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Persistence and Foliation Theory for Random Dynamical System and Their Application to Geometric Singular Perturbation

Ji Li

A dissertation submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

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ABSTRACT

Persistence and Foliation Theory for Random Dynamical System and Their Application to Geometric Singular Perturbation

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Persistence problem of compact invariant manifold under random perturbation is considered in this dissertation. Under uniformly small random perturbation and the condition of normal hyperbolicity, the original invariant manifold persists and becomes a random invariant manifold. The random counterpart has random local stable and unstable manifolds. They could be invariantly foliated thanks to the normal hyperbolicity.

Those underlie an extension of the geometric singular perturbation theory to the random case which means the slow manifold persists and becomes a random manifold so that the local global structure near the slow manifold persists under singular perturbation. A normal form for a random differential equation is obtained and this helps to prove a random version of the exchange lemma.

Keywords: Random Dynamical System, Random Manifolds, Singular Perturbation

Contents

Chapter 1. Introduction

To study the long term behavior of states is one of the main goal in dynamical system. It reflects the idea of understanding and prediction of a real evolving physical system. This study is mainly carried out by learning the invariant set or manifold of the dynamical system. Two natural questions arise when studying invariant manifolds. The first one is the problem of persistence. In the original modeling process, approximation or even conjecture are used to obtain an ideal dynamical system. However, the real system is always a perturbation of the ideal system. So the persistence problem under perturbation is well worth studying. The other question is what happens in the vicinity. Do points nearby converge to or diverge from the invariant manifold? Do some of them evolve at the same speed while others not? Could they be classified? All these questions concern the study of invariant foliation.

In this work, we initiate the study of the persistence problem of deterministic invariant manifolds under random perturbation. It is well known that the sufficient and necessary condition of persistence of deterministic invariant manifold under deterministic perturbation is the so called normal hyperbolicity, the definition of which is in the next chapter. We prove under random perturbation and the condition of normal hyperbolicity, that the invariant manifold persists and becomes a random invariant manifold, the definition of which is also given in the next chapter. We also prove there exists an invariant foliation for the random invariant manifold.

The following summarizes our main results:

Theorem. *Assume that $\psi(t)$ is a C^r flow, $r \geq 1$, and has a compact, connected C^r normally hyperbolic invariant manifold $\mathcal{M} \subset \mathbb{R}^n$. Then there exists $\rho > 0$ such that for any C^1 random flow $\phi(t, \omega)$ in $\mathbb{R}^n$ if*

$$||\phi(t,\omega) - \psi(t)||_{C^1} < \rho, \quad \textit{for } t \in [0,1], \omega \in \Omega,$$

then

(i) Persistence: $\phi(t,\omega)$ has a C^1 normally hyperbolic random invariant manifold $\tilde{\mathcal{M}}(\omega)$.

(ii) Smoothness: If $\phi(t,\omega)$ is C^r and the normal hyperbolicity is sufficiently strong (see Theorem 3.1) then $\tilde{\mathcal{M}}(\omega)$ is a C^r manifold diffeomorphic to $\mathcal{M}$ for each $\omega \in \Omega$.

(iii) Existence of Stable Manifolds: $\phi(t,\omega)$ has a stable manifold $\tilde{\mathcal{W}}^s(\omega)$ at $\tilde{\mathcal{M}}(\omega)$.

(iv) Existence of Unstable Manifolds: $\phi(t,\omega)$ has an unstable manifold $\tilde{\mathcal{W}}^u(\omega)$ at $\tilde{\mathcal{M}}(\omega)$.

(v) Foliation: The stable and unstable manifolds can be invariantly foliated based on the random manifold $\tilde{\mathcal{M}}$.

The above results are extensions of the classical deterministic results of Fenichel [F1], [F2] and [F3]. The precise statements will be in chapter 3.

One of the most important application of persistence of invariant manifold under random perturbation and the theory of invariant foliation is to reduce a random singular perturbation system to a lower dimensional relatively simpler system. We devote the last chapter of this dissertation to the study of singular perturbation systems with real noise, using our random persistence and foliation theory. We extend the classical deterministic geometric singular perturbation theory [F4] to the random case. We also prove a random version of the exchange lemma which describes the smooth configuration of an invariant manifold. The statement of our extension results is too long and too hard to show here in concisely. See the last chapter.

Invariant manifolds provide large scale structures that serve to organize the global phase space of a dynamical system. The classical theory of invariant manifold and invariant foliation for deterministic system has a long and rich history. The most general theory of compact normally hyperbolic invariant manifolds for finite dimensional dynamical systems was independently proved by Hirsch, Pugh and Shub [HPS1], [HPS2] and Fenichel [F1], [F2] and [F3]. The most general results for infinite dimensional systems are proved in [BLZ1],

[BLZ2] and [BLZ3] by Bates, Lu and Zeng. For a more comprehensive introduction for invariant manifold and invariant foliation theory of deterministic dynamical system, we refer the reader to these works.

Random dynamical systems take into account stochastic effects such as stochastic forcing, uncertain parameters, random sources or inputs, and random boundary conditions. They arise in the modeling of many phenomena in physics, biology, climatology, economics, etc. They are more capable of interpreting experimental data. Stochastic partial or ordinary differential equations are appropriate models for randomly influenced evolving systems. They both generate random dynamical systems. There are many local results for local random stable, unstable, and center manifolds of stationary solutions. For finite dimensional random dynamical systems, we refer to [C, BK, Bx, D, LQ, S, Wan, A, LiL]. The results on local invariant manifolds for infinite dimensional random dynamical systems can be found in [Ru, CLR, DSL1, DSL2, MS, MZZ, LS, GLS, CDLS, WD]. Random inertial manifolds were obtained in [BF, GC, DD]. Included in the previously cited work are discrete time dynamics with products of diffeomorphisms, in Euclidean space, on Riemannian manifolds, or in infinite-dimensional spaces, where applications to PDEs are considered. Various approaches have been used, the Lyapunov-Perron method in particular.

In [Wan], Wanner established the existence of invariant foliations for finite dimensional random dynamical systems in a neighborhood of a stationary solution and used the foliations to prove a Hartman-Grobman theorem for finite dimensional RDSs. Li and Lu [LiL] proved a stable and unstable foliation theorem and used it to establish a smooth linearization theorem (Sternberg type of theorem) for finite dimensional random dynamical systems. Pesin's result was established by Liu and Qian [LQ] for finite dimensional RDS and by Lian and Lu for infinite dimensional RDS. The local theory of invariant foliations for stochastic PDEs was obtained by Lu and Schmalfuss [LS].

The random persistence and foliation theory we present here are global results. The biggest difficulty about the random case is the measurability. Although measurability of random manifold could be described by measurable selection, it is not at all clear how to describe the measurability of the tangent bundle, without which no local charts on the random manifold make sense, neither does any kind of coordinate transform. Another difficulty is the local charts on random manifold. They change as ω changes which belongs to a non-compact set Ω. At present we do not have results along the lines of this paper for infinite-dimensional random semiflows, as one encounters with stochastic parabolic PDEs.

Persistence and invariant foliation results underlie geometric singular perturbation theory, which provides a powerful technique for analyzing systems involving multiple time scales which arise in applications such as neural networks and semi-conductors. Geometric singular perturbation theory gives a rigorous and well-defined way to reduce systems with small parameters to lower dimensional systems that are more easily analyzed. The theory was proved by Fenichel in [F4] in 1979. The proof is based on [F1, F2, F3]. It is a wonderful application of the deterministic persistence and invariant foliation theory. The geometric singular perturbation theory is largely used(more than 200 citations). Applications to problems from science and engineering have flourished, especially to the problem of signal transmission in nerve systems. Jones [Jo] provided a clear discussion about [F4] and made the geometric singular perturbation theory well understood to people.

In his lecture notes, Jones [Jo] included proofs of the geometric singular perturbation theory and application to a normal form. He also gave an important extension of the λ-lemma to the case where slow center directions are involved, which is known as the exchange lemma. See also [JK]. The exchange lemma is used in applications such as the study of nerve impulses and their propagation. These, mathematically, are called traveling pulses, which in moving coordinates are homoclinic solutions emanating from equilibria. Such solutions for the singular system and their persistence are of natural importance. One has to track the change of the smooth configuration of an invariant manifold as it passes a neighborhood of the slow manifold. The Exchange Lemma is a powerful tool designed for such investigations.

Many other papers address the same topic and make it more general. Jones and his coworkers [JKK] dealed with the case of multiple center directions and provide higher order and exponential estimates. Brunovsky [Brun1, Brun2] provided elementary proof of Jones' results, which also gives more insight to this problem. Liu [Liu1, Liu2] discussed the case of turning points which made this method applicable to the non-normally hyperbolic case. Schecter [S3, S2, S3] proved a general exchange lemma which contains all the above cases and a new case. He applied the general exchange lemma to study self-similar solutions of the Dafermos regularization. Our extensions deal with the random case by applying Brunovsky's method.

We hope our extensions of geometric singular perturbation theory and exchange lemma will lay the groundwork for applications and provide tools for studying random and stochastic systems.

Non-Technical Overview

Step 1. The basic idea for the existence of a normally hyperbolic random invariant manifold is due to Hadamard [H] and involves a graph transform to first construct a center-unstable manifold. One takes a Lipschitz graph over the unstable bundle at the deterministic normally-hyperbolic invariant manifold for $\psi(t)$ and maps it forward under $\phi(t,\omega)$ for large but fixed t for each $\omega \in \Omega$. Because each $\phi(t,\omega)$ is C^1-close to $\psi(t)$, and because the latter stretches the graph in the unstable direction while compressing in the normal direction, the image (for each ω) is again a Lipschitz graph. Furthermore, this mapping is a contraction on the space of such graphs and the resulting fixed graph (for each ω) is the center-unstable manifold. The center-stable manifold is obtained in the same way, by reversing time. The desired manifold is then the intersection of these two (for each ω).

Step 2. In order to prove the smoothness of the random unstable manifold, we formally differentiate the fixed point equation of the random unstable manifold to find out a functional equation which must be satisfied by the derivative of the random unstable manifold. Next, we prove the existence of a unique solution of that functional equation. Last, we prove the unique solution is indeed the derivative. For the measurability part, measurable selection is used. The biggest hurdle is the measurability of the tangent space. A metric introduced by Kato is used, which makes the discussion of measurability of the tangent space possible.

Step 3. The idea for the invariant foliation part is the same as the persistence part. The biggest difference is that instead of having local charts on a deterministic manifold, local charts on a random invariant manifold are introduced. Those charts should be related to each other for different $\omega \in \Omega$. We define local charts on the deterministic manifold and then 'induce' on the random manifold. This gives us well related local charts on the random invariant manifold. This kind of technique is also used in proving the asymptotical property of the invariant foliation.

Step 4. In order to prove the random geometric singular perturbation theory, the random persistence and foliation theory are used. We use the linearization to approximate the random singular system in a small neighborhood of the slow manifold. We then use a bump function to perturb the linear system a little bit so that the new system possesses overflowing and inflowing center-unstable and center-stable manifold, while the normal hyperbolicity of the slow manifold persists. Then the persistence and foliation theory can be used for the perturbed linear system which identify with the original system in a smaller neighborhood of the slow manifold. So all kinds of invariant manifolds and foliations of the original system exist.

Step 5. With the random geometric singular perturbation theory, the random singular system could be decoupled. The decoupled system could be analyzed quantitively to some degree. We consider a boundary value problem and prove the existence and uniqueness of the solution. The derivatives of the solution with respect to all arguments are also obtained. We get exponential order of the solution alone with its derivatives. Those exponential orders are exactly what we need to prove the random exchange lemma.

We organize this paper as follows: In chapter 2, we give the definition of some important terms and that of random dynamical system. In chapter 3, we state our main theorem. In chapter 4,we prove the existence of the random unstable manifold. In chapter 5, we prove all kinds of properties of the random unstable manifold and the persistence of normal hyperbolicity. In chapter 6, we prove the existence of the invariant foliation, the smoothness, measurability and the asymptotical property of the invariant foliation. In chapter 7, we discuss corresponding results in cases of overflowing and inflowing invariant manifold. In chapter 8, we prove a random version of the geometric singular perturbation theory and apply the theory to extend the exchange lemma to a random version.

Chapter 2. Notations and Preliminaries

2.1 Random Dynamical Systems

In this section, we review some of the basic concepts related to random dynamical systems in a Banach Space that are taken from Arnold [A]. Let $(\Omega, \mathcal{F}, P)$ be a probability space and X be a Banach space. Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$ endowed with their Borel $\sigma-$ algebra.

Definition 2.1.1. *A family $(\theta^t)_{t\in\mathbb{T}}$ of mappings from Ω into itself is called a metric dynamical system if*

(1) $(\omega, t) \to \theta^t\omega$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{T})$ measurable;

(2) $\theta^0 = id_\Omega$, the identity on Ω , $\theta^{t+s} = \theta^t \circ \theta^s$ for all $t, s \in \mathbb{T}$;

(3) θ^t preserves the probability measure P.

Definition 2.1.2. *A map*

$$\phi : \mathbb{T} \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \phi(t, \omega, x),$$

is called a random dynamical system (or a cocycle) on the Banach space X over a metric dynamical system $(\Omega, \mathcal{F}, P, \theta^t)_{t\in\mathbb{T}}$ if

(1) ϕ is $\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}(X)$-measurable;

(2) The mappings $\phi(t,\omega) := \phi(t,\omega,\cdot) : X \to X$ *form a cocycle over* θ^t*:*

$$\phi(0,\omega) = Id, \quad \textit{for all } \omega \in \Omega,$$

$$\phi(t+s,\omega) = \phi(t,\theta^s\omega) \circ \phi(s,\omega), \quad \textit{for all } t,s \in \mathbb{T}, \quad \omega \in \Omega.$$

An example is the solution operator for a random differential equation driven by a real noise:

$$\frac{dx}{dt} = f(\theta_t\omega, x), \tag{2.1.1}$$

where $x \in \mathbb{R}^d$, $f : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a measurable function and $f_\omega(t,\cdot) \equiv f(\theta_t\omega,\cdot) \in L_{loc}(\mathbb{R}, C_b^{0,1})$. (All through this paper, we suppose these conditions hold whenever there is a random differential equation of the form (2.1.1).)

Here, $(\Omega,\mathcal{F},P)$ is the classic Wiener space, i.e., $\Omega = \{\omega : \omega(\cdot) \in C(\mathbb{R},\mathbb{R}^d), \omega(0) = 0\}$ endowed with the open compact topology so that Ω is a Polish space and P is the Wiener measure. Define a measurable dynamical system θ^t on the probability space $(\Omega,\mathcal{F},P)$ by the Wiener shift $(\theta^t\omega)(\cdot) = \omega(t+\cdot) - \omega(t)$ for $t > 0$. It is well-known that P is invariant and ergodic under θ^t. This measurable dynamical system θ^t is also a metric dynamical system. This models the noise in the system, see [A], page 60 for details.

2.2 Normally Hyperbolic Random Invariant Manifolds

We first recall that a multifunction $\mathcal{M} = (\mathcal{M}(\omega))_{\omega\in\Omega}$ of nonempty closed sets $\mathcal{M}(\omega)$, $\omega \in \Omega$, contained in a separable Banach space X is called a random set if

$$\omega \mapsto \inf_{y\in\mathcal{M}(\omega)} ||x-y||$$

is a random variable for any $x \in X$. When each of the sets $\mathcal{M}(\omega)$ is a manifold, we call $\mathcal{M}$ a random manifold.

Definition 2.2.1. *A random manifold* $\mathcal{M}$ *is called a random invariant manifold for a ran-*

dom dynamical system $\phi(t,\omega)$ *if*

$$\phi(t,\omega,\mathcal{M}(\omega)) = \mathcal{M}(\theta^t\omega) \text{ for all } t \in \mathbb{R}, \omega \in \Omega$$

Definition 2.2.2. *A random manifold* $\bar{\mathcal{M}}(\omega) = \mathcal{M}(\omega) \cup \partial\mathcal{M}(\omega)$ *is called a random overflowing invariant manifold for a random dynamical system* $\phi(t,\omega)$ *if*

$$\phi(t,\omega,\mathcal{M}(\omega)) \supset \bar{\mathcal{M}}(\theta^t\omega) \text{ for all } t > 0, \omega \in \Omega$$

Definition 2.2.3. *A random manifold* $\bar{\mathcal{M}}(\omega) = \mathcal{M}(\omega) \cup \partial\mathcal{M}(\omega)$ *is called a random inflowing invariant manifold for a random dynamical system* $\phi(t,\omega)$ *if*

$$\phi(t,\omega,\mathcal{M}(\omega)) \supset \bar{\mathcal{M}}(\theta^t\omega) \text{ for all } t < 0, \omega \in \Omega$$

Definition 2.2.4. *A random invariant manifold* $\mathcal{M}$ *is said to be normally hyperbolic if for almost every* $\omega \in \Omega$ *and* $x \in \mathcal{M}(\omega)$, *there exists a splitting which is* C^0 *in* x *and measurable in* ω:

$$X = E^u(\omega,x) \oplus E^c(\omega,x) \oplus E^s(\omega,x)$$

of closed subspaces with associated projections $\Pi^u(\omega,x)$, $\Pi^c(\omega,x)$, *and* $\Pi^s(\omega,x)$ *such that*

(i) The splitting is invariant:

$$D_x\phi(t,\omega)(x)E^i(\omega,x) = E^i(\theta_t\omega, \phi(t,\omega)(x)), \quad \text{for } i = u, c,$$

and

$$D_x\phi(t,\omega)(x)E^s(\omega,x) \subset E^s(\theta_t\omega, \phi(t,\omega)(x)).$$

(ii) $D_x\phi(t,\omega)(x)\big|_{E^i(\omega,x)} : E^i(\omega,x) \to E^i(\theta_t\omega, \phi(t,\omega)(x))$ *is an isomorphism for* $i = u, c$. $E^c(\omega,x)$ *is the tangent space of* $\mathcal{M}(\omega)$ *at* x.

(iii) *There are* (θ, ϕ)*-invariant random variables* $\bar{\alpha}, \bar{\beta} : \mathcal{M} \to (0, \infty)$, $\bar{\alpha} < \bar{\beta}$, *and a tempered random variable* $K : \mathcal{M} \to [1, \infty)$ *such that*

$$||D_x\phi(t,\omega)(x)\Pi_s(\omega,x)|| \leq K(\omega,x)e^{-\bar{\beta}\omega,x)t} \quad \text{for } t \geq 0, \tag{2.2.1}$$

$$||D_x\phi(t,\omega)(x)\Pi_u(\omega,x)|| \leq K(\omega,x)e^{\bar{\beta}(\omega,x)t} \quad \text{for } t \leq 0, \tag{2.2.2}$$

$$||D_x\phi(t,\omega)(x)\Pi_c(\omega,x)|| \leq K(\omega,x)e^{\bar{\alpha}(\omega,x)|t|} \quad \text{for } -\infty < t < \infty. \tag{2.2.3}$$

Remark 1. We will also call a random set(non-invariant) **normally hyperbolic** if (i), (ii) and (iii) above hold on each orbit segment contained in the random set. Particularly, for a random overflowing invariant manifold, if there's only E^c and E^s in the splitting, we call the random overflowing invariant manifold normal (stably) hyperbolic. For a random inflowing invariant manifold, if there's only E^c and E^u in the splitting, we call the random inflowing invariant manifold normal (unstably) hyperbolic.

Chapter 3. Main Results

We consider a deterministic flow $\psi(t)(x) \equiv \psi(t,x)$ in $\mathbb{R}^n$ and its randomly perturbed (cocycle) counterpart $\phi(t,\omega)(x) \equiv \phi(t,\omega,x)$.

Let $\mathcal{M}$ be a compact connected C^r normally hyperbolic invariant manifold under the deterministic C^r flow $\psi(t)$ and let $T\mathbb{R}^n|\mathcal{M} = T\mathcal{M} \oplus E^s \oplus E^u$ be the associated splitting. We will sometimes use the notation E^c to denote the bundle $T\mathcal{M}$ and we will sometimes use E^i to deonote the subspace E^i_m $(i = c, s, u)$ at a point $m \in \mathcal{M}$ when the context makes the meaning obvious. Our main results are:

Theorem 3.0.1. *Assume that* $\psi(t)$ *is a* C^r *flow,* $r \geq 1$*, and has a compact, connected* C^r *normally hyperbolic invariant manifold* $\mathcal{M} \subset \mathbb{R}^n$*. Let the positive exponents related to the normal hyperbolicity be* $\alpha < \beta$ *in (2.1)-(2.3), which in this case are constant and* *deterministic. Then there exists $\rho > 0$ such that for any random C^1 flow $\phi(t, \omega)$ in $\mathbb{R}^n$ if*

$$||\phi(t, \omega) - \psi(t)||_{C^1} < \rho, \quad \text{for } t \in [0, 1], \omega \in \Omega,$$

then

(i) *Persistence: $\phi(t, \omega)$ has a C^1 normally hyperbolic random invariant manifold $\tilde{\mathcal{M}}(\omega)$,*

(ii) *Smoothness: If $\alpha < r\beta$ and $\phi(t, \omega)$ is C^r, then $\tilde{\mathcal{M}}(\omega)$ is a C^r manifold diffeomorphic to $\mathcal{M}$ for each $\omega \in \Omega$,*

(iii) *Existence of Stable Manifolds: $\phi(t, \omega)$ has a stable manifold $\tilde{\mathcal{W}}^s(\omega)$ at $\tilde{\mathcal{M}}(\omega)$,*

(iv) *Existence of Unstable Manifolds: $\phi(t, \omega)$ has an unstable manifold $\tilde{\mathcal{W}}^u(\omega)$ at $\tilde{\mathcal{M}}(\omega)$.*

Theorem 3.0.2. *Under the condition and result of theorem 3.0.1, there exists a unique C^{r-1} family of C^r submanifolds $\left\{\tilde{\mathcal{W}}^{uu}(\omega, x) \ : \ \omega \in \Omega, x \in \tilde{\mathcal{M}}(\omega)\right\}$ of $\tilde{\mathcal{W}}^u(\omega)$ satisfying:*

(1) *For each $(\omega, x) \in \Omega \times \tilde{\mathcal{M}}, \tilde{\mathcal{M}}(\omega) \cap \tilde{\mathcal{W}}^{uu}(\omega, x) = \{x\}$, $T_x\tilde{W}^{uu}(\omega, x) = \tilde{E}^u(\omega, x)$ and $\tilde{W}^{uu}(\omega, x)$ varies measurably with respect to (ω, x) in $\Omega \times \tilde{\mathcal{M}}$.*

(2) *If $x_1, \ x_2 \ \in \ \tilde{\mathcal{M}}(\omega), \ x_1 \ \neq \ x_2$, then $\tilde{\mathcal{W}}^{uu}(\omega, x_1) \cap \tilde{\mathcal{W}}^{uu}(\omega, x_2) \ = \ \emptyset$ and $\tilde{\mathcal{W}}^u(\omega) \ = \cup_{x \in \tilde{\mathcal{M}}(\omega)} \tilde{\mathcal{W}}^{uu}(\omega, x)$.*

(3) *For $x \in \tilde{\mathcal{M}}(\omega)$, $\phi(t, \omega)\big(\tilde{\mathcal{W}}^{uu}(\omega, x)\big) \subset \tilde{\mathcal{W}}^{uu}(\theta_t \omega, \phi(t, \omega)x)$ for $-t$ big enough.*

(4) *For $y \in \tilde{\mathcal{W}}^{uu}(\omega, x)$ and $x_1 \neq x \in \tilde{\mathcal{M}}(\omega)$ with $|\phi(t, \omega)(x_1) - \phi(t, \omega)(x)| \to 0$ as $t \to -\infty$, we have*

$$\frac{|\phi(t, \omega)(y) - \phi(t, \omega)(x)|}{|\phi(t, \omega)(y) - \phi(t, \omega)(x_1)|} \to 0$$

exponentially as $t \to -\infty$.

(5) *For $y_1, y_2 \in \tilde{\mathcal{W}}^{uu}(\omega, x)$, $|\phi(t, \omega)(y_1) - \phi(t, \omega)(y_2)| \to 0$ exponentially as $t \to -\infty$.*

(6) *$\tilde{\mathcal{W}}^{uu}(\theta^t \omega, x)$ is C^0 in t for each fixed (ω, x).*

Theorem 3.0.3. *Under the condition and result of theorem 3.0.1, there exists a unique C^{r-1} family of C^r submanifolds $\{\tilde{\mathcal{W}}^{ss}(\omega, x) : \omega \in \Omega, x \in \tilde{\mathcal{M}}(\omega)\}$ of $\tilde{\mathcal{W}}^s(\omega)$ satisfying:*

(1) For each $(\omega, x) \in \Omega \times \tilde{\mathcal{M}}, \tilde{\mathcal{M}}(\omega) \cap \tilde{\mathcal{W}}^{ss}(\omega, x) = \{x\}$, $T_x \tilde{W}^{ss}(\omega, x) = \tilde{E}^s(\omega, x)$ and $\tilde{W}^{ss}(\omega, x)$ varies measurably with respect to (ω, x) in $\Omega \times \tilde{\mathcal{M}}$.

(2) If x_1, $x_2 \in \tilde{\mathcal{M}}(\omega)$, $x_1 \neq x_2$, then $\tilde{\mathcal{W}}^{ss}(\omega, x_1) \cap \tilde{\mathcal{W}}^{ss}(\omega, x_2) = \emptyset$ and

$$\tilde{\mathcal{W}}^s(\omega) = \cup_{x \in \tilde{\mathcal{M}}(\omega)} \tilde{\mathcal{W}}^{ss}(\omega, x).$$

(3) For $x \in \tilde{\mathcal{M}}(\omega)$, $\phi(t, \omega)\big(\tilde{\mathcal{W}}^{ss}(\omega, x)\big) \subset \tilde{\mathcal{W}}^{ss}(\theta_t \omega, \phi(t, \omega)x)$ for t big enough.

(4) For $y \in \tilde{\mathcal{W}}^{ss}(\omega, x)$ and $x_1 \neq x \in \tilde{\mathcal{M}}(\omega)$ with $|\phi(t, \omega)(x_1) - \phi(t, \omega)(x)| \to 0$ as $t \to \infty$, we have

$$\frac{|\phi(t, \omega)(y) - \phi(t, \omega)(x)|}{|\phi(t, \omega)(y) - \phi(t, \omega)(x_1)|} \to 0$$

exponentially as $t \to +\infty$.

(5) For $y_1, y_2 \in \tilde{\mathcal{W}}^{ss}(\omega, x)$, $|\phi(t, \omega)(y_1) - \phi(t, \omega)(y_2)| \to 0$ exponentially as $t \to +\infty$.

(6) $\tilde{\mathcal{W}}^{ss}(\theta^t \omega, x)$ is C^0 in t for each fixed (ω, x).

We use the notation $B(0, r)$ to refer to the ball centered at 0 of radius r where the space is clear from the context. We will also use the notation $L|E$ to mean the restriction of the linear operator L to the subspace E. We do not place E as a subscript since often it has sub and superscripts.

Chapter 4. Existence of the Random Unstable Manifold

4.1 Basic Lemmas

We first recall some facts for deterministic systems taken from [F1]. Then we deduce the corresponding properties for random systems.

The first two propositions concern the hyperbolicity of the invariant manifold $\mathcal{M}$.

Proposition 4.1.1. *Normal hyperbolicity is independent of the metric.*

Proposition 4.1.2. *If $\mathcal{M}$ is normally hyperbolic under the flow $\psi(t)$*

(1) There exist positive constants $a < 1$ and c_1 such that

$$||D\psi(t)(\psi(-t)(m))|E^s|| < c_1 a^t$$

for all $m \in \mathcal{M}$ and $t \geq 0$ and

$$||D\psi(t)(\psi(-t)(m))|E^u|| < c_1 a^{-t}$$

for all $m \in \mathcal{M}$ and $t \leq 0$.

(2) If $\alpha < r\beta$, there exist $c_2 > 0$ and $r' > r$ such that

$$||D\psi(t)(\psi(-t)(m))|E^s||\, ||D(\psi(-t))(m)|E^c||^{r'} < c_2$$

for all $m \in \mathcal{M}$ and $t \geq 0$, and

$$||D\psi(t)(\psi(-t)(m))|E^u||\, ||D(\psi(-t))(m)|E^c||^{r'} < c_2$$

for all $m \in \mathcal{M}$ and $t \leq 0$.

Let the dimensions of E^s, E^u, and $\mathcal{M}$ be k, l, m, respectively, so that $k+l+m = n$, and let Π^s, Π^u, Π^c denote the projections corresponding to the splitting. The next two propositions are on the smooth approximation of normal bundles and the tubular neighborhood of $\mathcal{M}$, see [W].

Proposition 4.1.3. *There exist k and l-dimensional C^r subbundles $\tilde{E}^s$ and $\tilde{E}^u$ of $T\mathbb{R}^n|\mathcal{M}$ which are arbitrarily close to E^s and E^u, respectively, and such that $T\mathcal{M} \oplus \tilde{E}^i$ is invariant under $D\psi(t)$, i.e., $T\mathcal{M} \oplus \tilde{E}^i = T\mathcal{M} \oplus E^i$ for $i = s, u$.*

We have $T\mathbb{R}^n|\mathcal{M} = \tilde{E}^s \oplus \tilde{E}^u \oplus T\mathcal{M}$ and denote by $\tilde{\Pi}^s$, $\tilde{\Pi}^u$ and $\tilde{\Pi}^c$ the projections corresponding to the splitting. We sometimes use the notation $\tilde{E} = \tilde{E}^s \oplus \tilde{E}^u$.

Proposition 4.1.4. *There exists a C^r diffeomorphism γ from a neighborhood of the zero section of $\tilde{E}$ to a tubular neighborhood of $\mathcal{M}$ in $\mathbb{R}^n$.*

To construct the random unstable and stable manifolds, we first introduce the local coordinate charts in $\mathbb{R}^n$ near $\mathcal{M}$.

We use the normal bundle $\tilde{E}$ to define local charts in $\mathbb{R}^n$ near $\mathcal{M}$ as in [F1]. By compactness of $\mathcal{M}$, we can take finitely many points $p_1,\ p_2, \cdots, p_s$ of $\mathcal{M}$ with the following properties: for each p_j, there exists a coordinate neighborhood U^j of p_j in $\mathcal{M}$; $\{U^j\}$ covers $\mathcal{M}$; in each U^j, $\tilde{E}$ has an orthonormal basis which changes in a C^r manner. Moreover, we can shrink each U^j a little to $\tilde{U}^j$ such that $\overline{\tilde{U}^j} \subset U^j$ and $\{\tilde{U}^j\}$ still covers $\mathcal{M}$. Suppose $\{(\sigma_j, U_5^j(= U^j))\}$ is an atlas on $\mathcal{M}$ and $\sigma(U_5^j) = \mathcal{D}_5^j \subset \mathbb{R}^m$ is an open neighborhood of the origin in $\mathbb{R}^m$. We may take open subsets $\mathcal{D}_1^j$, $\mathcal{D}_2^j$, $\mathcal{D}_3^j$, $\mathcal{D}_4^j$ of $\mathcal{D}_5^j$ such that $\overline{\mathcal{D}_{i-1}^j} \subset \mathcal{D}_i^j$ for $i = 2, 3, 4, 5$ and $\overline{\tilde{U}^j} \subset \sigma_j^{-1}(\mathcal{D}_1^j)$. Taking $U_i^j = \sigma_j^{-1}(\mathcal{D}_i^j)$, U_i^j are open neighborhoods of p_j and $\tilde{U}^j \subset U_i^j$ for all $i = 1, 2, 3, 4, 5$. So $\{U_i^j\}_j$ covers $\mathcal{M}$. In fact, we have

$$\mathcal{M} = \cup_j U_1^j \subset \cup_j U_2^j \subset \cdots \subset \cup_j U_5^j = \mathcal{M}.$$

Define for small $\epsilon > 0$

$$\tilde{E}_\epsilon = \tilde{E}_\epsilon^s \oplus \tilde{E}_\epsilon^u := \{(m, \nu^s, \nu^u) \in \tilde{E} :\ ||\nu^s|| < \epsilon,\ ||\nu^u|| < \epsilon\}.$$

It follows from Proposition 4.1.4 that there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, $\tilde{E}_\epsilon$ is C^r diffeomorphic to a neighborhood V of $\mathcal{M}$ in $\mathbb{R}^n$. In other words, $\tilde{E}_\epsilon$ is a tubular neighborhood of $\mathcal{M}$. From now on, we view both $\psi(t, x)$ and $\phi(t, \omega, x)$ as flows on $\tilde{E}_\epsilon$ as well as on V.

Choose a C^r orthonormal basis on $\tilde{E}|U_5^j$ for each j. Thus, for $m \in U_5^j$ and $(m, \nu^s) \in \tilde{E}_\epsilon^s$, ν^s has coordinates $(x_1^s, \cdots, x_k^s)$ under the chosen orthonormal basis, and these depend smoothly on m. Define $\tau_j^s : \tilde{E}_\epsilon^s | U_5^j \to \mathbb{R}^k$ by

$$\tau_j^s(m, \nu^s) = x^s = (x_1^s, \cdots, x_k^s) \in \mathbb{R}^k,$$

and define τ_j^u similarly by

$$\tau_j^u(m, \nu^u) = x^u = (x_1^u, \cdots, x_l^u) \in \mathbb{R}^l.$$

We have

$$||x^s||_{\mathbb{R}^k} = ||\nu^s||, \ ||x^u||_{\mathbb{R}^l} = ||\nu^u||.$$

These give us C^r diffeomorphisms γ_j from $\tilde{E}_\epsilon | U_5^j$ to subsets of $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^k \oplus \mathbb{R}^l$, where

$$\gamma_j(m, \nu^s, \nu^u) = (x^c, x^s, x^u) = (\sigma_j(m), \tau_j^s(m, \nu^s), \tau_j^u(m, \nu^u)).$$

So $\{(\tilde{E}_\epsilon | U_5^j, \gamma_j)\}$ gives us an atlas on $\tilde{E}_\epsilon$.

Since $\mathcal{M}$ is compact, there exists $L_1 > 0$ such that $D\sigma_j$ and $D\sigma_j^{-1}$ are bounded by L_1 for all m and j.

Then for sufficiently large $T > 0$, we represent the random diffeomorphism $\phi(T, \omega)(\cdot)$ in the local charts by

$$(x^c, x^s, x^u) \to (f_{ij}^c(\omega, x^c, x^s, x^u), f_{ij}^s(\omega, x^c, x^s, x^u), f_{ij}^u(\omega, x^c, x^s, x^u)),$$

where

$$f_{ij}^l(\omega, x^c, x^s, x^u) = P_l \circ \gamma_j \circ \phi(T, \omega, \cdot) \circ \gamma_i^{-1}(x^c, x^s, x^u)$$

for $(x^c, x^s, x^u) \in \gamma_i \circ \tilde{E}_\epsilon | \overline{U_4^i \cap \psi(-T, U_4^j)}$ and $P_l(x^c, x^s, x^u) = x^l$ for $l = c, s, u$. Note that given i and (x^c, x^s, x^u), $f_{ij}(\omega, x^c, x^s, x^u)$ is defined only for some $j's$.

Denoting the differential operators $D_1 = D_{x^c}, D_2 = D_{x^s}, D_3 = D_{x^u}$, we have

$$\begin{aligned}
& D_2 f^s_{ij}(\omega, x^c, x^s, x^u) \\
= \quad & DP_l D\gamma_j D\phi(T,\omega) D_2\gamma^{-1}(x^c, x^s, x^u) \\
= \quad & DP_l D\gamma_j (\tilde{\Pi}^s D\phi(T,\omega)|\tilde{E}^s) D_2\gamma_i^{-1}(x^c, x^s, x^u).
\end{aligned}$$

Since $\mathcal{M}$ is compact, DP_l, $D\gamma_j$, $D_2\gamma^{-1}$ along with their inverses are uniformly bounded by a constant $L_2 > 0$. Then

$$||D_2 f^s_{ij}(\omega, x^c, x^s, x^u)|| \leq L_2^3 ||\tilde{\Pi}^s D\phi(T,\omega)|\tilde{E}^s||.$$

Similarly, we get for $k = 1, 2, 3$ and $l = c, s, u$,

$$||(D_k f^l_{ij}(\omega, x^c, x^s, x^u))^{\pm 1}|| \leq L ||\tilde{\Pi}^l D\phi(T,\omega)|\tilde{E}^l||,$$

for some $L > 0$. We have the following lemma:

Lemma 4.1.1. *There exists a $T > 0$ such that for any $\eta > 0$, if ϵ and ρ are small enough, then*

$$||(D_3 f^u_{ij}(\omega, x^c, x^s, x^u))^{-1}|| < \frac{1}{4} \tag{4.1.1}$$

$$||D_2 f^s_{ij}(\omega, x^c, x^s, x^u)|| < \frac{1}{2} \tag{4.1.2}$$

$$||(D_1 f^c_{ij}(\omega, x^c, x^s, x^u))^{-1}||^k ||D_2 f^s_{ij}(\omega, x^{c\prime}, x^{s\prime}, x^{u\prime})|| < \frac{1}{4}, \tag{4.1.3}$$

for $0 \leq k \leq r$, $x^{c\prime}$ close to x^c, and

$$||(D_3 f^c_{ij}(\omega, x^c, x^s, x^u))^{-1}|| < \eta. \tag{4.1.4}$$

$$||f^s_{ij}(\omega, x^c, x^s, x^u)|| < \eta, \;\; ||f^u_{ij}(\omega, x^c, x^s, x^u)|| < \eta \tag{4.1.5}$$

$$||D_1 f^s_{ij}(\omega, x^c, x^s, x^u)|| < \eta, \ \ ||D_3 f^s_{ij}(\omega, x^c, x^s, x^u)|| < \eta \tag{4.1.6}$$

$$||D_1 f^u_{ij}(\omega, x^c, x^s, x^u)|| < \eta, \ \ ||D_2 f^u_{ij}(\omega, x^c, x^s, x^u)|| < \eta. \tag{4.1.7}$$

Moreover, the norm of all first partial derivatives of f_{ij} and their inverses are bounded by some $Q > 0$.

Proof. By Proposition 4.1.2, there exists a $T > 0$ such that

$$L^r ||D\psi(-T)(m)|E^u|| < \frac{1}{16}$$

$$L^r ||D\psi(T)(\psi(-T)(m))|E^s|| < \frac{1}{8}$$

$$L^{r+1} ||D\psi(-T)(m)|T\mathcal{M}||^k ||D\psi(T)(\psi(-T, m))|E^s|| < \frac{1}{16}, \ 0 \leq k \leq r.$$

Since $\tilde{E}$ and E are arbitrarily close, and $||\phi(t, \omega) - \psi(t)||_{C^1} < \rho$ for all $t \in [0, 1]$ and $\omega \in \Omega$, if ρ is small enough, we have

$$L^r ||\tilde{\Pi}^u D\phi(-T, \omega)(m)|\tilde{E}^u|| < \frac{1}{8} \tag{4.1.8}$$

$$L^r ||\tilde{\Pi}^s D\phi(T, \omega)(\psi(-T, m))|\tilde{E}^s|| < \frac{1}{4} \tag{4.1.9}$$

$$L^{r+1} ||D\phi(-T, \omega)(m)|T\mathcal{M}||^k ||\tilde{\Pi}^s D\phi(T, \omega)(\psi(-T)(m))|\tilde{E}^s|| < \frac{1}{8}, \ 0 \leq k \leq r. \tag{4.1.10}$$

Note that we identified V with $E(\epsilon)$ and viewed $\psi(t)$ and $\phi(t, \omega)$ as flows in the bundles. So in terms of f^l_{ij},

$$||(D_3 f^u_{ij}(\omega, x^c, 0, 0))^{-1}|| < \frac{1}{8} \tag{4.1.11}$$

$$||D_2 f^s_{ij}(\omega, x^c, 0, 0)|| < \frac{1}{4} \tag{4.1.12}$$

$$||(D_1 f^c_{ij}(\omega, x^c, 0, 0))^{-1}||^k ||D_2 f^s_{ij}(\omega, x^c, 0, 0)|| < \frac{1}{8}, \ \ 0 \leq k \leq r. \tag{4.1.13}$$

We also have

$$||(D_3 f^c_{ij}(\omega, x^c, 0, 0))^{-1}|| < \frac{1}{2}\eta. \tag{4.1.14}$$

Choosing ϵ and ρ small enough, we have

$$||(D_3 f^u_{ij}(\omega, x^c, x^s, x^u))^{-1}|| < \frac{1}{4}$$

$$||D_2 f^s_{ij}(\omega, x^c, x^s, x^u)|| < \frac{1}{2}$$

$$||(D_1 f^c_{ij}(\omega, x^c, x^s, x^u))^{-1}||^k ||D_2 f^s_{ij}(\omega, x^{c\prime}, x^{s\prime}, x^{u\prime})|| < \frac{1}{4},$$

for $0 \le k \le r$, $x^{c\prime}$ close to x^c, and

$$||(D_3 f^c_{ij}(\omega, x^c, x^s, x^u))^{-1}|| < \eta.$$

By the invariance of $\mathcal{M}$ under $\psi(T)$, the invariance of E^u, E^s, $T\mathcal{M}$ under $D\psi(T)$ and the closeness of $\tilde{E}$ and E, if ϵ and ρ are small enough, we have

$$||f^s_{ij}(\omega, x^c, x^s, x^u)|| < \eta, \ \ ||f^u_{ij}(\omega, x^c, x^s, x^u)|| < \eta$$

$$||D_1 f^s_{ij}(\omega, x^c, x^s, x^u)|| < \eta, \ \ ||D_3 f^s_{ij}(\omega, x^c, x^s, x^u)|| < \eta$$

$$||D_1 f^u_{ij}(\omega, x^c, x^s, x^u)|| < \eta, \ \ ||D_2 f^u_{ij}(\omega, x^c, x^s, x^u)|| < \eta.$$

This completes the proof of the lemma. □

4.2 Existence of the Random Unstable Manifold

In this section, we will prove the existence of the random unstable manifold using the graph transform. The random unstable manifold will be constructed as a section of $\tilde{E}_\epsilon$ over $\tilde{E}^u_\epsilon$. We will define a transform on the space of all Lipschitz sections and prove the transform has a fixed point which gives the random unstable manifold.

Let X denote the space of sections of $\tilde{E}_\epsilon$ over $\tilde{E}^u_\epsilon$. For $u(\omega) \in X$, it is locally represented by $u_i(\omega)$:

$$u_i(\omega, x^c, x^u) = \tau_i^s \circ P^s \circ u(\omega) \circ (\sigma_i \times \tau_i^u)^{-1}(x^c, x^u)$$

where P^s is the fiber projection. For $\epsilon > 0$ fixed so that $\bar{B}_\epsilon(0) \subset \mathcal{D}_5^i$ for all j. Define

$$\operatorname{Lip} u(\omega) := \max_i \sup_{x^c, x^{c\prime} \in \mathcal{D}_3^i, ||x^u||, ||x^{u\prime}|| \leq \epsilon, (x^c, x^u) \neq (x^{c\prime}, x^{u\prime})} \frac{||u_i(\omega, x^c, x^u) - u_i(\omega, x^{c\prime}, x^{u\prime})||}{\max\{||x^c - x^{c\prime}||, ||x^u - x^{u\prime}||\}} \quad (4.2.1)$$

if it exists. Denote $X_\delta = \{u(\omega) \in X | \operatorname{Lip} u(\omega) \leq \delta\}$. Denote a random section by

$$u = \{u(\omega) \in X | \omega \in \Omega\}$$

and denote by S the set of all such $u = \{u(\omega) : \omega \in \Omega\}$. Define

$$\operatorname{Lip}\, u := \sup_\omega \operatorname{Lip} u(\omega),$$

and let

$$S_\delta = \{u \in S | \operatorname{Lip} u \leq \delta\}.$$

Define the norm on S_δ by

$$||u|| := \sup_\omega \max_i \sup_{x^c, x^u} |u_i(\omega, x^c, x^u)|.$$

Then the induced metric on S_δ makes it into a complete metric space. Let T be given by Lemma 3.1.

Proposition 4.2.1. *There is a unique $u \in S$ such that $\phi(t, \omega, \mathit{graph}\, u(\omega)) \supset \mathit{graph}\, u(\theta^t \omega)$ for all $t > T$ and $\omega \in \Omega$. Furthermore, $u \in S_\delta$.*

We need several lemmas to prove this proposition. We occasionally write $u(\omega)$ in place of $\operatorname{graph} u(\omega)$.

In the local coordinates, the mapping $u(\omega) \to \phi(T, \omega)(u(\omega))$ has the form

$$\begin{aligned}&(x^c, u_i(\omega, x^c, x^u), x^u)\\&\to (f_{ij}^c(\omega, x^c, u_i(\omega, x^c, x^u), x^u), f_{ij}^s(\omega, x^c, u_i(\omega, x^c, x^u), x^u), f_{ij}^u(\omega, x^c, u_i(\omega, x^c, x^u), x^u))\end{aligned}$$

Define an operator G on S_δ by

$$(Gu)_j(\omega, \xi^c, \xi^u) = f_{ij}^s(\theta^{-T}\omega, x^c, u_i(\theta^{-T}\omega, x^c, x^u), x^u),$$

where

$$\xi^c = f_{ij}^c(\theta^{-T}\omega, x^c, u_i(\theta^{-T}\omega, x^c, x^u), x^u),$$

$$\xi^u = f_{ij}^u(\theta^{-T}\omega, x^c, u_i(\theta^{-T}\omega, x^c, x^u), x^u).$$

In the following, we will show G is well defined. For convenience, we denote $x^{cu} = (x^c, x^u)$, $\xi^{cu} = (\xi^c, \xi^u)$, and write

$$f_{ij}^s(\omega, x^{cu}, x^s) = f_{ij}^s(\omega, x^c, x^s, x^u),$$

and

$$f_{ij}^{cu}(\omega, x^{cu}, x^s) = (f_{ij}^c(\omega, x^c, x^s, x^u), f_{ij}^u(\omega, x^c, x^s, x^u))^T.$$

We also use the norms

$$||x^{cu}|| = \max(||x^c||, ||x^u||),$$

$$||\xi^{cu}|| = \max(||\xi^c||, ||\xi^u||),$$

$$||f_{ij}^{cu}|| = \max(||f_{ij}^c||, ||f_{ij}^u||).$$

Thus, we may write the local representatives as

$$u_i(\omega, x^{cu}) = u_i(\omega, x^c, x^u)$$

$$(Gu)_j(\xi^{cu}) = f^s_{ij}(\omega, x^{cu}, u(\omega, x^{cu}))$$

with

$$\xi^{cu} = f^{cu}_{ij}(\omega, x^{cu}, u(\omega, x^{cu})).$$

Denote

$$A \equiv D_{1,3} f^s_{ij}(\omega, x^{cu}, x^s) \equiv (D_1 f^s_{ij}(\omega, x^c, x^s, x^u), D_3 f^s_{ij}(\omega, x^c, x^s, x^u)),$$

$$B \equiv D_2 f^s_{ij}(\omega, x^{cu}, x^s) = D_2 f^s_{ij}(\omega, x^c, x^s, x^u),$$

$$C \equiv D_{1,3} f^{cu}_{ij}(\omega, x^{cu}, x^s) = \begin{pmatrix} D_1 f^c_{ij}(\omega, x^c, x^s, x^u) & D_3 f^c_{ij}(\omega, x^c, x^s, x^u) \\ D_1 f^u_{ij}(\omega, x^c, x^s, x^u) & D_3 f^u_{ij}(\omega, x^c, x^s, x^u) \end{pmatrix},$$

and

$$E \equiv D_2 f^{cu}_{ij}(\omega, x^{cu}, x^s) = (D_2 f^c_{ij}(\omega, x^c, x^s, x^u), D_2 f^u_{ij}(\omega, x^c, x^s, x^u)).$$

Lemma 4.2.1. *For $\delta > 0$ small enough and any $0 < \epsilon < \epsilon_0$, there exists $\rho(\epsilon) > 0$ such that for $0 < \rho < \rho(\epsilon)$, G is well defined on S_δ.*

Proof. For fixed ω and i and any $x^{cu}, x^{cu\prime} \in \mathcal{D}^i_3 \times B(0, \epsilon)$, we have

$$\begin{aligned}
&||\xi^{cu} - \xi^{cu\prime}|| \\
&= ||f^{cu}_{ij}(\theta^{-T}\omega, x^{cu}, u_i(\theta^{-T}\omega, x^{cu})) - f^{cu}_{ij}(\theta^{-T}\omega, x^{cu\prime}, u_i(\theta^{-T}\omega, x^{cu\prime}))|| \\
&\geq \frac{1}{Q}||x^{cu} - x^{cu\prime}|| - Q\delta||x^{cu} - x^{cu\prime}|| \\
&> \frac{1}{2Q}||x^{cu} - x^{cu\prime}||,
\end{aligned}$$

which implies that $f^{cu}_{ij}(\theta^{-T}\omega, \cdot, u_i(\theta^{-T}\omega, \cdot))$ is one-to-one, thus is a homeomorphism from

$\mathcal{D}_3^i \times B(0,\epsilon)$ to its range. Note that although the Mean Value Theorem does not hold in multidimensional space, we still have the first inequality above since the matrix of partial derivatives obtained by using the Mean Value Theorem componentwise is close to the Jacobian. Such estimates will be used repeatedly in our analysis.

Letting $x^c = x^{c\prime}$ and using Lemma 4.1.1, we have

$$\begin{aligned}
&||\xi^u - \xi^{u\prime}|| \\
&= ||f_{ij}^u(\theta^{-T}\omega, x^c, u_i(\theta^{-T}\omega, x^c, x^u), x^u)) - f_{ij}^u(\theta^{-T}\omega, x^c, u_i(\theta^{-T}\omega, x^c, x^{u\prime}), x^{u\prime}))|| \\
&\geq ||(D_3 f_{ij}^u)^{-1}||^{-1}||x^u - x^{u\prime}|| - ||D_1 f_{ij}^u||||x^c - x^c|| \\
&\qquad - ||D_2 f_{ij}^u||||u_i(x^c, x^u) - u_i(x^c, x^{u\prime})|| + o(||x^{cu} - x^{cu\prime}||) \qquad (4.2.2)\\
&\geq 4||x^u - x^{u\prime}|| - 2Q\delta||x^{cu} - x^{cu\prime}|| \\
&= (4 - 2Q\delta)||x^u - x^{u\prime}|| \\
&> 3||x^u - x^{u\prime}||,
\end{aligned}$$

where by taking ϵ small enough, we have from $||x^u - x^{u\prime}|| < 2\epsilon$, that $o(||x^{cu} - x^{cu\prime}||) < Q\delta||x^{cu} - x^{cu\prime}||$.

For fixed $\delta, \epsilon > 0$, there exists $\rho(\epsilon) > 0$ such that for $0 < \rho < \rho(\epsilon)$, we have

$$||f_{ij}^u(\theta^{-T}\omega, x^c, u_i(\theta^{-T}\omega, x^c, 0), 0)|| < \frac{1}{3}\epsilon, \qquad (4.2.3)$$

which together with (4.2.2) yield that

$$||f_{ij}^u(\theta^{-T}\omega, x^c, u_i(\theta^{-T}\omega, x^c, x^u), x^u)|| > \epsilon$$

for $\frac{2}{3}\epsilon < ||x^u|| < \epsilon$. From this fact and that $f_{ij}^{cu}(\theta^{-T}\omega, \cdot, u_i(\theta^{-T}\omega, \cdot))$ is a homeomorphism, we conclude that $\cup_k \mathcal{D}_3^k \times B(0,\epsilon)$ is contained in the range of $f_{ij}^{cu}(\theta^{-T}\omega, \cdot, u_i(\theta^T\omega, \cdot))$ for i, j running through all possible choices. This shows G is well defined and completes the proof of the lemma. $\square$

Lemma 4.2.2. *For all* $0 \le k \le r$,

$$||B||\ ||C^{-1}||^k < \frac{1}{2}. \tag{4.2.4}$$

Proof. Using Lemma 4.1.1, we have

$$||B|| < \frac{1}{2}. \tag{4.2.5}$$

Writing

$$C = \begin{pmatrix} D_1 f^c_{ij}(\omega, x^c, x^s, x^u) & D_3 f^c_{ij}(\omega, x^c, x^s, x^u) \\ D_1 f^u_{ij}(\omega, x^c, x^s, x^u) & D_3 f^u_{ij}(\omega, x^c, x^s, x^u) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we have

$$C^{-1} = \begin{pmatrix} (a - bd^{-1}c)^{-1} & -(a - bd^{-1}c)^{-1}bd^{-1} \\ (ca^{-1}b - d)^{-1}ca^{-1} & -(ca^{-1}b - d)^{-1} \end{pmatrix}.$$

From Lemma 4.1.1, we know that

$$||a|| < Q,\ ||c|| < \eta,\ ||a^{-1}|| < Q,$$

$$||b|| < \eta,\ ||d|| < Q,\ ||d^{-1}|| < \frac{1}{4}.$$

Hence,

$$\begin{aligned}
||(C^{-1})_{11}|| &= ||(a - bd^{-1}c)^{-1}|| \\
&= ||(I - a^{-1}bd^{-1}c)^{-1}a^{-1}|| \\
&\leq ||a^{-1}||(1 + O(\eta^2)), \\
||(C^{-1})_{12}|| &= ||-(I - a^{-1}bd^{-1}c)^{-1}bd^{-1}a^{-1}|| \\
&\leq \frac{1}{4}Q\eta(1 + O(\eta^2)) = O(\eta), \\
||(C^{-1})_{21}|| &= ||(d^{-1}ca^{-1}b - I)^{-1}d^{-1}ca^{-1}|| \\
&\leq \frac{1}{4}Q\eta(1 + O(\eta^2)) = O(\eta), \\
||(C^{-1})_{22}|| &= ||-(d^{-1}ca^{-1}b - I)^{-1}d^{-1}|| \\
&\leq ||d^{-1}||(1 + O(\eta^2)) \leq \frac{1}{4}(1 + O(\eta^2))
\end{aligned}$$

which yield that for η sufficiently small,

$$||C^{-1}||^k < 2\max(||(D_1 f_{ij}^c)^{-1}||^k, \frac{1}{4}).$$

Therefore,

$$\begin{aligned}
||B||||C^{-1}||^k &< 2\max(||D_2 f_{ij}^s||||(D_1 f_{ij}^c)^{-1}||^k, \frac{1}{4}||D_2 f_{ij}^s||) \\
&< 2\max(\frac{1}{4}, \frac{1}{4} \times \frac{1}{2}) = \frac{1}{2}.
\end{aligned}$$

The proof of the lemma is complete. □

We note that

$$||A|| < \eta \tag{4.2.6}$$

and all A, B, C, E and their inverses are bounded by Q, perhaps choosing a larger value of Q.

Remark: A, B, C and E depend on $i, j, x^{cu}, x^s, \omega, \psi(T)$ and $\phi(T, \omega)$. But all the above estimates hold uniformly. Note that in (4.2.4) the arguments i, j, ω should be the same, while the arguments x^{cu}, x^s could vary as long as they vary only slightly.

Lemma 4.2.3. *For fixed δ small enough and any $0 < \epsilon < \epsilon_0$, there exists a $\rho(\epsilon) > 0$ such that for all $0 < \rho < \rho(\epsilon)$, G maps S_δ to S_δ.*

Proof. It is sufficient to show that if $u \in S_\delta$, then

$$||(Gu)_j(\omega, \xi^{cu}) - (Gu)_j(\omega, \xi^{cu\prime})|| \leq \delta||\xi^{cu} - \xi^{cu\prime}|| \tag{4.2.7}$$

for ξ^{cu} and $\xi^{cu\prime}$ in any sufficiently small neighborhood for all $\omega \in \Omega$ and j. This is because any two points $\xi^{cu}, \xi^{cu\prime} \in \mathcal{D}_3^j \times B(0, \epsilon)$ are connected by a line segment which is covered by finitely such neighborhoods and one can always choose a sequences of point $\xi_0 = \xi^{cu}, \xi_1, \cdots, \xi_m = \xi^{cu\prime}$ along the line segment such that for any $1 \leq k \leq m$, ξ_k and ξ_{k+1} are located in the same small neighborhood. Then

$$||(Gu)_j(\omega, \xi^{cu}) - (Gu)_j(\omega, \xi^{cu\prime})|| \leq \sum_k \delta||\xi_k - \xi_{k-1}|| = \delta||\xi_m - \xi_0|| = \delta||\xi^{cu} - \xi^{cu\prime}||.$$

Using Lemma 4.2.1, we have for $\xi^{cu\prime}$ near ξ^{cu}, there exists $x^{cu\prime} \in \mathcal{D}_3^i$ near x^{cu} such that

$$\xi^{cu} = f_{ij}^{cu}(\theta^{-T}\omega, x^{cu}, u_i(\theta^{-T}\omega, x^{cu}))$$

and

$$\xi^{cu\prime} = f_{ij}^{cu}(\theta^{-T}\omega, x^{cu\prime}, u_i(\theta^{-T}\omega, x^{cu\prime})).$$

Thus, we have

$$\begin{aligned}
&||(Gu)_j(\omega,\xi^{cu})-(Gu)_j(\omega,\xi^{cu\prime})|| \\
&=||f^s_{ij}(\theta^{-T}\omega,x^{cu},u_i(\theta^{-T}\omega,x^{cu}))-f^s_{ij}(\theta^{-T}\omega,x^{cu\prime},u_i(\theta^{-T}\omega,x^{cu\prime}))|| \\
&\le ||A||||x^{cu}-x^{cu\prime}||+||B||\delta||x^{cu}-x^{cu\prime}||+o(||x^{cu}-x^{cu\prime}||) \\
&\le (2\eta+\delta||B||)||x^{cu}-x^{cu\prime}||.
\end{aligned}$$

Here, we have taken $||x^{cu}-x^{cu\prime}||$ small enough so that $|o(||x^{cu}-x^{cu\prime}||)|<\eta||x^{cu}-x^{cu\prime}||$. Next we need to estimate $||\xi^{cu}-\xi^{cu\prime}||$ in terms of $||x^{cu}-x^{cu\prime}||$:

$$\begin{aligned}
||\xi^{cu}-\xi^{cu\prime}|| &= ||f^{cu}_{ij}(\theta^{-T}\omega,x^{cu},u_i(\theta^{-T}\omega,x^{cu}))-f^{cu}_{ij}(\theta^{-T}\omega,x^{cu\prime},u_i(\theta^{-T}\omega,x^{cu\prime}))|| \\
&\ge ||f^{cu}_{ij}(\theta^{-T}\omega,x^{cu},u_i(\theta^{-T}\omega,x^{cu}))-f^{cu}_{ij}(\theta^{-T}\omega,x^{cu\prime},u_i(\theta^{-T}\omega,x^{cu}))|| \\
&\quad -||f^{cu}_{ij}(\theta^{-T}\omega,x^{cu\prime},u_i(\theta^{-T}\omega,x^{cu}))-f^{cu}_{ij}(\theta^{-T}\omega,x^{cu\prime},u_i(\theta^{-T}\omega,x^{cu\prime}))|| \\
&\ge ||C^{-1}||^{-1}||x^{cu}-x^{cu\prime}||+o(||x^{cu}-x^{cu\prime}||)-||E||\delta||x^{cu}-x^{cu\prime}|| \\
&\ge ||C^{-1}||^{-1}(1-2\delta Q^2)||x^{cu}-x^{cu\prime}||.
\end{aligned}$$

Here again we have taken $||x^{cu}-x^{cu\prime}||$ small enough so that $|o(||x^{cu}-x^{cu\prime}||)|<\delta Q||x^{cu}-x^{cu\prime}||$. Then, we have

$$\begin{aligned}
||(Gu)_j(\omega,\xi^{cu})-(Gu)_j(\omega,\xi^{cu\prime})|| &\le (2\eta+\delta||B||)||x^{cu}-x^{cu\prime}|| \\
&\le (2\eta+\delta||B||)\tfrac{||C^{-1}||}{(1-2\delta Q^2)}||\xi^{cu}-\xi^{cu\prime}|| \\
&\le \tfrac{2\eta Q+\frac{1}{2}\delta}{1-2\delta Q^2}||\xi^{cu}-\xi^{cu\prime}|| \le \tfrac{3}{4}\delta||\xi^{cu}-\xi^{cu\prime}||,
\end{aligned}$$

provided $\delta<\frac{1}{12Q^2}$ and $\eta<\frac{\delta}{16Q}$. So G maps S_δ to S_δ. This completes the proof of the lemma. □

Lemma 4.2.4. *If $\delta,\eta>0$ are small enough, there exists $\epsilon(\eta)>0$ such that for any $0<\epsilon<\epsilon(\eta)$ there exists $\rho(\epsilon)>0$ such that for $0<\rho<\rho(\epsilon)$, G is a contraction on S_δ.*

Proof. Let $u, u' \in S_\delta$. Fix $\omega \in \Omega$ and let $\xi^{cu} \in (\sigma_j \times \tau_j^u)(\tilde{E}_\epsilon^u | U_3^j)$ be given. By Lemma 4.2.1, there exist i and $x^{cu}, x^{cu\prime} \in \mathcal{D}_3^i \times B(0, \epsilon)$ such that

$$\xi^{cu} = f_{ij}^c(\theta^{-T}\omega, x^{cu}, u_i(\theta^{-T}\omega, x^{cu})) = f_{ij}^c(\theta^{-T}\omega, x^{cu\prime}, u_i'(\theta^{-T}\omega, x^{cu\prime})).$$

Here, as long as ϵ is small, $x^{cu}, x^{cu\prime}$ are close enough to be in the same $\mathcal{D}_3^i \times B(0, \epsilon)$. Then if ϵ is small enough, we have $o(||x^{cu} - x^{cu\prime}||)$ and $o(||u - u'||)$ are so small that

$$\begin{aligned}
&||(Gu)_j(\omega, \xi^{cu}) - (Gu')_j(\omega, \xi^{cu})|| \\
&= ||f_{ij}^s(\theta^{-T}\omega, x^{cu}, u_i(\theta^{-T}\omega, x^{cu})) - f_{ij}^s(\theta^{-T}\omega, x^{cu\prime}, u_i'(\theta^{-T}\omega, x^{cu\prime}))|| \\
&\leq ||A|| ||x^{cu} - x^{cu\prime}|| + ||B|| ||u_i(\theta^{-T}\omega, x^{cu}) - u_i'(\theta^{-T}\omega, x^{cu})|| \\
&+ Q||u_i'(\theta^{-T}\omega, x^{cu}) - u_i'(\theta^{-T}\omega, x^{cu\prime})|| + o(||x^{cu} - x^{cu\prime}||) + o(||u - u'||) \\
&\leq (\eta + 2\delta Q)||x^{cu} - x^{cu\prime}|| + \tfrac{2}{3}||u_i(\theta^{-T}\omega) - u_i'(\theta^{-T}\omega)||.
\end{aligned}$$

Next, we need to estimate $||x^{cu} - x^{cu\prime}||$ in terms of $||u_i(\theta^{-T}\omega) - u_i'(\theta^{-T}\omega)||$. We first have

$$\begin{aligned}
&||f_{ij}^{cu}(\theta^{-T}\omega, x^{cu}, u_i(\theta^{-T}\omega, x^{cu})) - f_{ij}^{cu}(\theta^{-T}\omega, x^{cu\prime}, u_i(\theta^{-T}\omega, x^{cu}))|| \\
&\geq Q^{-1}||x^{cu} - x^{cu\prime}||
\end{aligned}$$

and

$$\begin{aligned}
&||f_{ij}^{cu}(\theta^{-T}\omega, x^{cu\prime}, u_i'(\theta^{-T}\omega, x^{cu\prime})) - f_{ij}^{cu}(\theta^{-T}\omega, x^{cu\prime}, u_i(\theta^{-T}\omega, x^{cu}))|| \\
&\leq Q(\delta||x^{cu} - x^{cu\prime}|| + ||u_i(\theta^{-T}\omega) - u_i'(\theta^{-T}\omega)||).
\end{aligned}$$

Noting that

$$\xi^{cu} = f_{ij}^{cu}(\theta^{-T}\omega, x^{cu}, u_i(\theta^{-T}\omega, x^{cu})) = f_{ij}^{cu}(\theta^{-T}\omega, x^{cu\prime}, u_i'(\theta^{-T}\omega, x^{cu\prime})),$$

we combine the above estimates to obtain

$$||x^{cu} - x^{cu\prime}|| \leq (2Q^2(\eta + 2\delta Q) + \frac{2}{3})||u_i(\theta^{-T}\omega) - u_i'(\theta^{-T}\omega)||.$$

Taking δ, η small enough, then we get

$$||(Gu)_j(\omega, \xi^{cu}) - (Gu')_j(\omega, \xi^{cu})|| \leq \frac{3}{4}||u_i(\theta^{-T}\omega) - u_i'(\theta^{-T}\omega)||.$$

$\square$

Proof of Proposition 4.2.1: Using Lemma 4.2.4, by the uniform contraction principle, G has a unique fixed point in S_δ, call it u. If $u' \in S$ is such that

$$\phi(T, \omega)(\operatorname{graph} u'(\omega)) \supset \operatorname{graph} u'(\theta^T\omega),$$

then the proof of Lemma 4.2.4 can be applied: Fix $\omega \in \Omega$ and let $\xi^{cu} \in (\sigma_j \times \tau_j^u)(\tilde{E}_\epsilon^u | U_3^j)$ be given. Since u' is invariant, there exist i and $x^{cu\prime} \in \mathcal{D}_3^i \times B(0, \epsilon)$ such that

$$\xi^{cu} = f_{ij}^{cu}(\theta^{-T}\omega, x^{cu}, u_i(\theta^{-T}\omega, x^{cu})) = f_{ij}^{cu}(\theta^{-T}\omega, x^{cu\prime}, u_i'(\theta^{-T}\omega, x^{cu\prime})).$$

Here, as long as ϵ is small, $x^{cu}, x^{cu\prime}$ are close enough to be in the same $\mathcal{D}_3^i \times B(0, \epsilon)$. Then

$$\begin{aligned} &||u_j(\omega, \xi^{cu}) - u_j'(\omega, \xi^{cu})|| \\ = &||f_{ij}^s(\theta^{-T}\omega, x^{cu}, u_i(\theta^{-T}\omega, x^{cu})) - f_{ij}^s(\theta^{-T}\omega, x^{cu\prime}, u_i'(\theta^{-T}\omega, x^{cu\prime}))|| \\ \leq &(\eta + 2\delta Q)||x^{cu} - x^{cu\prime}|| + \tfrac{2}{3}||u_i(\theta^{-T}\omega) - u_i'(\theta^{-T}\omega)||. \end{aligned}$$

Similarly, as in Lemma 4.2.4 we get

$$||x^{cu} - x^{cu\prime}|| \leq (2Q^2(\eta + 2\delta Q) + \frac{2}{3})||u_i(\theta^{-T}\omega) - u_i'(\theta^{-T}\omega)||.$$

So

$$||u_j(\omega, \xi^{cu}) - u_j'(\omega, \xi^{cu})|| \leq \frac{3}{4}||u_i(\theta^{-T}\omega) - u_i'(\theta^{-T}\omega)||.$$

This shows that $u = u'$, giving the uniqueness in S.

To show

$$\phi(t,\omega)(\operatorname{graph} u(\omega)) \supset \operatorname{graph} u(\theta^t\omega),$$

we use the uniqueness of u.

For any fixed $t > T$, we can define G' on S_δ based on t just as G is based on T. G' is well defined. Then we have $\phi(t,\omega)(\operatorname{graph} u(\omega)) \cap \tilde{E}_\epsilon$ is the graph of an element $\bar{u} \in S_\delta$ at the $\theta^t\omega$ fiber.

Since for each ω

$$\phi(T, \theta^{-T}\omega)(\operatorname{graph} u(\theta^{-T}\omega)) \supset \operatorname{graph} u(\omega) \tag{4.2.8}$$

we have

$$\begin{aligned} \phi(t,\omega)(\operatorname{graph} u(\omega)) &\subset \phi(t,\omega)\phi(T,\theta^{-T}\omega)(\operatorname{graph} u(\theta^{-T}\omega)) \\ &= \phi(T,\theta^{t-T}\omega)\phi(t,\theta^{-T}\omega)(\operatorname{graph} u(\theta^{-T}\omega)), \end{aligned}$$

which gives

$$\operatorname{graph} \bar{u}(\theta^t\omega) \subset \phi(T, \theta^{-T}(\theta^t\omega))(\operatorname{graph} \bar{u}(\theta^{-T}(\theta^t\omega))),$$

or equivalently:

$$\operatorname{graph} \bar{u}(\tilde{\omega}) \subset \phi(T, \theta^{-T}(\tilde{\omega}))(\operatorname{graph} \bar{u}(\theta^{-T}(\tilde{\omega}))). \tag{4.2.9}$$

Comparing (4.2.8) and (4.2.9) and using the uniqueness of the fixed point of G, we obtain

$$\bar{u} = u.$$

This completes the proof of the proposition.

$\square$

The graph of $u = \{u(\omega) : \omega \in \Omega\}$ gives us the random unstable manifold which is overflowing invariant under the random flow ϕ.

Chapter 5. Smoothness, Measurability and Normal Hyperbolic

In order to finish the proof of theorem 3.0.1, we need to prove the smoothness, and measurability of u.

5.1 Smoothness of the Random Unstable Manifolds

Let $u \in S_\delta$ denote the unique fixed point of G. So the graph of $u(\omega)$ is the unique random unstable manifold $\tilde{\mathcal{W}}^u(\omega)$. We prove u is C^r. We construct linear functions in the local coordinates as the candidates for $Du_i(\omega)$ and prove they are indeed the derivatives of $u_i(\omega)$.

In the local coordinates, for any fixed ω, $u(\omega)$ is represented by $u_i(\omega) : (\mathcal{D}_3^i \times B(0,\epsilon)) \to \mathbb{R}^k$, for $i = 1, \cdots, s$. If $u(\omega) \in C^1$, $Du_i(\omega)$ assigns to each point in $(\mathcal{D}_3^i \times B(0,\epsilon))$ a linear map from $\mathbb{R}^{n-k}$ to $\mathbb{R}^k$. Thus $Du(\omega)$ is represented by

$$v_i(\omega) \in C^0((\mathcal{D}_3^i \times B(0,\epsilon)), L(\mathbb{R}^{n-k}, \mathbb{R}^k)),$$

for $i = 1, \cdots, s$. The candidates for $Du(\omega)$ are of the form

$$v(\omega) = (v_1(\omega), \cdots, v_s(\omega)) \in \prod_{i=1}^{s} C^0((\mathcal{D}_3^i \times B(0,\epsilon)), L(\mathbb{R}^{n-k}, \mathbb{R}^k)),$$

and we denote the space of such mappings by $\bar{TS}$. If $v(\omega)$ is such an $s-$tuple, define

$$||v(\omega)|| = \max_{1 \le i \le s} \sup_{x^{cu} \in (\mathcal{D}_3^i \times B(0,\epsilon))} ||v_i(\omega, x^{cu})||, \tag{5.1.1}$$

if this exists, where $||\cdot||$ is the operator norm. For $v = \{v(\omega) \in \bar{TS} : \omega \in \Omega\}$, define

$$||v|| := \sup_{\omega} ||v(\omega)||. \tag{5.1.2}$$

Let TS be the space of all such random linear mappings v. To be more clear, any element v of TS can be first viewed as a function of $\omega \in \Omega$, and for any fixed ω, $v(\omega)$ is an s-tuple $(v_1(\omega), \cdots, v_s(\omega)) \in \prod_{i=1}^{s} C^0((\mathcal{D}_3^i \times B(0,\epsilon)), L(\mathbb{R}^{n-k}, \mathbb{R}^k))$. Define the norm $||\cdot||$ on TS by (5.1.1) and (5.1.2). Under this norm, TS is complete.

In terms of the local coordinates, $u(\omega)$ satisfies

$$u_j(\theta^T\omega, \xi^{cu}) = f_{ij}^{s}(\omega, x^{cu}, u_i(\omega, x^{cu}))$$

where $\xi^{cu} = f_{ij}^{cu}(\omega, x^{cu}, u_i(\omega, x^{cu}))$. Differentiating these formally, we have

$$v_j(\theta^T\omega, \xi^{cu})[C + E\, v_i(\omega, x^{cu})] = A + B\, v_i(\omega, x^{cu}),$$

where $A = D_{1,3} f_{ij}^{s}$, $B = D_2 f_{ij}^{s}$, $C = D_{1,3} f_{ij}^{cu}$ and $E = D_2 f_{ij}^{cu}$, the arguments of A, B, C, E are all $(\omega, x^{cu}, u_i(\omega, x^{cu}))$. So

$$v_j(\theta^T\omega, \xi^{cu}) = H_{ij}(\omega) v_i(\omega, x^{cu}),$$

where

$$H_{ij}(\omega) v_i(\omega, x^{cu}) = [A + B\, v_i(\omega, x^{cu})][C + E\, v_i(\omega, x^{cu})]^{-1}. \tag{5.1.3}$$

Thus, (5.1.3) induces an operator H on TS.

Remark In the definition of H_{ij}, the arguments of all A, B, C, E are all $(\omega, x^{cu}, u_i(\omega, x^{cu}))$, while later in the computation of contraction, the arguments of A, B, C, E may be different and in fact the matrices themselves may be different, being small perturbations of these Jacobians. However, this does not affect the estimates.

We want to prove that H has a fixed point $v(\omega)$. For technical reasons, we need a smooth partition of unity. Choose C^r functions: $h_i : \hat{U}_3^i \to [0,1]$ with support of $h_i \subset \hat{U}_2^i$ and $\sum_{i=1}^{s} h_i = 1$ on $\cup_i^s \hat{U}_1^i$. Here, $\hat{U}_k^i := (\sigma_i \times \tau_i^u)^{-1}(\mathcal{D}_k^i \times B(0,\epsilon))$. Define

$$v_i^0(\omega, x^{cu}) = 0, \ i = 1, 2, \cdots, s$$

and

$$v_j^{n+1}(\theta^T \omega, \xi^{cu}) = \sum_{i=1}^{s} h_i(m_-) H_{ij}(\omega) v_i^n(\omega, x^{cu}),$$

where

$$\phi(T, \omega, u(\omega, m_-)) = u(\theta^T \omega, \sigma_j^{-1}(\xi^{cu}))$$

Then $m_- \in \cup_i^s \hat{U}_1^i$ and $\sum_{i=1}^{s} h_i(m_-) = 1$. We will show $\{v^n\}$ converges to a fixed point of (5.1.3).

Proposition 5.1.1. *The sequence $\{v^n(\omega)\}$ converges to a solution of the equations*

$$v_j(\theta^T \omega) = \sum_{i=1}^{s} h_i \cdot H_{ij}(\omega) v_i(\omega). \tag{5.1.4}$$

First, we claim:

Lemma 5.1.1. $||v_i^n(\omega, x^{cu})|| < \delta$ *for all n and ω.*

Proof. It is sufficient to show

$$||H_{ij}(\omega) v_i^n(\omega)|| < \delta.$$

Letting $\delta > 0$ be such that $\delta Q^2 \ll 1$, for $||v_i^n(\omega)|| < \delta$ we have

$$\begin{aligned} ||[C + E\, v_i^n(\omega)]^{-1}|| &= ||[C\,(1 + C^{-1} E\, v_i^n)]^{-1}|| \\ &= ||[1 - (-C^{-1} E\, v_i^n)]^{-1} C^{-1}|| = ||\sum_{k=0}^{\infty} (-C^{-1} E\, v_i^n)^k C^{-1}|| \\ &\leq \sum_{k=0}^{\infty} ||C^{-1} E\, v_i^n||^k ||C^{-1}|| \leq \sum_{k=0}^{\infty} ||\delta Q^2||^k ||C^{-1}|| = \frac{||C^{-1}||}{1 - \delta Q^2}. \end{aligned}$$

Thus, using (5.1.3), by induction, we have

$$||H_{ij}(\omega)v_i^n(\omega)|| \leq (\eta + ||B||\delta)\frac{||C^{-1}||}{1-\delta Q^2} \leq \frac{Q\eta + \frac{1}{2}\delta}{1-\delta Q^2} \leq \delta$$

provided η is small enough. This completes the proof of the lemma. $\square$

Lemma 5.1.2. $||v^{n+1}(\theta^T\omega) - v^n(\theta^T\omega)|| \leq \lambda||v^n(\omega) - v^{n-1}(\omega)||$ *for some* $0 < \lambda < 1$.

Proof. It is sufficient to show

$$||H_{ij}(\omega)v_i^n(\omega) - H_{ij}(\omega)v_i^{n-1}(\omega)|| \leq \lambda||v^n(\omega) - v^{n-1}(\omega)|| \text{ for all } i, j.$$

First, we note that

$$\begin{aligned}
& H_{ij}(\omega)v_i^n(\omega) - H_{ij}(\omega)v_i^{n-1}(\omega) \\
= \; & [A + B\,v_i^n(\omega)][C + E\,v_i^n(\omega)]^{-1} \\
& -[A + B\,v_i^{n-1}(\omega)][C + E\,v_i^{n-1}(\omega)]^{-1} \\
= \; & (A + B\,v_i^n)(C + E\,v_i^n)^{-1}[(C + E\,v_i^{n-1}) - (C + E\,v_i^n)](C + E\,v_i^{n-1})^{-1} \\
& +[(A + B\,v_i^n) - (A + B\,v_i^{n-1})](C + E\,v_i^{n-1})^{-1}.
\end{aligned}$$

Estimating the above and using Lemma 4.2.2, we have

$$\begin{aligned}
& ||H_{ij}(\omega)v_i^n(\omega) - H_{ij}(\omega)v_i^{n-1}(\omega)|| \\
\leq \; & (\eta + ||B||\delta)\frac{||C^{-1}||}{1-\delta Q^2}Q||v_i^n - v_i^{n-1}||\frac{||C^{-1}||}{1-\delta Q^2} \\
& +||B||||v_i^n - v_i^{n-1}||\frac{||C^{-1}||}{1-\delta Q^2} \\
= \; & \Big((\eta + ||B||\delta)\frac{||C^{-1}||^2}{(1-\delta Q^2)^2}Q + \frac{\frac{1}{2}}{1-\delta Q^2}\Big)||v^n(\omega) - v^{n-1}(\omega)|| \\
= \; & \lambda||v^n(\omega) - v^{n-1}(\omega)||,
\end{aligned}$$

for all i, j. By choosing η and δ small enough, we have $\lambda < 1$. The proof is complete. $\square$

By the contraction principle, $\{v^n\}$ converges to v which satisfies (5.1.4).

We are ready to prove that v is the derivative of u.

Proposition 5.1.2. *For each $\xi^{cu} \in (\mathcal{D}_3^j \times B(0,\epsilon))$, $Du_j(\theta^T\omega, \xi^{cu})$ exists and equals $v_j(\theta^T\omega, \xi^{cu})$, hence, $u \in C^1$ and $v_j(\theta^T\omega, \xi^{cu}) = H_{ij}(\omega)v_i(\omega, x^{cu})$.*

Proof. For a fixed $\omega \in \Omega$, we define an increasing function $\gamma_\omega : (0,1) \to \mathbb{R}$,

$$\gamma_{\theta^T\omega}(a)$$
$$= \max_i \sup_{\xi^{cu},\xi^{cu\prime} \in (\mathcal{D}_3^i \times B(0,\epsilon)), 0<||\xi^{cu}-\xi^{cu\prime}||<a} \frac{||u_i(\theta^T\omega, \xi^{cu\prime}) - u_i(\theta^T\omega, \xi^{cu}) - v_i(\theta^T\omega, \xi^{cu})(\xi^{cu\prime} - \xi^{cu})||}{||\xi^{cu\prime} - \xi^{cu}||}.$$

Note that $\gamma_{\theta^T\omega}$ is bounded by 2δ.

We want to show $\gamma_\omega(a) \to 0$ as $a \to 0$. To prove this, we claim

Claim: *$\gamma_\omega(a)$ satisfies*

$$\gamma_{\theta^T\omega}(a) \le s\gamma_\omega(za) + r(\omega, a)$$

for small a, for some $0 \le s < 1$, $z > 1$, where $r(\omega, a)$ is a decreasing function which approaches zero as $a \to 0$ uniformly in $\omega \in \Omega$.

Proof of the claim: Let $\xi^{cu} \in (\mathcal{D}_3^j \times B(0,\epsilon))$. By Lemma 4.2.1 , we have

$$\xi^{cu} = f_{ij}^{cu}(\omega, x^{cu}, u_i(\omega, x^{cu}))$$

for some $x^{cu} \in (\mathcal{D}_2^i \times B(0,\epsilon))$. We choose $d \in (0,a)$ so small that if $\xi^{cu\prime} \in (\mathcal{D}_3^j \times B(0,\epsilon))$ with $||\xi^{cu} - \xi^{cu\prime}|| < d$, then there exist $x^{cu\prime} \in (\mathcal{D}_3^i \times B(0,\epsilon))$ such that $\xi^{cu\prime} = f_{ij}^c(\omega, x^{cu\prime}, u_i(\omega, x^{cu\prime}))$.

To show

$$\gamma_{\theta^T\omega}(a) \le s\gamma_\omega(za) + r(\omega, a),$$

it is sufficient to show that

$$||u_j(\theta^T\omega, \xi^{cu\prime}) - u_j(\theta^T\omega, \xi^{cu}) - H_{ij}(\omega)v_i(\theta^T\omega, \xi^{cu}) \cdot (\xi^{cu\prime} - \xi^{cu})|| \le [s\gamma_\omega(za) + r(a)]||\xi^{cu\prime} - \xi^{cu}||$$

for all $\xi^{cu}, \xi^{cu\prime}, i, j$ as above and $||\xi^{cu\prime} - \xi^{cu}|| \leq a < d$.

First, we have

$$\begin{aligned}
\xi^{cu\prime} - \xi^{cu} &= f_{ij}^{cu}(\omega, x^{cu\prime}, u_i(\omega, x^{cu\prime})) - f_{ij}^{cu}(\omega, x^{cu}, u_i(\omega, x^{cu})) \\
&= C(x^{cu\prime} - x^{cu}) + E(u_i(\omega, x^{cu\prime}) - u_i(\omega, x^{cu})) + o(||x^{cu\prime} - x^{cu}||),
\end{aligned}$$

$$\begin{aligned}
u_j(\theta^T\omega, \xi^{cu\prime}) - u_j(\theta^T\omega, \xi^{cu}) &= f_{ij}^{s}(\omega, x^{cu\prime}, u_i(\omega, x^{cu\prime})) - f_{ij}^{s}(\omega, x^{cu}, u_i(\omega, x^{cu})) \\
&= A(x^{cu\prime} - x^{cu}) + B(u_i(\omega, x^{cu\prime}) - u_i(\omega, x^{cu})) + o(||x^{cu\prime} - x^{cu}||).
\end{aligned}$$

Note that the quantity $o(||x^{cu} - x^{cu\prime}||)$ is uniformly in $\omega \in \Omega$. From the first equation it follows that

$$||x^{cu\prime} - x^{cu}|| \leq ||C^{-1}|| \frac{||\xi^{cu\prime} - \xi^{cu}||}{1 - 2\delta Q^2}$$

and that

$$\begin{aligned}
\xi^{cu\prime} - \xi^{cu} &= [C + E\, v_i(\omega, x^{cu})](x^{cu\prime} - x^{cu}) \\
&\quad + E[u_i(\omega, x^{cu\prime}) - u_i(\omega, x^{cu}) - v_i(\omega, x^{cu})(x^{cu\prime} - x^{cu})] + o(||x^{cu\prime} - x^{cu}||).
\end{aligned}$$

Hence,

$$
\begin{aligned}
&||u_j(\theta^T\omega, \xi^{cu\prime}) - u_j(\theta^T\omega, \xi^{cu}) - H_{ij}(\omega)v_i(\theta^T\omega, \xi^{cu})(\xi^{cu\prime} - \xi^{cu})|| \\
&= ||A(x^{cu\prime} - x^{cu}) + B(u_i(\omega, x^{cu\prime}) - u_i(\omega, x^{cu})) + o(||x^{cu\prime} - x^{cu}||) \\
&\quad - [A + B\, v_i(\omega, x^{cu})][C + E\, v_i(\omega, x^{cu})]^{-1} \cdot (\xi^{cu\prime} - \xi^{cu})|| \\
&= ||A(x^{cu\prime} - x^{cu}) + B[u_i(\omega, x^{cu\prime}) - u_i(\omega, x^{cu})] - [A + B\, v_i(\omega, x^{cu})](x^{cu\prime} - x^{cu}) \\
&\quad - [A + B\, v_i(\omega, x^{cu})][C + E\, v_i(\omega, x^{cu})]^{-1} E[u_i(\omega, x^{cu\prime}) - u_i(\omega, x^{cu}) \\
&\quad - v_i(\omega, x^{cu})(x^{cu\prime} - x^{cu})] + o(||x^{cu\prime} - x^{cu}||)|| \\
&= ||\{B - [A + B\, v_i(\omega, x^{cu})][C + E\, v_i(\omega, x^{cu})]^{-1} \cdot E\}[u_i(\omega, x^{cu\prime}) - u_i(\omega, x^{cu}) \\
&\quad - v_i(\omega, x^{cu})(x^{cu\prime} - x^{cu})] + o(||x^{cu\prime} - x^{cu}||)|| \\
&\le [||B|| + O(\eta + \delta)]\gamma_\omega(||x^{cu\prime} - x^{cu}||)||x^{cu\prime} - x^{cu}|| + ||o(||x^{cu\prime} - x^{cu}||)|| \\
&\le \frac{[||B|| + O(\eta + \delta)]||C^{-1}||}{1 - 2\delta Q^2}\gamma_\omega(||x^{cu\prime} - x^{cu}||)||\xi^{cu\prime} - \xi^{cu}|| + ||o(||x^{cu\prime} - x^{cu}||)||.
\end{aligned}
$$

Taking $s = \frac{||B||+O(\eta+\delta)||C^{-1}||}{1-2\delta Q^2}$, if η and δ are small enough, then $s < 1$. Let $z = \frac{Q}{1-2\delta Q^2}$. Then

$$||x^{cu\prime} - x^{cu}|| \le z||\xi^{cu\prime} - \xi^{cu}|| < zd < za.$$

Then

$$\gamma_\omega(||x^{cu\prime} - x^{cu}||) \le \gamma_\omega(za).$$

Since

$$||\xi^{cu\prime} - \xi^{cu}|| \ge z^{-1}||x^{cu\prime} - x^{cu}||,$$

$o(||x^{cu\prime} - x^{cu}||)$ can be bounded by $r(\omega, a)||\xi^{cu\prime} - \xi^{cu}||$ where $r(\omega, a) \to 0$ uniformly in $\omega \in \Omega$ as $a \to 0$. This shows

$$\gamma_{\theta^T\omega}(a) \le s\gamma_\omega(za) + r(\omega, a). \tag{5.1.5}$$

Note that z can be taken to be bigger than 1 because $\gamma_\omega(a)$ is increasing in a. This completes the proof of the claim.

We are ready to show $\gamma_\omega(a) \to 0$ as $a \to 0$.

Replace a successively by $az^{-1}, az^{-2}, \cdots, az^{-n}$, repalce ω successively by $\omega, \theta^T\omega, \cdots,$ $\theta^{(n-1)T}\omega$, weight the terms with $s^{n-1}, s^{n-2}, \cdots, 1$ and add them together to get:

$$\begin{aligned}\gamma_{\theta^{nT}\omega}(az^{-n}) &\leq s^n\gamma_\omega(a) + r(\theta^{(n-1)T\omega}, az^{-n}) + sr(\theta^{(n-2)T\omega}, az^{-n+1}) + \cdots + s^{n-1}r(\omega, az^{-1}) \\ &\leq s^n \cdot 2\delta + \frac{1}{1-s}\sup_\omega \max_{0\leq t\leq a} r(\omega, t).\end{aligned}$$

Since here ω is arbitrary, we get

$$\gamma_\omega(sz^{-n}) \leq 2\delta s^n + \frac{1}{1-s}\sup_\omega \max_{0\leq t\leq a} r(t).$$

It follows that $\gamma_\omega(a) \to 0$ as $a \to 0$.

So u is differentiable. Then from the definition of v^n we have $v^n \in C^0$. Since $v^n \to v$ uniformly, $v \in C^0$. Since $Du = v, u \in C^1$. $\square$

Next, we show $u \in C^r$. Knowing $u \in C^1$, by the construction of v^n, using the partition of unity $\{h_i\}$, we have $v^n \in C^1$ and

$$v_j^{n+1}(\theta^T\omega, \xi^{cu}) = \sum h_i[A + Bv_i^n(\omega, x^{cu})][C + Ev_i^n(\omega, x^{cu})]^{-1}.$$

Hence,

$$\begin{aligned}&Dv_j^{n+1}(\theta^T\omega, \xi^{cu})[C + E\, v_i(\omega, x^{cu})] \\ &= \sum h_i[B\, D\, v_i^n(\omega, x^{cu})(C + E\, v_i^n(\omega, x^{cu}))^{-1} \\ &\quad - (A + B\, v_i^n(\omega, x^{cu}))(C + E\, v_i^n(\omega, x^{cu}))^{-1} E\, D\, v_i^n(\omega, x^{cu})(C + E\, v_i^n(\omega, x^{cu}))^{-1}] \\ &\quad + (\text{terms not involving derivatives of } v^k \text{ or } v)\end{aligned}$$

which yields

$$
\begin{aligned}
&Dv_j^{n+1}(\theta^T\omega, \xi^{cu})\\
&= \sum h_i \Big(B\, D\, v_i^n(\omega, x^{cu})(C + E\, v_i^n(\omega, x^{cu}))^{-1}\\
&\quad - (A + B\, v_i^n(\omega, x^{cu}))(C + E\, v_i^n(\omega, x^{cu}))^{-1} E\, D\, v_i^n(\omega, x^{cu})(C + E\, v_i^n(\omega, x^{cu}))^{-1} \Big) \Big(C + E\, v_i(\omega, x^{cu}) \Big)^{-1}\\
&\quad + (\text{terms not involving derivatives of } v^k \text{ or } v).
\end{aligned}
$$

Then, using Lemma 4.2.2 and similar arguments to those in Lemma 5.1.1 and 5.1.2, one can show Dv^n is a Cauchy sequence. By the definition of v^n, $Dv^n \in C^0$. Hence Dv exists and equals the limit of Dv^n. So $v \in C^1$ and $u \in C^2$.

By using induction and Lemma 4.2.2, one can prove that $D^{k-1}v^n$ is a Cauchy sequence in C^0, and converges to $D^{k-1}v$ uniformly. So $v \in C^{k-1}$, i.e., $u \in C^k$. Thus, we have

Proposition 5.1.3. $u \in C^r$.

This completes the proof of the smoothness of the random unstable manifold $\tilde{\mathcal{W}}^u$. From the proof, we also obtain the following property, which will be used in the next section:

Proposition 5.1.4. *Each v_i is measurable in (ω, x^{cu}) jointly.*

Proof. From the definition of the sequence v_n, each v^{n+1} is defined by $H(\omega, v^n)$ in a local chart. $H(\omega, v)$ has the following *Caratheodory property*: $H(\cdot, v)$ is measurable for any fixed v, and $H(\omega, \cdot)$ is C^{r-1} for almost every $\omega \in \Omega$. So as long as v^0 is measurable in ω and C^{r-1} in x^{cu}, each v^n is measurable in ω and C^{r-1} in x^{cu}. Since v^n is measurable in ω and C^{r-1} in x^{cu}, it is measurable in (ω, x^{cu}) jointly (see [CV]). Since v^0 is chosen to be 0, all the above hold. Then the limit v of v^n is jointly measurable in (ω, x^{cu}). □

5.2 Measurability of the Unstable Manifold and its Tangent Space

In this section, we prove that the unique random unstable manifold is a random set and the tangent space of the random unstable manifold is measurable.

Let $u = \{u(\omega) : \omega \in \Omega\}$ be the unique fixed point of G in S_δ. For simplicity, we use $u(\omega)$ to denote both the section and the graph of the section.

Proposition 5.2.1. *$u(\omega)$ is a random set.*

Proof. Recall

$$\hat{U}_k^i \equiv (\sigma_i \times \tau_i^u)^{-1}(\mathcal{D}_k^i \times B(0, \epsilon)$$

and let

$$u^0(\omega) = \cup_{i=1}^s \hat{U}_3^i,$$

$$u^1(\omega) = \phi(T, \theta^{-T}\omega, u^0(\theta^{-T}\omega)) \cap V,$$

$$u^{n+1}(\omega) = \phi(T, \theta^{-T}\omega, u^n(\theta^{-T}\omega)) \cap V,$$

$$\cdots$$

$$u^n = \{u^n(\omega) : \omega \in \Omega\}.$$

Recall that V is a tubular neighborhood of M in $\mathbb{R}^n$ which is C^r diffeomorphic to $\tilde{E}_\epsilon$. Since we identify V with $\tilde{E}(\epsilon)$, $u^0(\omega)$ can be viewed as a set or the zero section. Here, we view $u^0(\omega)$ as a set. We have $V \in \mathcal{B}(\mathbb{R}^n)$. Since G is a uniform contraction on S_δ, we have

$$u^n \to u,$$

i.e.,

$$u^n(\omega) \to u(\omega)$$

uniformly in $\omega \in \Omega$.

We will show all $u^n(\omega)$ are random sets. Let $\{m_k\}$ be dense in $\cup_{i=1}^s \hat{U}_3^i$. Define

$$y_k^0(\omega) = m_k.$$

Let

$$y_k^1(\omega) = \phi(T, \theta^{-T}\omega, y_k^0(\theta^{-T}\omega)), \quad \cdots,$$

$$y_k^{n+1}(\omega) = \phi(T, \theta^{-T}\omega, y_k^n(\theta^{-T}\omega)).$$

Since $\phi(t, \omega, m)$ is $\mathcal{B}(\mathbb{R})\otimes\mathcal{F}\otimes\mathcal{B}(\mathbb{R}^n)-$measurable, we know that each $y_k^n(\omega)$ is $\mathcal{B}(\mathbb{R}^n)-$measurable. We also know that $\{y_k^n(\omega)\}_k \cap u^n(\omega)$ is dense in $u^n(\omega)$, since $\phi(-T, \theta^T\omega, \cdot)$ is continuous.

For any $x = x^{cu} \in \mathbb{R}^n$, define

$$r_x^n(\omega) = \inf_{y \in u^n(\omega)} |x - y|,$$

then r_x^n is a random variable. This is because for each fixed $a \in \mathbb{R}$,

$$\begin{aligned}
&\{\omega : r_x^n(\omega) \geq a\} \\
&= \{\omega : \inf_{y \in u^n(\omega)} |x - y| \geq a\} \\
&= \cap_k(\{\omega \,|\, y_k^n(\omega) \in V,\ |x - y_k^n(\omega)| \geq a\} \cup \{\omega \,|\, y_k^n(\omega) \in V^c\})
\end{aligned}$$

which is $\mathcal{F}-$measurable because of the measurability of $y_k^n(\omega)$ and V. So we have proved $r_x^n(\omega)$ is a random variable. Since n, x are arbitrary, $u^n(\omega)$ are random sets.

To show that $u(\omega)$ is a random set, let $x \in \mathbb{R}^n$ and define

$$r_x(\omega) = \inf_{y \in u(\omega)} |x - y|.$$

We want to show $r_x(\omega)$ is a random variable. Since

$$u^n(\omega) \to u(\omega)$$

uniformly for $\omega \in \Omega$ in C^0 norm, we have

$$r_x^n(\omega) \to r_x(\omega)$$

pointwise, which implies that $r_x(\omega)$ is a random variable, hence, u_ω is a random set. This completes the proof of the proposition. $\square$

To show the measurability of the tangent space $T\tilde{\mathcal{W}}^u$, we need to show

$$T\tilde{\mathcal{W}}^u : (\omega, m) \in \tilde{\mathcal{W}}^u \to T_{(\omega,m)}\tilde{\mathcal{W}}^u$$

is measurable. Since each $T_{(\omega,m)}\tilde{\mathcal{W}}^u$ is an $(n-k)$-dimensional subspace of $\mathbb{R}^n$, the map from $\tilde{\mathcal{W}}^u$ to the tangent space can be represented by

$$T\tilde{\mathcal{W}}^u : \tilde{\mathcal{W}}^u = \{(\omega, m) : \omega \in \Omega, m \in \tilde{\mathcal{W}}^u(\omega)\} \to K_{n-k},$$

$$(\omega, m) \mapsto T_{(\omega,m)}\tilde{\mathcal{W}}^u \in K_{n-k},$$

where K_{n-k} is the set of all $(n-k)$-dimensional subspaces of $\mathbb{R}^n$.

Consider the metric introduced by [K]. Let N_1, N_2 be linear subspace of $\mathbb{R}^n$ and let S_{N_1} be the unit sphere in N_1. Define, for $N_1 \neq \{0\} \neq N_2$

$$d_1(N_1, N_2) = \sup_{x \in S_{N_1}} dist(x, S_{N_2}),$$

$$d_2(N_1, N_2) = \max(d_1(N_1, N_2), d_2(N_1, N_2)).$$

Also define

$$d_2(0, N) = d_2(N, 0) = 1.$$

Then the set of all closed linear subspaces of $\mathbb{R}^n$ is a complete metric space with metric d_2 and K_{n-k} is also a complete metric space with this metric. See also [LL].

Corresponding to the local chart $\mathcal{D}_3^i \times B(0, \epsilon) \times \mathbb{R}^k$, for each $\omega \in \Omega$, there is an open set $\hat{U}_i(\omega)$ on $\tilde{\mathcal{W}}^u(\omega)$. We denote

$$\hat{U}_i := \{(\omega, m) : \omega \in \Omega, m \in \hat{U}_i(\omega)\}.$$

Then

$$\tilde{\mathcal{W}}^u = \hat{U}_1 \cup \hat{U}_2 \cup \cdots \cup \hat{U}_s.$$

Each $\hat{U}_i$ is a measurable subset of $\tilde{\mathcal{W}}^u$.

In the local chart $\mathcal{D}_3^i \times B(0, \epsilon) \times \mathbb{R}^k$, $\hat{U}_i$ is constructed as a collection of Lipschitz functions $u_i(\omega, x^{cu})$ over $\mathcal{D}_3^i \times B(0, \epsilon)$. $v_i(\omega, x^{cu})$ is the derivative of $u_i(\omega, x)$ which has values in $L(\mathbb{R}^{n-k}, \mathbb{R}^k)$. We use $\hat{v}_i(\omega, x^{cu})$ to denote the tangent space in the local chart. So $\hat{v}_i(\omega, x^{cu})$ has values in K_{n-k}. By an elementary computation, for any $\omega \in \Omega$, $x^{cu} \in \mathcal{D}_3^i \times B(0, \epsilon)$, the d_2 distance between $\hat{v}_i(\omega, x^{cu})$ and $\mathbb{R}^{n-k} \times \{0\}$ is at most $\sqrt{2 - \frac{2}{\sqrt{1+\delta^2}}}$, which, for convenience, will be denoted by δ^*. Then the range of $\hat{v}_i$ can be denoted by

$$\overline{B_{d_2}(\mathbb{R}^{n-k} \times \{0\}, \delta^*)},$$

the closed set containing all $n-k$ dimensional subspace of $\mathbb{R}^n$, whose d_2 distance to $\mathbb{R}^{n-k} \times \{0\}$ are no more than δ^*. Remember this set is in the chart $\mathcal{D}_3^i \times B(0, \epsilon) \times \mathbb{R}^k$.

In order for $T\tilde{\mathcal{W}}^u$ to be a well defined map, the range of all $T\tilde{\mathcal{W}}^u|\hat{U}_i$ should be in the same metric space. Or in other words, all values of $\hat{v}_i$ should be measured in a single chart.

Suppose the coordinate change from $\mathcal{D}_3^i \times B(0, \epsilon) \times \mathbb{R}^k$ to $\mathcal{D}_3^1 \times B(0, \epsilon) \times \mathbb{R}^k$ is represented by the invertible matrix A_{i1}. Then in the chart $\mathcal{D}_3^1 \times B(0, \epsilon) \times \mathbb{R}^k$, $\hat{v}$ is represented by

$(A_{11}\hat{v}_1, \cdots, A_{s1}\hat{v}_s)$. It has range

$$\cup_{i=1}^{s} A_{i1}\overline{B_{d_2}(\mathbb{R}^{n-k} \times \{0\}, \delta^*)}.$$

To prove the measurability of $\hat{v} = T\tilde{\mathcal{W}}^u$, it is enough to prove the measurability of each $T\tilde{\mathcal{W}}^u|\hat{U}_i$, or equivalently the measurability of each $A_{i1}\hat{v}_i$ as a function defined in the chart $\mathcal{D}_3^i \times B(0,\epsilon) \times \mathbb{R}^k$ taking value in K_{n-k} measured in the chart $\mathcal{D}_3^1 \times B(0,\epsilon) \times \mathbb{R}^k$:

$$\Omega \times \mathcal{D}_3^i \times B(0,\epsilon) \to A_{i1}\overline{B_{d_2}(\mathbb{R}^{n-k} \times \{0\}, \delta^*)}.$$

Since $A_{i,1}$ is an invertible matrix, it is a diffeomorphism from

$$\overline{B_{d_2}(\mathbb{R}^{n-k} \times \{0\}, \delta^*)}$$

to

$$A_{i1}\overline{B_{d_2}(\mathbb{R}^{n-k} \times \{0\}, \delta^*)}.$$

So it suffices to prove the measurability of $\hat{v}_i$ as a function defined in the chart $\mathcal{D}_3^i \times B(0,\epsilon) \times \mathbb{R}^k$ and taking values in K_{n-k} measured in the chart $\mathcal{D}_3^i \times B(0,\epsilon) \times \mathbb{R}^k$.

Proposition 5.2.2. *$\hat{v}_i$ is measurable.*

Proof. From section 5 we know v_i is measurable. We use the measurability of v_i to prove the measurability of $\hat{v}_i$.

The operator norm of each $v_i(\omega, x)$ is at most δ. Let $\overline{B(0,\delta)}$ be the closed ball centered at 0 with radius δ in the space $L(\mathbb{R}^{n-k}, \mathbb{R}^k)$. It induces a subset K_δ of K_{n-k}:

$$K_\delta := \{(a, \tilde{L}a) : a \in R^{n-k},\ \tilde{L} \in \overline{B(0,\delta)}\}.$$

We also have the representation

$$K_\delta = \overline{B_{d_2}(\mathbb{R}^{n-k} \times \{0\}, \delta^*)}.$$

This is because for any $\tilde{L} \in \overline{B(0,\delta)}$,

$$d_2(\mathbb{R}^{n-k} \times \{0\}, (I \times \tilde{L})(\mathbb{R}^{n-k})) = \sqrt{2 - \frac{2}{\sqrt{1 + ||\tilde{L}||^2}}},$$

which is no more than $\delta^* = \sqrt{2 - \frac{2}{\sqrt{1+\delta^2}}}$. On the other hand, for any $0 < \delta_1 \leq \delta$, and J an $n-k$ dimensional subspace of $\mathbb{R}^n$ whose d_2 distance to $\mathbb{R}^{n-k} \times \{0\}$ is δ_1, there exists an $\tilde{L}$ such that $||\tilde{L}|| = \delta_1$ and $\mathbb{R}^{n-k} \times \tilde{L}(\mathbb{R}^{n-k}) = J$. So we have

$$K_\delta = \overline{B_{d_2}(\mathbb{R}^{n-k} \times \{0\}, \delta^*)}.$$

Define a metric d_3 on K_δ by

$$d_3(\mathbb{R}^{n-k} \times L_1(\mathbb{R}^{n-k}), \mathbb{R}^{n-k} \times L_2(\mathbb{R}^{n-k})) := ||L_1 - L_2||_{L(\mathbb{R}^{n-k}, \mathbb{R}^k)}.$$

This makes K_δ into a metric space, and obviously we have

$$K_\delta = \overline{B_{d_3}(\mathbb{R}^{n-k} \times \{0\}, \delta)}.$$

Lemma 5.2.1. *d_2 and d_3 generate the same topology on K_δ.*

Proof. Let $D_n \subset K_\delta$. It is obvious that $d_2(D_n, D_0) \to 0$ if and only if $d_3(D_n, D_0) \to 0$. $\square$

We use the metric d_3 on K_δ. This makes K_δ isomorphic to $\overline{B(0,\delta)}$. So the measurability of $\hat{v}_i$ as a function to K_δ is equivalent to the measurability of v_i as a function to $\overline{B(0,\delta)}$. By Proposition 5.1.4, v_i is measurable. So $\hat{v}_i$ is measurable. $\square$

From the discussion before proposition 5.2.2, we have:

Proposition 5.2.3. *The tangent space of the random unstable manifold $\tilde{\mathcal{W}}^u$ is measurable.*

5.3 Stable Manifold and Random Invariant Manifold

Since the random flow $\phi(t, \omega)$ is invertible, by reversing the time and applying the results on the unstable manifold in previous sections, we have

Proposition 5.3.1. *There exists a unique C^r random stable manifold $\tilde{\mathcal{W}}^s(\omega)$ in the neighborhood V of $\mathcal{M}$. Moreover, it is a random set and its tangent space is measurable.*

Taking the intersection of $\tilde{\mathcal{W}}^u(\omega)$ and $\tilde{\mathcal{W}}^s(\omega)$, we get an invariant manifold $\tilde{\mathcal{M}}(\omega)$ of the random flow $\phi(t, \omega)$. From the proof of the existence of $\tilde{\mathcal{W}}^s(\omega)$ and $\tilde{\mathcal{W}}^u(\omega)$ we can find this intersection by taking a measurable m-dimensional random manifold $\mathcal{M}^0(\omega)$ on the random unstable manifold, as a graph over $\mathcal{M}$ and mapping it under the inverse flow. For example, we can take the random set which corresponds to $u_i(\omega, x^{cu}) = u_i(\omega, x^c, 0)$ in a local chart. Iterates of this random set approaches the random stable manifold uniformly under the inverse random flow. On the other hand, the random set stays in the random unstable manifold. So it converges to the intersection of the random stable and unstable manifold. In other words, the invariant manifold of intersection is the limit of $\mathcal{M}^0(\omega)$ under the inverse random flow. Since we have the measurability and smoothness of $u_i(\omega, x^c, 0)$ in local charts we know $\mathcal{M}^0(\omega)$ is measurable and smooth and since also the derivatives (tangent spaces) converge uniformly, then the limit $\tilde{\mathcal{M}}(\omega)$ is also measurable and smooth.

$\tilde{\mathcal{M}}(\omega)$ is obviously C^r diffeomorphic to $\mathcal{M}$ for each $\omega \in \Omega$ because it is the graph of a C^r section over the tangent bundle of $\mathcal{M}$.

The measurability of the tangent space of $\tilde{\mathcal{M}}(\omega)$ is also proved from the measurability and smoothness of the tangent space of $\mathcal{M}^0(\omega)$. From the measurability and smoothness of the tangent space of $\mathcal{M}^0(\omega)$, we get the measurability of the derivative map of $\mathcal{M}^0(\omega, m)$ in local charts using the *Caratheodory property* as we did in proposition 5.1.4. Then from uniform convergence of the tangent spaces, we get the measurability of the derivative map of $\mathcal{M}(\omega)$ in local charts. Applying the method of proving the measurability of the tangent space of the unstable manifold, which shows the equivalence between the measurability of the tangent space and the measurability of the tangent map, we obtain the measurability of the tangent space of the random invariant manifold $\mathcal{M}(\omega)$.

Summarizing the above, we get

Proposition 5.3.2. *There exists a unique C^r random invariant manifold $\tilde{\mathcal{M}}(\omega)$ in a neighborhood of $\mathcal{M}$. For each fixed $\omega \in \Omega$, $\tilde{\mathcal{M}}(\omega)$ is C^r diffeomorphic to $\mathcal{M}$. Moreover, it is a random set and its tangent space is measurable.*

5.4 Persistence of Normal Hyperbolicity

We show that $\tilde{\mathcal{M}}$ is normally hyperbolic to complete the proof of theorem 3.0.1. We prove the following proposition:

Proposition 5.4.1. *For each $x \in \tilde{\mathcal{M}}(\omega)$ there exists a splitting*

$$\mathbb{R}^n = E^u(\omega, x) \oplus E^c(\omega, x) \oplus E^s(\omega, x)$$

of closed subspaces with associated projections $\Pi^u(\omega, x)$, $\Pi^c(\omega, x)$, and $\Pi^s(\omega, x)$ such that

(i) The splitting is invariant:

$$D_x\phi(t, \omega)(x)E^i(\omega, x) = E^i(\theta_t\omega, \phi(t, \omega)(x)), \quad \textit{for } i = u, c, s$$

(ii) $D_x\phi(t, \omega)(x)\big|_{E^i(\omega,x)} : E^i(\omega, x) \to E^i(\theta_t\omega, \phi(t, \omega)(x))$ is an isomorphism for $i = u, c, s$. $E^c(\omega, x)$ is the tangent space of $\mathcal{M}(\omega)$ at x.

(iii) There are (θ, ϕ)-invariant random variables $\alpha, \beta : \mathcal{M} \to (0, \infty)$, $0 < \alpha < \beta$, and a

tempered random variable $K(\omega, x) : \mathcal{M} \to [1, \infty)$ such that

$$||D_x\phi(t,\omega)(x)\Pi^s(\omega,x)|| \le K(\omega,x)e^{-\beta(\omega,x)t} \quad \textit{for } t \ge 0, \tag{5.4.1}$$

$$||D_x\phi(t,\omega)(x)\Pi^u(\omega,x)|| \le K(\omega,x)e^{\beta(\omega,x)t} \quad \textit{for } t \le 0, \tag{5.4.2}$$

$$||D_x\phi(t,\omega)(x)\Pi^c(\omega,x)|| \le K(\omega,x)e^{\alpha(\omega,x)|t|} \quad \textit{for } -\infty < t < \infty. \tag{5.4.3}$$

Moreover, $E^i(\omega, x)$ are measurable in (ω, x) and C^{r-1} in x.

We prove the proposition by several lemmas.

Lemma 5.4.1. *For any $\omega \in \Omega$, there are subbundles $\tilde{E}^s(\omega)$ and $\tilde{E}^u(\omega)$ of $T\mathbb{R}^n|\tilde{\mathcal{M}}(\omega)$, which are uniformly close to $\tilde{E}^s$ and $\tilde{E}^u$ respectively, such that $\tilde{E}^s(\omega)$ is complementary to $T\tilde{\mathcal{M}}(\omega)$ in $T\tilde{\mathcal{W}}^s(\omega)|\tilde{\mathcal{M}}(\omega)$ and $\tilde{E}^u$ is complementary to $T\tilde{\mathcal{M}}(\omega)$ in $T\tilde{\mathcal{W}}^s(\omega)|\tilde{\mathcal{M}}(\omega)$.*

Proof. We first note that there exists an unstable manifold $\mathcal{W}^u$ of $\mathcal{M}$ under the deterministic flow $\psi(t)$. Both $\mathcal{W}^u$ and $\tilde{\mathcal{W}}^u(\omega)$ are constructed as sections over the same bundle. Since $\phi(t, \omega)$ and $\psi(t)$ are uniformly C^1 close, $\mathcal{W}^u$ and $\tilde{\mathcal{W}}^u(\omega)$ are C^1 close and diffeomorphic to each other. We denote the diffeomorphism between them by $u(\omega)$. Then $u(\omega)$ is C^1 close to the identity map. Taking the image of $\tilde{E}^u$ under the map $Du(\omega)$ to get $\tilde{E}^u(\omega)$, then $\tilde{E}^u(\omega)$ is uniformly close to $\tilde{E}^u$.

Similarly, we have $\tilde{E}^s(\omega)$ is uniformly close to $\tilde{E}^s$.

□

Since $\tilde{E}^i$ is arbitrarily close to E^i for $i = s, u$, we have that $\tilde{E}^i(\omega)$ are uniformly close to E^i. Since there is no invariance condition on $\tilde{E}^i(\omega)$, we can modify $\tilde{E}^i(\omega, m)$ such that they are C^r in m for each fixed $\omega \in \Omega$ and still stay close to E^i. This is from [W]. In any case, $\tilde{E}^i(\omega, m)$ is not necessarily measurable.

Define $\tilde{E}^c(\omega, m) \equiv T\tilde{\mathcal{M}}(\omega)$ and for $i = s, u, c$, $\tilde{\Pi}^i \equiv \tilde{\Pi}^i(\omega, m)$ the projections onto $\tilde{E}^i(\omega, m)$. Recall that $0 < \alpha < \beta$ are constants associated with the normal hyperbolicity of $\mathcal{M}$ with repect to $\psi(t)$.

Lemma 5.4.2. *(1) There exist positive constants $0 < a < 1$ and c_1 such that:*

$$||\tilde{\Pi}^s D\phi(t,\theta^{-t}\omega)(\phi(-t,\omega)(m))|\tilde{E}^s(\theta^{-t}\omega)|| < c_1 a^t$$

for all $m \in \tilde{\mathcal{M}}(\omega)$ and $t \geq 0$,

$$||\tilde{\Pi}^u D\phi(t,\theta^{-t}\omega)(\phi(-t,\omega)(m))|\tilde{E}^u(\theta^{-t}\omega)|| < c_1 a^t$$

for all $m \in \tilde{\mathcal{M}}(\omega)$ and $t \leq 0$.

(2) If $\alpha < r\beta$, there exist $c_2 > 0$ and $r' > r$ such that

$$||\tilde{\Pi}^s D\phi(t,\theta^{-t}\omega)(\phi(-t,\omega)(m))|\tilde{E}^s(\theta^{-t}\omega)||\,||D\phi(-t,\omega)(m)|\tilde{E}^c(\omega)||^{r'} < c_2$$

for all $m \in \tilde{\mathcal{M}}(\omega)$ and $t \geq 0$,

$$||\tilde{\Pi}^u D\phi(t,\theta^{-t}\omega)\phi(-t,\omega)(m)|\tilde{E}^u(\theta^{-t}\omega)||\,||D\phi(-t,\omega)(m)|\tilde{E}^c(\omega)||^{r'} < c_2$$

for all $m \in \tilde{\mathcal{M}}(\omega)$ and $t \leq 0$.

(3) If $\alpha < r\beta$, there exist $c_3 > 0$ and $r' > r$ such that

$$||\tilde{\Pi}^s D\phi(t,\theta^{-t}\omega)\phi(-t,\omega)(m)|\tilde{E}^s(\theta^{-t}\omega)||\,||D\phi(-t,\omega)(m)|\tilde{E}^c(\omega)||$$
$$||D\phi(t,\theta^{-t}\omega)\phi(-t,\omega)(m)|\tilde{E}^c(\theta^{-t}\omega)||^{r'-1} < c_3$$

for all $m \in \tilde{\mathcal{M}}(\omega)$ and $t \geq 0$,

$$||\tilde{\Pi}^u D\phi(t,\theta^{-t}\omega)\phi(-t,\omega)(m)|\tilde{E}^u(\theta^{-t}\omega)||\,||D\phi(-t,\omega)(m)|\tilde{E}^c(\omega)||$$
$$||D\phi(t,\theta^{-t}\omega)\phi(-t,\omega)(m)|\tilde{E}^c(\theta^{-t}\omega)||^{r'-1} < c_3$$

for all $m \in \tilde{\mathcal{M}}(\omega)$ and $t \leq 0$.

Proof. We prove this lemma using an idea extended from [F1].

We first prove (1). Since

$$||D\psi(t)\psi(-t)(m)|E^s|| \to 0 \text{ as } t \to \infty$$

for all $m \in \mathcal{M}$, for $m \in \mathcal{M}$ there exists $T(m) > 0$ such that

$$||D\psi(T(m))\psi(-T(m))(m)|E^s|| < 1.$$

Hence, there exists a neighborhood $U(m)$ of m in $\mathcal{M}$ such that for $m' \in U(m)$,

$$||D\psi(T(m))\psi(-T(m))(m')|E^s|| < 1.$$

Since $\mathcal{M}$ is compact, we may choose finitely many points $m_1, m_2, \cdots, m_N$ such that

$$\mathcal{M} \subset U(m_1) \cup U(m_2) \cup \cdots \cup U(m_N).$$

Choose $0 < a < 1$ such that

$$||D\psi(T(m_i))\psi(-T(m_i))(m')|E^s|| < a^{T(m_i)}$$

for $m' \in U(m_i)$. Since $\phi(t, \omega)$ and $\psi(t)$ are uniformly close, $\tilde{\mathcal{M}}(\omega)$ and $\mathcal{M}$ are uniformly close, and we have

$$||\tilde{\Pi}^s D\phi(T(m_i), \theta^{-T(m_i)}\omega)\phi(-T(m_i), \omega)(u(\omega, m'))|\tilde{E}^s(\theta^{-T(m_i)}\omega)|| < a^{T(m_i)}$$

for $m' \in U(m_i)$. Take $U(\omega, m_i) = u(\omega, U(m_i))$, then

$$\tilde{\mathcal{M}}(\omega) \subset U(\omega, m_1) \cup U(\omega, m_2) \cup \cdots \cup U(\omega, m_N)$$

and

$$||\tilde{\Pi}^s D\phi(T(m_i), \theta^{-T(m_i)}\omega)\phi(-T(m_i), \omega)(\tilde{m})|\tilde{E}^s(\theta^{-T(m_i)}\omega)|| < a^{T(m_i)}$$

for all $\omega \in \Omega$ and $\tilde{m} \in U(\omega, m_i)$.

Fix an arbitrary $\omega \in \Omega$. Let $m \in \tilde{\mathcal{M}}(\omega)$ be given. Choose a sequence of integers $i(1), i(2), \cdots$, as follows. Choose $i(1)$ such that $m \in U(\omega, m_{i(1)})$. If $i(1), i(2), \cdots, i(j)$ have been chosen, let $\tau(j) = T(m_{i(1)}) + \cdots + T(m_{i(j)})$. Choose $i(j+1)$ such that $\phi(-\tau(j), \omega)(m) \in U(\theta^{-\tau(j)}\omega, m_{i(j+1)})$. Let $t > 0$, it is possible to write $t = \tau(j) + r$ for some j and $0 \leq r < \max T(m_i)$. Then

$$\begin{aligned}
& ||\tilde{\Pi}^s D\phi(t, \theta^{-t}\omega)\phi(-t, \omega)(m)|\tilde{E}^s(\theta^{-t}\omega)|| \\
= \;& ||\tilde{\Pi}^s D\phi(T(m_{i(1)}), \theta^{-\tau(1)}\omega)\phi(-T(m_{i(1)}), \omega)(m)|\tilde{E}^s(\theta^{-\tau(1)}\omega) \\
& \tilde{\Pi}^s D\phi(T(m_{i(2)}), \theta^{-\tau(2)}\omega)\phi(-T(m_{i(2)}), \theta^{-\tau(1)}\omega)\phi(-\tau(1), \omega)(m)|\tilde{E}^s(\theta^{-\tau(2)}\omega) \\
& \tilde{\Pi}^s D\phi(T(m_{i(3)}), \theta^{-\tau(3)}\omega)\phi(-T(m_{i(3)}), \theta^{-\tau(2)}\omega)\phi(-\tau(2), \omega)(m)|\tilde{E}^s(\theta^{-\tau(3)}\omega) \\
& \cdots \\
& \tilde{\Pi}^s D\phi(r, \theta^{-t}\omega)\phi(-r, \theta^{-\tau(j)}\omega)\phi(-\tau(j), \omega)(m)|\tilde{E}^s(\theta^{-t}\omega)|| \\
\leq \;& a^{T(m_1)} \cdot a^{T(m_2)} \cdots a^{T(m_j)} \cdot a^r \cdot \\
& \frac{||\tilde{\Pi}^s D\phi(r, \theta^{-t}\omega)\phi(-r, \theta^{-\tau(j)}\omega)\phi(-\tau(j), \omega)(m)|\tilde{E}^s(\theta^{-t}\omega)||}{a^r} \\
\leq \;& c\, a^t,
\end{aligned}$$

where

$$c = \sup \frac{||\tilde{\Pi}^s D\phi(r, \theta^{-t}\omega)\phi(-r, \theta^{-\tau(j)}\omega)\phi(-\tau(j), \omega)(m)|\tilde{E}^s(\theta^{-t}\omega)||}{a^r},$$

and the supremum is taken over all $\omega \in \Omega, m \in \tilde{\mathcal{M}}(\omega)$ and $0 \leq r < \max T(m_i)$. Equivalently

$$c = \sup \frac{||\tilde{\Pi}^s D\phi(r, \theta^{-r}\omega')\phi(-r, \omega')(m')|\tilde{E}^s(\theta^{-r}\omega')||}{a^r},$$

and the supremum is taken over all $\omega' \in \Omega, m' \in \tilde{\mathcal{M}}(\omega')$ and $0 \leq r < \max T(m_i)$.

Since (2) and (3) follow the similar arguments, we omit the details. $\square$

Lemma 5.4.3. *There exist unique subbundles $E^s(\omega)$ and $E^u(\omega)$ of $T\mathbb{R}^n|\tilde{\mathcal{M}}(\omega)$, such that $E^s(\omega)$ is complementary to $T\tilde{\mathcal{M}}(\omega)$ in $T\tilde{\mathcal{W}}^s(\omega)|\tilde{\mathcal{M}}(\omega)$ and $E^u(\omega)$ is complementary to $T\tilde{\mathcal{M}}(\omega)$ in $T\tilde{\mathcal{W}}^u(\omega)|\tilde{\mathcal{M}}(\omega)$. Moreover, $E^i(\omega)$ $(i = s,\, u)$ is C^{r-1} and invariant under $D\phi(t,\omega)$ for any $t \in \mathbb{R}$ and $\omega \in \Omega$.*

Proof. We will show that for some K to be determined later, there exists a unique random subbundle $E^u(\omega)$ of $T\tilde{\mathcal{W}}^s(\omega)|\tilde{\mathcal{M}}(\omega)$ that is invariant under $D\phi(K,\omega)$. This will be sufficient to conclude the invariance under $D\phi(t,\omega)$ for any $t \in \mathbb{R}$. This is because once we proved the first, we know that $D\phi(t,\omega)(E^u(\omega))$ is also invariant under $D\phi(K,\omega)$. Then by the uniqueness we are done.

To start, we notice that any bundle complementary to $T\tilde{\mathcal{M}}(\omega)$ in $T\tilde{\mathcal{W}}^u(\omega)|\tilde{\mathcal{M}}(\omega)$ is the graph of a family of linear maps

$$h(\omega,m) : \tilde{E}^u(\omega,m) \to T\tilde{\mathcal{M}}(\omega,m),$$

and the bundle is invariant under $D\phi(K,\omega)$ if and only if

$$D\phi(K,\theta^{-K}\omega)h(\theta^{-K}\omega,\phi(-K,\omega,m)) = h(\omega,m)$$

for all $\omega \in \Omega$ and $m \in \tilde{\mathcal{M}}(\omega)$. Equivalently,

$$\begin{aligned} & h(\omega,m)\tilde{\Pi}^u D\phi(K,\theta^{-K}\omega,\phi(-K,\omega,m)) \cdot (\xi^{cu} + h(\theta^{-K}\omega,\phi(-K,\omega,m)))\xi^{cu} \\ = \;& \tilde{\Pi}^c D\phi(K,\theta^{-K}\omega,\phi(-K,\omega,m)) \cdot (\xi + h(\theta^{-K}\omega,\phi(-K,\omega,m)))\xi \end{aligned}$$

for all $\xi \in \tilde{E}^u(\theta^{-K}\omega,\phi(-K,\omega,m))$, for $\omega \in \Omega$, $m \in \tilde{\mathcal{M}}(\omega)$. Since $T\tilde{\mathcal{M}}(\omega)$ is invariant, we have

$$\tilde{\Pi}^u D\phi(K,\theta^{-K}\omega,\phi(-K\omega,m))h(\theta^{-K}\omega,\phi(-K,\omega,m)) = 0.$$

Suppressing ξ, we can write the functional equation for h as

$$\begin{aligned}
& h(\omega, m) \circ \tilde{\Pi}^u D\phi(K, \theta^{-K}\omega, \phi(-K, \omega, m))|\tilde{E}^u(\theta^{-K}\omega, \phi(-K, \omega, m)) \\
= \; & \tilde{\Pi}^c D\phi(K, \theta^{-K}\omega, \phi(-K, \omega, m))|\tilde{E}^u(\theta^{-K}\omega, \phi(-K, \omega, m)) \\
& + \tilde{\Pi}^c D\phi(K, \theta^{-K}\omega, \phi(-K, \omega, m)) \circ h(\theta^{-K}\omega, \phi(-K, \omega, m)))|\tilde{E}^u(\theta^{-K}\omega, \phi(-K, \omega, m)).
\end{aligned}$$

We have

$$[\tilde{\Pi}^u D\phi(K, \theta^{-K}\omega, \phi(-K, \omega, m))|\tilde{E}^u(\theta^{-K}\omega, \phi(-K, \omega, m))]^{-1} = \tilde{\Pi}^u D\phi(-K, \omega, m)|\tilde{E}^u.$$

Hence,

$$\begin{aligned}
& h(\omega, m) \\
= \; & \tilde{\Pi}^c D\phi(K, \theta^{-K}\omega, \phi(-K, \omega, m)) \circ \tilde{\Pi}^u D\phi(-K, \omega, m)|\tilde{E}^u \\
& + \tilde{\Pi}^c D\phi(K, \theta^{-K}\omega, \phi(-K, \omega, m)) \circ h(\theta^{-K}\omega, \phi(-K, \omega, m))) \circ \tilde{\Pi}^u D\phi(-K, \omega, m)|\tilde{E}^u.
\end{aligned}$$

This is a linear functional equation for h. We want to use the contraction mapping theorem to show it has a unique fixed point.

By Proposition 5.4.2 (1) and (2), we get

$$||D(\phi|\tilde{\mathcal{M}}(\theta^{-K}\omega))(K, \theta^{-K}\omega, \phi(-K, \omega, m))|| \, ||\tilde{\Pi}^u D\phi(-K, \omega, m)|\tilde{E}^u|| \leq \frac{1}{4} \tag{5.4.4}$$

$$\begin{aligned}
& ||D\phi(K, \theta^{-K}\omega, \phi(-K, \omega, m))|\tilde{E}^c(\theta^{-K}\omega)|| \, ||D\phi(-K, \omega, m)|\tilde{E}^c(\omega)||^k \, ||\tilde{\Pi}^u D\phi(-K, \omega, m)|\tilde{E}^u|| \\
& \leq \frac{1}{4}
\end{aligned} \tag{5.4.5}$$

for $1 \leq k \leq r-1$.

Define a space S, whose element $h \in S$ is of the form:

$$h = \{h(\omega, m) | \omega \in \Omega, m \in \tilde{\mathcal{M}}(\omega)\}.$$

Define the norm $|| \cdot ||_S$ by

$$||h||_S := \sup_{\omega} \max_{m \in \tilde{\mathcal{M}}(\omega)} ||h(\omega, m)||_{L(\tilde{E}^u(\omega,m), T\tilde{\mathcal{M}}(\omega,m))}.$$

Under this norm, S is a complete metric space. Let

$$h^0(\omega, m) \equiv 0 \in L(\tilde{E}^u(\omega, m), T\tilde{\mathcal{M}}(\omega, m)),$$

and define

$$\begin{aligned} & h^{n+1}(\omega, m) \\ = \ & \tilde{\Pi}^c D\phi(K, \theta^{-K}\omega, \phi(-K, \omega, m)) \circ \tilde{\Pi}^u D\phi(-K, \omega, m) | \tilde{E}^u \\ & + \tilde{\Pi}^c D\phi(K, \theta^{-K}\omega, \phi(-K, \omega, m)) \circ h^n(\theta^{-K}\omega, \phi(-K, \omega, m))) \circ \tilde{\Pi}^u D\phi(-K, \omega, m) | \tilde{E}^u. \end{aligned}$$

Then for all $n \geq 0$, and for any fixed ω, $h^n(\omega, m)$ is C^{r-1} in m . By (5.4.4), h^n is a Cauchy sequence. Suppose h is the unique limit of h^n. Then h satisfies

$$\begin{aligned} & h(\omega, m) \\ = \ & \tilde{\Pi}^c D\phi(K, \theta^{-K}\omega, \phi(-K, \omega, m)) \circ \tilde{\Pi}^u D\phi(-K, \omega, m) | \tilde{E}^u \\ & + \tilde{\Pi}^c D\phi(K, \theta^{-K}\omega, \phi(-K, \omega, m)) \circ h(\theta^{-K}\omega, \phi(-K, \omega, m))) \circ \tilde{\Pi}^u D\phi(-K, \omega, m) | \tilde{E}^u \end{aligned}$$

So $h(\omega, m)$ represents the unique invariant bundle $E(\omega, m)$. Since $h^n(\omega, m) \to h(\omega, m)$ uniformly, $h(\omega, m)$ is C^0 in m.

By (5.4.5), $D_m^k h^n(\omega, m)$ for $k \leq r-1$ is a Cauchy sequence in the corresponding space. So $D_m^k h^n(\omega, m)$ converges uniformly as n goes to ∞. Therefore, $D_m^k h(\omega, m)$ exists and equals the limit of $D_m^k h^n(\omega, m)$, which means $h(\omega, m)$ is C^{r-1} in m for any fixed ω. Hence, we obtain an unique C^{r-1} invariant bundle $E^u(\omega)$. Similarly, we have a unique C^{r-1} invariant bundle $E^s(\omega)$. □

From now on, we use $E^s(\omega)$ and $E^u(\omega)$ to denote the unique invariant bundle of lemma 5.4.3. Since $\psi(t)$ and $\phi(t, \omega)$ are uniformly close, and $\mathcal{M}$ and $\tilde{\mathcal{M}}(\omega)$ are uniformly close, we conclude that $E(\omega)$ are uniformly close to E. Since $\mathcal{M}$ is compact, the angle between $E(m)$ and $T_m\mathcal{M}$ is bounded away from 0. So also we have that the angle between $E(\omega, m)$ and $T_m\tilde{\mathcal{M}}(\omega)$ are uniformly bounded away from 0. So $||\Pi^s||$, $||\Pi^u||$, $\frac{1}{||\Pi^s||}$ and $\frac{1}{||\Pi^u||}$ are bounded. Suppose they are all bounded by $c_4 > 0$.

Lemma 5.4.4. *(5.4.1), (5.4.2) and (5.4.3) are satisfied for $E^s(\omega)$, $E^u(\omega)$ and $E^c(\omega) = T\tilde{\mathcal{M}}(\omega)$.*

Proof. For any $\nu \in T\tilde{\mathcal{W}}^s(\omega)|\tilde{\mathcal{M}}(\omega)$, we have

$$\nu - \Pi^s\nu \in T\tilde{\mathcal{M}}(\omega).$$

So $\tilde{\Pi}^s\Pi^s = \tilde{\Pi}^s$. Similarly, $\Pi^s\tilde{\Pi}^s = \Pi^s$.

For each fixed $\omega \in \Omega$, $m \in \tilde{\mathcal{M}}(\omega)$ and $\nu \in E^s(\omega, m)$, let $\tilde{\nu} = \tilde{\Pi}^s\nu$. Then $\tilde{\nu} - \nu \in T\tilde{\mathcal{M}}(m)$ and

$$\begin{aligned}
& c_4^{-1}|\tilde{\Pi}^s D\phi(-t,\omega)(\tilde{\nu})| \\
= \; & c_4^{-1}|\tilde{\Pi}^s D\phi(-t,\omega)(\tilde{\Pi}^s\nu)| = c_4^{-1}|\tilde{\Pi}^s D\phi(-t,\omega)(\nu)| \leq |\Pi^s\tilde{\Pi}^s D\phi(-t,\omega)(\nu)| \\
= \; & |\Pi^s D\phi(-t,\omega)(\nu)| \leq c_4|\tilde{\Pi}^s\Pi^s D\phi(-t,\omega)(\nu)| = c_4|\tilde{\Pi}^s D\phi(-t,\omega)(\nu)| \\
= \; & c_4|\tilde{\Pi}^s D\phi(-t,\omega)(\tilde{\Pi}^s\nu)| = c_4|\tilde{\Pi}^s D\phi(-t,\omega)(\tilde{\nu})|,
\end{aligned}$$

which gives us

$$c_4^{-1}|\tilde{\Pi}^s D\phi(-t,\omega)(\tilde{\nu})| \leq |\Pi^s D\phi(-t,\omega)(\nu)| \leq c_4|\tilde{\Pi}^s D\phi(-t,\omega)(\tilde{\nu})|. \tag{5.4.6}$$

From (5.4.6), all properties listed in lemma 5.4.2 persist if we change $\tilde{E}^s(\omega)$ and $\tilde{E}^u(\omega)$ to $E^s(\omega)$ and $E^u(\omega)$.

From (1) of lemma 5.4.2 , if we take $K(\omega,x) = c_1$ and $\beta(\omega,x) = -\log a$, then $\beta > 0$ and (θ,ϕ)-invariant, $K(\omega,x)$ is tempered, and (5.4.1) and (5.4.2) hold.

From the uniform closeness of $\phi(t,\omega)$ and $\psi(t)$, the normally hyperbolicity property of $\mathcal{M}$ under $\psi(t)$ and the uniform closeness of $\tilde{\mathcal{M}}(\omega)$ and $\mathcal{M}$, we have for some (θ,ϕ)-invariant random variable $\alpha : \tilde{M} \to (0,\infty)$ and tempered random variable $K(\omega,x) : \tilde{\mathcal{M}} \to [1,\infty)$ such that (5.4.3) hold.

What we need to prove is that $r\alpha < \beta$. From (2) of lemma 5.4.2, α and β can be chosen such that $r\alpha < \beta$. $\square$

Lemma 5.4.5. *$E^i(\omega,m)$ is measurable.*

Proof. In section 6,7, we proved the measurability of E^c. Here we only prove the measurability of E^u. For E^s, just follow exactly the same argument for E^u.

Take the orthogonal complement of $T_{(\omega,m)}\tilde{\mathcal{M}}$ in $T_{(\omega,m)}\tilde{\mathcal{W}}^u$ to get a normal bundle $\hat{E}^u(\omega,m)$. Since $T\tilde{\mathcal{M}}$ and $T\tilde{\mathcal{W}}^u$ are both measurable, $\hat{E}^u(\omega,m)$ is also measurable. Moreover, the angle between $\hat{E}^u(\omega,m)$ and $T_{(\omega,m)}\tilde{\mathcal{M}}$ are uniformly bounded away from 0. So by Lemma 5.4.4, all properties listed in lemma 5.4.2 hold, even though $\hat{E}(\omega,m)$ may not be close to E^u. So we can replace the bundle $\tilde{E}^u(\omega,m)$ in lemma 5.4.3 by $\hat{E}^u(\omega,m)$ and still get the unique invariant bundle $E^u(\omega,m)$ by the same argument. In this way, the contraction mapping in the functional equation of $h(\omega,x)$ is measurable in ω and C^{r-1} in x. As we got the measurability of v in proposition 5.1.4, we get the measurability of $h(\omega,x)$–the representation of $E^u(\omega,m)$. This completes the proof of the lemma.

$\square$

The bundles $E^c(\omega), E^s(\omega)$ and $E^u(\omega)$ satisfy all the conditions listed in proposition 5.4.1. So we have proved Proposition 5.4.1.

Summarizing Proposition 4.2.1, 5.1.3, 5.2.1, 5.2.3, 5.3.1,5.3.2 and 5.4.1, gives Theorem 3.0.1.

5.5 Last but not Least

we abstract from the property of normal hyperbolic a key lemma for later use:

Lemma 5.5.1. *(1) There exist positive constants $0 < a < 1$ and c_1 such that:*

$$||\Pi^s D\phi(t, \theta^{-t}\omega)(\phi(-t, \omega)(m))|E^s(\theta^{-t}\omega)|| < c_1 a^t$$

for all $m \in \tilde{\mathcal{M}}(\omega)$ and $t \geq 0$,

$$||\Pi^u D\phi(t, \theta^{-t}\omega)(\phi(-t, \omega)(m))|E^u(\theta^{-t}\omega)|| < c_1 a^t$$

for all $m \in \tilde{\mathcal{M}}(\omega)$ and $t \leq 0$.

(2) If $\alpha < r\beta$, there exist $c_2 > 0$ and $r' > r$ such that

$$||\Pi^s D\phi(t, \theta^{-t}\omega)(\phi(-t, \omega)(m))|E^s(\theta^{-t}\omega)|| \, ||D\phi(-t, \omega)(m)|E^c(\omega)||^{r'} < c_2$$

for all $m \in \tilde{\mathcal{M}}(\omega)$ and $t \geq 0$,

$$||\Pi^u D\phi(t, \theta^{-t}\omega)\phi(-t, \omega)(m)|E^u(\theta^{-t}\omega)|| \, ||D\phi(-t, \omega)(m)|E^c(\omega)||^{r'} < c_2$$

for all $m \in \tilde{\mathcal{M}}(\omega)$ and $t \leq 0$.

(3) If $\alpha < r\beta$, there exist $c_3 > 0$ and $r' > r$ such that

$$||\Pi^s D\phi(t, \theta^{-t}\omega)\phi(-t,\omega)(m)|E^s(\theta^{-t}\omega)||\,||D\phi(-t,\omega)(m)|E^c(\omega)||$$
$$||D\phi(t,\theta^{-t}\omega)\phi(-t,\omega)(m)|\tilde{E}^c(\theta^{-t}\omega)||^{r'-1} < c_3$$

for all $m \in \tilde{\mathcal{M}}(\omega)$ and $t \geq 0$,

$$||\Pi^u D\phi(t, \theta^{-t}\omega)\phi(-t,\omega)(m)|E^u(\theta^{-t}\omega)||\,||D\phi(-t,\omega)(m)|E^c(\omega)||$$
$$||D\phi(t,\theta^{-t}\omega)\phi(-t,\omega)(m)|\tilde{E}^c(\theta^{-t}\omega)||^{r'-1} < c_3$$

for all $m \in \tilde{\mathcal{M}}(\omega)$ and $t \leq 0$.

Proof. This follows exactly the same proof of lemma 5.4.2. □

Chapter 6. Invariant Foliation

6.1 Existence of the Invariant Foliation

In this section, we prove the existence of invariant foliation of the random unstable manifold. We will keep using the notations $\mathcal{M}$, $\tilde{\mathcal{M}}$, $\mathcal{W}$, $\tilde{\mathcal{W}}$, E^i(the deterministic bundles) and $E^i(\omega)$(the random bundles) to mean what they were in the previous sections. Others will be given new meanings when used.

We construct the invariant foliation in local coordinates on $\tilde{\mathcal{W}}^u(\omega)$. The basic idea is due to Hadamard [H] and involves a graph transform. First, we take a set of Lipschitz graphs in local charts, which pass through different points on the random invariant manifold $\tilde{\mathcal{M}}(\omega)$ and are contained in the random unstable manifold $\tilde{\mathcal{W}}^u(\omega)$. Then, we consider a random graph tranform on these graphs by $\phi(t,\omega)$ for some large fixed $t > 0$. We show that this random graph transform is a contraction on the space of such sets of graphs and the resulting fixed set of graphs gives us the invariant foliation of the random unstable manifold. By reversing the time, we get the invariant foliation of the random stable manifold. The technical hurdle is the construction of local charts. We need the local charts on $\tilde{\mathcal{W}}^u(\omega)$ for different $\omega \in \Omega$, while at the same time we need those charts are all related to each other for different ω. We overcome this difficulty by using the fact that the random unstable manifold $\tilde{\mathcal{W}}^u(\omega)$ and deterministic unstable manifold $\mathcal{W}^u$ are C^r diffeomorphic and close. Thus, we can define local coordinates on $\mathcal{W}^u$ and then induce local coordinates on $\tilde{\mathcal{W}}^u(\omega)$. We denote $i(\omega)$ the C^r diffeomorphism from $\mathcal{W}^u$ to $\tilde{\mathcal{W}}^u(\omega)$. For any fixed ω, $i(\omega)$ is C^r close to the identity map Id.

To define local coordinates on $\mathcal{W}^u$, we follow Fenichel's approach, see [F2] page 1122. Let exp be the exponential map. For each $m \in \mathcal{M}$ and $\nu \in T\mathcal{W}^u|\mathcal{M}$, let $exp_m(\nu)$ be the end point of the geodesic with initial point m and initial tangent vector ν. We borrow the following lemma from [F2]:

Lemma 6.1.1. *There exists $0 < \epsilon_1$ such that for each $m \in \mathcal{M}$,*

$$exp_m : \{\nu \in T_m\mathcal{W}^u|\mathcal{M} : |\nu| < \epsilon_1\} \to \mathcal{W}^u$$

is a diffeomorphism onto its range and its range lies in $\mathcal{W}^u$. Moreover, the derivative of the exponential map satisfies that $D\,exp_m(0)$ is the identity map and $||D\,exp_m(\nu)||$ and $||[D\,exp_m(\nu)]^{-1}||$ are arbitrarily close to 1 uniformly for $m \in \mathcal{M}$ and $|\nu| < \epsilon_1$.

Define

$$\Gamma(\omega) : T\tilde{\mathcal{W}}^u(\omega)|\tilde{\mathcal{M}}(\omega) \mapsto \tilde{\mathcal{W}}^u(\omega)$$

by

$$\Gamma(\omega) := i(\omega) \circ exp \circ [D\,i(\omega)]^{-1}.$$

This gives us local coordinates on $\tilde{\mathcal{W}}^u(\omega)$ near $\tilde{\mathcal{M}}(\omega)$. By the uniform closeness of $\psi(t)$ and $\phi(t, \omega)$, the uniform closeness of $\mathcal{W}^u$ and $\tilde{\mathcal{W}}^u(\omega)$ and Lemma 6.1.1, there exists $0 < \epsilon_2 < \epsilon_1$ such that $\Gamma(\omega)$ is well defined on $\{\nu \in T\tilde{\mathcal{W}}^u(\omega)|\tilde{\mathcal{M}}(\omega) : |\nu| < \epsilon_2\}$. Fix a $K > 0$ big enough.

There exists $0 < \epsilon_3 < \epsilon_2$ such that

$$\phi(K,\omega)\{\Gamma(\omega)(T\tilde{\mathcal{W}}^u(\omega)|\tilde{\mathcal{M}}(\omega))(\epsilon_3)\} \subset \Gamma(\theta^K\omega)(T\tilde{\mathcal{W}}^u(\theta^K\omega)|\tilde{\mathcal{M}}(\theta^K\omega))(\epsilon_2).$$

For any $\omega \in \Omega, m \in \tilde{\mathcal{M}}(\omega)$, let $\xi^c,\ \xi^u,\ x^c\ x^u$ denote elements of $T\tilde{\mathcal{M}}(\theta^{-K}\omega, \phi(-K,\omega)(m))$, $E^u(\theta^{-K}\omega, \phi(-K,\omega)(m))$, $T\tilde{\mathcal{M}}(\omega, m)$ and $E^u(\omega, m)$ respectively. We use (ξ^c, ξ^u) and (x^c, x^u) as coordinates near $\phi(-K,\omega)(m)$ and m. The map $\phi(K, \theta^{-K}\omega)$ has the form

$$(\xi^c, \xi^u) \ \mapsto \ (x^c, x^u) \ = \ (g^c(\xi^c, \xi^u), g^u(\xi^c, \xi^u)),$$

defined for $|\xi| = |\xi^c| + |\xi^u| < \epsilon_3$. Note that g^c and g^u depend on ξ^c, ξ^u as well as on m and ω. And there exists Q so large that all first partial derivatives of g^c and g^u along with their inverses are bounded by Q.

By lemma 5.5.1 we have

$$||D\phi(-K,\omega)(m)|E^u(\omega)|| < \frac{1}{4},$$

$$||D((\phi|\tilde{\mathcal{M}}(\theta^{-K}\omega))(K,\theta^{-K}\omega))\phi(-K,\omega)(m)||^k||D\phi(-K,\omega)(m)|E^u(\omega)|| < \frac{1}{4},$$

for any $0 \le k \le r$. In terms of g^c and g^u, the above give us

$$||[D_2g^u(0,0)]^{-1}|| = ||D\phi(-K,\omega)(m)|E^u(\omega)|| < \frac{1}{4},$$

$$||[D_2g^u(0,0)]^{-1}||||D_1g^c(0,0)||^k < \frac{1}{4}, \text{ for } 0 \le k \le r.$$

We also have from the invariance of $\tilde{\mathcal{M}}(\omega)$ and $E(\omega)$ the following

$$g^u(0,0) = 0, \quad g^c(0,0) = 0, \quad D_2g^c(0,0) = 0.$$

By the compactness of $\mathcal{M}$, the uniform closeness of $\psi(K)$ and $\phi(K,\omega)$ and the uniform closeness of $\tilde{\mathcal{W}}^u(\omega)$ and $\mathcal{W}^u$, for any $\beta > 0$ and $\gamma > 0$ there exists $0 < \epsilon_4 < \epsilon_3$ such that for all $\omega \in \Omega$ and $m \in \tilde{\mathcal{M}}(\omega)$, if $|\xi^c|,\ |\bar{\xi}^c|,\ |\xi^u|,\ |\bar{\xi}^u| \leq \epsilon_4$, then

$$||[D_2 g^u(\xi^c, \xi^u)]^{-1}|| < \frac{1}{3}, \tag{6.1.1}$$

$$||[D_2 g^u(\xi^c, \xi^u)]^{-1}|| ||D_1 g^c(\bar{\xi}^c, \bar{\xi}^u)||^k < \frac{1}{3}, \text{ for } 0 \leq k \leq r, \tag{6.1.2}$$

$$||g^u(\xi^c, \xi^u)|| < \gamma, \quad ||g^c(\xi^c, \xi^u)|| < \gamma, \quad ||D_2 g^c(\xi^c, \xi^u)|| < \gamma, \tag{6.1.3}$$

$$|g^u(\xi^c, \xi^u) - g^u(\xi^c, \bar{\xi}^u)| \geq [||[D_2 g^u(\xi^c, \xi^u)]^{-1}||^{-1} - \beta]\, |\xi^u - \bar{\xi}^u|, \tag{6.1.4}$$

$$|g^c(\xi^c, \xi^u) - g^c(\bar{\xi}^c, \xi^u)| \leq [||D_1 g^c(\xi^c, \xi^u)|| + \beta]\, |\xi^c - \bar{\xi}^c|. \tag{6.1.5}$$

For β sufficiently small, there exists a small positive δ_0 such that if

$$|\xi^c - \bar{\xi}^c| \leq \delta_0 |\xi^u - \bar{\xi}^u|,$$

then

$$\begin{aligned} &|g^u(\xi^c, \xi^u) - g^u(\bar{\xi}^c, \bar{\xi}^u)| \\ &\geq |g^u(\xi^c, \xi^u) - g^u(\xi^c, \bar{\xi}^u)| - |g^u(\xi^c, \bar{\xi}^u) - g^u(\bar{\xi}^c, \bar{\xi}^u)| \\ &\geq [||[D_2 g^u(\xi^c, \xi^u)]^{-1}||^{-1} - \beta]\, |\xi^u - \bar{\xi}^u| - Q|\xi^c - \bar{\xi}^c| \\ &\geq (3 - \beta - Q\delta_0)\, |\xi^u - \bar{\xi}^u| > 2|\xi^u - \bar{\xi}^u|. \end{aligned} \tag{6.1.6}$$

Let S denote the set of families of continuous maps $h = \{h(\omega, m) : \omega \in \Omega,\ m \in \tilde{\mathcal{M}}(\omega)\}$, where $h(m, \omega)(x^u) :\ E^u(m, \omega)(\epsilon_4) \to T\tilde{\mathcal{M}}(\omega, m)(\epsilon_4)$ is continuous in x^u and the base point m. Moreover, $h(m, \omega)(0) = 0$.

For $h \in S$, define

$$Lip\,h := \sup_{\omega\in\Omega} \max_{m\in\tilde{\mathcal{M}}(\omega)} \sup_{x^u,\bar{x}^u\in E^u(\omega,m)(\epsilon_4),x^u\neq\bar{x}^u} \frac{|h(\omega,m)(x^u)-h(\omega,m)(\bar{x}^u)|}{|x^u-\bar{x}^u|},$$

if it exists. Denote by S_δ the set of all Lipschitz h:

$$S_\delta := \{h \in S : Lip\,h \leq \delta\}.$$

Define a distance on S_δ,

$$d(h,h') \;=\; \sup\{\frac{|h(m,\omega)(x^u)-h'(m,\omega)(x^u)|}{|x^u|} \;:\; \omega\in\Omega,\; m\in\tilde{\mathcal{M}}(\omega),\; 0\neq x^u\in E(m,\omega)(\epsilon_4)\}.$$

The supremum exists because each term is bounded by 2δ. Under this metric, S_δ is complete. Moreover, convergence in S_δ implies uniform convergence.

We will construct a family $h \in S_\delta$ such that $\tilde{\mathcal{W}}^{uu}(\omega,m)$ is the graph of $h(\omega,m)$.

Proposition 6.1.1. *There exists a unique point in S_δ, which we denote by h. For any $t > K$, h satisfies the overflowing invariance condition:*

$$\phi(-t,\omega)(graph(h(\omega,m))) \subset graph(h(\theta^{-t}\omega,\phi(-t,\omega)(m))).$$

Proof. We first note that in local coordinates the above overflowing invariance condition is equivalent to the nonlinear functional equation:

$$h(\omega,m)(g^u(h(\theta^{-K}\omega,\phi(-K,\omega)(m))(\xi^u)),\xi^u) = g^c(h(\theta^{-K}\omega,\phi(-K,\omega)(m))(\xi^u)),\xi^u).$$

We will show that this functional equation has a unique solution in S_δ.

Define a map G on $S_\delta \to S$ as follows. For $h \in S_\delta,\; \omega\in\Omega,\; m\in\tilde{\mathcal{M}}(\omega)$,

$$(Gh)(m,\omega)(x^u) \;=\; g^c(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)$$

where

$$x^u = g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u),$$

$$m' = \phi(-K, \omega)(m).$$

The next lemma justifies this definition.

Lemma 6.1.2. *If δ and ϵ_4 are sufficiently small, for each $h \in S_\delta$, $\omega \in \Omega$ and $m \in \tilde{\mathcal{M}}(\omega)$, the map $\xi^u \mapsto g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)$ is one-to-one on $E(\theta^{-K}\omega, m')(\epsilon_4)$ and $E(\omega, m)(\epsilon_4)$ is contained in its range.*

Proof. By (6.1.6) we conclude that g^u is one-to-one. Then g^u is a continuous injection from an open subset of Euclidean space to Euclidean space of the same dimension. By invariance of domain, the range of g^u is open. Since $g^u(0,0) = 0$, there exists $c > 0$ such that $B(0,c)$ is contained in the range of g^u. Again from (6.1.6) we have that the pre-image of $B(0,c)$ is contained in $B(0,c/2)$ and that $B(0,\epsilon_4)$ is contained in the range of g^u. □

Lemma 6.1.3. *If δ and ϵ_4 are sufficiently small, G maps S_δ into S_δ.*

Proof. It is obvious that $(Gh)(\omega, m)(0) = 0$. We only need to estimate the Lipschitz constant of Gh. Let $x^u,\ \bar{x}^u \in E(\omega, m)(\epsilon)$ and define $\xi^u,\ \bar{\xi}^u$ by

$$x^u = g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u),$$

$$\bar{x}^u = g^u(h(\theta^{-K}\omega, m')(\bar{\xi}^u), \bar{\xi}^u),$$

which are well defined by lemma 6.1.2. Then

$$\begin{aligned}
|\bar{x}^u - x^u| &= |g^u(h(\theta^{-K}\omega, m')(\bar{\xi}^u), \bar{\xi}^u) - g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)| \\
&\geq |g^u(h(\theta^{-K}\omega, m')(\xi^u), \bar{\xi}^u) - g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)| \\
&\quad -|g^u(h(\theta^{-K}\omega, m')(\bar{\xi}^u), \bar{\xi}^u) - g^u(h(\theta^{-K}\omega, m')(\xi^u), \bar{\xi}^u)| \\
&\geq \{||[D_2 g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)]^{-1}||^{-1} - \beta\}|\bar{\xi}^u - \xi^u| \\
&\quad -Q|h(\theta^{-K}\omega, m')(\bar{\xi}^u) - h(\theta^{-K}\omega, m')(\xi^u)| \\
&\geq \{||[D_2 g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)]^{-1}||^{-1} - \beta - Q\delta\}|\bar{\xi}^u - \xi^u|.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&|(Gh)(\omega, m)(\bar{x}^u) - (Gh)(\omega, m)(x^u)| \\
= \ &|g^c(h(\theta^{-K}\omega, m')(\bar{\xi}^u), \bar{\xi}^u) - g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)| \\
\leq \ &[||D_1 g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)|| + \beta]|h(\theta^{-K}\omega, m')(\bar{\xi}^u) - h(\theta^{-K}\omega, m')(\xi^u)| + \gamma|\bar{\xi}^u - \xi^u| \\
\leq \ &\{[||D_1 g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)|| + \beta]\delta + \gamma\}|\bar{\xi}^u - \xi^u|.
\end{aligned}$$

So by (6.1.1) and (6.1.2), for δ small enough, choosing ϵ_4 and γ sufficiently small, we have $Lip\ Gh\ <\ \delta$. □

Lemma 6.1.4. *If δ and ϵ_4 are sufficiently small, G is a contraction on S_δ.*

Proof. Let $h,\ \hat{h} \in S_\delta$ and $x^u \in E(\omega, m)(\epsilon_4)$. Then, there exist $\xi^u,\ \hat{\xi}^u \in E(\theta^{-K}\omega, m')(\epsilon_4)$

such that $x^u = g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u) = g^u(\hat{h}(\theta^{-K}\omega, m')(\hat{\xi}^u), \hat{\xi}^u)$. By (6.1.6), we have

$$\begin{aligned}
2|\xi^u - \hat{\xi^u}| &\leq |g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u) - g^u(h(\theta^{-K}\omega, m')(\xi^u), \hat{\xi}^u)| \\
&= |g^u(\hat{h}(\theta^{-K}\omega, m')(\hat{\xi}^u), \hat{\xi}^u) - g^u(h(\theta^{-K}\omega, m')(\xi^u), \hat{\xi}^u)| \\
&\leq Q|\hat{h}(\theta^{-K}\omega, m')(\hat{\xi}^u) - h(\theta^{-K}\omega, m')(\xi^u)| \\
&\leq Q|\hat{h}(\theta^{-K}\omega, m')(\hat{\xi}^u) - \hat{h}(\theta^{-K}\omega, m')(\xi^u)| \\
&\quad +Q|\hat{h}(\theta^{-K}\omega, m')(\xi^u) - h(\theta^{-K}\omega, m')(\xi^u)| \\
&\leq Q\delta|\hat{\xi^u} - \xi^u| + Qd(\hat{h}, h)|\xi^u|.
\end{aligned}$$

Choosing δ such that $\delta < \frac{1}{Q}$, we have

$$|\xi^u - \hat{\xi^u}| \leq Qd(\hat{h}, h)|\xi^u|.$$

We also have

$$\begin{aligned}
& |(Gh)(\omega, m)(x^u) - (G\hat{h})(\omega, m)(x^u)| \\
= \;& |g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u) - g^c(\hat{h}(\theta^{-K}\omega, m')(\hat{\xi}^u), \hat{\xi}^u)| \\
\leq \;& |g^c(\hat{h}(\theta^{-K}\omega, m')(\hat{\xi}^u), \hat{\xi}^u) - g^c(\hat{h}(\theta^{-K}\omega, m')(\xi^u), \xi^u)| \\
& +|g^c(\hat{h}(\theta^{-K}\omega, m')(\xi^u), \xi^u) - g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)| \\
\leq \;& [(Q+\beta)\delta + \gamma]|\xi^u - \hat{\xi^u}| + [||D_1 g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)|| + \beta]d(h, \hat{h})|\xi^u| \\
\leq \;& (Q+\beta+\tau)\delta\ Qd(h, \hat{h})|\xi^u| + [||D_1 g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)|| + \beta]d(h, \hat{h})|\xi^u| \\
= \;& [||D_1 g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)|| + \beta + (Q+\beta+\theta)\delta\ Q]d(h, \hat{h})|\xi^u|
\end{aligned}$$

and

$$\begin{aligned}
|x^u| &= |g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)| \\
&= |g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u) - g^u(0,0)| \\
&\geq [||[D_2 g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)]^{-1}||^{-1} - \beta - Q\delta]|\xi^u|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{|(Gh)(\omega, m)(x^u) - (G\hat{h})(\omega, m)(x^u)|}{|x^u|} \\
\leq\ &\frac{||D_1 g^c(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)|| + \beta + (Q + \beta + \theta)\delta\ Q}{D_2 g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u)]^{-1}||^{-1} - \beta - Q\delta} d(h, \hat{h}).
\end{aligned}$$

Choosing δ and ϵ sufficiently small and using (6.1.2) the factor preceding $d(h, \hat{h})$ can be bounded by a constant $\lambda < 1$. Thus, we have

$$d(Gh, G\hat{h}) \leq \lambda d(h, \hat{h}).$$

This completes the proof of the lemma. □

By the contraction principle, there exists a unique fixed point h of G in S_δ. h satisfies:

$$\phi(-K, \omega)(\text{graph}(h(\omega, m))) \subset \text{graph}(h(\theta^{-K}\omega, \phi(-K, \omega)(m))).$$

For any fixed $t > K$ we can define G_1 just as we defined G for K. We know that G and G_1 commute. So we have

$$GG_1 h = G_1 G h = G_1 h.$$

By the uniqueness of G we conclude

$$G_1 h = h.$$

Or equivalently

$$\phi(-t,\omega)(\mathrm{graph}(h(\omega,m))) \subset \mathrm{graph}(h(\theta^{-t}\omega,\phi(-t,\omega)(m))).$$

This completes the proof of proposition 6.1.1. □

6.2 Smoothness

In this section, we prove two kinds of smoothness of the invariant foliation: the smoothness of each fiber and the smoothness in the base point. We also prove that as the base point changes, the fiber changes measurably.

To prove smoothness, we first differentiate the equation of the fixed point formally to find out the functional equation, which must be held by the real derivatives. Second, we show the functional equation has a unique solution in some space. Last, we show that unique solution is indeed the derivative.

6.2.1 Smoothness of the Fiber. By proving the following proposition, we show that each fiber of the invariant foliation is C^r.

Proposition 6.2.1. *For any $\omega \in \Omega$ and $m \in \tilde{\mathcal{M}}(\omega)$, $h(m,\omega)(x^u)$ is a C^r function in x^u and all derivatives $D^k h(m,\omega)(x^u), 1 \le k \le r$ are continuous in the base point m.*

Proof. First, we have

$$h(\omega,m)(g^u(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)) = g^c(h(\theta^{-K}\omega,m')(\xi^u),\xi^u). \tag{6.2.1}$$

Taking derivative formally on both sides of the above equation, we have

$$\begin{aligned} & Dh(\omega,m)(x^u)[D_1g^u(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)Dh(\theta^{-K}\omega,m')(\xi^u)+D_2g^u(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)] \\ = \ & D_1g^c(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)Dh(\theta^{-K}\omega,m')(\xi^u)+D_2g^c(h(\theta^{-K}\omega,m')(\xi^u),\xi^u). \end{aligned}$$

Thus, if $h(\omega, m)$ is differentiable, then we must have

$$\begin{aligned} & Dh(\omega,m)(x^u) \\ = \ & [D_1g^c(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)Dh(\theta^{-K}\omega,m')(\xi^u)+D_2g^c(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)] \\ & [D_1g^u(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)Dh(\theta^{-K}\omega,m')(\xi^u)+D_2g^u(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)]^{-1} \end{aligned}$$

where

$$x^u = g^u(h(\theta^{-K}\omega,m')(\xi^u),\xi^u),\ m' = \phi(-K,\omega)(m).$$

The candidate for Dh, which we denote by v, has the following form:

$$v = \{v(m,\omega) : \omega \in \Omega, m \in \tilde{\mathcal{M}}(\omega)\}.$$

For each $\omega \in \Omega, m \in \tilde{\mathcal{M}}(\omega)$, $v(m,\omega)(\cdot)$ is a continuous map from

$$E^u(m,\omega)(\epsilon_4) \to L(E^u(m,\omega), T\tilde{\mathcal{M}}(\omega,m)),$$

or equivalently,

$$v(m,\omega) \in C^0(E^u(m,\omega)(\epsilon_4), L(E^u(m,\omega), T\tilde{\mathcal{M}}(\omega,m))).$$

Let TS be the space of all such v. Define the norm $||\cdot||$ on TS by

$$||v|| = \sup_{\omega,m} \max_{x^u \in E^u(m,\omega)} ||v(m,\omega)(x^u)||.$$

Under this norm, TS is complete. We want to find an element $v \in TS$ such that

$$\begin{aligned} & v(\omega,m)(x^u) \\ = \; & [D_1 g^c(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)v(\theta^{-K}\omega,m')(\xi^u) + D_2 g^c(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)] \\ & [D_1 g^u(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)v(\theta^{-K}\omega,m')(\xi^u) + D_2 g^u(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)]^{-1} \end{aligned} \quad (6.2.2)$$

where

$$x^u = g^u(h(\theta^{-K}\omega,m')(\xi^u),\xi^u), \;\; m' = \phi(-K,\omega)(m).$$

We prove the functional equation 6.2.2 of v has a unique solution in TS.

Define a sequence $\{v^n\} \subset TS$ by induction: Let $v^0 \equiv 0$ and

$$\begin{aligned} & v^{n+1}(\omega,m)(x^u) \\ = \; & [D_1 g^c(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)v^n(\theta^{-K}\omega,m')(\xi^u) + D_2 g^c(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)] \\ & [D_1 g^u(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)v^n(\theta^{-K}\omega,m')(\xi^u) + D_2 g^u(h(\theta^{-K}\omega,m')(\xi^u),\xi^u)]^{-1} \end{aligned} \quad (6.2.3)$$

where

$$x^u = g^u(h(\theta^{-K}\omega,m')(\xi^u),\xi^u), \;\; m' = \phi(-K,\omega)(m).$$

We have the following two lemmas, the proof of which follow exactly the same as we did in section 5.1.

Lemma 6.2.1. $||v^n|| < \delta$ *for all* n.

Lemma 6.2.2. $||v^{n+1} - v^n|| \leq \lambda ||v^n - v^{n-1}||$ *for some* $0 < \lambda < 1$.

Hence, v^n converge to the unique solution v of equation 6.2.2. By the uniform convergence, $v(m,\omega)(x^u)$ is continuous in x^u and m.

The next lemma states that Proposition 6.2.1 holds for the case $k = 1$.

Lemma 6.2.3. *$Dh(m,\omega)(x^u) = v(m,\omega)(x^u)$.*

Proof. For a fixed $\omega \in \Omega$, we define an increasing function $\varrho_\omega : (0,1) \to \mathbb{R}$,

$$\varrho_\omega(a)$$
$$= \max_{m\in\tilde{\mathcal{M}}(\omega)} \sup_{x^u,x^{u\prime}\in E^u(m,\omega),0<||x^u-x^{u\prime}||<a} \frac{||h(m,\omega)(x^{u\prime}) - h(m,\omega)(x^u) - v(m,\omega)(x^u)(x^{u\prime} - x^u)||}{||x^{u\prime} - x^u||}.$$

Note that ϱ_ω is bounded by 2δ.

We want to show $\varrho_\omega(a) \to 0$ as $a \to 0$. To prove this, we claim

Claim: *$\varrho_\omega(a)$ satisfies*

$$\varrho_\omega(a) \leq \alpha\varrho_{\omega'}(\kappa a) + r(\theta^{-K}\omega, a)$$

for small a, where $r(\theta^{-K}\omega, a)$ is a decreasing function approaching to zero as $a \to 0$ uniformly with respect to $\omega \in \Omega$ and $0 \leq \alpha < 1$, $\kappa > 1$.

The proof of the claim follows exactly the same as we did in proposition 5.1.2.

Replace successively a by $a\kappa^{-1}, a\kappa^{-2}, \cdots, a\kappa^{-n}$ and ω by $\theta^K\omega, \theta^{2K}\omega, \cdots, \theta^{nK}\omega$ respectively and weight the terms with $1, \frac{1}{\alpha}, \cdots, \frac{1}{\alpha^{n-1}}$ and add them together to get:

$$\frac{1}{\alpha^{n-1}}\varrho_{\theta^{nK}\omega}(a\kappa^{-n}) \leq \alpha\varrho_\omega(a) + (1 + \frac{1}{\alpha} + \cdots + \frac{1}{\alpha^{n-1}}) \sup_\omega \max_{0\leq t\leq a} r(\omega, t).$$

Then since $\varrho_\omega(a) \leq 2\delta$, we have

$$\varrho_{\theta^{nK}\omega}(a\kappa^{-n}) \leq 2\alpha^n\delta + \frac{1}{1-\alpha} \sup_\omega \max_{0\leq t\leq a} r(\omega, t).$$

Since ω is arbitrary, we get

$$\varrho_\omega(a\kappa^{-n}) \leq 2\alpha^n\delta + \frac{1}{1-\alpha} \sup_\omega \max_{0\leq t\leq a} r(\omega, t).$$

It follows that $\varrho_\omega(a) \to 0$ as $a \to 0$.

So $h(m,\omega)(x^u)$ is C^1 in x^u and $Dh(m,\omega)(x^u)$ is continuous in the base point m. □

Lemma 6.2.4. *$D^{k-1}v(m,\omega)(x^u)$ exists for $2 \le k \le r$ and is continuous in x^u and m. Moreover $D^k h(m,\omega)(x^u) = D^{k-1}v(m,\omega)(x^u)$.*

Proof. By induction, it is easy to see $D^k v^n$ is a Cauchy sequence in the corresponding space, which implies the uniform convergence of $D^k v^n(m,\omega)$. Since v^n converge to v, we have that $D^k v(m,\omega)(x^u)$ exists and equals the uniform limit of $D^k v^n(m,\omega)(x^u)$. □

Combining Lemma 6.2.3 and Lemma 6.2.4 together gives Proposition 6.2.1. □

Proposition 6.2.2. *The graph of $h(\omega,m)$ is tangent to $E(\omega,m)$ at m.*

Proof. It is equivalent to show $Dh(\omega,m)(x^u)|_{x^u=0} = 0$. Since

$$|h(\omega,m)(x^u) - h(\omega,m)(\bar{x}^u)| \le \delta |x^u - \bar{x}^u|$$

we have

$$|Dh(\omega,m)(0)| \le \delta$$

for arbitrary $\delta > 0$. It follows that $Dh(\omega,m)(0) = 0$. □

6.2.2 Smoothness about Base Point. In this subsection, we prove the fiber changes C^{r-1} smoothly as the base point on the center manifold changes. We will introduce a new coordinate system, in which we prove the smoothness in the base point. The idea of our proof is the same as we did in the last subsection.

Proposition 6.2.3. *$h(\omega,m)$ is C^{r-1} in m for $m \in \tilde{\mathcal{M}}$.*

Proof. We need to prove the random C^0 manifold $\Sigma = \{\Sigma(\omega) : \omega \in \Omega\}$ defined by

$$\Sigma(\omega) = \{(m,p) | m \in \tilde{\mathcal{M}}(\omega), p \in \tilde{\mathcal{W}}^{uu}(\omega,m)\}$$

is a C^{r-1} submanifold of $\tilde{\mathcal{M}} \times \tilde{\mathcal{W}} = \{\tilde{\mathcal{M}}(\omega) \times \tilde{\mathcal{W}}(\omega) : \omega \in \Omega\}$.

Let $\tilde{\mathcal{M}}_* = \{\tilde{\mathcal{M}}_*(\omega) : \omega \in \Omega\}$ be the diagonal embedding of $\tilde{\mathcal{M}}$ to $\tilde{\mathcal{M}} \times \tilde{\mathcal{W}}^u$:

$$\tilde{\mathcal{M}}_*(\omega) := \{(m, m) | m \in \tilde{\mathcal{M}}(\omega)\}.$$

Then, $\tilde{\mathcal{M}}_*$ is a compact connected C^r random invariant manifold.

We embed $T\tilde{\mathcal{W}}^u(\omega)$ into $T(\tilde{\mathcal{M}}(\omega) \times \tilde{\mathcal{W}}^u(\omega))|\tilde{\mathcal{M}}_*(\omega)$ as follows. Let $\gamma(t)$ be a curve in $\tilde{\mathcal{W}}^u(\omega)$ such that $\gamma(0) = m \in \tilde{\mathcal{M}}(\omega)$. Then $\gamma^*(t) := (m, \gamma(t))$ is a curve in $\tilde{\mathcal{M}}(\omega) \times \tilde{\mathcal{W}}^u(\omega)$ such that $\gamma^*(0) = (m, m)$. The mapping $\gamma \to \gamma^*$ induces an injection $T\tilde{\mathcal{W}}^u(\omega)|\tilde{\mathcal{M}}(\omega) \to T(\tilde{\mathcal{M}}(\omega) \times \tilde{\mathcal{W}}^u(\omega))|\tilde{\mathcal{M}}_*(\omega)$. Let E^u_* and E^c_* be the image of E^u and $T\tilde{\mathcal{M}}$ under this injection, respectively. Then, we get the following splitting:

$$T(\tilde{\mathcal{M}}(\omega) \times \tilde{\mathcal{W}}^u(\omega))|\tilde{\mathcal{M}}_*(\omega) = T\tilde{\mathcal{M}}_*(\omega) \oplus E^u_*(\omega) \oplus E^c_*(\omega).$$

The embedding we have here is based on [F3].

To show proposition 6.2.3, we need local coordinates and partitions of unity along the line of section 4 and 5. We first present the proof in the case that $\tilde{\mathcal{M}}(\omega) \times \tilde{\mathcal{W}}^u(\omega)$ is a subset of a torus and $E^u_*(\omega)$ and $E^c_*(\omega)$ are trivial bundles. In other words, we have global coordinates. Later, we explain how this proof could be modified to fit the general case. The idea for this subsection is the same as it was in the previous subsection.

Denote the global coordinates by $(x^{cc}, x^u, x^c) \in T\tilde{\mathcal{M}}_*(\omega) \times E^u_*(\omega)(\epsilon_5) \times E^c_*(\omega)(\epsilon_5)$. The induced random flow $\phi^*(K, \omega) := (\phi(K, \omega), \phi(K, \omega))$ on $\tilde{\mathcal{M}}(\omega) \times \tilde{\mathcal{W}}^u(\omega)$ has the form:

$$(x^{cc}, x^u, x^c) \to (j^{cc}(x^{cc}, x^u, x^c), j^u(x^{cc}, x^u, x^c), j^c(x^{cc}, x^u, x^c)). \tag{6.2.4}$$

From the lemma 5.5.1 we have

$$||[D_2 j^u(x^{cc}, 0, 0)]^{-1}|| < \frac{1}{4}, \tag{6.2.5}$$

$$||[D_2 j^u(x^{cc},0,0)]^{-1}||||D_3 j^c(x^{cc},0,0)||^k < \frac{1}{4}, \tag{6.2.6}$$

$$||[D_1 j^{cc}(x^{cc},0,0)]^{-1}||^{k-1}||[D_2 j^u(x^{cc},0,0)]^{-1}||||D_3 j^c(x^{cc},0,0)|| < \frac{1}{4} \tag{6.2.7}$$

for $1 \leq k \leq r$. So for ϵ_5 small enough, we have

$$||[D_2 j^u(x^{cc},x^u,x^c)]^{-1}|| < \frac{1}{3}, \tag{6.2.8}$$

$$||[D_2 j^u(x^{cc},x^u,x^c)]^{-1}||||D_3 j^c(x^{cc},x^u,x^c)||^k < \frac{1}{3}, \tag{6.2.9}$$

$$||[D_1 j^{cc}(x^{cc},x^u,x^c)]^{-1}||^{k-1}||[D_2 j^u(x^{cc},x^u,x^c)]^{-1}||||D_3 j^c(x^{cc},x^u,x^c)|| < \frac{1}{3}. \tag{6.2.10}$$

From the invariance of $\tilde{\mathcal{M}}_*(\omega)$, we have

$$j^u(x^{cc},0,0) = 0, \;\; j^c(x^{cc},0,0) = 0 \tag{6.2.11}$$

and so

$$D_1 j^u(x^{cc},0,0) = 0, \;\; D_1 j^c(x^{cc},0,0) = 0. \tag{6.2.12}$$

By the invariance of $E^u_*(\omega)$ we get

$$D_2 j^c(x^{cc},0,0) = 0 \tag{6.2.13}$$

and

$$D_1 D_2 j^c(x^{cc},0,0) = 0. \tag{6.2.14}$$

Hence, for any small γ, choosing ϵ_5 small enough we have

$$|j^u(x^{cc},x^u,x^c)| < \gamma, \;\; |j^c(x^{cc},x^u,x^c)| < \gamma, \tag{6.2.15}$$

$$||D_1 j^u(x^{cc},x^u,x^c)|| < \gamma, \;\; ||D_1 j^c(x^{cc},x^u,x^c)|| < \gamma, \tag{6.2.16}$$

$$||D_2 j^c(x^{cc}, x^u, x^c)|| < \gamma, \;\; ||D_1 D_2 j^c(x^{cc}, x^u, x^c)|| < \gamma. \tag{6.2.17}$$

Moreover, we may suppose all first and second partial derivatives of j^{cc}, j^u, j^c are bounded by some $Q > 0$. Let this Q be large enough but finite such that it is the upper bound of all bounded terms which may come later.

We represent $\Sigma(\omega)$ by

$$h^*(\omega) : T\tilde{\mathcal{M}}_*(\omega, m^*) \times E^u_*(\omega, m^*) \to E^c_*(\omega, m^*).$$

From Lemma 6.2.1, $h^*(\omega)(x^{cc}, x^u)$ is C^r in x^u and $D_2^k h^*(\omega)(x^{cc}, x^u)$ is C^0 in x^{cc} for $0 \le k \le r$, and the following hold

$$h^*(\omega)(x^{cc}, 0) = 0, \; D_2 h^*(\omega)(x^{cc}, 0) = 0, \; ||D_2 h^*(\omega)(x^{cc}, x^u)|| \le \delta,$$

$$||D_2 h^*(\omega)(x^{cc}, x^u)|| = ||D_2 h^*(\omega)(x^{cc}, x^u) - D_2 h^*(\omega)(x^{cc}, 0)|| \le Q(\omega)|x^u|.$$

By the uniform C^r closeness of all $\tilde{\mathcal{M}}(\omega) \times \tilde{\mathcal{W}}^u(\omega)$ to each other, the uniform closeness of $\phi(K, \omega)$ to the deterministic $\psi(K)$ and the compactness, we get uniform estimate

$$||D_2 h^*(\omega)(x^{cc}, x^u)|| \le Q|x^u|. \tag{6.2.18}$$

From the invariance of $\Sigma(\omega)$ we obtain

$$h^*(\omega)(x^{cc}, x^u) = j^c(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)), \tag{6.2.19}$$

where

$$x^{cc} = j^{cc}(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)),$$

$$x^u = j^u(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)).$$

Taking the derivative with respect to ξ^{cc} formally on both side of (6.2.19) gives

$$\begin{aligned} & D_1h^*(\omega)[D_1j^{cc} + D_3j^{cc}D_1h^*(\theta^{-K}\omega)] + D_2h^*(\omega)[D_1j^u + D_3j^u D_1h^*(\theta^{-K}\omega)] \\ = \;& D_1j^c + D_3j^c D_1h^*(\theta^{-K}\omega), \end{aligned} \tag{6.2.20}$$

where the argument of h^*, j are clear from the context.

For any fixed $\omega \in \Omega$, let $v^*(\omega) \in C^0(T\tilde{\mathcal{M}}_*(\omega) \times E^u_*(\omega), L(T\tilde{\mathcal{M}}_*(\omega), E^c_*(\omega)))$ and $v^* = \{v^*(\omega) : \omega \in \Omega\}$. Define

$$||v^*||_{\text{LIP}} = \sup_{\omega} \sup_{x^u \neq 0} \frac{||v^*(\omega)(x^{cc}, x^u)||}{|x^u|}. \tag{6.2.21}$$

Let DS be the space of all such v. Define the norm on DS by (6.2.21). Under this norm, DS is a complete metric space. We prove the following functional equation of $v^* \in DS$ has a unique solution:

$$\begin{aligned} & v^*(\omega)(x^{cc}, x^u)[D_1j^{cc}(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)) \\ & \qquad + D_3j^{cc}(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u))v^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)] \\ & + D_2h^*(\omega)(x^{cc}, x^u)[D_1j^u(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)) \\ & \qquad + D_3j^u(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u))v^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)] \\ & = D_1j^c(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)) + D_3j^c(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u))v^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u), \end{aligned} \tag{6.2.22}$$

where

$$x^{cc} = j^{cc}(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)),$$

$$x^u = j^u(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{cc}, \xi^u)).$$

We follow the approach of Lemma 6.2.3.

Define a sequence $\{v_*^n\} \subset DS$ by

$$v_*^0 = 0,$$
$$v_*^{n+1}(\omega) = [D_1 j^c + D_3 j^c D_1 h^*(\theta^{-K}\omega) - D_2 h^*(\omega)(D_1 j^u + D_3 j^u v_*^n(\theta^{-K}\omega))]$$
$$[D_1 j^{cc} + D_3 j^{cc} v_*^n(\theta^{-K}\omega)]^{-1},$$

where the arguments of $h^*(\omega)$ and v_*^{n+1} are (x^{cc}, x^u), the arguments of $h^*(\theta^{-K}\omega)$ and v_*^n are (ξ^{cc}, ξ^u), the arguments of $D_1 j^{cc}, D_3 j^{cc}, D_1 j^u, D_3 j^u, D_1 j^c, D_3 j^c$ are $(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{xx}, \xi^u))$.

We prove $\{v_*^n\}$ is a Cauchy sequence in DS.

Lemma 6.2.5. $||v_*^{n+1}||_{LIP} \leq \delta$.

Proof. The proof of this lemma is straightforward following from (6.2.10), (6.2.15), (6.2.16), (6.2.17) and (6.2.18). □

Lemma 6.2.6. $||v_*^{n+1} - v_*^n||_{LIP} < \lambda ||v_*^n - v_*^{n-1}||_{LIP}$ *for some* $0 < \lambda < 1$.

Proof. First, we note that

$$\begin{aligned} & v_*^{n+1}(\omega)(x^{cc}, x^u)[D_1 j^{cc} + D_3 j^{cc} v_*^n(\theta^{-K}\omega)(\xi^{cc}, \xi^u)] \\ & \quad + D_2 h^*(\omega)[D_1 j^u + D_3 j^u v_*^n(\theta^{-K}\omega)(\xi^{cc}, \xi^u)] \\ & = D_1 j^c + D_3 j^c v_*^n(\theta^{-K}\omega)(\xi^{cc}, \xi^u), \end{aligned} \tag{6.2.23}$$

$$\begin{aligned} & v_*^n(\omega)(x^{cc}, x^u)[D_1 j^{cc} + D_3 j^{cc} v_*^{n-1}(\theta^{-K}\omega)(\xi^{cc}, \xi^u)] \\ & \quad + D_2 h^*(\omega)[D_1 j^u + D_3 j^u v_*^{n-1}(\theta^{-K}\omega)(\xi^{cc}, \xi^u)] \\ & = D_1 j^c + D_3 j^c v_*^{n-1}(\theta^{-K}\omega)(\xi^{cc}, \xi^u), \end{aligned} \tag{6.2.24}$$

where the arguments of $D_1 j^{cc}$, $D_3 j^{cc}$, $D_1 j^u$, $D_3 j^u$, $D_1 j^c$, $D_3 j^c$ are $(\xi^{cc}, \xi^u, h^*(\theta^{-K}\omega)(\xi^{xx}, \xi^u))$.

From (6.2.23)−(6.2.24), we get

$$\begin{aligned}
&v_*^{n+1}(\omega)(x^{cc},x^u) - v_*^n(\omega)(x^{cc},x^u)\\
&= [D_3 j^c - v_*^n(\theta^{-K}\omega)(\xi^{cc},\xi^u) D_3 j^{cc} - D_2 h^*(\omega)(x^{cc},x^u) D_3 j^u]\\
&\quad \cdot [v_*^n(\theta^{-K}\omega)(\xi^{cc},\xi^u) - v_*^{n-1}(\theta^{-K}\omega)(\xi^{cc},\xi^u)][D_1 j^{cc} + D_3 j^{cc} v_*^n(\theta^{-K}\omega)(\xi^{cc},\xi^u)]^{-1}.
\end{aligned} \tag{6.2.25}$$

We also have

$$\begin{aligned}
|x^u| &= |j^u(\xi^{cc},\xi^u,h(\theta^{-K}\omega)(\xi^{cc},\xi^u))|\\
&= |j^u(\xi^{cc},\xi^u,h(\theta^{-K}\omega)(\xi^{cc},\xi^u)) - j^u(\xi^{cc},0,h(\theta^{-K}\omega)(\xi^{cc},0))|\\
&\geq (||(D_2 j^u)^{-1}||^{-1} - \beta)|\xi^u| - Q\delta|\xi^u|.
\end{aligned} \tag{6.2.26}$$

From (6.2.10),(6.2.25) and (6.2.26), it is easy to get

$$\frac{||v_*^{n+1}(\omega)(x^{cc},x^u) - v_*^n(\omega)(x^{cc},x^u)||}{|x^u|} < \frac{1}{2}\frac{||v_*^n(\theta^{-K}\omega)(\xi^{cc},\xi^u) - v_*^{n-1}(\theta^{-K}\omega)(\xi^{cc},\xi^u)||}{|\xi^u|},$$

which gives us

$$||v_*^{n+1} - v_*^n||_{\mathrm{LIP}} \leq \frac{1}{2}||v_*^n - v_*^{n-1}||_{\mathrm{LIP}}.$$

□

Let v^* be the uniform limit of $\{v_*^n\}$. We prove that v^* is the partial derivative $D_1 h^*$ of h^*. Along the line of Lemma 6.2.3, we get that v^* is the partial derivative $D_1 h^*$ of h^*. We already have that $D_2 h^*$ exists and is C^0. So h^* is C^1 jointly in (m, x^u).

The rest of proposition 6.2.3 is straightforward, along the line of Lemma 6.2.1. So we complete the proof of proposition 6.2.3.

□

Remark. $h(\omega, m, x^u)$ is actually C^{r-1} jointly in (m, x^u).

Besides all the above smoothness properties, the invariant foliation has the following continuous property:

Proposition 6.2.4. *$\tilde{\mathcal{W}}^{uu}(\theta^t\omega, x)$ is C^0 in t for any fixed (ω, x).*

Proof. Suppose $m(0)$ is a point on the fiber $\tilde{\mathcal{W}}^{uu}(\omega, m)$ represented by $x^u(0)+h(\omega, m, x^u(0))$ in local coordinates. From the invariance property of the foliation, we have $\phi(t, \omega, m(0)) \in \tilde{\mathcal{W}}^{uu}(\theta^t\omega, \phi(t, \omega, m)$. So $\phi(t, \omega, m(0))$ can be represented in local coordinates by

$$x^u(t) + h(\theta^t\omega, \phi(t, \omega, m), x^u(t)).$$

Since $\phi(t, \omega, m(0))$ is C^0 in t, $x^u(t) + h(\theta^t\omega, \phi(t, \omega, m), x^u(t))$ is C^0 in t. Then since $x^u(t)$ and $\phi(t, \omega, m)$ are both C^0 in t, it must follow that $h(\theta^t\omega, m, x^u)$ is C^0 in t, which give us the conclusion of the proposition. $\square$

Remark. From the proof of the proposition, we conclude that the smoothness of the fibers $\tilde{\mathcal{W}}^{uu}(\theta^t\omega, x)$ in t for any fixed (ω, x) is the same as the smoothness of any orbit of the random system.

General Case. If no global chart exists, we construct local charts on $\tilde{\mathcal{M}} \times \tilde{\mathcal{W}}$ near $\tilde{\mathcal{M}}_*$ using a similar method as we did in section 3. Let $\mathcal{M} \times \mathcal{W}$ and $\mathcal{M}_*$ be the deterministic counterparts of $\tilde{\mathcal{M}} \times \tilde{\mathcal{W}}$ and $\tilde{\mathcal{M}}_*$ respectively. As in lemma 6.1.1, a similar lemma holds. In other words, lemma 6.1.1 persists if we replace $\mathcal{M}$ and $\mathcal{W}$ there by $\mathcal{M}_*$ and $\mathcal{M} \times \mathcal{W}$ respectively. Then the compactness of $\mathcal{M}_*$ gives us a local chart on $\mathcal{M} \times \mathcal{W}$ near $\mathcal{M}_*$. Then the uniform C^r closeness of $\tilde{\mathcal{M}}(\omega) \times \tilde{\mathcal{W}}(\omega)$ to $\mathcal{M} \times \mathcal{W}$ induces a local chart on $\tilde{\mathcal{M}} \times \tilde{\mathcal{W}}$ near $\tilde{\mathcal{M}}_*$. The method of induce is exactly the same as we did in section 3, after lemma 6.1.1.

In the local charts, the induced random flow $\phi^*(K, \omega)$ has exactly the same form as (6.2.4):

$$(x^{cc}, x^u, x^c) \to (j^{cc}(x^{cc}, x^u, x^c), j^u(x^{cc}, x^u, x^c), j^c(x^{cc}, x^u, x^c)),$$

with a different understanding that j depends on $m^* \in \tilde{\mathcal{M}}_*(\omega)$ as well. All the estimates are uniform about m and ω. So the proof is also adapted to the case of local charts.

6.3 Measurability of the Fibers

In this section, we prove that the fibers in the unique family $\{\tilde{\mathcal{W}}^{uu}(\omega, m) : \omega \in \Omega, m \in \tilde{\mathcal{M}}(\omega)\}$ change in a measurable way as ω changes.

What we need to do is to prove the representation $h^*(\omega, x^{cc}, x^u)$ of the unique family is measurable. The major difficult is that the coordinate system we used to construct the unique family depends on ω. In other words, the coordinates x^{cc} and x^u of $h^*(\omega, x^{cc}, x^u)$ depend on ω. It is very hard to prove the measurability of h^*. To overcome this problem, we use the measurability and smoothness of $\tilde{\mathcal{M}}_*(\omega), E_*^u(\omega)$ and $E_*^c(\omega)$ to construct ω-independent coordinates in $\mathbb{R}^m \oplus \mathbb{R}^l \oplus \mathbb{R}^m$.

Lemma 6.3.1. *There exists a coordinate system in which h^* has a new form $\tilde{h}^*(\omega, y^{cc}, y^u)$ with the following properties: y^{cc} and y^u are independent of ω; $\tilde{h}^*(\omega, y^{cc}, y^u)$ is C^r in y^u and C^{r-1} in (y^{cc}, y^u) jointly.*

Proof. Fix any $\omega_0 \in \Omega$, let $m_0(\omega_0) \in \tilde{\mathcal{M}}_*(\omega_0)$. Since for different $\omega \in \Omega$, all $\tilde{\mathcal{M}}_*(\omega)$ are C^r diffeomorphic to each other, we get a set of points $m_0(\omega) \in \tilde{\mathcal{M}}_*(\omega)$ corresponding to $m_0(\omega_0) \in \tilde{\mathcal{M}}_*(\omega_0)$. Then from the so called measurable selection, there exist measurable bases

$$\{\bar{e}_1^u(\omega, m_0(\omega)), \cdots, \bar{e}_l^u(\omega, m_0(\omega))\} \text{ and } \{\bar{e}_1^c(\omega, m_0(\omega)), \cdots, \bar{e}_m^c(\omega, m_0(\omega))\},$$

of the tangent spaces $E_*^u(\omega, m_0(\omega))$ and $E_*^c(\omega, m_0(\omega))$.

From C^{r-1} smoothness of $\tilde{\mathcal{M}}_*(\omega_0)$, $E_*^u(\omega_0)$ and $E_*^c(\omega_0)$, there exist bases of the bundles $E_*^u(\omega_0)$ and $E_*^c(\omega_0)$:

$$\{e_1^u(\omega_0, m(\omega_0)), \cdots, e_l^u(\omega_0, m(\omega_0))\} \text{ and } \{e_1^c(\omega_0, m(\omega_0)), \cdots, e_m^c(\omega_0, m(\omega_0))\},$$

which are C^{r-1} in $m(\omega_0)$ and satisfy:

$$e^i_k(\omega_0, m_0(\omega_0)) = \bar{e}^i_k(\omega_0, m_0(\omega_0)),$$

for $i = u(c)$ and $k = 1, \cdots, l(k)$.

For different but fixed $\omega_1 \in \Omega$, by the same reasoning, we get different C^{r-1} bases of the bundles $E^u_*(\omega_1)$ and $E^c_*(\omega_1)$:

$$\{e^u_1(\omega_1, m(\omega_1)), \cdots, e^u_l(\omega_1, m(\omega_1))\} \text{ and } \{e^c_1(\omega_1, m(\omega_1)), \cdots, e^c_m(\omega_1, m(\omega_1))\}.$$

Do this kind of construction for all $\omega \in \Omega$, then we get

$$\{e^u_1(\omega, m(\omega)), \cdots, e^u_l(\omega, m(\omega))\} \text{ and } \{e^c_1(\omega, m(\omega)), \cdots, e^c_m(\omega, m(\omega))\}.$$

Claim: *These bases are jointly measurable about $(\omega, m(\omega))$.*

Notice that for fixed m_0, they are measurable about ω. From the C^{r-1} smoothness, the measurability of the bundle and the C^{r-1} diffeomorphism of the bundles to each other and to the deterministic counterpart, for any fixed m_1(the definition of $m_1(\omega)$ are the same to that of $m_0(\omega)$), the bases at $(\omega, m_1(\omega))$ are measurable about ω because they could be viewed as a composition of a measurable C^{r-1} transition diffeomorphism, say $T_{m_0,m_1}(\omega, \cdot)$, with the bases at $(\omega, m_0(\omega))$. So they are measurable about ω and C^{r-1} about m. Then the claim follows.

Therefore, there exist a neighborhood of 0 in $\mathbb{R}^m \oplus \mathbb{R}^l \oplus \mathbb{R}^m$ and a map

$$T(\omega, \cdot) \,:\, \tilde{\mathcal{M}}(\omega) \times \tilde{\mathcal{W}}^u(\omega) \to D$$

such that $T(\omega, \cdot)$ is a C^{r-1} diffeomorphism for each ω and $T(\cdot, z)$, $T^{-1}(\cdot, z)$ are measurable for each $z \in \tilde{\mathcal{M}}(\omega) \times \tilde{\mathcal{W}}^u(\omega)$. Moreover, points on $\tilde{\mathcal{M}}(\omega) = \{(m, m)|m \in \tilde{\mathcal{M}}\}$ are mapped to $D \cap \mathbb{R}^m \times \{0\} \times \{0\}$ and $e^u_i(\omega, m)$ are mapped to unit vectors in the e_i directions in $\mathbb{R}^l$ and $e^c_j(\omega, m)$ are mapped to unit vectors in the e_j directions in $\mathbb{R}^m$.

h^* has the form $\tilde{h}^*(\omega, y^{cc}, y^u)$ in this new coordinate system D. Obviously, all the properties listed in the lemma are satisfied by $\tilde{h}^*$. □

Our next step is to prove that $\tilde{h}^*(\omega, y^{cc}, y^u)$ is measurable. The following lemma is from [CDLS]:

Lemma 6.3.2. *For a Polish space H, there is a mapping*

$$P : \Omega \times H \to H,$$

satisfying that $P(\omega, \cdot)$ is a homeomorphism for any $\omega \in \Omega$, and $P(\cdot, x), P^{-1}(\cdot, x)$ are measurable for any $x \in H$. If ϕ is a continuous random dynamical system, then so is ϕ' defined by

$$\phi'(t, \omega, x) := P(\theta^t \omega, \phi(t, \omega, P^{-1}(\omega, x)).$$

Recall that, $\phi^*(t, \omega, x)$ is the C^r random flow in the original ω-dependent coordinate system. Under the new ω-independent coordinate system, $\phi^*(t, \omega, x)$ has the form

$$\tilde{\phi}^*(t, \omega, y) = T(\theta^t \omega, \phi^*(t, \omega, T^{-1}(\omega) y)).$$

By the above lemma, $\tilde{\phi}^*(t, \omega, y)$ is a C^{r-1} random flow.

Next, we prove $\tilde{h}^*(\omega, y^{cc}, y^u)$ is measurable.

Proposition 6.3.1. *$\tilde{h}^*(\omega, y^{cc}, y^u)$ is C^{r-1} in (y^{cc}, y^u) and measurable in ω, so is measurable in (ω, y^{cc}, y^u).*

Proof. Recall that, in section 3, we constructed the invariant foliation by finding out the unique fixed point of a contraction mapping G on S_δ (graph transform). So the invariant foliation is the limit of any starting foliations(starting element of S_δ) under the mapping of G.

Suppose $h^0(\omega, x, m)$ is the representation of a starting foliation. After one iteration under the graph transform, $h^0(\omega, x, m)$ becomes $h^1(\omega, x, m)$. By Proposition 6.1.1, we have the relationship between $h^0(\omega, x, m)$ and $h^1(\omega, x, m)$: for $h^0 \in S_\delta,\ \omega \in \Omega,\ m \in \tilde{\mathcal{M}}(\omega)$,

$$h^1(m, \omega)(x^u) \ = \ g^c(h^0(\theta^{-K}\omega, m')(\xi^u), \xi^u)$$

where

$$x^u \ = \ g^u(h(\theta^{-K}\omega, m')(\xi^u), \xi^u),$$

$$m' \ = \ \phi(-K, \omega)(m).$$

From the above we see that as long as $\phi(t, \omega, x)$ is measurable in ω and C^r in x, g^u and g^c above are measurable in ω and C^r in other coordinates. Therefore, h^1 has the same measurability and smoothness properties as h^0.

Now, we consider it in the new coordinate system. We note that $\tilde{h}^*(\omega, y^{cc}, y^u)$ is the limit of a sequence $\tilde{h}^*_n(\omega, y^{cc}, y^u)$ which is generated by iterating the graph of $\tilde{h}^*_0(\omega, y^{cc}, y^u)$ under the graph transform G^*, where G^* is generated by the random flow $\tilde{\phi}^*(t, \omega, y)$.

Because $\tilde{\phi}^*(t, \omega, y)$ is a C^{r-1} random flow, i.e., measurable in ω and C^{r-1} in y, as long as we take $\tilde{h}^*_0(\omega, y^{cc}, y^u) \equiv 0$, which is C^{r-1} in (y^{cc}, y^u) and measurable in ω, we get the C^{r-1} smoothness and ω-measurability of all the sequence $\tilde{h}^*_n(\omega, y^{cc}, y^u)$. Therefore, the limit $\tilde{h}^*(\omega, y^{cc}, y^u)$ is also measurable in ω.

On the other hand, since the change of the coordinate system is given by $T(\omega, \cdot)$ which is a C^{r-1} diffeomorphism for each ω, $\tilde{h}^*(\omega, y^{cc}, y^u)$ is C^{r-1}. Therefore, $\tilde{h}^*(\omega, y^{cc}, y^u)$ is measurable in (ω, y^{cc}, y^u). $\square$

Summing up the results of this section, we get the following

Proposition 6.3.2. *The unique family of fibers $\{\tilde{\mathcal{W}}^{uu}(\omega, m) : \omega \in \Omega, m \in \tilde{\mathcal{M}}(\omega)\}$ is a C^{r-1} family of C^r manifolds and the fibers in it change measurably. Moreover, $\tilde{\mathcal{W}}^{uu}(\theta^t\omega, x)$ is C^0 in t for any fixed (ω, x).*

6.4 Asymptotic Property

In this section, we prove that the points on unstable fiber $\tilde{\mathcal{W}}^{uu}(\omega, m)$ are equivalent in a certain asymptotic sense and characterize the invariant foliation. The technical difficult is the lack of a uniformly metric (distance) on the random unstable(stable) manifold. To overcome this, we again use the C^r diffeomorphism and C^r closeness of the random unstable manifold to the corresponding deterministic ones to induce the one we need.

Since we will not use the smoothness in the base point nor the measurability of the invariant foliation in this section, we will use the coordinate system used in section 6.1.

Suppose $\hat{d}(\omega)(\cdot, \cdot)$ is the geodesic distance on $\tilde{\mathcal{W}}^u(\omega)$: For any $m, m' \in \tilde{\mathcal{W}}^u(\omega)$, $d(m, m')$ is the infimum of the lengths of piecewise smooth rectifiable curves joining m and m', if any such curve exists. Otherwise $d(m, m') = \infty$. Let $d(\cdot, \cdot)$ be the geodesic distance on $\mathcal{W}^u$. Then d induces a distance $\tilde{d}(\omega)$ on $\tilde{\mathcal{W}}^u(\omega)$ in the following manner:

$$\tilde{d}(\omega)(m, m') := \inf\{\text{length of } c(t)\}$$

for $c(t), t \in [0, a]$ a piecewise smooth rectifiable curves joining $i^{-1}(\omega, m)$ and $i^{-1}(\omega, m')$ in $\mathcal{W}^u$. Since $\mathcal{W}^u$ and $\tilde{\mathcal{W}}^u(\omega)$ are uniformly C^1 close, $i(\omega)$ are uniformly C^1 close to Id, we conclude that $\tilde{d}(\omega)$ is uniformly equivalent to $\hat{d}(\omega)$, the geodesic distance on $\tilde{\mathcal{W}}^u(\omega)$. Under $\tilde{d}$, we have

$$\tilde{d}(\omega)(\Gamma(\omega, m, \nu), m) = |Di^{-1}(\omega, m)\nu|.$$

Since $Di(\omega, m)$, $Di(\omega, m)^{-1}$ are uniformly close to the identity matrix transformation. So we can define another uniformly equivalent distance $d(\omega)$ on $\tilde{\mathcal{W}}^u(\omega)$ such that

$$d(\omega)(\Gamma(\omega, m, \nu), m) = |\nu|.$$

We will use $d(\omega)$ to measure the distance on $\tilde{\mathcal{W}}^u(\omega)$. To save notation, we use d for all $d(\omega)$. We have the following proposition which characterizes the fiber $\tilde{\mathcal{W}}^{uu}(\omega, m)$.

Proposition 6.4.1. *Suppose $m, m' \in \tilde{\mathcal{M}}(\omega)$, $p \in \tilde{\mathcal{W}}^{uu}(\omega, m)$ and $p' \in \tilde{\mathcal{W}}^{uu}(\omega, m')$, then*

(i) $d(\phi(-t,\omega)(p), \phi(-t,\omega)(m)) \to 0$ exponentially as $t \to \infty$;

(ii) If $m \neq m'$ and $d(\phi(-t,\omega)(m), \phi(-t,\omega)(m')) \to 0$ as $t \to \infty$, then

$$\frac{d(\phi(-t,\omega)(p), \phi(-t,\omega)(m))}{d(\phi(-t,\omega)(p'), \phi(-t,\omega)(m))} \to 0 \text{ as } t \to \infty,$$

$$\frac{d(\phi(-t,\omega)(p), \phi(-t,\omega)(m))}{d(\phi(-t,\omega)(p), \phi(-t,\omega)(m'))} \to 0 \text{ as } t \to \infty;$$

(iii) $\tilde{\mathcal{W}}^{uu}(\omega, m) \cap \tilde{\mathcal{W}}^{uu}(\omega, m') = \emptyset$ if $m \neq m'$;

(iv) $\tilde{\mathcal{W}}^{u}(\omega) = \cup_{m \in \tilde{\mathcal{M}}(\omega)} \tilde{\mathcal{W}}^{uu}(\omega, m)$.

Proof. From lemma 5.5.1 we have

$$||D\phi(-K,\omega)(m)|E^u(\omega)|| < \frac{1}{4}a_1^K$$

and

$$||D((\phi|\tilde{\mathcal{M}}(\theta^{-K}\omega))(K, \theta^{-K}\omega))\phi(-K,\omega)(m)||^r ||D\phi(-K,\omega)(m)|E^u(\omega)|| < \frac{1}{4}$$

which yield that for some $a_1 < a_2 < 1$,

$$||D\phi(-K,\omega)(m)|E^u(\omega)|| < \frac{1}{4}a_1^K < \frac{1}{4}a_2^K,$$

$$||D((\phi|\tilde{\mathcal{M}}(\theta^{-K}\omega))(K, \theta^{-K}\omega))\phi(-K,\omega)(m)||^k ||D\phi(-K,\omega)(m)|E^u(\omega)|| < \frac{1}{4}a_2^K,$$

where k is no larger than r.

Just as we get the estimates (6.1.1), (6.1.2) and (6.1.6), we get similar estimates:

$$||[D_2 g^u(\xi^c, \xi^u)]^{-1}|| < \frac{1}{3}a_1^K, \tag{6.4.1}$$

$$||[D_2 g^u(\xi^c, \xi^u)]^{-1}|| ||D_1 g^c(\bar{\xi}^c, \bar{\xi}^u)||^k < \frac{1}{3}a_2^K, \tag{6.4.2}$$

$$|g^u(h(\theta^{-K}\omega, x')(\xi^u), \xi^u)| > 2\frac{|\xi^u|}{a_1^K}, \tag{6.4.3}$$

where $m_1 = \phi(-K, \omega)(m)$. We have

$$d(\Gamma(\omega, m, \nu), m) = |\nu|$$

for $\omega \in \Omega$, $m \in \tilde{\mathcal{M}}(\omega)$ and $|\nu| < \epsilon_4$. If δ is sufficiently small, and $x^u \in E^u(\omega, m)(\epsilon_4)$, $x^c \in T\tilde{\mathcal{M}}(\omega, m)(\epsilon_4)$ such that $|x^c| \le \delta|x^u|$, for all $\omega \in \Omega$ and $m \in \tilde{\mathcal{M}}(\omega, m)$, then

$$\frac{3}{4}|x^u| \le d(\Gamma(\omega, m, (x^u, x^c), m) \le \frac{4}{3}|x^u|. \tag{6.4.4}$$

Moreover, without the condition that $|x^c| \le \delta|x^u|$, there is a constant c_5 such that

$$d(\Gamma(\omega, m, (x^u, x^c), m) \ge c_5|x^c|.$$

This is because the angle between $E^u(\omega, m)$ and $T\tilde{\mathcal{M}}(\omega, m)$ is bounded away from zero uniformly.

To prove part (i) and (ii), it is enough to let t approach to infinity through multiples of K.

(i) Let $p = \Gamma(\omega, m, (h(\omega, m)(x^u), x^u))$, $\phi(-K, \omega, p) = \Gamma(\theta^{-K}\omega, m_1, (h(\theta^{-K}\omega, m_1)(\xi^u), \xi^u))$. Then we have $x^u = g^u(h(\theta^{-K}\omega, m_1)(\xi^u), \xi^u)$ and by (6.4.3), (6.4.4),

$$\begin{aligned} d(\phi(-K, \omega)(p), \phi(-K, \omega)(m)) &\le \frac{4}{3}|\xi^u| \le \frac{4}{3}\cdot\frac{1}{2}a_1^K g^u(h(\theta^{-K}\omega, m_1)(\xi^u), \xi^u) \\ &= \frac{2}{3}a_1^K|x^u| \le \frac{8}{9}a_1^K d(p, m), \end{aligned}$$

which leads the conclusion of part (i).

(ii) There exists N large enough such that for $n \ge N$, $d(\phi(-nK, \omega)(m), \phi(-nK, \omega)(m'))$

are so small that $\phi(-nK,\omega)(m')$ can be represented in local coordinates near

$$m_n := \phi(-nK,\omega)(m)$$

as $(\hat{\xi}^c_n, \hat{\xi}^u_n)$, while $\phi(-nK,\omega)(p)$ is represented as (ξ^c_n, ξ^u_n) where $\xi^c_1 = \xi^c$, $\xi^u_1 = \xi^u$, $\hat{\xi}^c_1 = \hat{\xi}^c$, $\hat{\xi}^u_1 = \hat{\xi}^u$ and $\xi^c_0 = x^c$, $\xi^u_0 = x^u$ $\hat{\xi}^c_0 = \hat{x}^c$, $\hat{\xi}^u_0 = \hat{x}^u$ as we used before.

Without loss of generality, we may assume that for any $n \geq 0$, $\phi(-nK,\omega)(m')$ can be represented in local coordinates near m_n.

Since $m' \in \tilde{\mathcal{M}}(\omega)$ and $\tilde{\mathcal{M}}(\omega)$ is invariant under $\phi(-t,\omega)$, we have

$$|\hat{\xi}^c_n| \ > \ \delta|\hat{\xi}^u_n|$$

for any $n \geq 0$. Particularly,

$$|\hat{\xi}^c| \ > \ \delta|\hat{\xi}^u|.$$

Now

$$\begin{aligned}
|x^u| \ &= \ |g^u(h(\theta^{-K}\omega, m_1)(\xi^u), (\xi^u))| \\
&\geq \ |g^u(h(\theta^{-K}\omega, m_1)(\xi^u), (\xi^u)) - g^u(h(\theta^{-K}\omega, m_1)(\xi^u), 0)| \\
&\quad -|g^u(h(\theta^{-K}\omega, m_1)(\xi^u), 0) - g^u(0,0)| \\
&\geq \ [||[D_2 g^u(\xi^c, \xi^u)]^{-1}||^{-1} - \beta]|\xi^u| - Q\delta|\xi^u|
\end{aligned}$$

and

$$\begin{aligned}
|\hat{x}^c| &= |g^c(\hat{\xi}^c, \hat{\xi}^u)| \\
&\leq |g^c(\hat{\xi}^c, \hat{\xi}^u) - g^c(0, \hat{\xi}^u)| + |g^c(0, \hat{\xi}^u) - g^c(0,0)| \\
&\leq [||D_1 g^c(\hat{\xi}^c, \hat{\xi}^u)|| + \beta]|\hat{\xi}^c| + \gamma|\hat{\xi}^u| \\
&\leq [||D_1 g^c(\hat{\xi}^c, \hat{\xi}^u)|| + \beta + \frac{\gamma}{\delta}]|\hat{\xi}^c| \\
&= [||D_1 g^c(\hat{\xi}^c, \hat{\xi}^u)|| + \beta + \tau]|\hat{\xi}^c|.
\end{aligned}$$

So

$$\frac{|\xi^u|}{|\hat{\xi}^c|} \leq \frac{|x^u|}{|\hat{x}^c|} \; \frac{||D_1 g^c(\hat{\xi}^c, \hat{\xi}^u)|| + \beta + \tau}{||[D_2 g^u(\xi^c, \xi^u)]^{-1}||^{-1} - \beta - Q\delta} \leq \frac{|x^u|}{|\hat{x}^c|} a_2^K.$$

Similarly

$$\frac{|\xi_n^u|}{|\hat{\xi}_n^c|} \leq \frac{|\xi_{n-1}^u|}{|\hat{\xi}_{n-1}^c|} a_2^K \leq \; \cdots \; \leq \frac{|x^u|}{|\hat{x}^c|} a_2^{nK}.$$

Since

$$d(m', m) = |(\hat{x}^c, \hat{x}^u)| = \sqrt{|\hat{x}^c|^2 + |\hat{x}^u|^2} \leq \sqrt{\frac{1}{\delta^2} + 1} \; |\hat{x}^c| < (1 + \frac{1}{\delta})|\hat{x}^c|,$$

we conclude

$$\frac{d(\phi(-nK, \omega, p), \phi(-nK, \omega, m))}{d(\phi(-nK, \omega, m'), \phi(-nK, \omega, m))} \leq \frac{\frac{3}{4}|\xi_n^u|}{c_5|\hat{\xi}_n^c|} \leq \frac{3}{4c_5} a_2^{nK} \frac{|x^u|}{|\hat{x}^c|} \to 0 \text{ as } n \to \infty.$$

This gives us

$$\frac{d(\phi(-t, \omega)(p), \phi(t, \omega)(m))}{d(\phi(-t, \omega)(m'), \phi(-t, \omega)(m))} \to 0 \text{ exponentially as } t \to \infty. \tag{6.4.5}$$

Part (ii) is a conclusion of (6.4.5).

(iii) Suppose $q \in \tilde{\mathcal{W}}^{uu}(\omega, m) \cap \tilde{\mathcal{W}}^{uu}(\omega, m')$ for $m \neq m'$. By part (i),

$$d(\phi(-t, \omega)(q), \phi(-t, \omega)(m)) \to 0, \; d(\phi(-t, \omega)(q), \phi(-t, \omega)(m')) \to 0$$

as $t \to \infty$. Hence $d(\phi(-t,\omega)(m'), \phi(-t,\omega)(m)) \to 0$. Then by part (ii),

$$1 \leq \frac{d(\phi(-t,\omega)(q), \phi(-t,\omega)(m)) + d(\phi(-t,\omega)(q), \phi(-t,\omega)(m'))}{d(\phi(-t,\omega)(m'), \phi(-t,\omega)(m))} \to 0,$$

which is a contradiction.

(iv) Suppose (U, Φ) is a local chart on $\tilde{\mathcal{M}}(\omega)$ near a point $m \in \tilde{\mathcal{M}}(\omega)$ such that $E(\omega)$ has an C^{r-1} orthonormal basis in U. Let (V, Ψ) be a local chart on $\tilde{\mathcal{W}}^u(\omega)$ near m. Define a map $\chi : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^{m+l}$ by

$$\chi(x^c, x^u) = \Psi(\Gamma(\omega, \Phi^{-1}x^c, (h(\omega, \Phi^{-1}x^c)(x^u), x^u))).$$

Then this is a one-to-one continuous map from Euclidean space to Euclidean space with the same dimension. By invariance of domain this map is a homeomorphism. From this fact we conclude that

$$\tilde{\mathcal{W}}^u(\omega) = \cup_{m \in \tilde{\mathcal{M}}(\omega)} \tilde{\mathcal{W}}^{uu}(\omega, m).$$

$\square$

Putting proposition 6.1.1, 6.2.1, 6.2.2, 6.2.3, 6.3.2 and 6.4.1 together, we complete the proof of theorem 3.0.2.

By considering the time inverse flow, we have theorem 3.0.3.

Chapter 7. results for Overflowing and Inflowing Invariant Manifolds

In this chapter, we discuss the cases of overflowing and inflowing invariant manifolds. The proof of these results follows in the same fashion as the persistence of normally hyperbolic invariant manifolds with slight modifications. We will mainly discuss the conditions of theorem but not the proof of them.

Theorem 7.0.1. *Assume that $\psi(t)(x)$ is a C^r flow, $r \geq 1$, and has compact, connected C^r normally hyperbolic overflowing invariant manifold $\bar{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M} \subset \mathbb{R}^n$. Then there exists $\rho > 0$ such that for any C^r random flow $\phi(t, \omega, x)$ in $\mathbb{R}^n$ if*

$$||\phi(t,\omega) - \psi(t)||_{C^1} < \rho, \quad \textit{for } t \in [0,1], \omega \in \Omega,$$

then if $\alpha < r\beta$, $\phi(t,\omega)$ has a C^r normally hyperbolic random overflowing invariant manifold $\tilde{\mathcal{M}}(\omega)$ such that for each $\omega \in \Omega$, $\tilde{\mathcal{M}}(\omega)$ is C^r diffeomorphic to $\mathcal{M}$.

Proof. We enlarge the overflowing invariant manifold $\bar{\mathcal{M}}$ to $\bar{\mathcal{M}}_1 := \psi(1, \bar{\mathcal{M}})$ and $\bar{\mathcal{M}}_2 := \psi(2, \bar{\mathcal{M}})$ such that

$$\bar{\mathcal{M}} \subset \mathcal{M}_1 \subset \bar{\mathcal{M}}_1 \subset \mathcal{M}_2 \subset \bar{\mathcal{M}}_2,$$

where M_i, $i = 1, 2$, are the interiors of $\bar{M}_i$.

Following the same argument of the proof of Theorem 3.0.1, One can construct $\tilde{\mathcal{M}}(\omega)$ near $\mathcal{M}_1$ in a tubular neighborhood V of $\bar{\mathcal{M}}_2$ as a section of normal bundle. □

Since the normal direction contains only the stable direction, there is no need to combine the center and the unstable direction. So the proof for the persistence actually is shorter.

We have the following theorem for the inflowing manifolds.

Theorem 7.0.2. *Assume that $\psi(t)(x)$ is a C^r flow, $r \geq 1$, and has compact, connected C^r normally hyperbolic inflowing invariant manifold $\bar{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M} \subset \mathbb{R}^n$. Then there exists $\rho > 0$ such that for any C^r random flow $\phi(t, \omega, x)$ in $\mathbb{R}^n$ if*

$$||\phi(t,\omega) - \psi(t)||_{C^1} < \rho, \quad \textit{for } t \in [0,1], \omega \in \Omega,$$

then if $\alpha < r\beta$, $\phi(t,\omega)$ has a C^r normally hyperbolic random inflowing invariant manifold $\tilde{\mathcal{M}}(\omega)$ such that for each $\omega \in \Omega$, $\tilde{\mathcal{M}}(\omega)$ is C^r diffeomorphic to $\mathcal{M}$.

Remark 1. For the general overflowing case (with both extension and contraction in normal directions), if it is normally hyperbolic, an unstable manifold exists (see [F1] theorem 4) and persists under random perturbation.

Remark 2. For the general inflowing case, if it is normally hyperbolic, a stable manifold exists and persists under random perturbation.

Remark 3. Generally, for the overflowing(inflowing) case, we have neither the existence of a stable(unstable) manifold of $\mathcal{M}$ nor the persistence of $\mathcal{M}$. So we do not have the persistence of the normal hyperbolicity, either. However, we have the following theorem about the **persistence of normal hyperbolicity**:

Theorem 7.0.3. *Assume that $\psi(t)$ is a C^r flow, $r \geq 1$, and has compact, connected C^r normally hyperbolic overflowing(inflowing) invariant manifold $\bar{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M} \subset \mathbb{R}^n$ with $\alpha < r\beta$, $\mathcal{M}$ has both the stable and unstable manifold $\mathcal{W}^s$ and $\mathcal{W}^u$. Then there exists $\rho > 0$ such that for any random C^r flow $\phi(t,\omega)$ in $\mathbb{R}^n$ if*

$$||\phi(t,\omega) - \psi(t)||_{C^1} < \rho, \quad \text{for } t \in [0,1], \omega \in \Omega,$$

as long as $\phi(t,\omega)$ has compact, connected C^r random overflowing(inflowing) invariant manifold $\bar{\tilde{\mathcal{M}}}(\omega)$ with stable and unstable manifold $\tilde{\mathcal{W}}^s(\omega)$ and $\tilde{\mathcal{W}}^u(\omega)$ such that $\bar{\tilde{\mathcal{M}}}(\omega)$, $\tilde{\mathcal{W}}^s(\omega)$ and $\tilde{\mathcal{W}}^u(\omega)$ are C^1 close to $\bar{\mathcal{M}}$, $\mathcal{W}^s$ and $\mathcal{W}^u$, respectively. Then $\bar{\tilde{\mathcal{M}}}(\omega)$ is normally hyperbolic with constant $\alpha < r\beta$.

Proof. The proof follows the line of section 5.4. $\square$

Now, we discuss the foliation results for all cases discussed above.

Recall that, we use the expanding property of the random unstable manifold (contraction property of the random stable manifold) and the invariance of the center manifold to well define the operator on the space of all fibers. And actually, for the random unstable manifold, we only use the overflowing invariance of the center manifold, while for the random stable manifold, we only use the inflowing invariance of the center manifold. In other words, to construct the unique family of fibers, we demand the normal direction and the center direction have the same invariance property.

Under the conditions of theorem 7.0.1 and 7.0.2, the normal direction has a different invariance property than the center direction does, which implies that the neighborhood of the center manifold can not be foliated completely.

For the cases in Remark 1 (Remark 2), the normal direction has the same invariance property as the center direction does. However, we generally do not have the persistence of the center manifold. So generally we do not have a result about the foliation of the random unstable manifold based on the random center manifold. In order for the random unstable manifold in Remark 1(random stable manifold in Remark 2) to be foliated, some extra conditions should be given. We have the following 2 theorems for those 2 cases:

Theorem 7.0.4. *Assume the conditions of Theorem 7.0.3 hold for the overflowing case. Then there exists a unique C^{r-1} family of C^r submanifolds $\{\tilde{\mathcal{W}}^{uu}(\omega, x) : \omega \in \Omega, x \in \tilde{\mathcal{M}}(\omega)\}$ of $\tilde{\mathcal{W}}^u(\omega)$ satisfying:*

(1) For each $(\omega, x) \in \Omega \times \tilde{\mathcal{M}}, \tilde{\mathcal{M}}(\omega) \cap \tilde{\mathcal{W}}^{uu}(\omega, x) = \{x\}$, $T_x\tilde{\mathcal{W}}^{uu}(\omega, x) = E^u(\omega, x)$ and $\tilde{\mathcal{W}}^{uu}(\omega, x)$ varies measurably with respect to (ω, x) in $\Omega \times \tilde{\mathcal{M}}$.

(2) If $x_1,\ x_2 \in \tilde{\mathcal{M}}(\omega),\ x_1 \neq x_2$, then $\tilde{\mathcal{W}}^{uu}(\omega, x_1) \cap \tilde{\mathcal{W}}^{uu}(\omega, x_2) = \emptyset$ and $\tilde{\mathcal{W}}^u(\omega) = \cup_{x\in\tilde{\mathcal{M}}(\omega)}\tilde{\mathcal{W}}^{uu}(\omega, x)$.

(3) For $x \in \tilde{\mathcal{M}}(\omega)$, $\phi(t, \omega)\big(\tilde{\mathcal{W}}^{uu}(\omega, x)\big) \subset \tilde{\mathcal{W}}^{uu}(\theta_t\omega, \phi(t, \omega)x)$ for all $t > 0$ such that $\phi(t, \omega)x \in \tilde{\mathcal{M}}(\theta^t\omega)$.

(4) For $y \in \tilde{\mathcal{W}}^{uu}(\omega, x)$ and $x_1 \neq x \in \tilde{\mathcal{M}}(\omega)$ with $|\phi(t, \omega)(x_1) - \phi(t, \omega)(x)| \to 0$ as $t \to -\infty$, we have

$$\frac{|\phi(t, \omega)(y) - \phi(t, \omega)(x)|}{|\phi(t, \omega)(y) - \phi(t, \omega)(x_1)|} \to 0$$

exponentially as $t \to -\infty$.

(5) For $y_1, y_2 \in \tilde{\mathcal{W}}^{uu}(\omega, x)$, $|\phi(t, \omega)(y_1) - \phi(t, \omega)(y_2)| \to 0$ exponentially as $t \to -\infty$.

(6) $\tilde{\mathcal{W}}^{uu}(\theta^t\omega, x)$ *is* C^0 *in* t *for any fixed* (ω, x).

The next result is on the stable foliation.

Theorem 7.0.5. *Assume the conditions of theorem 7.0.3 hold for the inflowing case. Then, there exists a unique* C^{r-1} *family of* C^r *submanifolds* $\{\tilde{\mathcal{W}}^{ss}(\omega, x) \ : \ \omega \in \Omega, x \in \tilde{\mathcal{M}}(\omega)\}$ *of* $\tilde{\mathcal{W}}^s(\omega)$ *satisfying:*

(1) For each $(\omega, x) \in \Omega \times \tilde{\mathcal{M}}, \tilde{\mathcal{M}}(\omega) \cap \tilde{\mathcal{W}}^{ss}(\omega, x) = \{x\}$, $T_x\tilde{\mathcal{W}}^{ss}(\omega, x) = E^s(\omega, x)$ *and* $\tilde{\mathcal{W}}^{ss}(\omega, x)$ *varies measurably with respect to* (ω, x) *in* $\Omega \times \tilde{\mathcal{M}}$.

(2) If x_1, $x_2 \in \tilde{\mathcal{M}}(\omega)$, $x_1 \neq x_2$, *then* $\tilde{\mathcal{W}}^{ss}(\omega, x_1) \cap \tilde{\mathcal{W}}^{ss}(\omega, x_2) = \emptyset$ *and*

$$\tilde{\mathcal{W}}^s(\omega) = \cup_{x \in \tilde{\mathcal{M}}(\omega)} \tilde{\mathcal{W}}^{ss}(\omega, x).$$

(3) For $x \in \tilde{\mathcal{M}}(\omega)$, $\phi(t, \omega)\big(\tilde{\mathcal{W}}^{ss}(\omega, x)\big) \subset \tilde{\mathcal{W}}^{ss}(\theta_t\omega, \phi(t, \omega)x)$ *for all* $t < 0$ *such that* $\phi(t, \omega)x \in \tilde{\mathcal{M}}(\theta^t\omega)$.

(4) For $y \in \tilde{\mathcal{W}}^{ss}(\omega, x)$ *and* $x_1 \neq x \in \tilde{\mathcal{M}}(\omega)$ *with* $|\phi(t, \omega)(x_1) - \phi(t, \omega)(x)| \to 0$ *as* $t \to \infty$, *we have*

$$\frac{|\phi(t, \omega)(y) - \phi(t, \omega)(x)|}{|\phi(t, \omega)(y) - \phi(t, \omega)(x_1)|} \to 0$$

exponentially as $t \to +\infty$.

(5) For $y_1, y_2 \in \tilde{\mathcal{W}}^{ss}(\omega, x)$, $|\phi(t, \omega)(y_1) - \phi(t, \omega)(y_2)| \to 0$ *exponentially as* $t \to +\infty$.

(6) $\tilde{\mathcal{W}}^{ss}(\theta^t\omega, x)$ *is* C^0 *in* t *for any fixed* (ω, x).

Chapter 8. Application to Geometric Singular Perturbation

In this chapter, we build geometric singular perturbation theory under small random perturbation and apply this random geometric singular perturbation theory to prove a random version of the exchange lemma.

8.1 Random Geometric Singular Perturbation Theory

We recall first deterministic singular perturbation. Let k, m be positive integers, $n = k+m$, r be a positive integer or infinity, $r \geq 2$, χ be a C^r m-dimensional submanifold of $\mathbb{R}^n$. Generally, singular perturbation problem involves a C^r system of ordinary differential equations of the form

$$z' = h(z, \epsilon) \tag{8.1.1}$$

defined for $z \in \mathbb{R}^n$, $\epsilon \in (-\epsilon_0, \epsilon_0)$, subject to the condition

$$h(z, 0) = 0 \;\text{ for all } z \in \chi. \tag{8.1.2}$$

Consider a system in $\mathbb{R}^n$:

$$z' = h(z, \epsilon) + \epsilon H(\theta^t \omega, z, \epsilon), \tag{8.1.3}$$

where $\epsilon \in (-\epsilon_0, \epsilon_0), \omega \in \Omega$, h, H are C^r in (z, ϵ) for each $\omega \in \Omega$, subject to the condition

$$h(z, 0) = 0 \;\text{ for all } z \in \chi, \tag{8.1.4}$$

and the C^1 norm of $H(\theta^t \omega, z, \epsilon)$ are uniformly bounded for $t \in \mathbb{R}, \omega \in \Omega, z \in \mathbb{R}^n, \epsilon \in (-\epsilon_0, \epsilon_0)$. We call system (8.1.3) a random singular perturbation system.

The linearization of (8.1.3) for $\epsilon = 0$ at $z \in \chi$ is

$$\delta z' = D_1 h(z, 0) \delta z. \tag{8.1.5}$$

It follows from (8.1.4) that zero is an eigenvalue of $D_1 h(z, 0)$ of multiplicity at least m. We call these m zeros corresponding to the tangent space of χ the tangential eigenvalues, and call the remaining eigenvalues the normal eigenvalues. The long term properties of (8.1.3) are highly related to the normal eigenvalues.

We are interested in the case that all normal eigenvalues are nonzero, which motivates the following notation: Let $\chi^N \subset \chi$ be the open set where all the normal eigenvalues are nonzero. (For $z \in \chi^N$, the kernel of $D_1h(z,0)$ has a unique invariant complement, so there is a well-defined projection onto the kernel. We denote this projection by π^χ. The kernel and its invariant complement are C^{r-1}, so π^χ is also C^{r-1}.) Let $\chi^H \subset \chi$ be the open set where all the normal eigenvalues have nonzero real parts. For $z \in \chi^H$, the linearization of (8.1.3) for $\epsilon = 0$ normal to χ has a hyperbolic fixed point.

8.1.1 Terms and Notations. Let $\phi(t,\omega)$ be a random flow. For $V = \{V(\omega) : \omega \in \Omega\}$ a random manifold, define

$$I^+(V(\omega)) := \{p \in V(\omega) : \phi(t,\omega)(p) \in V(\theta^t\omega) \text{ for all } t \in [0,\infty)\}$$

$$I^-(V(\omega)) := \{p \in V(\omega) : \phi(t,\omega)(p) \in V(\theta^t\omega) \text{ for all } t \in (-\infty,0]\}$$

$$I(V(\omega)) := \{p \in V(\omega) : \phi(t,\omega)(p) \in V(\theta^t\omega) \text{ for all } t \in (-\infty,\infty)\},$$

and

$$I^+(V) := \{I^+(V(\omega)) : \omega \in \Omega\}$$

$$I^-(V) := \{I^-(V(\omega)) : \omega \in \Omega\}$$

$$I(V) := \{I(V(\omega)) : \omega \in \Omega\}.$$

Sometimes, we use $I^+_\epsilon, I^-_\epsilon$ and I_ϵ to mean the corresponding set under random flows $\phi^\epsilon(t,\omega)$ which are parameterized by ϵ.

Let $V_1 = \{V_1(\omega) : \omega \in \Omega\}$, $V_2 = \{V_2(\omega) : \omega \in \Omega\}$ such that for all $\omega \in \Omega$, $V_1(\omega) \subset V_2(\omega) \subset \mathbb{R}^n$. We say that V_1 is **invariant relative to** V_2 if orbit segments which leave V_1 also leave V_2. More precisely, this means that for any fixed ω, and all $p \in V_1(\omega)$, if $t_1 \geq 0$ and $\phi(s,\omega)(p) \in V_2(\theta^s\omega)$ for all $s \in [0,t_1]$, then $\phi(s,\omega)(p) \in V_1(\theta^s\omega)$ for all $s \in [0,t_1]$, and if $t_2 \leq 0$ and $\phi(s,\omega)(p) \in V_2(\theta^s\omega)$ for all $s \in [t_2,0]$, then $\phi(s,\omega)(p) \in V_1(\theta^s\omega)$ for all

$s \in [t_2, 0]$. Similarly, we define **relatively positively invariant** and **relatively negatively invariant**.

We say that a random set or random manifold $V_1 = \{V_1(\omega) : \omega \in \Omega\}$ is **locally invariant** if V_1 is invariant relative to some neighborhood $V_2 = \{V_2(\omega) : \omega \in \Omega\}$ of V_1, where by neighborhood we mean for each ω, $V_2(\omega)$ is open and $V_1(\omega) \subset V_2(\omega) \subset \mathbb{R}^n$. And similarly, we define **locally positively invariant** and **locally negatively invariant**.

Suppose $V = \{V(\omega) : \omega \in \Omega\}$ is locally positively invariant, and let $\{S(p, \omega) : \omega \in \Omega, p \in V(\omega)\}$ be a family of submanifolds of $\mathbb{R}^n$ parameterized by $\omega \in \Omega$ and $p \in V(\omega)$. We say that $\{S(p, \omega) : \omega \in \Omega, p \in V(\omega)\}$ is locally positively invariant if

$$\phi(t, \omega)(S(p, \omega)) \subset S(\phi(t, \omega)(p), \theta^t \omega)$$

for all $\omega \in \Omega, p \in V(\omega)$ and $t \geq 0$ such that $\phi(s, \omega)(p) \in V(\theta^s \omega)$ for all $s \in [0, t]$. Similarly, if $V = \{V(\omega) : \omega \in \Omega\}$ is locally negatively invariant, and $\{S(p, \omega) : \omega \in \Omega, p \in V(\omega)\}$ is a family of submanifolds of $\mathbb{R}^n$ parameterized by $\omega \in \Omega$ and $p \in V(\omega)$, we say that $\{S(p, \omega) : \omega \in \Omega, p \in V(\omega)\}$ is locally negatively invariant if

$$\phi(t, \omega)(S(p, \omega)) \subset S(\phi(t, \omega)(p), \theta^t \omega)$$

for all $\omega \in \Omega, p \in V(\omega)$ and $t \leq 0$ such that $\phi(s, \omega)(p) \in V(\theta^s \omega)$ for all $s \in [t, 0]$.

Suppose $\{V_\epsilon : \epsilon \in (-\epsilon_0, \epsilon_0)\}$ is a family of random submanifolds of $\mathbb{R}^n$ parameterized by ϵ. Let

$$V^*(\omega) = \{(p, \epsilon) : \epsilon \in (-\epsilon_0, \epsilon_0), p \in V_\epsilon(\omega)\},$$

$$V^* = \{V^*(\omega) : \omega \in \Omega\}.$$

We say that $\{V_\epsilon : \epsilon \in (-\epsilon_0, \epsilon_0)\}$ is a C^r **family of random submanifolds** of $\mathbb{R}^n$ if V^* is a C^r random submanifold of $\mathbb{R}^n \times (-\epsilon_0, \epsilon_0)$.

Suppose V is a locally invariant (or locally positively invariant or locally negatively invariant) random submanifold of $\mathbb{R}^n$, $\{S(p,\omega) : \omega \in \Omega, p \in V(\omega)\}$ is a family of random submanifolds of $\mathbb{R}^n$ parameterized by $\omega \in \Omega,\ p \in V(\omega)$. Let

$$S^*(\omega) = \{(p,p') : p \in V(\omega), p' \in S(p,\omega)\},$$

$$S^* = \{S^*(\omega) : \omega \in \Omega\}.$$

We say that $\{S(p,\omega) : \omega \in \Omega, p \in V(\omega)\}$ is a C^r **family of random submanifolds** if S^* is a random C^r submanifold of $\mathbb{R}^{2n}$.

Let $\{S(p,\omega,\epsilon) : \epsilon \in (-\epsilon_0,\epsilon_0), (p,\omega) \in V_\epsilon\}$ be a family of random submanifolds parameterized by $(p,\omega,\epsilon) \in V^*$. Let

$$S^*(\omega) = \{(p,p',\epsilon) : \epsilon \in (-\epsilon_0,\epsilon_0), (p,\omega) \in V_\epsilon, p' \in S(p,\omega,\epsilon)\},$$

$$S^* = \{S^*(\omega) : \omega \in \Omega\}.$$

We say that $\{S(p,\omega,\epsilon) : \epsilon \in (-\epsilon_0,\epsilon_0), (p,\omega) \in V_\epsilon\}$ is a C^r **family of random submanifolds** if S^* is a C^r random submanifold of $\mathbb{R}^{2n} \times (-\epsilon_0,\epsilon_0)$.

8.1.2 Theorem. We are ready to state the main theorems. Consider the system in $\mathbb{R}^n$:

$$z' = h(z,\epsilon) + \epsilon H(\theta^t\omega, z, \epsilon),$$

where $\epsilon \in (-\epsilon_0,\epsilon_0), \omega \in \Omega$, h, H are C^r in (z,ϵ) for each $\omega \in \Omega$, H is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R})$ measurable and for each fixed ω, H is C^0 in t. The following condition holds:

$$h(z,0) = 0 \ \text{ for all } z \in \chi.$$

Moreover the C^1 norm of $H(\theta^t\omega, z, \epsilon)$ are uniformly bounded for $t \in \mathbb{R}, \omega \in \Omega, z \in \mathbb{R}^n, \epsilon \in (-\epsilon_0,\epsilon_0)$. Let $\phi_\epsilon(t,\omega)$ be the generated random flow on $\mathbb{R}^n$ parameterized by $\epsilon \in (-\epsilon_0,\epsilon_0)$.

It is well known that for each ϵ, $\phi_\epsilon(t,\omega)$ is a continuous random dynamical system of class C^r. Moreover, $\phi_\epsilon(t,\omega)$ is C^1 in t. See [A].

Let k_1, k_2, k_3 be fixed integers such that $k = k_1 + k_2 + k_3$. Let $K \subset \chi^N$ be a compact connected manifold with boundary such that for all $z \in K$, $D_1h(z,0)$ has k_1 eigenvalues with negative real part, k_2 eigenvalues which are pure imaginary, and k_3 eigenvalues with positive real part.

For each $m \in K$, let E^s_m, E^c_m, E^u_m denote the invariant subspaces of $D_1h(z,0)$ associated with the eigenvalues of $D_1h(z,0)$ in the left half plane, on the imaginary axis, and in the right half plane, respectively. Note that the dimension of E^c_m is $k_2 + m$.

We call a manifold W^{cs} a center-stable manifold of K under $\phi^0(t,\omega)$ if W^{cs} is locally positively invariant under $\phi^0(t,\omega)$, and for all $m \in K$, W^{cs} is tangent to $E^s_m \oplus E^c_m$ at m. We define center-unstable manifold and center manifold the same way, with $E^s_m \oplus E^c_m$ replaced by $E^s_u \oplus E^c_m$ and E^c_m, respectively.

Theorem 8.1.1. *Let $K \subset \chi^N$ be a compact connected manifold with boundary such that for all $z \in K$, $D_1h(z,0)$ has k_1 eigenvalues with negative real part, k_2 eigenvalues which are pure imaginary, and k_3 eigenvalues with positive real part. Then*

(i) There is a C^r family of random manifolds $\{W^{cs}_\epsilon : \epsilon \in (-\epsilon_0, \epsilon_0)\}$ such that W^{cs}_ϵ is locally positively invariant under the flow $\phi_\epsilon(t,\omega)$ for each $\epsilon \in (-\epsilon_0, \epsilon_0)$ and W^{cs}_0 is a center-stable manifold of K under $\phi^0(t,\omega)$.

(ii) There is a C^r family of random manifolds $\{W^{cu}_\epsilon : \epsilon \in (-\epsilon_0, \epsilon_0)\}$ such that W^{cu}_ϵ is locally negatively invariant under the flow $\phi_\epsilon(t,\omega)$ for each $\epsilon \in (-\epsilon_0, \epsilon_0)$ and W^{cu}_0 is a center-unstable manifold of K under $\phi^0(t,\omega)$.

(iii) There is a C^r family of random manifolds $\{W^c_\epsilon : \epsilon \in (-\epsilon_0, \epsilon_0)\}$ such that W^c_ϵ is locally invariant under the flow $\phi_\epsilon(t,\omega)$ for each $\epsilon \in (-\epsilon_0, \epsilon_0)$ and W^{cs}_0 is a center manifold of K under $\phi^0(t,\omega)$.

There is a neighborhood U of K such that $I_\epsilon^+(U) \subset W_\epsilon^{cs}$, $I_\epsilon^-(U) \subset W_\epsilon^{cu}$, and $I_\epsilon(U) \subset W_\epsilon^c$ for each $\epsilon \in (-\epsilon_0, \epsilon_0)$. Moreover, the tangent space of the above random manifolds are all jointly measurable about (ω, z).

Theorem 8.1.2. *Under the condition and conclusion of theorem 8.1.1,*

(i) There is a C^{r-1} family $\{W_\epsilon^{ss}(p,\omega) : \epsilon \in (-\epsilon_0, \epsilon_0), \omega \in \Omega, p \in W_\epsilon^c(\omega)\}$ of C^r manifolds such that for each fixed $\epsilon \in (-\epsilon_0, \epsilon_0)$, the family $\{W_\epsilon^{ss}(p,\omega) : \omega \in \Omega, p \in W_\epsilon^c(\omega)\}$ is locally positively invariant, the fiber $W_0^{ss}(p,\omega) \equiv W_0^{ss}(p)$ is tangent to E_p^s at p for each $p \in K$. Moreover, for each fixed ϵ and ω, $p \in W_\epsilon^{ss}(p,\omega)$; for different $p, q \in W_\epsilon^c(\omega)$, $W_\epsilon^{ss}(p,\omega)$ and $W_\epsilon^{ss}(q,\omega)$ are disjoint and $\bigcup_{p \in W_\epsilon^c(\omega)} W_\epsilon^{ss}(p,\omega) = W_\epsilon^{cs}(\omega)$. Each $W_\epsilon^{ss}(p,\omega)$ intersects $W_\epsilon^c(\omega)$ transversely, in exactly one point p.

(ii) There is a C^{r-1} family $\{W_\epsilon^{uu}(p,\omega) : \epsilon \in (-\epsilon_0, \epsilon_0), \omega \in \Omega, p \in W_\epsilon^c(\omega)\}$ of C^r manifolds such that for each fixed $\epsilon \in (-\epsilon_0, \epsilon_0)$, the family $\{W_\epsilon^{uu}(p,\omega) : \omega \in \Omega, p \in W_\epsilon^c(\omega)\}$ is locally negatively invariant, the fiber $W_0^{uu}(p,\omega) \equiv W_0^{uu}(p)$ is tangent to E_p^u at p for each $p \in K$. Moreover, for each fixed ϵ and ω, $p \in W_\epsilon^{uu}(p,\omega)$; for different $p, q \in W_\epsilon^c(\omega)$, $W_\epsilon^{uu}(p,\omega)$ and $W_\epsilon^{uu}(q,\omega)$ are disjoint and $\bigcup_{p \in W_\epsilon^c(\omega)} W_\epsilon^{uu}(p,\omega) = W_\epsilon^{cu}(\omega)$. Each $W_\epsilon^{uu}(p,\omega)$ intersects $W_\epsilon^c(\omega)$ transversely, in exactly one point p.

8.1.3 Proof of Theorem 8.1.1 and 8.1.2.

The general idea to prove the main theorems is that in a compact region, the original system can be viewed as a small perturbation of the linearization of the $\epsilon = 0$ system. The latter can be modified a little bit to possesses a normally hyperbolic overflowing or inflowing invariant manifold which are in explicit form, and then the random invariant manifold theory can be applied.

It is convenient to view ϵ as a dummy variable in the phase space. So we consider the following system:

$$\begin{aligned} z' &= h(z,\epsilon) + \epsilon H(\theta^t\omega, z, \epsilon) \\ \epsilon' &= 0 \end{aligned} \tag{8.1.6}$$

Since K is compact, simply connected, we may choose global C^r coordinates $(x, y_1, y_2, y_3, \epsilon) = \Phi(z, \epsilon)$ near $K \times \{\epsilon = 0\}$ such that (8.1.6) takes the following form

$$\begin{aligned}
x' &= f(x, y_1, y_2, y_3, \theta^t\omega, \epsilon) \\
y_1' &= g_1(x, y_1, y_2, y_3, \theta^t\omega, \epsilon) \\
y_2' &= g_2(x, y_1, y_2, y_3, \theta^t\omega, \epsilon) \\
y_3' &= g_3(x, y_1, y_2, y_3, \theta^t\omega, \epsilon) \\
\epsilon' &= 0,
\end{aligned} \tag{8.1.7}$$

for x near some compact set $\tilde{K}$. (Here $\tilde{K}$ is a compact subset of $\mathbb{R}^m$. We may sometimes use this $\tilde{K}$ to mean the set $\{(x, 0) \in \mathbb{R}^n : x \in \tilde{K}\}$ or $\{(x, 0, 0) \in \mathbb{R}^{n+1} : x \in \tilde{K}\}$.) Moreover, for $x \in \tilde{K}$,

$$\begin{aligned}
&f(x, 0, 0, 0, \theta^t\omega, 0) = 0 \\
&g_i(x, 0, 0, 0, \theta^t\omega, 0) = 0 \text{ for } i = 1, 2, 3,
\end{aligned} \tag{8.1.8}$$

and

$$\frac{\partial(g_1, g_2, g_3)}{\partial(y_1, y_2, y_3)}\Big|_{\substack{y = 0 \\ \epsilon = 0}} = \begin{pmatrix} A_1(x) & 0 & 0 \\ 0 & A_2(x) & 0 \\ 0 & 0 & A_3(x) \end{pmatrix}, \tag{8.1.9}$$

where the eigenvalues of $A_1(x)$ are in the left half plane, the eigenvalues of $A_2(x)$ are on the imaginary axis, and the eigenvalues of $A_3(x)$ are in the right half plane. The dimension of A_1, A_2, A_3 are k_1, k_2, k_3.

Let δ_1 be a small positive number. Define $\tilde{K}_1 := \{x \in \mathbb{R}^m, d(x, \tilde{K}) \leq \delta_1\}$, $U_1 := \{(x, y_1, y_2, y_3, \epsilon) : x \in \tilde{K}_1, |y_i| \leq \delta_1, i = 1, 2, 3, |\epsilon| \leq \delta_1\}$. Then (8.1.7) is arbitrarily C^1-close to

$$
\begin{aligned}
x' &= 0 \\
y_1' &= A_2(x)y_1 \\
y_2' &= A_2(x)y_2 \\
y_3' &= A_3(x)y_3 \\
\epsilon' &= 0,
\end{aligned}
\tag{8.1.10}
$$

uniformly in U_1.

Let $\Lambda^s = \{(x, y_1, y_2, y_3, \epsilon) : y_3 = 0\}$, $\Lambda^u = \{(x, y_1, y_2, y_3, \epsilon) : y_1 = 0\}$, and $\Lambda = \Lambda^s \bigcap \Lambda^u$. The manifolds with corners $\Lambda^s \bigcap U_1, \Lambda^u \bigcap U_1$ and $\Lambda \bigcap U_1$ are compact. It is easy to check that under the system (8.1.10), $\Lambda \bigcap U_1$ is normally hyperbolic. However, $\Lambda \bigcap U_1$ is neither invariant nor overflowing invariant. We will modify the system (8.1.10) such that $\Lambda \bigcap U_1$ is overflowing invariant and still normally hyperbolic under the modified system.

Since $\tilde{K}$ is compact, there exist $l_1 < 0 < l_3$ such that for any $x \in \tilde{K}$, the largest of the real parts of the eigenvalues of $A_1(x)$ is smaller than l_1, and the smallest of the real parts of the eigenvalues of $A_3(x)$ is bigger than l_3. Choose C^r bases in $\mathbb{R}^{k_1}$, $\mathbb{R}^{k_2}$, $\mathbb{R}^{k_3}$ such that

$$
\begin{aligned}
< y_1, A_1(x)y_1 > &\leq (l_1 + \delta_2)|y_1|^2 \\
< y_2, A_2(x)y_2 > &\leq \delta_2|y_2|^2 \\
< y_2, -A_2(x)y_2 > &\leq \delta_2|y_2|^2 \\
< y_3, -A_3(x)y_3 > &\leq (-l_3 + \delta_2)|y_3|^2
\end{aligned}
$$

for all $y_1 \in \mathbb{R}^{k_1}$, $y_2 \in \mathbb{R}^{k_2}$, $y_3 \in \mathbb{R}^{k_3}$, where δ_2 is an arbitrary positive number. This can be achieved using the $\epsilon - jordan\, form$. So we have

$$||e^{A_1(x)t}|| \leq e^{(l_1+\delta_2)t} \text{ for all } t \geq 0$$

$$||e^{A_2(x)t}|| \leq e^{\delta_2 t} \text{ for all } t \in \mathbb{R}$$

$$||e^{A_3(x)t}|| \leq e^{(l_3-\delta_2)t} \text{ for all } t \leq 0.$$

To modify the system (8.1.10) and (8.1.7), let a_1, a_2, a_3, a_4 and a_5 be real numbers, satisfying $0 < a_5 < a_4 < a_3 < a_2 < a_1 = \delta_1$, and choose a C^∞ "bump" function $B : [0, \delta_1] \to \mathbb{R}$ such that $B(r) = 0$ for $r \in [0, a_4]$, $B(a_3) > 0, B(a_2) < 0, B(a_1) > 0$ and $B'(r) > 0$ for $r \in (a_4, a_3)$, $B'(r) < 0$ for $r \in (a_3, a_2)$, $B'(r) > 0$ for $r \in (a_2, a_1)$. Let $R : \tilde{K}_1 \to \mathbb{R}$ be defined by

$$R(x) = \begin{cases} 0, & x \in \tilde{K}, \\ d(x, \tilde{K}), & x \in \tilde{K}^c, \end{cases}$$

the modified systems are

$$\begin{aligned} x' &= f(x, y_1, y_2, y_3, \theta^t\omega, \epsilon) + \delta_3 B(R(x))x \\ y_1' &= g_1(x, y_1, y_2, y_3, \theta^t\omega, \epsilon) \\ y_2' &= g_2(x, y_1, y_2, y_3, \theta^t\omega, \epsilon) + \delta_3 B(|y_2|)y_2 \\ y_3' &= g_3(x, y_1, y_2, y_3, \theta^t\omega, \epsilon) \\ \epsilon' &= 0, \end{aligned} \tag{8.1.11}$$

and

$$\begin{aligned} x' &= \delta_3 B(R(x))x \\ y_1' &= A_2(x)y_1 \\ y_2' &= A_2(x)y_2 + \delta_3 B(|y_2|)y_2 \\ y_3' &= A_3(x)y_3 \\ \epsilon' &= 0, \end{aligned} \tag{8.1.12}$$

where δ_3 is a small positive number.

Let

$$\tilde{K}_i = \{x \in \tilde{K}_1 : d(x, \tilde{K}) \leq a_i\}$$

$$U_i = \{(x, y_1, y_2, y_3, \epsilon) : x \in \tilde{K}_i, |y_1| \leq a_i, |y_2| \leq a_i, |y_3| \leq a_i, |\epsilon| \leq \delta_1\}$$

for $i = 1, 2, 3, 4, 5$. Choose δ_1 small enough and $0 < \delta_2 \ll \delta_3 \ll \delta_1$, then $\Lambda^u \bigcap U_1$ is overflowing invariant under (8.1.12) and normally (stably) hyperbolic, while $\Lambda^s \bigcap U_2$ is inflowing invariant under (8.1.12) and normally (unstably) hyperbolic. (By choosing $\delta_2 \ll \delta_3$,we make y_2 expand or contract on $\Lambda^u \bigcap U_1$ or $\Lambda^s \bigcap U_2$, respectively. By choosing both δ_2, δ_3 small, we make the normal hyperbolicity strong enough to obtain enough smoothness.)

Now theorem 7.0.1 and 7.0.2 can be applied. Since δ_1 is small enough, the flow generated by (8.1.11) and (8.1.12) are uniformly C^1 close for bounded time t. So by theorem 7.0.1, there is a C^r random manifold $\tilde{\Lambda}^u = \{\tilde{\Lambda}^u(\omega) : \omega \in \Omega\}$ near $\Lambda^u \bigcap U_1$ which is overflowing invariant under (8.1.11), and by theorem 7.0.2, there is a C^r random manifold $\tilde{\Lambda}^s = \{\tilde{\Lambda}^s(\omega) : \omega \in \Omega\}$ near $\Lambda^s \bigcap U_2$ which is inflowing invariant under (8.1.11). Moreover, for system (8.1.11), $I^-(U_1) = \tilde{\Lambda}^u, I^+(U_2) = \tilde{\Lambda}^s$. Since $\tilde{K} \times \{\epsilon = 0\}$ consists of equilibria of (8.1.11), $\tilde{\Lambda}^u, \tilde{\Lambda}^s$ both contain $\tilde{K} \times \{\epsilon = 0\}$. Because the linearization of (8.1.11) and (8.1.12) coincide at points of $\tilde{K} \times \{\epsilon = 0\}$, the tangent space of $\tilde{\Lambda}^u$ and $\tilde{\Lambda}^s$ and that of Λ^u and Λ^s at points of $\tilde{K} \times \{\epsilon = 0\}$ coincide. Define

$$\tilde{\Lambda}^i_\epsilon := \{(x, y_1, y_2, y_3) : (x, y_1, y_2, y_3, \epsilon) \in \tilde{\Lambda}^i\} \text{ for } i = u, s$$

. Then

$$\tilde{\Lambda}^u_0 = \{(x, y_1, y_2, y_3) : (x, y_1, y_2, y_3, 0) \in \tilde{\Lambda}^u\}$$

is a center-unstable manifold of $\tilde{K}$ under system (8.1.11), and

$$\tilde{\Lambda}^s_0 = \{(x, y_1, y_2, y_3) : (x, y_1, y_2, y_3, 0) \in \tilde{\Lambda}^s\}$$

is a center-stable manifold of $\tilde{K}$ under system (8.1.11).

In order to apply the random foliation theory, consider $\tilde{\Lambda}^s$, $\tilde{\Lambda}^u$ and $\tilde{\Lambda}^s \bigcap \tilde{\Lambda}^u$. Recall that $\tilde{\Lambda}^s$ is C^1-close to $\Lambda \bigcap U_2$, $\tilde{\Lambda}^u$ is C^1-close to $\Lambda \bigcap U_1$, $\tilde{\Lambda}^s \bigcap \tilde{\Lambda}^u$ is C^1-close to $\Lambda \bigcap U_2$ and inflowing invariant under system (8.1.11). Since $\Lambda \bigcap U_2$ is inflowing invariant and normally hyperbolic, by theorem 7.0.3 $\tilde{\Lambda}^s \bigcap \tilde{\Lambda}^u$ is also normally hyperbolic. By theorem 7.0.5, for each $\omega \in \Omega$, there exists a C^{r-1} family of C^r submanifold $\{\tilde{\mathcal{W}}^{ss}(\omega, p) : p \in \tilde{\Lambda}^s \bigcap \tilde{\Lambda}^u(\omega)\}$ of $\tilde{\Lambda}^s(\omega)$ satisfying:

(1) For each ω and $p \in \tilde{\Lambda}^s \bigcap \tilde{\Lambda}^u(\omega), \tilde{\Lambda}^s \bigcap \tilde{\Lambda}^u(\omega) \cap \tilde{\mathcal{W}}^{ss}(\omega, p) = \{p\}$ and $\tilde{W}^{ss}(\omega, p)$ varies measurably with respect to (ω, p) in $\Omega \times \tilde{\mathcal{M}}$.

(2) If $p_1, p_2 \in \tilde{\Lambda}^s \bigcap \tilde{\Lambda}^u(\omega)$, $p_1 \neq p_2$, then $\tilde{\mathcal{W}}^{ss}(\omega, p_1) \cap \tilde{\mathcal{W}}^{ss}(\omega, p_2) = \emptyset$, $\cup_{x \in \tilde{\Lambda}^s \bigcap \tilde{\Lambda}^u(\omega)} \tilde{\mathcal{W}}^{ss}(\omega, p)$ is a neighborhood of $\tilde{\Lambda}^s \bigcap \tilde{\Lambda}^u(\omega)$ in $\tilde{\Lambda}^s(\omega)$, and if δ_1 is small enough, $\cup_{x \in \tilde{\Lambda}^s \bigcap \tilde{\Lambda}^u(\omega)} \tilde{\mathcal{W}}^{ss}(\omega, p)$ is equal to $\tilde{\Lambda}^s(\omega)$.

(3) For $p \in \tilde{\Lambda}^s \bigcap \tilde{\Lambda}^u(\omega)$, $\phi(t, \omega)\big(\tilde{\mathcal{W}}^{ss}(\omega, p)\big) \subset \tilde{\mathcal{W}}^{ss}(\theta_t \omega, \phi(t, \omega)p)$ for t big enough.

(4) For $q \in \tilde{\mathcal{W}}^{ss}(\omega, p)$ and $p_1 \neq p \in \tilde{\Lambda}^s \bigcap \tilde{\Lambda}^u(\omega)$ with $|\phi(t, \omega)(p_1) - \phi(t, \omega)(p)| \to 0$ as $t \to \infty$, we have
$$\frac{|\phi(t, \omega)(q) - \phi(t, \omega)(p)|}{|\phi(t, \omega)(q) - \phi(t, \omega)(p_1)|} \to 0$$
exponentially as $t \to +\infty$.

(5) For $q_1, q_2 \in \tilde{\mathcal{W}}^{ss}(\omega, p)$, $|\phi(t, \omega)(q_1) - \phi(t, \omega)(q_2)| \to 0$ exponentially as $t \to +\infty$.

Since $\tilde{\Lambda}^s \bigcap \tilde{\Lambda}^u$ is not overflowing invariant but inflowing, theorem 7.0.4 can not be applied directly. We consider the overflowing invariant manifold $\Lambda \bigcap U_3$. By theorem 7.0.1, there is a C^r random manifold $\bar{\Lambda}^u = \{\bar{\Lambda}^u(\omega) : \omega \in \Omega\}$ near $\Lambda^u \bigcap U_3$ which is overflowing invariant under (8.1.11) and contained in $\tilde{\Lambda}^u = \{\tilde{\Lambda}^u(\omega) : \omega \in \Omega\}$. Now $\bar{\Lambda}^u$ and $\bar{\Lambda}^u \bigcap \tilde{\Lambda}^s$ satisfy the conditions of theorem 7.0.4. So we get the C^{r-1} family of C^r submanifold $\{\tilde{\mathcal{W}}^{uu}(\omega, p) : p \in \bar{\Lambda}^u \bigcap \tilde{\Lambda}^s(\omega)\}$ of $\bar{\Lambda}^u(\omega)$ having similar properties as listed above.

Now we consider system (8.1.7). In U_4 and U_5, (8.1.11) and (8.1.7) coincide. Define for $p \in \bar{\Lambda}^u \bigcap \tilde{\Lambda}^s(\omega) \bigcap U_5$:

$$W^{uu}(\omega, p) := \tilde{\mathcal{W}}^{uu}(\omega, p) \bigcap U_4,$$

$$W^{uu}_\epsilon(\omega, p) := \{(x, y_1, y_2, y_3) : (x, y_1, y_2, y_3, \epsilon) \in W^{uu}(\omega, p)\}.$$

Define for $p \in \tilde{\Lambda}^u \bigcap \tilde{\Lambda}^s(\omega) \bigcap U_5$:

$$W^{ss}(\omega, p) := \tilde{\mathcal{W}}^{ss}(\omega, p) \bigcap U_4,$$

$$W^{ss}_\epsilon(\omega, p) := \{(x, y_1, y_2, y_3) : (x, y_1, y_2, y_3, \epsilon) \in W^{ss}(\omega, p)\}.$$

Then define

$$W^{cu}_\epsilon(\omega) := \bigcup_{p \in \bar{\Lambda}^u \bigcap \tilde{\Lambda}^s(\omega) \bigcap U_5} W^{uu}_\epsilon(\omega, p),$$

$$W^{cs}_\epsilon(\omega) := \bigcup_{p \in \tilde{\Lambda}^u \bigcap \tilde{\Lambda}^s(\omega) \bigcap U_5} W^{ss}_\epsilon(\omega, p).$$

Let $\epsilon_0 > 0$ be smaller than every constant used so far, and define

$$W^{cu}(\omega) := \{(x, y_1, y_2, y_3, \epsilon) : (x, y_1, y_2, y_3) \in W^{cu}_\epsilon(\omega),\ \epsilon \in (-\epsilon_0, \epsilon_0)\},$$

$$W^{cs}(\omega) := \{(x, y_1, y_2, y_3, \epsilon) : (x, y_1, y_2, y_3) \in W^{cs}_\epsilon(\omega),\ \epsilon \in (-\epsilon_0, \epsilon_0)\},$$

$$W^c(\omega) := W^{cu}(\omega) \bigcap W^{cs}(\omega).$$

Then for each ω, $W^{cu}(\omega)$ is a neighborhood of $W^c(\omega)$ in $\bar{\Lambda}^u$, $W^{cs}(\omega)$ is a neighborhood of $W^c(\omega)$ in $\tilde{\Lambda}^s$. Since $\bar{\Lambda}^u_0$ is a center-unstable manifold of $\tilde{K}$ under system (8.1.11), W^{cs}_0 is a center-unstable manifold of $\tilde{K}$ under system (8.1.7). Similarly, W^{cs}_0 is a center-stable manifold of $\tilde{K}$ under (8.1.7), W^c_0 is a center manifold of $\tilde{K}$ under (8.1.7).

From the definition of W^{ss}, W^{uu}, all properties listed in (1),(2),(3),(4),(5) above persist. In other words, W^{ss}, W^{uu} are the families of theorem 8.1.2.

8.2 Random Exchange Lemma

8.2.1 Normal Form. In applications, many singular perturbation problems take the following form:

$$\begin{aligned} x' = \left(\frac{dx}{d\tau}\right) &= f_1(x, y, \epsilon) \\ \epsilon y' &= f_2(x, y, \epsilon). \end{aligned} \tag{8.2.1}$$

We add real noise to the above system and get:

$$\begin{aligned} x' &= f_1(x, y, \epsilon) + \epsilon H_1(\theta^\tau \omega, x, y, \epsilon) \\ \epsilon y' &= f_2(x, y, \epsilon) + \epsilon H_2(\theta^\tau \omega, x, y, \epsilon), \end{aligned} \tag{8.2.2}$$

and a corresponding fast system with $t = \tau/\epsilon$ and $\dot{z} := \frac{dz}{dt}$:

$$\begin{aligned} \dot{x} &= \epsilon f_1(x, y, \epsilon) + \epsilon^2 H_1(\theta^\tau \omega, x, y, \epsilon) \\ \dot{y} &= f_2(x, y, \epsilon) + \epsilon H_2(\theta^\tau \omega, x, y, \epsilon). \end{aligned} \tag{8.2.3}$$

Our task is to decouple the above system near some specific region so that the new system could be analyzed quantitively. The tool is our random singular perturbation theory.

Let K be a bounded connected relatively open subset of $\{(x, y) | f_2(x, y, 0) = 0\}$. Assume for each point $(x, y) \in K$, the matrix

$$\begin{pmatrix} 0 & 0 \\ D_1 f_2(x, y, 0) & D_2 f_2(x, y, 0) \end{pmatrix}$$

has 0 as an eigenvalue of multiplicity exactly m. No other eigenvalues lie on the imaginary axis. So it corresponds to the case that $K \subset \chi^H$ which means that K is normally hyperbolic. Then we can decouple system (8.2.3) near K. We have the following

Theorem 8.2.1. *There exists a measurable C^{r-1} change of coordinates, such that in the new* *coordinates which we denote by (a, b, x), the system of equations (8.2.3) takes the following form:*

$$
\begin{aligned}
\dot{a} &= A(\theta^t\omega, a, b, x, \epsilon)a \\
\dot{b} &= B(\theta^t\omega, a, b, x, \epsilon)b \\
\dot{x} &= \epsilon[C(\theta^t\omega, x, \epsilon) + X(\theta^t\omega, a, b, x, \epsilon)ab]
\end{aligned}
\tag{8.2.4}
$$

near $\overline{K}$, for $a \in \mathbb{R}^{k_1}$, $b \in \mathbb{R}^{k_3}$, $x \in \mathbb{R}^m$, A, B are $k_1 \times k_1$ and $k_3 \times k_3$ matrix valued functions, A, B are C^{r-2} in (a, b, x, ϵ), C is C^{r-1} in (x, ϵ), X is C^{r-3} in (a, b, x, ϵ), all of A, B, C, X are C^0 in t for fixed ω, and jointly measurable. Moreover, the following hold:

$$
\begin{aligned}
(a, Aa) &\geq 4\alpha|a|^2 \\
(b, Bb) &\leq -4\alpha|b|^2
\end{aligned}
\tag{8.2.5}
$$

for some $\alpha > 0$.

Proof. Let $T\mathbb{R}^n|K = TK \oplus E^s \oplus E^u$ be the splitting corresponding to the normal hyperbolicity. Since K is relatively open and connected, it is contractible. It is well known that E^s and E^u are trivial bundles. Thus, we can set up a C^r coordinate system near K as follows: choose (a, b, x), $a \in \mathbb{R}^{k_1}, b \in \mathbb{R}^{k_3}, x \in \mathbb{R}^m$ such that

$$K = \{(a, b, x)|a = 0, b = 0\},$$

$$E^s = \{(a, b, x)|a = 0\},$$

$$E^u = \{(a, b, x)|b = 0\}.$$

In this new coordinate system, (8.2.3) takes the following form

$$\begin{aligned} \dot{a_1} &= A_1(\theta^t\omega, a_1, b_1, x_1, \epsilon) \\ \dot{b_1} &= B_1(\theta^t\omega, a_1, b_1, x_1, \epsilon) \\ \dot{x_1} &= \epsilon X_1(\theta^t\omega, a_1, b_1, x_1, \epsilon) \end{aligned} \tag{8.2.6}$$

where A_1, B_1, X_1 are C^r in $(a_1, b_1, x_1, \epsilon)$, C^0 in t for each fixed ω, and $\mathcal{F}\otimes\mathcal{B}(\mathbb{R}^{n+1})$ measurable.

It can be assumed from Theorem 8.1.1 that there exists a C^r family of C^r random locally invariant manifolds $M^\epsilon(\omega)(M^0(\omega) \equiv K)$, each having dimension m. And there exists a C^r family of C^r random overflowing invariant manifolds $W^u_\epsilon(\omega)$ and random inflowing invariant manifolds $W^s_\epsilon(\omega)$. Moreover, the C^r family of C^r random overflowing invariant manifolds $W^u_\epsilon(\omega)$ is given by graphs of functions in the coordinates (a_1, b_1, x_1), namely

$$W^u_\epsilon(\theta^t\omega) = \{(a_1, b_1, x_1) | b_1 = h^u(\theta^t\omega, a_1, x_1, \epsilon)\},$$

where $h^u(\theta^t\omega, a_1, x_1, \epsilon)$ is C^r in (a_1, x_1, ϵ), C^1 in t for each fixed ω, and $\mathcal{F}\otimes\mathcal{B}(\mathbb{R}^{k_1+m+1})$ measurable.

Make the following change of coordinates: $(a_1, b_1, x_1) \to (a_2, b_2, x_2)$ by:

$$\begin{aligned} a_2 &= a_1, \\ b_2 &= b_1 - h^u(\theta^t\omega, a_1, x_1, \epsilon), \\ x_2 &= x_1. \end{aligned}$$

Obviously, the transformation is invertible. In this new coordinates,

$$W^u_\epsilon(\omega) = \{(a_2, b_2, x_2) | b_2 = 0\}.$$

Consider the system in the new coordinates:

$$
\begin{aligned}
\dot{a_2} = \dot{a_1} &= A_1(\theta^t\omega, a_1, b_1, x_1, \epsilon) \\
&= A_1(\theta^t\omega, a_2, b_2 + h^u(\theta^t\omega, a_2, x_2, \epsilon), x_2, \epsilon) \\
&= A_2(\theta^t\omega, a_2, b_2, x_2, \epsilon) \\
\dot{b_2} &= \frac{d}{dt}(b_1 - h^u(\theta^t\omega, a_1, x_1, \epsilon)) \\
&= B_1(\theta^t\omega, a_2, b_2 + h^u(\theta^t\omega, a_2, x_2, \epsilon), x_2, \epsilon) - \frac{d}{dt}h^u(\theta^t\omega, a_1, x_1, \epsilon) \\
&= B_2(\theta^t\omega, a_2, b_2, x_2, \epsilon) \\
\dot{x_2} &= \epsilon X_2(\theta^t\omega, a_2, b_2, x_2, \epsilon)
\end{aligned}
$$

where, from the smoothness and measurability of A_1, B_1, X_1 and h^u, A_2, B_2, X_2 are C^r in $(a_2, b_2, x_2, \epsilon)$, C^0 in t for each fixed ω, and $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^{n+1})$ measurable.

Moreover, the C^r family of C^r random inflowing invariant manifolds $W^s_\epsilon(\omega)$ is given by graphs of functions in the coordinates (a_2, b_2, x_2), namely

$$W^s_\epsilon(\theta^t\omega) = \{(a_2, b_2, x_2) | a_2 = h^s(\theta^\tau\omega, b_2, x_2, \epsilon)\},$$

where $h^s(\theta^t\omega, b_2, x_2, \epsilon)$ is C^r in (b_2, x_1, ϵ), C^1 in t for each fixed ω, and $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^{k_3+m+1})$ measurable. Next we make the following change of coordinates: $(a_2, b_2, x_2) \to (a_3, b_3, x_3)$ by

$$
\begin{aligned}
&a_3 = a_2 - h^s(\theta^t\omega, b_2, x_2, \epsilon), \\
&b_3 = b_2, \\
&x_3 = x_2.
\end{aligned}
$$

Again, the transformation is invertible. In these new coordinates,

$$W^s_\epsilon(\omega) = \{(a_3, b_3, x_3) | a_3 = 0\},$$

$$W_\epsilon^u(\omega) = \{(a_3, b_3, x_3) | b_3 = 0\}.$$

The system of equations are now

$$\begin{aligned}
\dot{a_3} &= A_3(\theta^t\omega, a_3, b_3, x_3, \epsilon) \\
\dot{b_3} &= B_3(\theta^t\omega, a_3, b_3, x_3, \epsilon) \\
\dot{x_3} &= \epsilon X_3(\theta^t\omega, a_3, b_3, x_3, \epsilon)
\end{aligned} \tag{8.2.7}$$

and A_3, B_3, X_3 are C^r in $(a_3, b_3, x_3, \epsilon)$, C^0 in t for each fixed ω, and $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^{n+1})$ measurable.

From theorem 8.1.2, There is a C^{r-1} family $\{W_\epsilon^{uu}(p, \omega) : \epsilon \in (-\epsilon_0, \epsilon_0), \omega \in \Omega, p \in M_\epsilon(\omega)\}$ of C^r manifolds such that for each fixed $\epsilon \in (-\epsilon_0, \epsilon_0)$, the family $\{W_\epsilon^{uu}(p, \omega) : \omega \in \Omega, p \in M_\epsilon(\omega)\}$ is locally negatively invariant, the fiber $W_0^{uu}(p, \omega) \equiv W_0^{uu}(p)$ is tangent to E_p^u at p for each $p \in K$. Moreover, for each fixed ϵ and ω, $p \in W_\epsilon^{uu}(p, \omega)$; for different $p, q \in M_\epsilon(\omega)$, $W_\epsilon^{uu}(p, \omega)$ and $W_\epsilon^{uu}(q, \omega)$ are disjoint and $\bigcup_{p \in M_\epsilon(\omega)} W_\epsilon^{uu}(p, \omega) = W_\epsilon^{cu}(\omega)$. Each $W_\epsilon^{uu}(p, \omega)$ intersects $W_\epsilon^c(\omega)$ transversely, in exactly one point p. We call each manifold of the family an unstable fiber.

In the coordinates (a_3, b_3, x_3), the unstable fibers are represented by $h^{uu}(\theta^t\omega, a_3, x_3, \epsilon)$, for h^{uu} is C^r in (a_3, ϵ), C^{r-1} in (a_3, x_3, ϵ), C^1 in t for each fixed ω and $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^{k_1+m+1})$ measurable, in the following manner

$$(a_3, x_3) \to (a_3, h^{uu}(\theta^t\omega, a_3, x_3, \epsilon))$$

such that for fixed x_3, $(a_3, h^{uu}(\theta^t\omega, a_3, x_3, \epsilon))$ lies on the same fiber with base point $(0, x_3)$.

Now make the following change of coordinates: $(a_3, b_3, x_3) \to (a_4, b_4, x_4)$ by

$$\begin{aligned}
a_4 &= a_3 \\
b_4 &= b_3, \\
x_4 &= h^{uu}(\theta^t\omega, a_3, x_3, \epsilon).
\end{aligned}$$

This is a measurable C^{r-1} change.

The Jacobian $\frac{D(a_4,b_4,x_4)}{D(a_3,b_3,x_3)} = 1$ on $\{b_3 = 0, a_3 = 0\}$. Note the fact that

$$\begin{aligned} x_4 &= h^{uu}(\theta^t\omega, a_3, x_3, \epsilon) \\ &= x_3 + h^{uu}(\theta^t\omega, a_3, x_3, \epsilon) - h^{uu}(\theta^t\omega, 0, x_3, \epsilon) \\ &= x_3 + \tilde{h}^{uu}(\theta^t\omega, a_3, x_3, \epsilon)a_3. \end{aligned}$$

Since $h^{uu}(\theta^t\omega, a_3, x_3, \epsilon)$ are uniformly C^{r-1} close to $h^{uu}(\theta^t\omega, 0, x_3, 0)$, which is independent of ω, and K is bounded connected and relatively open, the above change of coordinates is uniformly invertible for a_3, b_3 small. Under the new coordinate system, the system of equations are

$$\begin{aligned} \dot{a_4} &= A_4(\theta^t\omega, a_4, b_4, x_4, \epsilon) \\ \dot{b_4} &= B_4(\theta^t\omega, a_4, b_4, x_4, \epsilon) \\ \dot{x_4} &= \epsilon X_4(\theta^t\omega, a_4, b_4, x_4, \epsilon) \end{aligned} \tag{8.2.8}$$

and A_4, B_4, X_4 are C^{r-1} in $(a_4, b_4, x_4, \epsilon)$, C^0 in t for each fixed ω, and $\mathcal{F}\otimes\mathcal{B}(\mathbb{R}^{n+1})$ measurable.

The unstable fibers $W^{uu}_\epsilon(\theta^t\omega, p)$ are represented by

$$W^{uu}_\epsilon(\theta^t\omega, p) = \{(a_4, b_4, x_4) | b_4 = 0, x_4 = \text{ constant }\}.$$

So by similar reason, by using the stable fibers, we could do another measurable C^{r-1} change of coordinates $(a_4, b_4, x_4) \to (a_5, b_5, x_5)$ such that under the coordinates (a_5, b_5, x_5), the stable fibers $W^{ss}_\epsilon(\omega, p)$ are represented by

$$W^{ss}_\epsilon(\omega, p) = \{(a_5, b_5, x_5) | a_5 = 0, x_5 = \text{ constant }\},$$

and the system of equations are

$$\begin{aligned}\dot{a_5} &= A_5(\theta^t\omega, a_5, b_5, x_5, \epsilon)\\ \dot{b_5} &= B_5(\theta^t\omega, a_5, b_5, x_5, \epsilon)\\ \dot{x_5} &= \epsilon X_5(\theta^t\omega, a_5, b_5, x_5, \epsilon)\end{aligned} \tag{8.2.9}$$

with A_5, B_5, X_5 C^{r-1} in $(a_5, b_5, x_5, \epsilon)$, C^0 in t for each fixed ω, and $\mathcal{F}\otimes\mathcal{B}(\mathbb{R}^{n+1})$ measurable.

We name these final coordinates (a_5, b_5, x_5) as (a, b, x). In a small neighborhood of K, say $\{|a| \leq \Delta, |b| \leq \Delta, x \in \overline{K}\}$, we have

$$W^u_\epsilon(\omega, p) = \{(a, b, x)|b = 0\},$$

$$W^s_\epsilon(\omega, p) = \{(a, b, x)|a = 0\},$$

$$W^{uu}_\epsilon(\omega, p) = \{(a, b, x)|b = 0, x = \text{ constant }\},$$

$$W^{ss}_\epsilon(\omega, p) = \{(a, b, x)|a = 0, x = \text{ constant }\}.$$

The system of equations are

$$\begin{aligned}\dot{a} &= \tilde{A}(\theta^t\omega, a, b, x, \epsilon)\\ \dot{b} &= \tilde{B}(\theta^t\omega, a, b, x, \epsilon)\\ \dot{x} &= \epsilon\tilde{X}(\theta^t\omega, a, b, x, \epsilon)\end{aligned} \tag{8.2.10}$$

with $\tilde{A}, \tilde{B}, \tilde{X}$ C^{r-1} in (a, b, x, ϵ), C^0 in t for each fixed ω, and $\mathcal{F}\otimes\mathcal{B}(\mathbb{R}^{n+1})$ measurable.

Note that $\{a = 0\}$ is the random stable manifold, which is invariant. It follows that

$$\tilde{A}(\theta^t\omega, 0, b, x, \epsilon) = 0.$$

So

$$\tilde{A}(\theta^t\omega, a, b, x, \epsilon) = A(\theta^t\omega, a, b, x, \epsilon)a,$$

for some $k_1 \times k_1$ matrix valued function A.

Similarly,

$$\tilde{B}(\theta^t\omega, a, b, x, \epsilon) = B(\theta^t\omega, a, b, x, \epsilon)b,$$

for some $k_3 \times k_3$ matrix valued function B. We have that A, B are C^{r-2} in (a, b, x, ϵ), C^0 in t for each fixed ω, and $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^{n+1})$ measurable.

Since

$$W_\epsilon^{uu}(\omega, p) = \{(a, b, x) | b = 0, x = \text{ constant } \},$$

and

$$W_\epsilon^{ss}(\omega, p) = \{(a, b, x) | a = 0, x = \text{ constant } \},$$

on $\{a = 0\}$ or $\{b = 0\}$, $\tilde{X}$ is independent of b or a respectively. It follows that

$$\tilde{X}(\theta^t\omega, a, b, x, \epsilon) = C(\theta^t\omega, x, \epsilon) + X(\theta^t\omega, a, b, x, \epsilon)ab,$$

for C C^{r-1} in (x, ϵ), X C^{r-3} in (a, b, x, ϵ), and both C and X are C^0 in t for each fixed ω, and measurable.

To prove (8.2.12), note that for $\epsilon = 0$, A, B are independent of ω. Using the $\epsilon-$Jordan form and by compactness of $\overline{K}$, there exists a C^{r-1} basis of $\mathbb{R}^{k_1}$ and $\mathbb{R}^{k_3}$, such that (8.2.12) hold for $\epsilon = 0$. Because $A(\theta^t\omega, a, b, x, \epsilon)$ and $B(\theta^t\omega, a, b, x, \epsilon)$ are uniformly C^0 close to $A(\theta^t\omega, 0, 0, x, 0)$ and $B(\theta^t\omega, 0, 0, x, 0)$, it follows that (8.2.12) also hold for small ϵ.

$\square$

8.2.2 Boundary Value Problem. In this section, we discuss a boundary value problem. All our discussion will be restricted to the compact region:

$$\Omega_K = \{|a| \leq \Delta, \; |b| \leq \Delta, \; x \in \bar{K}\}$$

for K a bounded connected relatively open subset of $\mathbb{R}^m$. The system of equations is the one we obtained in Theorem 8.2.1. For convenience, we write it down here:

$$\begin{aligned}
\dot{a} &= A(\theta^t\omega, a, b, x, \epsilon)a \\
\dot{b} &= B(\theta^t\omega, a, b, x, \epsilon)b \\
\dot{x} &= \epsilon[C(\theta^t\omega, x, \epsilon) + X(\theta^t\omega, a, b, x, \epsilon)ab]
\end{aligned} \tag{8.2.11}$$

for $a \in \mathbb{R}^{k_1}$, $b \in \mathbb{R}^{k_3}$, $x \in \mathbb{R}^m$, A, B are $k_1 \times k_1$ and $k_3 \times k_3$ matrix valued functions, A, B are C^{r-2} in (a, b, x, ϵ), C is C^{r-1} in (x, ϵ), X is C^{r-3} in (a, b, x, ϵ), all of A, B, C, X are C^0 in t for fixed ω, and jointly measurable. A, B satisfy:

$$\begin{aligned}
(a, Aa) &\geq 4\alpha|a|^2 \\
(b, Bb) &\leq -4\alpha|b|^2
\end{aligned} \tag{8.2.12}$$

for some $\alpha > 0$.

The boundary conditions are given as Silnikov type: the values $a(T), b(0), x(0)$ are given. In particular, the time T has the order of $O(\frac{1}{\epsilon})$. In the case that $|a(T)| = |b(0)| = \Delta$, the solution of the boundary value problem corresponds to a set of trajectories (parameterized by ω) which enter the compact region Ω_K and exit the region after a time of order $O(\frac{1}{\epsilon})$. So each solution is actually a random process.

The tool is contraction the mapping principle. First we write the system as the sum of a linear part and a nonlinear part. Then use the variation of constant formula to find out the mapping of which the unique solution is the fixed point. The key part is to prove this mapping is indeed a contraction.

For each $x^0 \in K$, there exists a random process which is generated by the slow flow

$$x' = C(\theta^{\frac{\tau}{\epsilon}}\omega, x, \epsilon)$$

on $\mathbb{R}^m$. Let $\rho_\epsilon(\tau, \omega, \cdot)$ be the random dynamical system generated by the slow flow.

One of the key points is to measure the difference of $x(t,\omega)$ and $\rho_\epsilon(\epsilon t,\omega,x^0)$, where $x(t,\omega)$ is the x component of a sample path, typically a sample path of a solution of the boundary value problem, which has x^0 as the beginning x coordinate. For convenience, we introduce a new dependent variable c to represent this difference:

$$c = x - \rho_\epsilon(\epsilon t,\omega,x^0).$$

So c satisfies:

$$\dot{c} = \bar{C}(\theta^t\omega,a,b,c,x^0,\epsilon)$$

where

$$\bar{C}(\theta^t\omega,a,b,c,x^0,\epsilon) = \epsilon[C(\theta^t\omega,\rho_\epsilon(\epsilon t,\omega,x^0)+c,\epsilon)-C(\theta^t\omega,\rho_\epsilon(\epsilon t,\omega,x^0),\epsilon)+X(\theta^t\omega,a,b,c,x^0,\epsilon)ab].$$

We will discuss the following boundary value problem:

$$\begin{aligned}
\dot{a} &= A(\theta^t\omega,a,b,x,\epsilon)a \\
\dot{b} &= B(\theta^t\omega,a,b,x,\epsilon)b \\
\dot{c} &= \bar{C}(\theta^t\omega,a,b,c,x^0,\epsilon) \\
a(T) = a^1, &\ b(0)=b^0, c(0)=0.
\end{aligned} \tag{8.2.13}$$

We will sometimes use y to mean the argument (a,b), and use z to mean (a,b,c). We use P_i to mean the projection to the i coordinate, for $i=a,b,c,y$. Also we will identify points $a\in\mathbb{R}^{k_1}, b\in\mathbb{R}^{k_3}$ with points $(a,0)$ and $(0,b)\in\mathbb{R}^{k_1+k_3}$.

Theorem 8.2.2. *Let $\bar{\tau}>0$ be fixed. Let $x^0\in K$ such that $\rho_0(\tau,\omega,x^0)\in K$ for $0\le\tau\le\bar{\tau}$. Fix any $\underline{\tau}\in(0,\bar{\tau})$. There exist $\epsilon_0>0,\Delta>0$ such that for any $0<\epsilon<\epsilon_0$, $T\in[\frac{\underline{\tau}}{\epsilon},\frac{\bar{\tau}}{\epsilon}]$, $|a^1|\le\Delta$, $|b^0|\le\Delta$, there exists a unique solution of (8.2.13) which we denote by Ψ. So*

$$\Psi = \Psi(a^1,b^0,x^0,T)$$

is a random process parameterized by (a^1, b_0, x_0, T). *Moreover,* Ψ *is* C^1 *in* (a^1, b_0, x_0, T)*, and the partial derivatives of* Ψ *satisfy:*

$$|(DP_a\Psi)(0)| = O(e^{-q/\epsilon}) \tag{8.2.14}$$

$$|(DP_b\Psi)(T)| = O(e^{-q/\epsilon}) \tag{8.2.15}$$

$$||DP_c\Psi|| = O(e^{-q/\epsilon}) \tag{8.2.16}$$

for some $q > 0$. *The norm* $||\cdot||$ *is equivalent to the maximum norm and is to be defined later.*

Proof. In order to use variation of constant formula, we write $A(\theta^t\omega, a, b, c, x^0, \epsilon)a$ and $B(\theta^t\omega, a, b, c, x_0, \epsilon)b$ as the sum of linear part:

$$\dot{a} = A(\theta^t\omega, 0, 0, \rho_\epsilon(\epsilon t, \omega, x^0), \epsilon)a + l_a(\theta^t\omega, a, b, c, x^0, \epsilon)$$

$$\dot{b} = B(\theta^t\omega, o, o, \rho_\epsilon(\epsilon t, \omega, x^0), \epsilon)b + l_b(\theta^t\omega, a, b, c, x^0, \epsilon)$$

where

$$l_a = [A(\theta^t\omega, a, b, x, \epsilon) - A(\theta^t\omega, 0, 0, \rho_\epsilon(\epsilon t, \omega, x^0), \epsilon)]a,$$

$$l_b = [B(\theta^t\omega, a, b, x, \epsilon) - B(\theta^t\omega, 0, 0, \rho_\epsilon(\epsilon t, \omega, x^0), \epsilon)]a.$$

The first order partial derivatives of l_a, l_b are all bound uniformly. We have

$$\begin{aligned} &|D_a l_a| = O(|z|) \\ &|D_i l_a| = O(|a|) \text{ for } i = b, c, x^0, \end{aligned} \tag{8.2.17}$$

$$\begin{aligned} &|D_b l_b| = O(|z|) \\ &|D_i l_b| = O(|b|) \text{ for } i = a, c, x^0. \end{aligned} \tag{8.2.18}$$

Consider first the linear system:

$$\dot{a} = A(\theta^t\omega, 0, 0, \rho_\epsilon(\epsilon t, \omega, x^0), \epsilon)a$$
$$\dot{b} = B(\theta^t\omega, 0, 0, \rho_\epsilon(\epsilon t, \omega, x^0), \epsilon)b.$$

Let $\Phi(t, t_0, \omega, x^0)$ be the fundamental matrix satisfying $\Phi(t_0, t_0) = I$. It follows that $\Phi(t, t_0)$ commutes with P_a, P_b. Moreover, by (8.2.12),

$$|\Phi(t, t_0)P_a| \leq e^{2\alpha(t-t_0)} \text{ for } t \leq t_0 \tag{8.2.19}$$

$$|\Phi(t, t_0)P_b| \leq e^{-2\alpha(t-t_0)} \text{ for } t \geq t_0 \tag{8.2.20}$$

From variation of constant formula, a random process

$$z(t, \omega) = (y(t, \omega), c(t, \omega)) = (a(t, \omega), b(t, \omega), c(t, \omega))$$

is a solution of (8.2.13) if and only if

$$\begin{aligned} y(t, \omega) =& \Phi(t, T, \omega, x^0)a^1 + \int_T^t \Phi(t, T, \omega, x^0)\Phi^{-1}(s, T, \omega, x^0)l_a(\theta^s\omega, z(s, \omega), x^0)ds \\ &+ \Phi(t, 0, \omega, x^0)b^0 + \int_0^t \Phi(t, 0, \omega, x^0)\Phi^{-1}(s, 0, \omega, x^0)l_b(\theta^s\omega, z(s, \omega), x^0)ds \qquad (8.2.21) \\ c(t, \omega) =& \int_0^t \bar{C}(\theta^s\omega, z(s), x^0)ds \end{aligned}$$

for all $0 \leq t \leq T$.

Let $S = S(T)$ be the space of continuous random process $z(t, \omega) = (a(t, \omega), b(t, \omega), c(t, \omega))$

on $[0, T]$ with the norm

$$||z|| = ||a|| + ||b|| + ||c||$$

$$||a|| = \sup_{0 \le t \le T} e^{\alpha(T-t)} |a(t)|$$

$$||b|| = \sup_{0 \le t \le T} e^{\alpha t} |b(t)|$$

$$||c|| = \sup_{0 \le t \le T} e^{-\kappa \epsilon t} |c(t)|,$$

where κ is a large number to be determined later. This is an equivalent norm to the maximum norm. With this norm, S is complete.

It is obvious that

$$|a(t)| \le ||a||,\ |b(t)| \le ||b||,\ |c(t)| \le e^{\kappa \bar{\tau}} ||c|| \tag{8.2.22}$$

for $0 \le t \le T$.

Define an operator G on S as follows. For any $z(t, \omega) \in S$, Gz is a random process defined by the right hand side of (8.2.21). Let

$$S_\Delta = \{z \in S : ||z|| \le 4\Delta\}.$$

We prove that for ϵ_0, Δ small enough, G maps S_Δ to itself and G is a contraction on S_Δ. Notice the fact that G is a composition of a nonlinear Nemytski operator and bounded linear maps, and the fact that the Nemytski operator is between functional spaces with supremum norms. Then G is C^1 differentiable and the derivatives are given by some explicit form similar to the form of G. So by proving $||DG||$ is uniformly small, we can show G is a contraction.

To prove $||DG||$ is small, we prove the a, b, c components of DG are all small. Consider first $P_a DG$. Let $z(t, \omega) \in S_\Delta$.

$$
\begin{aligned}
&|e^{\alpha(T-t)}P_aD_zG(z)\delta z| \\
=&|e^{\alpha(T-t)}\int_T^t \Phi(t,T,\omega,x^0)\Phi^{-1}(s,T,\omega,x^0)P_aD_zl_a(\theta^s\omega,z(s),x^0)\delta z(s)ds| \\
\leq &\int_t^T e^{\alpha(T-t)}e^{2\alpha(t-s)}|D_al_a(\theta^s\omega,z(s),x^0)\delta a(s) \\
&+D_bl_a(\theta^s\omega,z(s),x^0)\delta b(s)+D_cl_a(\theta^s\omega,z(s),x^0)\delta c(s)|ds \\
=&\int_t^T e^{\alpha(t-s)}[O(|z(s)|)|e^{\alpha(T-s)}\delta a(s) \\
&+O(|e^{\alpha(T-s)}a(s)|)(|\delta b(s)|+|\delta c(s)|)]ds \\
=&O((1+e^{\kappa\bar{\tau}})\Delta)||\delta a||+O(\Delta)(||\delta b||+e^{\kappa\bar{\tau}}||\delta c||) \\
=&O(e^{\kappa\bar{\tau}}\Delta)||\delta z||.
\end{aligned}
\tag{8.2.23}
$$

This shows

$$||D_zP_aG(z)||=O(\Delta). \tag{8.2.24}$$

By similar reasoning, we get

$$||D_zP_bG(z)||=O(\Delta). \tag{8.2.25}$$

Next we consider $||D_zP_cG(z)||$, which is no bigger than $||D_aP_cG(z)||+||D_bP_cG(z)||+||D_cP_cG(z)||$. We first prove $||D_aP_cG(z)||$ is small.

$$
\begin{aligned}
&|e^{-\kappa\epsilon t}(D_aP_cG(z)\delta a)(t)| \\
=&|\epsilon e^{-\kappa\epsilon t}\int_0^t [D_aX(\theta^s\omega,z(s))\delta a(s)a(s)b(s)+X(\theta^s\omega,z(s))\delta a(s)b(s)]ds| \\
=&\epsilon O(\int_0^T |\delta a(s)|\,|b(s)|ds) \\
=&\epsilon O(\int_0^T e^{-\alpha(T-s)}e^{-\alpha s}ds\,||\delta a||\,||b||) \\
=&O(\epsilon\Delta e^{-\frac{\alpha\tau}{\epsilon}})||\delta a||.
\end{aligned}
\tag{8.2.26}
$$

This shows

$$||D_a P_c G(z)|| = O(\Delta). \tag{8.2.27}$$

By similar reasoning, we get

$$||D_b P_c G(z)|| = O(\Delta). \tag{8.2.28}$$

Consider $D_c P_c G(z)$. We have

$$|e^{-\kappa\epsilon t}(D_c P_c G(z)\delta c)(t)| = O(\epsilon \int_0^t |e^{-\kappa\epsilon t}\delta c(s)|ds) = O(\epsilon \int_0^t e^{-\kappa\epsilon(t-s)}ds\,||\delta c||) = O(\frac{1}{\kappa}).$$

This shows

$$||D_c P_c G(z)|| = O(\frac{1}{\kappa}). \tag{8.2.29}$$

To sum up (8.2.24), (8.2.25), (8.2.27), (8.2.28) and (8.2.29), DG is small. We conclude that for Δ sufficiently small and κ sufficiently large, $||DG||$ could be made to be smaller than $\frac{1}{8}$.

We still need to prove that G maps S_Δ to itself. This will follow once we prove that $||G(z_0)|| \leq 3\Delta$ for some $z_0 \in S_\Delta$, since then we would have

$$||G(z)|| \leq ||G(z_0)|| + \frac{1}{8}||z - z_0|| \leq 4\Delta.$$

Consider $G(0)$.

$$G(0) = (\Phi(t, T, \omega, x^0)a^1, \Phi(t, 0, \omega, x^0)b^0, 0),$$

so

$$||G(0)|| \leq |a^1| + |b^0| < 3\Delta.$$

So G is a contraction on S_Δ. Let $\Psi = \Psi(a^1, b^0, x^0, T)$ be the unique fixed point.

Consider the derivative of Ψ. We have

$$\Psi(a^1, b^0, x^0, T) = G(\Psi(a^1, b^0, x^0, T), a^1, b^0, x^0, T).$$

To save notation, let $\mu \equiv (a^1, b^0, x^0, T)$. For $j \in \{a^1, b^0, x^0, T\}$, then by the chain rule

$$D_j\Psi(\mu) = D_zG(\Psi(\mu), \mu)D_j\Psi(\mu) + D_jG(\Psi(\mu), \mu).$$

We have already proved $||D_zG(\Psi(\mu), \mu)|| < \frac{1}{4}$. It follows that

$$||D_j\Psi(\mu)|| \leq ||[I - D_zG(\Psi(\mu), \mu)]^{-1}D_jG(\Psi(\mu), \mu)|| < 2||D_jG(\Psi(\mu), \mu)||.$$

Similarly, the above hold with Ψ, G replaced by $P_i\Psi, P_iG$:

$$||D_jP_i\Psi(\mu)|| < 2||D_jP_iG(\Psi(\mu), \mu)||,$$

for $i = a, b, c$.

First consider D_jP_aG:

$$D_{a^1}P_aG(z, \mu)(t) = \Phi(t, T, \omega, x^0) \tag{8.2.30}$$

$$D_{b^0}P_aG(z, \mu)(t) \equiv 0 \tag{8.2.31}$$

$$\begin{aligned} D_TP_aG(z, \mu)(t) =& D_T\Phi(t, T, \omega, x^0)a^1 + \Phi(t, T, \omega, x^0)l_a(\theta^T\omega, z(T), x^0) \\ &+ \int_T^t D_T[\Phi(t, T, \omega, x^0)\Phi^{-1}(s, T, \omega, x^0)]l_a(\theta^s\omega, z(s, \omega), x^0)ds \end{aligned} \tag{8.2.32}$$

$$D_{x^0}P_aG(z, \mu)(t) = D_{x^0}\Phi(t, T, \omega, x^0)a^1 + \int_T^t D_{x^0}[\Phi(t, T, \omega, x^0)\Phi^{-1}(s, T, \omega, x^0)l_a(\theta^s\omega, z(s, \omega), x^0)]ds \tag{8.2.33}$$

We have

$$\frac{d}{dt}\Phi(t,T,\omega,x^0) = A(\theta^t\omega,0,0,\rho_\epsilon(\epsilon t,\omega,x^0))\Phi(t,T,\omega,x^0)$$

Differentiate both sides with respect to T or x^0 we find out that

$$|D_i\Phi(t,T,\omega,x^0)P_a| = O(e^{-2\alpha(T-t)}) \tag{8.2.34}$$

for $i = x^0, T$.

By (8.2.30), (8.2.31), (8.2.32), (8.2.33), (8.2.34) and similar reasoning as we did in proving (8.2.24), we obtain the boundedness of $|e^{\alpha(T-t)}D_jP_aG(z,\mu)(t)|$. So $||D_jP_aG(z,\mu)||$ is bounded. It follows easily from the definition of the equivalent norm $||\cdot||$ that (8.2.14) holds.

Similarly, we obtain (8.2.15).

Now consider $D_jP_cG(z,\mu)$. Obviously, $P_cG(z,\mu)$ is independent of a^1, b^0, T. We only need to consider $D_{x^0}P_cG(z,\mu)$:

$$D_{x^0}P_cG(z,\mu)(t) = \int_0^t D_{x^0}C(\theta^s\omega,z(s),x^0)ds = \epsilon\int_0^t O(|c(s)| + |a(s)\,|b(s)|)ds. \tag{8.2.35}$$

We have

$$\begin{aligned}
|c(t)| =& |\int_0^t C(\theta^s\omega,z(s),x^0)ds| \\
=& \epsilon\int_0^t O(|c(s)| + |a(s)|\,|b(s)|)ds \\
=& O(\epsilon)\int_0^t |c(s)|ds + O(\epsilon)\int_0^t ||a||\,||b||e^{\alpha(s-T)}e^{-\alpha s}ds \\
=& O(\epsilon)\int_0^t |c(s)|ds + O(\epsilon\Delta^2)\int_0^t e^{-\alpha T}ds \\
=& O(\epsilon)\int_0^t |c(s)|ds + O(e^{-\alpha T}).
\end{aligned} \tag{8.2.36}$$

By Gronwall inequality

$$|c(t)| = O(e^{-\alpha T}e^{O(\epsilon)t}) = O(e^{-\frac{1}{2}\alpha T}) = O(e^{-\frac{1}{2}\alpha\frac{\tau}{\epsilon}}) = O(e^{-\frac{q}{\epsilon}}),$$

so

$$||c|| \leq |c(t)|e^{\kappa\bar{\tau}} = O(e^{-\frac{q}{\epsilon}}),$$

then by (8.2.35)

$$\begin{aligned} D_{x^0}P_cG(z,\mu)(t) &= \epsilon\int_0^t O(|c(s) + |a(s)|\,|b(s)|)ds \\ &= O(e^{-\frac{q}{\epsilon}}\epsilon T) + \epsilon\int_0^t O(\Delta^2)e^{-\alpha(T-s)}e^{-\alpha s}ds \\ &= O(e^{-\frac{q}{\epsilon}}) \end{aligned} \tag{8.2.37}$$

$$||D_{x^0}P_cG(z,\mu)|| \leq O(e^{-\frac{q}{\epsilon}})e^{\kappa\bar{\tau}} = O(e^{-\frac{q}{\epsilon}}).$$

So (8.2.16) also holds.

□

8.2.3 Random Exchange Lemma. Suppose $\Sigma = \{(a,b,x) : b = r(a,x,\omega)\}$ is a random invariant manifold of dimension $k_1 + p + 1$, for $0 \leq p \leq m-1$. Σ intersects the random stable manifold $\{a = 0\}$ transversally. Also, Σ intersects $\{|b| = \Delta\}$ transversally at

$$\Sigma_\Delta = \{(a,b,x) : |b| = \Delta,\, x = u(a,x^0,\omega)\},$$

where $\omega \in \tilde{\Omega} \subset \Omega$, and $x^0 \in J(\omega)$, a $p-$dimensional submanifold of K. Here $\tilde{\Omega}$ satisfies:

$$\bigcup_{\omega\in\tilde{\Omega},t\in\mathbb{R}} \theta^t\omega = \Omega,$$

and for $\omega_1 \neq \omega_2$, $\omega_1, \omega_2 \in \tilde{\Omega}$, we have $\theta^t\omega_1 \neq \theta^s\omega_2$ for any $t,s \in \mathbb{R}$. Moreover, suppose the transversality is at least uniformly of the order ϵ^l for some $l > 0$, so the derivatives of r and u are uniformly bounded below by $C\epsilon^l$.

Suppose there's no equilibrium point for the slow flow ρ_ϵ on K, or more generally suppose

$$|C(\theta^{\frac{\tau}{\epsilon}}\omega, x, \epsilon)| \geq C\epsilon^l.$$

Also, suppose $C(\theta^{\frac{\tau}{\epsilon}}\omega, x, \epsilon)$ is transversal to $J(\omega)$ for $\omega \in \tilde{\Omega}$ and the transversality is at least uniformly of the order ϵ^l, which means the angle:

$$\text{angle } (C(\theta^{\frac{\tau}{\epsilon}}\omega, x, \epsilon), T_{x^0}J(\omega)) \geq C\epsilon^l.$$

Then under the slow random flow ρ_ϵ, $J(\omega)$ evolves to become a $p+1$ dimensional random submanifold of $\mathbb{R}^m$.

Let $P(\omega) \in \Sigma_\Delta(\omega)$, and after a time $\tilde{T} \in (\frac{\underline{\tau}}{\epsilon}, \frac{\bar{\tau}}{\epsilon})$, $P(\omega)$ evolves under the random flow to a point $q(\omega) \in \{|a| = \Delta\}$. Between $(0, \tilde{T})$, it stays in Ω_K. Let $\bar{q}(\omega) = \rho_\epsilon(\epsilon\tilde{T}, x_0, \omega)$, we have the following random version of exchange lemma

Theorem 8.2.3. *There exists a neighborhood $U(\omega)$ of $\bar{q}(\omega)$ in $\rho_\epsilon(\epsilon t, J(\omega), \omega)$ and a C^1 function*

$$h(\omega) : \{|a| \leq \Delta\} \times U(\omega) \to \mathbb{R}^{k_3+m}$$

such that (a, b, x) is a point of $\Sigma(\omega)$ near $q(\omega)$ with

$$(a(T), b(T), x(T)) = \phi(\omega, T)(a^0, b^0, x^0)$$

for some $(a^0, b^0, x^0) \in \Sigma_\Delta$ and $\underline{\tau} < T < \bar{\tau}$ if and only if

$$(b, x) = h(\omega, a, \xi) \text{ and } \xi = \rho_\epsilon(\epsilon T, \omega, x^0).$$

Moreover,

$$|D^k P_b h(a, \xi)| = O(e^{-\frac{\lambda}{\epsilon}}) \tag{8.2.38}$$

$$|D^k(P_x h(a, \xi) - \xi)| = O(e^{-\frac{\lambda}{\epsilon}}) \tag{8.2.39}$$

for $k = 0$ *or* 1 *and some* $\lambda > 0$.

Proof. For each fixed (a^1, x^0, T), define a map $\eta(a^1, x^0, T)(\cdot) : \{|a^0| \leq \Delta\} \to \mathbb{R}^{k_1}$ as follows:

$$a^0 \mapsto P_a\Psi(a^1, r(a^0, u(a^0, x^0, \omega), \omega), x^0, T)(0).$$

Since the derivatives of r and u grow at most as powers of ϵ^{-1} as $\epsilon \to 0$ while those of $P_a\Psi(a^1, b^0, x^0, T)(0)$ are of the order $O(e^{-\frac{\lambda}{\epsilon}})$, it follows that $D_{a^0}\eta$ is uniformly small for ϵ sufficiently small. So η is a contraction.

Let η_f be the unique fixed point of η. Define h by

$$h(a, \xi, \omega) = P_{(b,x)}\phi(\omega, T)(\eta_f(a, x^0, T), r(\eta_f, u(\eta_f, x^0, \omega), \omega), u(\eta_f, x^0, \omega)),$$

then

$$h(a, \xi, \omega) = P_{(b,x)}\Psi(a, r(\eta_f(a, x^0, T), u(\eta_f, x^0, \omega), \omega), x^0, T)(T). \tag{8.2.40}$$

Thus h is well defined. From (8.2.15), (8.2.14), (8.2.40) and the chain rule, (8.2.38) and (8.2.39) are satisfied.

□

Bibliography

[A] L. Arnold, *Random Dynamical Systems*, Springer, New York, (1998).

[BK] I.U.Bronstein and A. Ya. Kopanskii, *Smooth invariant manifolds and normal forms*, vol.7, World Scientific Series on Nonlinear Science, Singapore, (1994).

[BLZ1] P. Bates, K. Lu, and C. Zeng, *Existence and persistence of invariant manifolds for semiflows in Banach space*, Memoirs of the AMS, **135** (1998).

[BLZ2] P. Bates, K. Lu, and C. Zeng, *Persistence of overflowing manifolds for semiflow*, Comm. Pure Appl. Math., **52** (1999), 983–1046.

[BLZ3] P. Bates, K. Lu, and C. Zeng, *Invariant foliations for semiflows near a normally hyperbolic invariant manifold*, Trans. Amer. Math. Soc., **352** (2000), 4641–4676.

[BF] A. Bensoussan and F. Flandoli. *Stochastic inertial manifold*, Stochastics Rep, 53(1–2):13–39, 1995.

[Bx] P. Boxler, *A stochastic version of center manifold theory*, Probab. Theory Related Fields 83 (1989), no. 4, 509545.

[Brun1] P. Brunovsky, *Tracking invariant manifolds without differential forms*, Acta Math. Univ. Comenianae Vol. LXV, 1(1996), 23-32.

[Brun2] P. Brunovsky, C^r *-inclination theorems for singularly perturbed equations*, J. Differential Equations 155(1999), 133-152.

[CDLS] T. Caraballo, J. Duan, K. Lu, and B. Schmalfuss, *Invariant manifolds for random and stochastic partial differential equations*, Adv. Nonlinear Stud. 10 (2010), no. 1, 2352.

[CLR] T. Caraballo, J. Langa and J. C. Robinson *A stochastic pitchfork bifurcation in a reaction-diffusion equation*, Proc. R. Soc. Lond. A **457** (2001), 2441-2453

[C] A. Carverhill, *A formula for the Lyapunov numbers of a stochastic flow. Application to a perturbation theorem*, Stochastics 14 (1985), no. 3, 209226.

[CV] C. Castaing and M. Valadier. *Convex Analysis and Measurable Multifunctions*. LNM 580. Springer–Verlag, Berlin–Heidelberg–New York, 1977.

[DD] G. Da Prato and A. Debussche. Construction of stochastic inertial manifolds using backward integration. *Stochastics Stochastics Rep.*, 59(3–4):305–324, 1996.

[D] S. Dahlke, *Invariant manifolds for products of random diffeomorphisms,* J. Dynam. Differential Equations 9 (1997), no. 2, 157210.

[Di] S. P. Diliberto, *Perturbation theorems for periodic surfaces, I*, Rend. Circ. Mat. Palermo Ser. 2, **9** (1960), 265–299.

[DSL1] J. Duan, K. Lu, and B. Schmalfuss. *Invariant manifolds for stochastic partial differential equations* Annals of Probability. **31**(2003), 2109-2135.

[DSL2] J. Duan, K. Lu and B. Schmalfuss, *Smooth stable and unstable manifolds for stochastic evolutionary equations*, J. Dynamics and Diff. Eqns. **16** (2004), 949-972.

[F1] N. Fenichel, *Persistence and smoothness of invariant manifolds for flows*, Indiana Univ. Math. Journal, **21** (1971), 193–226.

[F2] N. Fenichel, *Asymptotic stability with rate conditions*, Indiana Univ. Math. Journal, **23** (1974), 1109–1137.

[F3] N. Fenichel, *Asymptotic stability with rate conditions II*, Indiana Univ. Math. Journal, **26** (1977), 81–93.

[F4] N. Fenichel, *Geometric singular perturbation theory for ordinary differential equations*, J. Differential Equations, **31** (1979), 53–98.

[GLS] M. Garrido-Atienza, K. Lu, and B. Schmalfuss, *Unstable invariant manifolds for stochastic PDEs driven by a fractional Brownian motion*, J. Differential Equations 248 (2010), no. 7, 16371667.

[GC] T. V. Girya and I. D. Chueshov *Inertial manifolds and stationary measures for stochastically perturbed dissipative dynamical systems*, Sb. Math. 186(1):29–45, 1995.

[H] J. Hadamard, *Sur l'iteration et les solutions asymptotiques des equations differentielles*, Bull. Soc. Math. France 29 (1901), 224-228.

[HP] M.Hirsch and D. Pugh, *Stable manifolds and hyperbolic sets*, Glocal Analysis, Proc. Sympos, Pure Math, **14** (1970), Amer. Math. Soc., Providence, 133–163.

[HPS2] M. W. Hirsch, C.C. Pugh, and M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics, **583**, Springer-Verlag, New York, 1977.

[HPS1] M. W. Hirsch, C.C. Pugh and M. Shub, *Invariant manifolds*, Bull. AMS , **76** (1970), 1015–1019.

[JK] C. K. R. T. Jones and N. Kopell, *Tracking invariant manifolds with differential forms in singularly perturbed systems*, J. Differential Equations, **108** (1994), 64-88.

[JKK] C. K. R. T. Jones, T. Kaper and N. Kopell, *Tracking invariant manifolds up to exponentially small errors*, SIAM J. Math. Anal. 27(2) (1996) 558-577.

[Jo] C.K.R.T. Jones, *Geometric singular perturbation theory*, Lecture Notes in Mathematics, Springer(1994).

[K] T. Kato, *Perturbation theory for linear operators.* Die Grundlehren der mathematischen Wissenschaften, Band 132 Springer-Verlag New York, Inc., New York 1966.

[Li] X-B. Lin, *Shadowing lemma and singularly perturbed boundary value prblems*, SIAM J. Appl. Math. **49** (1989), 26–54.

[LiL] W. Li and K. Lu, *Sternberg theorems for random dynamical systems*, Comm. Pure Appl. Math. 58 (2005), no. 7, 941988.

[Liu1] Liu, W-S, *Exchange lemmas for singular perturbation problems with certain turning points*, J. Differential Equations 167(2000), no. 1, 134-180.

[Liu2] Liu, W-S, *Geometric singular perturbations for multiple turning points: invariant manifolds and exchange lemmas*, J. Dynam. Differential Equations 18 (2006), no. 3, 667-691.

[LL] Lian, Z and Lu, K, *Lyapunov Exponents and Invariant Manifolds for Infinite Dimensional Random Dynamical Systems in a Banach Space*, Memoirs of the AMS., 206 (2010), no.967, 106pp.

[LQ] Liu, P-D. and Qian, M. *Smooth ergodic theory of random dynamical systems*, Lecture Notes in Mathematics, 1606. Springer-Verlag, Berlin, 1995.

[LS] K. Lu and B. Schmalfuss, *Invariant manifolds for stochastic wave equations*, J. Differential Equations 236 (2007), no. 2, 460492.

[Ly] A. M. Lyapunov, *Probleme general de la stabilite du mouvement*, Annals Math. Studies **17** (1947), Princeton, N.J. (Original published in Russian, 1892).

[MS] S.-E. A. Mohammed and M. K. R. Scheutzow, *The stable manifold theorem for stochastic differential equations*, The Annals of Probability, 27(2): 615-652, 1999.

[MZZ] S-E. A. Mohammed, T. Zhang, and H. Zhao. *The stable manifold theorem for semilinear stochastic evolution equations and stochastic partial differential equations,* Memoirs of the American Mathematical Society, 196 (2008), no. 917, 105 pp.

[Ny] Y. Nyemark, *Integral manifolds for differential equations*, Izv. Visch. Uchebn. Zabed. Radiophisica **X** (1967), 321–334.

[PD] J. Palis and W. De Melo, *Geometric Theory of Dynamical Systems*, Springer-Verlag, New York, 1980.

[Pl] V. A. Pliss, *A reduction principle in the theory of the stability of motion*, Izv. Akad. Nauk SSSR, Mat. Ser. **28** (1964), 1279–1324.

[PS] V. A. Pliss and D. R. Sell, *Perturbations of attractors of differential equations*, J. Differential Equations **92** (1991), 100–124.

[P1] O. Perron, *Uber Stabilitat und asymptoticshes Verhalten der Integrale von Differentialgleichungssystemen*, Math. Z. **29** (1928), 129–160.

[P2] O. Perron, *Uber Stabilitat und asymptotisches Verhalten der Losungen eines Systems endlicher Differenzengleichungen*, J. Reine Angew. Math. **161** (1929), 41–64.

[P3] O. Perron, *Die Stabilitatsfrage bei Differentialgleichungen*, Math. Z. **32** (1930), 703–728.

[Ru] D. Ruelle. Characteristic exponents and invariant manifolds in Hilbert spaces. *Ann. of Math.*, 115:243–290, 1982.

[S3] S. Schecter and P. Szmolyan, *Persistence of rarefactions under Dafermos regulariztion: blow-up and an exchange lemma for gain-of-stability turning points*, SIAM J. Appl. Dyn. Syst. 8 (2009), no. 3, 822-853.

[S2] S. Schecter, *Exchange lemmas. II. General exchange lemma*, J. Differential Equations 245 (2008), no. 2, 411441.

[S3] S. Schecter, *Exchange lemmas. I. Deng's lemma*, J. Differential Equations 245 (2008), 392-410.

[S] B. Schmalfuss *A random fixed point theorem and the random graph transformation*, Journal of Mathematical Analysis and Applications, 225(1):91–113, 1998.

[Sa] R. Sacher, *A perturbation theorem for invariant manifolds and Holder continuity*, J. Math. Mech. **18** (1969), 705–762.

[Wa] C. E. Wayne, *Invariant manifolds for parabolic partial differential equations on unbounded domains*, Arch. Rat. Mech. Anal.

[WD] W. Wang and J. Duan, A dynamical approximation for stochastic partial differential equations. *J. Math. Phys.* **48**(2007), No. 10.

[Wan] T. Wanner. Linearization random dynamical systems. In C. Jones, U. Kirchgraber and H. O. Walther, editors, *Dynamics Reported*, Vol. 4, 203-269, Springer-Verlag, New York, 1995.

[Wh] H. Whitney, *Differential manifolds*, Ann. of Math., **37**(1936), 645-680.

[W] S. Wiggins, *Normally hyperbolic invariant manifolds in dynamical systems*, Springer-Verlag, New York, 1994.