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Poincaré Polynomial of FJRW Rings and the Group-Weights Conjecture

Julian Tay

A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of
Master of Science

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ABSTRACT

Poincaré Polynomial of FJRW Rings and the Group-Weights Conjecture

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FJRW-theory is a recent advancement in singularity theory arising from physics. The FJRW-theory is a graded vector space constructed from a quasihomogeneous weighted polynomial and symmetry group, but it has been conjectured that the theory only depends on the weights of the polynomial and the group. In this thesis, I prove this conjecture using Poincaré polynomials and Koszul complexes.

By constructing the Koszul complex of the state space, we have found an expression for the Poincaré polynomial of the state space for a given polynomial and associated group. This Poincaré polynomial is defined over the representation ring of a group in order for us to take $G$-invariants. It turns out that the construction of the Koszul complex is independent of the choice of polynomial, which proves our conjecture that two different polynomials with the same weights will have isomorphic FJRW rings as long as the associated groups are the same.

Keywords: Poincaré polynomial, FJRW theory, Group-Weights conjecture, Koszul complex
ACKNOWLEDGMENTS

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I would also like to thank my family for their support.
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7.1 Group-Weights Conjecture 

7.2 Using the Poincaré Polynomial 

7.3 Conclusion 

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Chapter 1. Introduction

Mirror symmetry is a phenomenon that occurs in geometry and physics. It describes relationships that show how a class of geometric objects (A-model) are related to a mirror dual class of objects (B-model). This thesis focuses on FJRW theory which provides the A-model construction of Landau-Ginzburg (LG) mirror symmetry [FJR12]. FJRW theory also contributes to the understanding of singularities which are often studied in algebraic geometry.

The A-model in LG mirror symmetry is a family of Frobenius algebras constructed from a quasihomogeneous (weighted) polynomial $W$ and a group $G$ of diagonal symmetries of $W$. Using a corresponding polynomial $W^T$ and group $G^T$, we can construct the B-model, which is also a family of Frobenius algebras. The LG conjecture is that

$$\mathcal{H}_{W,G} \cong \mathcal{Q}_{W^T,G^T}$$

where $\mathcal{H}_{W,G}$ is the FJRW A-model and $\mathcal{Q}_{W^T,G^T}$ is the B-model.

Verifying the LG conjecture is challenging because it requires the understanding of the higher structures of the A and B-models. The levels of structure of these models can be summarized as first graded vector space, second Frobenius algebra and third Frobenius manifold, in order from lowest to highest. The Frobenius manifold structures are difficult to compute and in some cases are still unknown. The LG conjecture has been verified in various special cases, and at various levels in papers such as [Kra09] and [FJJS11].

The Group-Weights conjecture is a property of FJRW A-models that has long been assumed to be true but never proved. It assumes that the FJRW A-model does not depend on the choice of $W$, but only the weights of $W$ and the given group.

Within the A-model it is known that both the Frobenius algebra and Frobenius manifold structures are deformation invariant and the axiom of deformation invariance tells us that the graded vector space structure, the weights of the polynomial and the symmetry group
determine the entire FJRW A-model, so the Groups-Weights conjecture depends only on showing that the graded vector space is determined by the group and weights alone.

**Definition 1.1.** A graded vector space $M$ is a direct sum $\bigoplus_{i=0}^{\infty} M_i$ where each $M_i$ is a finite-dimensional vector space. Every element in $M_i$ is defined to have degree $i$.

The Hilbert series is a tool used in understanding graded vector spaces. The Hilbert series (and finite equivalent, the Poincaré polynomial) is a formal power series $P(M) = \sum_{i=0}^{\infty} a_i t^i$ whose coefficients $a_i$ are the dimensions of the corresponding $M_i$.

The Milnor ring $Q_W$ is a key building block in the FJRW A-model. In [Arn74], we find a formula for the Poincaré polynomial of the Milnor ring $Q_W$. However, since the FJRW A-model involves taking $G$-invariants of the Milnor ring, the usual formula for the Poincaré polynomial does not apply to the FJRW construction.

In this thesis, we attempt to find a Poincaré polynomial that would help us also keep track of the $G$-invariants, which was something that was not possible with the usual Poincaré polynomial formula for the Milnor ring. Representation theory is very appropriate in this setting to understand the group action on a vector space. We derive the formula for the Poincaré polynomial of the Milnor ring in terms of representations. The Group-Weights conjecture turns out to be a corollary of the formula for the Poincaré polynomial, since the formula does not depend on the actual polynomial choice but on the weights of the polynomial and choice of group.
Chapter 2. FJRW-Theory

The construction of the graded vector space structure in FJRW-theory requires a polynomial and a symmetry group.

2.1 Admissible Polynomial

The polynomial that is used has two requirements, as stated in [FJR12]: The polynomial has to be non-degenerate and quasihomogeneous.

**Definition 2.1.** A polynomial $W$ is quasihomogeneous if there exists unique (up to scalar multiples) rational numbers $d, q_1, q_2, \ldots, q_n$ such that $W(\lambda^{q_1} x_1, \ldots, \lambda^{q_n} x_n) = \lambda^d W(x_1, \ldots, x_n)$ for all $\lambda \in \mathbb{C}$.

We will call $d$ the total weight of the polynomial $W$ and $q_i$ the weight of the variable $x_i$.

**Example 2.2.** For example, let $W = x^2 y + x y^4$. Since $W(\lambda^{q_x} x, \lambda^{q_y} y) = \lambda^{2q_x + q_y} x^2 y + \lambda^{q_x + 4q_y} x y^4$, we solve the equations

\[
\begin{align*}
2q_x + q_y &= d \\
q_x + 4q_y &= d
\end{align*}
\]

and get $q_x = 3$, $q_y = 1$ and $d = 7$. So $W$ has a total weight of 7. It is clear that the choice of $q_x$ and $q_y$ is unique up to scalar multiples of $(q_x, q_y, d)$.

**Definition 2.3.** A quasihomogeneous polynomial $W$ is non-degenerate if $W$ contains no monomials of the form $x_i x_j$ where $i \neq j$, and the only critical point of $W$ is at the origin, and the degrees of $W$ are uniquely determined up to scalar multiples.

The meaning of critical point in Definition 2.3 is the set of points where the partial derivatives all equal zero.

The definition of $W$ provides us with two related objects.
Definition 2.4. The \textit{Jacobian ideal} $\mathcal{J}$ of $W$ is the ideal generated by the partial derivatives of $W$, i.e. $\mathcal{J} = \left\langle \frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_n} \right\rangle$.

Definition 2.5. The \textit{Milnor ring} $\mathcal{Q}_W$ of $W$ is the quotient ring $\mathcal{Q}_W = \mathbb{C}[x_1, \ldots, x_n]/\mathcal{J}$.

Theorem 2.6. If $W$ is a non-degenerate quasihomogeneous polynomial, then $\mathcal{Q}_W$ is finite dimensional [Arn74].

2.2 Admissible Group

Definition 2.7. Given a polynomial, $W$, we define the \textit{maximal group of diagonal symmetries} as $G_W^{\text{max}} = \left\{ (\alpha_1, \ldots, \alpha_n) \subset (\mathbb{C}^*)^n | W(\alpha_1 x_1, \ldots, \alpha_n x_n) = W(x_1, \ldots, x_n) \right\}$

Since $W$ is quasihomogeneous, $W(\lambda q_1 x_1, \ldots, \lambda q_n x_n) = \lambda^d W(x_1, \ldots, x_n)$, and hence the element $J = (e^{(2\pi i q_1)/d}, \ldots, e^{(2\pi i q_n)/d})$ is an element of $G_W^{\text{max}}$.

Every group used in FJRW-theory is required to be a subgroup of $G_W^{\text{max}}$ containing $\langle J \rangle$. We call such groups “admissible”.

2.3 Invertible Polynomials

A subclass of the admissible polynomials that we are interested in are invertible polynomials.

Definition 2.8. Let $W$ be a non-degenerate quasihomogeneous polynomial with the same number of variables and monomials. Then we call $W$ \textit{invertible}. This name arises from the fact that if $W = \sum_{i=1}^{n} c_i \prod_{j=1}^{n} x_j^{a_{ij}}$ is non-degenerate, then $W$ is invertible if the exponent matrix $A = (a_{ij})$ is invertible.

In [KS92], it is shown that a non-degenerate invertible polynomial can be written as a sum of invertible polynomials of the following three atomic types:
• \( W_{\text{Fermat}} = x^a \) where \( a \) is a positive integer and \( a \geq 2 \).

• \( W_{\text{loop}} = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \ldots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1 \) where \( a_i \) is a positive integer and \( a_i \geq 2 \).

• \( W_{\text{chain}} = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \ldots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} \) where \( a_i \) is a positive integer and \( a_i \geq 2 \).

For most of this thesis, it is not required that the polynomials be invertible. This definition is useful in discussing different examples, especially in Chapter 3.

### 2.3.1 Weights of invertible polynomials.

Suppose we wanted to find the weights for an invertible polynomial. We would need each monomial to have the same total weight, or in other words,

\[
A \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} d \\ \vdots \\ d \end{bmatrix}
\]

Since \( A \) is invertible, fixing \( d \), there would be a unique solution for \( q_1, \ldots, q_n \), which is rational since the entries of \( A \) are integers.

### 2.4 Construction of the A-model state space

We use the definition in [Kra09] for the state space of the A-Model.

**Definition 2.9.** Let \( G \leq (\mathbb{C}^\times)^n \) and let \( G \) act on \( \mathbb{C}^n \) by coordinate-wise multiplication. For each \( h \in G \), we define \( \text{Fix}(h) \leq \mathbb{C}^n \) to be the fixed locus of \( h \).

**Definition 2.10.** Given a polynomial \( W \) and admissible group \( G \), for \( h \in G \), let \( x_{i_1}, \ldots, x_{i_{N_h}} \) be the coordinates of \( \mathbb{C}^n \) that are fixed by \( h \). Let \( \Omega^N_h(\text{Fix}(h)) \) be the space of all top-dimensional, holomorphic differential forms on the fixed locus \( \text{Fix}(h) \). We define

\[
\mathcal{H}_h := \Omega^N_h(\text{Fix}(h))/(dW|_{\text{Fix}(h)} \wedge \Omega^{N_h-1}).
\]
We call $\mathcal{H}_h$ the $h$-sector of $W$ and define the A-model state space to be

$$\mathcal{H}_{W,G} := \bigoplus_{h \in G} (\mathcal{H}_h)^G$$

where $(\cdot)^G$ refers to all the $G$-invariants.

The above definition is key in proving the main result of this thesis. However, it is usually easier to compute the FJRW state space using the following equivalent definition.

**Theorem 2.11.** Given a polynomial $W$ and admissible group $G$,

$$\Omega^{N_h}(\text{Fix}(h))/(dW|_{\text{Fix}h} \wedge \Omega^{N_h-1}) \cong \mathcal{Q}_{W|_{\text{Fix}h}} \cdot \omega$$

where $\text{Fix}h$ is the fixed locus of $h$, $N_h$ is the dimension of $\text{Fix}h$ and $\mathcal{Q}_{W|_{\text{Fix}h}}$ is the Milnor ring of $W$ restricted to $\text{Fix}h$. Let $\omega = dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_{N_h}}$ where $x_{i_1}, \ldots, x_{i_{N_h}}$ are the coordinates of $\mathbb{C}^N$ fixed by $h$ [Wal80a][Wal80b].

Each pair $(m, h)$ where $h \in G$ and $m \in (\mathcal{H}_h)^G$ is called a basis element of $\mathcal{H}_{W,G}$ and is denoted by $[m, h]$. 

**Example 2.12.** Let $W = x^3y + y^5$ and $G = \langle e^{\frac{2\pi i}{15}}, e^{\frac{2\pi i}{5}} \rangle$ be the generator of $G$.

Since $g^5 = \left(e^{\frac{2\pi i}{5}}, 1\right)$, we have $\text{Fix}g^5 = \{0\} \times \mathbb{C}_y$. This gives us $W|_{\text{Fix}g^5} = y^5$ and so $\mathcal{H}_{g^5} = \mathcal{Q}_{W|_{\text{Fix}g^5}}dy = (\mathbb{C}[y]/\langle 5y^4 \rangle) dy = \text{Span}_\mathbb{C}\{dy, ydy, y^2dy, y^3dy\}$. This is true also for $g^{10}$.

The element $g^{15} = 1$ is the identity $(1, 1)$ which has fixed locus $\text{Fix}1 = \mathbb{C}_x \times \mathbb{C}_y$. So $W|_{\text{Fix}1} = x^3y + y^5$ and $\mathcal{H}_1 = \mathcal{Q}_W dx \wedge dy$. For all the other non-trivial group elements, for example $g$, $W|_{\text{Fix}g} = 0$ and so $\mathcal{H}_g = \mathbb{C} = \langle 1 \rangle$.

We can observe the action of $G$ by looking at a basis of monomials. A basis for $\mathcal{H}_1 = \mathcal{Q}_W dx \wedge dy$ is $\{1 dx \wedge dy, y dx \wedge dy, x dx \wedge dy, y^2 dx \wedge dy, xy dx \wedge dy, x^2 dx \wedge dy, y^3 dx \wedge dy, xy^2 dx \wedge dy, y^4 dx \wedge dy, xy^3 dx \wedge dy, xy^4 dx \wedge dy\}$. Taking the element $g = \left(e^{\frac{2\pi i}{15}}, e^{\frac{2\pi i}{5}}\right)$,
we see for example:

\[ gxy^2 \, dx \wedge dy \mapsto e^{2\pi i \frac{14}{15}} x \left( e^{2\pi i \frac{1}{5}} \right)^2 y^2 e^{2\pi i \frac{14}{15}} \, dx \wedge e^{2\pi i \frac{1}{5}} \, dy \]

is not \( G \)-invariant. Checking each monomial, we see that \( (\mathcal{H}_1)^G = \text{Span}\{x^2 \, dx \wedge dy\} \).

In Tables 2.1, 2.2 and 2.3, we show the computation of three sectors in this example. First we compute the monomial basis for the sectors 1, \( g^5 \) and \( g \). The right column shows the action of \( g \) on this basis element. Since \( G \) is cyclic, if the monomial is invariant by the action of \( g \), then the monomial is invariant by the whole group \( G \).

From Table 2.1, \( x^2 \, dx \wedge dy \) is the only \( G \)-invariant monomial in \( \mathcal{H}_1 \). Table 2.2 tells us that from the \( g^5 \) sector, we get no \( G \)-invariants. Since \( \text{Fix}\, g^5 = \text{Fix}\, g^{10} \), we also have no \( G \)-invariants in the \( g^{10} \) sector. For \( g \) (and equivalently \( g^k \) where \( k = 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14 \)), we get the monomial 1 which is \( G \)-invariant.

Hence for this example, a basis for \( \mathcal{H}_{W,G} \) is

\[
\{[x^2 \, dx \wedge dy, 1], [1, g], [1, g^2], [1, g^3], [1, g^4], [1, g^6], [1, g^7], [1, g^8], [1, g^9], [1, g^{11}], [1, g^{12}], [1, g^{13}], [1, g^{14}]\}.
\]

**Definition 2.13.** Let \( W \) be a quasihomogeneous polynomial with weights \( \{q_i\} \). We define the **degree** of any monomial to be the weighted sum of the corresponding \( q_i \)'s.

**Example 2.14.** Let \( W = x^2 y + x y^4 \) and \( G = \left\langle \left( e^{2\pi i \frac{4}{7}}, e^{2\pi i \frac{1}{7}} \right) \right\rangle \). Every non-trivial \( g \in G \) has nonzero entries in both coordinates, and so for each \( g \)-sector, \( W|_{\text{Fix}\, g} = 0 \) and the \( g \)-sector \( \mathcal{H}_g \) is \( \mathbb{C} \).

However \( g^7 = 1 = (1,1) \), so \( W|_{\text{Fix}\, g^7} = W \), and thus the Milnor ring is \( \mathbb{C}[x, y]/\langle 2xy + y^4, x^2 + 4xy^3 \rangle = \text{Span}\{1, y, y^2, y^3, x, xy, x^2, xy^2, xy^3\} \), so \( \mathcal{H}_1 = \text{Span}\{1 \, dx \wedge dy, y \, dx \wedge dy, y^2 \, dx \wedge dy, y^3 \, dx \wedge dy, x \, dx \wedge dy, xy \, dx \wedge dy, xy^2 \, dx \wedge dy, xy^3 \, dx \wedge dy\} \).
<table>
<thead>
<tr>
<th>$\mathcal{H}_1$</th>
<th>$g$-action</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 , dx \wedge dy$</td>
<td>$e^{2\pi i \frac{2}{15}}$</td>
</tr>
<tr>
<td>$y , dx \wedge dy$</td>
<td>$e^{2\pi i \frac{5}{15}}$</td>
</tr>
<tr>
<td>$x , dx \wedge dy$</td>
<td>$e^{2\pi i \frac{1}{15}}$</td>
</tr>
<tr>
<td>$y^2 , dx \wedge dy$</td>
<td>$e^{2\pi i \frac{8}{15}}$</td>
</tr>
<tr>
<td>$xy , dx \wedge dy$</td>
<td>$e^{2\pi i \frac{4}{15}}$</td>
</tr>
<tr>
<td>$x^2 , dx \wedge dy$</td>
<td>$1$ (g-invariant)</td>
</tr>
<tr>
<td>$y^3 , dx \wedge dy$</td>
<td>$e^{2\pi i \frac{11}{15}}$</td>
</tr>
<tr>
<td>$xy^2 , dx \wedge dy$</td>
<td>$e^{2\pi i \frac{7}{15}}$</td>
</tr>
<tr>
<td>$y^4 , dx \wedge dy$</td>
<td>$e^{2\pi i \frac{14}{15}}$</td>
</tr>
<tr>
<td>$xy^3 , dx \wedge dy$</td>
<td>$e^{2\pi i \frac{10}{15}}$</td>
</tr>
<tr>
<td>$xy^4 , dx \wedge dy$</td>
<td>$e^{2\pi i \frac{13}{15}}$</td>
</tr>
</tbody>
</table>

Table 2.1: Monomial basis of $\mathcal{H}_1$ where $W = x^3 y + y^5$ and $1 = (1, 1)$

<table>
<thead>
<tr>
<th>$\mathcal{H}_{g^5}$</th>
<th>$g$-action</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 , dy$</td>
<td>$e^{2\pi i \frac{1}{5}}$</td>
</tr>
<tr>
<td>$y , dy$</td>
<td>$e^{2\pi i \frac{2}{5}}$</td>
</tr>
<tr>
<td>$y^2 , dy$</td>
<td>$e^{2\pi i \frac{4}{5}}$</td>
</tr>
<tr>
<td>$y^3 , dy$</td>
<td>$e^{2\pi i \frac{3}{5}}$</td>
</tr>
</tbody>
</table>

Table 2.2: Monomial basis of $\mathcal{H}_{g^5}$ where $W = x^3 y + y^5$ and $g^5 = (e^{2\pi i \frac{2}{3}}, 1)$

<table>
<thead>
<tr>
<th>$\mathcal{H}_g$</th>
<th>$g$-action</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

Table 2.3: Monomial basis of $\mathcal{H}_g$ where $W = x^3 y + y^5$ and $g = (e^{2\pi i \frac{14}{15}}, e^{2\pi i \frac{1}{5}})$
Table 2.4: Monomial basis of $\mathcal{H}_1$ of $x^2y + xy^4$ for $1 = (1, 1)$

The definition of degree as given in Definition 2.13 is the focus of this paper, however, this grading will not be preserved under the FJRW ring multiplication structure. The following grading, known as $W$-degree, is the actual grading for FJRW state space.

**Definition 2.15.** Let $h = (e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_m})$ be a group element where $0 \leq \theta_i < 1$. Then for $\alpha_h \in (H_h)^G$, the $W$-degree of $\alpha_h$ is $\deg_W(\alpha_h) := \dim \text{Fix } h + 2 \sum_{i=1}^m \left( \theta_i - \frac{q_i}{d} \right)$, where the $q_i$'s are the quasihomogeneous weights of $W$.

Note that the $W$-degree of a given element is just determined by the weights $q_i/d$ and the group element $h$. So to count the dimension of the space of elements of a given $W$-degree, it suffices to count the dimension of the space $(\mathcal{H}_h)^G$ for each $h \in G$. 

<table>
<thead>
<tr>
<th>Basis Element</th>
<th>W-Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\left[ 1, \left( e^{\frac{2\pi i}{7}}, e^{\frac{2\pi i}{7}} \right) \right]$</td>
<td>0</td>
</tr>
<tr>
<td>$\left[ 1, \left( e^{\frac{2\pi i}{7}}, e^{\frac{2\pi i}{7}} \right) \right]$</td>
<td>2/7</td>
</tr>
<tr>
<td>$\left[ 1, \left( e^{\frac{2\pi i}{7}}, e^{\frac{2\pi i}{7}} \right) \right]$</td>
<td>4/7</td>
</tr>
<tr>
<td>$\left[ y^3, (1, 1) \right]$</td>
<td>6/7</td>
</tr>
<tr>
<td>$\left[ x, (1, 1) \right]$</td>
<td>6/7</td>
</tr>
<tr>
<td>$\left[ 1, \left( e^{\frac{2\pi i}{7}}, e^{\frac{2\pi i}{7}} \right) \right]$</td>
<td>8/7</td>
</tr>
<tr>
<td>$\left[ 1, \left( e^{\frac{2\pi i}{7}}, e^{\frac{2\pi i}{7}} \right) \right]$</td>
<td>10/7</td>
</tr>
<tr>
<td>$\left[ 1, \left( e^{\frac{2\pi i}{7}}, e^{\frac{2\pi i}{7}} \right) \right]$</td>
<td>12/7</td>
</tr>
</tbody>
</table>

Table 2.5: Basis for $\mathcal{H}_{W,G}$ where $W = x^2 y + x y^4$ and $G = \langle e^{\frac{2\pi i}{7}}, e^{\frac{2\pi i}{7}} \rangle$

### 2.5 Construction of the FJRW Ring

From the state space, the ring structure of $\mathcal{H}_{W,G}$ is determined by objects known as genus-zero three-point correlators.

**Definition 2.16.** Given $r, s \in \mathcal{H}_{W,G}$, we define *ring multiplication* $\star$ as $r \star s := \sum_{\alpha, \beta} \langle r, s, \alpha \rangle_{0,3} \eta^{\alpha, \beta}$, where $\alpha, \beta$ are all possible elements in $\mathcal{H}_{W,G}$.

The three-point correlators, $\langle r, s, \alpha \rangle_{0,3}$ are not easy to compute and are usually computed using a set of so-called axioms.

*Deformation invariance* is an axiom of FJRW A-models which allow us to prove the Group-Weights conjecture without understanding the whole structure. Specifically, it says that if $G_1 = G_2$ as subgroups of $(\mathbb{C}^\times)^N$ and $\dim \mathcal{H}_{W_1,G_1} = \dim \mathcal{H}_{W_2,G_2}$, then the multiplicative
structures of two FJRW rings $\mathcal{H}_{W_1,G_1}$ and $\mathcal{H}_{W_2,G_2}$ will be the same

2.6 CONSTRUCTION OF THE B-MODEL STATE SPACE

The construction of the B-model state space is similar to the construction of the A-model. However, the difference is that the admissible groups in the B-model construction do not have to contain $\langle J \rangle$, but rather have to be subgroups of $SL_n(\mathbb{C})$.

Definition 2.17. Let $W$ be a non-degenerate quasihomogeneous polynomial. Let $G$ be a symmetry group of $W$ that contains $SL_n(\mathbb{C})$. For $g \in G$. Let $\text{Fix}(g) = \mathbb{C}^{N_g}$ with coordinates $x_{i_1}, \ldots, x_{i_{N_g}}$ where $N_g = \dim \text{Fix}(g)$ and let $\omega_g = dx_{i_1}, \ldots, dx_{i_{N_g}}$. We call $B_g = (\Omega_W|_{\text{Fix}(g)}) \omega_g$ an uprojected sector and the B-model state space consists of $G$-invariants of these sectors:

$$B_{W,G} = \bigoplus_{g \in G} (B_g)^G.$$
The Group-Weights conjecture is motivated by the fact that constructing FJRW rings seems to be independent of the choice of polynomial. The formula for the Poincaré polynomial of the Milnor ring describes its graded vector space structure. This formula is independent of the actual polynomial as it only uses the quasihomogeneous weights of the polynomial. However, it is not obvious that \( G \)-invariance is independent of the polynomial since the action of \( G \) is defined to act on individual variables.

**Conjecture (Group-Weights).** Let \( W_1 \) and \( W_2 \) be polynomials which have the same weights. Suppose \( G \leq G_{W_1}^{\text{max}} \) and \( G \leq G_{W_2}^{\text{max}} \). Then

\[
\mathcal{H}_{W_1,G} \cong \mathcal{H}_{W_2,G}
\]

as Frobenius manifolds.

### 3.1 Common Subgroups

A first approach to proving the Group-Weights conjecture is to study common admissible subgroups of polynomials with the same weights. If \( W_1 \) and \( W_2 \) are polynomials with the same weights system, then \( \langle J \rangle \leq G_{W_1}^{\text{max}} \) and \( \langle J \rangle \leq G_{W_2}^{\text{max}} \). Recall that an admissible group in the FJRW-theory is one that contains \( \langle J \rangle \). There are many examples of distinct polynomials with the same weights where the only admissible subgroup for both polynomials is \( \langle J \rangle \). In fact, we can show that for two-variable invertible polynomials, \( \langle J \rangle \) is the only admissible subgroup in common for distinct polynomials.

#### 3.1.1 Common Subgroups of Two Variable Invertible Polynomials

Recall from Section 2.3, invertible quasihomogeneous polynomials are sums of polynomials of atomic types. This means that given two invertible quasihomogeneous polynomials of degree two, they would be: a sum of two Fermat type polynomials, a chain type polynomial or a loop.
Theorem 3.1. Let $W_1$ and $W_2$ be two-variable invertible polynomials. The only admissible subgroup of $G_{W_1}^{\text{max}} \cap G_{W_2}^{\text{max}}$ is $\langle J \rangle$.

The proof follows from the following set of lemmas.

Lemma 3.2. If $W_{\text{Fermat}}$ is a two variable polynomial which is the sum of two Fermat’s, i.e. $W_{\text{Fermat}} = x^m + y^n$, then the weights of $W_{\text{Fermat}}$ are $q_x = 1/m$ and $q_y = 1/n$ with $d = 1$.

Proof. Let $W_{\text{Fermat}} = x^m + y^n$, where $m, n \geq 2$. We can solve for the weights of each variable to get

$$ q_x = \frac{1}{m}, q_y = \frac{1}{n}. $$

Lemma 3.3. If $W_{\text{loop}}$ is a two-variable polynomial that is a loop, i.e. $W_{\text{loop}} = x^a y + y^b x$, then the weights of $W_{\text{loop}}$ are $q_x = \frac{b - 1}{ab - 1}$ and $q_y = \frac{a - 1}{ab - 1}$ with $d = 1$.

Proof. Let $W_{\text{loop}}$ be a two variable polynomial which is a loop. Then $W = x^a y + y^b x$ where $a, b \geq 2$. Solving for the weights of each variable, we get

$$ aq_x + q_y = 1 $$

$$ q_x + bq_y = 1. $$

Solving this system of linear equations:

$$ q_x = \frac{b - 1}{ab - 1} $$

$$ q_y = \frac{a - 1}{ab - 1}. $$

Lemma 3.4. If $W_{\text{loop}}$ is a two-variable loop polynomial, i.e. $W_{\text{loop}} = x^a y + y^b x$, then the maximal symmetry group of $W_{\text{loop}}$ is $G_{W_{\text{loop}}}^{\text{max}} = \left\langle \left( \exp \left( -2\pi i \frac{1}{ab - 1} \right), \exp \left( 2\pi i \frac{a}{ab - 1} \right) \right) \right\rangle$. In particular, $G_{W_{\text{loop}}}^{\text{max}}$ is cyclic of order $ab - 1$. 

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Proof. Let \( G = \left\langle \left( \exp \left( -2\pi i \frac{1}{ab-1} \right), \exp \left( 2\pi i \frac{a}{ab-1} \right) \right) \right\rangle \). We see that \( G \) is a cyclic group of order \( ab - 1 \) that fixes \( W_{\text{loop}} \). Suppose there exists a \((e^{2\pi i \theta_1}, e^{2\pi i \theta_2}) \in G_{W_{\text{loop}}}^\max\). Since this element fixes the polynomial, we get

\[
a\theta_1 + \theta_2 \in \mathbb{Z}
\]
\[
\theta_1 + b\theta_2 \in \mathbb{Z}.
\]

Solving these, we get

\[
(ab - 1)\theta_1 \in \mathbb{Z}
\]
\[
(ab - 1)\theta_2 \in \mathbb{Z}.
\]

Hence both \( e^{2\pi i \theta_1} \) and \( e^{2\pi i \theta_2} \) are \((ab - 1)\)th roots of unity. Let \( \theta_1 \equiv \frac{r}{ab - 1} \mod \mathbb{Z} \) and \( \theta_2 \equiv \frac{s}{ab - 1} \mod \mathbb{Z} \) for \( r, s \in \mathbb{Z} \). Then

\[
ar + s \equiv 0 \mod \mathbb{Z}
\]
\[
\frac{s}{ab - 1} \equiv -ar \mod \mathbb{Z}.
\]

Hence \((e^{2\pi i \theta_1}, e^{2\pi i \theta_2}) = \left( \exp \left( -2\pi i \frac{1}{ab-1} \right), \exp \left( 2\pi i \frac{a}{ab-1} \right) \right)^{-r} \) and \( G_{W_{\text{loop}}}^\max \leq G \).

Thus \( G_{W_{\text{loop}}}^\max = \left\langle \left( \exp \left( -2\pi i \frac{1}{ab-1} \right), \exp \left( 2\pi i \frac{a}{ab-1} \right) \right) \right\rangle \). \( \square \)

Lemma 3.5. If \( W_{\text{chain}} \) is a two-variable polynomial that is a forward chain, i.e \( W_{\text{chain}} = x^h y + y^k \), then the weights of \( W_{\text{chain}} \) are \( q_x = \frac{k-1}{hk} \) and \( q_y = \frac{1}{k} \) with \( d = 1 \). Likewise, if \( W'_{\text{chain}} \) is a 2 variable polynomial that is a reverse chain, i.e \( W'_{\text{chain}} = x^h + y^k x \), then the weights of \( W'_{\text{chain}} \) are \( q_x = \frac{1}{h} \) and \( q_y = \frac{h-1}{hk} \) with \( d = 1 \).

Proof. Let \( W_{\text{chain}} = x^h y + y^k \) be a two-variable polynomial which is a forward chain, where
$h, k \geq 2$. Solving for the weights of each variable, we get

\begin{align*}
hq_x + q_y &= 1 \\
hq_y &= 1 \\
q_y &= \frac{1}{h} \\
q_x &= \frac{h - 1}{hk}.
\end{align*}

The weights for the reverse chain follows from the above calculations.

**Lemma 3.6.** If $W_{\text{chain}}$ is a two-variable chain polynomial, i.e. $W_{\text{chain}} = x^h y + y^k$, then the maximal symmetry group of $W_{\text{chain}}$ is $G_{W_{\text{chain}}}^{\text{max}} = \langle (\exp(-2\pi i \frac{1}{hk}), \exp(2\pi i \frac{1}{k})) \rangle$. In particular, $G_{W_{\text{chain}}}^{\text{max}}$ is cyclic of order $hk$.

**Proof.** Let $G = \langle (\exp(-2\pi i \frac{1}{hk}), \exp(2\pi i \frac{1}{k})) \rangle$. We see that $G$ is a cyclic group of order $hk$ that fixes $W_{\text{chain}}$. Suppose there exists a $(e^{2\pi i \theta_1}, e^{2\pi i \theta_2}) \in G_{W_{\text{chain}}}^{\text{max}}$. Since this element fixes the polynomial, we get

\begin{align*}
h\theta_1 + \theta_2 &\in \mathbb{Z} \\
k\theta_2 &\in \mathbb{Z}.
\end{align*}

Solving for $\theta_1$, we get

\begin{align*}
hk\theta_1 &\in \mathbb{Z}.
\end{align*}

Hence $e^{2\pi i \theta_1}$ is a $hk$th root of unity and $e^{2\pi i \theta_2}$ is a $k$th root of unity. Let $\theta_1 \equiv \frac{r}{hk} \mod \mathbb{Z}$ and $\theta_2 \equiv \frac{s}{k} \mod \mathbb{Z}$ for $r, s \in \mathbb{Z}$. Then

\begin{align*}
\frac{hr}{hk} + \frac{s}{k} &\equiv 0 \mod \mathbb{Z} \\
\frac{s}{k} &\equiv -\frac{r}{k} \mod \mathbb{Z}.
\end{align*}

Hence $(e^{2\pi i \theta_1}, e^{2\pi i \theta_2}) = \left(\exp\left(-2\pi i \frac{1}{hk}\right), \exp\left(2\pi i \frac{1}{k}\right)\right)^{-r}$ and $G_{W_{\text{chain}}}^{\text{max}} \leq G$. 15
Thus $G_{W_{\text{chain}}}^{\text{max}} = \left\langle \left( \exp \left( -2\pi i \frac{1}{\hbar k} \right), \exp \left( 2\pi i \frac{1}{k} \right) \right) \right\rangle$. \hfill \Box

Lemma 3.7. If $W_1$ and $W_2$ are two-variable polynomials with the same weights $(q_x, q_y)$, and $W_1$ is the sum of two Fermat’s $x^m + y^n$ and $W_2$ is a loop $x^a y + y^b x$, then $m = n$ and $a = b$. Also, $m = n = a + 1 = b + 1$.

Proof. If $W_2$ had the same weights as $W_1$, then

\[
\left( \frac{b - 1}{ab - 1} : \frac{a - 1}{ab - 1} \right) = \left( \frac{1}{m} : \frac{1}{n} \right).
\]

So we get that $b - 1|ab - 1$ and $a - 1|ab - 1$.

\[
b - 1|ab - 1 \implies b - 1|ab - 1 - (b - 1) \implies b - 1|b(a - 1) \implies b - 1|a - 1.
\]

\[
a - 1|ab - 1 \implies a - 1|ab - 1 - (a - 1) \implies a - 1|a(b - 1) \implies a - 1|b - 1 \implies a - 1 = b - 1 \implies a = b.
\]

And so

\[
\left( \frac{b - 1}{ab - 1} : \frac{a - 1}{ab - 1} \right) = \left( \frac{a - 1}{a^2 - 1} : \frac{a - 1}{a^2 - 1} \right) = \left( \frac{1}{a + 1} : \frac{1}{a + 1} \right).
\]

Since $W_1$ has the same weights vector as $W_2$, $m = n = a + 1 = b + 1$. \hfill \Box

Lemma 3.8. If $W_1$ and $W_2$ are two-variable polynomials with the same weights vector, and $W_1$ is the sum of two Fermat’s $x^m + y^n$ and $W_2$ is a loop $x^a y + y^b x$, then the only admissible group in common is the group generated by $J$. 

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Proof. By Lemma 3.7, we have weights \( x = \frac{1}{a+1} \) and \( q_y = \frac{1}{a+1} \). So \( G_{W_1}^{\max} \) is generated by \( \left( \exp \left( 2\pi i \frac{1}{a+1} \right), 1 \right) \) and \( \left( 1, \exp \left( 2\pi i \frac{1}{a+1} \right) \right) \), while \( G_{W_2}^{\max} \) is generated by \( \left( \exp \left( 2\pi i \frac{a}{a^2-1} \right), \exp \left( -2\pi i \frac{1}{a^2-1} \right) \right) \).

Each subgroup of \( G_{W_2}^{\max} \) is cyclic, and so if \( G_{W_1}^{\max} \) had a subgroup containing \( J \) in common with \( G_{W_2}^{\max} \), that subgroup would have to be cyclic. Any element of \( G_{W_1}^{\max} \) generates a subgroup which has order at most \( a+1 \), so any subgroup containing \( J \) has to have order \( a+1 \) and so would be \( \langle J \rangle \).

Lemma 3.9. If \( W_1 \) and \( W_3 \) are two variable polynomials with the same weights vector, and \( W_1 \) is the sum of two Fermat’s \( x^m + y^n \) and \( W_3 \) is a chain \( x^h y + y^k \), then \( m = ck \) for some constant \( ck \) and \( n = k \).

Proof. If \( W_3 \) had the same weights as \( W_1 \), then

\[
\left( \frac{k-1}{hk}, \frac{1}{k} \right) = \left( \frac{1}{m}, \frac{1}{n} \right)
\]

So we get that \( k-1|hk \). Since \( \gcd(k-1,k) = 1 \), \( k-1|h \), so let \( \frac{h}{k-1} = c \), then

\[
\left( \frac{k-1}{hk}, \frac{1}{k} \right) = \left( \frac{1}{ck'}, \frac{1}{k} \right)
\]

So we see that \( m = ck \) and \( n = k \). \( \square \)

Lemma 3.10. If \( W_1 \) and \( W_3 \) are two-variable polynomials with the same weights, and \( W_1 \) is the sum of two Fermat’s \( x^m + y^n \) and \( W_3 \) is a chain \( x^h y + y^k \), then the only admissible group in common is the group generated by \( J \).

Proof. By Lemma 3.9, we have that the weights would be \( q_x = \frac{1}{ck} \) and \( q_y = \frac{1}{k} \). The group \( G_{W_1}^{\max} \) is generated by \( \left( \exp \left( 2\pi i \frac{1}{ck} \right), 1 \right) \) and \( \left( 1, \exp \left( 2\pi i \frac{1}{k} \right) \right) \), while the group \( G_{W_3}^{\max} \) is generated by \( \left( -\exp \left( 2\pi i \frac{1}{hk} \right), \exp \left( 2\pi i \frac{1}{k} \right) \right) \).
Each subgroup of $G^\text{max}_W$ is cyclic, and so if $G^\text{max}_{W_1}$ had a subgroup containing $J$ in common with $G^\text{max}_{W_3}$, that subgroup would have to be cyclic. Any element of $G^\text{max}_{W_1}$ generates a subgroup which has order at most $ck$, so any subgroup containing $J$ has to have order $ck$ and so would be $\langle J \rangle$.

\begin{proof}
If $W_2$ and $W_3$ are two-variable polynomials with the same weights, and $W_2$ is a loop $(x^a y + y^b x)$ and $W_3$ is a chain $x^b y + y^k$, then $a = h$ and $q_x = \delta/k$ and $q_y = 1/k$ for some constant $\delta$.

\begin{align*}
\text{Proof.} \quad \text{If } W_3 \text{ had the same weights as } W_2, \text{ then } \left( \frac{b - 1}{ab - 1}, \frac{a - 1}{ab - 1} \right) = \left( \frac{k - 1}{hk}, \frac{1}{k} \right). \text{ Solving the right coordinates:}
\end{align*}

\begin{align*}
\frac{a - 1}{ab - 1} &= \frac{1}{k} \\
ab &= \frac{1}{k}.
\end{align*}

Since $\frac{k - 1}{hk} = \frac{b - 1}{ab - 1}$, substituting (3.1.1.1), we get

\begin{align*}
\frac{ab - 1}{a - 1} - 1 &= \frac{b - 1}{ab - 1} \\
\frac{ab - 1 - a + 1}{h(ab - 1)} &= \frac{b - 1}{ab - 1} \\
\frac{a(b - 1)}{h(ab - 1)} &= \frac{b - 1}{ab - 1} \\
\frac{a}{h} &= 1.
\end{align*}

So (3.1.1.2) tells us that $a = h$ and that we get that the weights would be $\left( \frac{b - 1}{hb - 1}, \frac{h - 1}{hb - 1} \right) = \left( \frac{k - 1}{hk}, \frac{1}{k} \right)$. Since $\frac{h - 1}{hb - 1} = \frac{1}{k}$,

\begin{align*}
hb - 1 &= kh - k \\
kh - hb &= k - 1 \\
h(k - b) &= k - 1.
\end{align*}

\end{proof}
So (3.1.1.3) tells us that \( h | k - 1 \), in fact \( \frac{k - 1}{h} = k - b \), so the constant \( \delta \) is \( k - b \). So the weights are \( q_x = \frac{\delta}{k} \) and \( q_y = \frac{1}{k} \).

**Lemma 3.12.** If \( W_2 \) and \( W_3 \) are two-variable polynomials with the same weights, and \( W_2 \) is a loop \( x^a y + y^b x \) and \( W_3 \) is a chain \( x^h y + y^k \), then the only admissible group in common is the group generated by \( J \).

**Proof.** By Lemma 3.11, we have that the weights \( q_x = \frac{\delta}{k} \) and \( q_y = \frac{1}{k} \). The group \( G_{W_2}^{\max} \) is generated by \( \left( \exp\left( 2\pi i \frac{b}{hb - 1} \right), \exp\left( -2\pi i \frac{1}{hb - 1} \right) \right) \), while the group \( G_{W_3}^{\max} \) is generated by \( \left( \exp\left( -2\pi i \frac{1}{hk} \right), \exp\left( 2\pi i \frac{1}{k} \right) \right) \).

All subgroups of \( G_{W_2}^{\max} \) and \( G_{W_3}^{\max} \) are cyclic. We know from Equation (3.1.1.1) that \( k | hb - 1 \), and \( \gcd(hb - 1, h) = 1 \), so \( \gcd(hb - 1, hk) = k \). So the only subgroups that \( G_{W_2}^{\max} \) and \( G_{W_3}^{\max} \) have in common is of order \( k \). If it is cyclic, and contains \( J \), it has to be \( \langle J \rangle \).

The preceding set of lemmas, together, prove Theorem 3.1.

**Theorem 3.13.** If \( W_1 \) and \( W_2 \) have the same weights vector, then \( \mathcal{H}_{W_1,\langle J \rangle} \cong \mathcal{H}_{W_2,\langle J \rangle} \) as graded vector spaces.

This theorem was proved in [Fra12]. Given two polynomials with the same weights, the number of basis elements in the \( g \)-sector is determined by the weights. The Poincaré polynomial tells us how many monomials of the same weights there are. Hence they have to be isomorphic.

In more than two variables it is easy to find examples where the only common subgroup is \( \langle J \rangle \).

**Example 3.14.** \( W_1 = x^{13} y + xy^5 + z^4 + w^2 \) and \( W_2 = x^{16} + y^4 z + z^2 w + w^2 \).

- \( J = \left( e^{2\pi i \frac{1}{14}}, e^{2\pi i \frac{3}{14}}, e^{2\pi i \frac{1}{2}}, e^{2\pi i \frac{1}{2}} \right) \)
- Common subgroup \( \langle J \rangle \).
Even though it was conjectured earlier on that the only common subgroup between the symmetry groups of distinct polynomials is \( \langle J \rangle \), I found examples in three or more variables that proved otherwise.

**Example 3.15.** \( W_1 = x^7y + xy^7 + z^4 + w^2 \) and \( W_2 = x^7y + y^8 + z^2w + w^2 \).

- \( J = \left( e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{1}{3}} \right) \)
- Common subgroup \( \langle \left( e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{1}{3}}, 0, 0 \right), \left( 0, 0, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{1}{3}} \right) \rangle \)

In this example, we see that both \( W_1 \) and \( W_2 \) are sums of invertible polynomials in terms of \( x, y \) and \( z, w \). So the common subgroup is not just \( \langle J \rangle \) but a direct product of the respective \( J \)'s of each invertible polynomial component.

**Example 3.16.** \( X^T_9 = x^3 + xy^2 + yz^2 \) and \( J^T_{10} = x^3 + y^3 + yz^2 \).

- \( J = \left( e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{1}{3}} \right) \).
- Common subgroup \( \left\langle \left( e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{1}{3}} \right) \right\rangle \)

Here \( X^T_9 \) is a chain type polynomial that is reversed while \( J^T_{10} \) is the sum of a chain type polynomial and a Fermat. The start of both chains is the common \( yz^2 \) term. This allows for the common subgroup to be slightly larger than \( J \).

**Example 3.17.** Chain type polynomials can be extended to arbitrary length to produce more examples of non-\( \langle J \rangle \) common subgroups.

\( W_1 = x^3y + y^3z + z^4 + w^4 \) and \( W_2 = x^3y + y^3z + z^3w + w^4 \).

- \( J = \left( e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{1}{4}} \right) \)
- Common subgroup \( \left\langle \left( e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{1}{4}} \right) \right\rangle \)

\( W_1 = x_1^3x_2 + x_2^3x_3 + x_3^3x_4 + x_4^3x_5 + x_5^4 \) and \( W_2 = x_1^3x_2 + x_2^3x_3 + x_3^4 + x_4^4 + x_5^4 \).

- \( J = \left( e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{1}{4}} \right) \)
- Common subgroup \( \left\langle \left( e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{1}{4}} \right) \right\rangle \)
3.2 FJRW Ring Multiplication and Deformation Invariance

Even for polynomials $W_1$ and $W_2$ where the common subgroup of $G_{W_1}^{max}$ and $G_{W_2}^{max}$ is known, finding an isomorphism between their respective FJRW rings is difficult. Recall from Section 2.5 that multiplication in FJRW rings require the hard task of finding so-called three-point correlators. We will not give details about these correlators here, but we describe several examples with their multiplicative structures.

Example 3.18. Let $V = x^9 + y^3$ and let $W = x^6y + y^3$. Then both $V$ and $W$ a non-degenerate polynomials with quasihomogeneous weights $q_x = 1$ and $q_y = 3$. Let $G$ be the group $G = \langle J \rangle = \langle e^{2\pi i \frac{2}{9}}, e^{2\pi i \frac{1}{3}} \rangle$.

<table>
<thead>
<tr>
<th>Name</th>
<th>Basis Element</th>
<th>W-Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>$1, (e^{2\pi i \frac{1}{9}}, e^{2\pi i \frac{1}{3}})$</td>
<td>0</td>
</tr>
<tr>
<td>$v_2$</td>
<td>$1, (e^{2\pi i \frac{1}{9}}, e^{2\pi i \frac{1}{3}})$</td>
<td>2/3</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$1, (e^{2\pi i \frac{1}{9}}, e^{2\pi i \frac{1}{3}})$</td>
<td>8/9</td>
</tr>
<tr>
<td>$v_4$</td>
<td>$[x^2y, (1, 1)]$</td>
<td>10/9</td>
</tr>
<tr>
<td>$v_5$</td>
<td>$[x^5, (1, 1)]$</td>
<td>10/9</td>
</tr>
<tr>
<td>$v_6$</td>
<td>$1, (e^{2\pi i \frac{1}{9}}, e^{2\pi i \frac{1}{3}})$</td>
<td>4/3</td>
</tr>
<tr>
<td>$v_7$</td>
<td>$1, (e^{2\pi i \frac{1}{9}}, e^{2\pi i \frac{1}{3}})$</td>
<td>14/9</td>
</tr>
<tr>
<td>$v_8$</td>
<td>$1, (e^{2\pi i \frac{1}{9}}, e^{2\pi i \frac{1}{3}})$</td>
<td>20/9</td>
</tr>
</tbody>
</table>

Table 3.1: Basis for $\mathcal{H}_{V,G}$ where $\mathcal{H}_{V,G}$ where $V = x^9 + y^3$ and $G = \langle e^{2\pi i \frac{1}{9}}, e^{2\pi i \frac{1}{3}} \rangle$
Table 3.2: Basis for $H_{W,G}$ where $W = x^6y + y^3$ and $G = \langle e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{1}{3}} \rangle$
Table 3.4: Multiplication Table for $\mathcal{H}_{W,G}$ where $W = x^6y + y^3$ and $G = \langle e^{2\pi i \frac{1}{9}}, e^{2\pi i \frac{2}{9}} \rangle$

In this example, we see that for the state space of both $\mathcal{H}_{V,G}$ and $\mathcal{H}_{W,G}$, there are tables of the same dimension in each degree. Comparing their multiplication tables, there is an obvious map between most of the elements in the ring. However, the problem arises for the elements with $W$-degree $\frac{10}{9}$.

Table 3.5: Multiplication Table for $\mathcal{H}_{V,G}$ for elements of $W$-degree $\frac{10}{9}$

\begin{table}[h]
\centering
\begin{tabular}{c|cccccccc}
\cdot & $v_4$ & $v_5$
\hline
$v_4$ & 0 & $\frac{1}{27}v_8$
$v_5$ & $\frac{1}{27}v_8$ & 0
\end{tabular}
\end{table}
\begin{array}{c|cc}
\cdot & w_4 & w_5 \\
\hline
w_4 & \frac{1}{18} w_8 & 0 \\
w_5 & 0 & -\frac{1}{6} w_8 \\
\end{array}

Table 3.6: Multiplication Table for \( \mathcal{H}_{W,G} \) for elements of \( W \)-degree \( \frac{10}{9} \)

An isomorphism of these spaces as Frobenius algebras will have to send \( v_4 \) and \( v_5 \) each to some linear combination of \( w_4 \) and \( w_5 \).

With more variables and higher degree polynomials, the examples become increasingly more complicated.

3.2.1 Deformation Invariance. One of the properties of the FJRW rings is deformation invariance. Deformation invariance tells us that the Frobenius manifold structure of the FJRW-theory (otherwise known as virtual cycle) is dependent only on the associated state space. In other words, if \( \mathcal{H}_{W_1,G_1} \cong \mathcal{H}_{W_2,G_2} \) as graded vector spaces, then \( \mathcal{H}_{W_1,G_1} \cong \mathcal{H}_{W_2,G_2} \) in the FJRW-theory. This allows us to avoid actually having to find a ring isomorphism.

In Chapter 6, I will prove the conjecture by proving that the graded vector space of a FJRW ring is defined only by the weights of \( W \) and the symmetry group \( G \). This will be a corollary of the computation of the Poincaré polynomial for \( \mathcal{H}_{W,G} \).
Chapter 4. Representation Theory

4.1 Representations

A group representation is a way of visualizing a group $G$ as a group of linear transformations of $n$-dimensional vector space. For the purposes of this thesis, we shall only consider $\mathbb{C}$-vector spaces.

**Definition 4.1.** A *representation* of $G$ over $\mathbb{C}$ is a homomorphism $\rho$ from $G$ to $GL(n, \mathbb{C})$, for some $n$. We call $n$ the *degree* of the representation $\rho$.

Another way of understanding representations is to think of $\mathbb{C}G$-modules, which are vector spaces that have a $G$ action on them.

**Definition 4.2.** Let $V$ be a vector space over $\mathbb{C}$ and let $G$ be a group. $V$ is a $\mathbb{C}G$-module if the action $gv$ for $v \in V$ and $g \in G$ satisfies the following conditions for all $u, v \in V$, $\lambda \in \mathbb{C}$ and $g, h \in G$.

1. $gv \in V$;
2. $(gh)v = g(hv)$;
3. $1v = v$;
4. $\lambda(gv) = g(\lambda v)$;
5. $g(u + v) = gu + gv$.

**Definition 4.3.** A $\mathbb{C}G$-submodule of $V$ is a vector subspace of $V$ which is a $\mathbb{C}G$-module.

**Example 4.4.** Consider the cyclic group $C_3 = \langle 1, g, g^2 \rangle$ [JL01].

Let $V = \text{Span}_\mathbb{C}\{v_1, v_2\}$, where $g$ acts by $\rho(g) = \begin{bmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{4\pi i/3} \end{bmatrix}$ and $g^2$ acts by $\rho(g^2) = \rho(g)^2 = \begin{bmatrix} e^{4\pi i/3} & 0 \\ 0 & e^{2\pi i/3} \end{bmatrix}$. Then $V$ is a $\mathbb{C}C_3$-module.

Since $gv_1 = e^{2\pi i/3}v_1$ and $gv_2 = e^{2\pi i^2/3}v_2$, and $g^2v_1 = e^{2\pi i^2/3}v_1$ and $g^2v_2 = e^{2\pi i^4/3}v_2$, we get two $\mathbb{C}C_3$-submodules of $V$ by Definition 4.3, namely $V_1 = \text{Span}_\mathbb{C}\{v_1\}$ and $V_2 = \text{Span}_\mathbb{C}\{v_2\}$. 

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Now we consider the group ring $\mathbb{C}C_3 = \text{Span}_{\mathbb{C}}\{1, g, g^2\}$. The group ring is also a $\mathbb{C}C_3$-module. From the group ring, we get a corresponding representation $\rho$. Since $g \cdot 1 = g$, $g \cdot g = g^2$ and $g \cdot g^2 = 1$, we get that $\rho(g) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

The group ring $\mathbb{C}C_3$ has three $\mathbb{C}C_3$-submodules:

- $U_0 = \text{Span}_{\mathbb{C}}\{1 + g + g^2\}$,
- $U_1 = \text{Span}_{\mathbb{C}}\{1 + e^{2\pi i \frac{2}{3}}g + e^{2\pi i \frac{1}{3}}g^2\}$ and
- $U_2 = \text{Span}_{\mathbb{C}}\{1 + e^{2\pi i \frac{1}{3}}g + e^{2\pi i \frac{2}{3}}g^2\}$. If we take the element $1 + e^{2\pi i \frac{2}{3}}g + e^{2\pi i \frac{1}{3}}g^2 \in U_1$ and act on it by $g$, we get $g + e^{2\pi i \frac{2}{3}}g^2 + e^{2\pi i \frac{1}{3}} = e^{2\pi i \frac{1}{3}}(1 + e^{2\pi i \frac{2}{3}}g + e^{2\pi i \frac{1}{3}}g^2)$. Hence we see that $U_1$ is isomorphic to $V_1$ in the earlier example.

**Definition 4.5.** A representation is *irreducible* if the corresponding $\mathbb{C}G$-module has no proper non-trivial $\mathbb{C}G$-submodules.

The following are three classical theorems in representation theory.

**Theorem 4.6.** Every irreducible $\mathbb{C}G$-module is isomorphic to a $\mathbb{C}G$-submodule of the group ring $\mathbb{C}G$.

**Theorem 4.7.** Let $G$ be an abelian group. Every irreducible $\mathbb{C}G$-module is one dimensional.

This property is useful for us since $G_{W_{\text{max}}}^\text{max}$ is abelian.

**Definition 4.8.** Let $\rho$ be a representation of $G$. The *character* of $\rho$ is a function $\chi : G \to \mathbb{C}$, where $\chi(g) = \text{tr} \rho(g)$ for each $g \in G$.

**Theorem 4.9.** Let $\rho$ be a representation of $G$. The dimension of the $\mathbb{C}G$-module corresponding to $\rho$ is $\text{tr} \rho(1_G) = \chi(1_G)$. 

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4.2 Representation Ring

**Definition 4.10.** Given representations \( \rho_1 : G \to GL(V_1) \) and \( \rho_2 : G \to GL(V_2) \) of \( G \), we define \( \rho_1 + \rho_2 \) and \( \rho_1 \times \rho_2 \) to be homomorphisms

\[
\rho_1 + \rho_2 : G \to GL(V_1 \oplus V_2) \\
\rho_1 \times \rho_2 : G \to GL(V_1 \otimes V_2)
\]

where \( (\rho_1 + \rho_2)(g) = \rho_1(g) \oplus \rho_2(g) \) and \( (\rho_1 \times \rho_2)(g) = \rho_1(g) \otimes \rho_2(g) \)

**Definition 4.11.** The representation ring \( R(G) \) is the ring generated by all representations of \( G \) over \( \mathbb{C} \) using the operations defined in Definition 4.10 and a formal additive inverse defined in the obvious way.

By Theorem 4.6, \( R(G) \) is generated by all irreducible representations of \( G \).

**Example 4.12.** Once again, we let \( G = C_3 \). Let \( V_1 \) and \( V_2 \) be the \( \mathbb{C}C_3 \)-submodules of \( V \) as in Example 4.3. Let \( \rho_1 \) and \( \rho_2 \) be representations corresponding to \( V_1 \) and \( V_2 \) respectively. \( \rho_1(g) = \begin{bmatrix} e^{2\pi i \frac{1}{3}} \\ \end{bmatrix} \) and \( \rho_2(g) = \begin{bmatrix} e^{2\pi i \frac{2}{3}} \\ \end{bmatrix} \). So \( \rho_1 + \rho_2(g) = \begin{bmatrix} e^{2\pi i \frac{1}{3}} & 0 \\ 0 & e^{2\pi i \frac{2}{3}} \\ \end{bmatrix} \).

\[
\rho_1 \times \rho_2(g)(v_1 \otimes v_2) = e^{2\pi i \frac{1}{3}} v_1 \otimes e^{2\pi i \frac{2}{3}} v_2 = v_1 \otimes v_2.
\]

So \( \rho_1 \times \rho_2 \) is the trivial representation \( 1_{R(C_3)} \).

The representation ring \( R(C_3) \) is the formal ring generated by \( 1, \rho_1, \rho_2 \). Since \( \rho_2 = \rho_2^3 \), we get that \( R(C_3) \) is generated as a ring by \( \rho_1 \) and satisfies \( \rho_1^3 = 1 \). Hence \( R(C_3) \) is isomorphic to the group ring \( \mathbb{Z}C_3 \).
Recall from the Introduction the following definition and theorem.

**Definition 5.1.** The Hilbert series of a graded vector space $M$ is a formal power series

$$P(M) = \sum_{i=0}^{\infty} a_i t^i,$$

where $a_i$ is the dimension of $M_i$.

When $P(M)$ is finite, we call $P(M)$ the Poincaré polynomial.

**Theorem 5.2.** Let $W$ be a quasihomogeneous polynomial with weights $d, q_1, q_2, \ldots, q_n$. The Hilbert series of the Milnor ring $Q_W$ is

$$P(Q_W) = \prod_{i=1}^{n} \frac{1 - t^{d-q_i}}{1 - t^{q_i}}.$$

Note that according to Theorem 2.6, $P(Q_W)$ is a polynomial when $W$ is non-degenerate.

We will prove a more refined version of Theorem 5.2 in Chapter 6.

Since the Poincaré polynomial tells us the dimension of each subspace of a certain degree, if we let $t = 1$, we would get the dimension of the entire space.

**Theorem 5.3.** The dimension of $Q_W$ is $\mu = \prod_{i=1}^{n} \left( \frac{d}{q_i} - 1 \right)$.

**Proof.** For each term in the product of $P(Q_W)$, we get that 1 is a root of both the numerator and the denominator.

$$\frac{1 - t^{d-q_i}}{1 - t^{q_i}} = \frac{(1-t)(1 + t + t^2 + \ldots + t^{d-q_i-1})}{(1-t)(1 + t + t^2 + \ldots + t^{q_i-1})} = \frac{1 + t + t^2 + \ldots + t^{d-q_i-1}}{1 + t + t^2 + \ldots + t^{q_i-1}}$$

Letting $t = 1$,

$$= \frac{d - q_i}{q_i} = \frac{d}{q_i} - 1. \quad \square$$
5.1 Representation-Valued Poincaré Polynomial

Definition 5.4. Let \( M = \bigoplus_{i=0}^{\infty} M_i \) be a graded \( \mathbb{C} \) vector space. Let \( G \) be a group which acts on \( M \) such that \( M \) is a \( \mathbb{C}G \)-module and each \( M_i \) is a \( \mathbb{C}G \)-submodule with corresponding representation \( \rho_i \) (where \( \rho_i \) is not necessarily irreducible). We define the representation-valued Hilbert series of \( M \) to be

\[
P(M, R(G)) = \sum_{i=1}^{\infty} \rho_i t^i
\]

where \( R(G) \) is the representation ring of \( G \).

Note that by taking the trace (character) of this expression at \( 1 \in G \), we get the usual Hilbert series.

This definition relies on the group \( G \) preserving the grading of \( M \), i.e. \( gm_i \in M_i \) for all \( m_i \in M_i \) and \( g \in G \). The action of the admissible groups on our state space multiplies a basis element by a complex root of unity as shown in Example 2.12 and thus preserves the grading. Hence, we can define the Hilbert series with representation for a FJRW state space. In fact, since the FJRW state space is finite dimensional, we can call it a representation-valued Poincaré polynomial.

5.2 Poincaré Polynomial on Exact Sequences

Definition 5.5. Let \( K \) be a sequence of vector spaces and linear transformations:

\[
0 \xrightarrow{d_0} N_1 \xrightarrow{d_1} N_2 \xrightarrow{d_2} \ldots \xrightarrow{d_{n-1}} N_{n-1} \xrightarrow{d_n} N_n \xrightarrow{d_{n+1}} \ldots
\]

the sequence is exact at \( N_i \) if \( \ker d_i = \text{Im} d_{i-1} \) and \( K \) is exact if it is exact at all \( N_i \).

Definition 5.6. Let \( d : M \to N \) be a linear transformation where \( M \) and \( N \) are graded vector spaces. We say that \( d \) has degree \( b \) if \( d(m_i) \) has degree \( i + b \) in \( N \) for every \( m_i \) of degree \( i \) in \( M \).
If \( b = 0 \), we say that \( d \) is degree-preserving.

**Proposition 5.7.** Let \( K \) be an exact sequence of vector spaces:

\[
0 \xrightarrow{d_0} N_1 \xrightarrow{d_1} N_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} N_n \xrightarrow{d_n} 0
\]

where each \( d_i \) has degree \( b_i \), then

\[
\sum_{i=1}^{n} (-1)^i \beta_i P(N_i) = 0 \text{ where } \beta_i = \sum_{j=i}^{n-1} b_j.
\]

**Proof.** For each vector space \( N_i \), we have \( N_i \cong \ker d_i \oplus \text{Im} d_i \). Suppose \( n \) is odd, then

\[
N_1 \oplus N_3 \oplus \ldots \oplus N_n \\
\cong \ker d_1 \oplus \text{Im} d_1 \oplus \ker d_3 \oplus \text{Im} d_3 \oplus \ker d_5 \oplus \ldots \oplus \text{Im} d_{n-2} \oplus \ker d_n \oplus \text{Im} d_n
\]

since \( \ker d_1 = \text{Im} d_0 = 0 \) and \( \text{Im} d_n = 0 \)

\[
\cong \ker d_2 \oplus \text{Im} d_2 \oplus \ker d_4 \oplus \text{Im} d_4 \oplus \ldots \oplus \ker d_{n-1} \oplus \text{Im} d_{n-1},
\]

by the condition of exactness in Definition 5.5

\[
\cong N_2 \oplus N_4 \oplus \cdots \oplus N_{n-1}.
\]

Similarly, if \( n \) is even, then

\[
N_1 \oplus N_3 \oplus \ldots \oplus N_{n-1} \\
\cong \ker d_1 \oplus \text{Im} d_1 \oplus \ker d_3 \oplus \text{Im} d_3 \oplus \ker d_5 \oplus \ldots \oplus \text{Im} d_{n-3} \oplus \ker d_{n-1} \oplus \text{Im} d_{n-1}
\]

since \( \ker d_1 = \text{Im} d_0 = 0 \)

\[
\cong \ker d_2 \oplus \text{Im} d_2 \oplus \ker d_4 \oplus \text{Im} d_4 \oplus \ldots \oplus \ker d_{n-2} \oplus \text{Im} d_{n-2} \oplus \ker d_n \oplus \text{Im} d_n,
\]

by the condition of exactness in Definition 5.5 and since \( \text{Im} d_n = 0 \)

\[
\cong N_2 \oplus N_4 \oplus \cdots \oplus N_n.
\]
We now use the following diagram to illustrate the logic for the rest of the proof.

\[
\begin{array}{cccc}
\text{ker} \ d_{n-2} & \text{Im} \ d_{n-2} & \text{ker} \ d_n \\
\downarrow & & \downarrow \\
\ldots & \text{Im} \ d_{n-3} & \text{ker} \ d_{n-1} & \text{Im} \ d_{n-1}
\end{array}
\]

We start from \( \text{ker} \ d_n = N_n \). From here, we know that \( P(N_n) = P(\text{ker} \ d_n) \). Using \( N_{n-1} = \ker d_{n-1} \oplus d_{n-1}^{-1}(\text{Im} \ d_{n-1}) \), we get \( P(N_{n-1}) = P(\ker d_{n-1}) + P(d_{n-1}^{-1}(\text{Im} \ d_{n-1})) \). To match the terms from \( N_{n-1} \) with the terms from \( N_n \), we multiply by \( t^{b_{n-1}} \) and get \( t^{b_{n-1}}P(N_{n-1}) = t^{b_{n-1}}P(\ker d_{n-1}) + t^{b_{n-1}}P(d_{n-1}^{-1}(\text{Im} \ d_{n-1})) = t^{b_{n-1}}P(\ker d_{n-1}) + P(\text{Im} \ d_{n-1}) \).

Next, from \( P(N_{n-2}) = P(\ker d_{n-2}) + P(d_{n-2}^{-1}(\text{Im} \ d_{n-2})) \), in order to match it with \( P(N_n) \), we multiply by \( t^{b_{n-2}}t^{b_{n-1}} \) since \( t^{b_{n-2}}t^{b_{n-1}}P(d_{n-2}^{-1}(\text{Im} \ d_{n-2})) = t^{b_{n-1}}P(\text{Im} \ d_{n-2}) = t^{b_{n-1}}P(\ker d_{n-1}) \).

So in order to match all of the terms, we have to multiply each \( P(N_i) \) by \( t^{\beta_i} \) where \( \beta_i \) is the sum of all the degrees of \( d_i \) proceeding \( N_i \) up to \( d_{n-1} \).

**Corollary 5.8.** Let \( K \) be an exact sequence of graded vector spaces:

\[
0 \overset{d_0}{\longrightarrow} N_1 \overset{d_1}{\longrightarrow} N_2 \overset{d_2}{\longrightarrow} \ldots \overset{d_{n-1}}{\longrightarrow} N_n \overset{d_n}{\longrightarrow} 0
\]

where each \( d_i \) is a degree-preserving map, then \( \sum_{i=1}^{n} (-1)^i P(N_i) = 0 \).

**Definition 5.9.** Let \( g \) have an action on sets \( M \) and \( N \). We say that the map \( \varphi : M \to N \) is **equivariant** if for \( m \in M \), \( g\varphi(m) = \varphi(gm) \).

**Proposition 5.10.** Let \( K \) be an exact sequence of graded vector spaces, with \( G \) acting on each \( N_i \) in a way that is degree-preserving:

\[
0 \overset{d_0}{\longrightarrow} N_1 \overset{d_1}{\longrightarrow} N_2 \overset{d_2}{\longrightarrow} \ldots \overset{d_{n-1}}{\longrightarrow} N_n \overset{d_n}{\longrightarrow} 0.
\]

If each \( d_i \) has degree \( b_i \) and is equivariant, then \( \sum_{i=1}^{n} (-1)^i t^{b_i} P(N_i, R(G)) = 0 \) where \( \beta_i = \sum_{i=1}^{n} b_i \).
\[ \sum_{j=i}^{n-1} b_j. \]

**Proof.** We follow from the proof of Proposition 5.7.

\[
\begin{array}{ccc}
\ker d_{n-2} & \text{Im } d_{n-2} & \ker d_n \\
\cdots & \text{Im } d_{n-3} & \ker d_{n-1} \text{ Im } d_{n-1}
\end{array}
\]

Once again we start from \( \ker d_n = N_n \). Since \( N_{n-1} = \ker d_{n-1} \oplus d_{n-1}^{-1}(\text{Im } d_{n-1}) \) and \( d_{n-1} \) is equivariant, both \( \ker d_{n-1} \) and \( d_{n-1}^{-1}(\text{Im } d_{n-1}) \) are \( \mathbb{C}G \)-submodules of \( N_{n-1} \). The representation of \( d_{n-1}^{-1}(\text{Im } d_{n-1}) \) is also the same as the representation of \( \ker d_n \). Once again, in order to match the terms from \( N_{n-1} \) with the terms from \( N_n \), we multiply by \( t^{b_{n-1}} \) and get

\[
t^{b_{n-1}} P(N_{n-1}, R_G(\mathbb{C})) = t^{b_{n-1}} P(\ker d_{n-1}) + t^{b_{n-1}} P(d_{n-1}^{-1}(\text{Im } d_{n-1}), R_G(\mathbb{C})) = t^{b_{n-1}} P(\ker d_{n-1}) + P(\text{Im } d_{n-1}, R_G(\mathbb{C})).
\]

Just as in the proof of Proposition 5.7, we continue matching terms, and since at each stage, the map \( d_i \) is equivariant, the representations all remain the same. \( \square \)
A useful tool in computing the Hilbert series of the FJRW state space is the Koszul complex. We use the fact that the partial derivatives of a non-degenerate polynomial forms a regular sequence. From this regular sequence, the Koszul complex gives us a long exact sequence where the final term in the sequence is the Milnor Ring.

6.1 Regular Sequence

Definition 6.1. Let $R$ be a ring and let $M$ be an $R$-module. A sequence of elements $r_1, \ldots, r_n \in R$ is called a regular sequence on $M$ if $(r_1, \ldots, r_n)M \neq M$, and for $i = 1, \ldots, n$, $r_i$ is a nonzerodivisor of $M/(r_1, \ldots, r_{i-1})M$.

Using the method given in [Stu93], we introduce the following terminology.

Definition 6.2. Let $R$ be an associative and commutative graded $k$-algebra, where $k$ is a field. A sequence of elements $r_1, \ldots, r_n \in R$ is called a homogeneous system of parameters (h.s.o.p.) of $R$ if $R/\langle r_1, \ldots, r_n \rangle$ is a finite dimensional $k$-vector space.

Theorem 6.3. Let $R$ be a Noetherian graded $k$-algebra. If $R$ has a h.s.o.p. that is a regular sequence, then any h.s.o.p. in $R$ is a regular sequence.

The proof of this theorem can be found in [Stu93] and [Sta79], and is further explained in [Pic06].

Proposition 6.4. Let $W$ be a nondegenerate polynomial in $\mathbb{C}[x_1, \ldots, x_n]$. Then the set of partial derivatives $\{W_{x_1}\}$ is a regular sequence on $\mathbb{C}[x_1, \ldots, x_n]$.

Proof. $x_1, \ldots, x_n$ is a h.s.o.p. and is also a regular sequence. Recall from Theorem 2.6 that the nondegeneracy of $W$ tells us that $\mathbb{C}[x_1, \ldots, x_n]/\langle W_{x_1}, \ldots, W_{x_n} \rangle$ is a finite dimensional $\mathbb{C}$ vector space and thus $W_{x_1}, \ldots, W_{x_n}$ is a h.s.o.p. Since $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian, by Theorem 6.3, $W_{x_1}, \ldots, W_{x_n}$ is a regular sequence. □
6.2 Koszul Complex

The Koszul complex that we are going to define is a sequence of graded vector spaces and this section gives us some definitions to help us understand its construction and properties.

**Definition 6.5.** Let $R$ be a commutative ring with unity. A sequence $K$ of $R$-modules:

\[
0 \xrightarrow{d_0} N_0 \xrightarrow{d_1} N_1 \xrightarrow{d_2} \ldots \xrightarrow{d_n} N_{n-1} \xrightarrow{d_n} N_n \xrightarrow{d_{n+1}} \ldots
\]

where each $d_i$ is a module homomorphism and $d_{n+1} \circ d_n = 0$ for all $n$ is called a complex. The $n^{th}$ cohomology group is defined to be

\[
H^n(K) = \ker d_{n+1} / \text{Im } d_n.
\]

To illustrate how the Koszul complex is constructed, we start with the Koszul complex in 1 and 2 variables. We denote the Koszul complex in one variable as $K(W_x dx)$ and it is the sequence below:

\[
0 \xrightarrow{} R \xrightarrow{r_x} R dx \xrightarrow{} 0,
\]

where $r_x(f) = f \wedge W_x dx$ for $f \in R = \mathbb{C}[x]$.

Certainly $r_x$ is injective since $W_x$ is a nonzerodivisor (i.e. $H^0(K(W_x dx)) = 0$). Now we compute $H^1(K(W_x dx)) = R dx/W_x R dx = R/(W_x) dx$. The Koszul complex can be extended to an exact sequence

\[
0 \xrightarrow{} R \xrightarrow{r_x} R dx \xrightarrow{} R/(W_x) dx \xrightarrow{} 0
\]

Now let $R = \mathbb{C}[x, y]$. The Koszul complex in 2 variables $K(W_x dx, W_y dy)$ is motivated
by the following complex

\[
\begin{array}{ccl}
0 & \longrightarrow & R \\
& \searrow_{r_x} & \swarrow_{r_y} R dx \\
& & \searrow_{r_y} R dx \\
0 & \longrightarrow & R dy \\
& & \swarrow_{r_x} \\
& & \searrow_{r_x} R dx \wedge dy \\
& & \longrightarrow 0
\end{array}
\]

where \( r_x(f) = f \wedge W_x dx \) and \( r_y(f) = f \wedge W_y dy \).

\( K(W_x dx, W_y dy) \) is the following complex

\[
\begin{array}{ccl}
0 & \longrightarrow & R \\
& \searrow_{s_1} & \swarrow_{s_2} R dx \oplus R dy \\
& & \searrow_{s_2} \\
& & \swarrow_{r_x} R dx \wedge dy \\
& & \longrightarrow 0
\end{array}
\]

where both \( s_1, s_2 \) are given by \( r \mapsto r \wedge dW \). That is

\[
s_1(r) = r \wedge dW \\
= r \wedge (W_x dx \oplus W_y dy) \\
= (r \wedge W_x dx, r \wedge W_y dy)
\]

And,

\[
s_2(a \, dx, b \, dy) = a \, dx \wedge (W_x dx + W_y dy) + a \, dy \wedge (W_x dx + W_y dy) \\
= aW_y \, dx \wedge dy + bW_x \, dx \wedge .
\]

\( H^0(K) = 0 \) since \( W_x \) and \( W_y \) are both nonzerodivisors making \( s_1 \) injective. The first cohomology group is \( \ker s_2/\text{Im } s_1 = \ker r_y/\text{Im } r_x \oplus \ker r_x/\text{Im } r_y \). From Proposition 6.4, we recall that the partial derivatives of the nondegenerate polynomial form a regular sequence which means that \( W_y \) is a nonzerodivisor in \( R/(W_x) \), so \( \ker r_y/\text{Im } r_x = 0 \). Likewise with \( \ker r_x/\text{Im } r_y \). Hence \( H^1(K) = 0 \). The image of the map \( s_2 \) is going to be sums of multiples of \( W_x \) and \( W_y \) which is the ideal generated by \( (W_x, W_y) \) in \( R \). Hence the second cohomology is going to \( R/(W_x, W_y) \, dx \wedge dy \). Once again we can extend \( K(W_x dx, W_y dy) \) to the exact
sequence

$$0 \longrightarrow R \xrightarrow{s_1} R dx \oplus R dy \xrightarrow{s_2} R dx \wedge dy \longrightarrow R/(W_x, W_y) dx \wedge dy \longrightarrow 0$$

To iterate the Koszul complex construction to more variables, we give a definition that describes $R dx \wedge dy$ in terms of $R dx \oplus R dy$.

**Definition 6.6.** Let $R = \mathbb{C}[x_1, x_2, \ldots, x_n]$. Let $N$ be the free module $\bigoplus_{i=1}^{n} R dx_i$. Then the **exterior algebra** is defined as $\wedge N = R \oplus N \oplus (N \otimes N) \oplus \ldots$ modulo the relations $dx \otimes dy = -dy \otimes dx$ and $dx \otimes dx = 0$ for all $dx$.

**Definition 6.7.** We define the $m$th **wedge product of** $N$ (denoted by $\wedge^m N$) to be the degree $m$ part of $\wedge N$.

Note that $\wedge^{n+k} N = 0$ for $k > 0$, and $\wedge^1 N = N$. Also, since $\wedge^k N = \bigoplus_{i_1 < i_2 < \ldots < i_k} R dx_{i_1} \wedge \ldots \wedge dx_{i_k}$, it contains $\binom{n}{k}$ copies of $R$.

So going back to $K(W_x dx, W_y dy)$, we get that $R dx \oplus R dy = N$ and $R dx \wedge dy = \wedge^2 N$ which gives

$$0 \longrightarrow R \xrightarrow{s_1} N \xrightarrow{s_2} \wedge^2 N \longrightarrow 0$$

The Koszul complex in $n$ variables is $K(W_x dx_1, W_x dx_2, \ldots, W_x dx_n)$, and is denoted as $K(W)$. It is the following sequence

$$0 \longrightarrow R \xrightarrow{s_1} N \xrightarrow{s_2} \wedge^2 N \xrightarrow{s_3} \ldots \xrightarrow{s_{n-1}} \wedge^{n-1} N \xrightarrow{s_n} \wedge^n N \longrightarrow 0$$

From [Eis95], we get that in the Koszul complex, $H^j(K(W)) = 0$ for $j < n$ and $H^n(K(W)) = \wedge^n N/(W_{x_1} dx_1, \ldots, W_{x_n} dx_n) = (R/(W_{x_1}, \ldots, W_{x_n})) dx_1 \wedge \ldots \wedge dx_n$ and so we get an exact sequence

$$0 \longrightarrow R \xrightarrow{s_1} N \xrightarrow{s_2} \ldots \xrightarrow{s_{n-1}} \wedge^{n-1} N \xrightarrow{s_n} \wedge^n N \longrightarrow (R/(W_{x_1}, \ldots, W_{x_n})) dx_1 \wedge \ldots \wedge dx_n \longrightarrow 0$$
Remark. Here, it is useful to note that by definition, $\wedge^k N = \Omega^k$ and so the $n^{th}$ cohomology (which is the last term) is $H_1$ as defined in Definition 2.10. To find $H_h$, for other $h \in G$, we just need to replace $W$ with $W|_{\text{Fix}_h}$ and let $n$ be the dimension of $\text{Fix}_h$. For each admissible group $G$ we get a natural $G$ action on each term of the Koszul complex that is given by multiplication on each variable. Since $dW = W_{x_1} \, dx_1 + W_{x_2} \, dx_2 + \ldots + W_{x_n} \, dx_n$, the action of $G$ on $dW$ is trivial since $G$ fixes the polynomial $W$. Hence the map $dW$ is equivariant.

6.3 Poincaré Polynomial of $(R/(W_{x_1}, \ldots, W_{x_n})) \, dx_1 \wedge \ldots \wedge dx_n$

Here we apply the formula in Proposition 5.10.

6.3.1 2 Variables. We first look at the two-variable case and use the previous tools to compute the representation-valued Poincaré polynomial of $\Omega^2/(dW \wedge \Omega^1)$ in two variables.

$$
\begin{array}{c}
0 \longrightarrow N_0 = R \xrightarrow{r_x} N_1 = R \, dx \xrightarrow{r_y} 0 \\
0 \longrightarrow N'_1 = R \, dy \xrightarrow{-r_y} N_2 = R \, dx \wedge dy \longrightarrow 0
\end{array}
$$

where $r_x(f) = f \wedge W_{x} \, dx$ and $r_y(f) = f \wedge W_{y} \, dy$.

We start from the final term $N_2$. This is a single copy of $R$. We let $q_x, q_y$ be the weights of $x$ and $y$ and $\rho_x, \rho_y$ be the representations of $G$ on $\text{Span}\{x\}$ and $\text{Span}\{y\}$ respectively. If we consider all possible monomials of in $R$, we get the Hilbert series $(1 + \rho_x t^{q_x} + \rho_x^2 t^{2q_x} + \ldots)(1 + \rho_y t^{q_y} + \rho_y^2 t^{2q_y} + \ldots)$ which can be simplified as a geometric series to $\frac{1}{1 - \rho_x t^{q_x}} \frac{1}{1 - \rho_y t^{q_y}}$. For all the terms in $N_2$, there is an extra $G$-action based on the $dx \wedge dy$, and so the Hilbert series for $N_2$ is $\frac{\rho_x \rho_y}{(1 - \rho_x t^{q_x})(1 - \rho_y t^{q_y})}$.

Looking to $N_1$, we also have a single copy of $R$ which gives us once again the denominator of $(1 - \rho_x t^{q_x})(1 - \rho_y t^{q_y})$. All the terms in $N_1$ have an extra $G$-action based on the $dx$ and so the numerator has a $\rho_x$ factor. Also, the map from $N_1$ to $N_2$ is multiplication by $W_{y} \, dy$ which has degree $d - q_y$ since taking the derivative of $W$ loses a single power of $y$. So the Hilbert series
for $N_1$ is \( \frac{\rho_x t^{d-q_x}}{(1 - \rho_x t^{d_x})(1 - \rho_y t^{q_y})} \). Likewise, the Hilbert series for $N_1'$ is \( \frac{\rho_y t^{d-q_y}}{(1 - \rho_x t^{d_x})(1 - \rho_y t^{q_y})} \).

Now for $N_0$. There are no attached $dx_i$'s, but now all the terms increase by the degree of $W_x W_y$ from $N_0$ to $N_2$. After accounting for that, we get that the Hilbert series for $N_0$ is

\[
\frac{\rho_x t^{d-q_x}}{(1 - \rho_x t^{d_x})(1 - \rho_y t^{q_y})}.
\]

If we include $H^2(K(W))$, we would get an exact sequence. Hence the Poincaré polynomial for $H^2(K(W))$ would be $P(N_2) - P(N_1) - P(N_1') + P(N_0)$.

\[
P(N_2) - P(N_1) - P(N_1') + P(N_0) = \frac{\rho_x \rho_y}{(1 - \rho_x t^{d_x})(1 - \rho_y t^{q_y})} \frac{\rho_x t^{d-q_x}}{(1 - \rho_x t^{d_x})(1 - \rho_y t^{q_y})} - \frac{\rho_y t^{d-q_y}}{(1 - \rho_x t^{d_x})(1 - \rho_y t^{q_y})} \frac{\rho_y t^{d-q_y}}{(1 - \rho_x t^{d_x})(1 - \rho_y t^{q_y})} + \frac{\rho_y t^{d-q_y}}{(1 - \rho_x t^{d_x})(1 - \rho_y t^{q_y})} \frac{\rho_y}{(1 - \rho_x t^{d_x})(1 - \rho_y t^{q_y})}.
\]

This gives us an explicit formula for the representation-valued Poincaré polynomial in two variables.

### 6.3.2 $n$ Variables

We now compute the representation-valued Poincaré polynomial in more variables. The term $\wedge^i N$ contains \( \binom{n}{i} \) copies of $R$. For $R$ itself, let $q_i$ be the weight of $x_i$ and let $\rho_i$ be the representation of the action of $G$ on $x_i$. Considering all possible monomials in $R$, we get that the Hilbert series for $R$ is \( (1 + \rho_1 t^{q_1} + \rho_1^2 t^{2q_1} + \ldots)(1 + \rho_2 t^{q_2} + \rho_2^2 t^{2q_2} + \ldots) \ldots (1 + \rho_n t^{q_n} + \rho_n^2 t^{2q_n} + \ldots) \). Since each factor is a geometric series, we can express \( (1 + \rho_i t^{q_i} + \rho_i^2 t^{2q_i} + \ldots) \) as \( \frac{1}{1 - \rho_i t^{q_i}} \).

Every copy of $R$ in $\wedge^i N$ would have the same denominator in the Poincaré polynomial, but the numerator would depend on the $dx_i$'s. For $R \ dx_{m_1} \wedge dx_{m_2} \wedge \ldots \wedge dx_{m_1}$, every term would have an extra action of $\rho_{m_1} \rho_{m_2} \ldots \rho_{m_i}$ due to the $dx_i$'s. At the same time, if we consider the degree of each term, based on the degree of the Koszul complex, we get that every term has an extra degree of \( d - q_{m_1'} + (d - q_{m_2'}) + \ldots + (d - q_{m_n'}) \), where the $m_i'$s are correspond to the other variables that are not part of \( \{x_{m_1}, \ldots, x_{m_i}\} \).

$\wedge^n N$ has numerator $\rho_1 \rho_2 \ldots \rho_n$. 

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\(^n-1 N\) has numerator \(\sum_{i} \rho_1 \rho_2 \ldots \rho_{i-1} \rho_{i+1} \ldots \rho_n t^{d-q_i}\).

\(^{n-k} N\) has numerator \(\sum_{i_1<i_2<\ldots<i_k} \rho_{i_1} \ldots \rho_{i_k} t^{\beta_{i_1,\ldots,i_k}}\) where \(\beta_{i_1,\ldots,i_k} = \sum_{j \notin \{i_1,\ldots,i_k\}} d - q_j\).

\(^0 N\) has numerator \(t \sum_i d - q_i\).

Taking \(P(\wedge^n N) - P(\wedge^{n-1} N) + \ldots + (-1)^n P(N)\) and simplifying it combinatorially, we get the product of \((\rho_1 - t^{d-q_1}) \ldots (\rho_n - t^{d-q_n})\). Dividing by the denominator, we get

\[\prod_{i=1}^{n} \frac{\rho_i - t^{d-q_i}}{1 - \rho_i t^{q_i}}.\]

**Theorem 6.8.** For any quasihomogeneous polynomial \(W\) and admissible group \(G\). The Poincaré polynomial for \(\mathcal{H}_h\) is \(\prod_{i=1}^{m} \frac{\rho_i - t^{d-q_i}}{1 - \rho_i t^{q_i}}\), where \(x_1,\ldots,x_m\) are the variables in \(\text{Fix} \, h\), \(q_1,\ldots,q_m\) are the corresponding weights of each variable and \(\rho_1 \ldots \rho_m\) are the representations of the action of \(G\) on the span of each variable.

Based on this theorem, Theorem 5.2 becomes a direct corollary.

**Corollary 6.9.** Let \(W\) be a non-degenerate quasihomogeneous polynomial with weights \(d,q_1,q_2,\ldots,q_n\). Then the Poincaré polynomial of the Milnor ring \(\mathcal{Q}_W\) without representation is

\[P(\mathcal{Q}_W) = \prod_{i=1}^{n} \frac{1 - t^{d-q_i}}{1 - t^{q_i}}.\]

**Proof.** Since we are going to lose all information regarding the group action, we can ignore all \(dx_i\)’s. The Milnor ring is isomorphic to \(\mathcal{H}_h\) where \(h\) is the trivial group element and thus we can use the representation of the trivial group, whose representation ring is isomorphic to the complex numbers. \(\square\)
7.1 Group-Weights Conjecture

The proof of the Group-Weights Conjecture follows almost directly from the description of the Poincaré polynomial in Theorem 6.8.

**Corollary 7.1.** Let $W_1$ and $W_2$ be polynomials which have the same weights. Suppose $G \leq G^\text{max}_{W_1}$ and $G \leq G^\text{max}_{W_2}$, then $\mathcal{H}_{W_1,G} \cong \mathcal{H}_{W_2,G}$ as graded vector spaces.

**Proof.** First, we recall that Definition 2.13 does not give us the graded vector space structure of the FJRW-Ring. Rather, the grading of the FJRW ring is based on $W$-degree.

By definition, $\mathcal{H}_{W_1,G} = \bigoplus_{h \in G} (\mathcal{H}_h)^G$ and $\mathcal{H}_{W_2,G} = \bigoplus_{h \in G} (\mathcal{H}_h)^G$ (note that $\mathcal{H}_h$ has a prime symbol because $\mathcal{H}_h$ depends on the choice of polynomial). For each $h \in G$, every element in $\mathcal{H}_h$ has the same degree, and so in order to prove isomorphism, all we have to show is that for each $h$, $\dim (\mathcal{H}_h)^G = \dim (\mathcal{H}_h')^G$.

The dimension of $(\mathcal{H}_h)^G$ is the dimension of the $G$-invariant representation of the Poincaré polynomial for $\mathcal{H}_h$. Since the formula for the Poincaré polynomial depends only upon the weights and the group chosen, $\dim (\mathcal{H}_h)^G = \dim (\mathcal{H}_h')^G$.

Recall from Section 2.5 that FJRW rings are deformation invariant which means that the Frobenius manifold structure of the FJRW-theory is dependent only on the associated state space. Hence by showing that $\mathcal{H}_{W_1,G_1} \cong \mathcal{H}_{W_2,G_2}$ as graded vector spaces, we have shown that $\mathcal{H}_{W_1,G_1} \cong \mathcal{H}_{W_2,G_2}$ in the FJRW-theory.

**Theorem (Group-Weights).** Let $W_1$ and $W_2$ be polynomials which have the same weights. Suppose $G \leq G^\text{max}_{W_1}$ and $G \leq G^\text{max}_{W_2}$, then $\mathcal{H}_{W_1,G} \cong \mathcal{H}_{W_2,G}$ as FJRW A-models.

Note that this theorem is a corollary to Corollary 7.1.
7.2 Using the Poincaré Polynomial

This section gives some insight as to the process of deriving the $G$-invariant portion of the Poincaré polynomial in Corollary 6.9. Here we focus on the sector $\mathcal{H}_1$ of the state space.

7.2.1 $W = x^n$. Let a Fermat atomic type polynomial be $W = x^n$ and with group $\langle e^{2\pi i \frac{1}{n}} \rangle$. We have that the weight of $x$ is 1 and the total weight of $W$ is $n$. So the Poincaré polynomial of $\mathcal{H}_1$ is $\frac{\rho - t^{n-1}}{1 - \rho t}$ where $\rho$ is the obvious representation on $\text{Span}\{x\}$. Note that $\rho^n = 1$.

The numerator factors: $(1 - \rho t)(\rho^{n-1}t^{n-2} + \rho^{n-2}t^{n-3} + \ldots + \rho^2 t + \rho)$. Hence we get the Poincaré polynomial of the trivial sector as $(\rho^{n-1}t^{n-2} + \rho^{n-2}t^{n-3} + \ldots + \rho^2 t + \rho)$, which has no $G$-invariant terms.

7.2.2 $W = x^2y + xy^4$. Let a loop type polynomial be $W = x^2y + xy^4$ with weights $q_x = 3$ and $q_y = 1$ as shown in Chapter 2. We let the group $G = \langle e^{2\pi i \frac{4}{7}}, e^{2\pi i \frac{1}{7}} \rangle \cong C_7$.

If we let the action of $G$ on the variable $y$ be represented by $\rho$, where $\rho^7 = 1$, then it follows that the action of $G$ on $x$ can be represented by $\rho^3$.

So the Poincaré polynomial of $\mathcal{H}_1$ is

$$\frac{\rho^3 - t^4}{1 - \rho^3 t^3} \frac{\rho - t^6}{1 - \rho t} = \frac{(1 - \rho t)(\rho^6 t^3 + \rho^5 t^2 + \rho^4 t + \rho^3)(1 - \rho^3 t^3)(\rho^4 t^3 + \rho)}{1 - \rho^3 t^3} \frac{1 - \rho t}{1 - \rho t}$$

$$= (\rho^6 t^3 + \rho^5 t^2 + \rho^4 t + \rho^3)(t^3 + \rho)$$

$$= \rho^3 t^6 + \rho^2 t^5 + \rho t^4 + 2t^3 + \rho^6 t^2 + \rho^5 t + \rho^4.$$

The term $2t^3$ is $G$-invariant and corresponds to the generators $y^3 \ dx \wedge dy$ and $x \ dx \wedge dy \in (\mathcal{H}_1)^G$, both of degree 3.

Now in general, we can find an expression for the Poincaré polynomial of a loop type polynomial.

Suppose that $x$ and $y$ are variables of $W$. Let $x$ and $y$ have weights $q_x$ and $q_y$ respectively and let $W$ have total weight $d$.

The term $x^a y$ is in the loop polynomial and so $aq_x + q_y = d$ or $aq_x = d - qy$. 

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Two of the terms in the product of the Poincaré polynomial of $W$ are $\frac{\rho - t^{d-q_y}}{1 - \rho t^{a_y}}$ and $\frac{\psi - t^{d-q_y}}{1 - \psi t^{a_y}}$, where $\rho$ and $\psi$ are representations of $x$ and $y$ respectively.

We can show that $1 - \rho t^{a_x}$ divides $\psi - t^{d-q_y}$ by first substituting $d - q_y$ with $a q_x$. Next, we note that since $G$ fixes $x^a y$, $\rho^a \psi = 1$ and thus $\psi = \rho^{-a}$.

$$\psi - t^{d-q_y} = \rho^{-a} - t^{a q_y} = (1 - \rho t^{a_y})(\rho^{-1} t^{a_x(a-1)} + \rho^{-2} t^{a_x(a-2)} + \ldots + \rho^{-(a-1)} t^{a_x} + \rho^{-a})$$

So if $W_{loop} = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \ldots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1$, with representations $\rho_1, \ldots, \rho_n$, then $1 - \rho_i t^{a_i}$ divides $\rho_{i+1} - t^{d-a_{i+1}}$ and we get that the Poincaré polynomial of the trivial sector of $W_{loop}$ is

$$\prod_{i=1}^{n} \left( \sum_{j=1}^{a_i} \rho_i^{-j} t^{a_i(a_i-j)} \right).$$

7.3 Conclusion

At shown in Section 7.1, we have proved the Group-Weights conjecture. However, in the process of proving this conjecture we also now have a Poincaré polynomial for $\mathcal{H}_h$ in terms of representations. This formula also works for finding the state space of the $B$-model. The argument for proving a similar Group-Weight conjecture for the $B$-model does not work since deformation invariance does not exist in the $B$-model.

As illustrated in Section 7.2, finding the $G$-invariance of the Poincaré polynomial with representation is not as easy as compared to the proof of Theorem 5.3 where we just let $t = 1$ in order to find the general dimension of the Milnor ring. The hope is that studying more examples will lead to a formula for the $G$-invariance of the Poincaré polynomial with representation.
Bibliography


