Dynamic Modeling, Trajectory Generation and Tracking for Towed Cable Systems

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Doctor of Philosophy

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ABSTRACT

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In this dissertation, we focus on the strategy that places and stabilizes the path of an aerial drogue, which is towed by a mothership aircraft using a long flexible cable, onto a horizontally flat orbit by maneuvering the mothership in the presence of wind. To achieve this goal, several studies for towed cable systems are conducted, which include the dynamic modeling for the cable, trajectory generation strategies for the mothership, trajectory-tracking control law design, and simulation and flight test implementations.

First, a discretized approximation method based on finite element and lumped mass is employed to establish the mathematical model for the towed cable system in the simulation. Two approaches, Gauss’s Principle and Newton’s second law, are utilized to derive the equations of motion for inelastic and elastic cables, respectively. The preliminary studies for several key parameters of the system are conducted to learn their sensitivities to the system motion in the steady state. Flight test results are used to validate the mathematical model as well as to determine an appropriate number of cable links.

Furthermore, differential flatness and model predictive control based methods are used to produce a mothership trajectory that leads the drogue onto a desired orbit. Different desired drogue orbits are utilized to generate required mothership trajectories in different wind conditions. The trajectory generation for a transitional flight in which the system flies from a straight and level flight into a circular orbit is also presented. The numerical results are presented to illustrate the required mothership orbits and its maneuverability in different wind conditions. A waypoint following based strategy for mothership to track its desired trajectory in flight test is developed. The flight test results are also presented to illustrate the effectiveness of the trajectory generation methods.

In addition, a nonlinear time-varying feedback control law is developed to regulate the mothership to follow the desired trajectory in the presence of wind. Cable tensions and wind disturbance are taken into account in the design model and Lyapunov based backstepping technique is employed to develop the controller. The mothership tracking error is proved to be capable of exponentially converging to an ultimate bound, which is a function of the upper limit of the unknown component of the wind. The simulation results are presented to validate the controller.

Finally, a trajectory-tracking strategy for unmanned aerial vehicles is developed where the autopilot is involved in the feedback controller design. The trajectory-tracking controller is derived based on a generalized design model using Lyapunov based backstepping. The augmentations of the design model and trajectory-tracking controller are conducted to involve
the autopilot in the closed-loop system. Lyapunov stability theory is used to guarantee the augmented controller is capable of driving the vehicle to exponentially converge to and follow the desired trajectory with the other states remaining bounded. Numerical and Software-In-the-Loop simulation results are presented to validate the augmented controller. This method presents a framework of implementing the developed trajectory-tracking controllers for unmanned aerial vehicles without any modification to the autopilot.

Keywords: towed cable system, trajectory generation, trajectory tracking, model predictive control, differential flatness, autopilot in the loop, aerial recovery
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Chapter 1

Introduction

1.1 Identification and Significance of the Problem

In the past decades, unmanned aircraft systems (UAS) have been developed in both military and civilian applications. While high-altitude, long-endurance UAS like the Predator and the Global Hawk provide persistent intelligence, surveillance, and reconnaissance (ISR) capabilities, they are a costly and limited resource that cannot be assigned specific tasks. At the other end of the spectrum, miniature air vehicles (MAVs), a class of fixed-wing aircraft with wingspans less than 5 feet, which are characterized by relatively low cost, superior portability, and in some cases, improved stealth, have the potential to open new application areas and broaden the availability of UAS technology. MAVs are typically battery powered, hand launched and belly landed, and therefore may not require a runway for take-off or landing.

The ability to deploy MAVs locally opens many opportunities. However, retrieving MAVs is problematic in certain scenarios. For instance, if a soldier deploys a backpackable MAV on the battlefield to gather time-critical ISR information, it is undesirable for the MAV to return to the soldier because this could disclose his or her location to the enemy. Another scenario of potential application of MAVs is that a large mothership deploys multiple MAVs in a remote location for ISR, wildfire monitoring, or other surveillance. Again for this application, retrieval of the MAV after it has performed its mission is difficult because target locations are often inaccessible, and the MAV may not have enough fuel to return to its home position. Similarly, in disaster areas that are too remote or dangerous, MAV search or monitoring platforms may not be recovered by ground personnel.

The relatively low cost of MAVs suggests that they may be expendable, thereby removing the need for recovery. However, even if the costs are low, MAVs still contain critical
and often classified technology that needs to be kept out of enemy hands. One option is to
destroy the MAV or damage the electronics so that it cannot be reused or reverse engineered.
However, most of the solutions that have been proposed require additional payload on the
MAV. Cost considerations, and the potential that MAV technology could still fall into enemy
hands, will limit the use of this technology.

Innovative recovery techniques are therefore critical to ubiquitous use of MAV tech-
nology. The primary difficulty with aerial recovery is the relative size and speed of the
mothership compared to the MAV. Aerial recovery is much like aerial refueling where the
goal is to extend the operational lifetime of the asset. However, in aerial refueling, the fighter
jet and the tanker can match their airspeeds, which is not possible with MAVs and larger
aircraft. One potential solution to this problem is to use helicopters for the recovery opera-
tion. However, helicopters produce significant prop wash making it difficult for the MAV to
operate in its vicinity.

Our approach is motivated by recent advances on the dynamics of towed cable sys-
tems, where a towplane drags a cable with a drogue at the end. In early work on this
problem, Skop and Choo [1] showed that if the towplane is in a constant-angular-rate orbit
of radius $R$, and the drogue has sufficient aerodynamic drag, then the motion of the drogue
has a stable orbit of radius $r \ll R$. Since the angular rates of the towplane and the drogue
are identical, the speed of the drogue will be significantly less than the speed of the towplane.
Murray [2] showed that the towplane-cable-drogue system is differentially flat with the po-
sition of the drogue being the flat output. In essence, this means that the trajectory of the
towplane is uniquely prescribed by the motion of the drogue. In more recent work, Williams
et al. [3, 4] gave a detailed description of the dynamics of circularly towed drogues and design
strategies for moving from one orbit configuration to another. The objective in [3, 4] was
precision pickup and delivery of payloads on the ground by a fixed-wing aircraft. Therefore,
the focus in [3, 4] and also in [2], was on minimizing the orbit radius of the drogue.

For aerial recovery of MAVs, we take a slightly different approach to the problem.
As shown in Fig. 1.0, rather than attempting to minimize the radius of the orbit of the
drogue, our objective will be to place the drogue onto a stable orbit whose radius $r$ is greater
than the minimum turning radius of the MAV. The basic scenario is that the mothership,
Figure 1.1: This figure shows the baseline concept for aerial recovery strategy. The mothership recovers a MAV by towing a long cable attached to a drogue. The drogue is actuated and can maneuver and communicate with the MAV to facilitate successful capture. The MAV uses vision-based guidance strategies to intercept the drogue.

which could either be unmanned (e.g., Predator), or manned (e.g., AC-130), tows a long cable attached to a drogue and enters an orbit designed to cause the towed body (drogue) to execute an orbit of smaller radius and lower speed (less than the nominal speed of the MAV). The MAV then enters the drogue orbit at its nominal airspeed and overtakes the drogue with a relatively low closing speed.

The objective of this dissertation is to explore innovative techniques that facilitate the MAV/drogue rendezvous by maneuvering the mothership so that the trajectory of the drogue is placed onto a horizontally flat orbit in the presence of wind disturbance. The remainder of this chapter is structured as follows. In Section 1.2, we review literature on the knowledge concerning recovery strategies for unmanned aerial vehicles (UAVs), modeling and control strategies of towed cable systems, and motion planning and control strategies for UAVs. The technical objectives and essential contributions are presented in Section 1.3 and the organization of this dissertation is introduced in Section 1.4.
1.2 Literature Review

In this section, comprehensive reviews are presented to introduce the previous related work on aerial recovery. To begin with, previous strategies used for the UAV recovery are introduced to study the lessons learned in both theoretical exploration and experimental implementation. In addition, detailed reviews of towed cable systems are presented to learn the challenges in dynamic modeling, motion planning and control of the systems. Finally, we review literature of trajectory tracking strategies for UAVs to explore an approach of designing control laws that takes the autopilot into consideration.

1.2.1 UAV Recovery Strategies

UAVs have been employed for a wide variety of military and civilian applications in the past decades. The increasingly critical technology and significant information gathered by UAVs call for protection and retrieval strategies after they complete their missions.

Wyllie [5] studied a recovery strategy using parachute systems for fixed wing UAVs to land on an unprepared terrain. This recovery system was assumed to be employed in a descent environment by virtue of the impact of wind. Some desirable characteristics of UAV recovery systems were proposed, such as safety, protection, accuracy, automation, mobility, reliability and repeatability. The relative merits of different parachute materials were assessed. Flight test implementations were also presented. The author concluded that a good wind estimation and accurate navigation techniques were essential to achieve the optimal deployment position. The parachute recovery system has the merits such as mobility and autonomy, with its ability to land on unprepared ground and relative low communication requirement with the ground station, while it will be subject to descent environment and inaccuracies due to the uncertainty of the wind, and will largely dictate the structural design of the airframe due to the higher landing loads.

Mullens et al. [6] presented an autonomous UAV mission system designed to launch, recover, refuel, rearm and re-launch a UAV. The proposed system, consisted of a vertical take-off and landing (VTOL) UAV and an unmanned ground vehicle (UGV), was used to expand the duration of small UAVs in remote operating area or dangerous field without increasing the risk to personnel. The authors discussed the technical characteristics of the
system prototype, lessons learned in the experiments and some of the military programs and applications. Three major phases of the mission including launch and recovery, refueling and landing were intensively studied, while no guidance and control techniques were introduced. This UGV based recovery platform increases the work load of controlling the UGV and also limits the working range of the UAV.

In recent years, strategies for moving vessels to retrieve and capture UAVs were investigated. The discussions were mainly centered on navigation and control algorithms for self-landing, and the mechanical structure of capturing devices.

Avanzini et al. [7] presented a recovery strategy for a VTOL aircraft to land on a moving ship. The ship motion in the altitude direction was assumed to be a sinusoid signal due to the sea wave. Trajectory generation based on optimal control was conducted off-line. A linearized system dynamic model was used to derive the control law in trajectory tracking. The hardware-in-the-loop (HIL) simulation results showed the ability of the navigation and control algorithms to drive the UAV onto a specified location in the absence of wind.

Khantsis et al. [8] described a genetic programming based control law design for landing a small fixed-wing UAV on a frigate ship deck. The UAV model, ship motion model and wind model were described. The evolutionary algorithm was introduced and examined using simulations. Although the algorithm was computationally expensive, the resulting controllers were able to drive UAVs with different starting positions to the target deck.

Wong et al. [9] studied the parameters that influenced successful ship recovery of a fixed-wing UAV using HIL simulation and flight test. The vessel’s course angle, velocity and the UAV airspeed were concluded to be the key parameters that influenced the miss distance.

Kahn [10] developed a vision-based target tracker and guidance law for small UAVs to be recovered by a net on a moving vessel. This approach particularly focused on the landing phase of the UAV within a safe area. The guidance law combined the telemetry information with machine vision technique. Simulation results showed the ability of the guidance law to recover the UAV with target hitting error within one wingspan in the presence of wind gust.

The inventions shown in patents in past years also expressed interests in capturing devices for UAVs retrieved by vessels. Watts et al. [11] presented a UAV capture system
coupled to the deck of a sea-faring vessel. The system included a single arresting line supported by a stanchion disposed on a rotatable boom. The target aircraft was equipped with an arresting hook [12]. Snediker [13] improved the capturing system in [11] by using a three-degree-of-freedom rotatable boom and a comb-like capture plate coupled to the boom. A ball-like mass coupled to a cord was equipped on the tail of the target UAV.

In the literature, discussion of the scenario of retrieving UAVs using aerial systems was absent. The concept of aerial recovery as shown in Fig. 1.0 was first introduced in [14], where a novel approach of deriving the equations of motion for a towed cable system and a motion control strategy for an active towed body were developed. In the successive years, publications related to aerial recovery can be found in [15–23]. The aerial recovery concept proposed a novel strategy of remotely and movably retrieving UAVs without landing them on specific locations. This dissertation summarizes our previous work in placing the system onto a stable and easy-to-follow orbit that facilitates the rendezvous of the MAV and the drogue in the presence of wind.

1.2.2 Towed Cable Systems

The system shown in Fig. 1.0 is a typical circularly towed cable (TC) system, which includes three components: a towing vehicle (mothership), a cable (string, tether) and a towed body (drogue). Depending on the working environment, towing vehicles are typically an aircraft in the air or a vessel in the sea. The cable used in the TC system is typically a long, thin and relative light-weight string connecting the towing vehicle and the towed body, a device typically with large drag relative to the cable and a small dimension and weight relative to the towing vehicle.

The ability of a cable to transmit forces and electrical signals over great distances allows users to extend their influence to the remote region inaccessible for human. In the past decades, TC systems have been studied in various applications. Such applications include payloads delivery and pickup [1, 24–28], tethered balloon, kite, aerostats and float [29–32], towing wire, pipe, antenna, radar or decoys [33–45], tether-connected munition system [46], towed aircraft [47], terrain following [48] and aerial refueling system [49]. A detailed summary of the review for aerial TC system can be found in [50].
In the literature, the studies of TC systems can be classified into topics like stability and equilibrium analysis, mathematical modeling, model validation, and control strategies. The successive subsections present the reviews in terms of these topics.

**Stability and Equilibrium Analysis**

The early studies of TC systems on stability and equilibrium analysis can be dated back to D. Bernoulli (1700-1782) and L. Euler (1707-1783). They studied the linearized solutions of the nonlinear dynamical equations and conducted an eigenvalue analysis on a TC system rotating around its longitudinal axis by neglecting the aerodynamic drag.

Modern studies began with Kolodner [51] who made a detailed mathematical study of the free whirling of a heavy chain with the tow-point fixed, and showed that if a given mode is considered, then above the linear critical speed the deflection is a continuous function of the speed of rotation. Barnes and Pothier [52] measured the drag coefficients of the towed cable in a wind tunnel with different angles of attack. Wu [53] extended Kolodner’s work [51] by analyzing the mode of the motion and the asymptotic solutions of a heavy string rotating at large angular speeds.

Skop and Choo [1] studied the equilibrium configuration for a cable towed in a circular path using a continuous dynamical model. The equilibrium and boundary conditions were computed by neglecting side and tangential aerodynamic drag forces. Russell and Anderson [54] studied the equilibrium configuration of a towed-body system by treating the cable as a rigid rod connected to a single point mass with zero fluid drag. They also extended their work in studying the stability of a circularly towed flexible cable system subject to fluid drag [55].

Nakagawa and Obata [56] studied the longitudinal stability of a TC system moving in a straight and level flight. The behavior around the steady state were classified into three types of motion modes that strongly depended on the conditions of the TC system. Etkin [57] investigated the eigenvalue analysis based on the linearization of the equations of motion and studied the stability of a body towed using a short cable in a straight and level flight. The author concluded that the lateral instability can be eliminated by attaching the cable forward of and above the center of gravity of the body.
Lambert and Nahon [31] conducted the stability analysis of a tethered aerostat by linearizing the nonlinear model using a finite difference approach in various wind conditions. The authors found that the stability of the system improved with increasing wind speed for all modes except the pendulum mode that has better stability with the longer cable at low wind speeds while with shorter cable at high wind speeds. Williams and Trivailo [3, 45] investigated the stability and equilibria for an aerial TC system flying in a circular orbit. The equilibrium and stability of practical configurations for three different aircraft were also conducted.

The early studies of stability analysis for TC systems were mainly conducted by using linearization technique and assuming that the environment was a uniform flow field in which the disturbances like wind were not present.

Williams and Trivailo [58] investigated the periodic solutions for TC systems towed in circular and elliptical paths in the presence of a horizontal cross-wind. The authors concluded that for a TC system with a 3 km long cable and a towing orbit with radius of 300 m and towing speed of 50 m/s, the significant effect of 5 m/s wind were only illustrated by the deflection of the towed-body from the origin (approximate 900 m in altitude and 2000 m in horizontal direction). However, from the simulation and flight test results in [16, 19, 59], for a TC system with a 100 m long cable and a towing orbit with radius of 100 m and towing speed of 14 m/s, the significant effect of 5 m/s wind were not only illustrated by the horizontal deflection (approximately 40 m) of the towed-body from the center of the orbit of the towing vehicle, but characterized by the large oscillation of the vertical motion (approximately 40 m in amplitude).

It is not hard to see that the periodic solution for TC systems towed in a circular orbit was typically affected by system parameters like the cable length, orbit radius, the towing speed, and the aerodynamic drag coefficient of the towed body. However, in the previous studies, few discussions were conducted to investigate the sensitivity of those parameters to the periodic solutions. In Chapter 2, a preliminary sensitivity study for these parameters are conducted using the flight test results.
Dynamic Modeling

Appropriate mathematical models for TC systems are significant foundations to develop control strategies. The central problem in modeling for TC systems is how the cable is treated. Basically, methods of mathematical modeling for cables in TC systems can be classified into two categories, continuous methods [1, 36, 40, 41, 47, 51, 53, 55, 56, 60–64] and discrete methods [3, 4, 14, 15, 18, 28, 30, 32, 34, 38, 43, 45, 46, 48, 57, 58, 65–72]. Each modeling class has subsets and the choice of which approach to utilize depends on various factors including solution accuracy, computation time, development time and cost, method of formulation, and personal preference [73].

Genin et al. [61] studied the coupling between transverse and longitudinal motion of the cable towed in a straight movement. Nonlinear dynamics of an extensible cable subjected to aerodynamic forces generated by a uniform flow field was used.

Choo and Casarella [74] summarized the merits and demerits of methods like characteristics, finite element, linearization, and lumped mass in analyzing the dynamics of cable-body systems. The authors concluded that finite element method was the most versatile.

Jones and Krausman [30] developed a six-degree-of-freedom dynamic simulation of a tethered aerostat using an elastic multi-link cable model. The behavior of the computer simulation is compared with the real data including wind disturbances. The authors concluded that the simulation was able to duplicate the behavior of the real system to a reasonable degree except for the large deviation of the aerostat in the wind direction and the larger cable tension. Cochran et al. [65] studied dynamic modeling for a TC system using a lumped-mass model of the cable. The towing vehicle is assumed unaffected by the towed vehicle. A short cable (less than 4 m) is used in the simulation and experiment. The results illustrated good match between the mathematical model and the real system.

Nakagawa and Obata [56] studied the longitudinal stability of a TC system moving in a straight and level flight using a continuous cable model. Different stable and unstable modes are studied in the simulation. Clifton et al. [36] studied the steady state of a TC system towed by an orbiting aircraft using a continuous and inelastic cable model. The accuracy of the model was tested by comparing simulation results with the flight test results, which illustrated a good match.
Murray [2] studied the trajectory generation problem using differential flatness for a TC system. The system dynamic model was based on lumped mass finite element. Etkin [57] developed a mathematical model using a flexible, extensible cable to study the stability analysis. Zhu and Rahn [75] derived equations of motion for the perturbed cable-drogue system from steady state. The vibrational equations were linearized and discretized using Galerkin’s method and the analysis of the effects of non-dimensional rotation speed, cable fluid drag, cable length, and point mass was also presented. Dreyer and Vuuren [76] studied the numerical solution of continuous and discrete models for an inflexible two-dimensional towed-cable system.

Kamman and Huston [39] initiate kinematic modeling of a variable length, multiple branch TC system parametrized by orientation angles of different frames. Tension force, pitch angle of the towed body and the depth changes during reel-in and pay-out are studied. Henderson et al. [38] employed a cable model based on finite element method to study the active control strategy of a towed body which is tugged by a large aircraft in a straight path. Chin et al. [62] the orbital movement of a towed body system in which the motion of the cable is described by a system of partial differential equations, and a six degree of freedom model used for the towed body. 500 m cable is used in the simulation.

Yamaguchi et al. [40] investigated the motion control of an undersea towed-cable system in which the cable dynamic was modeled using continuous Differential and Algebraic Equations (DAEs). The cable-body dynamic equations were linearized around the equilibrium, and the linear quadratic integral control method was applied to design the controller. Grost and Costello [46] developed the dynamic model of two projectiles connected using a flexible tether. Lambert and Nahon [31] developed the dynamic model of a tethered aerostat using a flexible, elastic cable. Williams and Trivailo [3] gave a detailed description of the dynamics of circularly TC system.

As proposed in [74], an easy, simple method that can solve any unsteady-state problem to a good accuracy and yet require only small amount of computation time is a need. Dreyer and Vuuren [76] concluded that although the continuous model yields more accurate results than the discrete model, the effort and cost of numerically integrating the continuous model
did not compare favorably with the relative ease and efficiency of solving the discrete model, which yields perhaps surprisingly accurate results.

In Chapter 2, the finite-element-based discrete method is employed to model the cable dynamics, and novel approaches are introduced to derive the equations of motion for TC systems with both elastic and inelastic cables.

Model Validation Using Experimental Data

In previous studies of towed body systems, experimental results were typically used to validate the mathematical model in the simulation [16, 33, 34, 36, 65, 73].

Cochran et al. [65] experimentally validated the theoretical model in a wind tunnel by comparing the lateral motions of the towed body in both experimental and simulation results. Short cables (1.5 – 3 m) and different wind speed conditions were used. Borst et al. [34] compared the drogue altitude and tension forces in flight test and simulation results in which the towing plane flew in an orbital path and a five mile long cable was used.

Hover [33] conducted the experiment in a test tank using a 1000 m long cable to study the control strategy of dynamic positioning of a towed pipe under water. Clifton et al. [36] conducted a flight test by commanding the towing plane on a circular path using a 20,000 ft long cable connected to the drogue. The drogue altitude variations were compared between flight test and simulation results. Williams et al. [73] presented experimental results using a rotated arm in a water tank towing different types of cable. Additional measurements were also taken using a 3 m long cable attached to a ceiling fan spinning at 72 rpm.

The experiments presented in the literature were conducted either using short cables, less than 10 m [65, 73], or long cables, more than 1000 m [33, 34, 36], and aerial TC systems are typically steered by manned aircraft, which make the experiments expensive and difficult to execute and repeat.

To set up a towed body system with long cable in an experiment is non-trivial. The towing vehicles described in the literature are typically large aerial vehicles flying at high airspeeds (more than 100 m/s). In our previous work [16], an unmanned towing vehicle and 100 m long cable were used to collect data for model validation. The purpose was to determine aerodynamic lift and drag coefficients for the drogue in the simulation. In [59],
the model validation was conducted by comparing trajectories of the drogue in flight test with those in simulations using models with different numbers of cable links. Chapter 2 summarizes the data validation work conducted for aerial recovery.

**Control Strategies**

After intensive studies of TC systems on stability analysis and mathematical modeling, in the past decade, control strategies for TC systems have received renewed interest. The typical control strategies for the motion of the towed body include towed body active control \[14, 35, 38, 41, 65\], cable length regulation \[16, 26, 48\], and towing vehicle maneuvering \[2–4, 15, 18, 33, 37, 40, 59, 68\].

In studies of towed body active control, Cochran et al. \[65\] studied the stable and control strategy for the towed body using its control surfaces. A maneuver (guidance) autopilot and a station-keeping autopilot were developed to achieve the switch between different trim conditions. Borst et al. \[34\] studied the stability strategy of avoiding altitude 'yoyo' oscillation using fuzzy logic control. Bourmistrov et al. \[35\] studied the control law for a maneuverable towed body to follow trajectories using inversion of nonlinear dynamic and kinematic equations of motion. The towing plane moved in a straight and level flight. Henderson et al. \[38\] studied the active control strategy to stable the motion of a towed body which is tugged by a large aircraft in a straight path.

Cable length regulation was also employed to control the motion of TC systems. Trivailo et al. \[26\] investigated motion control strategy for towed body using the cable length regulation in payload rendezvous. Optimization based method was employed to design the deploying rate controller for the cable. Williams \[48\] developed a cable winch controller using optimal control based method to achieve terrain following. The cable flexibility and elasticity were neglected in the controller design. In our previous work \[16\], an altitude stable strategy for the towed body using a proportional-derivative controller is developed to regulate the cable winch.

For control strategies using towed body active control and cable length regulation, additional efforts are needed to design specific mechanism for the towed body and the extra
device like winch. The more versatile strategy for motion control of TC systems is the towing vehicle maneuvering.

Hover [33] studied the control strategy of dynamic positioning of a towed pipe under the sea by moving the towing vehicle. Both nonlinear and linear control methods were investigated. The author concluded that the parameters for the nonlinear controller could be obtained by manipulating quantities in a linear framework. Murray [2] developed an approach to determine trajectories for the towing vehicle using differential flatness [77] for a TC system. Yamaguchi et al. [40] studied the motion control of an undersea TC system. The cable-body dynamic equations are linearized around the equilibrium and the linear quadratic integral control method was applied to design the controller. Williams [68] employed the differential flatness to produce the desired trajectory for the towing vehicle and developed a linear receding horizon control law for the towing vehicle to follow the calculated trajectory.

The differential flatness based method in the trajectory generation problem for the towing vehicle is limited by the mathematical model of the cable dynamics which is required to be differentially flat. In addition, the performance limits were not directly taken into consideration during the calculation.

Another popular approach to produce the desired trajectory for the towing vehicle is based on optimal control. Williams [69] employed optimal control based method to produce the trajectory for the towing vehicle so that the towed body can be stabled at certain position. Williams et al. [70] used the optimal control based method to solve the trajectory generation problem in multiple payload pickups using a TC system. In our previous work [59], an optimal trajectory generation method based on model predictive control (MPC) was developed. Different desired drogue trajectories were used to derive the required mothership orbits.

In the literature, towing vehicles were usually large manned aircraft or vessels whose motions were assumed to be unaffected by the cable. However, if smaller aircraft are used as towing vehicles, the cable tension may not be negligible when the mothership maneuvers.

In Chapters 3 and 4, we summarize our previous work in trajectory generation using both differential flatness and optimal control based methods. Furthermore, in Chapter 5, a
nonlinear feedback controller that involves the effect of cable tension on the mothership is developed to lead the mothership following desired trajectory.

### 1.2.3 Control Law Design with Autopilot In the Loop

In the literature, motion control problems for autonomous vehicles can be classified into two categories: path following and trajectory (reference) tracking. Path following problems are primarily concerned with the design of control laws that drive an object (mobile robot, ship, aircraft, etc.) to reach and follow a geometric path, defined as a function of certain path parameter \[78\]. The objective of trajectory tracking is to force the actual trajectory of the object to follow a reference signal, a given function of time.

The early studies of trajectory tracking for an autonomous vehicle can be found in \[79\], where the feedback linearization and Lyapunov based approaches were employed to develop the control laws for unicycle-type and two-steering-wheels mobile robots to follow a predefined path. The kinematic model of the vehicle was derived with respect to a Frenet-Serret frame which parametrized the vehicles relative to the followed path, in terms of distance and orientation.

A detailed review of developments in motion planning and control for nonholonomic system can be found in \[80\]. The authors summarized the generalized formats of system design models, developed methods for motion planning and the approaches in stabilizing the system using feedback control law. The motion planning and control of a car-trailer system was presented in \[81\], where the linearization about the desired trajectory was used to convert the nonlinear kinematic model into locally linear equations and LQR based motion control law was developed to achieve the objectives like parallel parking and docking.

The study of trajectory tracking for UAVs can be found in \[82\], where a generalized error dynamics was presented, and gain-scheduled control was used to drive the UAV follow a specific path. A motion control strategy for marine craft was presented in \[83\], where the objectives of both trajectory tracking and path following were combined by using Lagrange multiplier, and Lyapunov based backstepping was employed to derive the controller. A similar strategy using Lyapunov based backstepping for trajectory tracking of a hovercraft was presented in \[84, 85\], and the experimental results was presented in \[84\].
A motion control strategy accounting for parametric modeling uncertainty was presented in [86], where the Lyapunov based backstepping was used to derive the control law and the experimental results using a hovercraft were presented. A vector field based path following guidance law was presented in [87, 88], where straight-line and circular paths were employed to illustrate the algorithm and Lyapunov stability arguments were also presented.

In the previous studies of motion planning and control for UAVs, most work focused on the derivation of the control laws, while few discussions were presented about the implementation of the control strategy. In the implementation phase of the motion control for UAVs, the relative fixed interface structure of the autopilot may constrain the applicability of the control law.

An overview of the autopilots used on small fixed-wing UAVs can be found in [89]. A path following strategy for UAVs based on adaptive control were presented in [90, 91], which involved autopilot waypoint following capability in the closed-loop dynamic. The experimental results were presented in [91] to validate the adaptive controller.

In Chapter 6, a study in trajectory tracking for UAVs with autopilot in the closed-loop system is presented. A time-varying feedback controller is firstly developed using generalized system model. The combined system which involves the autopilot and UAV dynamics in the loop is employed to derive the augmented controller. The UAV trajectory driven by the resulting system is proved to converge exponentially to the desired trajectory in the sense of Lyapunov.

1.3 Technical Objectives and Contributions

In this Section, the technical objectives and main contributions are presented.

1.3.1 Technical Objectives

The main objective of this dissertation effort is to develop a strategy of stabilizing the drogue orbit by maneuvering the mothership so that the air-to-air rendezvous of the drogue and MAV is facilitated in the final phase of aerial recovery. To support the main objective, we need to conduct mathematical modeling, path planning and control studies,
and simulation and flight test implementation. The specific technical objectives are listed below.

Objective 1. Mathematical Modeling, Simulation, and Model Validation

Develop dynamic models that describe the interaction between the mothership, the cable and the drogue. Using this model perform trade studies and sensitivity analysis to select key system parameters for candidate mothership and drogue. Using flight test results to improve the accuracy and fidelity of the model.

Objective 2. Path Planning, Guidance, and Control of the Mothership

Develop numerically stable path planning algorithms for the mothership that transition the drogue to the vicinity of a stable orbit, and keep the drogue near the orbit in the presence of wind. Develop nonlinear time-varying feedback control law to regulate the mothership to follow the required orbit which puts the drogue onto the desired orbit.

Objective 3. Flight Experimentation and Demonstration

Validate and refine the drogue orbit control through a significant flight experimentation component.

1.3.2 Contributions

The major contributions of this dissertation are listed below.

• A discretized dynamic modeling method is developed for the cable using finite element and lumped mass method. Two approaches, Gauss’s Principle and Newton’s second law, are utilized to derive the equations of motion for inelastic and elastic cables using multi-link models, respectively. Simulation results show the effectiveness of the derived dynamic models.

• Trade studies are conducted in simulation to analyze the sensitivity of several key parameters. Experimental data are used to validate as well as to determine key parameters of the system.
• Differential flatness method is employed to develop the trajectory generation approach for the mothership to place the drogue onto a desired orbit. Different desired drogue orbits are utilized to generate the required mothership trajectories in different wind conditions. The precalculated desired orbits of the mothership in different wind conditions illustrate the requirement for the mothership maneuverability.

• A trajectory tracking control law based on waypoint following is developed to control the mothership to follow required orbits in the flight test. The flight test results show the effectiveness of the approach.

• A wind update strategy is developed which updates the mothership orbit according to the change in the wind. Flight test results validate the strategy.

• An optimal trajectory generation strategy is developed using model predictive control (MPC). A combined objective function with novel $L_1$-norm function is employed in the calculation. The resulting trajectory for the mothership takes the performance limits of the mothership into consideration during the calculation. This technique is able to generate a feasible trajectory for the mothership even though some variables have already reach their limits. The numerical results illustrates the effectiveness of the approach.

• A trajectory tracking control law for the mothership is developed by using Lyapunov based backstepping in the presence of wind. The cable tension and wind disturbances are taken into account in the mothership dynamics. The mothership tracking error is guaranteed to exponentially converge to an ultimate bound, which is a function of the limit of the unknown component of the wind. The simulation results validate the controller.

• A trajectory tracking control law design strategy for UAV is developed by involving the autopilot in the closed-loop system. Lyapunov based stability proof is conducted to guarantee the resulting controller is able to drive the UAV follow the desired trajectory and other states still remain bounded. The numerical and software-in-the-loop (SIL) simulation results validate the control strategy.
1.4 Dissertation Organization

The organization of this dissertation is as follows. In Chapter 2, the mathematical model of the towed cable system used in the simulation is developed, and the model validation are conducted using experimental data. Chapter 3 introduces motion planning and control strategies for the mothership using differential flatness method and waypoint following. Chapter 4 discussed trajectory generation strategies for the TC system using model predictive control. In Chapter 5, the trajectory tracking algorithm using Lyapunov based backstepping is developed for the mothership to follow the required trajectory. Chapter 6 discusses the redesign of the trajectory-tracking control law for the mothership with autopilot in the closed-loop system. Conclusions and future directions are presented in Chapter 7.
Chapter 2

Dynamic Modeling, Trade Studies and Model Validation

Mathematical models of towed-cable systems were established in the literature for both air and underwater environments. Several approaches to model cable dynamics have been described. Choo and Casarella [74] compared various methods and described the relative strengths and limitations of each. They concluded that, despite the heavy computational workload required for implementation, the finite element, or lumped mass technique is the most versatile of the methods studied. As recommended in [74], we also followed this approach in this Chapter. Using the finite element approach, the cable is modeled as a series of $N < \infty$ rigid links with lumped masses at joints. The drogue is considered as a point mass, which is the last joint of the cable.

In the literature, researchers developed the equations of motion for towed cable systems using Lagrange’s method [2, 32] and Kane’s equations [69, 73], which required that the internal and external forces are described explicitly, therefore, do not scale well to a large number of links. These methods resulted in dynamic models that are complicated and difficult to use for the purposes of simulation and control law design.

In this chapter, one major contribution is that we develop the mathematical model for a flexible and inelastic cable using Gauss’s principle, as described in the work of [92]. A similar approach was used in the context of path planning for UAVs in [93]. Gauss’s principle is well-suited to problems with complex internal forces, as seen in mothership-cable-drogue interactions. Rather than computing internal forces between cable links directly, the kinematic constraints are employed.

For a towed cable system with an elastic cable model, Gauss’s principle may not be suitable for the dynamic modeling. Newton’s second law is a fundamental and widely used tool to formulate equations of motion for dynamical systems. However, this method was
seldom used to establish the equations of motion for the cable in the literature. In this chapter, Newton’s second law is used to derive the equations of motion for a flexible and elastic cable.

Given a dynamic model of the towed cable system, a series of sensitivity analyses are needed for several key parameters like the length of the cable, the mass of the drogue, and the drag coefficient of the drogue to understand the effect of these parameters to the system behavior. The results of the sensitivity analyses also facilitate the flight test to start with a feasible setup. In this chapter, sensitivity analyses for several parameters are conducted to investigate the influence of these parameters to the system motion in the steady state.

Investigating the behavior of the towed cable system in flight tests is essential to validate the mathematical model. In this chapter, flight test data are compared with the simulation results to validate the dynamic model. The results using different number of cable links in the simulation are produced to illustrate the effect of the number of cable links to the accuracy of the mathematical model. Hardware configurations in the flight test are also introduced.

The organization of this chapter is as follows. In Section 2.1, the concept of Gauss’s principle is introduced. The derivation of the equations of motion for a flexible and inelastic cable using Gauss’s principle is conducted in Section 2.2. In Section 2.3, the derivation of the equations of motion for a flexible and elastic cable using Newton’s second law is developed. Section 2.4 introduced the applied forces on the cable-drogue system. Preliminary trade studies are conducted in Section 2.5. In Section 2.6, the experimental results are illustrated to validate the mathematical model.

2.1 Gauss’s Principle

Consider a system of \( n \) particles of masses \( m_1, m_2, ..., m_n \). Let the vector \( \mathbf{p}_i = (x_i, y_i, z_i)^T \in \mathbb{R}^3 \) represent the position of the \( i^{th} \) particle of the system in a rectangular inertial reference frame \([92]\). We assume that the \( i^{th} \) particle is subjected to a given impressed force \( \mathbf{F}_i(t) \), so that its acceleration without constraints would be given by the vector \( \mathbf{a}_i = \mathbf{F}_i(t) / m_i \). The three components of the vector \( \mathbf{a}_i \) correspond to the accelerations of
the \(i\)th particle driven by \(\mathbf{F}_i\) in the three mutually perpendicular coordinate directions. Thus the equations of motion without constraints on the particles of the system can be written as

\[
\mathbf{M} \ddot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \dot{\mathbf{x}}(t), t),
\]

where

\[
\begin{align*}
\mathbf{F}(t) &= (\mathbf{F}_1^T, \mathbf{F}_2^T, ..., \mathbf{F}_n^T)^T, \\
\mathbf{a}(t) &= (\mathbf{a}_1^T, \mathbf{a}_2^T, ..., \mathbf{a}_n^T)^T, \\
\mathbf{x}(t) &= (\mathbf{p}_1^T, \mathbf{p}_2^T, ..., \mathbf{p}_n^T)^T, \\
\mathbf{M} &= \text{Diag}(m_1, m_1, m_2, ..., m_n, m_n, m_n).
\end{align*}
\]

In the presence of constraints, the acceleration of each particle at time \(t\) will differ from \(\mathbf{a}(t)\). We denote this constrained acceleration by the 3\(n\)-vector \(\ddot{\mathbf{x}}(t) = (\ddot{\mathbf{p}}_1^T, \ddot{\mathbf{p}}_2^T, ..., \ddot{\mathbf{p}}_n^T)^T\). Gauss’s principle asserts that, among all the accelerations that the system can have at time \(t\) that are compatible with the constraints, the accelerations that actually occur are those that minimize

\[
G(\ddot{\mathbf{x}}) = (\ddot{\mathbf{x}} - \mathbf{a})^T \mathbf{M} (\ddot{\mathbf{x}} - \mathbf{a})^T = \left(\mathbf{M}^{\frac{1}{2}} \ddot{\mathbf{x}} - \mathbf{M}^{\frac{1}{2}} \mathbf{a}\right)^T \left(\mathbf{M}^{\frac{1}{2}} \ddot{\mathbf{x}} - \mathbf{M}^{\frac{1}{2}} \mathbf{a}\right). \tag{2.2}
\]

Assuming that \(q\) constraints can be expressed as linear equality relations between the accelerations of the particles of the system, the constraints will always be of the standard form

\[
\mathbf{A}(\dot{\mathbf{x}}, \mathbf{x}, t) \ddot{\mathbf{x}} = \mathbf{b}(\dot{\mathbf{x}}, \mathbf{x}, t), \tag{2.3}
\]

where the matrix \(\mathbf{A}\) is \(q\) by \(3n\) and the vector \(\mathbf{b}\) is an \(q\)-vector.

Minimizing (2.2) subject to the constraint (2.3) implies that at each instant of time \(t\), the actual acceleration of the system of \(n\) particles is given by

\[
\ddot{\mathbf{x}} = \mathbf{a} + \mathbf{M}^{-\frac{1}{2}} \left(\mathbf{A} \mathbf{M}^{-\frac{1}{2}}\right)^+ (\mathbf{b} - \mathbf{A} \mathbf{a}), \tag{2.4}
\]

where \((\cdot)^+\) is the unique Moore-Penrose inverse [92].
2.2 Derivation of Equations of Motion of the Cable Using Gauss’s Principle

Figure 2.0 depicts the cable-drogue system with the a flexible and inelastic cable modeled as $N$ rigid links. The forces acting on each link are lumped and applied at the joints, and the drogue is the last joint of the cable. Let $p_i = (x_i, y_i, z_i)^T \in \mathbb{R}^3, i = 1, 2, ..., N$ be the location of the $i$th link. The position of the mothership is $p_m = (x_m, y_m, z_m)^T \in \mathbb{R}^3$.

If the point masses associated with each link are unconstrained, then the dynamic equations describing their motion are

\[
\ddot{p}_i = a_i,
\]
\[
\ddot{p}_m = a_m,
\]

where $a_i$ and $a_m \in \mathbb{R}^3$ are the unconstrained accelerations driven by the applied forces in three dimensions. Alternatively, defining $x = (p_1^T, p_2^T, \ldots, p_N^T)^T$ and $a = (a_1^T, a_2^T, \ldots, a_N^T)^T$ gives

\[
\ddot{x} = a. \quad (2.5)
\]

Figure 2.1: $N$-link lumped mass representation of towed cable system with a flexible and inelastic cable. The forces acting on each link are lumped and applied on the joints (red dots). The mothership is the 0th joint (red dots) and the drogue is the last joint.
However, the motion of the point masses on an elastic cable associated with each link are constrained by the relationship

\[
\| \mathbf{p}_1 - \mathbf{p}_m \|^2 = \ell_0^2, \\
\| \mathbf{p}_{i+1} - \mathbf{p}_i \|^2 = \ell_0^2, \quad i = 1, 2, \ldots, N - 1,
\]

where \( \ell_0 = L_0/N \), and \( L_0 \) is the original cable length. These position constraints may also be expressed in the matrix form as

\[
\phi(x, \mathbf{p}_m) \triangleq \begin{pmatrix}
\| \mathbf{p}_1 - \mathbf{p}_m \|^2 - \ell_0^2 \\
\| \mathbf{p}_2 - \mathbf{p}_1 \|^2 - \ell_0^2 \\
\vdots \\
\| \mathbf{p}_N - \mathbf{p}_{N-1} \|^2 - \ell_0^2
\end{pmatrix} = 0. \tag{2.6}
\]

Differentiating (2.6) with respect to time results in the velocity constraints

\[
\psi(x, \mathbf{p}_m) \triangleq \begin{pmatrix}
(p_1 - p_m)^T(\dot{p}_1 - \dot{p}_m) \\
(p_2 - p_1)^T(\dot{p}_2 - \dot{p}_1) \\
\vdots \\
(p_N - p_{N-1})^T(\dot{p}_N - \dot{p}_{N-1})
\end{pmatrix} = 0. \tag{2.7}
\]

Assuming that the motion of the mothership \( (\mathbf{p}_m, \dot{\mathbf{p}}_m, \ddot{\mathbf{p}}_m) \) is known, the acceleration constraints can be written in matrix form as

\[
\mathbf{A}(x) \ddot{x} = \mathbf{b}(\dot{x}, \dot{\mathbf{p}}_m, \ddot{\mathbf{p}}_m), \tag{2.8}
\]

where

\[
\mathbf{A} = \begin{pmatrix}
(p_1 - p_m)^T & 0 & \cdots & 0 \\
-(p_2 - p_1)^T & (p_2 - p_1)^T & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & -(p_N - p_{N-1})^T & (p_N - p_{N-1})^T
\end{pmatrix},
\]
Based on Gauss’s principle, the actual acceleration of the cable-drogue system (2.5) subject to the constraints (2.8) is given by (2.4). The initial conditions for the system must be chosen such that both \( \phi(x, \mathbf{p}_m)^T = 0 \) and \( \psi(x, \mathbf{p}_m)^T = 0 \). As indicated by [93], one of the drawbacks of this method is that while solving (2.4), numerical errors may cause the constraints \( \phi(x, \mathbf{p}_m) \) and \( \psi(x, \mathbf{p}_m) \) to drift from zero. When this happens, Equation (2.4) no longer represents the physical dynamics of the cable. That is to say, no mechanism serves to drive the constraints back to zero. To mitigate this problem, Equation (2.4) is modified as [93]

\[
\ddot{x} = \mathbf{a} + M^{-1/2} (AM^{-1/2})^+ (\mathbf{b} - \mathbf{Aa}) - \gamma_1 \left( \frac{\partial \phi}{\partial \mathbf{x}} \right)^T \phi - \gamma_2 \left( \frac{\partial \psi}{\partial \mathbf{x}} \right)^T \psi,
\]

where \( \gamma_1 \) and \( \gamma_2 \) are positive constants that are tuned through simulation to give satisfactory convergence for the selected link lengths of the cable model. The additional two terms cause the ODE solution to decrease the gradient of the constraints until they are not violated. Selecting \( \gamma_1 \) and \( \gamma_2 \) properly guarantees that the modified equation approximately represents the dynamics of the constrained physical system. For example, for a 1000 m long cable modeled as 10 links (100 meters per link), \( \gamma_1 \) and \( \gamma_2 \) can be given the values of 0.05 and 0.002, respectively. The mass matrix \( M = \text{Diag}(m_l, m_l, ..., m_l, m_d, m_d, m_d) \in \mathbf{R}^{3N \times 3N} \), where \( m_c \) is the total mass of the cable, \( m_l = m_c/N \) is the unit mass of each link, and \( m_d \) is the mass of the drogue.

\[ 2.3 \quad \text{Derivation of Equations of Motion of the Cable Using Newton’s Second Law} \]

In the literature, the dynamics of towed-body systems are typically modeled by assuming that the cable is flexible and inelastic [2, 14, 35, 36, 65, 67]. However, in our own
flight tests we have observed that the cable stretched considerably [16]. An elastic model for the cable is therefore needed to match simulation results to flight results. Williams and Trivailo [3] developed the equations of motion of the cable by introducing an elastic model together with two attitude angles at each joint. In this section, we will develop the cable-drogue dynamics using an elastic model based on Newton’s second law. Figure 2.1 depicts a cable-drogue system with an \( N \)–link cable modeled as a finite number point mass nodes connected by springs. The forces acting on each link are lumped together and applied at the joint. The drogue is the last joint of the cable. Let \( \mathbf{p}_j \in \mathbb{R}^3 \), \( j = 1, 2, \ldots, N \), be the location of the \( j \)th joint. Based on Newton’s second law, dynamic equations for the drogue are given by

\[
    m_N \ddot{\mathbf{p}}_N = T_N + \Omega_N, \\
    \Omega_N = G_N + D_N + L_N,
\]

**Figure 2.2:** \( N \)-link lumped mass representation of towed cable system with a flexible and elastic cable. The forces acting on each link are lumped and applied on the joints (red dots). The mothership is the 0th joint (red dots) and the drogue is the last joint.
and equations of motion of other joints are given by

\[ m_{j-1} \ddot{p}_{j-1} = T_{j-1} + \Omega_{j-1} - T_j, \]
\[ \Omega_{j-1} = G_{j-1} + D_{j-1} + L_{j-1}, \]
\[ j = 2, 3, \cdots, N, \]

where \( m_j \) is the mass of the \( j \)'th joint. The elastic tension force exerted on the \( j \)'th joint is given by

\[ T_j = \frac{EA}{\ell_0} (\|p_{j-1} - p_j\| - \ell_0) \frac{p_{j-1} - p_j}{\|p_{j-1} - p_j\|}, \]

where \( E \) is the Young's modulus, \( A \) is the cross-sectional area of the cable, \( p_0 = p_m \) is the position of the mothership, and \( G_j, D_j, L_j \in \mathbb{R}^3 \) are the gravity, aerodynamic drag and lift forces corresponding to the \( j \)'th point mass, respectively.

### 2.4 Applied Forces on the Towed Cable System

The applied forces on each joint are assumed to consist of gravity and aerodynamic forces.

#### 2.4.1 Gravity

Letting \( e_3 \triangleq (0, 0, 1)^T \) be a unit vector in north-east-down (NED) coordinates, and \( g \) be the gravitational force exerted on a unit mass at Earth sea level, then the gravity forces exerted on the \( j \)'th joint \( G_j \) and the drogue \( G_{dr} \) are given by

\[ G_j = m_j g e_3, \quad \text{and} \quad G_{dr} = m_{dr} g e_3. \quad (2.9) \]

#### 2.4.2 Aerodynamic Forces on the Cable

The aerodynamic forces acting on the cable can be determined based on the cross-flow principle described in [3, 65]. Any components of the forces due to effects such as vortex shedding are not treated in the model. Letting \( \mathbf{l}_j \triangleq p_{j-1} - p_j, j = 1, 2, \cdots, N, \) be the
position vector between the \((j - 1)^{\text{th}}\) and \(j^{\text{th}}\) joints, the corresponding lift and drag forces are calculated using \(v^s_j\), the velocity vector of the \(j^{\text{th}}\) cable segment relative to the wind frame, which can be approximated from the motion of the adjacent joints as

\[
v^s_j = \frac{1}{2} [(\dot{p}_{j-1} - w_c) + (\dot{p}_j - w_c)],
\]

(2.10)

where \(w_c\) is the constant component of the wind expressed in the inertial frame. This approximation approach of the airspeed of the \(j^{\text{th}}\) cable segment requires the involvement of both the \((j - 1)^{\text{th}}\) and the \(j^{\text{th}}\) joints velocity in the inertial frame. Then, the angle of attack of the \(j^{\text{th}}\) segment \(\alpha_j\) can be computed as

\[
\alpha_j = \cos^{-1}\left(\frac{l_j \cdot v^s_j}{\|l_j\|\|v^s_j\|}\right)
\]

(2.11)

Letting \(M_{p_j}\) be the Mach number parallel to the \(j^{\text{th}}\) segment, and \(M_{n_j}\) be the Mach number normal to the \(j^{\text{th}}\) segment, the drag and lift coefficients of the \(j^{\text{th}}\) link are given by [3, 65].

\[
C_{D_j} = C_{f_j} + C_{n_j} \sin^2(\alpha_j),
\]

(2.12)

\[
C_{L_j} = C_{n_j} \sin^2(\alpha_j) \cos(\alpha_j),
\]

(2.13)

where

\[
C_{f_j} = \begin{cases} 
0.038 - 0.0425M_{p_j} & M_{p_j} < 0.4 \\
0.013 + 0.0395(M_{p_j} - 0.85)^2 & M_{p_j} \geq 0.4
\end{cases},
\]

(2.14)

\[
C_{n_j} = 1.17 + M_{n_j}/40 - M_{n_j}^2/4 + 5M_{n_j}^3/8, \quad j = 1, 2, \ldots, N - 1.
\]

(2.15)

The unit vectors defining the directions of the drag and lift forces in the inertial frame are given by

\[
e_{D_j} = -\frac{v^s_j}{\|v^s_j\|},
\]

(2.16)

\[
e_{L_j} = -\frac{(v^s_j \times l_j) \times v^s_j}{\|(v^s_j \times l_j) \times v^s_j\|},
\]

(2.17)
where \( \times \) denotes the cross product of two vectors. Letting \( \rho \) be the atmospheric density, \( \ell_j \) be the stretched length of the \( j \)th cable segment, and \( d \) be the diameter of the cable segment, the drag and lift forces acting on the \( j \)th segment are given by

\[
D_j = \frac{1}{2} \rho C_{D_j} \ell_j d \| \mathbf{v}_j^s \|^2 \mathbf{e}_{D_j}, \tag{2.18}
\]

\[
L_j = \frac{1}{2} \rho C_{L_j} \ell_j d \| \mathbf{v}_j^s \|^2 \mathbf{e}_{L_j}, \tag{2.19}
\]

These lift and drag vectors are assumed to be constant over each cable segment. Hence the aerodynamic forces are lumped and centered at the midpoint of the link. The aerodynamic forces exerted on the \( j \)th joint are given by

\[
F_{aero}^j = \frac{1}{2} \left[ (D_j + D_{j+1}) + (L_j + L_{j+1}) \right], \quad j = 1, 2, \ldots, N - 1.
\]

The aerodynamic force exerted on the drogue from the last segment of the cable is given by

\[
F_{aero}^N = \frac{1}{2} (D_N + L_N).
\]

### 2.4.3 Aerodynamic Forces on the Drogue

Letting \( C_{L_{dr}} \) and \( C_{D_{dr}} \) be the aerodynamic lift and drag coefficients of the drogue respectively, \( S_{dr} \) be the planform area of the drogue wing, \( \mathbf{v}_{dr}^s \) be the velocity vector of the drogue relative to the wind frame, \( \mathbf{e}_{L_{dr}} \) be the unit vector representing the direction of the aerodynamic lift force on the drogue, the aerodynamic lift and drag forces on the drogue are given by

\[
L_{dr} = \frac{1}{2} \rho C_{L_{dr}} S_{dr} \| \mathbf{v}_{dr}^s \|^2 \mathbf{e}_{L_{dr}}, \tag{2.20}
\]

\[
D_{dr} = - \frac{1}{2} \rho C_{D_{dr}} S_{dr} \| \mathbf{v}_{dr}^s \| \mathbf{v}_{dr}^s, \tag{2.21}
\]

where

\[
\mathbf{e}_{L_{dr}} = - \frac{\left( \mathbf{v}_{dr}^s \times \mathbf{e}_3 \right) \times \mathbf{v}_{dr}^s}{\| \left( \mathbf{v}_{dr}^s \times \mathbf{e}_3 \right) \times \mathbf{v}_{dr}^s \|}.
\]
2.5 Preliminary Trade Studies of the Towed Cable System in the Simulation

One of the purposes for developing an accurate simulation model is to determine which design parameters are critical for successful flight test demonstration of the aerial recovery. Preliminary trade studies conducted using an inelastic cable model in the simulation are illustrated in Figs. 2.2, 2.3 and 2.4 in the absence of wind. The equations of motion for the cable-drogue system are derived using Gauss’s principle.

Figure 2.2 indicates that the radius of the drogue orbit in the steady state is strongly dependent on its drag coefficient which ranges from 0.2 to 1.4, while the altitude of the drogue orbit is slight affected by the drag coefficient. This result informs our design decisions for drogue actuation and control. Figure 2.3 indicates that the radius of drogue orbit is also strongly dependent on the mass of the drogue which ranges from 0.159 kg to 2.659 kg, which implies that an abrupt change might happen after the MAV docks on the drogue, and this issue will need to be taken into consideration in the air-to-air rendezvous. Figure 2.4 indicates that increasing the drag coefficient with a fixed mass results in lower altitude and radius of the drogue orbit, and increasing the mass, which ranges from 0.159 kg to 2.659 kg with a fixed drag coefficient results in larger altitude and radius of the drogue orbit.

![Cable length vs. Drogue altitude with different drag coefficient](image1)

(a)

![Cable length vs. Drogue radius with different drag coefficient](image2)

(b)

Figure 2.3: Cable length vs. drogue altitude and radius with different drag coefficient.
2.6 Validation of the Mathematical Model Using Experimental Data

In this section, the fidelity of the mathematical model presented in the previous section was validated with flight data. Because the number of cable links used in the simulation determines the complexity of the equations of motion of the cable, and affects the computation time in the path planning algorithms, in this section, we focus on determining an appropriate number of cable links that strikes a compromise between the accuracy and complexity of the cable mathematical model. From the flight test results, we noticed that the cable (a fishing line) stretched considerably. Therefore, the mathematical model of the cable is derived using Newton’s method in this section such that the cable stretch is considered.
2.6.1 Hardware System Description

The hardware system used to collect experimental data consisted of four elements: a mothership UAS, a hemisphere-shaped drogue, a 100 m long cable and a ground station. The key parameters of the systems are shown in Table 2.0.

Table 2.1: System parameters in flight test

<table>
<thead>
<tr>
<th></th>
<th>Mothership</th>
<th>Drogue</th>
<th>Cable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass (kg)</td>
<td>1.76</td>
<td>0.32</td>
<td>0.02</td>
</tr>
<tr>
<td>$C_L$</td>
<td>0.28</td>
<td>0.01</td>
<td></td>
</tr>
<tr>
<td>$C_D$</td>
<td>0.06</td>
<td>0.42</td>
<td></td>
</tr>
<tr>
<td>Wing area (m$^2$)</td>
<td>0.307</td>
<td>4.02</td>
<td></td>
</tr>
<tr>
<td>Wing span (m)</td>
<td>1.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Diameter (cm)</td>
<td>30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Length (m)</td>
<td>85</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E$ (GPa)</td>
<td>1.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d$ (mm)</td>
<td>0.46</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The mothership, shown in Fig. 2.5 (a), was a fixed wing UAS with two 770 Watt battery-operated motors, and was equipped with a Kestrel 2 autopilot, as shown in Fig. 2.5 (d), and a radio modem to communicate with the ground station. To prevent the cable and drogue from exerting large forces during the landing phase, a cable release module was placed on the underside of the mothership, and was actuated from the ground station. The hemisphere-shaped drogue with 30-cm diameter used in the flight test was constructed of reinforced plastic, as shown in Fig. 2.5 (b). The drogue was equipped with a Kestrel 2 autopilot and radio modem for reporting its position and velocity to the ground station. The cable is a trichloroethylene fishing line, with 0.46 mm diameter and 20 lb maximal load, and the mass of a 100 m cable is approximate 20 g. The ground station consists of a desktop computer with Intel i5 processor running at 3.1 GHz with 8 GB RAM, a radio modem combox, and a remote controller. The ground station control software was Virtual Cockpit (VC), developed by Procerus Technologies\textsuperscript{1}.

\textsuperscript{1}http://procerusuav.com/
2.6.2 Flight Test

In the flight test, the mothership was commanded to follow a loiter of 100 m radius and a constant 125 m altitude with the airspeed commanded at 14 m/s. The results of the system trajectory are shown in Fig. 2.6. The top-down view of the system trajectory presented in Fig. 2.6 (a) shows that a circular mothership orbit resulted in a smaller circular orbit of the drogue. Because of the wind, the center of the drogue orbit shifted to the west. The East-Altitude view of the system trajectory presented in Fig. 2.6 (b) shows that the resulting drogue orbit was inclined because of the wind. The amplitude of the drogue’s altitude oscillation was approximately 20 m. The onboard measurement of GPS velocities of the mothership and drogue, the airspeed of the mothership are shown in Fig. 2.7 (a). It can be seen that the actual airspeed of the mothership essentially followed the commanded value,
and the GPS velocities of the mothership and drogue oscillated between 8 m/s and 20 m/s, which implies the average wind was approximately 6 m/s. Figure 2.7 (b) shows the wind estimation in the north and east directions, respectively. The direction of the wind matched the direction of the center shift of the drogue orbit, while the average magnitude of the wind (approximately 4 m/s) was smaller than the one implied in Fig. 2.7 (a) The difference may come from the GPS drifting error and the wind estimation error.

Figure 2.7: Trajectories of the mothership and drogue in the flight test.

2.6.3 Model Validation

To validate the mathematical model, we force the simulated mothership to follow the same trajectory as the actual mothership, and then compare the motion of the simulated drogue to the motion of the actual drogue. The number of links in the cable are increased until a suitable match is obtained.
(a) This figure shows the mothership airspeed (dash-dot line) essentially followed the commanded value (14 m/s), and the GPS velocities (ground speed) of the mothership (solid line) and drogue (dashed line) oscillated between 8 m/s and 20 m/s, which implies the wind was approximately 6 m/s.

(b) Wind estimation in north (solid line) and east (dashed line) directions. The direction of the wind matched the direction of the center shift of the drogue orbit, and the average magnitude of the wind was approximately 4 m/s.

**Figure 2.8:** Measurements obtained from Virtual Cockpit.

Using the parameters shown in Table 2.0, we conducted simulations using a different numbers of cable links. The airspeed of the mothership was selected as 14 m/s and the constant wind vector was selected as (0.5, -4) m/s in the north-east coordinate. Figure 2.8 overlays the mothership trajectories from both flight test and simulation using different number of cable links in north, east and altitude directions, respectively. It can be seen that the trajectory of the simulated mothership essentially matched the trajectory of the actual mothership during the flight test using different number of cable links. Figure 2.9 shows the trajectories of the simulated drogue using 1, 2 and 5 cable links, respectively, overlaid on the trajectory of the actual drogue during flight test. It can be seen that in the north and east directions, the simulation results essentially match the flight test results. In the altitude direction, as the number of cable links increased, the simulation results followed the flight results more accurately in the sense of phase, but the amplitude of the oscillation slightly decreased. Figure 2.10 shows the top-down and side views of the drogue trajectories from both simulation and the flight test. It can be seen from Fig. 2.9 and 2.10 that as the number of cable links increases, the radii of simulated drogue orbits increased which match the real
Figure 2.9: Comparison of the mothership trajectories in the flight test (solid line) and simulation (dashed line) using different number of cable links in north, east and altitude directions, respectively. It can be seen that the mothership trajectory in the simulation essentially followed the trajectory in the flight test.

Figure 2.10: Comparison of the drogue trajectories in the flight test (solid line) and simulation (dashed line) using a different number of cable links in north, east and altitude directions, respectively. In the north and east directions, the simulation trajectories essentially matched the flight test results. In the altitude direction, as the number of cable links increased, the simulation results followed the flight test results more accurately in the sense of phase, while the amplitude of the oscillation slightly decreased.

drogue orbit more precisely, while the amplitudes of simulated drogue’s altitude oscillations decreased which deviate from the flight test result. Thus, it can be seen that simulations with 1 to 5 cable links verified that an increased number of cable links did not significantly improve the accuracy of the predictions.
(a) Top-down view of drogue trajectories using different number of cable links in the simulation and flight test. As the number of cable links increased, the radius of the drogue orbit expanded. (b) Side view of drogue trajectories using different number of cable links in the simulation and flight test. As the number of cable links increased, the amplitude of the altitude oscillation slightly shrank.

Figure 2.11: Drogue trajectories in the flight test (solid line) and simulation using a different number of cable links (star-dot line for the 1-link model, dashed line for the 2-link model and dotted line for the 5-link model).

2.7 Conclusions

In this chapter, the multi-link cable-drogue dynamic model with a flexible and inelastic cable is established based on Gauss’s principle, which provides a succinct method to model the system with complex kinematic constraints. Newton’s second law is also employed to derive the equations of motion for a flexible and elastic cable. Trade studies based on the inelastic cable model are conducted in the simulation to investigate the sensitivity of several parameters to the system motion in the steady state. The results show that the drag coefficient and the mass of the drogue affect the radius of the drogue orbit more significant than the altitude of the drogue orbit in the steady state. The flight test data are used to validate the mathematical model in the simulation as well as to determine key parameters like the number of cable links used in the simulation. The results show that as the number of cable links increases, the radii of the drogue orbit match the flight test result more precisely, while amplitudes of the drogue altitude oscillations deviate from the flight test result. Thus, an increased number of cable links from 1 to 5 did not significantly improve the accuracy of
the predictions. We could select 2 as the number of cable links with the current hardware configuration to compromise between the lateral and longitudinal matches.
Chapter 3

Motion Planning and Control for the Mothership Based on Differential Flatness and Waypoint Following

Given a mathematical model with sufficient fidelity, we need a strategy to regulate the mothership so that the drogue trajectory follows a desired path. Existing methods for generating the desired trajectory for the towing vehicle (mothership) in towed cable systems can be classified into two categories: differential flatness based methods [2, 15, 18, 19, 68] and optimal control based methods [3, 48, 50, 69, 70, 72]. The concept of differential flatness has proved useful for path planning for complex nonlinear systems. Fliess et al. [77] introduced the characteristics of flat systems and apply the differential flatness theory to vertical take-off aircraft and articulated ground vehicles. Murray [2] presented a differential flatness based solution in which the motion of the system was parametrized using the motion of the towed-body as a flat output. However, Murray’s solution technique had numerical stability problems and was not further developed. A similar scheme of using differential flatness for motion planning of the mothership was discussed by Williams [68], and a control law was developed for the mothership based on its kinematic model.

In this chapter, we make use of the differential flatness property of the cable-drogue system to calculate the inverse dynamics relating a desired drogue orbit to the required orbit of the mothership. The differential flatness based method is typically applied to the discretized model of the cable, and is computationally inexpensive compared to traditional optimal control methods. However, it requires the equations of motion of the system to be differentially flat [77]. In the mathematical model developed in Chapter 2, the motion of the $j^{th}$ joint was related to the motions of the $(j - 1)^{th}$ and $(j + 1)^{th}$ joints, which makes it nontrivial to apply the differential flatness to calculate the required trajectory for the
mothership. Thus, in this chapter, a modification is made in deriving equations of motion for the cable, which makes the mathematical model of the cable is differentially flat.

After the desired mothership trajectory has been calculated, a strategy of implementing the trajectory tracking in the flight test is developed in this chapter. By virtue of the waypoint tracking capability of the autopilot equipped on UAVs, the desired orbital trajectory of the mothership can be converted to a series of waypoints with their airspeeds. The commanded waypoint can be calculated using the orbital angle of the current UAV position.

Because the constant component of the wind estimation is used in the calculation of desired mothership trajectory, a time varying wind requires an update of the desired mothership trajectory in the trajectory tracking. However, the towed cable system is a nonrigid object, which means that the cable connection delays the response of the drogue motion when the motion pattern of the mothership changes. Thus, the instant update of the desired mothership trajectory based on the wind update with large gust may result in an unstable drogue motion. In this chapter, a periodic wind update strategy is developed based on the assumption that the average of the wind does not change abruptly. The flight test results are also presented to validate the control strategy.

The remainder of the chapter is structured as follows. Section 3.1 introduces the concept of differential flatness and develops the trajectory generation algorithm for the mothership using differential flatness. Section 3.2 presents the numerical results of the trajectory generation of the mothership using two different desired drogue trajectories. Section 3.3 shows the flight test results in which the waypoint generation and tracking, and the wind update strategy were tested.

3.1 Trajectory Generation for the Mothership Using Differential Flatness

3.1.1 Differential Flatness

**Definition** A system with state equations $\dot{x} = f(x, u, t)$, where the states $x \in \mathbb{R}^n$ and the inputs $u \in \mathbb{R}^m$, is differentially flat if we can find a set of variables $y \in \mathbb{R}^m$ called
flat outputs and integers \( r \) and \( q \) such that

\[
\begin{align*}
y &= h(x, u, \dot{u}, \ddot{u}, \cdots, u^{(r)}), \\
x &= h_1(y, \dot{y}, \ddot{y}, \cdots, y^{(q)}), \\
u &= h_2(y, \dot{y}, \ddot{y}, \cdots, y^{(q+1)}),
\end{align*}
\]

satisfy the system state equations [77].

3.1.2 Trajectory Generation Using Differential Flatness

To enable the applicability of the differential-flatness based method to derive the mothership trajectory from a desired drogue trajectory using the same model developed in Chapter 2, the airspeed of the \( j \)th cable link other than Eq. (2.10) can be given by

\[
v_j^s = \dot{p}_j - w,
\]

where \( p_j \) is the position of the \( j \)th joint and \( w \) is the wind estimation expressed in the inertial frame. The aerodynamic forces of each link are assumed to be lumped at each joint.

Method 1

Supposing that the trajectory of the drogue \( p_{dr} = p_N \) is smooth, the required tension components in the \( N \)th link of the cable (at the end attached to the drogue) can be calculated from

\[
T_N = (m_{dr} + m_N) \ddot{p}_N - (G_{dr} + D_{dr} + L_{dr} + G_N + D_N + L_N),
\]

where \( m_N \) is the mass of the last joint, and \( D_N \) and \( L_N \) are calculated by using \( p_{N-1} \). Meanwhile, \( T_N \) is exerted by the stretch of the \( N \)th link and can be calculated by

\[
T_N = \frac{EA}{\ell_0} (\|p_{N-1} - p_N\| - \ell_0) \frac{p_{N-1} - p_N}{\|p_{N-1} - p_N\|}.
\]
Thus, $p_{N-1}$ can be calculated by solving the following equation

\[
(m_{dr} + m_N) \ddot{p}_N - (G_{dr} + D_{dr} + L_{dr} + G_N + D_N + L_N)
= \frac{EA}{\ell_0} \left( \|p_{N-1} - p_N\| - \ell_0 \right) \frac{p_{N-1} - p_N}{\|p_{N-1} - p_N\|}.
\]

**Method 2**

Supposing that the trajectory of the drogue $p_{dr} = p_N$ is smooth and supposing that the aerodynamic lift and drag forces of the $j$th link are lumped on the $(j - 1)$th joint, the required tension components in the $N$th link of the cable (at the end attached to the drogue) can be calculated from

\[
T_N = (m_{dr} + m_N) \ddot{p}_N - (G_{dr} + D_{dr} + L_{dr} + G_N).
\]  \hspace{1cm} (3.3)

The stretched length of the $j$th link $\ell_j$ can be calculated by

\[
\ell_j = \ell_0 \left( 1 + \frac{\|T_j\|}{EA} \right), \quad j = 2, 3, \ldots, N,
\]  \hspace{1cm} (3.4)

where $\| \cdot \|$ denotes the Euclidean norm. Then the location of the $(j - 1)$th point mass is related to the $j$th point mass by

\[
p_{j-1} = p_j + \ell_j \frac{T_j}{\|T_j\|}.
\]  \hspace{1cm} (3.5)

Consequently, the forces on the $(j - 1)$th point mass can be calculated as

\[
T_{j-1} = m_{j-1} \ddot{p}_{j-1} - (G_j + D_j + L_j) + T_j, \quad j = N, N-1, \ldots, 1.
\]  \hspace{1cm} (3.6)

At each time step, these equations are applied recursively to each link of the cable until the trajectory of the mothership is calculated.
3.2 Numerical Results

In this section, different desired drogue trajectories are employed to derive desired trajectories for the mothership.

3.2.1 Trajectory Formulation for a Circular Drogue Orbit

A typical circular drogue orbit can be used to derive the desired mothership trajectory. Without loss of the generality, the center of the drogue orbit is assumed to be the origin. Letting \( \mathbf{p}_{dr}^d (t) \triangleq (p_{dr_n}^d, p_{dr_e}^d, p_{dr_d}^d)^T \in \mathbb{R}^3 \) be the desired trajectory of the drogue in the NED frame, \( r_{dr}^d \) be the desired constant orbit radius of the drogue, \( \varphi \) be the orbital angle for a clockwise motion, and \( h_0 \) be the desired constant altitude of the drogue, the desired circular trajectory of the drogue can be written as

\[
\begin{align*}
p_{dr_n}^d (t) &= r_{dr}^d \cos \varphi (t), \quad (3.7) \\
p_{dr_e}^d (t) &= r_{dr}^d \sin \varphi (t), \quad (3.8) \\
p_{dr_d}^d (t) &= -h_0. \quad (3.9)
\end{align*}
\]

Letting \( \mathbf{v}_{dr}^s (t) \in \mathbb{R}^3 \) be the velocity vector of the drogue in the wind frame, and \( \mathbf{w} (t) \triangleq (w_n, w_e, w_d)^T \in \mathbb{R}^3 \) be the wind speed vector in the NED frame, the first and second order time derivatives of \( \mathbf{p}_{dr}^d (t) \) are given by

\[
\begin{align*}
\dot{\mathbf{p}}_{dr}^d &= \begin{pmatrix}
-r_{dr}^d \dot{\varphi} \sin \varphi \\
r_{dr}^d \dot{\varphi} \cos \varphi \\
0
\end{pmatrix} = \mathbf{v}_{dr}^s + \mathbf{w}, \quad (3.10)
\end{align*}
\]

\[
\begin{align*}
\ddot{\mathbf{p}}_{dr}^d &= \begin{pmatrix}
-r_{dr}^d (\ddot{\varphi} \sin \varphi + \dot{\varphi}^2 \cos \varphi) \\
r_{dr}^d (\ddot{\varphi} \cos \varphi - \dot{\varphi}^2 \sin \varphi) \\
0
\end{pmatrix}.
\end{align*}
\]
Assuming that the wind estimation $w_c$ is known, the velocity vector of the last joint of the cable (drogue) in the wind frame is given by

$$v_N = \dot{p}_N - w.$$ 

Then, the aerodynamic lift and drag forces of the drogue, $L_{dr}$ and $D_{dr}$ can be calculated using Eqs. (2.20) and (2.21). Assuming that the mass of the drogue is known, the gravity force of the drogue $G_{dr}$ can be calculated using Eq. (2.9). Supposing that the aerodynamic lift and drag forces of the $j^{th}$ link are lumped on the $(j-1)^{th}$ joint, the required tension force $T_N$ can be calculated using Eq. (3.3). Thus the position of the $(N-1)^{th}$ joint $p_{N-1}$ can be calculated by Eq. (3.5). Then, $\dot{p}_{N-1}$ and $\dot{\dot{p}}_{N-1}$ can be numerically calculated for the use of calculating the position of the $(N-2)^{th}$ joint $p_{N-2}$.

Overall, given the desired $r_{dr}^d$, $h_0$, $\varphi(t)$, and the estimation of the wind vector $w$, the desired trajectory for the mothership can be derived by applying the differential flatness based strategy recursively.

### 3.2.2 Desired Circular Orbit of the Drogue with Constant Ground Speed

The orbit angle for a clockwise motion can be written as $\varphi(t) = v_{dr}^o t/r_{dr}^d$, where $v_{dr}^o > 0$ is the desired constant ground speed of the drogue. Without loss of generality, we assume that the wind comes from the west. Figure 3.0 shows the desired mothership trajectories under different wind conditions by selecting $v_{dr}^o = 14 \text{ m/s}$, $h_0 = 100 \text{ m}$, $r_{dr}^d = 120 \text{ m}$, and using a 85 m long, 5-link cable model.

 Desired mothership orbits, as shown in Fig. 3.0 can be calculated using differential flatness. It can be seen from Fig. 3.0 (a) that as the wind speed increases, in order to lead the drogue path to follow the desired orbit, the desired mothership orbits shift to the west in the horizontal direction. Figure 3.0 (b) shows that the amplitude of the mothership’s altitude oscillation increases when the wind speed increases. The amplitudes of the oscillation increase from 15 m to 70 m when the wind speeds increase from 2 m/s to 10 m/s. Figure 3.0 (c) shows the variation trend of the desired mothership orbits in a 3D view. Figure 3.0 (d) shows the desired airspeed of the mothership versus the orbit angle. It can be seen that airspeed of the
(a) Top-down view of the system trajectories. As the wind speed (from west) increases, the desired mothership orbit shifts to the west.

(b) Side view of the system trajectories. As the wind speed increases, the amplitude of altitude oscillations of desired mothership orbits increases prominently.

(c) 3D view of the system trajectories.

(d) Desired mothership airspeed in different wind conditions. As the wind speed increases, the amplitude of the oscillation of desired mothership airspeeds increases prominently.

**Figure 3.1:** Desired mothership trajectories using desired drogue orbit with constant ground speed in different wind conditions.

mothership experiences increasingly larger oscillation when the wind speed increases. It can be seen that when the wind speed is greater than 6 m/s, the lowest required airspeed drops under 10 m/s while the greatest one increases to approximately 25 m/s.

3.2.3 Desired Circular Orbit of the Drogue with Constant Airspeed

Typically, the autopilot on the UAS is designed to regulate a constant airspeed. Therefore, in this section, a desired drogue trajectory with constant airspeed is used to calculate the desired mothership orbit. The drogue airspeed, $\|v_{dr}\|$, using Eq. (3.10) is given
by
\[ \| \mathbf{v}_{dr} \| = \sqrt{(-r_{dr} \dot{\varphi} \sin \varphi - w_n)^2 + (r_{dr} \dot{\varphi} \cos \varphi - w_e)^2 + w_a^2}. \] (3.11)

Therefore, given a desired drogue airspeed \( v_{dr}^a \), we have
\[ r_{dr}^2 \dot{\varphi}^2 + 2r_{dr} (w_n \sin \varphi - w_e \cos \varphi) \dot{\varphi} + w_n^2 + w_e^2 + w_a^2 - (v_{dr}^a)^2 = 0. \] (3.12)

Solving the quadratic equation for \( \dot{\varphi} \), the clockwise motion is given by
\[ \dot{\varphi} = \frac{w_e \cos \varphi - w_n \sin \varphi + \sqrt{(w_n \sin \varphi - w_e \cos \varphi)^2 - (w_n^2 + w_e^2 + w_a^2 - (v_{dr}^a)^2)}}{r_{dr}}. \] (3.13)

Figure 3.1 shows the desired mothership trajectories in different wind conditions using a desired drogue orbit with constant airspeed by selecting \( v_{dr}^a = 14 \) m/s, \( h_0 = 100 \) m, \( r_{dr}^d = 120 \) m, and using a 85 m long, 5-link cable model. It can be seen from Fig. 3.1 (a) that as the wind speed increases, the mothership orbit shifted to the west in the horizontal direction, which is similar to the results in Fig. 3.0 (a). It can be seen from Fig. 3.1 (b) that the amplitude of the oscillation of desired mothership orbits decreases compared with Fig. 3.0 (b). The amplitude of the oscillation was no more than 5 m for wind speed 10 m/s. Figure 3.1 (c) shows the variation trend of the desired mothership orbits in a 3D view. Figure 3.1 (d) shows the desired airspeed of the mothership versus the orbit angle. It can be seen that the amplitude of mothership’s airspeed oscillations also decreases compared with Fig. 3.0 (d). It can be seen that the lowest required airspeed is greater than the nominal desired airspeed of the drogue while the highest one increases to approximately 24 m/s.

For the purpose of implementation of the strategy of the drogue orbit control in the flight test, the performance limits of the mothership need to be accounted for. Based on the current hardware of the flight test, the airspeed range of the mothership is 10 – 20 m/s. Thus, the mothership is able to follow the desired trajectory shown in Fig. 3.1 in the presence of wind whose speed is smaller than 7 m/s. To increase the capability of the mothership to follow the desired trajectory in larger wind, the amplitudes of oscillations of desired mothership airsreads need to be decreased by changing the parameters of the desired drogue orbit. Because the desired drogue airspeed is required to be the nominal airspeed of the MAV, the
desired orbit radius of the drogue needs to be increased so that amplitudes of oscillations of the desired airspeed could decrease. Figure 3.2 shows the results using the desired orbit radius of the drogue as \( r_{dr}^d = 250 \text{ m} \). It can be seen that desired airspeeds of the mothership remain in \( 14 – 18 \text{ m/s} \). Amplitudes of oscillations of the desired mothership altitude also become smaller relative to Fig. 3.1 (b).
(a) Top-down view of the system trajectories. As the wind speed (from west) increases, the desired mothership orbit shift to the west.

(b) Side view of the system trajectories.

(c) 3D view of the system trajectories.

(d) Desired mothership airspeed in different wind conditions. As the wind speed increases, amplitudes of oscillations of desired mothership airspeeds increase. The largest amplitude is smaller than that in Fig. 3.1 (d).

**Figure 3.3:** Desired mothership trajectories using desired drogue orbit with constant airspeed and orbit radius \( r_{dr}^d = 250 \text{ m} \) in different wind conditions.

### 3.3 Flight Test Results

In this section, an approach to convert the desired mothership trajectory into a series of waypoints is developed. After that, a trajectory tracking method based on waypoint following is introduced. The flight test results are presented to validate the applicability of the motion planning and control strategies.
3.3.1 Waypoint Generation and Tracking

To implement the trajectory tracking for the mothership in the flight test, traditional controllers may be impractical because the autopilot can only take certain commands as inputs. In this section, a strategy is developed to produce a series of waypoints with corresponding airspeeds using the desired mothership trajectory.

Given the current position of the mothership, the procedure of computing the commanded waypoint is as follows:

1. Given a desired drogue orbit, the desired mothership trajectory can be numerically calculated using the differential-flatness-based method. Selecting an appropriate fixed step size in the calculation to guarantee the resulting mothership trajectory includes more than 361 discrete points in one period of the mothership orbit.

2. The center of the desired mothership orbit can be calculated by averaging the coordinates of selected points in Step 1.

3. The orbit angle \( \phi \) of each selected point on the orbit with respect to the orbit center can be calculated with round off and wrapped within \([-180, 180]\) degree.

4. A look-up table can be established with \( \phi \) ascending sorted in the first column. The corresponding desired coordinates in three dimensions, and airspeeds are listed in the successive columns.

5. Removing the rows with duplicates of \( \phi \), a look-up table with 361 rows ascending sorted in \( \phi \) can be obtained.

6. Given the current position of the mothership, its orbit angle \( \phi_p \) can be calculated. Define the orbit angle of the commanded waypoint as \( \phi_c \triangleq \phi_p + \lambda \delta \phi \), where \( \lambda = 1 \) signifies a clockwise orbit and \( \lambda = -1 \) signifies a counterclockwise orbit, and \( \delta \phi \) is a constant integer defined as the offset of the orbit angle of the commanded waypoint and the orbit angle of the current position. The offset integer \( \delta \phi \) is dependent by the step size of the controller and communication frequency, and its value is determined by trial and error.
Figure 3.4: Flight test results using the desired drogue orbit with constant ground speed.
3.3.2 Mothership Trajectory Tracking Using Waypoint Following

Results Using Desired Drogue Orbit with Constant Ground Speed

Given a desired drogue orbit with constant ground speed, flight test results of following the calculated mothership trajectory in the steady state are shown in Fig. 3.3. It is can be seen from Fig. 3.3 (a) (top-down view) and (b) (side view) that the actual mothership path was able to follow the desired trajectory with small tracking error in three dimensions. It also can be see from Fig. 3.3 (a) that the actual drogue path was able to follow the desired orbit precisely except for the western quarter circle of the orbit. The altitude tracking result of the drogue shown in Fig. 3.3 (b) illustrated oscillation with the amplitude of approximately 15 m. The error may come from multiple sources like the inaccuracy of the wind estimation, unknown wind estimation in the down direction and the uncertainty of the mathematical model used for the trajectory generation. It can be seen from Fig. 3.3 (c) and (d) that the waypoint following strategy works very well in manipulating the mothership to follow the desired position and airspeed. It can be seen from Fig. 3.3 (e) and (f) that before the time 60 s, the oscillation of the drogue altitude was less than 10 m, while, after 60 s, because the wind speed changed from \((-3, 2)\) m/s to \((-3.5, 2.5)\), m/s, the old desired mothership orbit which is calculated using wind speed \((-3, 2)\) m/s was unable to lead the drogue onto the desired orbit precisely, which resulted in an oscillation of the drogue altitude with the amplitude of approximately 15 m. Thus, a strategy of updating the desired mothership orbit based on the change of the wind is needed.

Wind Update Strategy

To update the desired mothership orbit based on the change of the wind speed and direction, the traditional instant feedback strategy is not applicable because the towed cable system is a non-rigid system, which means that the cable connection delays the response the drogue motion when the motion pattern of the mothership changes. In this section, the desired mothership orbit is updated once per orbit based on the change of wind estimation by assuming that the wind speed and direction do not change abruptly within each orbit.
The desired mothership orbit is updated once the mothership crosses the half plane \[88\] 

\[ \mathcal{H}(r, n) \triangleq \left\{ p_m \in \mathbb{R}^2 : (p_m - r)^T n \geq 0 \right\}, \]

where \( p_m \) is the position of the mothership in the horizontal plane, \( r \in \mathbb{R}^2 \) and a normal vector \( n \in \mathbb{R}^2 \) are used to define the position half plane. By selecting \( r = (-2, 0)^T \) and \( n = (0, -1)^T \), the half plane is defined as the plane which separates the third and forth quadrant.

To facilitate the wind update during the flight test, the desired trajectories of the mothership in different wind conditions were precalculated using different wind speeds with the same assumed direction (from the west). Each time when the mothership crosses the half plane, the desired mothership orbit is updated using the average wind update in the past circle. When the actual wind direction is not along the assumed direction, the precalculated mothership orbit will be rotated so that the desired mothership orbit corresponds to the actual wind condition.

**Results Using Desired Drogue Orbit with Constant Airspeed**

To facilitate the air-to-air rendezvous for drogue and seeker UAS in the flight test, a desired drogue orbit with constant airspeed and larger radius (250 m) was employed to generate the desired mothership orbit so that the seeker has less curved path to follow and the drogue has a relative stable airspeed which is easier for the seeker UAS to match.

Using the desired drogue orbit in Fig. 3.2 and the wind update strategy introduced in the previous sub-section, Figure 3.4 shows the flight test results using a desired drogue orbit with constant airspeed and a orbit radius of 250 m. The desired mothership orbit was updated at time 20, 100, 180, 260 s. It can be seen from Fig. 3.4 (a) and (b) that the actual mothership path was able to follow the desired trajectory with small tracking error in three dimensions except for a deviation in the southwest corner. This deviation came from a short delay in the communication between MATLAB/Simulink and VC. This phenomenon sometimes happened when MATLAB/Simulink started for first time. The altitude tracking result of the drogue shown in Fig. 3.3 (b) illustrated oscillation with an average amplitude
Figure 3.5: Flight test results using the desired drogue orbit with constant airspeed.

of approximately 10 m except for some large deviation. It can be seen from Fig. 3.4 (c) and (d) that the waypoint following strategy works very well in manipulating the mothership to follow the desired position and airspeed. It can be seen from Fig. 3.4 (d) that an obvious
deviation of the horizontal tracking happened at time period 50 − 100 s, which corresponded to the deviation in Fig. 3.4 (a) at the southwest corner. The reason could be the large wind gust (see Fig. 3.4 (f)) and the inherent communication lag between the autopilot and the combox in the multi-agent hardware structure. Figure 3.4 (e) shows that the amplitude of the oscillation of the drogue altitude hold within 10 m except for the large deviation at time 50 − 100 s which matches the results in Fig. 3.4 (a) and (d). Figure 3.4 (f) shows the wind estimation in north-east frame.

3.4 Conclusions

In this chapter, a trajectory generation algorithm based on differential flatness is developed for the mothership to place the drogue onto a desired circular orbit. The equations of motion of the cable are modified slightly to make the system dynamics differentially flat. The numerical results are used to illustrate desired mothership trajectories and its performance requirements like airspeed in different wind conditions using different desired drogue trajectories. The resulting mothership trajectories using the desired drogue orbit with constant airspeed and larger orbit radius express less requirement of the maneuver on its airspeed and altitude. To implement the trajectory tracking in the flight test, an open-loop waypoint following based approach was developed. The experimental results showed that the waypoint following approach was satisfactory in tracking the commanded waypoints and the airspeed. To take the wind change during the flight into consideration, a periodic trajectory update strategy is developed to update the precalculated mothership trajectory based on the wind prediction. The experimental results for the mothership to follow an orbit with a large radius (250 m) with wind update showed that the mothership was able to lead the drogue onto a desired orbit with the amplitude of the altitude oscillation less than 10 m for 5 min.
Chapter 4

Optimal Trajectory Generation for the Mothership Using Model Predictive Control

In the previous chapter, we introduced the trajectory generation algorithm using differential flatness. The differential flatness based method is typically applied to the discretized model of the cable, and is computationally inexpensive compared to the optimal control methods. However it requires the equations of motion of the system to be differentially flat \[77\]. Another limitation is that this method does not directly take into account the performance limitations of the system, so that the resulting trajectory of the towing vehicle might be impractical. In particular, the resulting trajectories may violate constraint limits on the manipulated variables (e.g. maximum available mothership thrust) or the controlled variables (e.g. tension limitations of the cable).

Optimal control based methods were also used to generate the desired trajectory for the towing vehicle \[3, 4, 69, 71, 72\]. Williams \[3, 69\] employed an optimal control method to find a periodic path for the towing vehicle in order to minimize the motion of the towed body subject to dynamic constraints. The problem was solved using sequential quadratic programming. Williams et al. \[4\] used simulated annealing to solve the optimal control problem in scheduling the orbit radius of the towing vehicle while the system transitions from a straight flight into an orbit. Williams et al. \[71\] used optimal control in determining the motion of the towing plane, as well as the cable deployment rate so that the towed body passed through a set of desired waypoints. Williams \[72\] extended his work to find an optimal elliptical orbit and cable deployment rate to compensate for crosswind disturbances.

Establishing an optimal motion of the towing vehicle subject to constraints using a discretized multi-link cable model is a complicated optimization problem with many states and degrees of freedom. However, discussions of the computational burden were seldom
The typical algorithm used in solving the problem are based on quadratic programming in which squared-error objectives are used. In this chapter, an approach based on Model Predictive Control (MPC) using the $L_1$-norm and squared error objectives are introduced and applied to perform the optimal trajectory generation of the towing vehicle.

MPC with linear models [94–96] has been widely used in industrial applications such as chemical plants and refineries [96]. Model predictive controllers rely on dynamic models of the process, most often linear empirical models obtained by system identification. Because many of these processes have either semi-batch characteristics or nonlinear behavior, the linear models are retrofitted with elements that approximate nonlinear control characteristics to ensure that the linear models are applicable over a wider range of operating conditions and disturbances. Some of the nonlinear process can be controlled by including gain scheduling, switching between multiple models depending on operating conditions, and other logical programming. Recent developments have permitted the direct application of nonlinear models in control applications [97].

Various types of linear models are used for linear MPC applications including finite impulse response, discrete or continuous state space, and Laplace domain functions. When dealing with nonlinear systems, practitioners have attempted to extend linear models through collections of piecewise linear models or linear time varying models. The linear models adapt with either time or as a function of the current state of the system. A more general form that is not dependent on model switching is collections of DAEs in open equation format. These equations may include equality or inequality constraints, integer variables, and differential elements. To apply DAEs to nonlinear models, different approaches have been studied and implemented in the literature, including simultaneous methods [98], decomposition methods [99, 100], efficient nonlinear programming solvers [101], improved estimation techniques [102–105], and large-scale techniques for applications to industrial systems [106, 107].

The objective function used in the control optimization problems are typically based on a weighted squared error or an $L_2$-norm form. A common challenge with real measured values is sensitivity to data outliers; the sensitivity to outliers is exacerbated by the squared error norm.
One novel contribution of this chapter is a new $L_1$-norm objective function for the optimal control problem. These objective forms have a number of advantages over traditional squared-error or $L_2$-norm objectives. The $L_1$-norm form is less sensitive to data outliers and better rejects measurement noise. As mentioned previously, many of the remaining challenges associated with implementing nonlinear models are due to the complexity of the numerical solution techniques and commercial and academic software has been developed to meet this challenge. This project uses the APMonitor Modeling Language [108] to model and solve the large-scale DAE system of equations. Filtered bias updating, Kalman filtering, Moving Horizon Estimation (MHE) and nonlinear MPC can be implemented in this web-services platform through interfaces to MATLAB or Python. In this paper, the optimal control problem of generating a desired trajectory for the mothership with performance limits to place the drogue trajectory onto a desired orbit will be solved by using a nonlinear MPC method. We utilize a combination objective functions in which a squared-error objective form is used in calculating the trajectory tracking error and $L_1$-norm objectives are used in regulating the constraints.

The remainder of the chapter is structured as follows. Section 4.1 introduces the formulation of nonlinear MPC and the combined objective function used in generating a constrained optimal trajectory for the mothership. Section 4.2 shows the numerical results of the optimal trajectory generation of the mothership using various desired drogue motions.

4.1 MPC Formulation

Given a desired trajectory for the drogue (e.g., a level circular orbit), a strategy is needed to generate a mothership trajectory that produces the desired drogue path. Differential flatness has been used in the trajectory generation of the towed cable system in [2, 15, 18, 68] by applying certain types of dynamic models of the cable. For the mathematical model presented in Chapter 2, it is nontrivial to calculate the desired mothership trajectory by using differential flatness. In addition, since the mothership has performance limits like airspeed, roll angle, and climbing rate, differential flatness based methods do not directly take these constraints into consideration. Furthermore, during transitions between
straight and level flight and orbital flight, the tension forces exerted on the cable should not exceed the loading limit of the cable.

In this section, we will introduce a strategy to produce an optimal trajectory for the mothership to place the drogue onto the desired orbit in the presence of system constraints. In the literature, when formulating the optimal control problem for generating the desired mothership trajectory, the nonlinear kinematic or dynamic models of the mothership were typically used, and the objective functions were usually based on squared error or \( L_2 \)-norm form. In this section, to reduce the computation time, we use linear dynamic equations of the mothership and a novel format of the objective function to formulate the MPC problem.

Letting \( v_m \in \mathbb{R}^3 \) be the mothership velocity relative to the wind frame, \( w_c \) be the constant component of the wind expressed in the inertial frame, and \( a_m \in \mathbb{R}^3 \) be the mothership acceleration in the inertial frame, which is selected as the input of the mothership dynamic, the equations of motion for the mothership are given by

\[
\dot{p}_m = v_m + w_c, \quad (4.1) \\
\dot{v}_m = a_m. \quad (4.2)
\]

The performance constraints of the mothership are typically given by the magnitude of the mothership airspeed \( V_a \), the heading angle \( \psi \), the bank angle \( \phi \), and the air mass referenced flight path angle \( \gamma_a \), which is defined as the angle from the inertial north-east plane to the velocity vector of the aircraft relative to the air mass. The kinematic equations of motion for the mothership are written as [88]

\[
\dot{p}_n = V_a \cos \psi \cos \gamma_a + w_n, \quad (4.3) \\
\dot{p}_e = V_a \sin \psi \cos \gamma_a + w_e, \quad (4.4) \\
\dot{p}_d = -V_a \sin \gamma_a + w_d, \quad (4.5) \\
\dot{\psi} = \frac{g}{V_a} \tan \phi. \quad (4.6)
\]
By comparing Eqs. (4.1) and (4.2), the constrained variables can be expressed as

\[
V_a = \|v_m\|, \\
\psi = \tan^{-1}\left( \frac{v_m(2)}{v_m(1)} \right), \\
\gamma_a = -\sin^{-1}\left( \frac{v_m(3)}{V_a} \right), \\
\dot{\psi} = a_m(2)v_m(1) - a_m(1)v_m(2) \|
\]

\[
V_a \parallel v_m \|^2, \\
\phi = \tan^{-1}\left( \frac{V_a g}{\dot{\psi}} \right).
\]

Letting \(p_{dt}(t) \in \mathbb{R}^3\) be the desired drogue trajectory, \(p_{dr}(t) \in \mathbb{R}^3\) be the actual drogue trajectory, \(\Phi \in \mathbb{R}\) be the objective function value, \(y \triangleq (y_0, \ldots, y_n)^T \in \mathbb{R}^n\) be the controlled variables, \(y_{hi}, y_{lo} \in \mathbb{R}^n\) be the higher and lower dead-bands of the controlled variables, \(w_{hi}, w_{lo} \in \mathbb{R}^n\) be the weights on the controlled variables outside the dead-bands, \(u \in \mathbb{R}^n\) be the inputs, \(c_y, c_u \in \mathbb{R}^n\) are cost weights of \(y\) and \(u\), \(f: \mathbb{R}^n \to \mathbb{R}^n\) represent the equations of motion, \(x \in \mathbb{R}^n\) be the states of the equations of motion, \(d \in \mathbb{R}^n\) be parameters or unmeasured disturbances, \(g_{iec}: \mathbb{R}^n \to \mathbb{R}^n\) represent the inequality constraints, \(\tau \in \mathbb{R}\) be the response time constant of the desired controlled variables, \(s_{ph}, s_{pl} \in \mathbb{R}^n\) be dead-band setpoints, and \(e_{hi}\) and \(e_{lo}\) be slack variables \(^1\) selected by the optimizer to penalize \(y\) above and below the dead-band, and given by

\[
e_{hi,i} = \begin{cases} 
  y_i - y_{hi,i} & y_i - y_{hi,i} \geq 0 \\
  0 & y_i - y_{hi,i} < 0
\end{cases}, \quad \text{and} \quad e_{lo,i} = \begin{cases} 
  y_{lo,i} - y_i & y_{lo,i} - y_i \geq 0 \\
  0 & y_{lo,i} - y_i < 0
\end{cases}, \quad i = 1, \ldots, n,
\]

\(^1\)http://apmonitor.com/wiki/index.php/Main/SlackVariables
then the trajectory generation problem can be posed as the following optimization problem:

\[
\min_{u(t_0, t_1)} \Phi = w_{hi}^T e_{hi} + w_{lo}^T e_{lo} + y^T c_y + u^T c_u \\
+ \int_{t_0}^{t_1} (p_{dr}(\delta) - p_{dr}^d(\delta))^T (p_{dr}(\delta) - p_{dr}^d(\delta)) \, d\delta, \quad (4.7a)
\]

s.t. \( f(\dot{x}, x, y, u, d) = 0, \) \( (4.7b) \)

\[
g_{iec}(\dot{x}, x, y, u, d) \geq 0, \) \( (4.7c) \)

\[
\tau \frac{\partial y_{hi}}{\partial t} + y_{hi} = sp_{hi}, \quad (4.7d)
\]

\[
\tau \frac{\partial y_{lo}}{\partial t} + y_{lo} = sp_{lo}. \quad (4.7e)
\]

A combination of \( L_1 \)-norm and squared-error objectives, shown in Eq. (4.7a), is used to accomplish multiple objectives. The controlled variables \( y \) are selected as the constraints of the system. The \( L_1 \)-norm objective was used to regulate high-priority constraints like the cable tension and the airspeed of the mothership. In this case, the controlled variables were not forced to follow a desired trajectory, but were constrained to remain within certain range of acceptable limits. The slack variables \( e_{hi} \) and \( e_{lo} \) are then used to regulate \( y \) to remain within dynamic dead-bands parametrized by \( y_{hi} \) and \( y_{lo} \). The squared-error (integration) term was used to penalize the trajectory tracking error of the drogue with a lower weighting that represented the lower priority of the tracking objective. Eqs. (4.7b) and (4.7c) are used to regulate the states to satisfy the equations of motion and some other inequality constraints.

Eqs. (4.7d) and (4.7e) are linear first order equations that define the regulations for controlled variables represented by either a dead-band or reference trajectory to the setpoints. The setpoints \( sp_{hi} \) and \( sp_{lo} \) are used to define regions that are not penalized in the objective function and are referred to as the “dead-band”. It is desirable to make the evolution of the controlled variables effectively approach setpoints at a specified rate so that excessive movements of manipulated variables or response overshoots of controlled variables can be avoided. Based on different control objectives, the initial conditions of \( y_{hi} \) and \( y_{lo} \), can be set to give a wider dead-band at the beginning of the simulation and only to enforce the steady
state response (and vice versa). Different initial conditions defining a wide or narrow dead-
band are the trade-off between precise steady-state response and precise dynamic evolution.
Therefore, reference trajectories of $y$ by specifying the values of $sp_{hi}$ and $sp_{lo}$ can be a step,
ramp or other dynamic signals.

4.2 Numerical Results

In this section, we use the MPC strategy described in Section 4.1 to compute open-
loop mothership trajectories, for a variety of desired drogue trajectories. The performance
limits of the mothership are selected as $\psi \in [-0.35, 0.35]$ rad/s, $V_a \in [10, 20]$ m/s and $\gamma_a \in
[-0.35, 0.35]$ rad. The MPC problem was solved using APMonitor Modeling Language [108].
A step size of 2 seconds was selected as a compromise between the computation time and
the accuracy of the results. The computer used to solve the optimization problem has an
AMD 64 core processor with 64 GB of RAM.

4.2.1 Desired Drogue Orbit with Constant Ground Speed

In the final phase of the aerial recovery scenario, the drogue must be placed onto an
orbit that can be easily followed by the MAV. In this section, the desired drogue trajectory
is a constant altitude circular orbit with constant ground speed. Let $r_{dr}^d$ be the desired
constant orbit radius of the drogue, $\varphi(t)$ be the orbital angle of the orbit measured from
the north, and $h_0$ be the desired constant altitude of the drogue, the desired circular orbit
of the drogue in three dimensions is given by

$$ p_{dr,n}^d(t) = r_{dr}^d \cos \varphi(t), \quad (4.8) $$
$$ p_{dr,e}^d(t) = r_{dr}^d \sin \varphi(t), \quad (4.9) $$
$$ p_{dr,a}^d(t) = -h_0. \quad (4.10) $$

Letting $v_{dr}^g$ be the desired constant ground speed of the drogue, the orbit angle for a clockwise
motion can be written as $\varphi(t) = v_{dr}^g t/r_{dr}$. Without loss of generality, the wind is assumed
from the west direction. A typical circular orbit for drogue can be parametrized by selecting
$v_{dr}^g = 12$ m/s, $h_0 = 100$ m, $r_{dr}^d = 100$ m, and the orbit period can be calculated by $T_p^g = \ldots$
Table 4.1: Solution results using the desired drogue orbit with constant ground speed

<table>
<thead>
<tr>
<th>Wind speed (m/s)</th>
<th>( \mathbf{p}_m^0 ) (m)</th>
<th>( \mathbf{v}_m^0 ) (m/s)</th>
<th>Solution Time (s)</th>
<th>( \Phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(-79, 90, -157)</td>
<td>(-10, -8, 0)</td>
<td>18.4</td>
<td>0.1</td>
</tr>
<tr>
<td>( (0, 5, 0)^T )</td>
<td>(-67, 55, -157)</td>
<td>(-9.4, -7.3, 2)</td>
<td>17.1</td>
<td>0.2</td>
</tr>
<tr>
<td>( (0, 10, 0)^T )</td>
<td>(-61, 35, -142)</td>
<td>(-9, -5, 2)</td>
<td>48.1</td>
<td>7965.1</td>
</tr>
</tbody>
</table>

\( \mathbf{p}_m^0 \) = initial mothership position; \( \mathbf{v}_m^0 \) = initial mothership velocity in NED frame.

\( 2\pi r_{dr}^d / v_{dr}^g \) = 52.36 s. The starting and end clocks are selected as \( t_0 = 0 \) and \( t_1 = 70 \text{ s} \), so that the resulting mothership trajectory has enough waypoints to produce an orbit. The optimal control solver is selected as IPOPT [109], which is an open-source Interior Point solver for solving nonlinear programming (NLP) problems included with the COIN-OR collection. The initial configuration and solution results are shown in Table 4.0. It can be seen that the objective function values were less than 0.5 when the wind speeds were less than 5 m/s, and increased to approximately 8000 when the wind speed increased to 10 m/s. This is because the existing performance limits of the mothership made the resulting optimal orbit unable to precisely place the drogue orbit onto the desired orbit.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{system_trajectory.png}
\caption{3-D view.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{system_trajectory_topdown.png}
\caption{Top-down view.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{system_trajectory_side.png}
\caption{Side view.}
\end{figure}

Figure 4.0 shows the different views of optimal system trajectories in the absence of wind. It can be seen that a horizontally flat desired drogue orbit resulted in a desired horizontally flat mothership orbit (dash-dot line). The actual drogue trajectory (dashed line) follows the desired orbit precisely. The cable (solid line) curved because of the aerodynamic forces exerted on the joint.

Figure 4.1: Optimal system trajectories in the absence of wind. A horizontally flat desired drogue orbit (triangle-dot line) resulted in a desired horizontally flat mothership orbit (dash-dot line). The actual drogue trajectory (dashed line) follows the desired orbit precisely. The cable (solid line) curved because of the aerodynamic forces exerted on the joint.
**Figure 4.2:** Optimal system trajectories using a desired drogue orbit with constant ground speed in the presence of 5 m/s wind from the west. A horizontally flat desired drogue orbit (triangle-dot line) resulted in an inclined desired mothership orbit (dash-dot line). The center of the mothership orbit shifted to the west of the center of the drogue orbit. The amplitude of the mothership’s altitude oscillation was approximately 40 m.

**Figure 4.3:** Evolution of constrained variables of the mothership using a desired drogue orbit with constant ground speed in the presence of 5 m/s wind from the west. All the variable values (solid lines) remained within their limits (dashed lines).
Figure 4.4: Optimal system trajectories using a desired drogue orbit with constant ground speed in the presence of 10 m/s wind from the west. The resulting optimal mothership orbit (dash-dot line) was unable to place the actual drogue trajectory (dashed line) onto the desired orbit (triangle-dot line) precisely because of the performance limits. The amplitude of the mothership’s altitude oscillation was approximately 70 m, while the amplitude of the drogue’s altitude oscillation was approximately 15 m.

Figure 4.5: Evolution of constrained variables of the mothership using a desired drogue orbit with constant ground speed in the presence of 10 m/s wind from the west. All variables (solid lines) reached the limits (dashed lines) during the simulation. The optimizer was able to produce an optimal trajectory for the mothership by keeping all the constrained variables within their limits.

wind. Because of the performance limits of the mothership, the resulting optimal orbit was unable to precisely place the drogue orbit onto the desired orbit, which results in a large value for the objective function in Table 4.0. The amplitude of the mothership’s altitude oscillation was approximately 70 m, while the amplitude of the drogue’s altitude oscillation was approximately 15 m. Figure 4.4 shows the evolutions of the constraints $V_a$, $\gamma_a$ and $\dot{\psi}$ of the mothership. It can be seen that although all the constrained variables reached their
limits during the flight, the optimizer was able to produce an optimal trajectory for the mothership to place the drogue orbit to essentially follow the desired orbit.

4.2.2 Desired Drogue Orbit with Constant Airspeed

Because the autopilot on the MAV is typically designed to regulate a constant airspeed, in this section, a desired drogue trajectory with constant airspeed is used to calculate the desired mothership orbit. The time derivative of \( \mathbf{p}_{dr}^d(t) \) using Eqs. (4.8) to (4.10) is given by

\[
\dot{\mathbf{p}}_{dr}^d = \begin{pmatrix}
-r_{dr}^d \dot{\phi} \sin \phi \\
r_{dr}^d \dot{\phi} \cos \phi \\
0
\end{pmatrix} = \mathbf{v}_{dr}^a + \mathbf{w}.
\]  

(4.11)

Using Eqs. (3.11)-(3.13), we can obtain a drogue orbit with constant airspeed. The orbital period \( T_{ap} \) can be calculated by

\[
T_{ap} = \int_0^{2\pi} \frac{1}{\dot{\phi}} \, d\phi.
\]  

(4.12)

It is not difficult to see that \( T_{ap} \) increases when the wind speed increases. In the presence of 5 m/s wind, \( T_{ap} \) can be calculated as 60.51 s by using Eq. (4.12). When the wind increases to 10 m/s, \( T_{ap} \) increases to 135.8 s. To guarantee that the optimal trajectory of the mothership has enough waypoints to produce an orbit, the starting and end times are selected as \( t_0 = 0 \) and \( t_1 = 70 \) s for the case of 5 m/s wind, and \( t_1 = 150 \) s for the case of 10 m/s wind. The initial configuration and solution results are shown in Table 4.1. It can be seen that when the wind speed increased to 10 m/s, the objective function value increased to 755.2. This is because the existing performance limits of the mothership made the resulting optimal orbit unable to precisely place the drogue orbit onto the desired orbit. The parameters of the desired drogue orbit were selected as \( r_{dr} = 100 \) m and \( v_{dr}^a = 12 \) m/s to compare the results with those in the previous section.

Figure 4.5 shows the different views of the optimal system trajectories in the presence of 5 m/s wind from the west. It can be seen that the offset of the mothership orbit center in Fig. 4.5 (b) is smaller than the one in Fig. 4.1 (b). It also can be seen that the inclination
Table 4.2: Solution results using the desired drogue orbit with constant airspeed

<table>
<thead>
<tr>
<th>Wind speed (m/s)</th>
<th>( \mathbf{p}_m^0 ) (m)</th>
<th>( \mathbf{v}_m^0 ) (m/s)</th>
<th>Solution Time (s)</th>
<th>( \Phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 5, 0) ( t )</td>
<td>((-74, 63, -159))</td>
<td>((-6, -9, 0))</td>
<td>18.8</td>
<td>17.9</td>
</tr>
<tr>
<td>(0, 10, 0) ( t )</td>
<td>((-20, 40, -159))</td>
<td>((-3, -4, 0))</td>
<td>39.8</td>
<td>755.2</td>
</tr>
</tbody>
</table>

\( \mathbf{p}_m^0 \) = initial mothership position; \( \mathbf{v}_m^0 \) = initial mothership velocity in NED frame.

of the mothership orbit in Fig. 4.5 (c) was much smaller than the one in Fig. 4.1 (c). The amplitude of the mothership’s altitude oscillation was approximately 10 m. Because the desired airspeed of the drogue was constant, it can be seen that the waypoints placed close together when the system is flying upwind (towards the west), and sparsely when the system was flying downwind (towards the east). Figure 4.6 shows the evolutions of the constrained variables \( V_a \), \( \gamma_a \) and \( \dot{\psi} \) of the mothership. It can be seen that the desired airspeed of mothership in Fig. 4.6 (a) reached its upper limit, while in the same wind condition, the desired airspeed in Fig. 4.2 (a) was still within the limits.

![3-D view](image1)
(a) 3-D view.

![Top-down view](image2)
(b) Top-down view.

![Side view](image3)
(c) Side view.

**Figure 4.6:** Optimal system trajectories using a desired drogue orbit with constant airspeed in the presence of 5 m/s wind from the west. The resulting optimal mothership orbit (dash-dot line) was larger than the one in Fig. 4.1. The amplitude of the mothership’s altitude oscillation was approximately 10 m, which was much smaller than the one in Fig. 4.1. The waypoints placed close together when the system was flying upwind (towards the west), and sparse when the system was flying downwind (towards the east).

Figure 4.7 shows the different views of the optimal system trajectories in the presence of 10 m/s wind from the west. It can be seen that the offset of the mothership orbit center and the inclination of the desired mothership orbit become larger than those in Fig. 4.5. The
Figure 4.7: Evolution of constrained variables of the mothership using a desired drogue orbit with constant airspeed in the presence of 5 m/s wind from the west. All the constrained variables (solid lines) remained within their limits (dashed lines) except for the airspeed.

resulting drogue orbit shows both horizontal and vertical offsets, which imply the constrained variables of the mothership reached their limits during the flight. The mothership trajectory in Fig. 4.7 (c) may look unsmooth, but this is simply an effect of quantifying the trajectory waypoints. Figure 4.8 shows the evolutions of the constrained variables \( V_a \), \( \gamma_a \) and \( \dot{\psi} \) of the mothership. It can be seen that the desired airspeed of the mothership reached the limit, while \( \gamma_a \) and \( \dot{\psi} \) still remained within their limits.
4.2.3 Transitions between Straight Level Flight and Orbital Flight

The transition between a straight and level flight and orbital flight (and vice versa) needs special attention for a towed cable system because the cable may become slack when the mothership turns and may experience large and sudden forces that may break the cable [4]. To prevent the tension forces exerted on the cable exceeding the loading limit of the cable, an optimal trajectory of the mothership is needed to keep the tension forces within their limits during the transition. In this section, we focus on the tow-in motion in which the system flies from a straight flight into an orbit. The tension forces on the cable are selected as additional constraints in the optimization with the limits $\| \mathbf{T}_i \| \in [0, 10] \text{ N}$, $i = 1, 2$. Optimal results in different wind conditions are presented in this section.

Letting $t_a$ be the time when transition started, the desired drogue trajectory in a tow-in motion can be written as

1. when $t \in [0, t_a]$, the straight-line trajectory of the drogue is given by

$$p_{dr_a}^d(t) = -v_{dr}^d t,$$
$$p_{dr_x}^d(t) = r_{dr}^d t,$$
$$p_{dr_d}^d(t) = -h_{0d}.$$
(2) when \( t \in (t_a, t_1] \), the circular trajectory of the drogue is given by

\[
\begin{align*}
    p^d_{dr_a} (t) &= -v^g_{dr} t_a + r^d_{dr} \cos \varphi (t), \\
    p^d_{dr_e} (t) &= r^d_{dr} \sin \varphi (t), \\
    p^d_{dr_d} (t) &= -h_0.
\end{align*}
\]

In this section, we use the desired circular drogue orbit with constant ground speed, i.e., \( \varphi (t) = v^g_{dr} t/r^d_{dr} \). We select \( r_{dr} = 100 \text{ m} \), \( v^g_{dr} = 12 \text{ m/s} \), \( t_a = 20 \text{ s} \) and \( t_1 = 80 \text{ s} \) in the optimization algorithm. The initial configuration and solution results are shown in Table 4.2. It can be seen that the solution times under different wind conditions were less than 30 s, while larger than those in Table 4.0 because a larger \( t_1 \) was selected. When the wind increased to 10 m/s, the objective function value increased to 1751. This is because the existing performance limits of the mothership made the resulting optimal orbit unable to precisely place the drogue orbit onto the desired orbit.

**Table 4.3:** Solution results of transitional flight in different wind conditions

<table>
<thead>
<tr>
<th>Wind speed (m/s)</th>
<th>( \mathbf{p}^0_m ) (m)</th>
<th>( \mathbf{v}^0_m ) (m/s)</th>
<th>Solution Time (s)</th>
<th>( \Phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>((-80, 100, -155))</td>
<td>((-14, 0, 0))</td>
<td>23.2</td>
<td>20.3</td>
</tr>
<tr>
<td>((0, 5, 0)^T)</td>
<td>((-80, 67, -155))</td>
<td>((-14, 0, 0))</td>
<td>20.1</td>
<td>21.6</td>
</tr>
<tr>
<td>((0, 10, 0)^T)</td>
<td>((-70, 41, -143))</td>
<td>((-14, 0, 0))</td>
<td>29.7</td>
<td>1751.0</td>
</tr>
</tbody>
</table>

\( \mathbf{p}^0_m \) = initial mothership position; \( \mathbf{v}^0_m \) = initial mothership velocity in NED frame.

Figure 4.9 shows the different views of the optimal system trajectories during a tow-in maneuver in the absence of wind. It can be seen that it takes one quarter circle for the mothership to complete the transition. The mothership trajectory had an altitude oscillation of approximately 20 m during the transition. Figure 4.10 shows the evolutions of the constrained variables \( V_a, \gamma_a \) and \( \dot{\psi} \) of the mothership. It can be seen that the airspeed reached its lower limits during the transition. The large oscillation of \( \gamma_a \) explained the altitude oscillation of the mothership during the transition. Figure 4.11 shows the evolution
of the tension forces on the cable in the transitional flight. The tension forces had a small oscillation (approximately 1 N), while remaining within their limits.

![3-D view.](image1)

![Top-down view.](image2)

![Side view.](image3)

**Figure 4.10:** Optimal system trajectories in the transitional flight in the absence of wind. The mothership trajectory (dash-dot line) had an obvious altitude oscillation (approximately 20 m) during the transition and converged quickly to the desired orbit which place the drogue trajectory (dashed line) into the desired orbit (triangle-dot line).

![Airspeed](image4)

![Flight path angle](image5)

![Heading rate](image6)

(a) Airspeed $V_a$ vs. time.  
(b) Flight path angle $\gamma_a$ vs. time.  
(c) Heading rate $\dot{\psi}$ vs. time.

**Figure 4.11:** Evolution of constrained variables of the mothership in the transitional flight in the absence of wind. The airspeed and flight path angle reached their limits during the transition.

Figure 4.12 shows the different views of the optimal system trajectories during a tow-in maneuver in the presence of 5 m/s wind from the west. It can be seen that the mothership trajectory had an altitude oscillation of approximately 15 m during the transition.
Figure 4.12: Evolution of the tension forces of the cable in the transitional flight in the absence of wind. The tension forces of the cable had a small oscillation during the transition, while they remained within their limits.

Figure 4.13: Optimal system trajectories in the transitional flight in the presence of 5 m/s wind from the west. The resulting optimal mothership orbit (dash-dot line) had an offset to the west of the drogue trajectory during the straight flight, experienced an altitude oscillation with amplitude of approximately 15 m during the transition, and converged to the desired orbit which place the drogue trajectory (dashed line) into the desired orbit (triangle-dot line).

Figure 4.13 shows the evolutions of the constrained variables $V_a$, $\gamma_a$ and $\dot{\psi}$ of the mothership. It can be seen that all the variables remained within the limits during the transition. Figure 4.14 shows the cable tension evolution during the transitional flight. It can be seen that the tension forces had an oscillation of approximately 2 N during the transition while they still remained within the limits.

Figure 4.15 shows the different views of the optimal system trajectories during a tow-in maneuver in the presence of 10 m/s wind from the west. It can be seen that the mothership trajectory had an altitude oscillation of approximately 10 m during the transition.
Figure 4.14: Evolution of constrained variables of the mothership in the transitional flight in the presence of 5 m/s wind from the west. All the variables remained within their limits. The large oscillation of the flight path angle (15 s to 30 s) explains the oscillation of the mothership altitude during the transition.

Figure 4.15: Evolution of the tension forces of the cable in the transitional flight in the presence of 5 m/s wind from the west. The tension forces of the cable had a small oscillation during the transition, while they remained within their limits.

Figure 4.16 shows the evolutions of the constrained variables $V_a$, $\gamma_a$ and $\dot{\psi}$ of the mothership. It can be seen that $V_a$ reached its upper bound while $\gamma_a$ and $\dot{\psi}$ remained within the limits during the transition. Therefore, the resulting optimal orbit unable to precisely place the drogue orbit onto the desired orbit, and this explains the large objective function value (1751) in Table 4.2. Figure 4.17 shows the cable tension evolution during the transitional flight. It can be seen that the tension forces had a major elevation of approximately 3 N, while they remained within the limits during the transition.
Figure 4.16: Optimal system trajectories in the transitional flight in the presence of 10 m/s wind from the west. The resulting optimal mothership orbit (dash-dot line) had a offset to the west of the drogue trajectory during the straight flight, experienced an altitude oscillation with amplitude of approximately 10 m during the transition, and converged to the desired orbit which place the drogue trajectory (dashed line) into the desired orbit (triangle-dot line).

Figure 4.17: Evolution of constrained variables of the mothership in the transitional flight in the presence of 10 m/s wind from the west. The airspeed reached its limit during the transition, while the flight path angle and heading rate remained within their limits. The large oscillation of the flight path angle (15 s to 30 s) explains the oscillation of the mothership altitude during the transition.

It can be seen that the altitude oscillation of the mothership increases as the wind speed increases. The tension forces only had small oscillations during the transition and remained within their limits.

4.3 Conclusions

This chapter presents a strategy for generating optimal trajectories for the constrained towing vehicle (mothership) of an aerially towed cable system using MPC. Two cable links
Figure 4.18: Evolution of the tension forces of the cable in the transitional flight in the presence of 10 m/s wind from the west. The tension forces of the cable had a major elevation during the transition, while they remained within their limits.

were chosen as compromise between accuracy and complexity. The optimization formulation using model predictive control was presented by introducing a combination of the squared-error and $L_1$-norm form objective function. The desired drogue motions of different patterns were employed to examine the strategy of the optimal trajectory generation. For the desired drogue orbits with constant ground speed, stronger wind required larger maneuvers on both the airspeed and flight path angle of the mothership, while for the desired drogue orbits with constant airspeed, stronger wind required larger maneuvers only on the airspeed of the mothership. In the transitional flight, as the wind increased, the altitude oscillation of the mothership during the transition decreased. The tension forces on the cable were also kept within the limits during the transition. This MPC-based optimal trajectory generation strategy can be a framework for specifying any arbitrary flight path of the towed body by optimizing the action of the towing vehicle subject to constraints and wind disturbances.
After the desired trajectory of the mothership is calculated using approaches in previous chapters, a trajectory-tracking controller is needed to drive the mothership to follow the calculated trajectory. In previous studies of towed cable systems, the dimension and weight of the cable and drogue were assumed small relative to the mothership. Therefore, the dynamics of the mothership was typically assumed to be unaffected by the tension force exerted from the cable. However, for a mothership with relatively low airspeed and thrust [16], the cable tension may exert a significant effect on the mothership maneuvering. In addition, the wind disturbance, typically $5 \sim 10$ m/s in the atmosphere, is also a significant component that affects the dynamics of airplanes with relatively low airspeed (e.g. MAV with 14 m/s nominal airspeed). However, the wind effect on the mothership maneuvering was seldom discussed in the literature.

One major contribution of this chapter is a nonlinear trajectory-tracking control law derived using Lyapunov-based backstepping approach [110] based on a dynamic design model of the mothership which takes both the cable tension and wind disturbances into consideration. This approach is inspired, in part, by the work of [111] and [86], the focus of which, however, was primarily on ground and water vehicles with low speeds and less degrees of freedom relative to aerial vehicles. Another contribution of this chapter is that the developed controller is able to guarantee that the tracking error of the mothership exponentially converges to an ultimate bound, which is a function of the upper limit of the unknown component of the wind.

The remainder of the chapter is structured as follows. Section 5.1 introduces the dynamic model of the mothership that is used to develop the trajectory-tracking control law.
Section 5.2 derives the nonlinear feedback control law using Lyapunov-based backstepping. Section 5.3 presents the simulation results.

5.1 Mothership Dynamics

Let \( p_m \triangleq (p_n, p_e, p_d)^T \) be the position of the mothership in the inertial NED coordinate frame, \( V_a \) be the magnitude of the mothership airspeed, \( \psi \) be the heading angle, \( \gamma_a \) be the air mass referenced flight path angle, which is defined as the angle from the inertial North-East plane to the velocity vector of the aircraft relative to the air mass, \( w_c \triangleq (w_n, w_e, w_d)^T \) be the constant component of the wind expressed in the inertial frame (which can be estimated), \( N_w \triangleq (N_n, N_e, N_d)^T \) be the noisy component of the wind where it is assumed that \( \|N_w\| \leq \bar{N} \) where \( \bar{N} > 0 \) is assumed known, \( \phi \) be the roll angle, \( m_m \) be the mass of the mothership, \( g \) be the gravitational constant at Earth sea level, \( u_n \triangleq \frac{L_m}{m_m g} \) be the (controlled) load factor, \( L_m \) and \( D_m \) be the aerodynamic lift and drag forces respectively, \( u_T \) be the thrust and \((F_V, F_\psi, F_\gamma_a)\) be the tension forces in the velocity frame. Then the equations of motion of the mothership in the presence of wind are given by

\[
\begin{align*}
\dot{p}_n &= V_a \cos \psi \cos \gamma_a + w_n + N_n, \quad (5.1) \\
\dot{p}_e &= V_a \sin \psi \cos \gamma_a + w_e + N_e, \quad (5.2) \\
\dot{p}_d &= -V_a \sin \gamma_a + w_d + N_d, \quad (5.3) \\
\dot{V}_a &= -g \sin \gamma_a - \frac{D_m}{m_m} + \frac{1}{m_m} u_T + \frac{F_V}{m_m}, \quad (5.4) \\
\dot{\gamma}_a &= -\frac{g}{V_a} \cos \gamma_a + \frac{g}{V_a} (\cos \phi) u_n + \frac{F_\gamma_a}{m_m V_a}, \quad (5.5) \\
\dot{\psi} &= \frac{L_m}{m_m V_a} \sin \phi + \frac{F_\psi}{m_m V_a \cos \gamma_a}, \quad (5.6) \\
\dot{\phi} &= u_\phi. \quad (5.7)
\end{align*}
\]

The control inputs are selected as the thrust \( u_T \), the load factor \( u_n \), and the roll angle rate \( u_\phi \). Let \( T_m \) be the components of tension in the inertial frame for the first cable element connected to the mothership and the tension forces in the inertial coordinate system can be
expressed in the velocity frame via the transformation \((F_V, F_\psi, -F_\gamma_a)^T = \Gamma \cdot T_m\), where

\[
\Gamma \triangleq \begin{pmatrix}
\cos \gamma_a \cos \psi & \cos \gamma_a \sin \psi & -\sin \gamma_a \\
-\sin \psi & \cos \psi & 0 \\
\sin \gamma_a \cos \psi & \sin \gamma_a \sin \psi & \cos \gamma_a
\end{pmatrix}.
\]

### 5.2 Mothership Trajectory-Tracking Control Law Design Using Backstepping

**Theorem 6.1** Consider the system with the dynamic equations (5.1) - (5.7), and suppose that \(V_a > 0\) and \(\gamma_a \in (-\frac{\pi}{2}, \frac{\pi}{2})\). Defining \(u_c \triangleq (u_T, u_n, \sin \phi)^T\),

\[
R \triangleq \begin{pmatrix}
-g \sin \gamma_a - \frac{D_m}{m_m} + \frac{F_v}{m_m} \\
-\frac{g}{V_a} \cos \gamma_a + \frac{F_{\gamma a}}{m_m V_a} \\
\frac{F_v}{m_m V_a \cos \gamma_a}
\end{pmatrix},
\quad
S \triangleq \begin{pmatrix}
\frac{1}{m_m} & 0 & 0 \\
0 & \frac{g}{V_a} \cos \phi & 0 \\
0 & 0 & \frac{L_m}{m_m V_a \cos \gamma_a}
\end{pmatrix},
\]

we can obtain

\[
\begin{pmatrix}
\dot{V}_a \\
\dot{\gamma}_a \\
\dot{\psi}
\end{pmatrix} = R + Su_c.
\]

Let

\[
e_m \triangleq p_m - \hat{p}_m, \quad (5.8)
\]

be the tracking error vector of the mothership in the inertial frame, and let

\[
V_m \triangleq \begin{pmatrix}
V_a \cos \psi \cos \gamma_a \\
V_a \sin \psi \cos \gamma_a \\
-V_a \sin \gamma_a
\end{pmatrix},
\]
be the velocity vector of the airframe relative to the surrounding air. Let \( k_1, k_2, \) and \( k_3 \) be constant positive control gains, and define

\[
\mathbf{Q} \triangleq \begin{pmatrix}
\cos \gamma_a \cos \psi & -V_a \sin \gamma_a \cos \psi & -V_a \cos \gamma_a \sin \psi \\
\cos \gamma_a \sin \psi & -V_a \sin \gamma_a \sin \psi & -V_a \cos \gamma_a \cos \psi \\
-\sin \gamma_a & -V_a \cos \gamma_a & 0
\end{pmatrix},
\]

\[
\mathbf{z}_{e}^{e} \triangleq -\mathbf{V}_m - \mathbf{w}_c + \dot{\mathbf{p}}^{c}_m - k_1 \mathbf{e}_m,
\]

\[\xi \triangleq -(\mathbf{Q} \mathbf{S})^{-1} (\mathbf{e}_m + \mathbf{Q} \mathbf{R} - \dot{\mathbf{p}}^{c}_m + k_1 \mathbf{e}_m - k_2 \mathbf{z}_{e}^{e}), \tag{5.9}\]

\[
\xi \triangleq -(\mathbf{Q} \mathbf{S})^{-1} (\mathbf{e}_m + \mathbf{Q} \mathbf{R} - \dot{\mathbf{p}}^{c}_m + k_1 \mathbf{e}_m - k_2 \mathbf{z}_{e}^{e}), \tag{5.10}\]

\[
\mathbf{z}_{\phi}^{e} \triangleq \sin \phi - \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \xi.
\]

Suppose that the desired time-varying mothership trajectory \( \mathbf{p}^{c}_m(t) = (p_n, p_e, p_d)^T \) is a sufficiently smooth with bounded derivatives. Then, if the control inputs are selected as

\[
\begin{pmatrix} u_T \\ u_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \xi, \tag{5.11}\]

\[
u_\phi = \frac{1}{\cos \phi} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \dot{\xi} + (\mathbf{z}_{e}^{e})^T \mathbf{Q} \mathbf{S} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - k_3 \mathbf{z}_{\phi}^{e}, \tag{5.12}\]

the tracking error \( \mathbf{e}_m \) is uniformly ultimate bounded and exponentially converges to a ultimate bound \( \bar{N}/\sqrt{\lambda \sigma} \), where

\[
\sigma = \min \left\{ 2k_1, (\min \{k_1, k_2, k_3\})^2 \right\}, \tag{5.13}\]

\[
\lambda = \min \left\{ 1, 2 \cdot \min \left\{ \left( k_1 - \frac{\sigma}{2} \right), k_2, k_3 \right\} \right\}. \tag{5.14}\]

\[
\mathbf{e}_m \xrightarrow{e.p.} \bar{N}/\sqrt{\lambda \sigma}
\]

Proof:
Error dynamics: The dynamic equation for the inertial tracking error is given by differentiating Equation (5.8) to obtain

\[ \dot{e}_m = \dot{p}_m - \dot{p}_m^c = V_m + w_c + N_w - \dot{p}_m^c. \]

Error convergence: Define the Lyapunov function candidate \( V_1 \triangleq \frac{1}{2} e_m^T e_m \), which has time derivative

\[ \dot{V}_1 = e_m^T (V_m + w_c + N_w - \dot{p}_m^c). \]

Introducing the error variable (5.9), we have

\[ \dot{V}_1 = -k_1 e_m^T e_m - e_m^T (z_m^e) + e_m^T N_w. \]

Backstepping for \( z_m^e \): Consider the augmented Lyapunov function candidate

\[ V_2 \triangleq V_1 + \frac{1}{2} (z_m^e)^T (z_m^e), \]

with time derivative

\[ \dot{V}_2 = -k_1 e_m^T e_m + (z_m^e)^T (-e_m - \dot{V}_m + \ddot{p}_m^c - k_1 \dot{e}_m) + e_m^T N_w. \]

From the mothership dynamic equations we have \( \dot{V}_m = Q (R + S u_c) \). Therefore

\[ \dot{V}_2 = -k_1 e_m^T e_m + (z_m^e)^T (-e_m - QR - Q S u_c + \ddot{p}_m^c - k_1 \dot{e}_m) + e_m^T N_w. \]

It can be seen that \( QS \) is invertible\(^1\), therefore using (5.10) and defining \( \eta \triangleq \sin \phi \) and supposing that \( u_T \) and \( u_n \) are selected as in (5.11), then the time derivative of the \( z_m^e \) is

\(^1\)Under the assumption that \( V_a > 0 \) and \( \gamma_a \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), the matrices Q and S will be full rank. The product of two full-rank matrices is also full rank.
\[ \dot{z}_m = u_\phi \cos \phi - \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \dot{\xi}, \text{ and} \]

\[
\begin{pmatrix} u_T \\ u_n \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

Thus

\[
\dot{V}_2 = -k_1 e_m^T e_m - k_2 (z_m^e)^T (z_m^e) + (z_m^e)^T \left( -QS \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) + e^T N_w.
\]

**Backstepping for \(z_m^\phi\):** Consider the augmented Lyapunov function candidate

\[
V_3 = V_2 + \frac{1}{2} (z_m^\phi)^2 = \frac{1}{2} \| e_m \|^2 + \frac{1}{2} \| z_m^e \|^2 + \frac{1}{2} (z_m^\phi)^2,
\]

with time derivative

\[
\dot{V}_3 = -k_1 e_m^T e_m - k_2 (z_m^e)^T (z_m^e) + z_m^\phi \left( u_\phi \cos \phi - \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \dot{\xi} - (z_m^e)^T QS \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) + e^T N_w.
\]

If \(u_\phi\) is given by (5.12), then the time derivative of \(V_3\) becomes

\[
\dot{V}_3 = -k_1 e_m^T e_m - k_2 (z_m^e)^T (z_m^e) - k_3 (z_m^\phi)^2 + e^T N_w.
\]

**Ultimate bound:** From Equation (5.16), we obtain

\[
\dot{V}_3 \leq -k_1 \| e_m \|^2 - k_2 \| z_m^e \|^2 - k_3 (z_m^\phi)^2 + \| e_m \| \| N_w \|. \]

Defining the vector \(Y \triangleq (e_m^T, (z_m^e)^T, z_m^\phi)^T\), and letting \(0 < \mu \leq \min \{ k_1, k_2, k_3 \}\), we have \(\dot{V}_3 \leq -\mu \| Y \|^2 + \| Y \| \| N_w \|. \) To use the term \(\mu \| Y \|^2\) to dominate \(\| Y \| \| N_w \|\), we rewrite
the foregoing inequality as

$$\dot{V}_3 \leq -\mu (1 - \theta) \|Y\|^2 - \mu \theta \|Y\|^2 + \|Y\| \|N_w\|,$$

where $0 < \theta < 1$. The term $-\mu \theta \|Y\|^2 + \|Y\| \|N_w\|$ will be less than zero if $\|Y\| \geq \frac{\|N_w\|}{\mu \theta}$.

From the input-to-state stability theorem [110], it can be concluded that

$$\|Y\| \leq \beta (\|Y(t_0)\|, t - t_0) + \gamma \left( \sup_{t_0 \leq \tau \leq t} N_w(\tau) \right),$$

where $\beta$ is a class $\mathcal{KL}$ function and $\gamma$ is a class $\mathcal{K}$ function defined as $\gamma(r) \triangleq \frac{r}{\mu \theta}$. Therefore, according to the Lyapunov stability theorem [110], $\|Y\|$ is uniformly ultimately bounded, and since $\|e_m\| \leq \|Y\|$, $\|e_m\|$ is also uniformly ultimately bounded within a ball of radius $\bar{N}/\mu \theta$.

**Exponential convergence:** By using the Young’s inequality\(^2\), it can be concluded that for any $\sigma > 0$

$$\dot{V}_3 \leq - \left( k_1 - \frac{\sigma}{2} \right) \|e_m\|^2 - k_2 \|z_{m}^e\|^2 - k_3 (z_{m}^\phi)^2 + \frac{1}{2\sigma} \|N_w\|^2 .$$

Suppose we choose $0 < \sigma < 2k_1$ so that the term $k_1 - \frac{\sigma}{2}$ is positive. Then we can conclude that there is a sufficiently small positive constant $\lambda$ satisfying

$$0 < \frac{\lambda}{2} \leq \min \left\{ \left( k_1 - \frac{\sigma}{2} \right), \ k_2, \ k_3 \right\},$$

such that

$$\dot{V}_3 \leq -\lambda V_3 + \frac{1}{2\sigma} \bar{N}^2,$$

therefore, it can be concluded from the Comparison Lemma [110] that

$$V_3(t) \leq e^{-\lambda t} V_3(0) + \frac{1}{2\lambda \sigma} \bar{N}^2, \ t \geq 0.$$

\(^2\)A special case of the Young’s inequality is $ab \leq (\gamma/2) a^2 + (1/2\gamma) b^2$, where $a, b \geq 0$, and $\gamma$ is any positive constant.
Then all signals remain bounded and therefore the solution exists globally. Moreover, $V_3$ converges to a ball of radius $\bar{N}^2/(2\lambda \sigma)$ and $e_m$ converges exponentially to a ball of radius $\bar{N}/\sqrt{\lambda \sigma}$ because of (5.15).

Selecting $\sigma$ and $\lambda$ using (5.13) and (5.14), we pick $\theta = \sqrt{\lambda}$ and $\mu = \sqrt{\sigma}$ to guarantee that the ultimate bound is $\bar{N}/\sqrt{\lambda \sigma}$.

### 5.3 Simulation Results

<table>
<thead>
<tr>
<th></th>
<th>Mothership</th>
<th>Drogue</th>
<th>Cable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass (kg)</td>
<td>1.76</td>
<td>0.11</td>
<td>0.01</td>
</tr>
<tr>
<td>$C_L$</td>
<td>0.28</td>
<td>0.01</td>
<td></td>
</tr>
<tr>
<td>$C_D$</td>
<td>0.06</td>
<td>0.03</td>
<td></td>
</tr>
<tr>
<td>Wing area (m$^2$)</td>
<td>0.307</td>
<td>30</td>
<td>1.904</td>
</tr>
<tr>
<td>Wing span (m)</td>
<td>1.4</td>
<td></td>
<td>0.41</td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E (GPa)</td>
<td></td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>d (mm)</td>
<td></td>
<td></td>
<td>0.41</td>
</tr>
</tbody>
</table>

The key parameters used in the simulation are shown in Table 5.0. The performance limits of the mothership are selected as $12 \leq V_a \leq 20 \text{m/s}$, $-35 \leq \phi \leq 35$ degree, $-15 \leq \gamma_a \leq 35$ degree. A desired circular trajectory of the drogue can be written in parametric form as $p_n^{dr}(t) = R^{dr} \sin \left( \frac{V^{dr}}{R^{dr}} ^t \right)$, $p_e^{dr}(t) = R^{dr} \cos \left( \frac{V^{dr}}{R^{dr}} ^t \right)$, $p_d^{dr}(t) = -100 \text{m}, t \in [0, +\infty)$, where $(p_n^{dr}, p_e^{dr}, p_d^{dr})^T$ are the coordinates of the drogue in NED coordinates, $R^{dr}$ and $V^{dr}$ are the desired orbit radius and ground speed of the drogue, respectively. For a constant wind speed vector $w_c = (5, 0, 0)^T \text{m/s}$, $R^{dr}$ and $V^{dr}$ can be selected as 110 m and 13 m/s. The control gains are tuned and selected as $k_1 = 0.3, k_2 = 4, k_3 = 20$. The superior bound on the wind gusts can be given by $N = 0.5 \text{m/s}$, therefore the ultimate bound can be calculated as 2.33 m. The initial positions of the mothership and drogue are $(0, 130, -150)^T \text{m}$ and $(0, 130, -40)^T \text{m}$, respectively in NED coordinates.

Figure 5.0 shows the North-East view of the mothership trajectory, in which it can be seen that the mothership follows the desired trajectory precisely after one circle of the orbit. Figure 5.1 shows the North-altitude view of the mothership trajectory, in which we can see the large inclined orbit of the mothership. As mentioned in [2], even though the mothership...
flies a flat orbit the presence of wind causes the orbit of the drogue to be inclined. Therefore, to compensate for the oscillation of the drogue altitude, the mothership must incline its orbit. Figure 5.2 shows the error of the mothership. Although the mothership has an original offset
from desired position, after 30 seconds, all the tracking errors converge to a neighborhood of the origin. Figure 5.3 shows the top-down view of the drogue trajectory. It can be seen that it takes longer for the drogue to converge to desired orbit than the mothership. This is because of the flexible connection between the mothership and the drogue, which delays the response of the drogue. Figure 5.4 gives the tracking errors of the drogue. It can be seen that after the mothership tracking errors converge to a neighborhood of the origin \((t > 30 \text{ s})\),
the drogue tracking errors also enter a neighborhood of the origin \((t > 40\, s)\). The delayed responding time (about 10 s in the current simulation) of the drogue will vary with the different initial positions of the mothership. Figure 5.5 shows the evolution of the roll angle \(\phi\), the path angle \(\gamma_a\), the thrust \(T\) and the load factor \(n\) of the mothership during flight. Because of the inclined orbit of the mothership, it has to fly up and down explaining the oscillation of path angle and the load factor. Since the constant wind dominates the noise, to keep a constant ground speed, the mothership regulates its airspeed to mitigate the effect of wind during flight explaining the oscillation of the mothership thrust. Figure 5.6 shows the magnitude of the mothership tracking error converges within its ultimate bound after 40 seconds.

5.4 Conclusions

In this chapter, a trajectory-tracking control law using Lyapunov-based backstepping is developed. The controller is derived using a dynamic model of the mothership which takes both the cable tension and wind disturbance into consideration. The simulation results show that the controller is able to drive the mothership tracking error to exponentially converge to an ultimate bound, which is a function of the upper limit of the unknown component of the wind.
Figure 5.6: Time evolution of the roll angle $\phi$, path angle $\gamma_a$, thrust $T$ and the load factor $n$ of the mothership in the presence of wind.

Figure 5.7: Time evolution of the magnitude of the tracking error of the mothership in the presence of wind.
Chapter 6

Trajectory-Tracking Control Law Design for UAVs with Autopilot in the Loop

In the previous chapter, a nonlinear feedback control law was derived for the moth- ership to follow the desired trajectory. However, the control inputs selected in the design model are typically difficult to be directly applied in the experiment, because the autopilot used to stabilize and control the vehicle, only accepts certain input commands. Strategies dealing with this issue are not found in the literature.

In this chapter, we develop an approach of designing the trajectory tracking control law for autonomous vehicles with the autopilot in the loop. The major contribution of this chapter is that the specific interface structure of the autopilot is taken into account in the controller design. To get the autopilot involved, the original designed trajectory tracking controller and design model only need to be augmented using an extra term, which is the function of the feedback error, so that the actual system trajectory is guaranteed to exponentially converge to the reference trajectory with all states bounded. A sufficient condition for qualified commands of the autopilot is also introduced. This strategy provide a framework to implement the developed controller on the experimental vehicle with autopilot in the loop without modifying the key structure of the design model and the original controller.

The remainder of the chapter is structured as follows. Section 6.1 introduces the formulation of the problem. Section 6.2 develops the control strategy of trajectory-tracking for UAVs with autopilot in the loop. Section 6.3 presents the numerical results using both kinematic and dynamic models. Section 6.4 shows the results of software-in-the-loop simulation.
6.1 Problem Formulation

This section formulates the trajectory tracking problem for UAVs with an autopilot in the closed-loop system. A typical closed-loop structure of the feedback stabilization and control in UAVs is shown in Fig. 6.0, where the autopilot performs the functions of stabilizing the vehicle as well as following the input commands, which may include airspeed, altitude, and attitude. A closed-loop structure of the feedback control for UAVs using a specific design model is shown in Fig. 6.1, where the design model stands for the equations of motion of UAVs (kinematic or dynamic models) and the controller is designed to achieve specific goals (e.g. trajectory tracking or path following).

In particular, given the autopilot, the grey box shown in Fig. 6.2 is fixed and cannot be changed. Most commercial autopilots like Kestrel, Piccolo, etc., are black box systems, in the sense that they implement specific autopilot functionality and we are constrained to work with that functionality. However the input/output model of the autopilot/UAV system, which is usually overly simplistic, may not be sufficient to capture the design requirements for the control scheme (e.g., the case in aerial recovery). Therefore, it becomes desirable to design a control strategy based on a medium fidelity model, and then to implement that strategy on the autopilot.
In this section, we are interested in the situation where the output of the controller are not the same variables as the input commands for the autopilot. In order to use the controller designed in Fig. 6.1 without any modifications to the autopilot, a closed-loop system combining the systems shown in Figs. 6.0 and 6.1 is proposed as depicted in Fig. 6.2, where the design model and controller are modified to drive the UAV position to exponentially converge to and follow the reference. The main objective of this section is to develop a strategy of augmenting the design model and the controller in Fig. 6.2 to achieve the trajectory tracking for UAVs.

6.2 Trajectory-Tracking Control Law Design with Autopilot in the Loop

This section develops the control law of trajectory tracking for UAVs using the closed-loop system in Fig. 6.2. Lyapunov based backstepping is employed to derive the controller which is proved to be capable of driving the UAV to exponentially converge to and follow the desired trajectory. The controller in Fig. 6.1 is firstly developed and the autopilot in Fig. 6.0 to follow the specific commands is developed after. An observer-based sufficient condition is derived to examine the qualification of the commands that guarantees the convergence. Finally, the augmentation law of the controller and design model is developed to complete the theory.

6.2.1 Trajectory-Tracking Control Law Design Using Design Models

A trajectory tracking controller shown in Fig. 6.1 is developed in this section using a specific design model.
Lemma 1: Defining the states $\hat{\eta} \in D_\eta \subset \mathbb{R}^3$, $\hat{\xi} \in D_\xi \subset \mathbb{R}^3$ and the system inputs $\hat{u} \in \mathbb{R}^3$, consider the nonlinear system

\begin{align*}
\dot{\hat{\eta}} &= f_1(\hat{\xi}), \quad (6.1) \\
\dot{\hat{\xi}} &= f_2(\hat{\xi}) + g_2(\hat{\xi}) \hat{u}, \quad (6.2)
\end{align*}

where $f_1 : D_\xi \mapsto \mathbb{R}^3$, $f_2 : D_\xi \mapsto \mathbb{R}^3$, and $g_2 : D_\xi \mapsto \mathbb{R}^3$ are smooth vector fields and locally Lipschitz in $\xi$. Assuming that the Lie derivative $[110] \mathcal{L}_{g_2} f_1(\hat{\xi})$ is invertible for $\hat{\xi}$ on $D_\xi$, and $\eta^d(t)$ is $C^2$ with bounded time-derivatives. If constants $\hat{k}_1$ and $\hat{k}_2$ are positive, and $\hat{u}$ is selected as

\begin{align*}
\hat{u} &= \left( \mathcal{L}_{g_2} f_1(\hat{\xi}) \right)^{-1} \left( \hat{\eta}^d - \mathcal{L}_{f_2} f_1(\hat{\xi}) - (\hat{k}_1 + \hat{k}_2) (\hat{\eta} - \eta^d) \right), \quad (6.3)
\end{align*}

then the tracking error $\hat{\eta} - \eta^d$ converges exponentially to the origin and $\hat{\xi}$ remains bounded.

Proof:

Step 1: Convergence of $\hat{\eta} \to \eta^d$.

Let $e_\eta \triangleq \hat{\eta} - \eta^d$ be the tracking error and the error dynamics is given by $\dot{e}_\eta = \hat{\eta} - \eta^d$. Define the Lyapunov function candidate $V_1 = \frac{1}{2} e_\eta^T e_\eta$, which has the time derivative

\begin{align*}
\dot{V}_1 &= e_\eta^T \dot{e}_\eta \nonumber \\
&= e_\eta^T \left( \dot{\eta} - \dot{\eta}^d \right). \quad (6.4)
\end{align*}

At this stage of the development, we consider $\dot{\eta}$ as a virtual control, where $\dot{V}_1$ can be made negative definite by setting $\dot{\eta}$ equals to $\dot{\eta}^d - \hat{k}_1 e_\eta$. Introducing the error variable

\begin{align*}
z &\triangleq \dot{\eta}^d - \hat{k}_1 e_\eta - \dot{\eta}, \quad (6.5)
\end{align*}

and adding and subtracting $z$ in (6.4) gives

\begin{align*}
\dot{V}_1 &= -\hat{k}_1 e_\eta^T e_\eta - e_\eta^T z.
\end{align*}
Consider the augmented Lyapunov function candidate
\[ V_2 \triangleq V_1 + \frac{1}{2} z^T z, \]
with the time derivative
\[ \dot{V}_2 = -\hat{k}_1 e_{\dot{\eta}}^T e_{\dot{\eta}} + z^T (-e_{\dot{\eta}} + \dot{z}) \]
\[ = -\hat{k}_1 e_{\dot{\eta}}^T e_{\dot{\eta}} + z^T \left( -e_{\dot{\eta}} + i\ddot{\eta} - \dot{k}_1 \dot{\dot{\eta}} - \mathcal{L}_{f_2} f_1 - \mathcal{L}_{g_2} f_1 \hat{u} \right). \]

If \( \hat{u} \) is given by (6.3), the time derivative of \( V_2 \) becomes
\[ \dot{V}_2 = -\hat{k}_1 e_{\dot{\eta}}^T e_{\dot{\eta}} - \hat{k}_2 z^T z. \] (6.6)

Based on Lyapunov stability theory [110], it can be concluded that \( e_{\dot{\eta}} \) and \( z \) exponentially converge to the origin.

**Step 2:** Boundedness of \( \hat{\xi} \).

From Eqs. (6.5) and (6.6) and the assumption that \( \dot{\eta}^d \) is bounded, we have \( e_{\dot{\eta}} \) and \( z \) are bounded, and then we have \( f_1(\hat{\xi}) \) is bounded. Since \( f_1 \) is locally Lipschitz in \( \hat{\xi} \), we can conclude that \( \hat{\xi} \) is bounded.

\[ \square \]

### 6.2.2 Control Law Design for the Autopilot

A feedback control law for autopilot to follow the desired commands, shown in Fig. 6.0, is developed in this section. The input commands are denoted as \( \hat{\xi} \) in this section.

**Lemma 2:** Defining the states \( \xi \in D_\xi \subset \mathbb{R}^3 \) and system inputs \( u \in \mathbb{R}^3 \), consider the nonlinear system
\[ \dot{\xi} = f_2(\xi) + g_2(\xi) u, \]
with the assumption that the function $g_2(x)$ is invertible for $x$ on $D_\xi$. Suppose that $\hat{\xi}(t) \in D_\xi$ is $C^2$ with bounded time-derivatives. If the constant $k$ is positive, and $u$ is selected as

$$u = g_2^{-1}(\xi) \left( -f_2(\xi) + \dot{\xi} - k(\xi - \dot{\xi}) \right), \quad (6.7)$$

then the tracking error $\xi - \hat{\xi}$ converges exponentially to the origin.

**Proof:**

Let $e_\xi \triangleq \xi - \hat{\xi}$, then the error dynamic is given by $\dot{e}_\xi = \dot{\xi} - \dot{\hat{\xi}}$. Define the Lyapunov function candidate $V_3 = \frac{1}{2} e_\xi^T e_\xi$, which has the time derivative

$$\dot{V}_3 = e_\xi^T \dot{e}_\xi$$

$$= e_\xi^T \left( f_2 + g_2 u - \dot{\xi} \right). \quad (6.8)$$

If $u$ is given by (6.7), the time derivative of $V_3$ becomes

$$\dot{V}_3 = -k e_\xi^T e_\xi.$$

Based on Lyapunov stability theory, it can concluded that $e_\xi$ exponentially converges to the origin.
6.2.3 Sufficient Conditions for Qualified Commands of the Autopilot

The commands used in the previous subsection is selected as \( \hat{\xi} \). However, different autopilots may accept different commands which may not always be \( \hat{\xi} \). In this section, we propose an observer-based sufficient condition that can be used to examine whether the commands of autopilot are qualified to guarantee that \( \xi \rightarrow \hat{\xi} \).

**Lemma 3:** Suppose the commands that autopilot takes from the output of the design model are \( \hat{y} \), which is a function of the system states \( \hat{x} \triangleq \left( \hat{\eta}^T, \hat{\xi}^T \right)^T \), i.e., \( \hat{y} \triangleq H (\hat{x}) \in \mathbb{R}^3 \), and the corresponding variables in the UAV dynamic are \( y \triangleq H (x) \in \mathbb{R}^3 \). Then \( y \rightarrow \hat{y} \) implies that \( \xi \rightarrow \hat{\xi} \), if the system (6.1) and (6.2) with output function \( \hat{y} = H (\hat{x}) \) is observable with respect to \( \hat{\xi} \).

**Proof:**

If the system (6.1) and (6.2) with output function \( \hat{y} = H (\hat{x}) \) is observable respective to \( \hat{\xi} \), it is implied that \( \hat{\xi} \) can be reconstructed from \( \hat{y} \). Thus, if it is guaranteed that \( y \rightarrow \hat{y} \), it can be obtained that \( \xi \rightarrow \hat{\xi} \).

\[ \square \]

6.2.4 Control Law Design Using Combined System

A control strategy, shown in Fig. 6.2, is presented in this section by augmenting the design model and controller developed in Lemma 1. The UAV dynamics are selected as the copy of the design model, and the autopilot control law is selected as the one developed in Lemma 2. The combined systems using specific equations of motion are shown in Fig. 6.3. The commands sent to the autopilot are selected as \( \hat{\xi} \).

**Theorem 1:** Defining the states \( \tilde{\eta}, \eta \in D_\eta \subset \mathbb{R}^3 \), \( \tilde{\xi}, \xi \in D_\xi \subset \mathbb{R}^3 \), and the system inputs \( \tilde{u}', u \in \mathbb{R}^3 \), consider the nonlinear systems described in Fig. 6.3 with the assumption that the Lie derivative \( L_{g_2}f_1 (x) \) is invertible for \( x \) on \( D_\xi \). Suppose that \( \eta^d(t) \) is \( C^2 \) with bounded time-derivatives. If constants \( k_1, k_2 \) and \( \ell \) are positive, the extra term of the design model is selected as

\[ \delta (t) = \ell (\eta - \tilde{\eta}) \]
\( \mathbf{u} \) is selected in (6.3), \( \mathbf{u} \) is selected in (6.7), and the extra term of the developed controller \( \nu(t) \) is selected as
\[
\nu = -\left( \mathcal{L}_{g_2} f_1(\hat{\xi}) \right)^{-1} \left( \dot{\delta} + \hat{k}_2 \delta \right),
\]
then the tracking error \( \eta - \eta^d \) and \( \dot{\eta} - \eta^d \) converge exponentially to the origin and \( \xi \) and \( \dot{\xi} \) remain bounded for all \( t \geq 0 \).

**Proof:**

The convergence of \( \xi \) to \( \hat{\xi} \) follows from Lemma 2.

The convergence of \( \eta \) to \( \dot{\eta} \) can be shown as follows.

Let \( \mathbf{e}_\eta \triangleq \eta - \dot{\eta} \) and the error dynamic of \( \mathbf{e}_\eta \) is given by
\[
\dot{\mathbf{e}}_\eta = \dot{\eta} - \dot{\eta} = f_1(\xi) - f_1(\hat{\xi}) - \ell \mathbf{e}_\eta.
\]
Because \( \xi \) exponentially converges to \( \hat{\xi} \), it can be concluded that \( f_1(\xi) \) exponentially converges to \( f_1(\hat{\xi}) \). Also, because \( \ell \) is a positive constant, it can be concluded that the tracking error \( \mathbf{e}_\eta \) exponentially converges to the origin since the linear system \( \dot{\mathbf{e}}_\eta = -\ell \mathbf{e}_\eta \) is input to state stable [110].

The convergence of \( \dot{\eta} \rightarrow \eta^d \) can be shown as follow.

Let \( \mathbf{e}_\eta \triangleq \dot{\eta} - \eta^d \) and the error dynamic of \( \mathbf{e}_\eta \) is given by \( \dot{\mathbf{e}}_\eta = \dot{\eta} - \eta^d \). Define the Lyapunov function candidate \( V_4 \triangleq \frac{1}{2} \mathbf{e}_\eta^T \mathbf{e}_\eta \), which has the time derivative
\[
\dot{V}_4 = \mathbf{e}_\eta^T \left( \dot{\eta} - \eta^d \right) = \mathbf{e}_\eta^T \left( f_1(\hat{\xi}) + \delta - \eta^d \right).
\]
Introducing the error variable
\[
\hat{\mathbf{z}} \triangleq -f_1(\hat{\xi}) - \delta + \dot{\eta}^d - \hat{k}_1 \mathbf{e}_\eta,
\]
and adding and subtracting $\hat{z}$ in (6.10) gives

$$\dot{V}_4 = -\hat{k}_1 e_{\eta}^T e_{\eta} - e_{\eta}^T \hat{z}.$$ 

Consider the augmented Lyapunov function candidate

$$V_5 \triangleq V_4 + \frac{1}{2} \hat{z}^T \hat{z},$$

with time derivative

$$\dot{V}_5 = -\hat{k}_1 e_{\eta}^T e_{\eta} + \hat{z}^T \left( -e_{\eta} + \hat{z} \right)$$

$$= -\hat{k}_1 e_{\eta}^T e_{\eta} + \hat{z}^T \left( -e_{\eta} + \eta^d - \hat{k}_1 e_{\eta} \right)$$

$$-\hat{\delta} - \mathcal{L}_{f_2} \dot{f}_1 - \mathcal{L}_{g_2} \dot{f}_1 \left( \hat{u} + \nu \right).$$

If $\hat{u}$ is selected in (6.3) and $\nu$ is selected in (6.9), it can be obtained

$$\dot{V}_5 = -\hat{k}_1 e_{\eta}^T e_{\eta} - \hat{k}_2 \hat{z}^T \hat{z}.$$ 

Based on Lyapunov stability theorem [110] it can be concluded that $e_{\eta}$ exponentially converges to the origin.

The boundedness of $\xi$ and $\hat{\xi}$ can be shown as follow.

Because

$$\hat{z} = -f_1 \left( \hat{\xi} \right) - \delta + \eta^d - \hat{k}_1 e_{\eta},$$

and $\delta$ exponentially converges to the origin, it can be proved that $\hat{\xi}$ is bounded by using the similar strategy in Lemma 1. From Lemma 2, it can be concluded that $\xi$ exponentially converges to $\hat{\xi}$, so $\xi$ is also bounded.

6.3 Numerical Results

In this section, simulation results are shown to validate the augmented controller developed in the previous section. Both the kinematic and dynamic models are employed to validate the controllers. The reference trajectory is selected as an inclined circular orbit.
Letting $R$ be the desired orbit radius in the horizontal plane, $V_d$ be the desired constant ground speed, $h_0$ be the desired average altitude and $h_c$ be the desired magnitude of the altitude oscillation, the reference trajectory $\eta^d$ in North-East-Down (NED) frame is given by

$$
\eta^d(t) = \begin{pmatrix}
R \sin(\omega_0 t) \\
R \cos(\omega_0 t) \\
-h_0 + h_c \sin(\omega_0 t)
\end{pmatrix},
$$

where $\omega_0 = V_d^d/R$. In this paper, $R$ is selected as 200 m, $V_d$ is selected as 14 m/s, $h_0$ is selected as 200 m and $h_c$ is selected as 10 m.

### 6.3.1 Simulation Results Using Kinematic Model

Letting $(p_n, p_e, p_d)^T$ be the position of the UAV in NED frame, $V_a$ be the airspeed, $\psi$ be the heading angle, $\gamma_a$ be the air mass referenced flight path angle, which is defined as the angle from the inertial North-East plane to the velocity vector of the aircraft relative to the air mass $(w_n, w_e, w_d)^T$ be the constant wind vector in NED frame, and airspeed rate $u_{V_a}$, heading angular rate $u_\psi$ and flight path angular rate $u_{\gamma_a}$ be the control inputs, the equations of motion of the UAV can be written as

$$
\dot{p}_n = V_a \cos\psi \cos\gamma_a + w_n, 
$$

$$
\dot{p}_e = V_a \sin\psi \cos\gamma_a + w_e, 
$$

$$
\dot{p}_d = -V_a \sin\gamma_a + w_d, 
$$

$$
\dot{V}_a = u_{V_a}, 
$$

$$
\dot{\psi} = u_\psi, 
$$

$$
\dot{\gamma}_a = u_{\gamma_a}.
$$
Table 6.1: Initial configuration of the system

<table>
<thead>
<tr>
<th>Item</th>
<th>UAV Dynamics</th>
<th>Design Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_0$, m</td>
<td>(0, 260, −180)</td>
<td>(0, 220, −150)</td>
</tr>
<tr>
<td>$V_{a0}$, m/s</td>
<td>20</td>
<td>12</td>
</tr>
<tr>
<td>$\psi_0$, rad</td>
<td>$\frac{\pi}{2}$</td>
<td>$\pi$</td>
</tr>
<tr>
<td>$\gamma_{a0}$, rad</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\phi_0$ rad</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Comparing (6.12) to (6.17) with the equations in Fig. 6.3, it can be obtained that $\hat{\eta} = (p_n, p_e, p_d)^T$, $\hat{\xi} = (V_a, \psi, \gamma_a)^T$, $\hat{\mathbf{u}} = (u_{V_a}, u_\psi, u_{\gamma_a})^T$, $f_2(\cdot) = 0$, $g_2(\cdot) = \mathbf{I}_3$ and

$$f_1(\cdot) = \begin{pmatrix} V_a \cos \psi \cos \gamma_a + w_n \\ V_a \sin \psi \cos \gamma_a + w_e \\ -V_a \sin \gamma_a + w_d \end{pmatrix}.$$  

In the simulation, the control gains are selected as $k = 2$, $\hat{k}_1 = 0.5$, $\hat{k}_2 = 2$ and $\ell = 1$. Letting the initial position, airspeed, heading angle and flight path angle of the UAV be $p_0$, $V_{a0}$, $\psi_0$ and $\gamma_{a0}$, respectively, the initial configuration of the system in the simulation is listed in Table 6.0. Without loss of generality The constant wind vector is selected as $(3, 0, 0)^T$ m/s. The performance limits of the UAV are selected as $\dot{\psi} \in [-40, 40]$ deg/s, $V_a \in [10, 20]$ m/s, $\gamma_a \in [-15, 15]$ deg and $\phi \in [-35, 35]$ deg.

The simulation results are shown in Fig. 6.4. The UAV trajectories in the 3-D and top-down views are shown in Fig. 6.4 (a) and (b), respectively. It can be seen that the actual trajectory of the UAV started from a position outside the desired orbit with an initial distance error approximately 60 m and converged to the desired trajectory after approximately a half circle. Fig. 6.4 (c) shows the tracking error between the UAV trajectory and the reference in North, East, and altitude directions. It can be seen that all the errors converged to zero after 40 s. Fig. 6.4 (d) shows the evolution of the states $V_a$, $\psi$ and $\gamma_a$. The oscillation of $V_a$ is due to the presence of wind and the oscillation of $\gamma_a$ matched the desired altitude oscillation.
6.3.2 Simulation Results Using Dynamic Model

Although models that include kinematic relationships may be suitable for certain control objectives, models that include dynamic effects are required for other purposes. For example, the dynamics of the mothership used in the aerial recovery are affected by the tension force exerted from the cable. If this tension force cannot be neglected for the mothership, the dynamic model needs to be utilized to design trajectory-tracking control law for the mothership.

Letting $m$ be the UAV mass, $g$ be the gravitational constant at Earth sea level, $L$ and $D$ be the aerodynamic lift and drag forces, respectively, $\phi$ be the roll angle, the control inputs be the thrust $u_r$, the load factor $u_n \triangleq \frac{L}{mg}$ and roll angular rate $u_\phi$, the dynamic model
of the UAV can be written as

\[ \begin{align*}
\dot{p}_n &= V_a \cos \psi \cos \gamma_a + w_n, \\
\dot{p}_e &= V_a \sin \psi \cos \gamma_a + w_e, \\
\dot{p}_d &= -V_a \sin \gamma_a + w_d, \\
\dot{V}_a &= \frac{u\tau - D}{m} - g \sin \gamma_a, \\
\dot{\psi} &= \frac{L \sin \phi}{mV_a \cos \gamma_a}, \\
\dot{\gamma}_a &= \frac{g}{V_a} (u_n \cos \phi - \cos \gamma_a), \\
\dot{\phi} &= u_\phi.
\end{align*} \]

To match the UAV dynamic model to 6.1 and 6.2, Equation (6.21) through (6.23) are rearranged as

\[
\begin{pmatrix}
\dot{V}_a \\
\dot{\gamma}_a \\
\dot{\psi}
\end{pmatrix} =
\begin{pmatrix}
-\frac{D}{m} - g \sin \gamma_a \\
-\frac{g}{V_a} \cos \gamma_a \\
0
\end{pmatrix} +
\begin{pmatrix}
\frac{1}{m} & 0 & 0 \\
0 & \frac{g}{V_a} \cos \phi & 0 \\
0 & 0 & \frac{L}{mV_a \cos \gamma_a}
\end{pmatrix}
\begin{pmatrix}
u_{\tau} \\
u_n \\
\mu
\end{pmatrix},
\]

where \( \mu \triangleq \sin \phi \) is the virtual control input. Then it can be obtained that

\[
\begin{pmatrix}
\dot{V}_a \\
\dot{\gamma}_a \\
\dot{\psi}
\end{pmatrix} =
\begin{pmatrix}
-\frac{D}{m} - g \sin \gamma_a \\
-\frac{g}{V_a} \cos \gamma_a \\
0
\end{pmatrix}, \quad g_1 (\cdot) =
\begin{pmatrix}
\frac{1}{m} & 0 & 0 \\
0 & \frac{g}{V_a} \cos \phi & 0 \\
0 & 0 & \frac{L}{mV_a \cos \gamma_a}
\end{pmatrix}, \quad g_2 (\cdot) =
\begin{pmatrix}
u_{\tau} \\
u_n \\
\mu
\end{pmatrix},
\]

Then the roll angular rate can be calculated by

\[
\dot{\phi} = (\sin^{-1}(\dot{\mu}(3)) - \phi) / T_s,
\]

where \( T_s \) is the sample time of the simulation.
Figure 6.6: Simulation results using dynamic model.

In the simulation, the control gains are selected as $k = 2$, $\hat{k}_1 = 0.2$, $\hat{k}_2 = 0.5$, $\ell = 0.5$. Letting the initial roll angle be $\phi_0$, the initial configuration of the system in the simulation is listed in Table 6.0. The constant wind vector is selected as $(3, 0, 0)^T$ m/s.

The simulation results are shown in Fig. 6.5 which was very similar to the results using a kinematic model shown in Fig. 6.4. It can be seen that the convergence of the actual trajectory was slightly slower than that in Fig. 6.4. The oscillation of the roll angle is due to the presence of wind.

6.4 Software-In-the-Loop Simulation Results

In this section, a software-in-the-loop (SIL) simulation is conducted to validate the control strategy developed in this chapter. The experimental platform consists of MATLAB, MEX-function interface, Virtual Cockpit and a simulator, Aviones. The communication structure is very similar to the one used in Chapter 3. For the current Kestrel Autopilot, the commands that autopilot accepts are airspeed $V_a$, course angle $\chi$ and the altitude $-p_d$. The course angle $\chi$ can be expressed as a function of heading angle $\psi$, air mass referenced
flight path angle $\gamma_a$ and the wind estimation, which is given by [88]

$$
\chi = \tan^{-1} \left( \frac{V_a \sin \psi \cos \gamma_a + w_e}{V_a \cos \psi \cos \gamma_a + w_n} \right).
$$

We can prove that the commands $\left( \hat{V}_a, \hat{\chi}, -\hat{p}_d \right)^T$ are qualified outputs that guarantee $\xi \to \hat{\xi}$ using Lemma 3 (See Appendix A).

The communication rate between MATLAB and VC is approximately 5 Hz, so the step size in MATLAB is selected as 0.2 s. Because of the communication lag between the MATLAB, Virtual Cockpit and the simulator, the control gain $\ell$ was tuned and selected as 0.1 to compromise between the tracking error of the actual and simulated UAV and tracking error of the simulated UAV and the desired trajectory. A wind of 3 m/s from the west is added in Aviones.

### 6.4.1 Circular Orbit Tracking

In this section, we use a circular orbit as a desired trajectory in the SIL simulation. In Eq. (6.11), we select $R$ as 200 m, $V^d$ as 15 m/s, $h_0$ as 100 m and $h_c$ as 5 m. A constant wind $(0, 3, 0)$ m/s is given in the simulator.

Figure 6.6 shows the SIL simulation results of tracking a circular orbit. Figure 6.6 (a) presents the top-down view of system trajectories. It can be seen that the actual trajectory (line with “Asterisk”) started from approximately $(-60, -40)$ and quickly converged to the desired orbit (solid line). The UAV trajectory obtained from the design model (“diamond” dots) also quickly converges to the desired orbit. Figure 6.6 (b) shows the side view of system trajectories. It can be seen that the actual UAV altitude is able to essentially track the desired orbit. Figure 6.6 (c) shows the tracking errors in the north, east and altitude direction, respectively. It can be seen that tracking errors converge to the neighborhood of the origin and oscillate within $\pm 10$ m in three directions, respectively. Figure 6.6 (d) illustrates the distance between the actual UAV position and the desired trajectory. It can be seen that the distance error converge to a small value (approximately 15 m) from a large initial error (approximately 250 m) and remain within (15 m). Figure 6.6 (e) shows the commanded autopilot inputs and the actual value of the UAV. It can be seen that the
Figure 6.7: SIL simulation results in tracking a circular orbit.

altitude tracker has about 10s lag. However, the airspeed and course angle trackers work more precisely. Figure 6.6 (f) shows the wind estimation of VC. The wind estimation has small offsets in both north and east directions. Besides the communication lag, the wind estimation error is another key that results in large tracking errors.
6.4.2 Figure-8 Orbit Tracking

To further explore the effect of the control strategy, a figure-8 trajectory is employed as the desired trajectory. Equations characterizing a typical figure-8 trajectory can be given by

\[ \eta^d(t) = \begin{pmatrix} R_a \sin (\varphi(t)) \\ R_b \sin (2 \cdot \varphi(t)) \\ -h_0 + h_c \sin (\varphi(t)) \end{pmatrix}. \]

To guarantee that the actual UAV is able to track the desired trajectory under its performance limits, an updated law for \( \varphi \) is given to make sure the desired trajectory has a constant airspeed in the presence of wind. The time derivative of \( \eta^d(t) \) can be calculated as

\[ \dot{\eta}^d = \begin{pmatrix} R_a \dot{\varphi} \sin \varphi \\ 2R_b \dot{\varphi} \cos (2\varphi) \\ h_c \dot{\varphi} \cos \varphi \end{pmatrix} = \mathbf{v}_a + \mathbf{w}, \tag{6.25} \]

where \( \mathbf{v}_a \) is the airspeed vector in the wind frame. Thus, the airspeed magnitude of the UAV \( V_a = \| \mathbf{v}_a \| \) is calculated as

\[ V_a = \sqrt{(R_a \dot{\varphi} \sin \varphi - w_n)^2 + (2R_b \dot{\varphi} \cos (2\varphi) - w_e)^2 + (h_c \dot{\varphi} \cos \varphi - w_d)^2}. \]

Given a desired UAV airspeed \( V_a^d \), we can solving the quadratic equation for \( \dot{\varphi} \). In this section, \( V_a^d \) is selected as 14 m/s, \( R_a \) and \( R_b \) are selected as 250 m 100 m, respectively.

Figure 6.7 shows the SIL simulation results of tracking a figure-8 orbit. Figure 6.7 (a) presents the top-down view of system trajectories. It can be seen that the actual trajectory (line with “Asterisk”) started from approximately (−80, 20) and is able to essentially follow the desired orbit (solid line). The UAV trajectory obtained from the design model (“diamond” dots) quickly converges to the desired orbit. Tracking errors of curved segments are larger the straight part. Figure 6.7 (b) shows the side view of system trajectories. It can be seen that the actual UAV altitude is able to essentially track the desired orbit. Figure 6.7 (c) shows the tracking errors in the north, east and altitude direction, respectively. It can be seen that tracking errors oscillate within ±50 m in the north and east directions, respectively.
while ±5 m in the altitude direction. The figure-8 orbit is more complex than the circular orbit with large curvature of the curved segment, which requires more maneuverability of the vehicle. Figure 6.7 (d) illustrates the distance between the actual UAV position and the desired trajectory. It can be seen that the distance error oscillates within ±50 m which
is much larger than that in Fig. 6.6 (d). Figure 6.7 (e) shows the commanded autopilot inputs and the actual value of the UAV. It can be seen that the altitude tracker becomes smaller than that in Fig. 6.6 (e), and the airspeed and course angle trackers work precisely. Figure 6.7 (f) shows the wind estimation of VC. The wind estimation has small consistent offsets in both north and east directions. It can be seen that, for more complex desired trajectory, this control strategy needs high bandwidth. Onboard embedded implementation could be an effective way.

6.5 Conclusions

This chapter presented a trajectory tracking strategy for UAVs with autopilot in the closed-loop. Lyapunov based backstepping was used to derive the trajectory-tracking controller using a design model with generalized equations of motion. To involve the autopilot in the closed-loop system, the developed trajectory-tracking controller and the design model were augmented. The updated controller was also proved using Lyapunov stability theory to be capable of driving the vehicle to exponentially converge to and follow the reference by keeping the other states bounded. The simulations using both kinematic and dynamic models were conducted to validate the developed controllers. The numerical results showed that the augmented controller was capable of achieving the trajectory-tracking objectives. The SIL simulation results showed that this trajectory tracking control strategy is able to drive the UAV to essentially track the desired circular and figure-8 trajectories. More complex trajectories like figure-8 orbit requires higher communication bandwidth.
Chapter 7

Conclusions and Future Directions

7.1 Conclusions

In this dissertation, we focus on the strategy that places and stabilizes the path of an aerial drogue, towed by a mothership aircraft using a long flexible cable, on a horizontally flat orbit by maneuvering the mothership in the presence of wind. To achieve this goal, several studies for towed cable systems are conducted, which include the dynamic modeling for the cable, trajectory generation strategies for the mothership, trajectory-tracking control law design, and simulation and flight test implementations. The main contributions and results are summarized as follows.

In Chapter 2, a discretized approximation method based on finite element and lumped mass is employed to model the cable dynamics. One contribution of this chapter is that two approaches, Gauss’s Principle and Newton’s second law, are utilized to derive the equations of motion for inelastic and elastic cables, respectively. For inelastic cables, Gauss’s Principle provides a succinct method to model the system dynamics with complex kinematic constraints by neglecting the internal forces. For elastic cables, Newton’s second law provides an effective and efficient method to formulate equations of motion for multi-link cable dynamics with internal forces like cable tensions and external aerodynamic forces. Because the cable using in the flight test is a fishing line, which expresses considerable stretch during the flight, The mathematical model for the cable-drogue system is derived using Newton’s second law in the rest of the dissertation. The simulation results show the essential match to the flight test results in both horizontal and vertical directions, respectively. The results also illustrate that an increased number of cable links from 1 to 5 does not significantly improve the accuracy of the predictions for the current configuration of the system.
In Chapter 3, differential flatness theory is employed to develop the trajectory generation approach for the mothership to place the drogue onto the desired orbit based on the mathematical model developed in the previous chapter. One contribution of this chapter is that different desired drogue orbits are utilized to generate the required mothership trajectories in different wind conditions. The numerical results show that using the desired drogue orbits with constant airspeed, the resulting mothership orbits present smaller altitude oscillations and larger airspeed oscillations than using the drogue orbits with constant ground speed. Another contribution of this chapter is that a wind update strategy is developed which is capable of updating the precalculated required mothership orbit according to the change of the wind estimation. The differential flatness based method has the advantages like it is computationally inexpensive for discretized model of the cable, however, it requires the system dynamic to be differentially flat and does not directly take performance limits of the mothership into consideration. The flight test results show that the amplitude of drogue altitude oscillation can be stabled within 10 m in the presence of wind of approximately 4 m/s. The tracking error of the drogue may result from several reasons like the inaccuracy of the model, imprecise and delayed wind estimation, wind gust and the tracking error of the mothership.

In Chapter 4, model predictive control (MPC) is utilized to develop the trajectory generation approach for the mothership to lead the drogue onto the desired orbit. The MPC method is based on optimization theory which can be applied to most dynamical systems and is able to take the performance limits of the mothership into consideration. One contribution of this chapter is that a combination objective function is developed in which a squared-error objective is used to calculate the trajectory tracking error and $L_1$-norm objectives are used to regulate the constraints. Different desired drogue trajectories are also used to produce the required mothership trajectories in different wind conditions. The numerical results show that even in severe wind conditions (e.g. wind speed greater than 10 m/s), the MPC based method is able to produce trajectories that guarantee that the mothership maneuvers within its performance limits as well as placing the drogue essentially onto the desired orbit. Trajectory generation for a transitional flight in which the system flies from a straight and level flight into a circular orbit, is also presented. The cable tension is added as an additional
constraint so that the tension force variates within its limits during the transition. Primary drawbacks of the optimization based method include that it is computationally expensive and that it is difficult to configure relative to the method based on differential flatness. The possibility of an unsuccessful solution may also limit this method from being applied in the real time. This MPC-based optimal trajectory generation strategy can be a framework for specifying any arbitrary flight path of the towed body by optimizing the action of the towing vehicle subject to constraints and wind disturbance.

In Chapter 5, a trajectory tracking control law for the mothership is developed using Lyapunov based backstepping in the presence of wind. One contribution of this chapter is that the cable tension and wind disturbance are taken into account in the mothership dynamics and the mothership tracking error is guaranteed to exponentially converge to an ultimate bound, which is a function of a superior limit of the unknown component of the wind.

In Chapter 6, we develop an approach of designing the trajectory tracking control law for autonomous vehicles with the autopilot in the loop. One contribution of this chapter is that the specific dynamic of the autopilot is taken into account in the controller design. The original designed trajectory tracking controller and design model only need to be augmented using an extra term which is a function of the feedback error, so that the actual system trajectory is guaranteed to exponentially converge to the origin with all states bounded. A sufficient condition for qualified commands of the autopilot is also developed. This strategy provide a framework to implement the developed controller on the experimental vehicle without modifying the key structure of the design model and the controller.

7.2 Future Directions

A number of potential topics may be explored in the future work based on the contributions and results of this dissertation. In this section, we propose several possible directions.

Two methods are employed in Chapter 2, to derive the equations of motion for the cable dynamics. The flight test data are used to determine an appropriate number of cable links. However, it is nontrivial to determine reasonable values for key parameters like drag coefficient of the drogue, cable length, and Young’s modulus. Therefore, an estimation based
strategy is needed that utilizes the experimental data to determine values for parameters in the simulation.

Existing methods that solve the motion planning problem for towed cable systems are typically based on optimization and differential flatness, as discussed in Chapter 3 and 4. However, these two methods are primarily used in offline calculations. Because the differential flatness based method does not take the performance limits and constraints like cable tension into consideration, for realtime application, it may not be suitable to produce required trajectories for towing vehicles with low maneuverability. The optimization based method has drawbacks like computationally expensive and possibly infeasible solutions, which may be impractical in real-time application. Therefore, a real-time trajectory generation strategy for towed cable systems is needed.

In this dissertation, the trajectory tracking objective for the drogue is achieved by assuming that the drogue motion is able to follow the desired trajectory if the mothership tracks the required trajectory. This is actually an open-loop control strategy. The traditional closed-loop control strategy is hard to apply to the towed cable system because the cable connection between the mothership and drogue delays the drogue response to the mothership maneuvering. However, periodic feedback control may be a potential approach that is able to solve the problem. One possible way is to linearize the system dynamics around the desired drogue trajectory. A periodic update strategy for the mothership motion is also needed by feeding back the target control variables of the drogue (e.g. the oscillation of the drogue altitude).

In Chapter 3, the experimental results show that the mothership is able to essentially follow the calculated trajectory in the flight test using waypoint following. The current experimental platform that implements developed control law in the flight test consists of MATLAB, MEX-function interface, Virtual Cockpit and the autopilot. The developed controller is implemented in MATLAB to compute the control inputs using the telemetry data. MEX-function interface is used to transfer the telemetry data and the control inputs between Virtual Cockpit and MATLAB. Virtual Cockpit is employed to control the airplane by sending the commands to the autopilot. However, the current communication frequency between MATLAB and Virtual Cockpit is approximately 5 Hz, and the commands that Virtual Cock-
pit accepts are typically variables like altitude, course angle and airspeed. The course angle and altitude following in the autopilot belong to the outer loop control with low bandwidth so that the maneuverability of the vehicle is attenuated. Therefore, the accessibility to inner loop controlled variables like bank angle and pitch angle are needed. It is also required that the communication frequency between MATLAB and Virtual Cockpit increase to at least 25 Hz, which is the communicate rate between the Virtual Cockpit and the autopilot.
Bibliography


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Appendix A

Observability Proof of Specific Autopilot Commands

Firstly, the definition of the observability is proposed as follows [112].

Consider nonlinear multi-output systems with \( s \) outputs \((y = (y_1, \cdots, y_s)^T)\)

\[
\dot{x} = f(x) + g(x, u), \\
y = H(x) = \begin{pmatrix} h_1(x) \\ \vdots \\ h_s(x) \end{pmatrix},
\]

(A.1)

where \( h_1, \cdots, h_s \) are smooth functions with \( h_i(0) = 0, \ 1 \leq i \leq s \) and \( dh_1, \cdots, dh_s \) are linearly independent in \( \mathbb{R}^n \), \( f \) is a smooth vector field with \( f(0) = 0 \), and \( g(x, u) \) are smooth functions with \( g(x,0) = 0, \ \forall x \in \mathbb{R}^n \).

**Definition 1** A set of observability indices \( \{k_1, \cdots, k_s\} \) is uniquely associated to system (A.1) with \( u = 0 \) that is

\[
\dot{x} = f(x), \\
y_i = h_i(x), \ 1 \leq i \leq s,
\]
as follows:

\[
k_i = \text{card} \{ s_j \geq i : j \geq 0 \}, \ 1 \leq i \leq s,
\]

with

\[
\begin{align*}
s_0 &= \text{rank} \{ dh_i(x) : 1 \leq i \leq s \}, \\
\vdots & \\
s_k &= \text{rank} \{ dh_i(x), \cdots, d(L^k h_i(x)) : 1 \leq i \leq s \} - \text{rank} \{ dh_i(x), \cdots, d(L^{k-1} h_i(x)) : 1 \leq i \leq s \}, \\
\vdots & \\
s_{n-1} &= \text{rank} \{ dh_i(x), \cdots, d(L^{n-1} h_i(x)) : 1 \leq i \leq s \} - \text{rank} \{ dh_i(x), \cdots, d(L^{n-2} h_i(x)) : 1 \leq i \leq s \}.
\end{align*}
\]

**Definition 2** System (A.1) is said to be locally observable in \( D \) (or observable in \( \mathbb{R}^n \)) if

\[
\text{rank} \{ dh_i(x), \cdots, d(L^{k_i-1} h_i) : 1 \leq i \leq s \} = n,
\]

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for every $x \in D$ (for every $x \in \mathbb{R}^n$).

**Assumption 1** The horizontal component of wind speed is less than the minimal horizontal component of the UAV airspeed.

**Lemma 4** System (6.1) and (6.2) with output $\left( -\hat{p}_d, \hat{V}_a, \hat{\chi} \right)^T$ is observable with respect to $\hat{\xi} = \left( \hat{V}_a, \hat{\psi}, \hat{\gamma}_a \right)^T$.

**Proof:**
Consider the following equations extracting from the kinematic model of the system

\[
\begin{align*}
\dot{\hat{p}}_d &= -\hat{V}_a \sin \hat{\gamma}_a - \ell (p_d - \hat{p}_d), \\
\dot{\hat{V}}_a &= \hat{u}_V, \\
\dot{\hat{\psi}} &= \hat{u}_\psi, \\
\dot{\hat{\gamma}}_a &= \hat{u}_{\gamma_a}.
\end{align*}
\] (A.2)

Letting $x = \left( \hat{p}_d, \hat{V}_a, \hat{\psi}, \hat{\gamma}_a \right)^T$ and $u = (\hat{u}_V, \hat{u}_\psi, \hat{u}_{\gamma_a})^T$, we can obtain

\[
\begin{align*}
\dot{x}_1 &= -x_2 \sin x_4 - \ell (p_d - x_1), \\
\dot{x}_2 &= \hat{u}_1, \\
\dot{x}_3 &= \hat{u}_2, \\
\dot{x}_4 &= \hat{u}_3,
\end{align*}
\] (A.6)

\[
y = H (x) = \begin{pmatrix} h_1 (x) \\ h_2 (x) \\ h_3 (x) \end{pmatrix},
\] (A.10)

where

\[
\begin{align*}
h_1 (x) &= -x_1, \\
h_2 (x) &= x_2, \\
h_3 (x) &= \tan^{-1} \left( \frac{x_2 s_{x_3 c_{x_4}} + w_e}{x_2 c_{x_3 c_{x_4}} + w_n} \right),
\end{align*}
\]

in which $s_* \triangleq \sin *$ and $c_* \triangleq \cos *$.

$dh_1, \cdots, dh_3$ are linearly independent in $D$.

From Eqs. (A.6) to (A.9), the differentials of $h_1 (x)$ and $h_2 (x)$ are given by [112]

\[
\begin{align*}
\text{dh}_1 &= -dx_1, \\
\text{dh}_2 &= dx_2.
\end{align*}
\]

Letting $A \triangleq x_2 s_{x_3 c_{x_4}} + w_e$ and $B \triangleq x_2 c_{x_3 c_{x_4}} + w_n$, the differential of $h_3 (x)$ is given by

\[
\text{dh}_3 = \frac{1}{1 + \frac{A^2}{B^2}} \cdot \frac{1}{B^2} \cdot (dA \cdot B - A \cdot dB),
\] (A.11)
where
\[
\begin{align*}
dA & = s_{x_3} c_{x_4} dx_2 + x_2 c_{x_3} c_{x_4} dx_3 - x_2 s_{x_3} s_{x_4} dx_4, \\
 dB & = c_{x_3} c_{x_4} dx_2 - x_2 s_{x_3} c_{x_4} dx_3 - x_2 c_{x_3} s_{x_4} dx_4.
\end{align*}
\]

Then, the simplified Eq. (A.11) is given by
\[
dh_3 = \frac{1}{A^2 + B^2} (a_2 dx_2 + a_3 dx_3 + a_4 dx_4),
\]

where
\[
\begin{align*}
a_2 & = s_{x_3} c_{x_4} w_n - c_{x_3} c_{x_4} w_e, \\
a_3 & = x_2^2 c_{x_4} + x_2 c_{x_4} (c_{x_3} w_n + s_{x_3} w_e), \\
a_4 & = -x_2 s_{x_3} s_{x_4} w_n + x_2 c_{x_3} s_{x_4} w_e. \\
\end{align*}
\]

(A.12)

It can be seen that, the term \(A^2 + B^2\) is actually the horizontal component of the UAV ground speed. From Assumption 1, we can obtain \(A^2 + B^2 > 0\) for \(x \in D\). Rearranging Eq. (A.12), we have
\[
a_3 = x_2 c_{x_4} \left( x_2 c_{x_4} + \sqrt{w_n^2 + w_e^2} \sin (x_3 - \varphi) \right),
\]

where
\[
\tan \varphi = \frac{w_n}{w_e}.
\]

For \(x \in D\), we have \(x_2 > 0\) and \(c_{x_4} > 0\). Then from Assumption 1, it can be obtained that \(x_2 c_{x_4} > \sqrt{w_n^2 + w_e^2}\) and
\[
a_3 > x_2 c_{x_4} \left( x_2 c_{x_4} - \sqrt{w_n^2 + w_e^2} \right) > 0.
\]

Therefore rank \(\{dh_1, dh_2, dh_3\} = 3\), which implies that \(dh_1, \ldots, dh_3\) are linearly independent in \(D\).

**System described by Eqs. (A.6) to (A.10) is observable.**

From Eqs. (A.6) to (A.9) we can obtain
\[
f = (-x_2 \sin x_4 - \ell (p_d - x_1)) \frac{\partial}{\partial x_1}.
\]

The Lie derivatives are given by
\[
\begin{align*}
\mathcal{L}_f h_1 & = -x_2 \sin x_4 - \ell (p_d - x_1), \\
\mathcal{L}_f h_2 & = 0, \\
\mathcal{L}_f h_3 & = 0.
\end{align*}
\]
Furthermore, we can calculate
\[
\begin{align*}
  d (\mathcal{L}_f h_1) &= -\sin x_4 dx_2 - x_2 \cos x_4 dx_4 + \ell dx_1, \\
  d (\mathcal{L}_f h_2) &= 0, \\
  d (\mathcal{L}_f h_3) &= 0,
\end{align*}
\]
and
\[
\mathcal{L}_f^2 h_1 = -\ell x \sin x_4 - \ell^2 (p_d - x_1) \quad \text{and} \quad \mathcal{L}_f^2 h_2 = \mathcal{L}_f^2 h_3 = 0.
\]
Thus, we can obtain
\[
\mathcal{L}_f^i h_1 = -\ell^{i-1} x_2 \sin x_4 - \ell^i (p_d - x_1),
\]
and
\[
d (\mathcal{L}_f^i h_1) = -\ell^{i-1} \sin x_4 dx_2 + (-\ell)^{i-1} x_2 \cos x_4 dx_4 + \ell^i dx_1, \quad i = 1, 2, 3, 4.
\]
From Definition 2, we can obtain
\[
s_0 = 3, \quad s_1 = 1 \quad \text{and} \quad s_2 = s_3 = 0.
\]
Thus we have
\[
k_1 = 2, \quad k_2 = 1 \quad \text{and} \quad k_3 = 1.
\]
Therefore, we obtain
\[
\text{rank} \{ dh_i (x), \ldots, d (\mathcal{L}_f^{i-1} h_i) : 1 \leq i \leq 3 \} = 4.
\]
\hfill \Box