \neq \tilde{x}\tilde{y}$. But when $\tilde{x}\tilde{y}^{-1} \sim \tilde{x}$, we showed above that $\tilde{x}^{-1}\tilde{y} \not\sim \tilde{y}$, so that $\tilde{x}^{-1}\tilde{y} \in N$ only if $\tilde{x}^{-1}\tilde{y} = \tilde{x}\tilde{y}$ or $\tilde{x}^{-1}\tilde{y} = \tilde{y}\tilde{x}$. This, in conjunction with the fact that

$|M \cap N| = 2$, implies that $|N| = 2$, so that $N = \{\tilde{x}\tilde{y}, \tilde{y}\tilde{x}\}$ with $\tilde{x}\tilde{y} \neq \tilde{y}\tilde{x}$. And so we must have $\tilde{y}\tilde{x} = \tilde{x}\tilde{y}^{-1}$.

When $|N \cap M| = 2$, then $M \cap N = \{\tilde{x}\tilde{y}, \tilde{y}\tilde{x}\}$ so that $\widetilde{xy} \in \{\tilde{x}\tilde{y}, \tilde{y}\tilde{x}\}$. However, we found that we have $|M \cap N| = 2$ only if the elements $\tilde{x}, \tilde{y}$ satisfy certain conditions.

In case A, we had $M = \{\tilde{x}\tilde{y}, \tilde{y}\tilde{x}\}$ because $\tilde{y}\tilde{x} \sim \tilde{y}^{-1}$ with $N = \{\tilde{x}\tilde{y}, \tilde{x}^{-1}\tilde{y}\}$ and $\tilde{y}\tilde{x} = \tilde{x}^{-1}\tilde{y}$. In this case, $\tilde{x}, \tilde{y}$ must satisfy the following conditions:

(a) $\tilde{x}^2 \neq 1$, because $\tilde{x}\tilde{y} \neq \tilde{x}^{-1}\tilde{y}$.

(b) $\tilde{y}\tilde{x} = \tilde{x}^{-1}\tilde{y}$. Inverting both sides and multiplying on the right by $\tilde{y}$ we get $\tilde{x}^{-1}\tilde{y}^{-1}\tilde{y} = \tilde{y}^{-1}\tilde{x}\tilde{y}$, and this is equivalent to $\tilde{x}^{\tilde{y}} = \tilde{x}^{-1}$.

(c) $\tilde{y} \sim \tilde{y}^{-1}$ because $\tilde{y} \sim \tilde{x}^{-1}\tilde{y} = \tilde{y}\tilde{x} \sim \tilde{y}^{-1}$.

(d) $\tilde{y} \sim \tilde{x}\tilde{y} \sim \tilde{x}^{-1}\tilde{y} \sim (\tilde{x}^{-1})^2\tilde{y}$, because $\tilde{y} \sim \tilde{x}^{-1}\tilde{y} = \tilde{y}\tilde{x} \sim \tilde{y}^{-1}$ and $\tilde{y}^{\tilde{x}} = (\tilde{x}^{-1})^2 y$.

In case B, we had $M = \{\tilde{x}\tilde{y}, \tilde{x}\tilde{y}^{-1}\}$ with $\tilde{x}\tilde{y} \neq \tilde{x}\tilde{y}^{-1}$, and $\tilde{x}\tilde{y}^{-1} \sim \tilde{x}$. We showed $N = \{\tilde{x}\tilde{y}, \tilde{y}\tilde{x}\}$, with $\tilde{x}\tilde{y} \neq \tilde{y}\tilde{x}$, so that $\tilde{y}\tilde{x} = \tilde{x}\tilde{y}^{-1}$. In this case, for similar reasons as above, $\tilde{x}, \tilde{y}$ must satisfy the following:

(a) $\tilde{y}^2 \neq 1$.

(b) $\tilde{y}^{\tilde{x}} = \tilde{y}^{-1}$, which is equivalent to $\tilde{x}^{\tilde{y}} = (\tilde{y})^{-2}\tilde{x}$.

(c) $\tilde{x} \sim \tilde{x}^{-1}$.

(d) $\tilde{x} \sim \tilde{x}\tilde{y} \sim \tilde{x}\tilde{y}^{-1} \sim \tilde{x}\tilde{y}^2$.

These are a pretty restrictive list of conditions. And, they are symmetric. By that I mean that if (u, v) satisfies the second set of conditions, with $\tilde{x} = u$, $\tilde{y} = v$, then (v, u) satisfies the first set of conditions with $v = \tilde{x}, u = \tilde{y}$. We recall the following definition of ND pair, which makes it easier to keep track of constraints:

Definition 3.19. A pair (u, v) with $u, v \in H$, $u \not\sim v$, is called an *ND pair* (not determined) if u, v satisfy the following list of conditions:

- $uv \neq vu$, which implies that u is not a power of v and v is not a power of u.

- $v^2 \neq 1$.

- $v^u = v^{-1}$, which implies $u^v = uv^2$.

- $u \sim uv \sim uv^{-1} \sim uv^2 \sim u^{-1}$.

Using this terminology, we summarize what we have done with the following lemma.

Lemma 3.20. *Let G, H be finite groups and $\psi : G \to H$ a UTCCI map. Let $x, y \in G$ such that $x \not\sim y$, $x \not\sim y^{-1}$. If $(\tilde{x}, \tilde{y})$ and $(\tilde{y}, \tilde{x})$ are not ND pairs, then $\widetilde{xy} = \tilde{x}\tilde{y}$. If $(\tilde{x}, \tilde{y})$ or $(\tilde{y}, \tilde{x})$ is an ND pair, then $\widetilde{xy} \in \{\tilde{x}\tilde{y}, \tilde{y}\tilde{x}\}$.* □

Also, we note that ND pairs exist. For example, consider the elements $\alpha = (12), \beta = (123)$ in S_3. It is easy to verify that:

- $\alpha\beta \neq \beta\alpha$

- $\beta^2 = (132) \neq 1$

- $\beta^\alpha = (132) = \beta^{-1}$

- $\alpha \sim \alpha\beta \sim \alpha\beta^2$

So that $((12), (123))$ is an ND pair.

When we combine Lemma 3.20 with Lemma 3.14 we get the following:

Theorem 3.21. *Let G, H be finite groups with $\psi : G \to H$ a UTCCI map. Let $x \not\sim y$, $x, y \in G$. If $(\psi(x), \psi(y))$ and $(\psi(y), \psi(x))$ are not ND pairs, then $\psi(xy) = \psi(x)\psi(y)$. If $(\psi(x), \psi(y))$ or $(\psi(y), \psi(x))$ is an ND pair, then $\psi(xy) \in \{\psi(x)\psi(y), \psi(y)\psi(x)\}$.*

Proof. Let G, H be finite groups, and $\psi : G \to H$ a UTCCI map. Let $x, y \in G$, with $x \not\sim y$. We showed in Lemma 3.14 that when $x \not\sim x^{-1}$ and $y \not\sim y^{-1}$ we have $\psi(xy) = \psi(x)\psi(y)$. When either $x \sim x^{-1}$ or $y \sim y^{-1}$, then $x \not\sim y^{-1}$, and we showed in Lemma 3.20 that unless one of $(\tilde{x}, \tilde{y})$ or $(\tilde{y}, \tilde{x})$ is an ND pair, then we have $\widetilde{xy} = \tilde{x}\tilde{y}$. When one of $(\tilde{x}, \tilde{y})$ or $(\tilde{y}, \tilde{x})$ is an ND pair then $\widetilde{xy} \in \{\tilde{x}\tilde{y}, \tilde{y}\tilde{x}\}$. □

3.4 The 'non-determined' incomplete Cayley Table of generalized dihedral groups of order $2n$, n odd

Here is an example of the NDICT for the group S_3, where again we are intentionally not giving traditional labels to the elements of the group.

S_3	e	s	t	u	v	w
e	e	s	t	u	v	w
s	s	t	1	$\{v,w\}$	$\{u,w\}$	$\{u,v\}$
t	t	1	s	$\{v,w\}$	$\{u,w\}$	$\{u,v\}$
u	u	$\{v,w\}$	$\{v,w\}$	1		
v	v	$\{u,w\}$	$\{u,w\}$		1	
w	w	$\{u,v\}$	$\{u,v\}$			1

Corollary 3.22. *Let G be a finite abelian group. Then the NDICT of G is the Cayley Table of G.*

Proof. Let $g \in G$, where G is a finite abelian group. Then the only element of g^G is g and g^2 is determined by the NDICT. For all other elements $h \in G$, $gh = hg$ is determined by the NDICT of G. $\square$

There are two ways to fill in the NDICT of S_3 as the Cayley table of a group. They are:

S_3	e	s	t	u	v	w
e	e	s	t	u	v	w
s	s	t	1	v	w	u
t	t	1	s	w	u	v
u	u	w	v	1	t	s
v	v	u	w	s	1	t
w	w	v	u	t	s	1

S_3	e	s	t	u	v	w
e	e	s	t	u	v	w
s	s	t	1	w	u	v
t	t	1	s	v	w	u
u	u	v	w	1	s	t
v	v	w	u	t	1	s
w	w	u	v	s	t	1

This scenario is not unique to S_3. Let $G = N \rtimes C_2$ be a generalized dihedral group where N is abelian of odd order and $C_2 = \langle t | t^2 = 1 \rangle$ acts by inversion on N.

In order to compute the NDICT of G, fix $g, h \in N$. Then $g^G = \{g, g^{-1}\}$, so the g, h entry in the NDICT is gh (if $g \sim h$ then gh is known because one is a power of the other). Also, $g^{ht} = g^t = g^{-1}$ and $(ht)^G = Nt$ is a single class of involutions, so $ht \sim ght$, and (ht, g) is an ND pair and the g, ht entry of the NDICT is the set $\{ght, htg\}$. If $g = h$, then $gt = ht$ and the product $ghht = 1$ is known. When $gt \neq ht$, the (gt, ht) entry must be $\emptyset$ because $gt \sim ht$ and neither is a power of the other.

Next we want to show that there is not a unique way to fill in the NDICT of G as the Cayley table of G when G is generalized dihedral of order $2n$, n odd. For $g \in N$, define $\psi : G \to G$ by $\psi(gt^i) = t^i g$, where $i \in \{0, 1\}$. We show that $\psi : G \to G$ is an NDICT map.

Because N is abelian, $\psi(gh) = gh = \psi(g)\psi(h)$. For the ND pair (g, ht), we have $\psi(g(ht)) = tgh = h^{-1}tg = \psi(ht)\psi(g)$ and also $\psi((ht)g) = \psi(g^{-1}ht) = tg^{-1}h = gh^{-1}t = \psi(g)\psi(ht)$. This shows that $\psi(\{ght, htg\}) = \{\psi(g)\psi(ht), \psi(ht)\psi(g)\}$.

The existence of this map will prevent us from showing that an NDICT map is a CICT map. However, this is not a problem, because we have the following:

Theorem 3.23. *Let $G = N \rtimes C_2$ be a generalized dihedral group of order $2n$, n odd. Then G is determined by the NDICT of G*

This follows from the earlier remark that every product of elements in N is determined by the NDICT of G, and N determines G.

3.5 The NDICT determines finite groups G with $|G| < 256$

In the next chapter we show that when G is a finite group with $|G| \geq 256$ which is not generalized dihedral of order $2n$, n odd, then an NDICT map $G \to H$ is actually a CICT map. In order to do this, we show that the NDICT of such groups can only be completed in one way as the CICT of a group. We do this by showing that the NDICT in fact determines xy when (x, y) is an ND pair and $|G| \geq 256$. We will show the following:

Theorem 3.24. *Let G and H be finite groups with $|G| \geq 256$ and G,H not generalized dihedral of order $2n$, n odd. If $\psi : G \to H$ an NDICT map, ψ is an isomorphism.*

In this section we discuss groups of order smaller than 256. Actually we let GAP do the heavy lifting. In her dissertation [22], Ellen Skrzipczyk showed that the smallest 2-groups which were Brauer pairs were of order 2^8. And when one rules out 2-groups, it is easy to write code to verify that there are no other Brauer pairs of order less than 256.

At the end of this section we show (Lemma 3.29) that if two groups have the same NDICT they are a Brauer pair. From that result it will follow that:

Theorem 3.25. *If G, H are finite groups which have the same NDICT and $|G| < 256$, then G is determined by its NDICT.*

From Theorems 3.24, 3.23, and 3.25 we have

Theorem 3.26. *If G, H are finite groups and $\psi : G \to H$ is an NDICT map, then G and H are isomorphic.*

Corollary 3.27. *If there exists a UTCCI map $\psi : G \to H$, then G and H are isomorphic.*

Corollary 3.28. *The 3-S-ring of a finite group G determines G.*

Next we prove Lemma 3.29. This Lemma also allows us to state that Theorem 3.26 gives another answer to Brauer's question.

Lemma 3.29. *Let G, H be finite groups which have the same NDICT. Then G and H are a Brauer pair.*

Proof. Let $\psi : G \to H$ be a NDICT map. Then by definition ψ respects powers. It remains to show that ψ induces an isomorphism of centralizer rings. Let $\{C_1, C_2, \cdots, C_n\}$ be the conjugacy classes of G. It suffices to show that the structure constants λ_{ijk} defined by $\overline{C}_i\overline{C}_j = \sum_{k=1}^{n} \lambda_{ijk}\overline{C}_k$ are determined by the NDICT of G. Fix C_i. Then $\overline{C}_i\overline{G} = |C_i|\overline{G} = \sum_{k=1}^{n} |C_i|\overline{C}_k$. It follows that if all λ_{ijk} for $i \neq j$ are known, then one can determine the structure constants

λ_{iik}. And when $i \neq j$, then λ_{ijk} is determined by the NDICT. If we don't know the product gh for a elements $g \in C_i, h \in C_j$, then we know the set $\{gh, hg\}$ and these elements are conjugate, so we know how many elements in $\overline{C}_i\overline{C}_j$ are in each C_k and can figure out the coefficient λ_{ijk}. Thus, we can determine λ_{ijk} from the NDICT of G, and the centralizer algebra is determined. □

Chapter 4. The NDICT of a finite group G when $|G| \geq 256$

As our main result of this chapter we prove the following:

Theorem 4.1. *Let G be a finite group with $|G| \geq 256$ which is not dihedral of order $2n$, n odd. If (x, y) is an ND pair, then xy is determined by the NDICT of G.*

This implies that $\psi(gh) = \psi(g)\psi(h)$ when $g \not\sim h$, from which we get the following:

Theorem 4.2. *Let G and H be finite groups with $|G| \geq 256$ and G, H not dihedral of order $2n$, n odd, with $\psi : G \to H$ an NDICT map. Then ψ is a CICT isomorphism.*

From this result and Theorem 2.7 we get Theorem 3.24.

Our method is constructive: Let G be an FC group for which the NDICT of G is known. We fix $g \in G$ and consider the g^{th} 'row' of the NDICT of G. If we have a single entry z in the h column of the NDICT, then we know that $gh = z$, and that either g is a power of h, or h is a power of g, or that $g \not\sim h$ and (g, h) and (h, g) are not ND pairs. We are going to leverage the products which are determined by the NDICT to find the product xy for ND pairs (x, y). If we can show for all ND pairs (x, y) that the product xy is determined by the NDICT, then it follows that the CICT of G is determined by the NDICT.

Throughout this chapter G is a finite group unless otherwise mentioned.

4.1 Definitions and some Lemmas

Definition 4.3. Let G be a finite group and $x \in G \setminus \{1\}$. We define:

(i) $C_x = \{g \in x^G | g$ and x are not powers of each other$\}$.

(ii) $L_x = \{g \in G | (x, g)$ is an ND pair$\}$.

(iii) $R_x = \{g \in G | (g, x)$ is an ND pair$\}$.

We will also use the following facts about the sets C_x, L_x, R_x:

Lemma 4.4. *Let G be an FC group and $x \in G \setminus \{1\}$. We have the following:*

(a) $C_x \subseteq x^G \setminus \{x, x^{-1}\}$ and $|C_x| \leq |x^G| - 1$. If $x \neq x^{-1}$, then $|C_x| \leq |x^G| - 2$.

(b) $L_x \subseteq x^{-1}x^G \setminus \{1, x^{-2}\}$ and $|L_x| \leq |x^G| - 1$. If $x \neq x^{-1}$, then $|L_x| \leq |x^G| - 2$.

(c) If $x^g = x^{-1}$, then $R_x \subseteq C_G(x)g$ and $|R_x| \leq |C_G(x)|$.

(d) If x is an involution, then $R_x = \emptyset$ and $|R_x| = 0$.

(e) If x has odd order, then $L_x = \emptyset$ and $|L_x| = 0$.

(f) If $h \sim x$, then $C_x \cup C_h \subseteq x^G$ and $|C_x \cup C_h| \leq |x^G|$.

(g) If x is an involution, then $L_x = \{g \in x^{-1}x^G | g^2 \neq 1\}$.

(h) If $g \in G \setminus (C_x \cup L_x \cup R_x)$, then xg and gx are determined by the NDICT of G.

(i) If (x, y) is an ND pair, then $R_y \subseteq C_G(y)x$ and $|R_y| \leq |C_G(y)|$.

Proof. Let $x, g \in G \setminus \{1\}$, where G is an FC group.

(a) Because x and x^{-1} are powers of x, by definition of C_x we have $C_x \subseteq x^G \setminus \{x, x^{-1}\}$

(b) Let $x^G = \{x, xg_2, xg_3, \ldots, xg_k\}$ where $|x^G| = k$. If (x, g) is an ND pair, then $x \sim xg$, so $g \in \{g_2, g_3, \ldots, g_k\} = (x^{-1}x^G) \setminus \{1\}$. From the definition of ND pair, we know that $x \sim x^{-1}$. If $x \neq x^{-1}$, then without loss of generality, we assume $xg_2 = x^{-1}$ so that $g_2 = x^{-2}$. Because x^{-2} is a power of x, so we know that (x, x^{-2}) is not an ND pair. Also, when $x = x^{-1}$, then $x^2 = 1$. So we have $L_x \subseteq x^{-1}x^G \setminus \{1, x^{-2}\}$.

(c) If (g, x) is an ND pair, then, among other conditions, we know $x^g = x^{-1}$. For $h \in G$, it follows from the orbit stabilizer relationship that $h \in C_G(x)g$ exactly when $x^h = x^{-1}$. It follows that $R_x \subseteq C_G(x)g$, and $|R_x| \leq |C_G(x)|$.

(d) By definition, if x is an involution, then (g, x) cannot be an ND pair for any $g \in G$.

(e) By definition, if x has odd order, then (x, g) cannot be an ND pair for any $g \in G$.

(f) When $x^G = h^G$, then $C_x \cup C_h \subseteq x^G \cup h^G = x^G$.

(g) Suppose x an involution, and that $x \sim xg$ for some $g \in G \setminus \{1\}$. Then xg is also an involution, so that $xgxg = 1$. Multiplying by g^{-1} on the right, we see that $xgx = g^{-1}$ so that $g^x = g^{-1}$. From this it also follows that $(xg)^x = x^x g^x = xg^{-1}$, from which it follows that $x \sim xg \sim xg^{-1}$. So if g is not an involution, then (x, g) is an ND pair.

(h) These are exactly the elements $g \in G$ which for which $g \not\sim x$, (x, g) is not ND, and (g, x) is not ND. It follows from Lemma 3.20 that for such a g, the product xg is determined by the NDICT.

(i) If (x, y) is an ND pair, then $y^x = y^{-1}$, and so we have $\{g \in G | y^g = y^{-1}\} = C_G(y)x$. If $g \in R_y$, then $y^g = y^{-1}$, so $R_y \subseteq C_G(y)x$.

□

Lemma 4.5. *If (x, y) form an ND pair, then $x \sim xy^k$ for all $k \in \mathbb{Z}$, i.e. $x\langle y \rangle \subseteq x^G$. Also $o(y) \leq |x^G|$.*

Proof. We know $x^y = xy^2$. Fix an integer k. Then $(xy^k)^y = x^y (y^k)^y = (xy^2)y^k = xy^{k+2}$. But we have $x \sim xy^1$ and $x \sim xy^2$, so by induction we have $x \sim xy^k$ for all integers k. □

4.2 Example

We are now going to do an example. Let $G = A_5$, the alternating group of degree 5. Let $x = (12)(34)$ and $y = (125)$. They are not powers of each other and it is straightforward to check that $y^x = y^{-1}$ and $x \sim xy \sim xy^2$, so that (x, y) is an ND pair. Our set $\{xy, yx\}$ is the set $\{(25)(34), (15)(35)\}$.

The class x^G consists of all the involutions in A_5. The class x^G has 15 elements, and straightforward calculation shows that:

$$x^{-1}x^G = \{1, (354), (345), (152), (125), (14325), (13542), (12453), (15234),$$
$$(14532), (15243), (13425), (12354), (14)(23), (13)(24)\}.$$

From this we can see that $C_x = x^G \setminus \{x\}$. Because (14)(23) and (13)(24) are involutions, $(x, (14)(23))$ and $(x, (13)(24))$ are not ND pairs. We can verify directly the results of Lemma 4.4(g): $|L_x| = 15 - 3 = 12$, the 15 elements of $xx^G = x^{-1}x^G$ minus identity and the two involutions.

One thing this example highlights is the fact that knowing that (x, h) is an ND pair tells us nothing about $\{g|(x, g)$ is an ND pair$\}$. For example, there is an ND pair (x, h) with an element h from every non-identity, non-x^G conjugacy class of G. Also, we have $(x, (354))$ and $(x, (152))$ both ND pairs, with $(354) \sim (152)$, but neither a power of the other.

If we consider the $x = (12)(34)$ row of the NDICT of A_5, we find 14 columns for which the entry is $\emptyset$, corresponding to the elements of $C_x = x^G \setminus \{x\}$. There are 12 columns in which we get paired entries $\{g_1, g_2\}$ corresponding to $h \in L_x$, where $\{g_1, g_2\} = \{xh, hx\}$. And there are 34 columns for which we have the single entry xh in the column h. Similarly, when we look at the x column of the NDICT of G, there will be 14 rows with $\emptyset$ as the entry corresponding to the 14 entries of C_x, 12 entries which are pairs $\{hx, xh\}$ corresponding to the 12 entries of L_x, and 34 rows which contain a single element.

Now consider $y = (125)$. Because (x, y) is an ND pair, we know that the (x, y) entry of the NDICT is the set $\{g_1, g_2\} = \{(25)(34), (15)(34)\}$. In order to show that the NDICT determines xy, we do not need to work from scratch to find xy as we did in the CICT case. We just need to show that the NDICT determines which of these two elements can be xy.

The way we will do this is to pick an element g_1 and analyze the g_1 column in the NDICT. For the sake of this example, we choose $g_1 = (25)(34)$. When we look at the g_1 column of the NDICT, we will again find 14 spots which have no entry, corresponding to the elements of $C_{g_1} = x^G \setminus \{g_1\}$, and 12 spots which have entry a set of two elements, corresponding to the elements of L_{g_1}. Also, $R_{g_1} = \emptyset$. These facts can be verified directly or by referring to Lemma 4.4.

We begin with those elements $g \in G$ for which the entry in the x and g_1 columns of the NDICT is a single element gx and gg_1 respectively. In set notation, we are considering the set $G \setminus (C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1})$.

Just using the results of Lemma 4.4 we get $|C_x \cup C_{g_1}| \leq 15$, $R_x = R_{g_1} = \emptyset$. So we know by rough estimation that $|C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}| \leq |x^G \cup L_x \cup L_{g_1}| \leq 15 + 12 + 12$. So simply by size considerations, there are at least 21 elements $g \in G$ for which both gx and gg_1 can be calculated.

In this example, we can (and will) calculate the actual size of the set $C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}$, but in general we can't, and groups with classes this big will require special techniques. In this example, we want to 'finish up' our demonstration of our main method.

We calculated $x^{-1}x^G$, so we know L_x. It is also straightforward to calculate $g_1^{-1}(g_1^G) = (25)(34)x^G$ to find L_{g_1}.

$$\begin{aligned}(25)(34)x^G =& \{1, (143), (134), (152), (125), (14253), (12345), (15432), (13524), \\ & (14523), (15342), (12435), (13254), (23)(45), (24)(35)\}.\end{aligned}$$

With this, we see by inspection that $|C_x \cup C_{g_1} \cup L_x \cup L_{g_1}| = 37$. (Recall $R_x = R_{g_1} = \emptyset$.) So, there are actually 23 elements $g \in G$ for which the entries in the x and g_1 columns are single elements, so that gx and gg_1 are determined from the NDICT.

Now consider the 23 elements gx, $g \in G \setminus (C_x \cup C_{g_1} \cup L_x \cup L_{g_1})$, and look at the y column for each of the rows corresponding to these elements. If, for some gx, there exists an entry $(gx)y$, the NDICT will determine the entry xy. This is because we have computed $(gx)y$ and

gg_1 for an 'arbitrary' $g_1 \in \{xy, yx\}$. If $(gx)y = gg_1$, then $g_1 = xy$. If $gg_1 \neq (gx)y$, then we know that $g_1 = yx$ and $g_2 = xy$. But in either case, this shows that the NDICT determines xy.

In general, if we can do this for any ND pair (g, h) we will have shown that there is only one way that we can fill out the CICT of G using the NDICT of G.

Next we show, using a numerical argument, the existence of a g for which gx and gg_1 are determined as a single entry in the NDICT, such that $gx \notin (C_y \cup L_y \cup R_y)$.

For this example, we know that $y = (125)$ has class size 20. We also know that $y \neq y^{-1}$ and y^{-1} is the only element in G that y is a power of. So $C_y = y^G \setminus \{y, y^{-1}\}$ and there are $20 - 2 = 18$ elements in C_y. Because y has odd order, it follows from Lemma 4.4(e) that $L_y = \emptyset$. We also know that $|y^G| = |G|/3$ so that $|C_G(y)| = 3$. It follows by Lemma 4.4(c) that $|R_y| \leq 3$.

From this, we determine that the y column of G has at most 18 empty entries and at most 3 pair entries, and for all other rows the entry is a single element. So, for at least two of the 23 gx with $g \in G \setminus (C_x \cup C_{g_1} \cup L_x \cup L_{g_1} \cup R_x \cup R_{g_1})$, there will be a single entry $(gx)y$ in the y column. This means that for $x = (12)(34)$ and $y = (125)$, the product xy is determined by the NDICT of A_5.

Because A_5 is not a Brauer pair with any group, we know that A_5 is determined by the NDICT. However, this example allowed us to demonstrate some of the methods we will use in the general proof.

4.3 Formalizing the ideas of the example

Given an ND pair (x, y) in a finite group G, our main method will be to fix $g_1 \in \{xy, yx\}$ and then show the existence of a $g \in G$ for which we can calculate gg_1, gx, and $(gx)y$. The existence of such a g shows that the product xy is determined by the information of the NDICT. We will call this *working from the left.* Occasionally, we will want to *work from the right* by finding an $h \in G$ for which g_1h, yh, and $x(yh)$ can be calculated. Showing the existence of such a g or h will often involve a size argument, and occasionally a little ingenuity as well.

Recall that when we work from the left we are looking for an element $g \in G$ for which we can determine gx, gg_1, and $(gx)y$ from the NDICT. The elements of $C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}$ are not candidates for g because for $h \in C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}$ one of the products hx or hg_1 is not determined by the NDICT of G. For the elements of $G \setminus (C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1})$, the products gx and gg_1 are determined by the NDICT. Thus, we want to show that for some element h,

$$h \in x\left[G \setminus (C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1})\right],$$

the product hy can be calculated. Because $C_y \cup L_y \cup R_y$ is the set of elements $h \in G$ for which hy is not determined by the NDICT of G, there exists a $g \in G$ for which gx, gg_1, and $(gx)y$ can be calculated if

$$x\left[G \setminus (C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1})\right] \setminus (C_y \cup L_y \cup R_y) \neq \emptyset.$$

Thus we have shown the following:

Lemma 4.6. *Let G be a finite group for which the NDICT of G is given. Let (x, y) be an ND pair in G and let $g_1 \in \{xy, yx\}$. If $x[G \setminus (C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1})] \setminus (C_y \cup L_y \cup R_y) \neq \emptyset$, then xy is determined by the NDICT of G.* □

Lemma 4.7. *Let G be a finite group for which the NDICT of G is given. Let (x, y) be an ND pair in G and let $g_1 \in \{xy, yx\}$. If there exists $g \in G$ for which yg, $g_1 g$ and $x(yg)$ can be determined from the NDICT of G, then xy is determined by the NDICT of G.* □

Often we will use a size argument and the following corollaries will be helpful:

Corollary 4.8. *Let (x, y) be an ND pair. If*

$$|x\left[C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}\right]| + |(C_y \cup L_y \cup R_y)| < n,$$

then xy is determined by the NDICT of G.

Proof. This follows from the Lemma 4.6 once we notice that

$$\begin{aligned} x\left[G \setminus (C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1})\right] \setminus (C_y \cup L_y \cup R_y) &= \\ G \setminus \left[x(C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}) \cup (C_y \cup L_y \cup R_y)\right]. \end{aligned}$$

□

Corollary 4.9. *Let G be a finite group with $|G| = n$. If $|C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}| + |C_y \cup L_y \cup R_y| \leq \left(1 - \frac{1}{a}\right) n + b$ for some positive real numbers a, b for which $ab < n$, then the NDICT of G determines xy.*

Proof. If $ab < n$ then $b < \frac{n}{a}$ so $b - \frac{n}{a} + n < n$ and so $\left(1 - \frac{1}{a}\right) n + b < n$ and the result follows from Corollary 4.8 □

The following technical lemma will allow us to do several class sizes at the same time.

Lemma 4.10. *Let n, l be positive integers with $n > l^2$. If $l \leq t \leq n/l$, then $t + n/t \leq l + n/l$.*

Proof. Fix n, l positive integers with $n^2 > l$. Consider the function $f(t) = \frac{n}{t} + t$, $l \leq t \leq \frac{n}{l}$. We have $f'(t) = 1 - \frac{n}{t^2}$, so the only critical point is at $t = \sqrt{n}$. Also $f''(t) = \frac{2n}{t^3}$ is always positive for $l \leq t \leq \frac{n}{l}$. So the maximum for $f(t)$ on the interval $[l, \frac{n}{l}]$ occurs at one of the endpoints. Evaluating at the endpoints, we get $f(l) = l + \frac{n}{l} = f(\frac{n}{l})$. So $f(t) \leq l + \frac{n}{l}$ when $l \leq t \leq \frac{n}{l}$. □

With our tools in hand and the example out of the way, we are ready to dig into the proof.

4.4 The first two cases

In this section we begin the proof of Theorem 4.1, that the NDICT of a finite group G with $|G| \geq 256$ determines the product xy for any ND pair (x, y) when G is not generalized dihedral of order $2n$, n odd. The proof is by cases on $k = |x^G|$ and $m = |y^G|$.

Because we are working by cases in two variables, we will keep track of what we prove in a table, indexed by $k = |x^G|$ and $m = |y^G|$. When a case is proven, we put the label of that case in the correct class sizes. So, for example, the letters A in the table below indicates that in Case A we show that the NDICT of G determines the product xy whenever (x, y) is an ND pair for which $6 \leq k \leq n/6$ or $6 \leq m \leq n/6$. Because $|G| \geq 256$, we have $9 < \frac{n}{9}$, so there is no ambiguity on the chart.

$k \backslash m$	2	3	4	5	6	7	8	9	> 9	$< \frac{n}{9}$	$\frac{n}{9}$	$\frac{n}{8}$	$\frac{n}{7}$	$\frac{n}{6}$	$\frac{n}{5}$	$\frac{n}{4}$	$\frac{n}{3}$	$\frac{n}{2}$
2																		
3																		
4																		
5																		
6					A	A	A	A	A	A	A	A	A	A				
> 6					A	A	A	A	A	A	A	A	A	A				
$< n/6$					A	A	A	A	A	A	A	A	A	A				
$n/6$					A	A	A	A	A	A	A	A	A	A				
$n/5$																		
$n/4$																		
$n/3$																		
$n/2$																		

Proof. Let G be a finite group, $|G| \geq 256$, for which the NDICT is known. Let (x, y) be an ND pair in G. Let $k = |x^G|$, $m = |y^G|$, and $n = |G|$. From the NDICT we know the set $\{xy, yx\}$. We fix $g_1 \in \{xy, yx\}$.

Case A: $6 \leq k \leq n/6$ and $6 \leq m \leq n/6$. In this case we know from Lemma 4.10 that $k + n/k \leq 6 + n/6$ and $m + n/m \leq 6 + n/6$.

Consider the set $C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}$. By Lemma 4.4(f), we have $|C_x \cup C_{g_1}| \leq k$. Also, by Lemma 4.4(b) we have $|L_x| \leq k - 1$ and $|L_{g_1}| \leq k - 1$. Finally, by Lemma 4.4(c) we have $|R_x| \le n/k$ and $|R_{g_1}| \le n/k$. This gives us

$$\begin{aligned}
|C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}| &\le |C_x \cup C_{g_1}| + |L_x| + |R_x| + |L_{g_1}| + |R_{g_1}| \\
&\le k + (k-1) + \frac{n}{k} + (k-1) + \frac{n}{k} \\
&= 3k + \frac{2n}{k} - 2 = k + 2\left(k + \frac{n}{k}\right) - 2 \\
&\le \frac{n}{6} + 2\left(6 + \frac{n}{6}\right) - 2 = \frac{n}{2} + 10.
\end{aligned}$$

We also have an upper bound on the size of the set $C_y \cup L_y \cup R_y$. By Lemma 4.4(a), we have $|C_y| \le m-1$. Also, by Lemma 4.4(b) we have $|L_x| \le m-1$. Finally, by Lemma 4.4(c) we have $|R_y| \le n/m$. So we have:

$$\begin{aligned}
|C_y \cup L_y \cup R_y| &\le |C_y| + |L_y| + |R_y| \\
&\le (m-1) + (m-1) + \frac{n}{m} \\
&= \left(m + \frac{n}{m}\right) + m - 2 \\
&\le 6 + \frac{n}{6} + \frac{n}{6} - 2 \\
&= \frac{n}{3} + 4.
\end{aligned}$$

This gives us

$$\begin{aligned}
|C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}| + |C_y \cup L_y \cup R_y| \le \quad & n/2 + 10 + n/3 + 4 \\
&= 5n/6 + 14 = (1 - 1/6)n + 14.
\end{aligned}$$

And because $6 \cdot 14 < 256$, xy is determined by Corollary 4.9. $\square$

Because of the restrictions placed on (x, y) based on their relationship as an ND pair, there are several cases for which there are no ND pairs (x, y) for which $k = |x^G|$ or $m = |y^G|$. For brevity, we combine these into a single case.

Case B: $k = 2$, n/k odd, $m = n/2$, or m odd. Suppose, by way of contradiction, that (x, y) is an ND pair and n/k is odd. Then x has even order (definition of ND pair), so $C_G(x)$

also has even order. But $C_G(x) = n/k$ is odd, a contradiction. In particular, because (x, y) is an ND pair, $k \neq n/3, n/5$.

Suppose, by way of contradiction, that $k = 2$. Then y is not an involution (definition of ND pair), so $y \neq y^{-1}$ and $|\langle y \rangle| > 2$. By Lemma 4.5, $x\langle y \rangle \subseteq x^G$. So $|x^G| = k > 2$, a contradiction. In particular, because (x, y) is an ND pair, we know $k \neq 2$.

Suppose, by way of contradiction, that $m = n/2$. Again, we know that y is not an involution, so that $|C_G(y)| \geq o(y) \geq 3$. Thus $|y^G| = |G|/|C_G(y)| \leq n/3$, a contradiction. So, because (x, y) is an ND pair we have $m \neq n/2$.

Finally, suppose, by way of contradiction, that m is odd. Because (x, y) is an ND pair, $y \sim y^{-1}$ and y is not an involution. Thus $|y^G| = m$ is even, a contradiction. In particular, because (x, y) is an ND pair we know $m \neq 3, 5, 7, 9$.

Thus, when $k = 2$, n/k is odd, $m = n/2$, or m is odd, there are no ND pairs (x, y) with $|x^G| = k$ of $|y^G| = m$ and so vacuously the NDICT determines xy for ND pairs with these class sizes. $\square$

The completed table below gives the flow of the rest of the proof. We note that on this table, there will be many cases (e.g. the case for which $k = 6$ and $m = 7$) that are proven more than once. When this occurs, the case is labeled by the first occurrence in the proof.

$k \backslash m$	2	3	4	5	6	7	8	9	> 9	$< \frac{n}{9}$	$\frac{n}{9}$	$\frac{n}{8}$	$\frac{n}{7}$	$\frac{n}{6}$	$\frac{n}{5}$	$\frac{n}{4}$	$\frac{n}{3}$	$\frac{n}{2}$
2	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
3	C	B	C	B	C	B	C	B	C	C	C	C	C	C	C	C	C	B
4	F	B	D	B	D	B	D	B	D	D	D	D	D	D	D	D	D	B
5	F	B	E	B	E	B	E	B	E	E	E	E	E	E	E	E	E	B
6	F	B	G	B	A	A	A	A	A	A	A	A	A	A	J	K	L	B
> 6	F	B	G	B	A	A	A	A	A	A	A	A	A	A	J	K	L	B
$< n/6$	F	B	G	B	A	A	A	A	A	A	A	A	A	A	J	K	L	B
$n/6$	F	B	G	B	A	A	A	A	A	A	A	A	A	A	J	K	L	B
$n/5$	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
$n/4$	F	B	N	B	M	B	M	B	M	M	M	P	M	Q	M	R	S	B
$n/3$	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B
$n/2$	H	B	H	B	H	B	H	B	H	H	H	H	H	H	H	H	H	B

For many of the remaining cases, a size argument will also suffice. However, with larger class sizes and/or larger centralizers, the sets C_x, L_x, R_x, also get larger. In these cases we have more information about $x^G, y^G, C_G(x), C_G(y)$ which we can use to find a g for which we can work from the right or from the left to determine xy.

4.5 All remaining cases except $k = n/4$

Case C: $k = 3$. Because (x, y) is an ND pair, we know that $x \sim x^{-1}$ and y is not an involution. Because $|x^G|$ is odd and $x \sim x^{-1}$, it follows that x is an involution. So from Lemma 4.4(d), we know that $R_x = R_{g_1} = \emptyset$. Also, because $|x^G| = 3$, and we know from Lemma 4.5 that $x\langle y\rangle \subseteq x^G$, we have $o(y) = 3$. Also, straightforward calculations show that $x^G = \{x, xy, xy^2\} = \{g_1, g_1 y, g_1 y^2\}$ regardless of whether we chose $g_1 = xy$ or $g_1 = yx$, so that $L_x = L_{g_1} = \{y, y^2\}$.

Because $x^2 = 1$ and $y^x = y^{-1}$, or equivalently $xy = y^{-1}x$, it is straightforward to show that $x^G x^G = \{1, y, y^2\}$. Thus $\{1, y, y^2\}$ is a product of sets which are normal in G and so must itself be a normal set and hence a union of conjugacy classes. And since $y^x = y^2$, we must have $y^G = \{y, y^2\}$. This gives us a subgroup $N = C_G(y)$ for which $[G : N] = 2$, so that $N \lhd G$. We know that x does not centralize y, so that $x \notin N$. More precisely, since N is normal in G, it follows that $x^G \cap N = \emptyset$. By definition $C_x \subset x^G$, $C_{g_1} \subset x^G$, so it follows that $(C_x \cup C_{g_1}) \cap N = \emptyset$.

In this case we are going to work from the right. Because $n \geq 256$ and $o(y) = 3$, we know that $N \setminus \langle y \rangle \neq \emptyset$. Fix $g \in N \setminus \langle y \rangle$. We can calculate yg because $g \not\sim y$ and $gy = yg$. We also will have $yg \in N \setminus \langle y \rangle$. As shown earlier, we have $R_x = R_{g_1} = \emptyset$ and $L_x = L_{g_1} = \{y, y^2\}$. Together this shows that $yg \in G \setminus (C_x \cup L_x \cup R_x)$ and $g \in G \setminus (C_{g_1} \cup L_{g_1} \cup R_{g_1})$. Thus $x(yg)$ and $g_1 g$ can be determined from the NDICT by Lemma 4.4(g). And, by Lemma 4.7 the product xy is determined by the NDICT in this case. □

In Case C, we used the fact that $n \geq 256$ to force the existence of an element in $N \setminus \langle y \rangle$ which we could use to determine xy. But all we really need is $n > 6$. The case $N = \langle y \rangle$ corresponds to the example of the NDICT of S_3 discussed earlier. In that example, we saw that the NDICT did not determine xy for the ND pair (x, y) where x was an involution and y had order 3 (but the NDICT still determined the group).

Case D: $k = 4$, $m \neq n/2$. From Lemma 4.5 we know that $x\langle y \rangle \subseteq x^G$. From the definition of ND pair we know y is not an involution. So y can only have order 3 or 4. We write $x^G = \{x, xy, xy^{-1}, z\}$ for some $z \in G$ where if $o(y) = 4$ we have $z = xy^2$. We consider first the subcase where x is an involution, and then the subcase where $x \neq x^{-1}$.

When x is an involution, a size argument suffices. We have $|C_x \cup C_{g_2}| \leq 4$, $|L_x| \leq 3$ and $|L_{g_1}| \leq 3$, and also $|R_x| = |R_{g_1}| = 0$ by Lemma 4.4, so that

$$|C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}| \leq 4 + 3 + 3 + 0 + 0 = 10.$$

When we consider y, we have $2 \leq m \leq n/3$, $|C_y| \leq m - 2$, $|L_x| \leq m - 2$, and $|R_y| \leq n/m$

so that

$$|C_y \cup L_y \cup R_y| \le m + (m + n/m) - 4 \le n/3 + (n/2 + 2) - 4 = 5n/6 - 2.$$

This gives us $|C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}| + |C_y \cup L_y \cup R_y| \le 5n/6 + 8$. And since $6 \cdot 8 < 256$, by Lemma 4.9 we know that the product xy is determined when x is an involution and $k = 4$.

When x is not an involution, we have $x \sim x^{-1}$ and $x \ne x^{-1}$, so that $x^{-1} \in x^G \setminus \{x\} = \{xy, xy^{-1}, z\}$. Because (x, y) is ND, we know x is not a power of y, and y is not a power of x. Because $x^{-1} = xy$ implies $x^2 = y^{-1}$, and $x^{-1} = xy^{-1}$ implies $y = x^2$, we must have $x^{-1} = z$. Also xy and xy^{-1} must be inverses, hence $1 = (xy)(xy^{-1}) = x(yx)y^{-1} = x(xy^{-1})y^{-1} = x^2(y^{-1})^2$, so that $x^2 = y^2$. If $o(y) = 3$ then $x^4 = (y^2)^2 = y$, a contradiction. So, when x is not an involution, we must in fact have $x^2 = y^2$, $o(x) = o(y) = 4$, and $x^G = \{x, xy, xy^2, xy^{-1}\}$.

Using the facts that $y^x = y^{-1}$ and $x^2 = y^2$, it follows from straightforward calculations that $x^G x^G = \{x^2, x^2y, x^2y^2, x^2y^3\} = \{1, y, y^2, y^3\}$. Thus $\langle y \rangle$ is a product of sets which are normal in G and hence $\langle y \rangle$ is a normal subgroup of G. We have $y^x = y^{-1}$ and we know $o(y) \ne o(y^2)$, so we have $y^G = \{y, y^3\}$. This implies that $[G : C_G(y)] = 2$, so that $C_G(y) \lhd G$. And $n \ge 256 > 10$ so that $C_G(y) \setminus \langle y \rangle \ne \emptyset$. Fix $g \in C_G(y) \setminus \langle y \rangle$. Arguments similar to those of Case C show that we can determine yg, $x(yg)$ and g_1g from the NDICT, so that in the case where x is not an involution, xy is determined from the NDICT by Lemma 4.7. $\square$

Case E: $k = 5$ and m even. Because (x, y) is an ND pair, we know $x \sim x^{-1}$. But $k = |x^G|$ is odd, so x must be an involution. By Lemma 4.5 we have $x\langle y \rangle \subseteq x^G$. And y is not an involution, so $o(y) \in \{3, 4, 5\}$. We write $x^G = \{x, xy, xy^{-1}, xz, xt\}$ for some $z, t \in G$. Here we have not ruled out the possibility that $xz = xy^2$, etc.

We want to bound $m = |y^G|$. Because $x^2 = 1$ and $y^x = y^{-1}$, we get the following multiplication table for x^G times x^G.

	x	xy	xy^{-1}	xz	xh
x	1	y	y^{-1}	z	h
xy	y^{-1}	1	y	$y^{-1}z$	$y^{-1}h$
xy^{-1}	y	y^2	1	yz	yh
xz	z^x	z^xy	z^xy^{-1}	1	z^xh
xh	h^x	h^xy	h^xy^{-1}	h^xz	1

We note that y occurs as an entry of this table at least 3 times, so that each element of y^G must also occur at least 3 times as an entry of the table. So $3m \leq 20$ and $m \leq 6$.

By Lemma 4.4 we have $|C_x \cup C_{g_1}| \leq 5$. $|L_x| \leq 4$, $|L_{g_1}| \leq 4$, $|R_x| = |R_{g_1}| = 0$, so that $|C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}| \leq 13$.

Also, by Lemma 4.4 we have $|C_y| \leq m-2$, $|L_y| \leq m-2$, $|R_x| \leq n/m$. And $m \in \{2,4,6\}$ so that $|C_y \cup L_y \cup R_y| \leq m-2+m-2+n/m \leq 4+4+n/2 = n/2+8$.

Thus $|C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}| + |C_y \cup L_y \cup R_y| \leq 13+n/2+8 = n/2+21$ and $2 \cdot 21 < 256$, so xy is determined by Corollary 4.9. □

Case F: $m = 2$**,** $k \leq n/4$**.** Because $y^x = y^{-1}$, in this case we must have $y^G = \{y, y^{-1}\}$. We know $[G : C_G(y)] = 2$ and so $N = C_G(y)$ is normal in G, and $x \notin N$ because x and y do not commute. So that $x^G \cap N = \emptyset$.

For every $g \in N \setminus y^G$, we know $g \not\sim y$ and $yg = gy$, so that yg is determined. Also, since $y^G = \{y, y^{-1}\}$, we know yg for any $g \in y^G$. So the NDICT determines yg for any $g \in N$.

Fix $g \in N$. Then $g \notin x^G$ and $g \not\sim gg_1$, so $g \notin R_{g_1}$. Similarly, because $yg \in N$, we have $yg \notin x^G$ and $yg \notin R_x$. This is true for any $g \in N$. From Lemma 4.4(b) we have $|L_x| \leq k-1$ and $|L_{g_1}| \leq k-1$. So $|N \setminus (C_x \cup C_{g_1} \cup L_x \cup L_{g_1} \cup R_x \cup R_{g_1})| \geq n/2 - 2(k-1) \geq n/2 - 2(n/4) + 2 = 2$. So there exists $g \in N$ for which yg, $x(yg)$ and g_1g can be calculated, and it follows from Lemma 4.7 that xy is determined in this case. □

Case G: $m = 4$**,** $6 \leq k \leq n/6$**.** This case follows from a size argument. We have

$|C_x \cup C_{g_1}| \leq k$, $|L_x| \leq k-1$, $|L_{g_1}| \leq k-1$, $|R_x| \leq n/k$, $|R_{g_1}| \leq n/k$ and $|C_y| \leq 2$, $|L_y| \leq 2$, $|R_y| \leq n/4$. This gives us the bound we need:

$$\begin{aligned}|C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}| + |C_y \cup L_y \cup R_y| &\leq k + 2(k + n/k) - 2 + 4 + n/4 \\ &\leq n/6 + 2(6 + n/6) + 2 + n/4 \\ &= 3n/4 + 14.\end{aligned}$$

And $4 \cdot 14 < 256$, so xy is determined by Corollary 4.9.

Case H: $|x^G| = \frac{|G|}{2}$. In this case $G = N \rtimes C_2$ is generalized dihedral with N abelian of odd order. As shown in Theorem 3.23, G is determined by the NDICT in this case.

Case J: $m = \frac{n}{5}$, $6 \leq k \leq \frac{n}{6}$. When $m = n/5$ then $o(y) = 5$. We have $|C_x \cup C_{g_1}| \leq k$, $|L_x| \leq k-1$, $|L_{g_1}| \leq k-1$, $|R_x| \leq n/k$, $|R_{g_1}| \leq n/k$ and $|C_y| \leq n/5 - 1$, $|L_y| = 0$, $|R_y| \leq 5$. This gives us the bound:

$$\begin{aligned}|C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}| + |C_y \cup L_y \cup R_y| &\leq k + 2(k + n/k) - 2 + 4 + n/5 \\ &\leq n/6 + 2(6 + n/6) + 2 + n/5 \\ &= \frac{7n}{10} + 14.\end{aligned}$$

And $(10/3) \cdot 14 < 256$, so xy is determined by Corollary 4.9. □

Case K: $m = \frac{n}{4}$, $6 \leq k \leq \frac{n}{6}$. For this case, we consider separately the subcases $k = 6$, $7 \leq k \leq n/7$, $k = n/6$.

Subcase: $k = 6$ follows from a pure size argument. We have $|C_x \cup C_{g_1}| \leq 6$, $|L_x| \leq 5$, $|L_{g_1}| \leq 5$, $|R_x| \leq n/6$, $|R_{g_1}| \leq n/6$. Also, we have $|C_y| \leq n/4 - 1$, $|L_y| \leq n/4 - 1$, $|R_y| \leq 4$. This gives us the bound:

$$\begin{aligned}|C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}| + |C_y \cup L_y \cup R_y| &\leq 16 + \frac{n}{3} + \frac{n}{2} + 2 \\ &= \frac{5n}{6} + 18.\end{aligned}$$

And $6 \cdot 18 < 256$, so xy is determined by Corollary 4.9.

Subcase: $7 \leq k \leq n/7$ also follows from a pure size argument. We have $|C_x \cup C_{g_1}| \leq k$, $|L_x| \leq k-1$, $|L_{g_1}| \leq k-1$, $|R_x| \leq n/k$, $|R_{g_1}| \leq n/k$ and $|C_y| \leq n/4-2$, $|L_y| \leq n/4-2$, $|R_y| \leq 4$ by Lemma 4.4. This gives us the bound:

$$\begin{aligned} |C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}| + |C_y \cup L_y \cup R_y| &\leq k + 2(k + n/k) - 2 + 2 + n/2 \\ &\leq n/7 + 2(7 + n/7) + n/2 \\ &= \frac{13n}{14} + 12. \end{aligned}$$

And $12 \cdot 14 = 256 < n$, so xy is determined.

Subcase: $k = n/6$, however, requires a little finesse, and then a size argument. The finesse involves showing that x must be an involution. Because $|C_G(x)| = 6$, we know x has order 2, 3, or 6, but (x, y) is an ND pair, so $o(x) \neq 3$. Suppose, by way of contradiction, that $o(x) = 6$. Then $y^{x^2} = (y^x)^x = (y^{-1})^x = y$ so $x^2 \in C_G(y)$. But $o(x^2) = 3$ and $|C_G(y)| = 4$, a contradiction. It follows that x must be an involution.

So, we have $|C_x \cup C_{g_1}| \leq n/6$, $|L_x| \leq n/6-1$, $|L_{g_1}| \leq n/6-1$, $|R_x| = |R_{g_1}| = 0$ and $|C_y| \leq n/4-2$, $|L_y| \leq n/4-2$, $|R_y| \leq 4$. This gives us the bound:

$$\begin{aligned} |C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}| + |C_y \cup L_y \cup R_y| &\leq 3(n/6) - 2 + 2(n/4) - 4 + 4 \\ &= n - 2 < n. \end{aligned}$$

Thus xy is determined by Lemma 4.8. $\square$

Case L: $m = \frac{n}{3}$**,** $6 \leq k \leq \frac{n}{6}$**.** This case also follows from a size argument. When $m = n/3$ then $o(y) = 3$. We have $|C_x \cup C_{g_1}| \leq k$, $|L_x| \leq k-1$, $|L_{g_1}| \leq k-1$, $|R_x| \leq n/k$, $|R_{g_1}| \leq n/k$

and $|C_y| \leq n/3 - 1$, $|L_y| = 0$, $|R_y| \leq 3$. This gives us the bound:

$$\begin{aligned}|C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}| + |C_y \cup L_y \cup R_y| &\leq k + 2(k + n/k) - 2 + 4 + n/3 \\ &\leq n/6 + 2(6 + n/6) + 2 + n/3 \\ &= \frac{5n}{6} + 12.\end{aligned}$$

And $6 \cdot 12 < 256$, so xy is determined by Corollary 4.9. □

4.6 The final $k = \frac{n}{4}$ cases

In this section, we conclude the proof of Theorem 4.1. We show that when $k = n/4$ and m is one of $4, 6, 8, n/9, n/8, n/7, n/6, n/5, n/4, n/3$ or $10 \leq m \leq n/10$, then the product xy is determined for the ND pair (x, y).

Throughout this section, (x, y) is an ND pair with $k = |x^G|$, $m = |y^G|$. Also, $g_1 \in \{xy, yx\}$, so $g_1^{-1} \in \{yx^{-1}, y^{-1}x^{-1}\}$. We let $t = o(y)$ and define $\alpha \in \{1, -1\}$ by $g_1^{-1} = y^{\alpha}x^{-1}$. All cases will use the fact that when $k = n/4$, then $o(x) \in \{2, 4\}$, and $C_G(x)$ is a group of order 4, so it is either cyclic of order 4 or $C_G(x) = \{1, x, xs, s\}$ for some involution $s \in G$.

Finite groups with a self centralizing subgroup of order 4 have been classified by Wong [28] who built on work by Suzuki [23], and the groups that we are considering in this section definitely fall in this category. But, in the 'algorithmic' spirit of this proof, I am going to, when possible, show the existence of the element that allows us to fill in the entry of the table without referencing these results.

In this section we are mainly working from the left, but we pay closer attention to C_x, L_x, R_x, etc. in order to find better size estimates for the set $x(C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}) \cup (C_y \cup L_y \cup R_y)$. In particular, we get:

Proposition 4.11. *Let (x, y) be an ND pair with $|x^G| = n/4$, $|y^G| = m$, and $t = o(y)$. Then for $s = |x(C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}) \cup (C_y \cup L_y \cup R_y)|$ we have the following bounds:*

(i) If $o(x) = 2$ and $o(y)$ is odd, then $s \leq \frac{3n}{4} + m + \frac{n}{m} - 2t - 2$.

(ii) If $o(x) = 2$ and $o(y)$ is even, then $s \leq \frac{3n}{4} + 2m + \frac{n}{m} - 2t - 4$.

(iii) If $o(x) = 4$ and $o(y)$ is odd, then $s \leq \frac{3n}{4} + m + \frac{n}{m} - 4t + 6$.

(iv) If $o(x) = 4$ and $o(y) = 4$, then $s \leq \frac{3n}{4} + 2m + \frac{n}{m} - 12$.

(v) If $o(x) = 4$ and $o(y) > 4$ is even, then $s \leq \frac{3n}{4} + 2m + \frac{n}{m} - 2t + 4$.

Proof. Let G be a finite group and (x, y) an ND pair of elements in G with $|x^G| = n/4$, $|y^G| = m$, and $t = o(y)$, we let $F = C_x \cup C_{g_1} \cup R_x \cup R_{g_1} \cup L_x \cup L_{g_1}$ and $E = C_y \cup R_y \cup L_y$. With this notation, we have $s = |xF \cup E|$.

In this proof we use repeatedly the fact that when A, B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.

Case 1: $o(x) = 2$ and $o(y)$ is odd. We write

$$x^G = \{x, xy, xy^2, \ldots, xy^{t-1}, xh_1, xh_2, \ldots, xh_q\},$$

where $q = k - t$ and the xy^i, xh_j are all distinct, so that

$$x^{-1}x^G = \{1, y, y^2, \ldots, y^{t-1}, h_1, h_2, \ldots, h_q\} \text{ and}$$

$$g_1^{-1}x^G = \{1, y, y^2, \ldots, y^{t-1}, y^\alpha h_1, y^\alpha h_2, \ldots, y^\alpha h_q\}.$$

In this case, because $o(x) = 2$, we know $R_x = R_{g_1} = \emptyset$ by Lemma 4.4(d). Also, $C_x \cup C_{g_1} \subseteq x^G$, by Lemma 4.4(f) and $L_x \subseteq x^{-1}x^G \setminus \{1\}$, $L_{g_1} \subseteq g_1^{-1}x^G \setminus \{1\}$ by Lemma 4.4(b). We note that $\{y, y^2, \ldots, y^{t-1}\} \subseteq (x^{-1}x^G \setminus \{1\}) \cap (g_1^{-1}x^G \setminus \{1\})$, so $|(x^{-1}x^G \setminus \{1\}) \cup (g_1^{-1}x^G \setminus \{1\})| \leq (n/4 - 1) + (n/4 - 1) - (t - 1) = n/2 - t - 1$.

We let $T = x^G \cup (x^{-1}x^G \setminus \{1\}) \cup (g_1^{-1}x^G \setminus \{1\})$. Then $F \subseteq T$ and $\{y, y^2, \ldots, y^{t-1}\} \subseteq T$. Also,

$$|T| \leq |x^G| + |(x^{-1}x^G \setminus \{1\}) \cup (g_1^{-1}x^G \setminus \{1\})| \leq \frac{n}{4} + (\frac{n}{2} - t - 1) = \frac{3n}{4} - t - 1.$$

It then follows that $xF \subseteq xT$, $\{xy, xy^2, \ldots, xy^{t-1}\} \subseteq xT$, and $|xT| = |T| \leq 3n/4 - t - 1$.

Now we consider $E = C_y \cup L_y \cup R_y$. By Lemma 4.4(a), $C_y \subseteq y^G \setminus \{y, y^{-1}\}$ and, because y has odd order $L_y = \emptyset$, by Lemma 4.4(e). Also, $R_y \subseteq C_G(y)x$ by Lemma 4.4(i). Because $\langle y \rangle \in C_G(y)$, we have $\{xy, xy^2, \ldots, xy^{t-1}\} \subseteq C_G(y)x$. Let $S = (y^G \setminus \{y, y^{-1}\}) \cup C_G(y)x$. Then $E \subseteq S$, $\{xy, xy^2, \ldots, xy^{t-1}\} \subseteq S$, and $|S| = |(y^G \setminus \{y, y^{-1}\}) \cup C_G(y)x| \leq m - 2 + n/m$.

Finally, consider the set $xT \cup S$, which certainly contains $xF \cup E$. Because

$$\{xy, xy^2, \ldots, xy^{t-1}\} \subseteq xT \cap S,$$

it follows that

$$|xT \cup S| \leq (\frac{3n}{4} - t - 1) + (m - 2 + \frac{n}{m}) - (t - 1) = \frac{3n}{4} + \frac{n}{m} - m - 2t - 2.$$

And because $xF \cup E \subseteq xT \cup S$ this shows that $s = |xF \cup E| \leq 3n/4 + n/m + m - 2t - 2$ in this case. □

Case 2: $o(x) = 2$ and $o(y)$ is even. In this case we have $o(y) = t = 2r$ for some positive integer r. Because $(y^r)^x = y^{-r} = y^r$ we know that y^r and x commute and (x, y^r) is not an ND pair. We know $x \neq y^r$, because x and y do not commute.

In this case, we still write

$$x^G = \{x, xy, xy^2, \ldots, xy^{t-1}, xh_1, xh_2, \ldots, xh_q\},$$

for $q = k - t$, where the elements xy^i, xh_j are all distinct, so that

$$x^{-1}x^G = \{1, y, y^2, \ldots, y^{t-1}, h_1, h_2, \ldots, h_q\} \text{ and}$$
$$g_1^{-1}x^G = \{1, y, y^2, \ldots, y^{t-1}, y^\alpha h_1, y^\alpha h_2, \ldots, y^\alpha h_q\}.$$

Because $o(x) = 2$, we have $R_x = R_{g_1} = \emptyset$ by Lemma 4.4(d), $C_x \cup C_{g_1} \subseteq x^G$ by Lemma 4.4(f), and $L_x \subseteq x^{-1}x^G \setminus \{1, y^r\}$, $L_{g_1} \subseteq g_1^{-1}x^G \setminus \{1, y^r\}$ by Lemma 4.4(g). Also

$\{y, y^2, \ldots, y^{t-1}\} \setminus \{y^r\} \subseteq (x^{-1}x^G \setminus \{1, y^r\}) \cap (g_1^{-1}x^G \setminus \{1, y^r\})$, so that

$$|(x^{-1}x^G \setminus \{1, y^r\}) \cup (g_1^{-1}x^G \setminus \{1, y^r\})| \leq (n/4 - 2) + (n/4 - 2) - (t - 2) = n/2 - t - 2.$$

In this case, we let $T = x^G \cup (x^{-1}x^G \setminus \{1, y^r\}) \cup (g_1^{-1}x^G \setminus \{1, y^r\})$ and have $F \subseteq T$, $\{y, y^2, \ldots, y^{t-1}\} \setminus \{y^r\} \subseteq T$, so that

$$|T| \leq |x^G| + |(x^{-1}x^G \setminus \{1\}) \cup (g_1^{-1}x^G \setminus \{1\})| \leq n/4 + (n/4 - t - 2) = 3n/4 - t - 2.$$

We also have $xF \subseteq xT$, $\{xy, xy^2, \ldots, xy^{t-1}\} \setminus \{xy^r\} \subseteq xT$ and $|xT| = |T| \leq 3n/4 - t - 2$.

Now we consider $E = C_y \cup L_y \cup R_y$. By Lemma 4.4 (a) we have $C_y \subseteq y^G \setminus \{y, y^{-1}\}$ and, because y has even order, from Lemma 4.4(b) we know that $L_y \subseteq y^{-1}y^G \setminus \{1, y^2\}$. We set $S = \big(y^G \setminus \{y, y^{-1}\}\big) \cup \big(y^{-1}y^G \setminus \{1, y^2\}\big) \cup C_G(y)x$, so that $E \subseteq S$ and we have

$$|S| \leq (m - 2) + (m - 2) + n/m = n/m + 2m - 4.$$

Finally, we consider the set $xT \cup S$. Because $\{xy, xy^2, \ldots, xy^{t-1}\} \setminus \{y^r\} \subseteq xT \cap S$, we know

$$|xT \cup S| \leq (\frac{3n}{4} - t - 2) + (\frac{n}{m} + 2m - 4) - (t - 2) = \frac{3n}{4} + \frac{n}{m} + 2m - 2t - 4.$$

And because $xF \cup E \subseteq xT \cup S$ we have $s \leq 3n/4 + n/m + 2m - 2t - 4$ in this case. $\square$

Case 3: $o(x) = 4$ and $o(y)$ is odd. As a consequence of (x, y) being an ND pair, we have $x^{-1} = x^3 \in x^G$. In fact for all i, $0 \leq i \leq t - 1$, we have $(x^3y^i)(xy^i) = x^3(xy^{-i})y^i = x^4 = 1$, so $x^3y^i = (xy^i)^{-1} \in x^G$. Because $g_1 \in \{xy, yx\}$, and $(xy)^2 = (yx)^2 = x^2$, we know that $g_1^2 = x^2$. Also $y^{x^2} = (y^x)^x = (y^{-1})^x = y$ so that $x^2 \in C_G(y)$. Thus we can write

$$x^G = \{x, xy, xy^2, \ldots, xy^{t-1}, x^3, x^3y, \ldots, x^3y^{t-1}, xh_1, xh_2, \ldots, xh_q\},$$

for $q = k - 2t$, where the elements xy^i, x^3y^i, and xh_j are all distinct, so that

$$x^{-1}x^G = \{1, y, y^2, \ldots, y^{t-1}, x^2, x^2y, \ldots, x^2y^{t-1}, h_1, h_2, \ldots, h_q\} \text{ and}$$
$$g_1^{-1}x^G = \{1, y, y^2, \ldots, y^{t-1}, x^2, x^2y, \ldots, x^2y^{t-1}, y^\alpha h_1, y^\alpha h_2, \ldots, y^\alpha h_q\}. \tag{4.1}$$

We have $L_x \subseteq x^{-1}x^G \setminus \{1, x^2\}$ and $L_{g_1} \subseteq g_1^{-1}x^G \setminus \{1, x^2\}$ by Lemma 4.4(b). From Lemma 4.4 (c) we know that $|R_x| \leq 4$, $|R_{g_1}| \leq 4$. And, as always, $C_x \cup C_{g_1} \subseteq x^G$, by Lemma 4.4(f). We note that $\{y, y^2, \ldots, y^{t-1}, x^2y, x^2y^2, \ldots, x^2y^{t-1}\} \subseteq (x^{-1}x^G \setminus \{1, x^2\}) \cap (g_1^{-1}x^G \setminus \{1, x^2\})$, so that $|(x^{-1}x^G \setminus \{1, x^2\}) \cup (g_1^{-2}x^G \setminus \{1\})| \leq (n/4 - 2) + (n/4 - 2) - (2t - 2) = n/2 - 2t - 2$. Let $T = x^G \cup (x^{-1}x^G \setminus \{1, x^2\}) \cup (g_1^{-1}x^G \setminus \{1, x^2\}) \cup R_x \cup R_{g_1}$.

Then we have $F \subseteq T$ and

$$\begin{aligned} |T| &\leq |x^G| + |(x^{-1}x^G \setminus \{1, x^2\}) \cup (g_1^{-1}x^G \setminus \{1, x^2\})| + |R_x| + |R_{g_1}| \\ &\leq n/4 + (n/2 - 2t - 2) + 4 + 4 = 3n/4 - 2t + 6. \end{aligned}$$

Also, we have $xF \subseteq xT$, $\{xy, xy^2, \ldots, xy^{t-1}, x^3y, x^3y^2, \ldots, x^3y^{t-1}\} \subseteq xT$, and $|xT| \leq 3n/4 - 2t + 6$.

Now we consider $E = C_y \cup L_y \cup R_y$. By Lemma 4.4(a) we have $C_y \subseteq y^G \setminus \{y, y^{-1}\}$ and, because y has odd order we have $L_y = \emptyset$ by Lemma 4.4(e). Also, $R_y \subseteq C_G(y)x$ by Lemma 4.4(i). Let $S = (y^G \setminus \{y, y^{-1}\}) \cup C_G(y)x$. Because $\langle y, x^2 \rangle \in C_G(y)$, we have $\{xy, xy^2, \ldots, xy^{t-1}, x^3y, x^3y^2, \ldots, x^3y^{t-1}\} \subseteq C_G(y)x \subseteq S$. Also, we have $E \subseteq S$, and $|S| \leq m - 2n + n/m$.

Finally, consider the set $xT \cup S$. Because $\{xy, xy^2, \ldots, xy^{t-1}, x^2y, x^2y^2, \ldots, x^2y^{t-1}\} \subseteq xT \cap S$, we have

$$|xT \cup S| \leq (\frac{3n}{4} - 2t + 6) + (m - 2 + \frac{n}{m}) - (2t - 2) = \frac{3n}{4} + \frac{n}{m} + m - 4t + 6.$$

And because $xF \cup E \subseteq xT \cup S$ we have $s \leq 3n/4 + n/m + m - 4t + 6$ in this case. $\square$

Case 4: $o(x) = 4$ and $o(y) = 4$. This case must be considered separately, because when $o(x) = o(y) = 4$, we have $(y^2)^x = y^{-2} = y^2$, so that $y^2 \in C_G(x)$ and hence $y^2 = x^2$. Also, we note that $x^y = xy^2 = x^3 = x^{-1}$ and $(xy)^y = x^{-1}y = xx^2y = xy^3 = yx = (xy)^{-1}$. We will use this fact later when determining R_x and R_{g_1}. We can write

$$x^G = \{x, xy, x^3, x^3y, xh_1, xh_2, \ldots, xh_q\},$$

for $q = k-4$, where the elements $x, xy, x^3 = xy^2, x^3y = xy^3$, and xh_j are all distinct, so that

$$\begin{aligned} x^{-1}x^G &= \{1, y, y^2, y^3, h_1, h_2, \ldots, h_q\} \text{ and} \\ g_1^{-1}x^G &= \{1, y, y^2, y^3, y^\alpha h_1, y^\alpha h_2, \ldots, y^\alpha h_q\}. \end{aligned} \tag{4.2}$$

Noting that $x^2 = y^2$, we see that $L_x \subseteq x^{-1}x^G \setminus \{1, y^2\}$ and $L_{g_1} \subseteq g_1^{-1}x^G \setminus \{1, y^2\}$ by Lemma 4.4(b) and also we have $C_x \cup C_{g_1} \subseteq x^G$, by Lemma 4.4(f). From Lemma 4.4 (c) we know that $R_x \subseteq C_G(x)y$, $R_{g_1} \subseteq C_G(g_1)y$. But if $g \in C_G(x)y = \{y, xy, x^2y, x^3y\}$, then $g \not\sim gx$, and if $g \in C_G(g_1)y = \{y, xy^2, x^2y, x\}$, then $g \not\sim gg_1$. So we have $R_x = R_{g_1} = \emptyset$. Let $T = \left(x^{-1}x^G \cup g_1^{-1}x^G\right) \setminus \{1, x^2\} \cup x^G$. By inspection, we see that $R_x \cup R_{g_1} \subseteq T$, so that in fact $F \subseteq T$.

Because $\{y, y^3\} \subseteq (x^{-1}x^G \setminus \{1, x^2\}) \cap (g_1^{-1}x^G \setminus \{1, x^2\})$, we have $|(x^{-1}x^G \setminus \{1, x^2\}) \cup (g_1^{-1}x^G \setminus \{1, x^2\})| \leq (n/4 - 2) + (n/4 - 2) - 2 = n/2 - 6$, and

$$\begin{aligned} |T| &\leq |x^G| + |(x^{-1}x^G \setminus \{1, x^2\}) \cup (g_1^{-1}x^G \setminus \{1, x^2\})| \\ &\leq n/4 + (n/2 - 6) = 3n/4 - 6. \end{aligned}$$

Also, $xF \subseteq xT$, $\{xy, xy^3\} \subseteq xT$, and $|xT| \leq 3n/4 - 6$.

Now we consider $E = C_y \cup L_y \cup R_y$. By Lemma 4.4(a) we have $C_y \subseteq y^G \setminus \{y, y^{-1}\}$ and, by Lemma 4.4 (b) we have $L_y \subseteq y^{-1}y^G \setminus \{1, y^2\}$. Also, $R_y \subseteq C_G(y)x$ by Lemma 4.4(i). Because $\langle y \rangle \in C_G(y)$, we have $\{xy, xy^3\} \subseteq C_G(y)x$. Let $S = (y^G \setminus \{y, y^{-1}\}) \cup y^{-1}y^G \setminus \{1, y^2\} \cup C_G(y)x$, so that $E \subseteq S$, $\{xy, xy^3\} \subseteq S$, and $|S| \le (m-2) + (m-2) + n/m = n/m + 2m - 4$.

Finally, consider the set $xT \cup S$. Because $\{xy, xy^3\} \subseteq xT \cap S$, we have

$$|xT \cup S| \le (\frac{3n}{4} - 6) + (2m + \frac{n}{m} - 4) - 2 = \frac{3n}{4} + \frac{n}{m} + 2m - 12.$$

And because $xF \cup E \subseteq T \cup S$ we have $s \le 3n/4 + n/m + 2m - 12$ in this case. □

Case 5: $o(x) = 4$ and $o(y) > 4$ is even. We let $o(y) = t = 2r$ for some non-negative integer r. We see that $(y^r)^x = y^{-r} = y^r$, so that y^r and x commute. Because $y^r \in C_G(x) = \langle x \rangle$, we have $y^r = x^2$ and $x^{-1} = xx^2 = xy^r$. Thus, in this case we can write

$$x^G = \{x, xy, xy^2, \ldots, xy^{t-1}, xh_1, xh_2, \ldots, xh_q\},$$

for $q = k - t$, where the elements xy^i, xh_j are all distinct, so that

$$x^{-1}x^G = \{1, y, y^2, \ldots, y^{t-1}, h_1, h_2, \ldots, h_q\} \text{ and}$$
$$g_1^{-1}x^G = \{1, y, y^2, \ldots, y^{t-1}, y^\alpha h_1, y^\alpha h_2, \ldots, y^\alpha h_q\}.$$

Again in this case, we have $|R_x| \le 4$, $|R_{g_1}| \le 4$ by Lemma 4.4(c), $C_x \cup C_{g_1} \subseteq x^G$, by Lemma 4.4(f), and $L_x \subseteq x^{-1}x^G \setminus \{1, y^r\}$, $L_{g_1} \subseteq g_1^{-1}x^G \setminus \{1, y^r\}$ by Lemma 4.4(b), where we have $y^r = x^2$. Also $\{y, y^2, \ldots, y^{t-1}\} \setminus \{y^r\} \subseteq (x^{-1}x^G \setminus \{1, y^r\}) \cap (g_1^{-1}x^G \setminus \{1, y^r\})$, so that $|(x^{-1}x^G \setminus \{1, y^r\}) \cup (g_1^{-1}x^G \setminus \{1, y^r\})| \le (n/4 - 2) + (n/4 - 2) - (t-2) = n/2 - t - 2$.

So in this case we let $T = x^G \cup (x^{-1}x^G \setminus \{1, y^r\}) \cup (g_1^{-1}x^G \setminus \{1, y^r\})$ and have $F \subseteq T$, where

$$|T| \le |x^G| + |(x^{-1}x^G \cup g_1^{-1}x^G \setminus \{1, y^r\})| + 8 \le \frac{n}{4} + (\frac{n}{4} - t - 2) + 8 = \frac{3n}{4} - t + 6.$$

Also, we note that $xF \subseteq xT$, $\{xy, xy^2, \ldots, xy^{t-1}\} \setminus \{xy^r\} \subseteq xT$ and $|xT| = |T| \le 3n/4 - t + 6$.

Now we consider $E = C_y \cup L_y \cup R_y$. By Lemma 4.4 (a) we have $C_y \subseteq y^G \setminus \{y, y^{-1}\}$ and, because y has even order from Lemma 4.4(b) it follows that $L_y \subseteq y^{-1}y^G \setminus \{1, y^2\}$, so that $|L_y| \leq m - 2$. From Lemma 4.4 (c) we have $R_y \subseteq C_G(y)x$. We set $S = y^G \setminus \{y, y^{-1}\} \cup y^{-1}y^G \setminus \{1, y^2\} \cup C_G(y)x$, so that $E \subseteq S$ and $|S| \leq (m-2) + (m-2) + n/m = n/m + 2m - 4$.

Finally, consider the set $xT \cup S$. We have $\{xy, xy^2, \ldots, xy^{t-1}\} \subseteq C_G(y)x \subseteq S$, so that $\{xy, xy^2, \ldots, xy^{t-1}\} \setminus \{y^r\} \subseteq xT \cap S$, from which it follows that

$$|xT \cup S| \leq (\frac{3n}{4} - t + 6) + (\frac{n}{m} + 2m - 4) - (t - 2) = \frac{3n}{4} + \frac{n}{m} + 2m - 2t + 4.$$

And because $xF \cup E \subseteq xT \cup S$ we have $s \leq 3n/4 + n/m + 2m - 2t + 4$ in this case.

This concludes consideration of all cases. $\square$

We are now ready to prove Case M, which we do in three subcases.

Case M1: $k = n/4, m = 4, 6, 8, 10 \leq m \leq n/10$, excluding $m = 4$, $o(y) = 6$, $o(x) = 4$. For the cases m=4,6,8 and $10 \leq m \leq n/10$ we cannot in general say anything about the order of y. We are going to work these cases in parallel and record our work in a table. The rows of the table are labelled in the first column by cases, based on the orders of x and y. In the second column, we use the facts that when $o(y)$ is odd, then $t \geq 3$ and when $o(y)$ is even, then $t \geq 4$, and the results of Proposition 4.11 to get bounds for $s = |x\,[C_x \cup L_x \cup R_x \cup C_{g_1} \cup L_{g_1} \cup R_{g_1}] \cup (C_y \cup L_y \cup R_y)|$ in each of those cases. These bounds are listed in the second column of the table. In the third, fourth, and fifth columns, we plug in $m = 4, 6, 8$ respectively to get the bounds for s corresponding to each m. We apply Lemma 4.10, which tells us that if $10 \leq m \leq n/10$, then $m + n/m \leq 10 + n/10$, to get the bounds on s which are listed in the last column.

Case	General	$m=4$	$m=6$	$m=8$	$10 \leq m \leq \frac{n}{10}$
$o(x)=2$, $o(y)$ odd	$\frac{3n}{4}+\frac{n}{m}+m-8$	$n-4$	$\frac{11n}{12}-2$	$\frac{7n}{8}$	$\frac{17n}{20}+2$
$o(x)=2$, $o(y)$ even	$\frac{3n}{4}+\frac{n}{m}+2m-12$	$n-4$	$\frac{11n}{12}$	$\frac{7n}{8}+4$	$\frac{19n}{20}-2$
$o(x)=4$, $o(y)$ odd	$\frac{3n}{4}+\frac{n}{m}+m-6$	$n-2$	$\frac{11n}{12}$	$\frac{7n}{8}+2$	$\frac{17n}{20}+4$
$o(x)=4$, $o(y)=4$	$\frac{3n}{4}+\frac{n}{m}+2m-11$	$n-3$	$\frac{11n}{12}+1$	$\frac{7n}{8}+5$	$\frac{19n}{20}-1$
$o(x)=4$, $o(y)=6$	$\frac{3n}{4}+\frac{n}{m}+2m-8$	$\mathbf{n}$	$\frac{11n}{12}+4$	$\frac{7n}{8}+8$	$\frac{19n}{20}+2$
$o(x)=4$, $o(y)\geq 6$ even	$\frac{3n}{4}+\frac{n}{m}+2m-10$	$n-2$	$\frac{11n}{12}+2$	$\frac{7n}{8}+6$	$\frac{19n}{20}$

Using this table, we see by Lemma 4.8 that when (x,y) is an ND pair in a finite group G with $|G|>64$, such that $|x^G|=4$, $|y^G|=m$, and $m\in\{4,6,8,10,\ldots,n/10\}$, then xy is determined by the NDICT of G by Lemma 4.8 unless $o(x)=4$, $o(y)=6$ and $m=4$. We consider the case $o(x)=4$, $o(y)=6$ and $m=4$ later in this section.

Case M2: $k=n/4, m=n/9,n/8,n/7,n/6,n/5$, excluding $m=n/6$, $o(y)=6$. In these cases, we do know something about the order of y. When $m=n/9$, then y has order 3 or 4. When $m=n/8$, then y has order 8 or 4. When $m=n/7$ then $o(y)=7$. When $m=n/6$ then y has order 3 or 6. And finally, when $m=n/5$, then $o(y)=5$. We are going to build another table, with rows corresponding to different cases based on the orders of x and y, and columns corresponding to values of m. For some values of $o(y)$, it will be impossible to have certain m values, and when this occurs, we will put an N/A in the table. For example, when $o(y)=4$, then it is impossible to have $m=n/5$. Also, in this table, when $o(y)$ is known, we use that value for t, and when $o(y)$ is odd, we use $t\geq 3$. From Proposition 4.11 we get the following bounds on s.

Case	General	$m = \frac{n}{9}$	$m = \frac{n}{8}$	$m = \frac{n}{7}$	$m = \frac{n}{6}$	$m = \frac{n}{5}$
$o(x) = 2$, $o(y)$ odd	$\frac{3n}{4} + \frac{n}{m} + m - 8$	$\frac{31n}{36} + 1$	N/A	$\frac{25n}{28} - 1$	$\frac{11n}{12} - 2$	$\frac{19n}{20} - 3$
$o(x) = 2$, $o(y) = 4$	$\frac{3n}{4} + \frac{n}{m} + 2m - 12$	N/A	n-4	N/A	N/A	N/A
$o(x) = 2$, $o(y) = 6$	$\frac{3n}{4} + \frac{n}{m} + 2m - 16$	N/A	n-8	N/A	$\mathbf{\frac{13n}{12} - 10}$	N/A
$o(x) = 2$, $o(y) = 8$	$\frac{3n}{4} + \frac{n}{m} + 2m - 20$	N/A	n-12	N/A	N/A	N/A
$o(x) = 4$, $o(y)$ odd	$\frac{3n}{4} + \frac{n}{m} + m - 6$	$\frac{31n}{36} + 4$	N/A	$\frac{25n}{28} + 1$	$\frac{11n}{12}$	$\frac{19n}{20}$
$o(x) = 4$, $o(y) = 4$	$\frac{3n}{4} + \frac{n}{m} + 2m - 12$	N/A	$n - 4$	N/A	N/A	N/A
$o(x) = 4$, $o(y) = 6$	$\frac{3n}{4} + \frac{n}{m} + 2m - 8$	N/A	N/A	N/A	$\mathbf{\frac{13n}{12}}$	N/A
$o(x) = 4$, $o(y) = 8$	$\frac{3n}{4} + \frac{n}{m} + 2m - 12$	N/A	$n - 4$	N/A	N/A	N/A

Using this table, we see by Lemma 4.8 that when (x, y) is an ND pair in a finite group G with $|G| > 64$, such that $|x^G| = 4$, $|y^G| = m$, $m \in \{n/9, n/8, n/7, n/6, n/5\}$, then xy is determined by the NDICT of G unless $o(y) = 6$ and $m = n/6$.

Case M3: $k = n/4, m = 4$, and $o(y) = 6, o(x) = 4$ or $k = n/4, m = n/6$, and $o(y) = 6$. We consider the subcase $m = 4$, and then the subcase $m = n/6$.

First, we suppose $m = 4$, $o(x) = 4$, $o(y) = 6$. We know that y^2 is determined by the NDICT of G. Next we show that $g_1 y^2$ is determined by the NDICT.

Because $o(y^2) = 3$ is odd, and $o(g_1) = o(x) = 4$, we know both that $y^2 \not\sim g_1$ and that (y^2, g_1) cannot be an ND pair. So the only way that $g_1 y^2$ can fail to be determined is if (g_1, y^2) is an ND pair. Suppose (g_1, y^2) is an ND pair. Because $C_G(y) \subseteq C_G(y^2)$, and $|C_G(y)| = n/4$, we know that $|C_G(y^2)| \geq n/4$ and so $|(y^2)^G| \subseteq \{1, 2, 4\}$. By assumption, (g_1, y^2) is an ND pair so y^2 is not central. If $|(y^2)^G| = 2$, then the ND pair (g_1, y^2) satisfies $|g_1^G| = n/4$, $|(y^2)^G| = 2$, and it follows from case F of this proof that $g_1 y^2$ is determined by the NDICT. If $|(y^2)^G| = 4$, then by Case M1 applied to the ND pair (g_1, y^2), we know that $g_1 y^2$ is determined by the NDICT of G, so that in this case $g_1 y^2$ is also determined by the NDICT of G, so that $g_1 y^2$ is alwayss determined by the NDICT.

We recall that $g_1 \in \{xy, yx = xy^{-1}\}$, and that this set is determined by the NDICT of G. If $g_1 = xy^{-1}$, then $g_1 y^2 = xy^{-1} y^2 = xy = g_2$. If $g_1 = xy$ then $g_1 y^2 = xyy^2 = xy^3 \neq g_2 = xy^{-1}$.

Thus, when g_1y^2 is known, we know whether $g_1 = xy$ or $g_1 = xy^{-1}$, so that xy is determined by the NDICT of G in this case.

Now, we consider the case where $m = n/6, o(y) = 6, o(x) \in \{2, 4\}$. Again we show that g_1y^2 is determined by the NDICT of G. Then it will follow from the above argument that xy is determined. Because $o(y^2) = 3$, we know $y^2 \not\sim g_1$ and (y^2, g_1) is not an ND pair. Thus g_1y^2 is determined unless (g_1, y^2) is an ND pair. Suppose (g_1, y^2) is an ND pair. We have $|g_1^G| = n/4$ and $|(y^2)^G| \le |y^G| \le n/6$. It follows from one of the cases M1, M2, B or F that g_1y^2 is determined by the NDICT. This shows that we can always determine g_1y^2 in this subcase, so that xy is determined by the NDICT in this case as well.

Thus, in either subcase xy is determined by the NDICT of G. $\square$

In our next two cases, we use the classifications mentioned in the introduction. Groups with a self centralizing subgroup of order 4 were classified by Suzuki [23]. He shows:

Theorem 4.12 (Suzuki)**.** *Let G be a finite group containing an element y of order 4. If y commutes only with its powers, then either G contains a normal subgroup of index 2 which does not contain y, or G contains an abelian normal subgroup G_0 of odd order such that the factor group G/G_0 is one of the following groups: SL(2,3), SL(2,5), LF(2,7), or the alternating groups A_6 or A_7.*

Case N: $k = n/4$ and $m = n/4$. We use Theorem 4.12 to show that no finite groups G, $|G| \ge 256$, has elements $x, y \in G$ for which (x, y) is an ND pair and $k = m = n/4$.

When $k = m = n/4$ we know that $\sim \langle y\rangle x$, $y^x = y^{-1}$, $|y^G| = n/4$, $o(y) = 4$ and $|x^G| = n/4$. We note that $y^{x^2} = (y^x)^x = (y^{-1})^x = y$, so that $x^2 \in C_G(y) = \langle y\rangle$ so that $x^2 = 1$ or $x^2 = y^2$.

Subcase 1: G contains a normal subgroup N of index 2 with $y \notin N$. Suppose $x^G \subseteq N$. Then $x(xy) = y \in N^2$, a contradiction. Suppose $x^G \not\subseteq N$. Then $yx \in G \setminus N = yN$ so $x \in N$, a contradiction.

Subcase 2: G has an abelian normal subgroup G_0 such that the factor group G/G_0 is one of the groups $SL(2,3)$, $SL(2,5)$, $LF(2,7) = PSL(2,7)$, A_6, and A_7.

The groups $SL(2,3)$, $SL(2,5)$, $LF(2,7) = PSL(2,7)$, A_6, and A_7 are well understood.

They all have exactly $n/4$ elements of order 4, and have no more than $n/8$ involutions. The $n/4$ elements of order 4 correspond to the elements of y^G, so for these groups it is impossible to have x^G of size $n/4$. This gives a contradiction.

Thus, it is impossible for a group to have (x, y) an ND pair with $|x^G| = |y^G| = n/4$. $\square$

For the final case we use another classification theorem. Groups with a self centralizing subgroup of order 3 were classified by Feit and Thompson [6]. They show

Theorem 4.13 (Feit,Thompson)**.** *Let G be a finite group which contains a self-centralizing subgroup of order 3. Then one of the following statements is true.*

(i) G contains a nilpotent normal subgroup N such that G/N is isomorphic to either A_3 or S_3.

(ii) G contains a normal subgroup N which is a 2-group such that G/N is isomorphic to A_5.

(iii) G is isomorphic to $PSL(2, 7)$.

Case O: $k = n/4$ and $m = n/3$.

Because $y^{x^2} = (y^{-1})^x = y$, we have $x^2 \in C_G(y) = \{1, y, y^2\}$, so that in fact we must have $o(x) = 2$ in this case. Also, we must have $C_G(x) = \{1, x, t, xt\}$ where $o(t) = 2$. To see this, suppose by way of contradiction that $C_G(x) = \langle t \rangle$ where $t^2 = x$. Then $C_G(t) = C_G(x)$, but $x \not\sim t$, so that we have two classes of size $n/4$ in addition to the class of y which has size $n/3$. Because $C_G(t) = \langle t \rangle$, we can apply Suzuki's result (Theorem 4.12). If G has a normal subgroup N of index 2 that does not contain t, then $x = t^2 \in N$, so $xy \in N$, and $x(xy) = y \in N$, so $y \in N$ and hence we also have $y^G \subseteq N$. But this implies $|N| \geq |y^G| + |x^G| = n/4 + n/3$, a contradiction. And, as we mentioned earlier, the groups $SL(2, 3)$, $SL(2, 5)$, $LF(2, 7) = PSL(2, 7)$, A_6, and A_7 do not have $n/4$ involutions, and neither can any group which has them as a quotient. Thus, we must have $C_G(x) = \{1, x, t, xt\}$ where $o(t) = 2$.

We consider the cases of Theorem 4.13 separately to determine those finite groups, G, which have elements $x, y \in G$ for which $x \sim \langle y \rangle x$, $y^x = y^{-2}$, $|y^G| = n/3$ and $|x^G| = n/4$.

Subcase 1: $G/N = S_3$ for N nilpotent.

By size considerations $x^G \not\subseteq N$, $y^G \not\subseteq N$. Because $x, xy, xy^2 \in x^G$ and $y, y^2 \in y^G$, it follows that $y, y^2, xy, xy^2, x \notin N$, so that the cosets $N, yN, y^2N, xN, xyN, xy^2N$ of N must be distinct. The set $K = N \cup yN \cup y^2N$ is closed under products, and so is a subgroup of G. Because it has 2 in G it is normal, so that in fact $K = N \cup y^G$. We know that $C_G(x) = \{1, x, xt, t\}$ for some involution t, and we can assume without loss of generality that $t \in K$. (If $t, x \in G \setminus K$, then $xt \in K$, and we relabel.)

We work from the right with $t \in K$ to show that xy can be determined in this case. We know by order consideration that $y \not\sim t$, and because $t^2 = 1$ we also know that (y, t) is not an ND pair. Because $y^g = y^{-1}$ implies $g \in \{x, xy, xy^2\}$, and $t \notin \{x, xy, xy^2\}$, it follows from Lemma 4.4(c) that (t, y) is not an ND pair. Thus, yt is determined by the NDICT of G.

Because $t \in K$ and $g_1 \in G \setminus K$, we know that $g_1 \not\sim t$. Because t and g_1 are involutions, (t, g_1) and (g_1, t) cannot be ND pairs. Thus $g_1 t$ is determined by the NDICT of G.

We noted earlier that $y^t \neq y^{-1}$. From this it follows that $(yt)^2 = ytyt = yy^t \neq 1$, so that $yt \not\sim x$. Because $t \notin C_G(y)$, we know that $(yt)^x = y^{-1}t \neq ty^{-1} = (yt)^{-1}$, so (x, yt) cannot be an ND pair. And (yt, x) cannot be an ND pair because x is an involution. So $x(yt)$ is determined by the NDICT of G. Thus $x(yt)$ and $g_1 t$ are determined from the NDICT of G, so xy is determined by the NDICT of G by Theorem 4.7.

Subcase 2: $G/N = A_3$.

In this case, we know $g^3 \in N$ for all $g \in G$. Because $x^3 = x$, it follows that $x \in N$ and also $x^G \in N$. From this it follows that $xy \in N$ and $x(xy) = y \in N$. From this it follows that $y^G \subset N$, a contradiction, because $|y^G \cup x^G| > n/3 = |N|$. Thus, it is not possible for a group of this type to have an ND pair (x, y) with $|x^G| = n/4$ and $|y^G| = n/3$.

Subcase 3: $G/N = A_5$, N a 2-group.

By size considerations $x^G \not\subseteq N$, $y^G \not\subseteq N$. If $N = \{1\}$ so that $G = A_5$, then $|G| < 256$.

(We did this case in the example in section 4.2.) If $N \neq \{1\}$, consider $C_G(x) = \langle x, t\rangle$ where $x^2 = t^2 = (xt)^2 = 1$. Because x normalizes N, $\langle N, x\rangle$ is a 2-group, and so it has center. That center can only be $\langle t\rangle$ or $\langle xt\rangle$, and it follows that either $t \in N$ or $xt \in N$. Without loss of generality, we assume $t \in N$.

We can work from the left with t to determine xy. We know that $t \not\sim x$ because $t \in N$, $x \notin N$, and that (x, t) and (t, x) are not ND pairs because $t^2 = x^2 = 1$. So the product tx is determined. Next, we know that $t \not\sim g_1$ because $t \in N$, $g_1 \notin N$, and that (t, g_1) and (g_1, t) are not ND pairs because $t^2 = g_1^2 = 1$. So tg_1 is also determined by the NDICT of G. Finally, we have $tx \not\sim y$ because tx and y have different orders and (y, tx) is not an ND pair because xt is an involution. Also, (tx, y) is not an ND pair because $t \notin C_G(y)$, which implies that $y^{tx} = y^{xt} = (y^{-1})^t \neq y^{-1}$.

Because tx, tg_1 and $(tx)y$ are determined by the NDICT, it follows from Lemma 4.7 that xy is determined by the NDICT when $G/N = A_5$ for some normal 2-group N.

Subcase 4: $G = PSL(2, 7)$. This group has order 168 and so is not in consideration in this section.

This concludes consideration of all groups with a class of size $n/3$ and shows that when there is an ND pair (x, y) with $|x^G| = n/4$, $|y^G| = n/3$, the product xy is determined by the NDICT of G. □

As this was our final case, we have shown that for any ND pair (x, y) of G, G a finite group with $|G| \geq 256$, G not generalized dihedral of order $2n$, n odd, that the product xy is determined by the NDICT of G. □

Chapter 5. Character Theory of 2-S-rings

In this chapter we present some results on the characters of the 2-S-rings of finite groups.

Classically, an S-ring (or Schur-ring) T on a finite group G is a subring of the group ring $\mathbb{Z}G$ characterized by a partition $G = \Gamma_1 \cup \cdots \cup \Gamma_t$ of G into non-empty trivially intersecting subsets of G, called *principal sets* with the following properties:

(i) $\Gamma_1 = \{e\}$

(ii) For every $i \in \{1, \ldots, t\}$ there is a $j \in \{1, \ldots, t\}$ such that $\Gamma_i^{-1} := \{g^{-1} | g \in \Gamma_i\} = \Gamma_j$.

(iii) The elements $\tau_i = \sum_{g \in \Gamma_i} g$, $(i \in \{1, \ldots, t\})$ are a $\mathbb{Z}$ basis for T.

The definition we have used in this paper of a Schur-ring is what classically would have been called a Schur-algebra:

Definition 5.1 (Tamaschke)**.** Let T be a Schur-ring on the finite group G. Then the set $\mathbb{C}T$ of all linear combinations of the T-class sums with complex numbers as coefficients forms a subalgebra of the group algebra $\mathbb{C}G$ which is called a *Schur-algebra* on G over $\mathbb{C}$.

Thus, k-S-rings as we have defined them are actually Schur-algebras and not Schur-rings. But we continue to call them rings by convention.

If $F : \mathbb{C}T \to M_n(\mathbb{C})$ is a representation (ring homomorphism) of $\mathbb{C}T$ (as a $\mathbb{C}$-algebra), then the character of the representation is defined as follows:

Definition 5.2 (Tamaschke)**.** For any representation $F : \mathbb{C}T \to M_n(\mathbb{C})$ the complex valued function $\phi : g \mapsto \text{trace}\left(F\left(\frac{\overline{C_i}}{|C_i|}\right)\right)$ where $g \in C_i$, is called the *T-character of G related to F*. The character is called irreducible if the representation is irreducible.

We are also going to consider the relationship of characters of the 2-S-ring of G both to the characters of G^2 and to the Frobenius 2-characters.

Much of the work in this section was motivated by attempting to understand the characters of the 2-S-ring of S_3, including their relationship to the characters of S_3^2 and the 2-character of S_3. We present that example in the first section.

In the next section we prove generalized results for the character tables of 1- and 2-S-rings of finite groups. We build on the work on character tables of S-rings done by Olaf Tamaschke [24], who was a graduate student of Helmut Wielandt.

We apply some of those results in the third section, in which we determine the character table of the 2-S-ring of G where G is dihedral of order $2n$, n odd.

Finally, we show that each Frobenius 2-character of a finite group corresponds in a natural way to a character of the 2-S-ring of G.

5.1 An example with S_3

In this section we consider the 2-S-ring of the group $G = S_3 := \langle a, b | a^3 = b^2 = 1, a^b = a^2 \rangle$. Calculations are suppressed in this section to facilitate understanding the big picture, but all calculations are carried out in a later section when we determine the character table of the 2-S-ring of generalized dihedral groups of order $2n$, n odd.

The group S_3 has three 1-classes: $\{e\}, \{a, a^2\}, \{b, ab, a^2b\}$, and three pairwise non-isomorphic representations ρ_1, ρ_2, ρ_3 corresponding to the characters χ_1, χ_2, χ_3 of the character table of S_3.

$$\left[\begin{array}{c|ccc} & e & a & b \\ \hline \chi_1 & 1 & 1 & 1 \\ \chi_2 & 1 & 1 & -1 \\ \chi_3 & 2 & -1 & 0 \end{array}\right]$$

Each representation ρ_i of S_3, induces a representation of $Z(\mathbb{C}G)$. The ring $Z(\mathbb{C}G)$ has $\mathbb{C}$ basis $\{e, a + a^2, b + ab + a^2b\}$, and all irreducible representations of this ring are linear. In particular, the irreducible group representation ρ_3 induces a reducible representation of $Z(\mathbb{C}G)$ which is two copies of the same linear representation of $Z(\mathbb{C}G)$. This is a well known consequence of Schur's Lemma.

The 1-S-ring and $Z(\mathbb{C}G)$ are the same ring, but we think of the standard basis of the 1-S-ring as being the average class sums. Thus, for this example we have $\tau_0 = 1$, $\tau_1 = \frac{1}{2}(a + a^2)$, $\tau_2 = \frac{1}{3}(b + ab + a^2b)$ as the basis of $\mathfrak{S}_G^{(1)}$.

For $CT(\mathfrak{S}_{S_3}^{(1)})$ we have the character table

$$\left[\begin{array}{rr|rrr} \text{Class Sizes} & & 1 & 2 & 3 \\ & z_i & e & a & b \\ \hline \chi_1 & 1 & 1 & 1 & 1 \\ \chi_2 & 1 & 1 & 1 & -1 \\ \chi_3 & 4 & 1 & -1/2 & 0 \end{array}\right].$$

Let ρ_r be the regular representation of S_3. Then $\hat{\rho}_r$ is a degree six representation of $\mathbb{C}G$, which reduces as $\hat{\rho}_r = \chi_1 + \chi_2 + 4\chi_3$, so that z_i is the coefficient of χ_i in this sum. We also see that $\langle \chi_3, \chi_3 \rangle = \frac{1}{6}\left[1 + 2(\frac{1}{4}) + 0\right] = \frac{1}{4} = \frac{1}{z_3}$. These characters are orthogonal (because the irreducible characters of G are orthogonal), but this calculation shows that they are not 'orthonormal.' However, they do have the desirable characteristic that for the linear character $\chi_i : G \to \mathbb{C} : g \mapsto \chi(g)$, the map $\hat{\chi}_i : \mathfrak{S}_G^{(1)} \to \mathbb{C}$ is a representation of the 1-S-ring.

For S_3, the 2-S-ring is commutative. There are many groups, however, which have non-commutative 2-S-rings. There are relatively few groups with commutative 3-S-ring. We discuss groups with commutative 3-S-rings in Chapter 6.

It is easy to verify that S_3 has eight 2-classes:

$$\begin{aligned}
C_1 &= \{(e,e)\}; \\
C_2 &= \{(e,a),(e,a^2),(a,e),(a^2,e)\}; \\
C_3 &= \{(a,a),(a^2,a^2)\}; \\
C_4 &= \{(a,a^2),(a^2,a)\}; \\
C_5 &= \{(e,b),(e,a^2b),(e,ab),(b,e),(a^2b,e),(ab,e)\}; \\
C_6 &= \{(b,b)(a^2b,a^2b),(ab,ab)\}; \\
C_7 &= \{(b,a)(a^2b,a),(ab,a),(b,a^2)(a^2b,a^2),(ab,a^2),(a,b),(a,a^2b),(a,ab),(a^2,b), \\
&\qquad (a^2,a^2b),(a^2,ab)\}; \\
C_8 &= \{(b,a^2b),(b,ab),(a^2b,b),(a^2b,ab),(ab,b),(ab,a^2b)\}.
\end{aligned}$$

The character table of the 2-S-ring of S_3 is given below. The z_i have meaning similar to their meaning in the 1-S-ring case, which we discuss below.

Size of 2-class		1	2	2	4	3	6	6	12
	z_i	(e,e)	(a,a^2)	(a,a)	(e,a)	(b,b)	(e,b)	(b,ab)	(a,ab)
ψ_1	1	1	1	1	1	1	1	1	1
ψ_2	1	1	1	1	1	1	-1	1	-1
ψ_3	2	1	1	1	1	-1	0	-1	0
ψ_4	4	1	$-1/2$	1	$-1/2$	1	0	$-1/2$	0
ψ_5	4	1	$-1/2$	1	$-1/2$	-1	0	$1/2$	0
ψ_6	8	1	1	$-1/2$	$-1/2$	0	0	0	0
ψ_7	8	1	$-1/2$	$-1/2$	$1/4$	0	$1/2$	0	$-1/4$
ψ_8	8	1	$-1/2$	$-1/2$	$1/4$	0	$-1/2$	0	$1/4$

These characters also have the property that $\widehat{\psi_i} : \mathfrak{S}^{(2)}_{S_3} \to \mathbb{C}$ is a representation of the 2-S-ring. We want to understand the relationship of characters of the 2-S-ring of S_3 both to the characters of $S_3^2 = S_3 \times S_3$ and to the Frobenius 2-characters of S_3.

We first discuss the relationship to the Frobenius 2-character. We see that for any $(g,h) \in S_3^2$, $\psi_5(g,h) = \frac{1}{2}\chi_3^{(2)}(g,h)$, so that the scaled Frobenius 2-character is in fact the character of a representation of S_3. We show in section 5.4that the Frobenius 2-character always corresponds to a representation of the 2-S-ring.

Recall that for an $r \times s$ matrix $A = (a_{ij})$, and $p \times q$ matrix B the tensor product $A \otimes B$ is the $rp \times sq$ matrix defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1s}B \\ a_{21}B & a_{22}B & \cdots & a_{2s}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1}B & a_{r2}B & \cdots & a_{rs}B \end{bmatrix}.$$

The group S_3^2 has 9 irreducible representations. These representations can be obtained by taking pairwise tensor products of the representations ρ_1, ρ_2, ρ_3 of S^3 [11, Theorem 19.18]. We use $\chi_i \otimes \chi_j$ to denote the character of $\rho_i \otimes \rho_j$, so that we have $\chi_i \otimes \chi_j(g, h) = \chi_i(g)\chi_j(h)$. Then S_3^2 has the following character table:

S_3^2	(e,e)	(e,a)	(e,b)	(a,e)	(a,a)	(a,b)	(b,e)	(b,a)	(b,b)
$\chi_1 \otimes \chi_1$	1	1	1	1	1	1	1	1	1
$\chi_1 \otimes \chi_2$	1	1	-1	1	1	-1	1	1	-1
$\chi_2 \otimes \chi_1$	1	1	1	1	1	1	-1	-1	-1
$\chi_2 \otimes \chi_2$	1	1	-1	1	1	-1	-1	-1	1
$\chi_3 \otimes \chi_3$	4	-2	0	-2	1	0	0	0	0
$\chi_1 \otimes \chi_3$	2	-1	0	2	-1	0	2	-1	0
$\chi_3 \otimes \chi_1$	2	2	2	-1	-1	-1	0	0	0
$\chi_2 \otimes \chi_3$	2	-1	0	2	-1	0	-2	1	0
$\chi_3 \otimes \chi_2$	2	2	-2	-1	-1	1	0	0	0

If we take the representation $\rho_i \otimes \rho_j$ of S_3^2 and extend to representations of the group algebra $Z(\mathbb{C}S_3^2)$, then restrict to representations of the 2-S-ring of S_3, we get a representation $\widehat{\rho_i \otimes \rho_j}$ of the 2-S-ring. With a slight abuse of notation, we will denote the character of $\widehat{\rho_i \otimes \rho_j}$ by $\widehat{\chi_i \otimes \chi_j}$.

In this way we get the following (not necessarily irreducible) characters of the 2-S-ring.

	(e,e)	(a,a^2)	(a,a)	(e,a)	(b,b)	(e,b)	(b,ab)	(a,ab)
$\widehat{\chi_1 \otimes \chi_1}$	1	1	1	1	1	1	1	1
$\widehat{\chi_1 \otimes \chi_2}$	1	1	1	1	-1	0	-1	0
$\widehat{\chi_2 \otimes \chi_1}$	1	1	1	1	-1	0	-1	0
$\widehat{\chi_2 \otimes \chi_2}$	1	1	1	1	1	-1	1	-1
$\widehat{\chi_3 \otimes \chi_3}$	4	1	1	-2	0	0	0	0
$\widehat{\chi_1 \otimes \chi_3}$	2	-1	-1	$1/2$	0	1	0	$-1/2$
$\widehat{\chi_3 \otimes \chi_1}$	2	-1	-1	$1/2$	0	1	0	$-1/2$
$\widehat{\chi_2 \otimes \chi_3}$	2	-1	-1	$1/2$	0	-1	0	$1/2$
$\widehat{\chi_3 \otimes \chi_2}$	2	-1	-1	$1/2$	0	-1	0	$1/2$

We see that $\widehat{\chi_i \otimes \chi_j}(g,h) = \frac{1}{2}\left(\chi_i(g)\chi_j(h) + \chi_i(h)\chi_j(g)\right)$, which doesn't surprise us considering the symmetric nature of the 2-classes. It is also straightforward to verify that:

$$
\begin{aligned}
\widehat{\chi_1 \otimes \chi_1} &= \psi_1; \\
\widehat{\chi_2 \otimes \chi_2} &= \psi_2; \\
\widehat{\chi_1 \otimes \chi_2} &= \psi_3; \\
\widehat{\chi_2 \otimes \chi_1} &= \psi_3; \\
\widehat{\chi_3 \otimes \chi_3} &= \psi_4 + \psi_5 + 2\psi_6; \\
\widehat{\chi_1 \otimes \chi_3} &= 2\psi_7; \\
\widehat{\chi_3 \otimes \chi_1} &= 2\psi_7; \\
\widehat{\chi_2 \otimes \chi_3} &= 2\psi_8; \\
\widehat{\chi_3 \otimes \chi_2} &= 2\psi_8.
\end{aligned}
$$

The regular representation ρ_r of S_3 decomposes as a direct sum $\rho_r = \rho_1 + \rho_2 + 2\rho_3$. The regular representation $\rho_r^2 = \rho_r \otimes \rho_r$ of S_3^2 also decomposes as a sum of irreducible representations

$$\rho_r^2 = \rho_1 \otimes \rho_1 + \rho_1 \otimes \rho_2 + \rho_2 \otimes \rho_1 + \rho_2 \otimes \rho_2 + 2\rho_1 \otimes \rho_3 + 2\rho_3 \otimes \rho_1 + 2\rho_2 \otimes \rho_3 + 2\rho_3 \otimes \rho_2 + 4\rho_3 \otimes \rho_3.$$

And $\hat{\rho}_r^2$ also decomposes as a direct sum of the $\widehat{\rho_i \otimes \rho_j}$:

$$\hat{\rho}_r^2 = \widehat{\rho_1 \otimes \rho_1} + 2\widehat{\rho_1 \otimes \rho_2} + \widehat{\rho_2 \otimes \rho_2} + 4\widehat{\rho_1 \otimes \rho_3} + 4\widehat{\rho_2 \otimes \rho_3} + 4\widehat{\rho_3 \otimes \rho_3}.$$

Using the table above to finish decomposing the regular representation, we can further decompose $\hat{\rho}_r^2$ as a sum of linear representations (characters):

$$\hat{\rho}_r^2 = \psi_1 + \psi_2 + 2\psi_3 + 4(\psi_4 + \psi_5 + 2\psi_6) + 8\psi_7 + 8\psi_8.$$

Thus, the z_i in the character table of $\mathfrak{S}_G^{(2)}$ are just the coefficient of ψ_i in the above sum. If we calculate, for example, $\langle \psi_8, \psi_8 \rangle$ we get

$$\langle \psi_8, \psi_8 \rangle = \frac{1}{36}\left(1 \cdot 1 + 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{16} + 3 \cdot 0 + 6 \cdot \frac{1}{4} + 6 \cdot 0 + 12 \cdot \frac{1}{16}\right) = \frac{1}{8} = \frac{1}{z_8}.$$

It is true for all the characters of the 2-S-ring of S_3 that $\langle \psi_i, \psi_j \rangle = \delta_{ij} \frac{1}{z_j}$.

This work sheds light on work done previously by Ken Johnson in [12]. In that paper Johnson defines the *extended* 2*-characters* of a finite group G as follows:

Definition 5.3. Let G be a finite group with $\mathrm{Irr}(G) = \{\theta_1, \dots, \theta_r\}$. For g, h in G we define

(i) $\theta_i^{(2)}(g,h) = \theta_i(g)\theta_i(h) - \theta_i(gh)$, for each θ_i of degree > 1.

(ii) $\theta_i^{(2,+)}(g,h) = \theta_i(g)\theta_i(h) + \theta_i(gh)$, for each θ_i of degree > 1.

(iii) $\theta_i \circ \theta_j(g,h) = \theta_i(g)\theta_j(h) + \theta_i(h)\theta_j(g)$, for θ_i, θ_j distinct.

These functions are the *extended* 2*-characters* of G.

He shows that these functions are constant on 2-classes and notes [12, Proposition 2.2, 2.3] that the extended 2-characters are pairwise orthogonal, in the sense that if ψ, ϕ are two distinct extended 2-characters, then

$$\sum_{x \in G^2} \psi(x)\overline{\phi(x)} = 0.$$

He also defines the 2-character table of a finite group G using the extended 2-characters.

Definition 5.4. Let G be a finite group. The 2-*character table of* G has rows indexed by the extended 2-characters and columns labeled by an element in each 2-class, with entries the extended 2-character values.

He also defines a norm $||\psi|| = \sum_{x \in G^2} |\psi(x)|^2$, for ψ an extended 2-character.

We give the 2-character table of $S_3 = D_6 = \langle a, b | a^3 = b^2 = 1, a^b = a^2 \rangle$. This 2-character table is not exactly the character table given in [12], but has been modified according to corrections annotated by Ken Johnson.

Class Order		1	2	2	4	3	6	6	12
Representative		(e,e)	(a,a^2)	(a,a)	(e,a)	(b,b)	(e,b)	(b,ab)	(a,ab)
Character	Norm								
$\frac{1}{2}\chi_1 \circ \chi_1$	36	1	1	1	1	1	1	1	1
$\chi_1 \circ \chi_2$	72	2	2	2	2	-2	0	-2	0
$\chi_1 \circ \chi_3$	72	4	-2	-2	1	0	2	0	-1
$\frac{1}{2}\chi_2 \circ \chi_2$	36	1	1	1	1	1	-1	1	-1
$\chi_2 \circ \chi_3$	72	4	-2	-2	1	0	-2	0	1
$\chi_3^{(2,+)}$	108	6	3	0	-3	2	0	-1	0
$\chi_3^{(2)}$	36	2	-1	2	-1	-2	0	1	0
Orthogonal Complement	54	0	3	-3	0	-2	0	1	0

The paper does not assign any type of meaning to the norm, nor does it explain why he chose certain multiples of extended k-characters for certain rows. He notes that when using only the extended 2-characters, this 2-character table is not square, and finds an orthogonal complement using row orthogonality. The paper also gives a character table for D_8 and explains why D_8 and Q_8 do not have the same 2-character table. The 3-character table is also defined.

When we compare the 2-character table of S_3 to the character table of $\mathfrak{S}_{S_3}^{(2)}$, however, we see that, with the exception of $\chi_3^{(2,+)}$, each of the extended 2-characters is a multiple of an irreducible character of the 2-S-ring. Also $\chi_3^{(2,+)} = 2(\psi_4 + 2\psi_6)$ is a sum of multiples of characters, and also corresponds to a representation of the $\mathfrak{S}_{S_3}^{(2)}$.

Using this information, we see that in fact there is a missing row on this 2-character table because $\chi_3^{(2,+)}$ corresponds to a sum of irreducible representations, and hence does not correspond to an irreducible character. Essentially, in this table the character of $\chi_3 \otimes \chi_3$ has been split into two characters, one of which (the Frobenius character) corresponds to an irreducible representation, so that $\chi_3^{(2,+)}$ is just 'everything else.'

In the remainder of this chapter, our primary goal is to generalize the results of this example as much as possible, and also, when we fail to get a generalization, to give a counter example. In particular, our work with the character tables of 2-S-rings will allow us to determine the extended 2-characters, in particular the Frobenius 2-characters, as characters of representations of $S_G^{(2)}$ in a natural way.

Monica Vazarani [25] also did extensive work with extended k-characters. She found various representations which had as characters the extended k-characters. However, she also produced the same 'non-square' 2-character table for S_3 as Ken Johnson.

As we mentioned, there are groups with non-commutative 2-S-rings, and for these groups we will not get a square character table for the 2-S-ring of the group. However, when the 2-S-ring of a group is commutative, then the 2-S-ring will have a square character table.

5.2 Character tables of G, G^2, $\mathfrak{S}_G^{(1)}$ and $\mathfrak{S}_G^{(2)}$

In this section, we formalize some of the 'coincidences' of the last chapter. Many of our results will rely heavily on the work of Olaf Tamaschke in [24] regarding the character theory of S-rings.

Throughout this section G is a finite group and T is an S-ring of $G = C_1 \cup C_2 \cup \cdots \cup C_s$ where the C_i are the principal sets. We let $\mathbb{C}T$ be the associated Schur-algebra, which will have standard basis $\left\{\tau_j = \frac{C_j}{|C_j|}\right\}$, $1 \le j \le s$.

Also, we let ρ_r denote the regular representation of G, and for ρ a representation (either of G or T), we let χ_ρ represent the character of the representation.

We let $\{F_i\}$, $1 \le i \le r$, be a complete set of pairwise non-isomorphic irreducible representations of $\mathbb{C}T$, and y_i denote the dimension of the representation. For each F_i we define z_i to be the multiplicity of F_i as a factor of $\widehat{\rho_r}$, so that $\widehat{\rho_r} = \sum_{i=0}^{r} z_i F_i$. Also, we let ϕ_i be the character of G rel T associated with F_i.

Because it is possible that T is not a commutative ring, the number of principal sets, s, need not be equal to r, the number of distinct irreducible representations of T.

Every Schur-algebra $\mathbb{C}T$ is semisimple [26, p.386, footnote]. Thus, every representation of $\mathbb{C}T$ is completely reducible, and hence each character of $\mathbb{C}T$ (T character of G) is a linear combination of the irreducible characters of $\mathbb{C}T$ with non-negative integral coefficients [24, p. 342]. Also, it follows from semisimplicity that $\dim(\mathbb{C}T) = \sum_{i=1}^{r} y_i^2$.

The following result is due to Tamaschke [24, Theorem 1.5]. We use the notation we have introduced to make the statements clearer:

Theorem 5.5 (Tamaschke)**.** *Let G be a finite group and T an S-ring over G. If F_i, F_j are irreducible representations of T and ϕ_i, ϕ_j are their characters of G rel T, then*

$$\langle \phi_i, \phi_j \rangle = \delta_{ij} \frac{y_i}{z_j}.$$

In the S_3 example, we saw that the multiplicity of each irreducible representation of $\mathfrak{S}_{S_3}^{(2)}$ as a factor of the regular representation of S_3^2 was equal to the reciprocal of the norm of the representation. This theorem generalizes that fact.

In the 1-S-ring case, the relationship between representations of G and representations of the 1-S-ring is well understood.

Proposition 5.6. *Let ρ be a degree n representation of G. Then $\hat{\rho}$ is a completely reducible representation of $\mathfrak{S}_G^{(1)}$, with $\hat{\rho}\left(\frac{\overline{C_i}}{|C_i|}\right) = \frac{1}{n}\chi_\rho(g_i) I_n$, where $g_i \in C_i$.*

Proof. When C_i is a conjugacy class of a finite group G, then $\overline{C_i}$ is central in $\mathbb{C}G$, and so by Schur's Lemma $\rho(\overline{C_i})$ must be a scalar matrix. It is easy to verify that one must have $\rho\left(\overline{C_i}\right) = \frac{1}{n}|C_i|\chi_\rho(g_i)I_n$, where g_i in C_i. For more details, see [5, pp. 233-238]. $\square$

Corollary 5.7. *If each row of the character table of G is divided by the degree of the character in that row, then the result is $CT(\mathfrak{S}_G^{(1)})$.*

Proof. Let $\tau_i = \frac{\overline{C_i}}{|C_i|}$, $g_i \in C_i$. From Proposition 5.6 we see that $\tau_i \mapsto \frac{1}{n}\chi_\rho(g_i)$ is a linear representation of $\mathfrak{S}_G^{(1)}$ and hence $g \mapsto \frac{1}{n}\chi_\rho(g)$ is a linear character of $\mathfrak{S}_G^{(1)}$. Because the dimension of the 1-S-ring equals the number of conjugacy classes, this is a complete set of irreducible representations. $\square$

For the 2-S-ring, the relationship of the character table of $\mathfrak{S}_G^{(2)}$ and G^2 is not so straightforward. We can always find characters of $\mathfrak{S}_G^{(2)}$ from the character table of G^2, but in general, as we saw in the example, these characters are not irreducible characters.

For the remainder of the section, we let $\{\rho_1, \ldots, \rho_t\}$ be a complete set of irreducible representations of G and let $\{\chi_1, \ldots, \chi_t\}$ be the associated irreducible characters of G.

At this point, we want to discuss the three types of 2-classes that can occur for any finite group G. We let $K(g, h)$ denotes the 2-class of the element $(g, h) \in G^2$.

First, there is the diagonal type class $D = K(g, g)$. In the discussion, it will be useful to write $\overline{D} = \frac{1}{2}\sum[(x, x)+(x, x)]$ where the sum is over all $x \in D$. Next, there is the 'conjugate' type class $C = K(g, h)$ where $g \sim h$, $g \neq h$. Because of the $S_2 2$ action of $\tilde{G}_2 = G \times S_2$ on G^2, $(x, y) \in C$ if and only if $(y, x) \in C$ and $(x, y) \neq (y, x)$ for $(x, y) \in C$. We can write $\overline{C} = \frac{1}{2}\sum[(x, y) + (y, x)]$ where the sum is over all $(x, y) \in C$. Finally, we have classes $B = K(g, h)$ where $g \not\sim h$. We define $B_g = \{(x, y) \in B | x \sim g\}$. For this type of class we have $\overline{B} = \sum\limits_{(x,y)\in B_g} [(x, y) + (y, x)]$. Thus we have:

Lemma 5.8. *Fix $(g, h) \in G^2$ and let $K = K(g, h)$, $K_g = \{(x, y) \in K | x \sim g\}$. Then for the* *average class sum* τ *of* $K(g,h)$ *we have*

$$\tau = \frac{\overline{K}}{|K|} = \frac{1}{2|K_g|} \sum_{(x,y)\in K_g} [(x,y)+(y,x)]. \quad \square$$

This lemma allows us to use the same format to represent any of the three types of classes and allows us to avoid using cases in the proofs of this section.

Fix $(g,h) \in G^2$, and let $B = K(g,h)$, $\tau = \frac{\overline{B}}{|B|}$. Then

$$\rho_i \otimes \rho_j(\tau) = \frac{1}{2|B_g|} \sum_{(x,y)\in B_g} [\rho_i(x)\otimes\rho_j(y) + \rho_i(y)\otimes\rho_j(x)].$$

And because $\mathrm{tr}(M\otimes N) = \mathrm{tr}(M)\mathrm{tr}(N)$ for any square matrices M, N, it follows that:

$$\begin{aligned}
\mathrm{tr}(\rho_i \otimes \rho_j(\tau)) &= \tfrac{1}{2|B_g|} \sum_{(x,y)\in B_g} [\mathrm{tr}(\rho_i(x)\otimes\rho_j(y)) + \mathrm{tr}(\rho_i(y)\otimes\rho_j(x))] \\
&= \tfrac{1}{2|B_g|} \sum_{(x,y)\in B_g} [\mathrm{tr}(\rho_i(x))\mathrm{tr}(\rho_j(y)) + \mathrm{tr}(\rho_i(y)\mathrm{tr}(\rho_j(x))] \\
&= \tfrac{1}{2|B_g|} \sum_{(x,y)\in B_g} [\chi_i(x)\chi_j(y) + \chi_i(y)\chi_j(x)] \\
&= \tfrac{1}{2|B_g|} \sum_{(x,y)\in B_g} [\chi_i(g)\chi_j(h) + \chi_i(h)\chi_j(g)] \\
&= \tfrac{1}{2|B_g|}|B_g|[\chi_i(g)\chi_j(h) + \chi_i(h)\chi_j(g)] \\
&= \tfrac{1}{2}[\chi_i(g)\chi_j(h) + \chi_i(h)\chi_j(g)].
\end{aligned}$$

We have shown the following:

Proposition 5.9. *With the notations defined above we have*

$$\widehat{\chi_i \otimes \chi_j}(g,h) = \frac{1}{2}[\chi_i(g)\chi_j(h) + \chi_i(h)\chi_j(g)]. \quad \square$$

Corollary 5.10. *If* χ_i, χ_j *are characters of a finite group* G*, then the generalized* 2*-character* $\chi_i \circ \chi_j$ *is a character of the* 2*-S-ring of* G*.* $\square$

If θ is a linear representation of G and ρ is any other representation, it is also straightforward to calculate the representation $\widehat{\theta \otimes \rho}$.

Lemma 5.11. *Let G be a finite group. If $\theta : G \to \mathbb{C}$ is a linear representation of G and ρ is a degree n representation of G, then $\widehat{\theta \otimes \rho}(g,h) = \frac{1}{2n}(\theta(g)\chi_\rho(h) + \chi_\rho(g)\theta(h))I_n$, where χ_ρ is the character of ρ.*

Proof. Fix $g, h \in G$. We use the notation θ, ρ to denote both the representations of G and the representation of $\mathbb{C}G$, where θ is a linear representation and ρ is a representation of degree n.

Fix $g, h \in G$. We let $A = K(g,h)$ and let $\tau = \frac{1}{2|A_g|} \sum\limits_{(x,y)\in A_g} [(x,y) + (y,x)]$ be the average class sum of A. (Recall that $A_g = \{(x,y) \in A | x \sim g\}$. Then we have

$$\begin{aligned}
\theta \otimes \rho(\tau) &= \tfrac{1}{2|A_g|} \sum_{(x,y)\in A_g} [\theta(x) \otimes \rho(y) + \theta(y) \otimes \rho(x)] \\
&= \tfrac{1}{2|A_g|} \sum_{(x,y)\in A_g} [\theta(x)\rho(y) + \theta(y)\rho(x)] \\
&= \tfrac{1}{2|A_g|} \sum_{(x,y)\in A_g} [\theta(g)\rho(y) + \theta(h)\rho(x)] \\
&= \tfrac{1}{2|A_g|} \left(\sum_{(x,y)\in A_g} \theta(g)\rho(y) + \sum_{(x,y)\in A} \theta(h)\rho(x) \right) \\
&= \tfrac{1}{2|A_g|} \left(\theta(g) \sum_{(x,y)\in A_g} \rho(y) + \theta(h) \sum_{(x,y)\in A_g} \rho(x) \right) \\
&= \tfrac{1}{2|A_g|} \left(\theta(g) \tfrac{|A^G|}{|h^G|} \rho(\overline{h^G}) + \theta(h) \tfrac{|A_g|}{|g^G|} \rho(\overline{g^G}) \right) \\
&= \tfrac{1}{2|A_g|} \left(\theta(g) \tfrac{|A^G|}{|h^G|} \left(\tfrac{|h^G|}{n} \chi_\rho(h) I_n \right) + \theta(h) \tfrac{|A_g|}{|g^G|} \left(\tfrac{|g^G|}{n} \chi_\rho(g) I_n \right) \right) \\
&= \tfrac{1}{2n} [\theta(g)\chi_\rho(h) + \theta(h)\chi_\rho(g)] I_n.
\end{aligned}$$

□

In the next section we apply these results to help us determine the character table of the 2-S-ring of dihedral groups of order $2n$, n odd.

5.3 2-character table of Dihedral groups of order $2n$, n odd

Throughout this section, $G = \langle a, b | a^n = b^2 = 1, a^b = a^{-1}\rangle$ is the dihedral group of order $2n$ with $n > 1$ odd. We let $m = \frac{n-1}{2}$, $N = \langle a\rangle \triangleleft G$, $\langle a\rangle b = Nb = G \setminus N$, and $\tilde{G}_2 = G \times S_2$, where $S_2 = \langle\sigma\rangle$, $\sigma = (12)$. We think of $\tilde{G}_2$ as acting on G^2 where G acts by diagonal conjugation and S_2 acts by permuting elements, so that the 2-classes are the orbits of G^2 under the action of $\tilde{G}_2$. We will use the information about the classes, character table, and representations of D_{2n} given in [11, pp. 181-182].

We will determine the character table of the 2-S-ring of G. Because Theorem 5.5 guarantees that each irreducible representation of $\mathfrak{S}_G^{(2)}$ occurs as a factor of $\hat{\rho}_r^2$, after we have determined the 2-classes of G, we will start with the irreducible representations of G^2 in order to determine the irreducible representations of $\mathfrak{S}_G^{(2)}$.

Because n is odd, $G = D_{2n}$ has $m + 2 = (n + 3)/2$ conjugacy classes. They are

$$\{e\}, \{a^r, a^{-r}\}(1 \leq r \leq m), \{b, ab, \ldots, a^n b\}.$$

Also, G has 2 linear representations. They are

$$\begin{aligned} \rho_1 :&\quad G \to \mathbb{C}^* : g \mapsto 1; \\ \rho_2 :&\quad G \to \mathbb{C}^* : a^r b^j \mapsto (-1)^j. \end{aligned}$$

Let $\xi = e^{2\pi i/n}$. For $1 \leq k \leq n$, we let

$$A = \begin{bmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{bmatrix}, A_k = A^k = \begin{bmatrix} \xi^k & 0 \\ 0 & \xi^{-k} \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For $1 \leq k \leq m$ there are irreducible degree 2 representations of G defined by

$$\nu_k : G \to GL_2(\mathbb{C}) : a^r b^j \mapsto A_k^r B^j,$$

and these $m+2$ representations ρ_1, ρ_2, ν_k, $1 \le k \le (n-1)/2$ are a complete set of irreducible representations of G.

Thus for $G = D_{2n}$, $n > 1$ odd, we have the following character table, where $\alpha_{kr} = \xi^{kr} + \xi^{-kr} = \chi_k(a^r)$:

$\lvert g_i^G \rvert$	1	2	n
g_i	1	$a^r (1 \le r \le m)$	b
ψ_1	1	1	1
ψ_2	1	1	-1
χ_k	2	α_{kr}	0

In order to find the 2-classes of G, we consider first the 2-classes in $N \times N$. Consider $(a^r, a^s) \in N \times N$, where $1 \le r, s \le n-1$. For any $g \in N$ we know $(a^r, a^s)^g = (a^r, a^s)$, and for $h \in Nb$ we know $(a^r, a^s)^h = (a^{-r}, a^{-s})$, so the orbit of the G action of diagonal conjugation is $\{(a^r, a^s), (a^{-r}, a^{-s})\}$. Also, $(a^r, a^s)^\sigma = (a^s, a^r)$, so the $\tilde{G}_2$ orbit of (a^r, a^s) is $\{(a^r, a^s), (a^{-r}, a^{-s}), (a^s, a^r), (a^{-s}, a^{-r})\}$.

As we consider the possibilities that $s = r$, $s = -r$, $s \ne r$, $s = 0$, or $r = 0$, we see that in addition to the diagonal classes $K(e, e)$, $K(a^r, a^r)$, $(1 \le r \le m)$ in $N \times N$, we get the following 2-classes in $N \times N$ for $1 \le r \le n$, $1 \le |s| < r$:

$$
\begin{aligned}
K(a^r, a^{-r}) &= \{(a^r, a^{-r}), (a^{-r}, a^r)\}; \\
K(a^r, a^s) &= \{(a^r, a^s), (a^{-r}, a^{-s}), (a^s, a^r), (a^{-s}, a^{-r})\}; \\
K(e, a^r) &= \{(e, a^r), (e, a^{-r}), (a^r, e), (a^{-r}, e)\}.
\end{aligned}
$$

It is straightforward to verify that every element of $N \times N$ is in exactly one of these classes.

Now we consider the 2-classes in $(N \times Nb) \cup (Nb \times N)$. By 1.28(i) we have $K(e, b) = \{(e, a^i b), (a^i b, e)\}_{i=0}^{n}$ with $|K(e, b)| = 2n$. For $1 \le r, s \le m$, $(a^r, b)^{a^s} = (a^r, a^{-2s} b)$. Because n is odd, it follow that $(a^r, b)^N = \{(a^r, a^k b)\}_{k=0}^{n-1} = \{a^r\} \times Nb$. Also, $(a^r, a^k b)^b = (a^{-r}, a^{-k} b)$, so $(a^r, b)^{Nb} = \{(a^{-r}, a^k b)\}_{k=0}^{n-1} = \{a^{-r}\} \times Nb$. When we consider the S_2 action, we see that

$K(a^r, b) = (\{a^r, a^{-r}\} \times Nb) \cup (Nb \times \{a^r, a^{-r}\})$. Because each element of $(N \times Nb) \cup (Nb \times N)$ is in $K(a^r, b)$ for some r, $0 \leq r \leq m$, these classes are all the 2-classes in $(N \times Nb) \cup (Nb \times N)$.

Now we consider the 2-classes in $Nb \times Nb$. We will have the diagonal class $K(b, b)$. Consider $(b, a^r b) \in Nb \times Nb$, for $1 \leq r \leq m$. We know that $(b, a^r b)^{a^s} = (a^{-2s} b, a^{r-2s} b)$, so $(b, a^r)^N = \{(a^k b, a^{r+k} b)\}_{k=0}^{n-1}$. Also $(a^k b, a^{k+r} b)^b = (a^{-k} b, a^{-k-r} b)$, so that we have $(b, a^r)^{Nb} = \{(a^k b, a^{k-r} b)\}_{k=0}^{n-1}$. Finally, note that the set $\{(a^k b, a^{r+k} b)\}_{k=0}^{n-1} \cup \{(a^k b, a^{k-r} b)\}_{k=0}^{n-1}$ is closed under the S_2 action, so that $K(b, a^r b) = \{(a^k b, a^{r+k} b), (a^k b, a^{k-r} b)\}_{k=0}^{n-1}$. For a fixed r, $1 \leq r \leq m$, we get $K(b, a^r b) = \{(a^i b, a^{i+r} b)\} \cup \{(a^i b, a^{i-r} b)\}$ which has size $2n$. Because each element of $Nb \times Nb$ is an element of $K(b, a^r b)$ for some $r, 0 \leq r \leq m$, these are all the 2-classes of $Nb \times Nb$.

For $1 \leq r \leq m$, $1 \leq |s| < r$, we define

$$
\begin{aligned}
C_1 &= K(e,e), & \tau_1 &= (e,e); \\
C_{2,r} &= K(e,a^r), & \tau_{2,r} &= \tfrac{1}{4}\left[(e,a^r) + (e,a^{-r}) + (a^r,e) + (a^{-r},e)\right]; \\
C_{3,r} &= K(a^r,a^r), & \tau_{3,r} &= \tfrac{1}{2}\left[(a^r,a^r) + (a^{-r},a^{-r})\right]; \\
C_{4,r} &= K(a^r,a^{-r}), & \tau_{4,r} &= \tfrac{1}{2}\left[(a^r,a^{-r}) + (a^{-r},a^r)\right]; \\
C_{5,r,s} &= K(a^r,a^s), & \tau_{5,r,s} &= \tfrac{1}{4}\left[(a^r,a^s) + (a^{-r},a^{-s}) + (a^s,a^r) + (a^{-s},a^{-r})\right]; \\
C_6 &= K(e,b), & \tau_6 &= \tfrac{1}{2n}\sum_{t=0}^{n-1}[(e,a^t b) + (a^t b,e)]; \\
C_{7,r} &= K(b,a^r), & \tau_{7,r} &= \tfrac{1}{4n}\sum_{t=0}^{n-1}[(a^t b,a^r) + (a^t b,a^{-r}) + (a^r,a^t b) + (a^{-r},a^t b)]; \\
C_8 &= K(b,b), & \tau_8 &= \tfrac{1}{n}\sum_{t=0}^{n-1}[(a^t b,a^t b)]; \\
C_{9,r} &= K(b,a^r b), & \tau_{9,r} &= \tfrac{1}{2n}\sum_{t=0}^{n-1}[(a^t b,a^{t+r} b) + (a^t b,a^{t-r} b)].
\end{aligned}
$$

Proposition 5.12. *The 2-S-ring of $G = D_{2n}$ is generated by the average class sums listed above where $1 \leq r \leq m$ and $1 \leq |s| < r$. It has dimension $m^2 + 4m + 3$.*

Proof. By definition the τ_i form a basis. Because $1 \leq r \leq m$ and $1 \leq |s| \leq r$, there are $m(m-1)$ basis elements $\tau_{5,r,s}$. Thus $\mathfrak{S}_G^{(2)}$ has dimension $1 + m + m + m + m(m-1) + 1 + m + 1 + m = m^2 + 4m + 3$. $\square$

We now want to find the irreducible representations of $\mathfrak{S}_G^{(2)}$. The group G^2 has $(m+2)^2$ irreducible representations, which are $\rho_i\otimes\rho_j$, $\rho_i\otimes\nu_k$, $\nu_k\otimes\rho_i$, $\nu_k\otimes\nu_l$, $1\leq i,j\leq 2$, $1\leq k,l\leq m$. These representations extend to representations of $\mathbb{C}G^2$, which in turn can be restricted to representations $\widehat{\rho_i\otimes\rho_j}$, $\widehat{\rho_i\otimes\nu_k}$, $\widehat{\nu_k\otimes\rho_i}$, $\widehat{\nu_k\otimes\nu_l}$, of $\mathfrak{S}_G^{(2)}$. Thus we find the representations $\widehat{\rho_i\otimes\rho_j}$, $\widehat{\rho_i\otimes\nu_k}$, $\widehat{\nu_k\otimes\rho_i}$, $\widehat{\nu_k\otimes\nu_l}$, for $1\leq i,j\leq 2$, $1\leq k,l\leq m$, by evaluating $\rho_i\otimes\rho_j$, $\rho_i\otimes\nu_k$, etc., on these average class sums.

We can use Lemma 5.11 to determine the representations for $\widehat{\rho_i\otimes\rho_j}(\tau_\alpha)=\widehat{\rho_j\otimes\rho_i}(\tau_\alpha)$, $\widehat{\rho_i\otimes\nu_k}(\tau_\alpha)=\widehat{\nu_k\otimes\rho_i}(\tau_\alpha)$, for $1\leq i,j\leq 2$, $1\leq k\leq m$ using the values of the character table of D_{2n}. For $\rho_i\otimes\rho_j$ we get the following linear representations/characters:

	1	4	2	2	4	$2n$	$4n$	n	$2n$
	τ_1	$\tau_{2,r}$	$\tau_{3,r}$	$\tau_{4,r}$	$\tau_{5,r,s}$	τ_6	$\tau_{7,r}$	τ_8	$\tau_{9,r}$
$\widehat{\rho_1\otimes\rho_1}$	1	1	1	1	1	1	1	1	1
$\widehat{\rho_1\otimes\rho_2}$	1	1	1	1	1	0	0	-1	-1
$\widehat{\rho_2\otimes\rho_2}$	1	1	1	1	1	0	0	1	1

Before going on to compute the other representations, we pause to compute some inner products.

Let ϕ, χ be $\mathfrak{S}_G^{(2)}$ characters of G^2, and $C_1,C_2,\ldots C_s$ be the 2-classes of G. Fix $(g_i,h_i)\in C_i$. The inner product is defined to be

$$\langle\phi,\chi\rangle=\frac{1}{|G^2|}\sum_{(g,h)\in G^2}\phi(g)\overline{\chi(h)}.$$

But because the ϕ, χ are constant on 2-classes, we can write

$$\begin{aligned}\langle\phi,\chi\rangle &= \tfrac{1}{|G^2|}\left(\sum_{i=1}^{s}\sum_{(g,h)\in C_i}\phi(g)\overline{\chi(h)}\right)\\ &= \tfrac{1}{|G^2|}\left(\sum_{i=1}^{s}|C_i|\phi(g_i)\overline{\chi(h_i)}\right),\end{aligned}$$

This simplifies the calculations that we will make to compute inner products. Because the character values $\widehat{\rho_i \otimes \rho_j}(\tau_{k,r})$, $k \in \{2,3,4,7,9\}$, and $\widehat{\rho_i \otimes \rho_j}(\tau_{5,r,s})$, $i,j \in \{1,2\}$, do not depend on r or s we can further simplify the calculation of inner products in this case, multiplying the number of classes of each of the nine 'types' by $|C_i|\phi(g)\overline{\chi(h)}$, (where (g,h) is a class representative) then taking the sum over the nine types of 2-classes.

We get

$$\begin{aligned}\langle \widehat{\rho_1 \otimes \rho_1}, \widehat{\rho_1 \otimes \rho_1}\rangle = \quad & \frac{1}{4n^2}(1 \cdot 1 + 4 \cdot 1 \cdot m + 2 \cdot 1 \cdot m + 2 \cdot 1 \cdot m + 4 \cdot 1 \cdot \tfrac{m(m-1)}{2} \\ & +4 \cdot 1 \cdot \tfrac{m(m-1)}{2} + 2n \cdot 1 \cdot 1 + 4n \cdot 1 \cdot m + n \cdot 1 \cdot 1 + 2n \cdot 1 \cdot m),\end{aligned}$$

so that

$$\begin{aligned}\langle \widehat{\rho_1 \otimes \rho_1}, \widehat{\rho_1 \otimes \rho_1}\rangle &= \frac{1}{4n^2}(1 + 8m + 8\tfrac{m(m-1)}{2} + 2n + 4nm + n + 2nm \\ &= \frac{1}{4n^2}(1 + 8m + 4m^2 - 4m + (3 + 6m)(2m+1)) \\ &= \frac{1}{4n^2}(1 + 4m + 4m^2 + 12m^2 + 12m + 3) \\ &= \tfrac{4(4m^2+4m+1)}{4n^2} = 1.\end{aligned}$$

Also, we have

$$\begin{aligned}\langle \widehat{\rho_1 \otimes \rho_1}, \widehat{\rho_1 \otimes \rho_2}\rangle = \quad & \frac{1}{4n^2}(1 \cdot 1 \cdot 1 + 4 \cdot 1 \cdot m + 2 \cdot 1 \cdot m + 2 \cdot 1 \cdot m + 4 \cdot 1 \cdot \tfrac{m(m-1)}{2} \\ & +4 \cdot 1 \cdot \tfrac{m(m-1)}{2} + 2n \cdot 0 \cdot 1 + 4n \cdot 0 \cdot m + n \cdot -1 \cdot 1 + 2n \cdot -1 \cdot m),\end{aligned}$$

so that

$$\begin{aligned}\langle \widehat{\rho_1 \otimes \rho_1}, \widehat{\rho_1 \otimes \rho_2}\rangle &= \tfrac{1}{4n^2}\left(1 + 8m + 8\tfrac{m(m-1)}{2} - n - 2nm\right) \\ &= \tfrac{1+4m+4m^2-(1+2m)(2m+1)}{4n^2} = 0.\end{aligned}$$

And finally

$$\begin{aligned}\langle \widehat{\rho_1 \otimes \rho_2}, \widehat{\rho_1 \otimes \rho_2}\rangle &= \frac{1}{4n^2}(1 \cdot 1 \cdot 1 + 4 \cdot 1 \cdot m + 2 \cdot 1 \cdot m + 2 \cdot 1 \cdot m + 4 \cdot 1 \cdot \tfrac{m(m-1)}{2} \\ & \quad +4 \cdot 1 \cdot \tfrac{m(m-1)}{2} + 2n \cdot 0 \cdot 1 + 4n \cdot 0 \cdot m + n \cdot 1 \cdot 1 + 2n \cdot 1 \cdot m),\end{aligned}$$

so that

$$
\begin{aligned}
\langle \widehat{\rho_1 \otimes \rho_2}, \widehat{\rho_1 \otimes \rho_2} \rangle &= (1 + 8m + 8m(m-1)/2 + n + 2nm)/(4n^2) \\
&= 1 + 4m + 4m^2 + (1+2m)(2m+1))/(4n^2) = 2n^2/(4n^2) \\
&= 1/2.
\end{aligned}
$$

This agrees with the results of Theorem 5.5.

Next we consider the degree 2 representations, $\rho_1 \otimes \nu_k$, $\rho_2 \otimes \nu_k$, for $1 \leq k \leq m$. These representations can also be determined using the results of Lemma 5.11.

First we have

$$
\begin{aligned}
\rho_1 \otimes \nu_k(\tau_1) &= I_2, 1 \leq k \leq m; \\
\rho_2 \otimes \nu_k(\tau_1) &= I_2, 1 \leq k \leq m.
\end{aligned}
$$

For the average class sum $\tau_{2,r} = \frac{1}{4}\left[(e, a^r) + (e, a^{-r}) + (a^r, e) + (a^{-r}, e)\right]$ we get

$$
\begin{aligned}
\rho_1 \otimes \nu_k(\tau_{2,r}) &= \tfrac{1}{2\cdot 2}(\psi_1(e)\chi_k(a^r) + \psi_1(a^r)(\chi_k(e))I_2 = \tfrac{1}{4}(\alpha_{kr} + 2)I_2; \\
\rho_2 \otimes \nu_k(\tau_{2,r}) &= \tfrac{1}{4}(\psi_2(e)\chi_k(a^r) + \psi_2(a^r)(\chi_k(e))I_2 = \tfrac{1}{4}(\alpha_{kr} + 2)I_2.
\end{aligned}
$$

For $\tau_{3,r}$, $\tau_{4,r}$, $\tau_{5,r,s}$, we get the following:

$$
\begin{aligned}
\rho_1 \otimes \nu_k(\tau_{3,r}) &= \tfrac{1}{4}(\psi_2(a^r)\chi_k(a^r) + \psi_2(a^r)(\chi_k(a^r))I_2 = \tfrac{1}{2}(\alpha_{kr})I_2; \\
\rho_2 \otimes \nu_k(\tau_{3,r}) &= \tfrac{1}{2}(\alpha_{kr})I_2.
\end{aligned}
$$

$$
\begin{aligned}
\rho_1 \otimes \nu_k(\tau_{4,r}) &= \tfrac{1}{4}(\psi_1(a^r)\chi_k(a^{-r}) + \psi_1(a^{-r})\chi_k(a^r))I_2 = \tfrac{1}{2}(\alpha_{kr})I_2; \\
\rho_2 \otimes \nu_k(\tau_{4,r}) &= \tfrac{1}{2}(\alpha_{kr})I_2.
\end{aligned}
$$

$$
\begin{aligned}
\rho_1 \otimes \nu_k(\tau_{5,r,x}) &= \tfrac{1}{4}(\psi_1(a^r)\chi_k(a^s) + \psi_1(a^s)\chi_k(a^r))I_2 = \tfrac{1}{4}(\alpha_{kr} + \alpha_{ks})I_2; \\
\rho_2 \otimes \nu_k(\tau_{5,r,x}) &= \tfrac{1}{4}(\alpha_{kr} + \alpha_{ks})I_2.
\end{aligned}
$$

For the class $\tau_6 = \frac{1}{2n}\sum_{t=0}^{n-1}[(e, a^t b) + (a^t b, e)]$,

$$\begin{aligned}
\rho_1 \otimes \nu_k(\tau_6) &= \tfrac{1}{4}(\psi_1(e)\chi_k(a^r b) + \psi_1(a^r b)(\chi_k(e))I_{=}\tfrac{1}{4}(0+2)I_2 = \tfrac{1}{2}I_2. \\
\rho_2 \otimes \nu_k(\tau_6) &= -\tfrac{1}{2}I_2.
\end{aligned}$$

Then, $\tau_{7,r} = \frac{1}{4n}\sum_{t=0}^{n-1}[(a^t b, a^r) + (a^t b, a^{-r}) + (a^r, a^t b) + (a^{-r}, a^t b)]$, and so

$$\begin{aligned}
\rho_1 \otimes \nu_k(\tau_{7,r}) &= \tfrac{1}{4}\alpha_{kr}I_2; \\
\rho_2 \otimes \nu_k(\tau_{7,r}) &= -\tfrac{1}{4}\alpha_{kr}I_2.
\end{aligned}$$

Finally, for $\tau_8 = \frac{1}{n}\sum_{t=0}^{n-1}[(a^t b, a^t b)]$ and $\tau_{9,r} = \frac{1}{2n}\sum_{t=0}^{n-1}[(a^t b, a^{t+r} b) + (a^t b, a^{t-r} b)]$, we have

$$\begin{aligned}
\rho_1 \otimes \nu_k(\tau_8) &= \rho_1 \otimes \nu_k(\tau_{9,r}) &= 0; \\
\rho_2 \otimes \nu_k(\tau_8) &= \rho_2 \otimes \nu_k(\tau_{9,r}) &= 0.
\end{aligned}$$

The representations $\widehat{\rho_1 \otimes \nu_k}$, $\widehat{\rho_2 \otimes \nu_k}$, for $1 \le k \le m$ are all scalar matrices, and so each is a sum of two copies of the same linear representation, say $\widehat{\rho_i \otimes \nu_k} = 2\psi_{i,k}$. Thus we have $2m$ more linear characters $\psi_{1,k}$, $\psi_{2,k}$ to add to our 2-S-ring character table.

	1	4	2	2	4	$2n$	$4n$	n	$2n$
	τ_1	$\tau_{2,r}$	$\tau_{3,r}$	$\tau_{4,r}$	$\tau_{5,r,s}$	τ_6	$\tau_{7,r}$	τ_8	$\tau_{9,r}$
$\widehat{\rho_1 \otimes \rho_1}$	1	1	1	1	1	1	1	1	1
$\widehat{\rho_1 \otimes \rho_2}$	1	1	1	1	1	0	0	-1	-1
$\widehat{\rho_2 \otimes \rho_2}$	1	1	1	1	1	0	0	1	1
$\psi_{1,k}$	1	$\frac{1}{2} + \frac{1}{4}\alpha_{kr}$	$\frac{1}{2}\alpha_{kr}$	$\frac{1}{2}\alpha_{kr}$	$\frac{1}{4}(\alpha_{kr} + \alpha_{ks})$	$\frac{1}{2}$	$\frac{1}{4}\alpha_{kr}$	0	0
$\psi_{2,k}$	1	$\frac{1}{2} + \frac{1}{4}\alpha_{kr}$	$\frac{1}{2}\alpha_{kr}$	$\frac{1}{2}\alpha_{kr}$	$\frac{1}{4}(\alpha_{kr} + \alpha_{ks})$	$-\frac{1}{2}$	$-\frac{1}{4}\alpha_{kr}$	0	0

Finally, we consider the degree 4 representations. We consider the cases $\nu_k \otimes \nu_k$ and $\nu_k \otimes \nu_l$ for $k \ne l$ separately, where $1 \le k \le m$. We will write $\mathrm{D}(a_1, a_2, a_3, a_4)$ for the diagonal matrix with entries a_1, a_2, a_3, a_4 on the main diagonal. Throughout, $1 \leq k, l \leq m$, $k \neq l$.

First, for $(a^r b^i, a^s b^j) \in G^2$ we have:

$$\nu_k \otimes \nu_l(a^r b^i, a^s b^j) = A_k^r B^i \otimes A_l^s B^j, 1 \leq k, l \leq m.$$

To simplify calculations we note that

$$A_k^r \otimes A_l^s = \begin{bmatrix} \xi^{kr}\xi^{ls} & 0 & 0 & 0 \\ 0 & \xi^{kr}\xi^{-ls} & 0 & 0 \\ 0 & 0 & \xi^{-kr}\xi^{ls} & 0 \\ 0 & 0 & 0 & \xi^{-kr}\xi^{-ls} \end{bmatrix};$$

$$A_k^{-r} \otimes A_l^{-s} = \begin{bmatrix} \xi^{-kr}\xi^{-ls} & 0 & 0 & 0 \\ 0 & \xi^{-kr}\xi^{ls} & 0 & 0 \\ 0 & 0 & \xi^{kr}\xi^{-ls} & 0 \\ 0 & 0 & 0 & \xi^{kr}\xi^{ls} \end{bmatrix}.$$

Or,

$$A_k^r \otimes A_l^s + A_k^{-r} \otimes A_l^{-s} \quad = \mathrm{D}(\alpha_{kr+ls}, \alpha_{kr-ls}, \alpha_{kr-ls}, \alpha_{kr+ls}).$$

For $\tau_1 = (e, e)$ we get

$$\begin{aligned} \nu_k \otimes \nu_l(\tau_1) &= I_4; \\ \nu_k \otimes \nu_k(\tau_1) &= I_4. \end{aligned}$$

For $\tau_{2,r} = \frac{1}{4}\left[(e, a^r) + (e, a^{-r}) + (a^r, e) + (a^{-r}, e)\right]$ we get

$$\begin{aligned} \nu_k \otimes \nu_l(\tau_{2,r}) &= \nu_k \otimes \nu_l(\tfrac{1}{4}\left[(e, a^r) + (e, a^{-r}) + (a^r, e) + (a^{-r}, e)\right]) \\ &= \tfrac{1}{4}(I \otimes A_l^r + I \otimes A_l^{-r} + A_k^r \otimes I + A_k^{-r} \otimes I) \\ &= \tfrac{1}{4}(\xi^{lr} + \xi^{-lr} + \xi^{kr} + \xi^{-kr})I_4 = \tfrac{1}{4}(\alpha_{kr} + \alpha_{lr})I_4; \\ \nu_k \otimes \nu_k(\tau_{2,r}) &= \tfrac{1}{2}\alpha_{kr} I_4. \end{aligned}$$

The results for $\tau_{3,r}$ and $\tau_{4,r}$are obtained similarly:

$$\begin{aligned}
\nu_k \otimes \nu_l(\tau_{3,r}) &= \tfrac{1}{2}\mathrm{D}(\alpha_{kr+lr}, \alpha_{kr-lr}, \alpha_{kr-lr}, \alpha_{kr+lr});\\
\nu_k \otimes \nu_k(\tau_{3,r}) &= \tfrac{1}{2}\mathrm{D}(\alpha_{2kr}, 2, 2, \alpha_{2kr});\\
\nu_k \otimes \nu_l(\tau_{4,r}) &= \tfrac{1}{2}\mathrm{D}(\alpha_{kr-lr}, \alpha_{kr+lr}, \alpha_{kr+lr}, \alpha_{kr-lr});\\
\nu_k \otimes \nu_k(\tau_{4,r}) &= \tfrac{1}{2}\mathrm{D}(2, \alpha_{2kr}, \alpha_{2kr}, 2).
\end{aligned}$$

For $\tau_{5,r,s}$ we get

$$\begin{aligned}
\nu_k \otimes \nu_l(\tau_{5,r,s}) &= \tfrac{1}{4}\mathrm{D}(\alpha_{kr+ls} + \alpha_{ks+lr}, \alpha_{kr-ls} + \alpha_{ks-lr}, \alpha_{kr-ls} + \alpha_{ks-lr}, \alpha_{kr+ls} + \alpha_{ks+lr}));\\
\nu_k \otimes \nu_k(\tau_{5,r,s}) &= \tfrac{1}{2}\mathrm{D}(\alpha_{kr+ks}, \alpha_{kr-ks}, \alpha_{kr-ks}, \alpha_{kr+ks}).
\end{aligned}$$

Next, we consider the class $\tau_6 = \frac{1}{2n}\sum_{t=0}^{n-1}[(e, a^t b) + (a^t b, e)]$. Recall that $A_k^t = \mathrm{diag}(\xi^{kt}, \xi^{-kt})$, so that

$$\sum_{t=0}^{n-1} A_k^t = \mathrm{diag}\left(\sum_{t=0}^{n-1}\xi^k t, \sum_{t=0}^{n-1}\xi^{-kt}\right) = 0_2.$$

Also, for any $q \times q$ matrices M_t, and any matrix N, it is a property of tensor products that

$$\sum_{t=0}^{n-1}(M_t \otimes N) = \left(\sum_{t=0}^{n-1} M_t\right) \otimes N.$$

Then we use these two facts to help us simplify the next few cases:

$$\begin{aligned}
\nu_k \otimes \nu_l(\tau_6) &= \nu_k \otimes \nu_l\big(\tfrac{1}{2n}\sum_{t=0}^{n-1}[(e, a^t b) + (a^t b, e)]\big) = \tfrac{1}{2n}\sum_{t=0}^{n-1}[I \otimes A_l^t B + A_k^t B \otimes I] = 0_4;\\
\nu_k \otimes \nu_k(\tau_6) &= 0_4.
\end{aligned}$$

Also, for $\tau_{7,r} = \frac{1}{4n}\sum_{t=0}^{n-1}[(a^t b, a^r) + (a^t b, a^{-r}) + (a^r, a^t b) + (a^{-r}, a^t b)]$:

$$\begin{aligned}
\nu_k \otimes \nu_l(\tau_{7,r}) &= \nu_k \otimes \nu_l\left(\frac{1}{4n}\sum_{t=0}^{n-1}[(a^t b, a^r) + (a^t b, a^{-r}) + (a^r, a^t b) + (a^{-r}, a^t b)]\right) \\
&= \frac{1}{4n}\sum_{t=0}^{n-1}[(A_k^t B \otimes A_l^r) + (A_k^t B \otimes A_l^{-r}) + (A_k^r \otimes A_l^t B) + (A_k^{-r} \otimes A_l^t B)] = 0_4; \\
\nu_k \otimes \nu_k(\tau_{7,r}) &= 0_4.
\end{aligned}$$

To simplify calculations for the τ_8 case we note that

$$A_k^t B \otimes A_l^t B = \begin{bmatrix} 0 & 0 & 0 & \xi^{kt+lt} \\ 0 & 0 & \xi^{kt-lt} & 0 \\ 0 & \xi^{-kt+lt} & 0 & 0 \\ \xi^{-kt-lt} & 0 & 0 & 0 \end{bmatrix}.$$

And $1 \leq k, l \leq m$, so when $l \neq k$, then $\xi^{k+l}, \xi^{k-l} \neq 1$ are non-identity nth roots of unity (usually not primitive). Also, ξ^{2kr} is a non-identity nth root of unity, so we have

$$A_k^t B \otimes A_k^t B = \begin{bmatrix} 0 & 0 & 0 & \xi^{2kt} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \xi^{-2kt} & 0 & 0 & 0 \end{bmatrix}.$$

For $\tau_8 = \frac{1}{n}\sum_{t=0}^{n-1}[(a^t b, a^t b)]$, we have

$$\begin{aligned}
\nu_k \otimes \nu_l(\tau_8) &= \nu_k \otimes \nu_l\left(\frac{1}{n}\sum_{t=0}^{n-1}(a^t b, a^t b)\right) = \frac{1}{n}\sum_{t=0}^{n-1} A_k^t B \otimes A_l^t B = 0; \\
\nu_k \otimes \nu_k(\tau_8) &= \nu_k \otimes \nu_k\left(\frac{1}{n}\sum_{t=0}^{n-1}(a^t b, a^t b)\right) = \frac{1}{n}\sum_{t=0}^{n-1} A_k^t B \otimes A_k^t B \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Finally, for $\tau_{9,r} = \frac{1}{4n}\sum_{t=0}^{n-1}[(a^t b, a^{t+r} b) + (a^t b, a^{t-r} b) + (a^{t+r}, a^t b) + (a^{t-r}, a^t b)]$, we have

$$\begin{aligned}
\nu_k \otimes \nu_l(\tau_{9,r}) &= \nu_k \otimes \nu_l\left(\frac{1}{4n}\sum_{t=0}^{n-1}[(a^t b, a^{t+r} b) + (a^t b, a^{t-r} b) + (a^{t+r} b, a^t b) + (a^{t-r} b, a^t b)]\right) \\
&= \frac{1}{4n}\sum_{t=0}^{n-1}[A_k^t B \otimes A_l^{t+r} B + A_k^t B \otimes A_l^{t-r} B + A_k^{t+r} B \otimes A_l^t B + A_k^{t-r} B \otimes A_l^t B] \\
&= 0_4; \\
\nu_k \otimes \nu_k(\tau_{9,r}) &= \nu_k \otimes \nu_k\left(\frac{1}{4n}\sum_{t=0}^{n-1}[(a^t b, a^{t+r} b) + (a^t b, a^{t-r} b) + (a^{t+r} b, a^t b) + (a^{t-r} b, a^t b)]\right) \\
&= \frac{1}{4n}\sum_{t=0}^{n-1}[A_k^t B \otimes A_k^{t+r} B + A_k^t B \otimes A_k^{t-r} B + A_k^{t+r} B \otimes A_k^t B + A_k^{t-r} B \otimes A_k^t B] \\
&= \frac{1}{4n}\sum_{t=0}^{n-1}[(A_k^t B \otimes A_k^t B)(I \otimes A^{kr}) + (A_k^t B \otimes A_k^t B)(I \otimes A^{-kr}) \\
&\quad + (A_k^t B \otimes A_k^t B)(A^{kr} \otimes I) + (A_k^t B \otimes A_k^t B)(A^{-kr} \otimes I)] \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\alpha_{kr} & 0 \\ 0 & \frac{1}{2}\alpha_{kr} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Because we are interested in finding a complete set of irreducible representations for D_{2n}, n odd, we need to find the invariant subspaces of these representations.

First we consider $\nu_k \otimes \nu_l$ for $k \neq l$:

$$
\begin{aligned}
\nu_k \otimes \nu_l(\tau_1) &= I_4; \\
\nu_k \otimes \nu_l(\tau_{2,r}) &= \tfrac{1}{4}(\alpha_{kr} + \alpha_{lr}) I_4; \\
\nu_k \otimes \nu_l(\tau_{3,r}) &= \tfrac{1}{2}\mathrm{D}(\alpha_{kr+lr}, \alpha_{kr-lr}, \alpha_{kr-lr}, \alpha_{kr+lr}); \\
\nu_k \otimes \nu_l(\tau_{4,r}) &= \tfrac{1}{2}\mathrm{D}(\alpha_{kr-lr}, \alpha_{kr+lr}, \alpha_{kr+lr}, \alpha_{kr-lr}); \\
\nu_k \otimes \nu_l(\tau_{5,r,s}) &= \tfrac{1}{4}\mathrm{D}(\alpha_{kr+ls} + \alpha_{ks+lr}, \alpha_{kr-ls} + \alpha_{ks-lr}, \alpha_{kr-ls} + \alpha_{ks-lr}, \alpha_{kr+ls} + \alpha_{ks+lr})); \\
\nu_k \otimes \nu_l(\tau_6) &= 0_4; \\
\nu_k \otimes \nu_l(\tau_{7,r}) &= 0_4; \\
\nu_k \otimes \nu_l(\tau_8) &= 0_4; \\
\nu_k \otimes \nu_l(\tau_{9,r}) &= 0_4.
\end{aligned}
$$

These matrices are all diagonal, and by observation, we see that these representations are the sum of two copies each of two distinct linear representations, which we denote $\chi_{k,l,1}, \chi_{k,l,2}$.

	1	4	2	2	4	$2n$	$4n$	n	$2n$
	τ_1	$\tau_{2,r}$	$\tau_{3,r}$	$\tau_{4,r}$	$\tau_{5,r,s}$	τ_6	$\tau_{7,r}$	τ_8	$\tau_{9,r}$
$\chi_{k,l,1}$	1	$\frac{1}{4}(\alpha_{kr} + \alpha_{lr})$	$\frac{1}{2}(\alpha_{kr+lr})$	$\frac{1}{2}(\alpha_{kr-lr})$	$\frac{1}{4}(\alpha_{kr+ls} + \alpha_{ks+lr})$	0	0	0	0
$\chi_{k,l,2}$	1	$\frac{1}{4}(\alpha_{kr} + \alpha_{lr})$	$\frac{1}{2}(\alpha_{kr-lr})$	$\frac{1}{2}(\alpha_{kr+lr})$	$\frac{1}{4}(\alpha_{kr-ls} + \alpha_{ks-lr})$	0	0	0	0

In the $k = l$ case, we will see that $\widehat{\nu_k \otimes \nu_k}$ is also a sum of four linear representations.

$$\begin{aligned}
\nu_k \otimes \nu_k(\tau_1) &= I_4; \\
\nu_k \otimes \nu_k(\tau_{2,r}) &= \tfrac{1}{2}\alpha_{kr} I_4; \\
\nu_k \otimes \nu_k(\tau_{3,r}) &= \mathrm{D}(\tfrac{1}{2}\alpha_{2kr}, 1, 1, \tfrac{1}{2}\alpha_{2kr}); \\
\nu_k \otimes \nu_k(\tau_{4,r}) &= \mathrm{D}(1, \tfrac{1}{2}\alpha_{2kr}, \tfrac{1}{2}\alpha_{2kr}, 1); \\
\nu_k \otimes \nu_k(\tau_{5,r,s}) &= \tfrac{1}{2}\mathrm{D}(\alpha_{kr+ks}, \alpha_{kr-ks}, \alpha_{kr-ks}, \alpha_{kr+ks})); \\
\nu_k \otimes \nu_k(\tau_6) &= 0_4; \\
\nu_k \otimes \nu_k(\tau_{7,r}) &= 0_4; \\
\nu_k \otimes \nu_k(\tau_8) &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \\
\nu_k \otimes \nu_k(\tau_{9,r}) &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\alpha_{kr} & 0 \\ 0 & \frac{1}{2}\alpha_{kr} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Because each of the four vectors $v_1 = (0,1,1,0)$, $v_2 = (0,1,-1,0)$, $v_3 = (1,0,0,0)$, $v_4 = (0,0,0,1)$ are eigenvectors of each matrix in the representation, they are four 1-dimensional modules. The collection of eigenvalues corresponding to these eigenspaces are irreducible characters/representations of $\mathfrak{S}_G^{(2)}$. For each vector v_1 we let $\chi_{k,k,i}$ be the character corresponding to the module span(v_i). We note that $\chi_{k,k,3} = \chi_{k,k,4}$, so we only include one of these two in our final character table.

	1	4	2	2	4	$2n$	$4n$	n	$2n$
	τ_1	$\tau_{2,r}$	$\tau_{3,r}$	$\tau_{4,r}$	$\tau_{5,r,s}$	τ_6	$\tau_{7,r}$	τ_8	$\tau_{9,r}$
$\widehat{\rho_1 \otimes \rho_1}$	1	1	1	1	1	1	1	1	1
$\widehat{\rho_1 \otimes \rho_2}$	1	1	1	1	1	0	0	-1	-1
$\widehat{\rho_2 \otimes \rho_2}$	1	1	1	1	1	0	0	1	1
$\psi_{1,k}$	1	$\frac{1}{2}+\frac{1}{4}\alpha_{kr}$	$\frac{1}{2}\alpha_{kr}$	$\frac{1}{2}\alpha_{kr}$	$\frac{1}{4}(\alpha_{kr}+\alpha_{ks})$	$\frac{1}{2}$	$\frac{1}{4}\alpha_{kr}$	0	0
$\psi_{2,k}$	1	$\frac{1}{2}+\frac{1}{4}\alpha_{kr}$	$\frac{1}{2}\alpha_{kr}$	$\frac{1}{2}\alpha_{kr}$	$\frac{1}{4}(\alpha_{kr}+\alpha_{ks})$	$-\frac{1}{2}$	$-\frac{1}{4}\alpha_{kr}$	0	0
$\chi_{k,l,1}$	1	$\frac{1}{4}(\alpha_{kr}+\alpha_{lr})$	$\frac{1}{2}(\alpha_{kr+lr})$	$\frac{1}{2}(\alpha_{kr-lr})$	β_{krls}	0	0	0	0
$\chi_{k,l,2}$	1	$\frac{1}{4}(\alpha_{kr}+\alpha_{lr})$	$\frac{1}{2}(\alpha_{kr-lr})$	$\frac{1}{2}(\alpha_{kr+lr})$	γ_{krls}	0	0	0	0
$\chi_{k,k,1}$	1	$\frac{1}{2}\alpha_{kr}$	1	$\frac{1}{2}\alpha_{2kr}$	$\frac{1}{2}\alpha_{kr-ks}$	0	0	1	$\frac{1}{2}\alpha_{kr}$
$\chi_{k,k,2}$	1	$\frac{1}{2}\alpha_{kr}$	1	$\frac{1}{2}\alpha_{2kr}$	$\frac{1}{2}\alpha_{kr-ks}$	0	0	-1	$-\frac{1}{2}\alpha_{kr}$
$\chi_{k,k,3}$	1	$\frac{1}{2}\alpha_{kr}$	$\frac{1}{2}\alpha_{2kr}$	1	$\frac{1}{2}\alpha_{kr+ks}$	0	0	0	0

Where $1 \le r \le m$, $1 \le |s| < r$, $1 \le k \le m$, $1 \le l < m$, and $\beta_{krls} = \frac{1}{4}(\alpha_{kr+ls}+\alpha_{ks+lr})$, $\gamma_{krls} = \frac{1}{4}(\alpha_{kr-ls}+\alpha_{ks-lr})$. This gives us $3+m+m+m(m-1)/2+m(m-1)/2+m+m+m = m^2+4m+3$ (irreducible) linear representations of $\mathfrak{S}_G^{(2)}$, which was a ring of dimension m^2+4m+3.

Voila! We have the 2-character table for an infinite family of groups. Because all representations of $\mathfrak{S}_G^{(2)}$ are linear, this ring is commutative.

5.4 On the representation of $\mathfrak{S}_G^{(2)}$ corresponding to a Frobenius 2-character

In this section we finish the proof that all extended 2-characters of a finite group G correspond to representations of the 2-S-ring of G. Recall that if $\mathrm{Irr}(G) = \{\theta_1, \theta_2, \ldots, \theta_k\}$, then the extended 2-characters are the functions:

(i) $\theta_i^{(2)}(g,h) = \theta_i(g)\theta_i(h) - \theta_i(gh)$, for each θ_i of degree > 1.

(ii) $\theta_i^{(2,+)}(g,h) = \theta_i(g)\theta_i(h) + \theta_i(gh)$, for each θ_i of degree > 1.

(iii) $\theta_i \circ \theta_j(g,h) = \theta_i(g)\theta_j(h) + \theta_i(h)\theta_j(g)$, for θ_i, θ_j distinct.

In Theorem 5.9 we showed that if ρ_i, ρ_j are representations corresponding to characters θ_i, θ_j, then $\frac{1}{2}\theta_i \circ \theta_j(g,h)$ is the $\mathfrak{S}_G^{(2)}$ character of G^2 corresponding to the representation $\widehat{\rho_i \otimes \rho_j}$.

Throughout this section, let G be a finite group. Let V be an n dimensional $\mathbb{C}G$ module with character χ. Fix $B = \{e_1, e_2, \ldots, e_n\}$ as a basis of V. We are consistently going to work with our fixed basis B in this section. In particular, $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, etc.

Now consider $V \otimes V$. The symmetric basis elements $\{e_i \otimes e_j + e_j \otimes e_i, 1 \le i \le j \le n\}$ of $V \otimes V$ form a basis for a subspace $S = S(V \otimes V)$ of $V \otimes V$, and the anti-symmetric basis elements $\{e_i \otimes e_j - e_j \otimes e_i, 1 \le i < j \le n\}$ form a basis for $A = A(V \otimes V)$. We prove:

Theorem 5.13. *Let G be a finite group and V a n-dimensional $\mathbb{C}G$ module with character χ, $n \ge 2$. Then $A(V \otimes V)$ is an $S_G^{(2)}$ module with character $\frac{1}{2}\chi^{(2)}(g,h)$.*

Thus, for any non-linear representation ρ of G, the representation $\widehat{\rho \otimes \rho}$ is a reducible representation of $\mathfrak{S}_G^{(2)}$. When ρ is a linear representation, then $\widehat{\rho \otimes \rho}$ is also a linear representation, and $A(V \otimes V) = \emptyset$. If χ is a linear character, then $\chi^{(2)} = 0$.

Because $V \otimes V$ is a direct sum of S and A, and $\chi^{(2,+)} + \chi^{(2)} = \widehat{\chi \otimes \chi}$, it follows that $\chi^{(2,+)}$ is the character of the $\mathfrak{S}_G^{(2)}$ module $S(V \otimes V)$. It follows from this result and Corollary 5.10 that:

Theorem 5.14. *Every extended 2-character of a finite group G corresponds to a representation of the 2-S-ring of G.* □

Theorem 5.14 isn't surprising. It is well known that G acts on $V \otimes V$ by $g(e_i \otimes e_j) = ge_i \otimes ge_j$, and the character χ^2 of G can be broken up as the sum of a symmetric part and an antisymmetric part by considering the action of G on the symmetric and antisymmetric parts of $V \otimes V$ respectively. (See, for example, [11, p. 196-208]). We show that under the (standard) action of G^2 on $V \otimes V$ defined by $(g,h)(e_i \otimes e_j) = g(e_i) \otimes h(e_j)$, A is an $\mathfrak{S}_G^{(2)}$ module with character $\frac{1}{2}\chi^{(2)}$, where $\chi^{(2)}$ is the Frobenius 2-character.

Lemma 5.15. *Let V be a $\mathbb{C}G$-module and let A be the anti-symmetric part of $V \otimes V$. Then A is an $\mathfrak{S}_G^{(2)}$-module with the action inherited from the $\mathbb{C}G$ action.*

Proof. We note that for any $(x,y) \in G^2$,

$$[(x,y)+(y,x)](e_i \otimes e_j - e_j \otimes e_i) = (xe_i \otimes ye_j - ye_j \otimes xe_i) + (ye_i \otimes xe_j - xe_j \otimes ye_i) \in A.$$

If τ is the average class sum of $B = K(g,h) \in G^2$, then by Lemma 5.8 we have

$$\tau(e_i \otimes e_j - e_j \otimes e_i) = \frac{1}{2|B_g|} \sum_{(x,y)\in B_g} [(x,y)+(y,x)](e_i \otimes e_j - e_j \otimes e_i).$$

Thus, $\tau(e_i \otimes e_j - e_j \otimes e_i)$ is a linear combination of elements of A, and so is in A. $\square$

In order to calculate character values we use the basic fact about tensor products that if $A = (a_{ij}), B = (b_{ij})$ are $n \times n$ matrices, then

$$(A \otimes B)(e_i \otimes e_j) = \sum_{r=1}^{n} \sum_{s=1}^{n} a_{ri} b_{sj} e_r \otimes e_s. \text{ (***)}$$

Using this fact, we can prove the following proposition:

Proposition 5.16. *Let $A = (a_{ij}), B = (b_{ij})$ be $n \times n$ matrices. Then*

$$(A \otimes B + B \otimes A)(e_i \otimes e_j - e_j \otimes e_i) =$$
$$\sum_{1 \le r < s \le n} (a_{ri}b_{sj} + b_{ri}a_{sj} - a_{rj}b_{si} - b_{rj}a_{si})(e_r \otimes e_s - e_s \otimes e_r).$$

Proof. We set

$$P = (A \otimes B + B \otimes A)(e_i \otimes e_j - e_j \otimes e_i).$$

Distributing, we get

$$P = (A \otimes B)(e_i \otimes e_j) + (B \otimes A)(e_i \otimes e_j) - (A \otimes B)(e_j \otimes e_j) - (B \otimes A)(e_j \otimes e_i).$$

It follows from (***) that we have

$$P = \sum_{r=1}^{n} \sum_{s=1}^{n} [a_{ri}b_{sj} + b_{ri}a_{sj} - (a_{rj}b_{si} + b_{rj}a_{si})] e_r \otimes e_s.$$

Next, we note that when $r = s$, then the coefficient $a_{ri}b_{sj} + b_{ri}a_{sj} - (a_{rj}b_{si} + b_{rj}a_{si})$ is 0, so we have

$$\begin{aligned} P &= \sum_{r=1}^{n} \sum_{s=1}^{n} [a_{ri}b_{sj} + b_{ri}a_{sj} - (a_{rj}b_{si} + b_{rj}a_{si})] e_r \otimes e_s \\ &= \sum_{1 \le r < s \le n} [a_{ri}b_{sj} + b_{ri}a_{sj} - (a_{rj}b_{si} + b_{rj}a_{si})] e_r \otimes e_s \\ &\quad + \sum_{1 \le s < r \le n} [a_{ri}b_{sj} + b_{ri}a_{sj} - (a_{rj}b_{si} + b_{rj}a_{si})] e_r \otimes e_s. \end{aligned}$$

We can rewrite the second term using a change of variables to get:

$$\sum_{1 \le s < r \le n} [a_{ri}b_{sj} + b_{ri}a_{sj} - (a_{rj}b_{si} + b_{rj}a_{si})] e_r \otimes e_s = \sum_{1 \le r < s \le n} [a_{si}b_{rj} + b_{si}a_{rj} - (a_{sj}b_{ri} + b_{sj}a_{ri})] e_s \otimes e_r,$$

and when we substitute this in to the equation above, we get

$$\begin{aligned} P = & \sum_{1 \le r < s \le n} [a_{ri}b_{sj} + b_{ri}a_{sj} - (a_{rj}b_{si} + b_{rj}a_{si})] e_r \otimes e_s \\ & \quad + \sum_{1 \le r < s \le n} [a_{si}b_{rj} + b_{si}a_{rj} - (a_{sj}b_{ri} + b_{sj}a_{ri})] e_s \otimes e_r. \end{aligned}$$

Factoring a negative out of the second term, we get

$$\begin{aligned} P = & \sum_{1 \le r < s \le n} [a_{ri}b_{sj} + b_{ri}a_{sj} - (a_{rj}b_{si} + b_{rj}a_{si})] e_r \otimes e_s \\ & \quad - \sum_{1 \le r < s \le n} [a_{sj}b_{ri} + b_{sj}a_{ri} - (a_{si}b_{rj} + b_{si}a_{rj})] e_s \otimes e_r. \end{aligned}$$

But these coefficients are equal, so in fact we have

$$P = \sum_{1 \le r < s \le n} (a_{ri}b_{sj} + b_{ri}a_{sj} - a_{rj}b_{si} - b_{rj}a_{si})(e_r \otimes e_s - e_s \otimes e_r).$$

And because $P = (A \otimes B + B \otimes A)(e_i \otimes e_j - e_j \otimes e_i)$, this concludes the proof. □

We are now ready to prove Theorem 5.13.

Proof. Let χ be the character of a representation ρ corresponding to an n-dimensional module V, $n \ge 2$. We have shown that $A(V \otimes V)$ is a $\mathfrak{S}_G^{(2)}$ module. We let $\hat{\chi}$ denote the $\mathfrak{S}_G^{(2)}$ character of G^2 corresponding to this module.

Fix $(g, h) \in G^2$ and let $B = K(g, h)$ so that $B_g = \{(x, y) \in B | x \sim g\}$. We write $B_g = \{(g_1, h_1), (g_2, h_2), \ldots, (g_m, h_m)\}$, and let $\rho(g_i) = A_i$, $\rho(h_i) = B_i$, $\rho(g) = A$, $\rho(h) = B$. We let $A_t = (a_{ij}^{(t)})$, $B_t = (b_{ij}^{(t)})$.

By Lemma 5.8 the average class sum of B is $\tau = \frac{1}{2m} \sum_{t=1}^{m} [(g_t, h_t) + (h_t, g_t)]$, and

$$\rho(\tau) = \frac{1}{2m} \sum_{t=1}^{m} (A_t \otimes B_t + B_t \otimes A_t).$$

By Proposition 5.16 we have

$$\rho(\tau)(e_i \otimes e_j - e_j \otimes e_i) = \frac{1}{2m} \sum_{t=1}^{m} \sum_{1 \le r < s \le n} (a_{ri}^{(t)} b_{sj}^{(t)} + b_{ri}^{(t)} a_{sj}^{(t)} - a_{rj}^{(t)} b_{si}^{(t)} - b_{rj}^{(t)} a_{si}^{(t)})(e_r \otimes e_s - e_s \otimes e_r).$$

We are looking for the character $\hat{\chi}$ associated to this module, which is the trace of this representation. Since $A(V \otimes V)$ has basis $\{e_i \otimes e_j - e_j \otimes e_i | 1 \le j < i \le n\}$, we will get $\hat{\chi}(\tau)$ by taking the sum of the coefficient of $e_i \otimes e_j - e_j \otimes e_i$ in $\rho(\tau)(e_i \otimes e_j - e_j \otimes e_i)$ over all basis elements $\{e_i \otimes e_j - e_j \otimes e_i | 1 \le j < i \le n\}$.

This coefficient is

$$\frac{1}{2m} \sum_{t=1}^{m} (a_{ii}^{(t)} b_{jj}^{(t)} + b_{ii}^{(t)} a_{jj}^{(t)} - a_{ij}^{(t)} b_{ji}^{(t)} - b_{ij}^{(t)} a_{ji}^{(t)}),$$

so

$$\begin{aligned}
\hat{\chi}(\tau) &= \sum_{1\le i<j\le n} \tfrac{1}{2m}\sum_{t=1}^{m}(a_{ii}^{(t)}b_{jj}^{(t)} + b_{ii}^{(t)}a_{jj}^{(t)} - a_{ij}^{(t)}b_{ji}^{(t)} - b_{ij}^{(t)}a_{ji}^{(t)}) \\
&= \tfrac{1}{2}\left(\sum_{1\le i<j\le n} \tfrac{1}{2m}\sum_{t=1}^{m}(a_{ii}^{(t)}b_{jj}^{(t)} + b_{ii}^{(t)}a_{jj}^{(t)} - a_{ij}^{(t)}b_{ji}^{(t)} - b_{ij}^{(t)}a_{ji}^{(t)})\right) \\
&\quad + \tfrac{1}{2}\left(\sum_{1\le j<i\le n} \tfrac{1}{2m}\sum_{t=1}^{m}(a_{jj}^{(t)}b_{ii}^{(t)} + b_{jj}^{(t)}a_{ii}^{(t)} - a_{ji}^{(t)}b_{ij}^{(t)} - b_{ij}^{(t)}a_{ij}^{(t)})\right)
\end{aligned}$$

Where the final two terms are equal, but we have changed variables, relabeling i with j and j with i in the second term.

When $i = j$, $a_{ii}^{(t)}b_{jj}^{(t)} + b_{ii}^{(t)}a_{jj}^{(t)} - a_{ij}^{(t)}b_{ji}^{(t)} - b_{ij}^{(t)}a_{ji}^{(t)} = a_{ii}^{(t)}b_{ii}^{(t)} + b_{ii}^{(t)}a_{ii}^{(t)} - a_{ii}^{(t)}b_{ii}^{(t)} - b_{ii}^{(t)}a_{ii}^{(t)} = 0$, so we can write

$$\begin{aligned}
\hat{\chi}(\tau) &= \tfrac{1}{2}\left(\sum_{1\le i,j\le n} \tfrac{1}{2m}\sum_{t=1}^{m}(a_{ii}^{(t)}b_{jj}^{(t)} + b_{ii}^{(t)}a_{jj}^{(t)} - a_{ij}^{(t)}b_{ji}^{(t)} - b_{ij}^{(t)}a_{ji}^{(t)})\right) \\
&= \tfrac{1}{4m}\left(\sum_{t=1}^{m}\sum_{1\le i,j\le n}(a_{ii}^{(t)}b_{jj}^{(t)} + b_{ii}^{(t)}a_{jj}^{(t)} - a_{ij}^{(t)}b_{ji}^{(t)} - b_{ij}^{(t)}a_{ji}^{(t)})\right).
\end{aligned}$$

Now we fix t and consider the sum $\sum_{1\le i,j\le n}\left(a_{ii}^{(t)}b_{jj}^{(t)} + b_{ii}^{(t)}a_{jj}^{(t)} - a_{ij}^{(t)}b_{ji}^{(t)} - b_{ij}^{(t)}a_{ji}^{(t)}\right)$.

Because

$$\operatorname{tr}(A_t) = \sum_{i=1}^{n} a_{ii}^{(t)},\ \operatorname{tr}(B_t) = \sum_{j=1}^{n} b_{jj}^{(t)},\ \text{and}\ \operatorname{tr}(A_tB_t) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(t)}b_{ji}^{(t)},$$

we see that

$$\sum_{1\le i,j\le n}(a_{ii}^{(t)}b_{jj}^{(t)} + b_{ii}^{(t)}a_{jj}^{(t)} - a_{ij}^{(t)}b_{ji}^{(t)} - b_{ij}^{(t)}a_{ji}^{(t)}) = \operatorname{tr}(A_t)\operatorname{tr}(B_t) + \operatorname{tr}(B_t)\operatorname{tr}(A_t) - \operatorname{tr}(A_tB_t) - \operatorname{tr}(B_tA_t).$$

But $(g_t, h_t) = (g,h)^\alpha$ for some $\alpha \in G$, so $g_t \sim g$, $h_t \sim h$, and $g_th_t \sim gh$, all via conjugation by α. Thus, A_t, B_t, and A_tB_t are similar to $A = \rho(g)$, $B = \rho(h)$, and AB respectively. So

$$\sum_{1\le i,j\le n}(a_{ii}^{(t)}b_{jj}^{(t)} + b_{ii}^{(t)}a_{jj}^{(t)} - a_{ij}^{(t)}b_{ji}^{(t)} - b_{ij}^{(t)}a_{ji}^{(t)}) = 2[\operatorname{tr}(A)\operatorname{tr}(B) - \operatorname{tr}(AB)].$$

Because this is true for all t, $1 \leq t \leq m$, we have

$$\begin{aligned}
\hat{\chi}(\tau) &= \frac{1}{4m}\sum_{t=1}^{m}\sum_{1\leq i,j\leq n}\left(a_{ii}^{(t)}b_{jj}^{(t)} + b_{ii}^{(t)}a_{jj}^{(t)} - a_{ij}^{(t)}b_{ji}^{(t)} - b_{ij}^{(t)}a_{ji}^{(t)}\right) \\
&= \frac{1}{4m}\sum_{t=1}^{m} 2[\mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB)] = \frac{1}{2}\left[\chi(g)\chi(h) - \chi(gh)\right].
\end{aligned}$$

So the 2-character $\chi^{(2)}$ is the character of the module $A(V \otimes V)$. $\square$

5.5 The Character Table of a Commutative k-S-ring.

If $\mathfrak{S}_G^{(k)}$ is commutative, then it is possible to find the character table of the k-S-ring by solving equations determined by the structure constants of the $\mathfrak{S}_G^{(k)}$.

In his work, Frobenius shows [4, Prop 3.2] that if T is an associative, commutative algebra, of dimension r, then there are exactly r non-trivial solutions $v_i = (v_{i1}, v_{i2}, \ldots, v_{ir}), 1 \leq i \leq r$, over the complex numbers to the set of equations

$$x_i x_j = \sum_k \lambda_{ijk} x_k,$$

where the λ_{ijk} are the structure constants of the algebra and the x_u are independent commuting variables. He also shows that these solution vectors are linearly independent.

For an arbitrary (not necessarily commutative) S-ring T we can still look at the set of complex solutions of these equations, however one may not get a full set of r linearly independent such solutions. If the solutions are $v_i = (v_{i1}, v_{i2}, \ldots, v_{ir}), 1 \leq i \leq s \leq r$, then one gets a matrix of solutions of size $s \times r$ that we denote by $\mathfrak{M}(T)$. We note that if the sizes of the principal sets of $\mathfrak{S}$ are $n_1, \ldots, n_r$, then $(n_1, n_2, \ldots, n_r)$ is a solution. We always make this the first row.

If the S-ring in question is $\mathfrak{S}_G^{(k)}$, then we will denote the matrix one obtains by $\mathfrak{CT}_G^{(k)}$.

Example 5.17. For $G = S_3, a = (1,2,3), b = (1,2)$ and the 2-classes with representatives in the order

$$(e,e),(e,a),(e,b),(a,a),(a,a^{-1}),(a,b),(b,b),(b,ab),$$

we have:

$$\mathfrak{CT}^{(2)}_{S_3} = \begin{bmatrix} 1 & 4 & 6 & 2 & 2 & 12 & 3 & 6 \\ 1 & 4 & -6 & 2 & 2 & -12 & 3 & 6 \\ 1 & 4 & 0 & 2 & 2 & 0 & -3 & -6 \\ 1 & -2 & 0 & 2 & -1 & 0 & -3 & 3 \\ 1 & -2 & 0 & 2 & -1 & 0 & 3 & -3 \\ 1 & -2 & 0 & -1 & 2 & 0 & 0 & 0 \\ 1 & 1 & -3 & -1 & -1 & 3 & 0 & 0 \\ 1 & 1 & 3 & -1 & -1 & -3 & 0 & 0 \end{bmatrix}$$

If G is commutative, this matrix will give us a normalized version of the character table $CT(\mathfrak{S}^{(k)}_G)$ of $\mathfrak{S}^{(k)}_G$. To get the character table we need only divide the entries in each column by the size of the principal set corresponding to that column.

The following is a generalization of the fact that the information that is contained in the character table of a group G is the same as the information contained in $Z(\mathbb{C}G)$:

Theorem 5.18. *Let $k \geq 1$. If G has a commutative k-S-ring, then $\mathfrak{CT}^{(k)}_G$ and $\mathfrak{S}^{(k)}_G$ determine each other and so contain the same information about G (in the weaker sense of being algebraically isomorphic, see example 1.12).*

Proof. We let λ_{ijm} be the structure constants for $\mathfrak{S}^{(k)}_G$ (relative to a fixed ordering of the basis of k-classes), which is assumed to be a commutative ring. Note, as before, that [4, Prop 3.2] shows that there are exactly r non-trivial solutions $v_i = (v_{i1}, v_{i2}, \ldots, v_{ir}) \in \mathbb{C}^r, i \leq r$, over the complex numbers to the set of equations

$$x_i x_j = \sum_m \lambda_{ijm} x_m,$$

where the x_u are commuting independent variables. It was also shown that these solution vectors are linearly independent, so that the matrix $V = (v_{ij})$ is non-singular.

Consider the column vectors $V_j = (v_{1j}, v_{2j}, \dots, v_{rj})^T$. Then the above shows that $V_1, \dots, V_r$ are linearly independent and span $\mathbb{C}^r$. For two column vectors $U = (u_1, \dots, u_r)$, $W = (w_1, \dots, w_r) \in \mathbb{C}^r$ we let $U \circ W = (u_1 w_1, u_2 w_2, \dots, u_r w_r)$. Then the fact that each v_i is a solution shows that each V_j is a solution vector in the following sense:

$$V_i \circ V_j = \sum_m \lambda_{ijm} V_m.$$

Since the V_i form a basis, the λ_{ijm} are completely determined by the basis $V_1, \dots, V_r$. Thus $\mathfrak{CT}_G^{(k)}$ determines the structure constants for $\mathfrak{S}_G^{(k)}$, as required. Conversely, by definition $\mathfrak{S}_G^{(k)}$ determines $\mathfrak{CT}_G^{(k)}$. □

It should be noted that there are groups G with $\mathfrak{S}_G^{(2)}$ not commutative, the Frobenius group of order 20 being such an example. However, when $\mathfrak{S}_G^{(2)}$ is commutative, this gives us a method of computing the character table using MAGMA.

Chapter 6. Finite groups with commutative k-S-rings

It is well known that the 1-S-ring of any finite group is commutative. There exist finite groups with non-commutative 2-S-rings, for example the Frobenius group of order 20. Finite groups with commutative 2-S-rings have not been completely classified. However, finite groups with commutative 3-S-rings are understood. In this chapter, we classify finite groups with commutative 3-S-rings.

We will need the following definition:

Definition 6.1. A finite group G will be called a 3-*S-ring group* if for all ordered pairs $x, y \in G$ we have one of:

(1) $xy = yx$;

(2) x and y are conjugate;

(3) $x^y = x^{-1}$;

(4) $y^x = y^{-1}$.

First, we show that the notion of a 3-S-ring-group arises naturally when studying groups with commutative 3-S-ring.

Proposition 6.2. *Let G be a group with commutative 3-S-ring. Then G is a 3-S-ring group.*

Proof. Let $g, h \in G$. We wish to show that the pair g, h satisfies one of (1),...,(4). We consider the elements $x = (g, 1, g), y = (h, h, 1) \in G^3$ and let $A = K^{(3)}(x), B = K^{(3)}(y)$ denote their 3-classes. We have $xy = (gh, h, g) \in AB$, so that (gh, h, g) is a term of $\overline{AB}$. But because $\mathfrak{S}_G^{(3)}$ is commutative, we know (gh, h, g) is a term of $\overline{BA}$ as well, so that $(gh, h, g) \in BA$.

Now the elements of $A = K^{(3)}(x)$ have the form

$$(i)\ (g^a, 1, g^a), \quad (ii)\ (g^a, g^a, 1), \quad (iii)\ (1, g^a, g^a),$$

for some $a \in G$, and the elements of $B = K^{(3)}(y)$ have the form

$$(i')\ (h^b, 1, h^b), \quad (ii')\ (h^b, h^b, 1), \quad (iii')\ (1, h^b, h^b),$$

for some $b \in G$. It follows that $xy = (gh, g, h)$ occurs in BA as one of the following possible products:

Case (i), (i'): here

$$(h^b, 1, h^b)(g^a, 1, g^a) = (gh, h, g),$$

and so $h = 1$, giving $[g, h] = 1$.

Case (i), (ii'): here

$$(h^b, h^b, 1)(g^a, 1, g^a) = (gh, h, g),$$

and so $h^b g^a = gh, h^b = h, g^a = g$, giving $[g, h] = 1$.

Case (i), (iii'): here

$$(1, h^b, h^b)(g^a, 1, g^a) = (gh, h, g),$$

and so $g^a = gh, h^b = h, h^b g^a = g$ giving $g^a = h^{-1}g = gh$, so that $h^g = h^{-1}$.

Case (ii), (i'): here

$$(h^b, 1, h^b)(g^a, g^a, 1) = (gh, h, g),$$

and so $g^a = h$, giving $g \sim h$.

Case (ii), (ii'): here

$$(h^b, h^b, 1)(g^a, g^a, 1) = (gh, h, g),$$

and so $g = 1$, giving $[g, h] = 1$.

Case (ii), (iii'): here

$$(1, h^b, h^b)(g^a, g^a, 1) = (gh, h, g),$$

and so $h^b = g$, giving $g \sim h$.

Case (iii), (i'): here

$$(h^b, 1, h^b)(1, g^a, g^a) = (gh, h, g),$$

and so $g^a = h$, giving $g \sim h$.

Case (iii), (ii'): here

$$(h^b, h^b, 1)(1, g^a, g^a) = (gh, h, g),$$

and so $h^b = gh, h^b g^a = h, g^a = g$, giving $gh = h^b = hg^{-1}$. We thus have $g^h = g^{-1}$.

Case (iii), (iii'): here

$$(1, h^b, h^b)(1, g^a, g^a) = (gh, h, g),$$

and so $gh = 1$. We thus have $[g, h] = 1$.

This concludes consideration of all cases. $\square$

6.1 Classification of 3-S-ring groups

Now we characterize 3-S-ring groups. In particular, we prove:

Theorem 6.3. *A finite group G is a 3-S-ring group if G satisfies one of the following conditions:*

(i) G is abelian;

(ii) G is generalized dihedral of order $2n$, n odd. Recall that a group G is generalized dihedral if $G \cong N \rtimes C_2$, where N is a finite abelian group and $C_2 = \langle t \rangle$ acts on $y \in N$ by $y^t = y^{-1}$.

(iii) $G \cong Q_8 \times C_2^r$, where C_2 is the cyclic group, $r \geq 0$, and Q_8 is the quaternion group of order 8.

We note that each of the above groups have irreducible characters of degree at most 2, such groups having been characterized by Amitsur [1].

We will be constantly referring to conditions (1)-(4) of Definition 6.1 for an ordered pair of elements $x, y \in G$. We let $\sim_H$ denote conjugacy in the subgroup H of G.

Proof. Suppose that G is a group satisfying (i), (ii) or (iii) of Theorem 6.3. We show that G is a 3-S-ring group, i.e. that any pair $x, y \in G$ satisfies one of (1),(2),(3) or (4). This is clear if G is abelian.

Assume that G satisfies (ii). Then we have $G \cong N \rtimes C_2$ for some abelian subgroup N and $C_2 = \langle t \rangle$. Because N has odd order, we know that $G \setminus N$ is a single class of G, and any element $x \in G \setminus N$ can be written $x = x't$ for some $x' \in N$. If $g, h \in N$, then $[g, h] = 1$. If $g, h \in G \setminus N$, then $g \sim h$. If $g \in N, h \in G \setminus N$, then $h = h't$ for some $h' \in N$ and $g^h = g^{h't} = g^t = g^{-1}$. So G is a 3-S-ring group.

Lastly, we now suppose that $G \cong Q_8 \times C_2^r$, where $Q_8 = \langle x, y | x^y = x^{-1}, y^x = y^{-1}, x^4 = y^4 = 1 \rangle$ for some $x, y \in G$. If $u, v \in G = Q_8 \times C_2^r$, where $[u, v] \neq 1$, then it is easy to see that we can assume $u = xu_1, v = yv_1$, where $u_1, v_1 \in Z(G)$. Then $u_1^2 = v_1^2 = 1$ and we have

$u^v = (xu_1)^{yv_1} = x^y u_1 = x^{-1}u_1^{-1} = u^{-1}$. Thus any $u, v \in G$ either commute or satisfy one of $u^v = u^{-1}$ or $v^u = v^{-1}$.

Thus each of the groups listed is a 3-S-ring-group. $\square$

In any group, let y be any element of odd order and suppose $x^y = x^{-1}$ for some $x \neq 1$. Then $x = x^{y^2} = x^{y^4} = \cdots = x^y$, a contradiction. Thus we have:

Proposition 6.4. *If $x, y \in G$, G a 3-S-ring group, and $o(y)$ is odd, then either $x \sim y$, $[x, y] = 1$, or $y^x = y^{-1}$. If both x and y have odd order and $x \not\sim y$, then $[x, y] = 1$.* $\square$

Let $N \subseteq G$ be the set of elements having odd order. Using the observations of the proposition 6.4, we can prove the following:

Proposition 6.5. *Let G be a 3-S-ring group of even order. If $y \in G$ has odd order, then y is central or $y^G = \{y, y^{-1}\}$.*

Proof. If $x \in G \setminus y^G$, then $[x, y] = 1$ or $y^x = y^{-1}$.

Suppose $y \not\sim y^{-1}$. Then every element of G either commutes with y or is conjugate to y, i.e. $G = y^G \cup C_G(y)$. If $C_G(y) = G$, then y is central. If not, then by size consideration we must have $|y^G| = |C_G(y)| = |G|/2$, and so $|G| = |y^G||C_G(y)| = |G|^2/4$ and $|G| = 4$, a contradiction, because G is not abelian.

Now suppose $y^x = y^{-1}$ for some $x \in G$. Then for any $g \in G$ either $[g, y] = 1$, $y \sim g$, or $y^g = y^{=1}$, so that $G = y^G \cup C_G(y) \cup xC_G(y)$. We know y is not central, so $C_G(y)$ is a proper subgroup of G. Suppose by way of contradiction that $|y^G| > 2$. Because y has odd order, we know $|C_G(y)| \geq 3$ and $|y^G| \leq |G|/3$. Also, $y \sim y^{-1}$, so we must have $|y^G| \geq 4$ from which it follows that $|C_G(y)| \leq |G|/4$. And $G = y^G \cup C_G(y) \cup xC_G(y)$ so we have $|G| \leq |G|/3 + |G|/4 + |G|/4$, which gives the contradiction. $\square$

Corollary 6.6. *If N is the set of elements of odd order, then N is an abelian, normal subgroup of G.*

Proof. Let $x, y \in G$ be elements of odd order. It follows from Proposition 6.5 that $y^x \in \{y, y^{-1}\}$. From Proposition 6.4 we know $y^x \neq y^{-1}$, so in fact x and y commute. It follows both that N is a subgroup, and that N is abelian. $\square$

We will also need the following lemma:

Lemma 6.7. *Let $x \in G \setminus N$. Then there is $\varepsilon = \varepsilon(x) = \pm 1$ such that $y^x = y^\varepsilon$ for all $y \in N$.*

Proof. Suppose that there are $y_1, y_2 \in N \setminus \{1\}$ such that $y_1^x = y_1, y_2^x = y_2^{-1}$. Then $(y_1y_2)^x = y_1y_2^{-1}$. But by Lemma 6.4 we have either (a) $(y_1y_2)^x = y_1y_2$; or (b) $(y_1y_2)^x = (y_1y_2)^{-1}$. If we have (a), then $y_1y_2^{-1} = y_1y_2$, shows that $y_2^2 = 1$. But $y_2 \in N, y_2 \neq 1$, is a contradiction.

If we have (b), then $y_1y_2^{-1} = y_2^{-1}y_1^{-1}$, showing that $y_1^{y_2^{-1}} = y_1^{-1}$, which again gives a contradiction, by Proposition 6.4. $\square$

We are now ready to prove the following:

Theorem 6.8. *If G is a 3-S-group, then one of the following is true:*

(i) G is abelian;

(ii) G is generalized dihedral of order $2n$, n odd.

(iii) $G \cong Q_8 \times C_2^r$, where C_2 is the cyclic group, $r \geq 0$, and Q_8 is the quaternion group of order 8.

Proof. Let $S_2 = S_2(G)$ be a Sylow 2-subgroup of G. The proof is by induction on $|S_2| \geq 1$. The case where $|S_2| = 1$ is covered by Corollary 6.6. This starts the induction. So now assume that we have a 3-S-ring group G with $S_2(G) \neq \{1\}$ and that the result is true for such groups with smaller S_2. We may also assume that G is not abelian.

First consider the situation where G is a 2-group. Let $z \in Z(G), |z| = 2$. Then $G/\langle z\rangle$ is also a 3-S-ring group with smaller S_2. Thus the induction shows that either (A) $G/\langle z\rangle$ is abelian; or (B) $G/\langle z\rangle \cong Q_8 \times C_2^r, r \geq 0$.

Suppose we have (A). Then the classes of G are either central elements or cosets of the central subgroup $\langle z\rangle$. In particular, elements of the same class commute. Thus we can write G as a disjoint union of classes:

$$G = Z(G) \cup g_1\langle z\rangle \cup g_2\langle z\rangle \cup \cdots \cup g_s\langle z\rangle.$$

Lemma 6.9. *(i) If $|g_i| = |g_j| = 2, i \neq j$, then $[g_i, g_j]) = 1$.*

(ii) If $|g_i| = 2, |g_j| > 2$, then $[g_i, g_j] = 1$.

In particular, all involutions are central in G.

Proof. (i) If, for the pair of involutions g_i, g_j, we have (1), (3) or (4), then $[g_i, g_j] = 1$. However we cannot have (2), since $i \neq j$. This gives (i).

(ii) For the pair g_i, g_j we cannot have (2), and (3) implies that $[g_i, g_j] = 1$.

So suppose that we have (4): $g_2^{g_1} = g_2^{-1}$. We next note that $g_1g_2\langle z\rangle$ is either a central set or is a conjugacy class. If g_1g_2 is central then $(g_1, g_1g_2) = 1$, showing that $[g_1, g_2] = 1$. So now suppose that $g_1g_2\langle z\rangle$ is a class. Note that in fact $g_1g_2\langle z\rangle \neq g_1\langle z\rangle$. Thus the pair g_1, g_1g_2 does not satisfy (2) (in G). If g_1, g_2 satisfies (3), i.e. $g_1^{g_1g_2} = g_1^{-1} = g_1$, then we have $(g_1, g_2) = 1$, as required. Lastly, if g_1, g_1g_2 satisfies (4), then

$$(g_1g_2)^{g_1} = g_2^{-1}g_1^{-1} = g_2^{-1}g_1.$$

Then using $g_2^{g_1} = g_2^{-1}$ we see that this gives $g_1g_2^{-1} = g_1g_2^{g_1} = (g_1g_2)^{g_1} = g_2^{-1}g_1$, showing that $[g_1, g_2] = 1$. □

Lemma 6.10. *Let $x, y \in G$ where $(x, y) \neq 1$. Then we have*

$$x^4 = y^4 = 1, \quad x^2 = y^2 = z, \quad [x, z], \quad [y, z], \quad x^y = x^{-1}, \quad y^x = y^{-1}. \quad (6.1)$$

and $\langle x, y\rangle \cong Q_8$.

Proof. If we can show these relations, then certainly $\langle x, y\rangle \cong Q_8$. Since $z \in Z(G)$ we have $[x, z] = [y, z] = 1$. $\square$

For the pair x, y of Lemma 6.10we do not have (1). If we have (2), $x \sim y$, then $G/\langle z\rangle$ abelian means that either $x = y$ or $x = yz$, and in either case we have $[x, y] = 1$, a contradiction. Thus we must have $x^y = x^{-1}$ or $y^x = y^{-1}$. By symmetry there is no loss in assuming $x^y = x^{-1}$. But $x^y \neq x$ implies that $x^y = xz$. Then we have $xz = x^y = x^{-1}$, giving $x^2 = z$ and $x^4 = 1$.

Now consider the pair yx, y^x. If we have $(yx, y^x) = 1$ for this pair, then one gets $[x, y] = 1$. If we have (2): $yx \sim y^x$, then we have $yx = y^x z$ and so $[yx, y^x] = 1$ again. If we have (3) for this pair, then

$$x^{-1}y^{-1}x \cdot yx \cdot x^{-1}yx = x^{-1}y^{-1},$$

and so $z = x^2 = y^{-2}$ and we are done, since $y^x = y^{-1}$ follows.

If we have (4), then

$$x^{-1}y^{-1} \cdot x^{-1}yx \cdot yx = x^{-1}y^{-1}x,$$

which gives $y^x = y^{-1}$, which in turn gives $y^2 = z$ and $y^4 = 1$. $\square$

Corollary 6.11. *Let $x, y \in G$ where $[x, y] \neq 1$. Let $u \in C_G(\langle x, y\rangle)$. Then $u^2 = 1$. In particular $Z(G) = C_G(\langle x, y\rangle)$ is an elementary 2-group.*

Proof. By Lemma 6.10 we see that x, y satisfy (6.1). Consider the pair xu, yu. This pair cannot satisfy (1) or (2). If we have (3), then $(xu)^{yu} = u^{-1}x^{-1}$ gives $x^y u = x^{-1}u^{-2}$. But from the above we have $x^y = x^{-1}$, and so $u^2 = 1$. We similarly obtain $u^2 = 1$ if we have (4) for this pair. This shows that $C_G(\langle x, y\rangle)$ has exponent 2 and so is an elementary 2-group.

We clearly have $Z(G) \subseteq C_G(\langle x, y\rangle)$, and if $u \in C_G(\langle x, y\rangle)$, then $u^2 = 1$ and so Lemma 6.9 shows that $u \in Z(G)$. $\square$

Proposition 6.12. *Let $x, y \in G$ where $(x, y) \neq 1$. Then $G = \langle x, y, C_G(x, y)\rangle \cong Q_8 \times C_2^r$.*

Proof. Lemma 6.10 shows that x, y satisfy (6.1). Let $w \in G \setminus \langle x, y, C_G(x, y)\rangle$; then one of $[x, w], [y, w]$ is non-trivial. Assume, without loss, that $[w, x] \neq 1$. By Lemma 6.10 we have the relations

$$w^4 = 1, \quad w^2 = z, \quad (w, z), \quad x^w = x^{-1}, \quad w^x = w^{-1}. \quad (6.2)$$

If we also have $[y, w] \neq 1$, then by Lemma 6.10 we have the relations (6.1) and

$$y^w = x^{-1}, \quad w^y = w^{-1}. \quad (6.3)$$

The group satisfying (6.1), (6.2) and (6.3) has the property that $xyz \in C_G(x, y)$ and so $w \in \langle x, y, C_G(x, y)\rangle$, a contradiction.

If $[w, y] = 1$, then the group satisfying (6.1), (6.2) and $[y, w] = 1$ has $yw \in C_G(x, y)$ and so $w \in \langle x, y, C_G(x, y)\rangle$, a contradiction. This shows that $G = \langle x, y, C_G(x, y)\rangle$.

Now from Lemma 6.11 we have $Z(G) = C_G(\langle x, y\rangle) = C_2^{r+1}, r \geq 0$. Now $Z(\langle x, y\rangle) = \langle z\rangle \subset Z(G)$ and so we may write $Z(G) = \langle z\rangle \times C_2^r$; it follows that $G = \langle x, y, C_G(x, y)\rangle = \langle x, y\rangle \times C_2^r$, as required. $\square$

This concludes consideration of (A).

Now suppose that we have (B): Since $G/\langle z\rangle = Q_8 \times C_2^r$ there are $x, y \in G$ such that $\pi(\langle x, y\rangle) = Q_8$. Here $\pi : G \to G/\langle z\rangle$ is the projection. Thus $H = \langle x, y, z\rangle$ is a normal subgroup of G of order 16, where $H/\langle z\rangle \cong Q_8$ for some $z \in Z(H)$. One can check that the only possibilities for H are: (I) $G = Q_8 \times C_2$; and (II) the group

$$J = \langle x, y, u, v | x^2 = v, y^2 = u, u^2, v^2, y^x = yu, [u, x], [u, y][v, x], [v, y]\rangle.$$

We now look at each case separately:

(I) $H = Q_8 \times C_2 = \langle x, y\rangle \times \langle u\rangle$. Then the only possibilities for z are (a) $z = u$ or (b) $z = ux^2$, since these are the only elements of H of order 2 with $H/\langle z\rangle$ non-abelian. In both cases we see that x, y satisfy the relations $x^4 = y^4 = 1, x^y = x^{-1}, y^x = y^{-1}$.

Lemma 6.13. *If $u \in C_G(\langle x, y\rangle)$, then $u^2 = 1$. In particular $C_G(\langle x, y\rangle) \cong C_2^s$.*

Proof. If $u \in H$, then $u \in Z(H) = \langle x^2, z\rangle$ and we certainly have $u^2 = 1$.

If $u \notin H$, then we consider the pair ux, uy. Now $ux \not\sim uy$, as one can see by considering the quotient $G \to Q_8 \times C_2^r \to Q_8$. Clearly we have $[ux, uy] \neq 1$. If we have (3), then $(ux)^{uy} = u^{-1}x^{-1}$ gives $ux^{-1} = ux^y = (ux)^{uy} = u^{-1}x^{-1}$, giving $u^2 = 1$. Condition (4) similarly gives $u^2 = 1$.

The last statement follows from the fact that $C_G(\langle x, y\rangle)$ has exponent 2. $\square$

Lemma 6.14. $G = \langle x, y, C_G(\langle x, y\rangle)\rangle$.

Proof. Let $u \in G \setminus \langle x, y, C_G(\langle x, y\rangle)\rangle$. Then either $[x, u] \neq 1$ or $[y, u] \neq 1$. Assume without loss that $[x, u] \neq 1$. We also have $u \notin H$, so that $u \not\sim w$ for all $w \in H \triangleleft G$. Thus the pair x, u satisfies (3) or (4). Further, the pair y, u satisfies one of (1), (3), (4). We consider the six cases so determined.

(3), (1): $x^u = x^{-1}, y^u = y$. Here we have $u^x = x^2u = y^2u$, so that $(yu)^x = y^{-1}y^2u = yu$. Thus $yu \in C_G(\langle x, y, \rangle)$ and so $u \in \langle x, y, C_G(\langle x, y\rangle)\rangle$, a contradiction

(3), (3): $x^u = x^{-1}, y^u = y^{-1}$. Here we have $(xyu)^x = xy^{-1}(y^2u) = xyu$ and $(xyu)^y = x^{-1}yx^2u = xyu$, giving $xyu \in C_G(\langle x, y\rangle)$ and so $u \in \langle x, y, C_G(\langle x, y\rangle)\rangle$.

(3), (4): $x^u = x^{-1}, u^y = u^{-1}$. Here we consider the pair xy, u: If $(xy, u) = 1$, then $xyu \in C_G(\langle x, y\rangle)$; if $(xy)^u = (xy)^{-1}$, then $yu \in C_G(\langle x, y\rangle)$. Thus in each case we get $u \in \langle x, y, C_G(\langle x, y\rangle)\rangle$. We are left with the case $u^{xy} = u^{-1}$. Here we consider the pair u, yu: if we impose any of the relations (1), (3), (4) on u, yu, then we get $|\langle x, y\rangle| = 4$, a contradiction.

(4), (1): $u^x = u^{-1}, y^u = y$. Here we consider the pair x, xu: if $[x, xu] = 1$, then $u \in C_G(\langle x, y\rangle)$. If $x^{xu} = x^{-1}$ or $(xu)^x = (xu)^{-1}$, then $yu \in C_G(\langle x, y\rangle)$. Thus in each case we get $u \in \langle x, y, C_G(\langle x, y\rangle)\rangle$.

(4), (3): $u^x = u^{-1}, y^u = y^{-1}$. Here we consider the pair xy^{-1}, u: if $[xy^{-1}, u] = 1$, then $xyu \in C_G(\langle x, y\rangle)$. If $(xy^{-1})^u = (xy^{-1})^{-1}$, then $xu \in C_G(\langle x, y\rangle)$. If $u^{xy^{-1}} = u^{-1}$, then $|\langle x, y\rangle| = 4$, a contradiction. Thus again we get $u \in \langle x, y, C_G(\langle x, y\rangle)\rangle$.

(4), (4): $u^x = u^{-1}, u^y = u^{-1}$. Here we consider the pair x, xu: if $[x, xu] = 1$, then $u \in C_G(\langle x, y\rangle)$. If $x^{xu} = x^{-1}$ or $(xu)^x = (xu)^{-1}$, then $xyu \in C_G(\langle x, y\rangle)$.

This concludes consideration of all cases. □

It follows easily from the fact that $H = Q_8 \times C_2 = \langle x, y\rangle \times \langle u\rangle \triangleleft G$, together with Lemmas 6.13 and 6.14 that $G = Q_8 \times C_2^r$.

(II) $H = J$. Here one can check that if the pair x, xy satisfies any of (1), (3), (4), then $|H| = 8$, a contradiction. However $x \sim_H xy$ implies $x \sim_{Q_8} xy$, a contradiction. Thus the case $H = J$ does not happen.

This concludes consideration of the case where G is a 2-group.

Now for the situation where G is not a 2-group: $|N| > 1$. We continue to assume that G is non-abelian. From Lemma 6.7 we obtain a homomorphism $\varepsilon : S_2 \to \{\pm 1\}, x \mapsto \varepsilon(x)$. Let $K = \ker \varepsilon$.

If $K = S_2$, then $G = S_2 \times N$. Since G is not abelian, we see that S_2 is not abelian. But $G/N \cong S_2$ is also a 3-S-ring group and so $S_2 = Q_8 \times C_2^r$ by what we have done above. Let $Q_8 = \langle x, y\rangle$, where x, y satisfy (6.1). Let $u \in N, u \neq 1$. Then it is easy to see that the pair xu, yu does not satisfy (1) or (2). If it satisfies (3), then $(xu)^{yu} = x^{-1}u^{-1}$, giving $x^{-1}u = x^{-1}u^{-1}$, so that $u^2 = 1$, a contradiction. Similarly (4) gives the same contradiction. Thus $K \neq S_2$. Thus there is some element $x \in S_2$ with $\varepsilon(x) = -1$.

We may also assume that $S_2 \neq \{1\}$, since G is not abelian.

If $K = \{1\}$, then $S_2 = C_2 = \langle x\rangle$, for the $x \in S_2$ chosen above with $\varepsilon(x) = -1$; this gives us the result that we want: such a group is generalized dihedral of order $2n$, $n = |N|$ odd. Thus we may assume that $K \neq \{1\}$, so that there is some $z \in K \cap Z(S_2), |z| = 2$.

Note that z centralizes elements of N, and is central in S_2, and so $z \in Z(G)$. Note that $G/\langle z\rangle$ is also a 3-S-ring group, and so the inductive hypothesis shows that either

(i) $G/\langle z\rangle \cong N \rtimes C_2$ (where the action of C_2 is inversion); or

(ii) $G/\langle z\rangle$ is abelian.

In case (i) we have $C_2 = \langle x'\rangle, x' \in S_2(G/\langle z\rangle)$. Let $x \in S_2(G)$ such that $x' = x\langle z\rangle$. There are two possibilities for $\langle x, z\rangle$:

(a) $\langle x, z\rangle \cong C_4$; or

(b) $\langle x, z\rangle \cong C_2^2$.

Since G is not abelian and $z \in K$ we see that we have $\varepsilon(x) = -1$. We now deal with the following cases:

Case (i) and (a): Here we have $G = \langle x, N\rangle$ and we also have $\langle x, z\rangle = \langle x\rangle$ and $z = x^2$. We now consider the pair x, xyz for any $y \in N, y \neq 1$. We note that $(x, y) \neq 1$ and so $(x, xyz) \neq 1$. Thus we do not have (1) for the pair x, xyz. If we have $x \sim_G xyz$, then there is $g \in G = \langle x, N\rangle$ such that $x^g = xyz$. Now g can be written as $g = x^a y_1, y_1 \in N$, and so we have

$$x^{y_1} = x^{x^a y_1} = x^g = xyz.$$

Thus $x^{-1}y_1^{-1}xy_1 = yz$, which gives $y_1^2 = yz$. But yz has even order gives $y_1 = 1$, and so $x = x^{y_1} = xyz$, a contradiction; thus we do not have (2) for the pair x, xyz.

Now (3) for the pair x, xyz is $x^{xyz} = x^{-1} = xz$, which gives $x^y = xz$ so that $x^{-1}y^{-1}xy = z$. But this is $y^2 = z$, a contradiction.

If we have (4), then $(xyz)^x = z^{-1}y^{-1}x^{-1} = zy^{-1}x^{-1}$, so that $xy^{-1}z = zy^{-1}x^{-1}$. This gives $xy^{-1} = y^{-1}x^{-1}$ and so

$$y^{-1} = x^{-1}y^{-1}x^{-1} = x^{-1}y^{-1}x \cdot x^2 = yz,$$

giving $y^{-2} = z$, a contradiction.

Case (i) and (b): We leave the details to the reader as this is similar to the above, the only difference being that you get $y^2 = 1$ as the contradiction in (3) and (4).

Case (ii): Since G is not abelian there is some $x \in S_2$ such that $\varepsilon(x) = -1$. Let $y \in N, y \neq 1$.

Since $G/\langle z\rangle$ is abelian we have $y^x\langle z\rangle = y\langle z\rangle \in G/\langle z\rangle$ and so

$$y^{-1} = y^x \in \{y, yz\}.$$

The two cases $y^{-1} = y$ and $y^{-1} = yz$ both give contradictions.

This concludes the proof of (B) and so of Theorem 6.8. □

6.2 The 3-S-ring groups with commutative 3-S-rings

We have shown that all groups with commutative 3-S-rings are 3-S-ring groups, and have classified those groups. To finish our proof, we classify those 3-S-ring groups which have commutative 3-S-rings. First we show the following:

Lemma 6.15. *A group of the form $G = Q_8 \times C_2^r$ does not have a commutative 3-S-ring.*

Proof. Let $\pi : G = Q_8 \times C_2^r \to Q_8$ be the projection. Then π induces a homomorphism of 3-S-rings, $\pi : \mathfrak{S}^{(3)}(G) \to \mathfrak{S}^{(3)}(Q_8)$. Since $\pi(\mathfrak{S}^{(3)}(G)) = \mathfrak{S}^{(3)}(Q_8)$ we need only show that $\mathfrak{S}^{(3)}(Q_8)$ is not commutative. Suppose that $Q_8 = \langle x, y\rangle$, where x, y satisfy the relations (6.1). Let α be the 3-class of $(x, x, 1)$ and β be the 3-class of $(xy, xy, 1)$. Then one can check that $\alpha\beta \neq \beta\alpha$. □

Lemma 6.16. *A 3-S-ring group G not of the form $G = Q_8 \times C_2^r$ has a commutative 3-S-ring.*

Proof. This is clearly true if G is abelian. Thus we may now assume that G is not abelian.

So from Theorem 6.8 that we have $G = N \rtimes C_2$, where N is an abelian group of odd order and that $C_2 = \langle x\rangle$ acts on N by inversion. Elements of G will be written $nx^\varepsilon, n \in N, \varepsilon = 0, 1$. If $\alpha \in G^3$, then $K^{(3)}(\alpha)$ contains an element of one of the following four types:

$$(A) : (n_1, n_2, n_3), \quad (B) : (n_1x, n_2, n_3), \quad (C) : (n_1x, n_2x, n_3), \quad (D) : (n_1x, n_2x, n_3x).$$

Here $n_i \in N, i = 1, 2, 3$.

Let $\alpha, \beta \in G^3$. We thus have some cases to consider to show that $K^{(3)}(\alpha)K^{(3)}(\beta) = K^{(3)}(\beta)K^{(3)}(\alpha)$:

Case: (A) $\times$ (A). Here $\alpha = (n_1, n_2, n_3), \beta = (n_1', n_2', n_3')$ and in this case we have $\alpha\beta = \beta\alpha$, so we certainly have $K^{(3)}(\alpha)K^{(3)}(\beta) = K^{(3)}(\beta)K^{(3)}(\alpha)$.

Case: (A) $\times$ (B). Here $\alpha = (n_1, n_2, n_3), \beta = (n_1'x, n_2', n_3')$. To prove this case we just need to show that $\alpha\beta \in K^{(3)}(\beta) \cdot K^{(3)}(\alpha)$. Now for $n \in N$ we have $x^n = n^{-2}x$ and so

$$(n_1'x, n_2', n_3')^n = (n_1'n^{-2}x, n_2', n_3'),$$

Since $|N|$ is odd and N is abelian the map $n \mapsto n^2$ gives a surjection of N, and so the element $n \in N$ can be chosen so that

$$\beta^n\alpha = (n_1'n^{-2}x, n_2', n_3')(n_1, n_2, n_3) \text{ is equal to } (n_1n_1'x, n_2n_2', n_3n_3') = \alpha\beta.$$

Case: (A) $\times$ (C). Here $\alpha = (n_1, n_2, n_3), \beta = (n_1'x, n_2'x, n_3')$. Let $\beta' = \beta x$, so that β' has type (B). Then from the above case $(A) \times (B)$ we have $\alpha\beta' = \beta'\alpha$. Thus we have

$$\alpha\beta = \alpha\beta'x = \beta'\alpha x = \beta'x \cdot x\alpha x = \beta\alpha^x \in K^{(3)}(\beta) \cdot K^{(3)}(\alpha), \quad (6.4)$$

as required.

Case: (A) $\times$ (D). Here $\alpha = (n_1, n_2, n_3), \beta = (n_1'x, n_2'x, n_3'x)$. Let $\beta' = \beta x$, so that β' has type (A). Then from the case $(A) \times (A)$ we have $\alpha\beta' = \beta'\alpha$. Thus (6.4) again gives this case.

Case: (B) $\times$ (B). Here we need to consider subcases:

(i) $\alpha = (n_1x, n_2, n_3), \beta = (n_1'x, n_2', n_3')$. Then $\alpha\beta = (n_1(n_1')^{-1}, n_2n_2', n_3n_3')$. But

$$\beta^n\alpha = (n_1'n^{-2}x, n_2', n_3')(n_1x, n_2, n_3) = (n_1'n^{-2}n_1^{-1}, n_2n_2', n_3n_3'),$$

and we can find $n \in N$ such that this is equal to $\alpha\beta$, as required.

(ii) $\alpha = (n_1 x, n_2, n_3), \beta = (n'_1, n'_2 x, n'_3)$. For $n, m \in N$ we have

$$\beta^m \alpha^n = (n'_1, n'_2 m^{-2} x, n'_3)(n_1 n^{-2} x, n_2, n_3) = (n'_1 n_1 n^{-2} x, n'_2 m^{-2} n_2^{-1} x, n_3 n'_3),$$

and we can choose $n, m \in N$ such that this is equal to $\alpha\beta = (n_1 (n'_1)^{-1} x, n_2 n'_2 x, n_3 n'_3)$.

Case: (B) $\times$ (C). Here $\alpha = (n_1 x, n_2, n_3), \beta = (n'_1 x, n'_2 x, n'_3)$. Let $\beta' = \beta x$. Then β' has type (B) and so we have $\alpha\beta' = \beta'\alpha$. The result now follows from (6.4).

The remainder of the cases can be proved by reducing to cases that we have already considered, and then using (6.4). □

From Proposition 6.2 and Lemmas 6.15 and 6.16 it follows that:

Theorem 6.17. *A finite group G has commutative 3-S-ring if and only if G is abelian or G is generalized dihedral of order $2n$, n odd.*

Chapter 7. Conclusion and Unanswered Questions

In this paper we showed that the 3-S-ring determines a finite group and that the NDICT determines an FC group. We also showed that a UTCCI map between FC groups is an isomorphism. In addition, we referred to the group determinant and the 1−, 2−, and 3-characters, which also determine a finite group. We did not explore the relationship between UTCCI maps of a finite group and the 1-, 2-, and 3-characters of the group, but would like to understand whether it is possible to show directly that either one determines the other.

It is shown in [12] that D_8 and Q_8 do not have the same 2-character table. We showed both that D_8 and Q_8 have the same 2-S-ring, and that the extended 2-characters correspond to representations of the 2-S-ring. This apparent contradiction is explained by the fact that the character table of the 2-S-ring is missing the labeling as to **which** of the characters of the 2-S-ring corresponds to the Frobenius 2-character of the group. We would like to understand what trait(s) of a character of a 2-S-ring make it a candidate to be a Frobenius 2-character.

We discussed the extended 2-characters and their relationship to characters of the 2-S-ring. If it can be shown that the 3-character corresponds to a representation of the 3-S-ring, then it might be possible to show that the 3-S-ring determines the group by relying on the result in [8] that 1-,2-, and 3-characters determine the group. We say maybe because of the labeling issues which arise in the 2-S-ring case which may or may not carry over to the 3-S-ring case. As with 2-S-rings, we do not know if there is a way to determine whether a character of the 3-S-ring is the Frobenius 3-character based on some inherent characteristic(s).

We showed that the CICT of a finite group corresponded to the Cayley Table of a unique group. We wonder what can be said about a loop which results from filling in the CICT of a group as the multiplication table of a loop.

We classified finite groups with commutative 3-S-rings were classified, but only touched on commutativity of 2-S-rings. It would be interesting to classify the finite or FC groups which have a commutative 2-S-ring. When a group has a non-commutative 2-S-ring, and we find all complex solutions to the equations we get from the structure constants of the k-S-ring, are there solutions corresponding to non-linear irreducible representations? What is the relationship? We also wonder if, given a non-commutative 2-S-ring, there is a subring of the 2-S-ring which is commutative and which is constant on k-classes, and whether it would be fruitful to study such subrings.

Finally, we have very few results on what the k-characters and k-S-rings tell us about finite groups. For example, Steve Humphries [9] has shown that the 2-S-ring and 2-characters together determine the derived length of a group. One wonders if it would be possible to get information about normal groups, Sylow p-subgroups, derived length, or other groups characteristics from the 2-S-ring character table, etc. It is our hope that with future study the characters of the 2- or 3-S-ring of a finite group may be useful in proving more general results about groups.

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