Investigation of Internal Wave Spectra Due to Observed Interactions

Benjamin Lee Hillyard

Brigham Young University - Provo

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ABSTRACT
Investigation of Internal Wave Spectra Due to Observed Interactions

Benjamin L. Hillyard
Department of Mechanical Engineering, BYU
Master of Science

Observational data are analyzed and decomposed to reveal internal ocean waves and their interactions with one another. Particularly, the interaction of small-scale internal waves with a large inertia wave packet is examined. Using the governing internal wave equations, an analysis is made of the energy propagation of a small scale internal wave with a large-scale inertia wave. With that, an assessment is made of the frequency of occurrence of various encounter types. Next, the possibility of energy transfer during an interaction is explored. The relative energy of the small wave before interacting with the large-scale inertia wave is calculated and compared to the relative energy during and after the interaction. Performing this analysis on multiple wave-wave interactions seen within the observational data set provides a look into the behavior of these wave types. Additionally, the dissipation within each of the corresponding time-space regions is calculated, giving an alternative explanation other than energy transfer among waves for the disparity in energy. Dissipation estimates and energy results are extrapolated to create a general energy transfer and dissipation estimate in the ocean resultant from these interaction types. A two dimensional non-linear method presents a comparison between the observational data findings and the expected computed result. From there, conclusions are drawn synthesizing the results from the observational and numerical analyses. It was concluded that for observational small waves propagating in the same direction as the background shear, a loss was seen in the wave’s energy. For interactions wherein the small wave propagated in the opposite direction, the observational small wave energy increased through the interaction. Within the numerical findings, the small wave energy in same direction interactions was partially lost while the small wave energy in opposite direction interactions was both lost and gained depending on the encounter type which encounter types could be confirmed in observations. The dissipation analysis showed the greatest dissipation during the interaction between a small wave and background shear so the gains seen occurred when the types of encounters expecting a gain were present.

Keywords: Benjamin Hillyard, Julie Vanderhoff, internal waves, group velocity, dissipation
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# TABLE OF CONTENTS

LIST OF TABLES ............................................................... vi

LIST OF FIGURES ........................................................... viii

Chapter 1  Introduction .................................................. 1
  1.0.1 Motivation ......................................................... 3
  1.0.2 Internal Wave Properties ....................................... 3
  1.0.3 Internal Wave Characteristics .................................. 4
  1.0.4 Internal Wave Generation ....................................... 6

Chapter 2  Literature Review ........................................... 13
  2.1 Literature Review ................................................... 13
    2.1.1 General Internal Waves Research ......................... 13
    2.1.2 Observations ................................................ 13
    2.1.3 Small Wave Steady Shear Interactions .................. 17
    2.1.4 Wave Steepening and Breaking ........................... 19
    2.1.5 Dissipation ................................................ 20

Chapter 3  Methods ....................................................... 23
  3.1 Theory .............................................................. 23
    3.1.1 Governing Equations ........................................ 23
    3.1.2 Idealized Problem .......................................... 25
    3.1.3 2D Non-linear Model ....................................... 26
  3.2 Observations ....................................................... 28
    3.2.1 Wave Field ................................................ 28
  3.3 Energy Estimate .................................................. 35
    3.3.1 Determining Above, Middle and Below Time/Space Regions 35
    3.3.2 Dissipation Estimate ...................................... 39

Chapter 4  Results and Discussion .................................. 45
  4.1 Numerical Results ................................................. 45
  4.2 Observational Results ........................................... 50
    4.2.1 Energy Spectra .............................................. 50
    4.2.2 Energy Analysis ............................................ 55
    4.2.3 Dissipation Analysis ..................................... 70

Chapter 5  Conclusion .................................................... 75
  5.1 Summary ........................................................... 75
    5.1.1 Research Goals .............................................. 76
    5.1.2 Conclusions ................................................ 76
    5.1.3 Future Work ................................................ 78
REFERENCES ................................................................. 79

Appendix A Deriving Natural Frequency Equation ......................... 83

Appendix B Polarization Relations ........................................... 85
  B.1 Polarization Relations .................................................. 85

Appendix C Deriving Dispersion Relation .................................. 89
LIST OF TABLES

4.1  Tabulated values for the changes in energy within a small wave when propagating through a background shear for numerical simulations. ........................................... 50
4.2  Tabulated values for the dissipation for numerical simulations. .......................... 50
4.3  Tabulated values for the changes in energy within a small wave when propagating through a background shear from the observational data. Values represent 1 day averages ................................................................. 60
4.4  Tabulated values for the changes in energy within a small wave when propagating through a background shear from the observational data. Values represent 1 day averages. Before to during and during to after the interaction values shown .......................... 61
4.5  Tabulated values for the changes in energy within a small wave when propagating through a background shear from the observational data. Values represent 17 day averages ................................................................. 64
4.6  Tabulated values for the changes in energy within a small wave when propagating through a background shear from the observational data. Values represent 17 day averages. Before to during and during to after the interaction values shown .......................... 64
4.7  Adjusted average energies of the small wave through the assumed interaction with a background shear from the observational data. This represents the composite between the 1-day and 17-day averages ....................................................... 65
4.8  Tabulated values for the changes in energy within a small wave when propagating through a background shear from the observational data. Values represent the differences between the 1 and 17 day averages. Before to during and during to after the interaction values shown ....................................................... 66
LIST OF FIGURES

1.1 A time-series view of a set of isopycnals from relative stability (Figure a) to relative levels of overturning and breaking (Figures b-d). ................................. 2
1.2 A schematic diagram illustrating the generation of an internal wave. ................. 4
1.3 A mock diagram of a two dimensional internal wave enveloped by a Gaussian. Propagation direction of phases and group speed shown. ................................. 5
1.4 Still frames from two videos of oscillating cylinders in a stably stratified fluid. The cylinder in Figure a was oscillated at a frequency of 0.125s-1 and the cylinder in Figure b was oscillated at a frequency of 0.2s-1 ................................. 8
1.5 A conceptualized representation of a critical level. A small wave approaches and interacts with a non-zero mean shear from above. The critical level is reached where the phase speed of the packet moves at the same rate as the mean flow. The small wave is rotated in the process and becomes a part of the flow ............................. 10

2.1 An example of a first kind encounter generated using ray theory. The solid individual lines together represent a packet of energy from a small wave propagating into a background shear given by the filled-in ellipses. ................................. 18
2.2 An example of a second kind encounter generated using ray theory. The solid individual lines together represent a packet of energy from a small wave propagating into a background shear given by the filled-in ellipses. ................................. 18
2.3 An example of a third kind encounter generated using ray theory. The solid individual lines together represent a packet of energy from a small wave propagating into a background shear given by the filled-in ellipses. ................................. 19

3.1 A simplified representation of a gravity wave and inertia wave interaction .............. 26
3.2 An aerial view of the Kaena Ridge, the location in the ocean where observational data was gathered as part of the Hawaiian Ocean Mixing Experiment. ......................... 29
3.3 A diagram of CTD’s and Doppler sonar used in the HOME study ........................ 30
3.4 Vertical displacement plot obtained through observational data .............................. 31
3.5 Vertical displacement plot obtained through observational data (zoom) ................. 32
3.6 Observational strain as derived from observational data. The strain was filtered for upward and downward propagation .............................................. 33
3.7 Observational shear as derived from observational data ........................................ 34
3.8 Axes showing the quadrants used to determine upward and downward motion in the Fourier transformed data .................................................. 35
3.9 Observational shears as derived from observational data. The shear was filtered for upward and downward propagating waves as shown .............................. 36
3.10 Horizontal velocity plots filtered for upward and downward energy ...................... 37
3.11 The first of three sets of figures depicting upward and downward background shears within the observational data. These figures display the time/space locations of the cases analyzed when determining energy gains/losses and dissipation before, during and after an interaction ........................................ 41
3.12 The second of three sets of figures depicting upward and downward background shears within the observational data. These figures display the time/space locations of the cases analyzed when determining energy gains/losses and dissipation before, during and after an interaction ................................................................. 42

3.13 The third of three sets of figures depicting upward and downward background shears within the observational data. These figures display the time/space locations of the cases analyzed when determining energy gains/losses and dissipation before, during and after an interaction ................................................................. 43

4.1 Idealized shear and perturbation density plots of a numerical simulation for a first kind encounter. White boxes show the locations of space/time regions examined for before, during and after an interaction between a background shear and a small internal gravity wave ................................................................. 46

4.2 Perturbation density plot of a numerical simulation showing multiple exits of the small wave from the inertia shear ................................................................. 47

4.3 Idealized shear and perturbation density plots of a numerical simulation for a third kind encounter. White boxes show the locations of space/time regions examined for before, during and after an interaction between a background shear and a small internal gravity wave ................................................................. 49

4.4 An energy spectrum of horizontal velocity with respect to frequency and an energy spectrum of horizontal velocity with respect to wavenumber. ................................................................. 51

4.5 An energy spectrum of the shear showing peaks within the spectrum at the coriolis, diurnal, semidiurnal and high frequencies present within the water column and an energy spectrum of the shear with respect to vertical wavenumber. ................................................................. 53

4.6 An energy spectrum of vertical velocity with respect to frequency and an energy spectrum of vertical velocity with respect to wavenumber. ................................................................. 54

4.7 Group velocity distribution plot displaying the various group velocities of small waves within the observational data compared to the estimated phase speed of the inertia wave ................................................................. 56

4.8 A plot showing the second case time/space regions selected within the observational shear ................................................................. 57

4.9 A sequence of plots showing the spectral energy for downward high frequency waves within the time/space regions selected for Case 2. ................................................................. 58

4.10 A sequence of plots showing the spectral energy for upward high frequency waves within the time/space regions selected for Case 2. ................................................................. 59

4.11 A plot showing total energy versus depth in the water column ................................................................. 62

4.12 A total energy plot of the energy of the shear within each of the time/space regions and filtered for upward and downward propagation. ................................................................. 67

4.13 A sequence of plots showing the spectral energy for Case 1 and Case 7 to compare the behavior of each case’s energy spectrum ................................................................. 68

4.14 A sequence of plots showing the spectral energy for Case 7 and Case 8 to compare the behavior of each case’s energy spectrum ................................................................. 69

4.15 A plot showing the average dissipation with depth within the water column. ................................................................. 71

4.16 A barchart showing the dissipation before, during and after the interaction within each case and for 1 day and 17 days. ................................................................. 72
4.17 A bar chart showing the dissipation before, during and after the interaction as a composite of the 1 and 17 day values.
CHAPTER 1. INTRODUCTION

Internal waves are present ubiquitously in the ocean and atmosphere and are associated with millions of watts of energy, the evolution of which is of great importance to the scientific community. Understanding the evolution of this energy leads to knowledge of the ocean’s heat and energy budget, ocean circulation patterns and global flows. Much of the millions of watts of energy is dissipated through internal waves overturning and breaking, where breaking is defined as turbulence production and irreversible energy dissipation [1]. Internal wave breaking that takes place is much like free surface wave breaking where the amplitude of the wave reaches a critical point wherein the crest of the wave overturns causing an instability.

One motivation for studying internal waves is to understand the cascade of energy from large to small scales to piece together the driving dissipative mechanisms. Knowing that there is approximately two terawatts of energy input into the ocean and atmosphere through surface winds, tidal structures and the earth’s rotation, the same amount of energy is expected to be dissipated. Currently, it is believed tidal dissipation occurs through the scattering of surface tides into internal waves [2], [3], although early suppositions pegged bottom friction in shallow seas as the major sink for tidal energy [4], [5], [6]. Egbert [4] shows one terawatt of dissipation occurring in the deep ocean, generally near areas of rough topography. The life-cycle of any remaining or unaccounted for energy may coincide with the generation and propagation of internal waves which may eventually lead to ocean mixing as they overturn and break due to local interactions with other waves.

Figure 1.1 illustrates the propagation and breaking process by displaying a time-series of isopycnals, or lines of constant density, as energy moves through a fluid eventually leading to breaking. These lines of constant density look much like free surface waves. As heavier fluid is moved over lighter fluid due to the energy propagation, the wave becomes unstable and breaks as shown in the top portion of the last two subfigures of Figure 1.1 as is indicated by the pro-
gressive steepening of the lines through time. In each subfigure, the vertical represents depth, \( z \), and horizontal, length, \( x \). The top time equivalence on each subfigure indicates the relative non-dimensional time. From (a) to (d), the isopycnals, represented by solid lines, eventually become too steep, causing heavier fluid to be on top of lighter fluid, and break, creating mixing across isopycnals as illustrated in the top portion of (c) and (d). Within interactions of the type studied in this research, the steepening of waves may occur given the proper conditions.

![Figure 1.1: This figure shows a time-series cross-sectional view of a set of isopycnals from relative stability, Figure (a), to relative levels of overturning and breaking in Figures (b-d). Variable \( t \) gives the time each figure within the time series was taken. Taken and adapted from Liu (2010) [7]](image-url)
1.0.1 Motivation

The mixing that takes place vertically between isopycnals, or diapycnal mixing, governs much of the activity in the ocean. Several effects may be seen as a result: warmer oceanic near-surface regions combine with cooler lower waters, nutrients are brought from deeper depths up to areas swimming with a multitude of biological beneficiaries, pollutants are distributed from their sources, clear air turbulence is created [8] and momentum and energy are transported where they would otherwise experience a cessation of movement. These phenomena help to maintain heat and energy budgets within the ocean and atmosphere, often affecting other areas of scientific research besides just internal wave study as these circulations cause exchanges of energy. The effects of internal waves can be seen in engineering applications related to acoustic signals, submersibles and marine cabling [9]. Due to their presence in a stratified fluid, these waves call all parts of the ocean and atmosphere home, giving geophysicists, marine biologists, oceanographers, astrophysicists and the like ample motivation for their study.

1.0.2 Internal Wave Properties

Simply speaking, an internal wave is the propagation of energy due to a perturbation of a fluid volume in a stably stratified fluid such as the ocean or atmosphere. A stably stratified fluid is one in which density increases with increasing depth and is defined by a reference density, gravity and the density gradient with respect to depth. A particle perturbed from its neutrally buoyant position, unforced will, oscillate at the buoyancy, natural or Brunt-Väisälä frequency. That frequency is represented by Equation 1.1. The derivation of the equation for natural frequency can be seen in Appendix A.

\[
N^2 = -\frac{g \partial \rho}{\rho_0 \partial z}
\]  

(1.1)

As the buoyancy frequency within a fluid increases, the fluid becomes more resistant to overturning. Because no internal waves can exist with an excitation frequency above the natural frequency of the fluid, as the natural frequency increases, the number of waves possible increases as well. Any disturbance to this stable stratification causes a volume of fluid particles to perturb or displace from their neutrally buoyant position, forcing them into surrounding fluid with a relatively
higher or lower density. A schematic diagram illustrating the concept of natural oscillation can be found in Figure 1.2. There, the horizontal axis represents density with \( \rho_0 \) and \( \rho_1 \) being a reference density and perturbed density respectively while the vertical axis is height with \( z_0 \) and \( z_1 \) a reference height and perturbed height respectively. After being released, buoyancy forces cause the fluid volume to move back towards its neutrally buoyant position, giving it some amount of kinetic energy. The fluid volume overshots its initial position thanks to the newly acquired kinetic energy, moving again into an environment of differing density. This battle between motion and buoyancy forces causes an oscillation of the fluid at the natural frequency which results in the generation of internal waves. If the oscillation is forced slower than the natural frequency, internal waves are generated at progressively shallower angles from the vertical.

![Schematic Diagram](image)

Figure 1.2: This schematic diagram illustrates the generation of an internal wave. \( F_B \) represents the buoyant force, \( \rho_0 \) and \( \rho_1 \) are the initial and perturbed density respectively, \( V \) is the particle volume, \( g \) is gravity and \( W \) is the particle weight. From Blackhurst [10].

1.0.3 Internal Wave Characteristics

As with acoustic or surface waves, internal waves are described using similar terminology although a few deviations exist. Please refer to Figure 1.3 for the next portion of the discussion. The wave’s spatial size is defined by its wavelength and amplitude, the former being the distance
from crest to neighboring crest while the latter is one half the height from crest to neighboring
trough. A wavenumber of the wave is inversely related to the wavelength by Equation 1.2

\[ \lambda = \frac{2\pi}{k} \]  

(1.2)

where \( k \) is the wavenumber and \( \lambda \) is the wavelength. However, because internal gravity waves are
three-dimensional in nature, they are described in all three cartesian coordinate axes directions and
therefore have a wavelength in each of the three directions, and consequentially, a wavenumber in
these directions as well: \( k, l, \) and \( m \) for \( x, y \) and \( z \) respectively where \( z \) is positive upward. The
waves in Figure 1.3 are enveloped by a Gaussian curve and are a pictoral estimate of a wave being
generated at one location. Vector \( C \) indicates the direction of propagation of the phases of the
wave while vector \( C_g \) points in the direction of the group speed or energy propagation. These two
vectors are always perpendicular to each other.

![Figure 1.3: This is a mock diagram of a two dimensional internal wave enveloped by a Gaussian
denoted by the dashed line. Vector \( C \) indicates the direction of propagation of the phases of the
wave while vector \( C_g \) points in the direction of the group speed or energy propagation. Vector
\( \lambda \) refers to the wavelength in one dimension, for instance \( x \) as indicated by the dotted line, and
is written as a vector to reiterate the two other wavelengths in the other two directions. Variable
\( a \) indicates the amplitude of the wave. \( C_g \) can be seen measured from the vertical (gravitational
direction) by angle \( \theta \). From Blackhurst [10].](image)
Kinetic energy associated with internal waves is of a high magnitude due to the scales involved. It is estimated through Moum [11] that for every meter of coastline, the internal waves within Moum’s research contained \(O(0.1)\) megajoules per meter. In the vertical, these waves are governed by the ocean depth, restricting their vertical wavelength from meters to a few kilometers. Horizontally these waves are given much more freedom extending anywhere from hundreds to tens of thousands of meters. It may be noted that although the energy of these waves travels at relatively slow speeds, on the order of millimeters per second or on the order of 100 m/day, the associated energy is of such high magnitude because it is found at large spatial scales or volumes.

The frequencies of internal waves are bound by the natural frequency of the fluid as the upper limit and the Coriolis frequency (the local rotation rate of the Earth), defined by Equation 1.3

\[
f = 2\Omega \sin(\phi)
\]

where \(\Omega\) is the rotation rate of the Earth and \(\phi\) is the degree latitude of interest, as the lower limit. The relationship between the intrinsic frequency of the wave, \(\omega\), and \(k\) is called the dispersion relation as it defines how waves of different wavenumber propagate at different speeds and is found using the \(x\) and \(y\) momentum equations as well as continuity (shown in Appendix B). It takes into account flows on a global scale and can be found in two forms, Equation 1.4 which accounts for global rotation and Equation 1.5 which disregards any global rotation. Figure 1.3 shows the direction of propagation of energy defined to be with \(C_g\) whereas the phases of the wave propagate in the orthogonal direction or with \(C\).

\[
\omega^2 = \frac{N^2 (k^2 + l^2) + f^2 m^2}{k^2 + l^2 + m^2}
\]  

(1.4)

\[
\omega^2 = \frac{N^2 (k^2 + l^2)}{k^2 + l^2 + m^2}
\]  

(1.5)

**1.0.4 Internal Wave Generation**

Any perturbation to the fluid must be at an excitation frequency below that of the natural frequency of the fluid for internal waves to be generated. Any frequency of excitation above the
natural frequency of the fluid is too fast for the fluid to respond and results mainly in turbulence. Additionally, the frequency at which waves are generated determines the propagation angle relative to the vertical (defined positive opposite of gravity). With decreasing frequency, the angle of propagation (from the vertical) increases. In an attempt to illustrate this concept, a series of experiments were conducted using a horizontal rod which was oscillated vertically within a stably stratified fluid and posted online. This can be seen in Figures 1.4(a) and 1.4(b). In these figures, the cylinder is oscillated at a period of 8 seconds and 5 seconds in 1.4(a) and 1.4(b) respectively. It can be seen that the wave beams or propagating energy are closer to the vertical (opposite of gravity) in the case where the cylinder is oscillated more quickly. The dark and light shades within each figure represent phases within the internal wave. Equation 1.6 shows the dependency of the angle of propagation on the natural frequency of the fluid and the excitation frequency of the disturbance.

\[ \theta = \cos^{-1}\left(\frac{\omega}{N}\right) \]  

(1.6)

Within Equation 1.6, the angle \( \theta \) is measured from the vertical to the centerline of the propagating ray or line of wave energy propagation. Any change in angle is the result of altering either the excitation frequency or the fluid in which the wave propagates (and thus, the natural frequency).

Perturbations to the stably stratified fluid can originate from many sources. The most common internal wave generation mechanisms are surface winds and internal currents or tides passing over a non-uniform floor. With ocean currents, seawater passes over highly variable sea-bottom topography causing dense fluid to be forced into a region of less dense fluid. As this phenomenon occurs throughout the ocean floor, internal waves are generated simultaneously and at multiple frequencies and scales. Also, as wind passes over the ocean’s surface, kinetic energy is transferred from the wind to water, exciting fluid perturbations on the surface and causing internal waves to propagate downwards through the mixed layer, a depth from the surface of the ocean to several tens of meters down in which the density gradient varies from the density gradient of the several thousands of meters below. As the Earth rotates, rotational energy is transferred to the oceans, setting up inertial waves. Further, large-scale ocean mixing can cause a mixed region wherein volumes of fluid of varying density are distributed randomly. This region collapses to its neutrally buoyant
(a) A still frame from a video depicting an oscillating cylinder in a stably stratified fluid. The cylinder in the video was oscillated at a frequency of 0.125s$^{-1}$.

(b) A still frame from a video depicting an oscillating cylinder in a stably stratified fluid. The cylinder in the video was oscillated at a frequency of 0.2s$^{-1}$.

Figure 1.4: Figures a and b show vertically oscillating cylinders at a frequency of 0.125s$^{-1}$ and 0.2s$^{-1}$ respectively. It may be noted that the cylinder oscillating at a lower frequency (a) creates internal wave beams with an angle of propagation closer to the horizontal. Used from Internal Gravity Waves [12].

depth and eventually restratification occurs. During this process, as the isopycnals realign, internal waves are created by the hasty realignment. Staquet and Sommeria (2002) mention other less common sources of generation including jets adjustment to equilibrium, unsteady vortical flows and rapid turbulent patch formation by dynamical instability [1], all of which generate internal waves by disturbing the stably stratified medium.

The high quantity of internal waves within the ocean and atmosphere forces a myriad of interactions between not only waves, but other phenomena. In reality, the timescale of the lifespan of internal waves is anywhere from a few seconds to several days depending on each wave’s particular encounter(s) with other ocean energetics. Over the course of its lifetime, an internal wave may encounter a tidal current or inertial shear, vortex, mixed region or other internal waves. Within each interaction, the parameters of the internal wave may change, exchanging energy, and possibly leading to overturning or breaking.

Internal wave interactions involve many types of phenomena. One common type of internal wave interaction involves a small internal wave propagating through local currents, or tide currents.
in a specific location, which can be thought of as a time-independent background flow. If the background flow increases with height and the small wave approaches from below, it is stretched and rotated due to the shear until its vertical velocity approaches zero, lending the wave’s energy to the background. This location where the small wave phase speed matches the translational velocity of the mean flow is called a critical level. In effect, the small wave wavelength increases infinitely and the wave becomes part of the background. For large amplitude waves, the steepness of the wave causes breaking before absorption occurs. The energy transfer involved in the critical layer is an important reason for their study. In 1989 and 1995, Winters and D’Asaro [13], [14] numerically studied critical levels as a small wave or wave packet interacted with a horizontal shear flow as shown in Figure 1.5. Initial conditions were set for the wave packet and initialized in a region void of shear. As the packet propagated down into the shear, the vast majority of the packet’s energy was absorbed by the background. It was shown that the absorption of the packet accelerated the mean flow.

In cases where the small wave approaches in a direction opposite that of the inertia wave, a turning point may occur wherein the approaching small wave is reflected. After reflection, the energy and wavenumber of the reflected wave remains the same but now propagates with a component at an opposite vertical direction.

Another common interaction is between a time-dependent shear and an internal gravity wave. These interactions vary significantly from a critical layer. Due to time dependence, various refractions are possible and will be discussed further in Chapter 2. Using numerical simulations, the commonplace practice is to track the changes in frequency and vertical wavenumber, telling more about the energy of the wave because of the direct relationship between frequency, wavenumber and group velocity. Through the interaction, it is hypothesized that those properties are changed by the refractions that take place due to the time-dependence of the background. The outcome of the interaction is an exchange of energy with the possibility of breaking.

A few objectives of this research are set forth. It is believed that results from this particular study add to the present knowledge of the breakdown of energy from large to small scales. The objectives are:

- Define a local wave field from observations, both high and low frequencies to determine what phenomena are present. Uncover the upward and downward propagating high fre-
Figure 1.5: Conceptualized representation of a critical level. A small wave approaches a non-zero mean shear from above. The critical level is reached where the phase speed of the packet moves at the same rate as the mean flow. The small wave is rotated in the process and becomes a part of the flow. Taken from Winters, et al. [13]

frequency motions, group velocities of small waves, and the prominent wave encounter type within the bounds of the available data as well as plotting the background shear to estimate common inertial waves. Using the observational data, shear plots are produced to identify time and space coordinates of prominent shear bands which correspond to internal waves. Decomposing the data further allows the author to analyze high frequency motions deemed small waves as if traveling through the shear bands found earlier. As will be discussed later, interactions between a short wave and inertia wave (from the low frequency observations) can be broken down into 3 distinct types. Within each type, the small wave involved behaves differently throughout the interaction.

- Calculate the expected energy lost or gained through simulated interactions and within cases for numerical simulations. The numerical simulations approximate the outcome for each
encounter type with the small wave approaching from both above and below the shear band. Within each encounter type, the small wave characteristics are altered to provide a few cases of the same kind encounter. The numerical model employed within this research will be described in more detail within the Methods section.

• Discover the general trend in energy transfer, if any, between the small and large internal waves involved in an interaction in numerical simulations and observational data. Compare the results from the observational data analysis with numerical simulations.

• Analyze the dissipation in the wave field. Along with energy transfer, the dissipation produced within the regions used to calculate the energy estimates will be determined. By picking out the shear, calculating a relative energy before, during and after a supposed interaction and then inspecting the results for any energy loss or gain, it may be inferred whether or not there was energy given or taken over the life of the interaction. As is inherent with observational data, idealized cases are never found meaning the location of the shear bands and perceived small wave energy may be subjective but deviations from what would be found in a numerical simulation will be taken into account.
CHAPTER 2. LITERATURE REVIEW

2.1 Literature Review

2.1.1 General Internal Waves Research

The study of internal waves has been conducted for many decades. Lighthill gives an introduction of internal waves to the technically minded. His discussion takes the reader through an explanation with a couple primary aims: understanding wave dispersion and illustrating general theory by explaining internal waves in a stratified fluid along with more simple wave properties and how these waves may be modeled using ray theory [15]. Other publications reviewing the basic properties of internal waves are included in works by Gill [16] and LeBlond et al. [17]. In 2002, Staquet et al. gave a detailed description of wave breaking leading to turbulence, elaborating on the mechanisms of steepening of these waves as well as the final process moving towards small-scale turbulence. They concluded that within a final instability which would eventually lead to breaking, there is a common overturning event which occurs in various cases [1]. Fritts et al. discuss atmospheric gravity waves in detail, enumerating their effects and behavior as they are generated, propagate and eventually dissipate [18]. Further, basic explanation on internal waves is given by Sutherland [8]. He presents various motivations for studying internal waves, an introduction to stratified fluids, internal wave equation assumptions, and the theory behind their examination. Casaday [19] looked at the initial parameters, amplitude in many cases, of a small-scale gravity wave interacting with a large-scale inertial wave within the atmosphere to determine what conditions lead to a turning point or critical level in the small waves.

2.1.2 Observations

Researchers have been collecting data to study internal waves for decades. One method uses plunging equipment aboard a sea-going vessel to collect internal wave parameters with depth
(depth profile). In 1975, Robert Pinkel described a process whereby data of the upper ocean were taken and how the data were used to further internal wave knowledge [20]. In this particular study, three data sets of temperature versus depth were taken, two of which were near California and one near Hawaii, through 440 vertical meters of the ocean from a starting point 60 meters below the ocean’s surface. The Floating Instrument Platform or FLIP, a horizontal to vertical manned stable platform buoy, was deployed to assist in gathering the data.

Garrett et al. [21] analyzed observational data to characterize the general internal wave spectrum and other general properties of the ocean. In one of his earliest publications, he shows the variation in buoyancy frequency as a function of depth and shows it to be approximately constant for the deep ocean far from boundaries. Further, a discussion of the distribution of space and time scales of internal waves is put forth. When plotting the energy density in frequency and wavenumber space derived from moored and towed measurements, a few consistencies were observed among all data. A spike was seen at the inertial frequency in shear spectra indicating the presence of inertial driven flows, and a steady drop-off of energy from low to high frequencies. Later, Garrett et al. reviewed internal processes in the ocean including critical layers, wave-wave interactions, solitons and dissipation [22]. Within wave-wave interactions, it is noted that a triad of internal waves can transfer action (which is defined as the energy of each of the waves divided by their respective frequencies) between themselves through the equations of motion’s non-linear terms.

A study called the Hawaiian Ocean Mixing Experiment (HOME) was conducted in the early 2000’s off the coast of Hawaii to analyze the mixing occurring due to tides over steep topography. Measurements were taken in what were called the nearfield and farfield, the former being directly over Kaena Ridge, an ocean ridge between Oahu and Kauai, and the latter 430 km from the ridge in the South-west direction. The data collected during the experiment have been the basis of research for a growing group of researchers, including, in part, for this author. Details on this study can be found in the Methods section.

Rainville et al. [23] outlines the use of HOME data for calculating the energy flux due to internal tides as a function of frequency and depth. A portion of the data used for this thesis is the same as was gathered in the research described in this paper and its gathering is described in more detail in the Methods section of this thesis. Here, temperature, density and conductivity profiles
were developed using Conductivity, Temperature and Density or CTD devices at depths of 0-400 and 400-800 meters. The profiles were obtained over approximately 6 weeks time, again using FLIP as in the study mentioned previously. It was found that the internal wave energy fluxes of significance are found at frequencies associated with the diurnal and semi-diurnal tides. Additionally, energy fluxes in the diurnal frequency band account for 15% to 20% of the semidiurnal energy flux.

Another method of data harvesting is moored instrumentation. In a publication by M. H. Alford [24], a moored profiler array gathered temperature, salinity and horizontal velocities at 6 locations spread from 25 – 37°N. This instrumentation allowed the observers to document the long-range propagation of the internal tide northward emanating from the Hawaiian Ridge. The objective of the study was to examine the long-range propagation of the semidiurnal internal tide in a northward direction from the location of data harvesting and any potential for parametric subharmonic instability or PSI, a resonant triad interaction. PSI is where energy is exchanged from low wavenumber, high frequency waves to high wavenumber, low frequency waves. The results indicated PSI generation at a critical latitude of 28.8°N but that it did not act to deplete significant energy from the internal tide elsewhere. Gathering data and using it in this fashion is a method also used within this research to look at wave-wave interactions.

**Various Interactions**

Propagating internal waves encounter various phenomena in their lifetime. A study conducted by Godoy, et al [25] details the outcome of an interaction between a small internal gravity wave generated by an oscillating cylinder and a pancake dipole vortex created by a gate of variable height in front of a rotating flap. It was discovered that the small internal gravity waves in some cases were shifted to a turning point, dictated by the configuration of the interaction where the internal gravity wave frequency tended closer to the natural frequency of the stratified fluid. Another possible outcome showed critical layers in which the frequency of the internal gravity wave tended to zero. Overturning and breaking of the small wave was possible although the nature of the experiment would not predict either outcome. Billant, et al [26] investigated the instability associated with a vertical columnar pair in a stratified fluid. It was found that in a strongly-stratified fluid, two columnar vorticies develop zigzag instabilities antisymmetrically along the separation line.
between the two vorticities, bending the vortex pair as a whole and developing the potential for internal waves.

Another interaction type involves a small gravity wave approaching a weakly stratified mixed region. Brown [27] investigated the transmission of internal gravity waves through vertically varying background shear and a stratified fluid with changing vertical gradients. Critical levels were observed in a portion of the simulations. Gregory, et al [28] noted changes to the properties of the wave after an interaction with a weakly stratified mixed region comparing experimental and numerical results. Their results may help to predict the evolution of internal wave sources in the atmosphere.

Blackhurst [10] examines the interaction between a small-scale internal gravity wave with a translating vortex dipole. While critical levels were seen, it was found that the initial conditions of the gravity wave affected how the gravity wave parameters were changed.

As mentioned before, this research is geared to study wave-wave interactions, specifically internal gravity wave interactions with time-dependent background flows. Although this research focuses on small wave interactions with time-dependent backgrounds, understanding the interactions with time-independent backgrounds is included for comparison. Winters et al. [13] emphasized the importance of their study noting that any flows involved in a critical level show significant energy transfer between the waves and the background and that these critical levels are locations of potential energy transfer to turbulence or the wave itself. Using a 2-D non-linear model, their objective was to explore any nonlinear behavior present near the critical level forced through prescribed packets and shear flow. They found that for small amplitude internal waves, their energy was almost completely absorbed at the critical level and an acceleration was seen at the critical level. Garrett [22] describes general internal processes using critical layers saying that a typical phase speed for the internal wave spectrum is about 2 m/s at the lowest mode with a decrease in speed with an increase in wavenumber. Therefore, the waves with the highest vertical wavenumber can be assumed to encounter critical levels and exchange energy or break because they are so slow. Javam, et al [29] used the full unsteady Navier-Stokes equations under the SIMPLE scheme on a non-staggered grid to analyze the interaction of two wave beams.
2.1.3 Small Wave Steady Shear Interactions

Wave-wave interactions involving a small gravity wave with a time-dependent background have been detailed by many researchers due to their similarities to actual oceanic phenomena. The time-dependent and time-independent interactions differ in part in that critical levels are not found in the time-dependent cases and the small gravity wave exhibits oscillating changes to its frequency and wavenumber as it is refracted in the background. In 1986 Broutman presented his research on small wave interactions with a single progressive inertial wave modeled first as infinitely sinusoidal and subsequently as a time-dependent packet [30]. It was found that the properties of the small wave are changed through the interaction leaving it to exit the background with different properties instead of being absorbed by the background. Understanding the change in frequency and vertical group speed of a small wave from before to after an interaction can suggest an energy exchange during the interaction or possible dissipation present. Vanderhoff, et al [31] employed a 2-D nonlinear solver along with linear theory ray tracing to expand the work of Broutman [32] and Broutman, et al [33] in the consideration of a statistically significant number of single-packet interactions at greater detail all while in the search for clarification of effects of refraction on internal wave amplitudes and spectra. The current work is guided by a similar approach, although here numerical simulations are juxtaposed with observational findings.

Rediscussed within Vanderhoff’s work are the Broutman-coined encounter types wherein the approaching group velocity of the small wave, $C_g$, compared to the inertia wave phase speed, $C$, categorizes any interaction of this type as one of three encounters. If $C_g/C >> 1$, a first-kind encounter is expected. Here the short wave quickly approaches the slowly propagating phases of the inertial wave. The angle of propagation of the small wave is relatively steep here which directly relates to the group velocity. Within these types of encounters, it may be noted that the expected energy after the interaction is about equal to or may be less than the energy of the short wave before the interaction. The short wave leaves at approximately the same group velocity as it came in with. Figure 2.1 shows a first kind encounter in plot form. Each line represents a ray, following the energy of the small wave propagating into a background shear given by the filled-in ellipses. These ellipses are regions of strong refraction, caustics, where the vertical group velocity of the short wave is equal to the phase speed of the inertial wave. The horizontal is non-dimensionalized time and the vertical is non-dimensionalized depth based on initial wavelength.
Figure 2.1: An example of a first kind encounter generated using ray theory. The solid individual lines together represent a packet of energy from a small wave propagating into a background shear given by the filled-in ellipses. These ellipses are regions of strong refraction, caustics, where the vertical group velocity of the short wave is equal to the phase speed of the inertial wave. The horizontal is non-dimensionalized time and the vertical is non-dimensionalized depth based on initial wavelength. From Vanderhoff [31]

In cases where \( C_g/C \sim 1 \), a second kind encounter is present. Second kind encounters have similar results as in first kind encounters where the energy of the exiting wave is about the same as what is started with and the approach velocity is similar to the exit velocity. Figure 2.2 shows a second kind encounter in plot form.

Figure 2.2: An example of a second kind encounter generated using ray theory. The solid individual lines together represent a packet of energy from a small wave propagating into a background shear given by the filled-in ellipses. The horizontal is non-dimensionalized time and the vertical is non-dimensionalized depth based on initial wavelength. From Vanderhoff [31]

Lastly, where \( C_g/C << 1 \), a third kind encounter would be found. It is in these encounter types that the small wave starts out with a relatively low energy but leaves with more due to an
overall increase in frequency of the waves. Within the interaction, energy from the background is given to the small wave, increasing its exit velocity. Figure 2.3 shows a third kind encounter in plot form. These are important to this work because each encounter type may help determine the transfer of energy present in the ocean using the observational data given.

Figure 2.3: An example of a third kind encounter generated using ray theory. The solid individual lines together represent a packet of energy from a small wave propagating into a background shear given by the filled-in ellipses. The horizontal is non-dimensionalized time and the vertical is non-dimensionalized depth based on initial wavelength. From Vanderhoff [31]

In a later publication, Vanderhoff, et al. [34] examined the refraction of a spectrum of short internal waves, or waves with a relatively short horizontal wavelength, due to the propagation through an inertia wave packet. They found that a spectrum of trapped short waves is affected more significantly by the interaction with a short, rapidly varying inertia wave packet than by the interaction with an inertia wave packet of opposite characteristics. This means that the short wave spectrum broadens quicker because the inertial wave is more spatially confined, pointing out how time-dependent flows can alter a wave’s parameters if the two interact with each other.

2.1.4 Wave Steepening and Breaking

The evolution of an internal wave is governed by the phenomena it encounters during its lifespan and its current characteristics as it approaches each interaction. In some cases, a propagating internal wave will steepen and break, dissipating the energy of the wave. This occurs through the interactions with other phenomena. For instance, an interaction with another gravity wave may cause the amplitude of an internal gravity wave to dramatically increase. With that increase in
amplitude, the wave may become unstable or susceptible to overturning and breaking. However, upon exiting the interaction to then again interact with a background shear, the properties of the gravity wave may be altered, possibly returning the amplitude of the wave to a stable magnitude. If, in any of the interactions the internal wave may experience, the steepness of the wave reaches a critical magnitude, the wave will break. The steepness is a combination of the amplitude of the wave and the vertical wavenumber. Liu, et al [7] illustrates this concept graphically. A portion of this illustration from his work is displayed in Figure 1.1.

Latorre [35] investigated the steepening mechanisms associated with the interaction of two internal waves wherein the energy of each wave propagated in opposite directions. One result from the study gave an instability chart based on the Richardson’s number displaying the various parameters associated with an increased probability of wave breaking.

2.1.5 Dissipation

A few instances may give rise to dissipation within a wave. One instance is simply in propagation. As an internal wave propagates, it can lose energy, albeit relatively small amounts. This can be seen as internal waves are generated on the sea-bottom topography and propagate upwards. As they propagate upwards, energy is dissipated in the propagation due to viscous effects. Another instance of energy dissipation is from the interaction of a wave with some other phenomenon. This could be in the form of an inertial shear, vortex, mixed region or other energetic. During and after the interaction, energy may be dissipated through breaking or less significantly, viscous dissipation.

A number of authors have focused on estimating energy dissipation within the ocean. General discussion is provided for those interested in an introduction, the basic mechanics and scales of oceanic turbulence in a work by Smyth [36]. St. Laurent examined energy dissipation to smaller scale motions of internal tides by employing internal wave theory [37]. Waite used numerical simulations to investigate stably stratified, vortically forced turbulence [38]. Gregg et al. [39] attempted to measure the viscous dissipation rate and parameterize it using larger-scale variables. Later, Garratt [40] further investigated the claimed first order link between internal waves and turbulence by Gregg. The rate of dissipation in the thermocline is studied by Yamazaki and Lueck [41]. They concluded that the dissipation rate estimates developed 23 years earlier by Gurvich et al. [42] were
valid only under specific circumstances and the method set forth by Gurvich was more accurate than the conventional method at the time. Phillips [43] suggests an estimate of dissipation applicable to cases where the energy and vertical wavenumber is known. This expression will be discussed in further detail in the Methods section.
CHAPTER 3. METHODS

3.1 Theory

3.1.1 Governing Equations

The governing equations used to characterize internal waves are the Navier-Stokes equations, Equations 3.1-3.3, in each coordinate direction as well as the continuity equation, Equation 3.4. Gravity is directed downwards.

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \tag{3.1}
\]

\[
\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \tag{3.2}
\]

\[
\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \tag{3.3}
\]

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \tag{3.4}
\]

A few terms are further defined, including some from the Buoyancy Frequency equation, Equation 1.1. Equation 3.5 shows the governing equation for pressure within a stratified fluid where \( \tilde{p} \) is the hydrostatic pressure calculation dependent on the specific weight of the fluid and the depth within the fluid and \( p' \) is the perturbation pressure.

\[
p = \tilde{p} + p' \tag{3.5}
\]
The specific weight of the fluid is the density multiplied by gravity, the density being dependent on a few factors. One is a reference density, \( \rho_0 \) taken at some point within the fluid, two is the average density \( \rho_{\text{ave}} \) which is a function of depth and \( \rho' \), the perturbation density. The equation for density in its full form is shown in Equation 3.6.

\[
\rho = \rho_0 + \rho_{\text{ave}} + \rho'
\]  

(3.6)

Using the Boussinesq approximation which assumes density perturbations \( \rho' \) are small relative to the reference density \( \rho_0 \), Equations 3.1-3.3 can be reduced. The Boussinesq approximation deems \( \rho' \) significant to the problem only if it is multiplied by gravity. Thus, with the Boussinesq approximation as well as assuming incompressible and inviscid conditions, the Navier-Stokes equations reduce to Equations B.2-3.10 after linearization and assuming \( u, v \) and \( w \) are small (see Appendix B for further derivation).

\[
\frac{\partial u'}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x}
\]  

(3.7)

\[
\frac{\partial v'}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial y}
\]  

(3.8)

\[
\frac{\partial w'}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial z} + \frac{\rho'}{\rho_0} g_z
\]  

(3.9)

\[
0 = \frac{\partial \rho'}{\partial t} + w' \frac{\partial \rho}{\partial z}
\]  

(3.10)

Together, the equations are known as the linear Boussinesq equations and they define velocities, perturbation density and pressure. The dispersion relation described in the introduction portion of this write up is derived from the linear Boussinesq equations. The derivation for this is shown in Appendix C.
### 3.1.2 Idealized Problem

As was stated in the Literature Review section of this publication, the scope of this research is investigating wave-wave interactions between a small internal gravity wave and a background inertial shear where the small wave approaches either from above or below the inertial wave. The background inertia wave can be thought of as bounded by an ideal Gaussian envelope in space as it inevitably will be spatially bounded realistically. The Gaussian envelope is described by Equation 3.11.

\[
U + iV = U_0 e^{-\left(\frac{z}{L}\right)^2} e^{i(Mz - ft)} \tag{3.11}
\]

In this equation, \(L\) is the envelope length-scale, \(U_0\) is the maximum horizontal velocity at the center, and \(z\) is the vertical location of the packet taken from the center of the envelope. The inertia wave frequency is the Coriolis frequency and the inertia wave group velocity is assumed to be zero in all calculations because the horizontal wavelengths are significantly larger than the vertical. This can be seen through the calculation of the group velocity. The group velocity is shown in Equation 3.12. Within the mid-frequency approximation, \(f << \omega << N\).

\[
C_{gz} = \frac{\delta \omega}{\delta M} \sim \frac{Nk}{M^2} \sim \frac{N\lambda_z}{\lambda_x} \tag{3.12}
\]

Within Equation 3.12, \(\lambda\) is the wavelength as indicated and other variables are as described earlier. From this equation, it can be seen with a large horizontal wavelength typical of inertial shears, the vertical group velocity tends to null.

The vertical phase speed is \(\omega/M\) and the vertical wavenumber is related to the vertical wavelength, \(\lambda_z\), by Equation 3.13.

\[
M = \frac{2\pi}{\lambda_z} \tag{3.13}
\]

To visualize this, a simplified representation is displayed in Figure 3.1. This simplified representation shows the common interaction type studied in this research. Here, \(z\) is positive upwards with the remaining \(x\) and \(y\) axes on the horizontal and into and out of the page respectively. A small internal gravity wave approaches a large inertia background wave from above or below (in Figure 3.1 from below) at a group velocity \(C_g\) upwards while its phases are downward. The group veloc-
ity of the inertia wave is assumed to be zero and its phases propagate in the positive z direction. Because the small wave phase speed $C$ is downward in $z$ and the inertia wave phases are upward in $z$, this interaction is considered an opposite direction interaction by the author. An interaction in which the phases of both the small wave and inertia shear are oriented in the same direction will be referred to as a same direction interaction. The small wave moves both horizontally and vertically, its wavenumbers $k$ and $m$ indicating the packet dimensions. The vertical wavenumber of the small wave is negative since the wave is propagating upwards. As the small wave propagates and eventually interacts with the background, its frequency is altered as it is refracted by internal wave phases and may be calculated by the dispersion relation, Equation 1.5, discussed in the introduction.

Figure 3.1: A simplified representation of a gravity wave and inertia wave interaction is shown. From Latorre (2012) [35]

3.1.3 2D Non-linear Model

A comparison is to be made between the observational results found in the HOME data with numerical simulations. Because the 2D non-linear model is not linear, the output is more similar to
the observational results than would be a linear model because it can capture any non-linearities. Also, the 2D non-linear model can reveal true amplitudes. These two abilities are important to ensure that the evaluation is a fair comparison between the observations and numerical output, although it must be kept in mind that since it is 2D, there is no breaking, turbulence or dissipation captured, as these are all 3D phenomenon. The model comes from the Navier-Stokes equations in their vorticity stream-function form or as Equation 3.14-3.16

\[ u' = -\frac{\partial \psi}{\partial z} \]  
(3.14)

\[ w' = \frac{\partial \psi}{\partial x} \]  
(3.15)

\[ q = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x} \]  
(3.16)

Here, \( u = (u', 0, w') \) because of the two components of velocity in the velocity field. In their vorticity stream-function form, these fully non-linear, inviscid Boussinesq equations are as follows (Equations 3.17-3.20):

\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = -q \]  
(3.17)

\[ \frac{\partial q}{\partial t} + J(\psi, q) - \frac{\partial \sigma}{\partial x} - f \frac{\partial v}{\partial z} = 0 \]  
(3.18)

\[ \frac{\partial v}{\partial t} + J(\psi, v) - fu = 0 \]  
(3.19)

\[ \frac{\partial \sigma}{\partial t} + J(\psi, \sigma) - N^2 w = 0 \]  
(3.20)

The variable \( q \) is the y-component of vorticity and \( J(\psi, q) \) is the Jacobian with respect to \((x, z)\). The scaled density perturbation is \( \sigma = \rho' / \rho_o \) noting \( g \) is gravitational constant and \( \rho' \) is the density perturbation. Each variable used is calculated assuming no dependence on \( y \). A fourth order Runge-Kutta scheme inside a time-stepping loop solves Equations 3.17-3.20, where periodic
boundary conditions in the x- and z-directions are established. The grid size is 512 points vertically with 16 in the horizontal. Horizontally, a limit was placed contingent on the size of the small wave. These specifications can be found in Vanderhoff, et al [34].

3.2 Observations

3.2.1 Wave Field

The observational data used for this research come from the Hawaiian Ocean Mixing Experiment. Measurements were taken in what were called the nearfield and farfield, the former being directly over Kaena Ridge, an ocean ridge between Oahu and Kauai, and the latter 430 km from the ridge in the South-west direction in an attempt to understand internal wave propagation across distances. Figure 3.2 illustrates the location of the nearfield site. Kaena Ridge provides bottom topography where a significant amount of barotropic tidal energy, or energy within the large-scale tides driven by the gravitational forces of the sun and moon, is found [44] resulting in continuous internal wave generation. As Rainville (2006) describes, the ridge is an isolated topographic feature, giving an environment free from foreign interference or phenomena affecting the generation, propagation and evolution of the internal tide [23]. The Floating Instrument Platform (FLIP) was used as a stationary base to gather data, minimizing the vertical fluctuations due to surface movements.

For a closer look at FLIP, refer to an online resource on its conception and operation [45]. A Deep-8 eight-beam Doppler sonar, an upward/downward-directed sonar, sitting at 450 m below the surface operated constantly for four and a half weeks with two rapidly profiling CTD’s (Conductivity Temperature and Density) surveying the depth from 0 to 800 meters down [46]. Measurements were taken with a step of 2 meters giving approximately 9400 profiles over the four and a half weeks and within the depths specified [47]. Each CTD employed was a Sea-Bird Electronics, Inc., SBE11 CTD with diving depth capability of 6800 meters. Please reference Figure 3.3.

The temperature and conductivity measurements obtained through the study are used to find a density with depth while the pressure recorded gives the depth of each measurement. These data allow for an observational analysis to define shear flows and vertical velocities or high frequency motions. Looking at the density measurements juxtaposed with time gives the calculation of a
vertical displacement which represents a vertical movement of fluid. That vertical displacement per step in $z$, defined to be positive upwards, gives a vertical velocity measurement. To find the shear, the derivative of the horizontal velocities given in the data set is taken with respect to the vertical, $z$, where the along ridge and cross ridge shear are given by using the along ridge and cross ridge velocities. Figure 3.4 shows a vertical displacement map over one week obtained from observational data. Waves may clearly be seen as the crests and troughs illustrated by the hot and cold colors oscillate through time. These oscillations have semi-diurnal frequencies as relatively clear hot and cold bands repeat twice a day. As the semi-diurnal tide flows over the Kaena Ridge, these internal waves are produced at the semi-diurnal frequency. These are not the only frequencies present though. A zoom-in on Figure 3.4, displayed in Figure 3.5, shows higher frequency motion present. This signifies there are many internal waves present at various frequencies and scales. The
horizontal axis is in yeardays, in other words, the day of the year with January 1 being day 1 and the vertical is depth measured in meters. While obtaining data, measuring began 80 meters below the ocean’s surface. The magnitude of the displacement is indicated on a colorbar with warmer colors representing the greatest magnitude.

Defining the high frequency wavefield within the water column further, the strain is plotted. Figures 3.6(a) and 3.6(b) show the strain within the wavefield for one week and through approximately 700 meters of the water column. These plots illustrate the presence of internal waves. As internal waves propagate through the wavefield, the resulting strain within the wavefield evidences the energy movement and corresponding phases. The elliptical regions highlight the downward and upward phases within the plots. The same time and space regions highlighted here are highlighted in the observational shear in Figure 3.7. It would be expected to see more prominent shear bands in those time and space locations where high frequency motions are present. This is illustrated through these ellipses as definitive high frequency motions are shown in the strain plot and correspond with strong shear, in the same direction, in the shear plot.
Figure 3.4: Vertical displacement is shown here from observational data. The horizontal displays time in yeardays and the vertical shows depth in meters. Displacement values are shown in millimeters. The outlined region is shown in Figure 3.5.

The observational data is used to produce shear plots within the water column surveyed. Initially, the shear within the wave field is produced. Figure 3.7 shows the shear within the wave field for three weeks and through the water column depth (approximately 700 meters). This plot is made assuming the horizontal velocity field is given by Equation 3.21

\[ U = u + vi \]  
(3.21)

where \( U \) is the complex-valued horizontal velocity used in the creation and analysis of the wave-field, \( u \) is the real-valued horizontal velocity in the North/South direction and \( v \) is the real-valued horizontal velocity in the East/West direction. Using \( U \) instead of either \( u \) or \( v \) or a combination
of the real values of both definitively defines the horizontal wave field such that when determining the direction of propagation (explained in Figure 3.8), an accurate representation of the horizontal velocity is used.

Any data lying in quadrants I or III of the resultant Fourier transformed data was considered upward phases (downward energy) and the remaining two were considered downward phases. Refer to Figure 3.8 for each quadrant’s characteristics. By definition, any negative wavenumber within the observational data corresponds with upward propagating energy within the wavefield.

Figure 3.5: A zoom-in of the outlined region in Figure 3.4 is shown here. The horizontal displays time in yeardays and the vertical shows depth in meters. Displacement values are shown in millimeters.
Figure 3.6: Observational strain as derived from observational data. The strain was filtered for upward and downward propagation of energy. Elliptical outlined regions highlight the downward or upward phases within the plots. The horizontal is time and measured in yeardays while the vertical is depth and measured in meters.
Figure 3.7: Observational shear as derived from observational data. The ellipses outline the same time and space region highlighted by the ellipses in Figures 3.6(a) through 3.6(b). The horizontal is time in yeardays while the vertical is depth in meters.

Figures 3.9(a) and 3.9(b) show the shear filtered for upward and downward energy propagation respectively. Looking closely at the shear, one may notice the prominent shear bands repeated approximately once a day with crests and troughs indicated by the extreme warmer and cooler colors. It is apparent that the shear energy in Figure 3.9(a) is moving downwards since the phases are propagating upwards. To visualize the phase direction or C, follow the deep red color, moving upward with time. Understanding the group velocity direction can be seen by selecting a prominent band of shear and observing it through time to find it’s direction. Follow a group of deep red and blue and notice it is moving downwards. The arrows on Figure 3.9(a) shows this concept. To get a feel for the size of these waves, the wavelength can be estimated by assuming the
Figure 3.8: Axes showing the quadrants used to determine upward and downward motion in the Fourier transformed data. Positive wavenumber M is upwards while positive frequency $\omega$ is to the right. Quadrants I and III contain all the data for waves with phases in the upward direction and energy in the downward direction.

bounds of either a crest or trough and reading across to the depth scale. Looking at Figure 3.9(a) around day 270 shows a red/blue crest/trough spanning approximately 100 meters from 450 to 550 meters. This represents one vertical wavelength. The occurrence of one cycle of the inertial wave bands shows that these are inertial waves as a frequency of one day is an inertial frequency.

Looking at the horizontal velocity, u, defining the low frequency wave field, Figures 3.10(a)-3.10(b) show the horizontal velocity filtered upward and downward. Within these plots, the tides can be viewed as long striations which extend longitudinally with time.

### 3.3 Energy Estimate

#### 3.3.1 Determining Above, Middle and Below Time/Space Regions

To characterize the energy transfer between a small wave and a background inertial wave, the steps described further were taken. Plots were generated both with a numerical model and observational data of a background inertia wave. Regions were selected above, within and below
Figure 3.9: Observational shear as derived from observational data. The shear was filtered for upward and downward propagating phases, Figures a and b respectively. The horizontal is time and measured in yeardays while the vertical is depth and measured in meters.
Figure 3.10: Horizontal velocity plots show the upward and downward filtered horizontal velocity $u$ in m/s. The horizontal is time and measured in yeardays while the vertical is depth and measured in meters.
an inertia wave in the plots in which a representative energy of the small wave would be found. Then, an average was taken of those energy values to yield the average energy within the selected regions. Using a simple difference evaluation, the percent difference in energy was found for all instances of an assumed interaction of a small wave with an inertia wave. This process is described further in this section and in the Results section.

To meet the objectives of this study, a method for determining locations within the shear where the type of interaction currently studied would be found was devised to give comparable results to what would be seen in numerical simulations. Assuming that the general behavior of small waves within any particular region in the observational data would be the same from day to day, time/space regions above, below and in the presence of the shear, referred to as time/space regions, were to be isolated to gather energy and dissipation estimates. The idea is that knowing the average energy in those three locations, above, below and middle, would allow the researcher to determine, depending on the direction of the waves involved, whether there was a gain or loss of energy or if there was no energy transfer at all from before to after the interaction. The size of the time/space regions temporally was one day in time for the one day estimates and 17 days in time for the average estimates used to normalize the one day estimates. Spatially, the time/space regions extended for 50 meters in the above, middle and below locations. The assignment of the time/space regions within the numerical plots was made assuming spatial bounds within the shear and perturbation density plots and is explained within Section 4.1. Mainly, the sizes of the time/space regions differ within each case but an average is taken to give an overall average value.

Energy was calculated using the vertical velocity magnitudes within each time/space region. It was assumed that the vertical velocity represented the high frequency wave energy well. For a small wave traveling upwards or in the positive z direction, the average energy found within a time/space region below a shear band would be considered the average energy within the small wave before the interaction. And likewise, the average energy found in a time/space region above the shear band would be considered the average energy after the interaction. Similarly, for a small wave propagating downwards, the average energy in a time/space region above and below a shear band would be considered the average energy before and after the interaction respectively.

The direction of the small wave was determined through a Fourier transform. The vertical velocity was first isolated within the observational data set giving the energy that would be found
in small waves. With the vertical velocity at hand, a Fourier transform took the data from the
time/space dimension to frequency/wavenumber space. While in this state, the data was filtered
for high frequencies assuming 6 cycles/day to be considered the cutoff and then into upward and
downward motions. The value of 6 cycles/day is generally accepted within the field.

In choosing space/time locations within the shear, a few conditions were to be met. First,
it should be remembered that both the upward and downward shear plots were used in conjunction
with each other. Therefore, when selecting an upward or downward band of shear to be the back-
ground for the small wave to propagate through, ideally, the magnitude of the shear propagating in
the opposite direction in the same time/space location would be minimal. If a shear band of either
an upward or downward direction were selected and the shear of the opposite direction was of as
great a magnitude in the same location, the interaction type could not be characterized as either
an interaction with an upward or downward shear. Next, in the same yearday period, a time/space
region was selected both above and below the shear and in a region where minimal shear would
be found. This would ensure that the supposed small waves present before and after an interaction
would not be influenced by the presence of any shear. In effect, three regions are selected, three
for the shear in one direction and three for the shear in the opposite direction, making a case or
an individual interaction of the type studied with both upward and downward small waves. Ten
cases were created for this research. Figures 3.11(a)-3.13(b) show the individual cases selected by
highlighting the regions within each case with a black box. These figures are included to reference
each time/space region chosen within the observational data. The cases are not in any order and
have been combined a few to a page to conserve space.

Since the data that is used in this research was obtained from one location in the vast ocean,
namely at the Kaena Ridge which sits off the North-west coast of O’ahu, Hawaii, it only represents
a view into the internal wave dynamics at that particular global position. However, since Kaena
Ridge mirrors other locations within the ocean, it is assumed that the results obtained here can be
applied to a wide array of internal wave interactions throughout the ocean.

3.3.2 Dissipation Estimate

As explained in the Literature Review section, the dissipation model used in this research
is determined by an estimation proposed by Phillips [43]. In using this estimate, it must be stated
that Phillips assumes $m >> 1^{-1}$ which is generally true for high frequency waves and where $m$ is the vertical wavenumber with $l$ as a horizontal wavenumber. With $\varepsilon$ as the dissipation estimate, the equation given for dissipation is shown in Equation 3.22

$$\varepsilon = 2\nu \int_0^\infty m^2 E(m) dm$$

Because the values for dissipation obtained are merely for comparison only, the constant in front of the integral is dropped as well as the assumed constant $\nu$. From there, to find the dissipation in any time/space region, the sum of the energy within that region multiplied by the square of the vertical wavenumber at which the energy is found is calculated. The dissipation estimates were made using the same time and space regions as outlined in the shear for the energy calculation or in Figures 3.11(a)-3.13(b). Phillips assumes the turbulent energy spectrum, $E(k)$, to represent the distribution of contributions to one half the average horizontal velocity squared with respect to wavenumber. These one day values are compared to 17 day averages to find the relative dissipation. This gives values for the change in dissipation relative to no interaction taking place.
Figure 3.11: Downward and upward energy shear plots respectively are shown. Each set of three vertical boxes outlines one particular case observed within the observational data. The number to the side indicates which case is which. A time/space region above is considered the box closest to the surface or a lower value on the vertical scale whereas a time/space region below is closer to the sea bottom or a high value on the vertical scale.
Figure 3.12: Downward and upward energy shear plots respectively are shown. Each set of three vertical boxes outlines one particular case observed within the observational data. The number to the side indicates which case is which. A time/space region above is considered the box closest to the surface or a lower value on the vertical scale whereas a time/space region below is closer to the sea bottom or a high value on the vertical scale.
Figure 3.13: Downward and upward energy shear plots respectively are shown. Each set of three vertical boxes outlines one particular case observed within the observational data. The number to the side indicates which case is which. A time/space region above is considered the box closest to the surface or a lower value on the vertical scale whereas a time/space region below is closer to the sea bottom or a high value on the vertical scale.
4.1 Numerical Results

Using an existing 2-D non-linear solver, first and third kind encounters (explained in Section 2.1.3) were simulated using parameters of observed conditions. The shear conditions included the shear scales taken from inertial wave estimates in the observations. Within the first and third kind encounters, the average energy before, during and after the simulated interaction between a small wave and inertia wave was calculated and then compared to observed conditions.

Finding the average energy before, during and after the interaction was a similar process in each situation. The calculation before the interaction was determined by assigning a region in time and space known to be outside of the inertial wave envelope which could easily be seen in a plot of the shear and where an accurate average of the small wave energy would be found (as dictated by the perturbation density plot- see Figure 4.1(b)). The spatial bounds of the time/space region within the shear wave bound by a visual inspection estimating the vertical bounds of the shear bands seen. The temporal bounds of the time/space region within the shear was determined by expanding the rectangular region only large enough to include all the effects of the small wave within the perturbation density plot (seen in Figure 4.1(b)). The spatial and temporal bounds of the time/space regions above and below the shear were found by expanding a rectangular region large enough to include all the visible effects of the small wave within the perturbation density plot but small enough to exclude any zero values surrounding the small wave (dark blue on Figure 4.1(b)). The total energy was divided by the number of sample points to give an average energy within that particular region. Figure 4.1(a) shows a plot of the background shear the small wave was to propagate through with the time/space regions that were used highlighted and Figure 4.1(b) shows a perturbation density plot wherein the time/space regions used are highlighted.

To find the average energy after the interaction, the same process was used but proved to be more difficult as the small wave, in some cases, partially exited the interaction with the background
Figure 4.1: The background shear (a) and perturbation density (b) of the numerical simulation for a first kind encounter. The white boxes indicate regions examined before, during and after the interaction. The arrow in Figure 4.1(a) indicates the region described in the second paragraph of this section. In this case, $C_{gz0}/C = 16$ and $Ti = 2\pi/\omega$ where $\omega$ is the frequency of the small wave.

at various times through the course of the simulation. However, in cases where necessary, the average energy at multiple regions was analyzed and used to determine an overall representative value of the average energy rather than simply finding the average energy of one of the exited waves. Due to the periodic boundary condition, averaging over all time is not accurate. An example of this process can be seen in Figure 4.2 where the region above the shear shows the small wave exiting at various moments in time. Lastly, the average energy during the interaction was found, the bounds of which were estimated through visual inspection, by calculating the energy of the small wave within the confines of the background and averaging over the number of points taken into account.

As percent differences were calculated for the various time/space regions within a case, the second time/space region, time-wise, was subtracted from the first and was subsequently divided by the average energy from the first time/space region. For example, when determining the average energy gain/loss from before to after the interaction in the time/space regions, the average energy in the time/space region after the interaction, approximately $6.0790e-008$ (m/s)$^2$, was subtracted from the average energy in the time/space region before the interaction, approximately
Figure 4.2: This figure shows the regions chosen to find the average energy below, in the middle of and above the shear band. The small wave exits the interaction at multiple points in time.

$7.8017 \times 10^{-7} \text{ (m/s)}^2$, and divided by the average energy in the time/space region before the interaction, approximately $7.8017 \times 10^{-7} \text{ (m/s)}^2$. Multiple encounters of the same kind were averaged to produce the results given next.

Within the first kind same direction encounters, the average small wave energy before the interaction compared to after decreased by approximately 50%. Breaking the interaction down further, it was found that the average energy during the interaction was 55% of the average energy before the interaction. Then, on average there was 26% less average energy after the interaction compared to during. However, within opposite direction interactions, there was 82% less energy after the interaction compared to before, with 60% less during the interaction compared to before and 80% less after the interaction compared to during. When analyzing these two scenarios, it can be seen that the overall average energy decrease was greater for the interaction in which the small wave energy was in the opposite direction as the energy of the background.
For third kind encounters, average energy differences varied from those of the first kind. When examining third kind same direction encounters, the average energy after the interaction decreased by close to 100% compared to before, with energy increasing many orders of magnitude during the interaction compared to before and decreasing by nearly 100% after the interaction compared to during. However, for opposite direction interactions, the overall average energy increased significantly, 201% on average. Therefore, in comparing the interactions of third kind encounters, the overall average energy of waves traveling in the same direction decreases whereas the overall average energy increased for interactions in which the waves traveled in opposite directions. This may be explained by the expected caustics during the interaction for opposite propagating wave types. As introduced earlier, waves here start propagating slowly but are refracted to high frequencies and propagate faster, often leaving the interaction at a higher velocity. This scenario is most likely the reason the small waves are expected to gain energy.

Now, looking at the dissipation for each encounter type, a comparison may be drawn between dissipation levels and overall average energy changes. As a general observation, the dissipation value from before to after the interaction decreased in each encounter type and independent of energy propagation direction of each wave involved. Breaking this analysis down into the various encounter types and propagation directions, it was found that for the first kind encounters in which both waves were propagating in the same direction, the average dissipation decreased by 41% across the interaction. When looking strictly at the average dissipation before, during and after the interaction, it was found that there was a 53% decrease from before to during and a nearly 100% gain from during to after the interaction. For first kind encounters wherein the waves involved were propagating in opposite directions, the average dissipation decreased by around 90%, a significantly greater percentage than found for the previously described scenario. From before to during the interaction, the dissipation in this case increased significantly, by as much as tens of thousands of percentages whereas from during to after the interaction, there was a 41% decrease in dissipation. So dissipation is greatest during the interaction and least after an interaction. A summary of these findings are found in Table 4.2.

For third kind encounters, the average dissipation estimate differences from before to after an interaction are as follows: for interactions involving waves propagating in the same direction, the average dissipation increased by 100% whereas in interactions with waves propagating in op-
Figure 4.3: shows the background shear and perturbation density of the numerical simulation for a third kind encounter. The white boxes indicate regions examined before, during and after the interaction.

In opposite directions, the average dissipation decreased by approximately 89%. Within the first type, the dissipation increased quite significantly from before to during the interaction but lost everything from during to after the interaction. However, in the second type, the dissipation levels were less consistent from before to during but saw a constant 95% decrease from during to after. It can be seen that when the waves are propagating the same direction, the overall average energy decreases while the average dissipation increases. For waves propagating in opposite directions, there is a significant energy gain while the dissipation decreases by nearly 100%. From before to during the interaction, approximately 60% of the original energy was lost on average.

These numerical results take into account multiple simulations of the same encounter type but with changes only occurring in the initial parameters. Therefore, the values gleaned from the simulations are a result of taking into account each run (two for every scenario) and not just a single case.
Table 4.1: shows the changes in energy within the small wave when it is traveling in the same direction as the background as well as the opposite direction for numerical simulations. There should be cohesion between these results and those found when analyzing observational data.

<table>
<thead>
<tr>
<th>Encounter type and direction</th>
<th>Energy gain/loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Kind Same Direction</td>
<td>50% loss</td>
</tr>
<tr>
<td>First Kind Opposite Direction</td>
<td>82% loss</td>
</tr>
<tr>
<td>Third Kind Same Direction</td>
<td>100% loss</td>
</tr>
<tr>
<td>Third Kind Opposite Direction</td>
<td>201% gain</td>
</tr>
</tbody>
</table>

Table 4.2: The tabulated values for the dissipation for numerical simulations.

<table>
<thead>
<tr>
<th>Encounter type and direction</th>
<th>Dissipation % change</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Kind Same Direction</td>
<td>41% loss</td>
</tr>
<tr>
<td>First Kind Opposite Direction</td>
<td>90% loss</td>
</tr>
<tr>
<td>Third Kind Same Direction</td>
<td>100% gain</td>
</tr>
<tr>
<td>Third Kind Opposite Direction</td>
<td>89% loss</td>
</tr>
</tbody>
</table>

4.2 Observational Results

4.2.1 Energy Spectra

A parameter of interest within the water column is the horizontal velocity. Figure 4.4(a) shows the energy spectrum with frequency of the horizontal velocity $U$ squared. This value is squared because in creating the spectral plot, the shifted Fourier coefficients are multiplied by the conjugate of the shifted coefficients. Because it is closely linked with the shear, it appears similar to the frequency spectrum of the shear (Figure 4.5(a), explained next). Although here, the semi-diurnal frequency is greater than the coriolis and diurnal frequencies showing a strong dependence on the tidal motions and less on the Earth’s rotation. The tides are horizontal motions, but fairly uniform and do not generate a shear. Thus, a big peak is expected in the horizontal velocity at the diurnal (once a day) and semidiurnal (twice a day) tides. The energy spectrum relative to wavenumber in Figure 4.4(b) also displays a common shape seen in ocean spectra where more energy is concentrated at low wavenumbers. These two plots as well as each subsequent plot produced displaying frequency and wavenumber spectra were generated using several hundred points
in space and several thousand points in time yielding a statistically significant result. Additionally, the results corresponded well with those found in previous research, Garrett, et al [21] for example.

Figure 4.4: The first figure shows an energy spectrum of horizontal velocity showing peaks within the spectrum at the coriolis, diurnal, semidiurnal, 6 cycles per day and the natural frequency. The spectrum is marked with vertical lines at these frequencies starting left and moving to the right respectively as stated and the second figure shows an energy spectrum of horizontal velocity with respect to wavenumber.
An energy spectrum of the shear can be seen to display the inherent frequencies within the internal waves present. Looking at Figure 4.5(a), one can see the coriolis, diurnal, semidiurnal and high frequencies. These (with the exception of the high frequencies) are evidenced by the peaks seen in the spectrum. For instance, if one was to follow the spectrum until the coriolis frequency (approximately 0.7 cycles per day), a peak is seen. The diurnal frequency (1 cycle per day) is also represented by a peak close to the coriolis frequency. Then from there, the semi-diurnal frequency can be seen (2 cycles per day) and harmonics of all previously mentioned frequencies. High frequency motions, defined to be about 6 to 10 cycles per day or larger, fall off at a prototypical slope seen commonly in the ocean. The slope of the trend downward is -3 on a log/log plot as described by Garrett, et al [21]. At frequencies greater than N, the slope is slightly different. Here, internal waves are not generated. Within the shear spectra, peaks are not as large at the tidal frequency because the shear is a result of uniform flow.

Looking next at an energy spectrum with respect to vertical wavenumber, the wavenumbers containing the greatest energy can be seen. Figure 4.5(b) shows the energy spectrum with respect to wavenumber for the shear. The greatest energy concentration is found at the lowest wavenumbers or the largest wavelengths which is an expected result. As discussed earlier, the shear is generated by the rotation of the Earth and the strong consistent winds at the surface, two movements on a large scale. Having the greatest energy at the largest wavelengths thus is expected. The line represents the corresponding wavenumber for a wavelength of 100 meters, the approximated wavelength of the inertia wave from the description of Figure 3.9(a).

While it is important to observe the energy spectra of the low frequency motions (i.e, the shear), the high frequency motion energy spectra is also important to uncover the phenomena within the wave field. Figure 4.6(a) shows the energy spectrum of the high frequency vertical velocity. It is noted that the spectrum begins at the higher frequencies verifying that it is high frequency and tapers off consistently similar to Figure 4.5(a). Few prominent peaks are seen as the high frequency motion energy is typically evenly distributed across the higher frequencies [21].

Taking a look at the energy spectrum of the vertical velocity in Figure 4.6(b) with wavenumber, the viewer may see a resemblance of this figure with Figure 4.4(b) with a few exceptions. The magnitude of energy associated with the vertical velocity is much greater than the shear. The common slope of -3 as discussed by Garrett, et al [21] is seen. Also, the energy spectrum of the vertical
Figure 4.5: The first figure shows an energy spectrum of shear showing peaks within the spectrum at the coriolis (f), diurnal (D), semidiurnal (M2), 6 cycles per day and the natural frequency (N). The spectrum is marked with vertical lines at these frequencies starting left and moving to the right respectively as stated and the second figure shows an energy spectrum of the shear with respect to vertical wavenumber. The equivalent wavenumber corresponding with a wavelength of 100 meters (the estimated vertical wavelength of the observational shear) is marked with a vertical line.

velocity shows the greatest energy at the lowest wavenumbers but a steady decline in energy with increasing wavenumber as opposed to the energy spectrum of the shear where the energy is nearly level with increasing wavenumber after which a significant decline is seen. There is a near-constant decline in energy with increasing wavenumber while the greatest energy is at the least values of

53
Figure 4.6: The first figure shows an energy spectrum of vertical velocity with respect to frequency. 6 cycles per day and the natural frequency is marked with vertical lines starting from the left and the second figure shows an energy spectrum of vertical velocity with respect to wavenumber.

With both previous frequency spectrums of velocity, there is a consistent decline in energy with increasing frequency.
4.2.2 Energy Analysis

Using the equations derived in the previous section, an estimation of the representative encounter types was found for the observations from HOME. Using the frequency and wavenumber values for each point in the data set, an equivalent horizontal wavenumber, \( k \), was found through the dispersion relation, Equation 1.4. Then, with values for \( k \) and \( m \) at each point, the corresponding group velocity was found. A series of ranges of group velocities, or bins, was set up and each group velocity value was added to the bin of its corresponding group velocity. The group velocities were all made positive. When compared to the phase speed of the inertial wave, results are shown in Figure 4.7. The results from this analysis showed a strong majority of encounter types within the observed data set to be first kind. Of all the interactions within the wave field, over 99% were found to be first kind, less than a fraction of a percent were second kind and even less third kind encounters resulted. As stated before, first kind encounters involved minimal net energy exchange from before to after an interaction whereas third kind encounters see a significant increase in energy from before to after the interaction. Although first kind encounters tend to dominate the spectrum, a further analysis of Figure 4.7 highlights the relatively high energy found within second and third kind encounters compared to first kind.

A closer look at the group velocity plot shows the majority of points lie at values greater than the phase speed of the inertia wave, approximately 140 m/day. There are a few points that lie at group speeds below the phase speed of the inertia wave and at relatively high energy. This may be important so the spectra of particular regions will be decomposed if the results do not match expectations.

The observational data was filtered and examined to isolate instances of significant upward and downward shear, revealing regions similar to those in the numerical analysis. Those regions were examined for similar trends in energy loss or gain. As the upward and downward propagating background shear plots were used, regions of strong upward or downward shear were found in which either the downward or upward shear plot respectively showed little activity in the same region of time and space as covered in the Methods section. Then, using those regions assuming an interaction with a small internal wave, the energy differences were found between before, during and after the assumed interaction. A table of the results found are included in Table 4.3.
Figure 4.7: The group velocity distribution of high frequency waves for the water column data over one week. The line shown represents the approximate phase speed of the inertial wave.

To illustrate this more clearly, take for instance Figure 4.8. Here there are two shear plots of the same time and space, the left one filtered for downward energy and the right for upward energy as labeled. All the time/space regions outlined with a box show relatively low magnitude shear except the middle time/space region in the upward energy shear plot. This particular time/space region with higher magnitude energy is where the assumed interaction took place. With a relatively low magnitude shear in the same time/space region in the downward energy plot, it was known that this would be an interaction with a upward propagating inertial wave. Now, because this case focused on a upward propagating inertial wave, two possible interactions are analyzed. The first
Figure 4.8: A plot showing the second case time/space regions selected within the observational shear with the left being shear with energy downward and the right is shear with energy upward. It may be noticed that all the time/space regions shown show relatively low magnitude shear except the middle time/space region in the upward energy shear plot.

is an interaction with a small wave from below propagating upwards in the same direction as the shear, or a same direction interaction as described before. The second is an interaction with a small wave from above propagating downwards in the opposite direction of the shear, or an opposite direction interaction. This was accomplished as the high frequency energy commonly associated with small waves was filtered for upward and downward propagating energy and estimated within each of the regions, as was accomplished in the numerical simulations. When evaluated separately, the opportunity to analyze these two interaction types was made possible. After an analysis of the data, it was found that within this case, the small wave lost 50% of its energy after propagating through the inertial shear in the same direction whereas when propagating in the opposite direction, the small wave gained 45% more energy through the interaction with the shear. Please reference Table 4.3 for the overall results of this case and all other cases evaluated over one day.
Figure 4.9: A sequence of plots showing the spectral energy for downward high frequency waves within the time/space regions selected for Case 2. All show general peaks near the 7-10 cycles/day frequency with tapering off with increasing frequency.

Looking specifically at a single case, Case 2, a spectral energy plot may be made for each time space region outlined. Figures 4.9(a)-4.9(c) show the spectral energy plots for Case 2 starting with the spectral energy plot for the energy before the interaction in Figure 4.9(a), the spectral energy plot for the energy during the interaction in Figure 4.9(b) to the spectral energy plot for the energy after the interaction in Figure 4.9(c). It may be noted that the plots show a spike in energy near the 7-10 cycles/day frequency range with a general taper downward with increasing frequency, a trend corresponding closely with the findings of Garrett, et al [21]. With higher frequencies generally being of first kind and lower of third kind, a gain in energy would be expected with this
interaction as the lower frequencies show larger peaks in energy. If tracking the lower frequency peak from before to after, or the first plot to the last plot, one may see a general increase in energy or broadening of the low frequency spectrum. Temporarily, there is a gain from before to during the interaction in the high frequencies but it is lost moving to after the interaction. Further results will show there is a 45% gain in energy in the small wave in this case.

Figure 4.10: A sequence of plots showing the spectral energy for upward high frequency waves within the time/space regions selected for Case 2. All show general peaks near the 7-10 cycles/day frequency with tapering off with increasing frequency.

Similar to the discussion for Figures 4.9(a)-4.9(c), Figures 4.10(a)-4.10(c) show spectral energy plots for Case 2 time/space regions but for upward energy small waves. As before, the first
Table 4.3: shows the changes in energy within the small wave when it is traveling in the same direction as the background as well as the opposite direction within the observational data.

<table>
<thead>
<tr>
<th>Case</th>
<th>Shear Direction</th>
<th>Same Direction</th>
<th>Opposite Direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Downward energy</td>
<td>1% loss</td>
<td>24% gain</td>
</tr>
<tr>
<td>2</td>
<td>Upward energy</td>
<td>13% loss</td>
<td>17% gain</td>
</tr>
<tr>
<td>3</td>
<td>Downward energy</td>
<td>31% loss</td>
<td>10% gain</td>
</tr>
<tr>
<td>4</td>
<td>Upward energy</td>
<td>47% gain</td>
<td>8% gain</td>
</tr>
<tr>
<td>5</td>
<td>Downward energy</td>
<td>36% loss</td>
<td>75% gain</td>
</tr>
<tr>
<td>6</td>
<td>Upward energy</td>
<td>13% gain</td>
<td>0%</td>
</tr>
<tr>
<td>7</td>
<td>Upward energy</td>
<td>30% loss</td>
<td>38% gain</td>
</tr>
<tr>
<td>8</td>
<td>Downward energy</td>
<td>49% gain</td>
<td>35% loss</td>
</tr>
<tr>
<td>9</td>
<td>Downward energy</td>
<td>38% loss</td>
<td>55% gain</td>
</tr>
<tr>
<td>10</td>
<td>Upward energy</td>
<td>35% loss</td>
<td>148% gain</td>
</tr>
</tbody>
</table>

of the figures, Figure 4.10(a) shows the spectral energy before the interaction while Figure 4.10(c) shows the spectral energy after the interaction has taken place. Here, each plot shows a spike in energy near the 7-10 cycles/day frequency range with a general taper downward with increasing frequency. From before to during the interaction, the spectrum increases in energy magnitude in all frequencies but especially at the higher frequencies. During to after the interaction shows a decrease again in frequencies but here, mainly at the higher frequencies spelling a decrease in frequency across the interaction. Further results show a 50% loss in energy from the small wave.

The results in Table 4.3 show gains and losses within these cases, but only over one day. These values were obtained by subtracting the energy value from the initial time/space region from the energy value in the final time/space region and dividing by the energy value in the initial time/space region. Here, initial means either before or during while final means either during or after depending on the two values being calculated. Due to the complexity of phenomena throughout the ocean, the total energy distribution within the water column is not uniform. A plot showing total energy with depth is displayed in Figure 4.11. From this figure, it can be seen that the total energy within the water column increases fairly steadily with depth with the exception of a point of decline around 400 meters. With internal waves being mainly generated at the sea-bottom topography, one would expect to see greater magnitude energy near the bottom. Because the generation mechanisms inherently contain variability, the plot does not show consistency at the deepest depths but coalesces with decreasing depth in the water column. Not only does the
Table 4.4: shows the changes in energy within the small wave when it is traveling in the same direction as the background as well as the opposite direction within the observational data. Before to during and during to after the interaction values are shown. Energy differences are the results of using one day of data only.

<table>
<thead>
<tr>
<th>Case</th>
<th>Same Direction Before/During</th>
<th>Same Direction During/After</th>
<th>Opposite Direction Before/During</th>
<th>Opposite Direction During/After</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>61% gain</td>
<td>39% loss</td>
<td>98% gain</td>
<td>37% loss</td>
</tr>
<tr>
<td>2</td>
<td>20% loss</td>
<td>10% gain</td>
<td>72% gain</td>
<td>32% loss</td>
</tr>
<tr>
<td>3</td>
<td>25% gain</td>
<td>45% loss</td>
<td>64% gain</td>
<td>19% loss</td>
</tr>
<tr>
<td>4</td>
<td>62% gain</td>
<td>9% loss</td>
<td>110% gain</td>
<td>49% loss</td>
</tr>
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<td>5</td>
<td>0% —</td>
<td>36% loss</td>
<td>74% gain</td>
<td>0% —</td>
</tr>
<tr>
<td>6</td>
<td>41% gain</td>
<td>20% loss</td>
<td>24% gain</td>
<td>20% loss</td>
</tr>
<tr>
<td>7</td>
<td>7% loss</td>
<td>25% loss</td>
<td>39% gain</td>
<td>1% loss</td>
</tr>
<tr>
<td>8</td>
<td>8% gain</td>
<td>38% gain</td>
<td>41% loss</td>
<td>9% loss</td>
</tr>
<tr>
<td>9</td>
<td>31% loss</td>
<td>10% loss</td>
<td>6% loss</td>
<td>64% gain</td>
</tr>
<tr>
<td>10</td>
<td>17% loss</td>
<td>22% loss</td>
<td>92% gain</td>
<td>29% gain</td>
</tr>
</tbody>
</table>

energy vary with depth, but also from week to week. It can be seen that the energy with depth trend for one week is almost always greater than the energy with depth trend for two weeks. The variability between weeks is explained by the spring and neap tides. As the sun and moon have less gravitational effect on the tidal motions through the tidal cycle, the neap tide presents itself. During times of greater effect, the spring tide occurs. With greater gravitational influence on the tide, the ocean’s internal energy increases and the stronger tides generate more internal waves. However, with a decrease in depth in the water column, the energy looks to be about the same between the plots because the ocean fairly quickly dampens the excess energy input. The units on energy within this plot are (m/s)^2. Already having found the energy spectrum of vertical velocity, its value squared was convenient and provided a sufficiently accurate value for the total energy for high frequency waves. The actual total energy could be found by calculating the sum of the potential and kinetic energies within the water column but would be unneeded as a relative understanding was the goal.

The procedure for making Table 4.3 was used to generate Table 4.4. Within this table, the gains and losses from before to during and from during to after the interaction of a small wave with an inertial wave was considered just as they were in the numerical simulations. Table 4.4 only takes into account those energy differences for one day of data.
Figure 4.11: A plot showing total energy versus depth in the water column is shown. From this figure, it can be seen that the total energy within the water column increases fairly steadily with depth with the exception of a point of decline around 400 meters.

So, to decipher the general trend in energy within these same regions of space (referencing Figure 4.8 again, it can be seen the middle region was selected from 300-350 meters, the above box from 125-175 meters and the below box from 500-550 meters) over an extended period of time is necessary to accurately conclude how much energy is lost or gained through an interaction. Therefore, the same procedure described previously was taken to find the general trend in energy for each case but this time over a period of 17 days. Averaging over many possible interactions within the regions of interest and estimating the energy changes was independent of there being an inertial wave present. The results from this analysis are shown in Table 4.5.
In looking at these results, a few notes may be made. First, although the averages for the energies in the same and opposite directions are dissimilar, the averages may not represent the true phenomenon occurring before, during and after the interaction. This is because there is a myriad of other phenomena most likely occurring in that same time/space location not displayed in the shear results. However, it is assumed that whatever phenomena are occurring in those particular time/space locations also occur throughout the observational data. Although the average is only over a few weeks, it is the best currently available and therefore some regions will be inaccurate.

Next, conditions surrounding the shear may play a role in determining the energy differences for each case. For instance, there may be an upward and downward shear in a location where a downward only shear was desired but due to lack of other options, that location must be selected. With that being the case, the results may not truly show the effects of only an upward or only a downward shear on a small wave. Also, the shear in one case may not be as strong as in another, although both cases are treated as though the magnitude does not matter. If these are not taken into account, an accurate showing of the results may not be possible. These effects will be further discussed when the results are being compared.

Third, from case to case, the relative depths differ at which the before and after average energies were calculated, leaving room for error if in fact, more or less energy would be expected closer to or farther from the assumed interaction depth.

As a solution to some of these issues associated with the observational results, each case was analyzed for its intrinsic energy values over a relatively longer period to compare to the previously discussed results. This was done by expanding the time interval of the average energy from one day to about 2 and one half weeks giving an estimate of the expected energy at each of the particular depths examined each case. Then, with the expected energies over those 17 days, a comparison could be made with the results of having only taken into account one day. The average energy values over the 17 days can be seen in Table 4.5:

Comparing the two values, derived from one day and 17 days, a new value for the average energy is found. Take, for instance, the Up Energy value in the first case, 19%. From the analysis, it is known that this is a 19% loss. Comparing that to the value found using only one day and within the same case, 14% loss, a composite value of a relative 5% gain due to the inertial wave presence is uncovered. It is considered a gain because the average energy percentage loss over
Table 4.5: shows the average energies associated with each case but taken over 17 days to compare to the previously found results (only 1 day).

<table>
<thead>
<tr>
<th>Case</th>
<th>Shear Direction</th>
<th>Same Direction</th>
<th>Opposite Direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Downward energy</td>
<td>5% gain</td>
<td>100% gain</td>
</tr>
<tr>
<td>2</td>
<td>Upward energy</td>
<td>37% gain</td>
<td>28% loss</td>
</tr>
<tr>
<td>3</td>
<td>Downward energy</td>
<td>24% loss</td>
<td>65% gain</td>
</tr>
<tr>
<td>4</td>
<td>Upward energy</td>
<td>62% gain</td>
<td>24% loss</td>
</tr>
<tr>
<td>5</td>
<td>Downward energy</td>
<td>1% loss</td>
<td>67% gain</td>
</tr>
<tr>
<td>6</td>
<td>Upward energy</td>
<td>75% gain</td>
<td>39% loss</td>
</tr>
<tr>
<td>7</td>
<td>Upward energy</td>
<td>68% gain</td>
<td>33% loss</td>
</tr>
<tr>
<td>8</td>
<td>Downward energy</td>
<td>15% loss</td>
<td>15% loss</td>
</tr>
<tr>
<td>9</td>
<td>Downward energy</td>
<td>53% gain</td>
<td>22% gain</td>
</tr>
<tr>
<td>10</td>
<td>Upward energy</td>
<td>47% gain</td>
<td>49% loss</td>
</tr>
</tbody>
</table>

Table 4.6: shows the changes in energy within the small wave when it is traveling in the same direction as the background as well as the opposite direction within the observational data. Before to during and during to after the interaction values are shown. Values represent 17 day averages.

<table>
<thead>
<tr>
<th>Case</th>
<th>Same Direction Before/During</th>
<th>Same Direction During/After</th>
<th>Opposite Direction Before/During</th>
<th>Opposite Direction During/After</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>70% gain</td>
<td>38% loss</td>
<td>67% gain</td>
<td>20% gain</td>
</tr>
<tr>
<td>2</td>
<td>2% loss</td>
<td>40% gain</td>
<td>20% gain</td>
<td>40% loss</td>
</tr>
<tr>
<td>3</td>
<td>2% gain</td>
<td>25% loss</td>
<td>69% gain</td>
<td>3% loss</td>
</tr>
<tr>
<td>4</td>
<td>68% gain</td>
<td>3% loss</td>
<td>21% gain</td>
<td>37% loss</td>
</tr>
<tr>
<td>5</td>
<td>80% gain</td>
<td>45% loss</td>
<td>22% gain</td>
<td>37% gain</td>
</tr>
<tr>
<td>6</td>
<td>3% loss</td>
<td>81% gain</td>
<td>20% loss</td>
<td>23% loss</td>
</tr>
<tr>
<td>7</td>
<td>60% gain</td>
<td>5% gain</td>
<td>56% gain</td>
<td>57% loss</td>
</tr>
<tr>
<td>8</td>
<td>10% loss</td>
<td>6% loss</td>
<td>16% loss</td>
<td>0% –</td>
</tr>
<tr>
<td>9</td>
<td>17% gain</td>
<td>31% gain</td>
<td>3% gain</td>
<td>18% gain</td>
</tr>
<tr>
<td>10</td>
<td>60% gain</td>
<td>8% loss</td>
<td>19% gain</td>
<td>57% loss</td>
</tr>
</tbody>
</table>

the 17 days is greater than the average energy percentage loss over the one day, showing that the energy within the day examined did not lose as much as would be expected on average or in other words, a relative energy gain. Performing a similar analysis for all the other cases yields the set of values represented in Table 4.7. This is the relative energy exchange due to the small wave interaction with the inertial wave.
Table 4.7: shows the average energies associated with each case comparing the 1-day and 17-day analysis.

<table>
<thead>
<tr>
<th>Case</th>
<th>Same Direction Energy Difference</th>
<th>Opposite Direction Energy Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6% loss</td>
<td>76% loss</td>
</tr>
<tr>
<td>2</td>
<td>50% loss</td>
<td>45% gain</td>
</tr>
<tr>
<td>3</td>
<td>7% loss</td>
<td>55% loss</td>
</tr>
<tr>
<td>4</td>
<td>15% loss</td>
<td>32% gain</td>
</tr>
<tr>
<td>5</td>
<td>35% loss</td>
<td>8% gain</td>
</tr>
<tr>
<td>6</td>
<td>62% loss</td>
<td>39% gain</td>
</tr>
<tr>
<td>7</td>
<td>98% loss</td>
<td>71% gain</td>
</tr>
<tr>
<td>8</td>
<td>64% gain</td>
<td>20% loss</td>
</tr>
<tr>
<td>9</td>
<td>91% loss</td>
<td>33% gain</td>
</tr>
<tr>
<td>10</td>
<td>82% loss</td>
<td>197% gain</td>
</tr>
</tbody>
</table>

As before, the procedure for making Table 4.5 was used to generate Table 4.6. Within this table, the gains and losses from before to during and from during to after the interaction of a small wave with an inertial wave was considered just as they were in the numerical simulations. Table 4.4 takes into account those energy differences for 17 days of data.

The results from this analysis point to similarities within the numerical and observational findings. As a general trend, the observational cases in which the two waves involved traveled in opposite directions, an overall gain in energy was experienced from before to after the interaction reminiscent of third kind encounters being present. In cases where the two waves were traveling in the same direction, an overall energy loss was seen. Comparing that to the numerical results, for first kind encounters and third, waves propagating in the same direction, there was a decrease in energy from before to after the interaction. Thus we see the third kind gains more importance and dominates the flow here. The two compare well and help to define the ocean better.

Once again, the procedure for making Table 4.7 was used to generate Table 4.8. Within this table, the gains and losses from before to during and from during to after the interaction of a small wave with an inertial wave was considered just as they were in the numerical simulations. Table 4.8 takes into account those energy differences for both the 1 and 17 days of data.

The total energy of the shear for up and down propagation within each time/space region was calculated and plotted in Figure 4.12. It can be seen that for many of the cases, the energy magnitude of either the middle up or middle down scenarios showed the most energy overall.
Table 4.8: shows the changes in energy within the small wave when it is traveling in the same direction as the background as well as the opposite direction within the observational data. Before to during and during to after the interaction values are shown. Values represent the difference between the 1 and 17 day averages.

<table>
<thead>
<tr>
<th>Case</th>
<th>Same Direction Before/During</th>
<th>Same Direction During/After</th>
<th>Opposite Direction Before/During</th>
<th>Opposite Direction During/After</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9% loss</td>
<td>1% loss</td>
<td>31% gain</td>
<td>57% loss</td>
</tr>
<tr>
<td>2</td>
<td>18% loss</td>
<td>30% loss</td>
<td>52% gain</td>
<td>8% gain</td>
</tr>
<tr>
<td>3</td>
<td>23% gain</td>
<td>20% loss</td>
<td>35% loss</td>
<td>16% loss</td>
</tr>
<tr>
<td>4</td>
<td>6% loss</td>
<td>6% loss</td>
<td>89% gain</td>
<td>12% loss</td>
</tr>
<tr>
<td>5</td>
<td>80% loss</td>
<td>9% gain</td>
<td>52% gain</td>
<td>37% loss</td>
</tr>
<tr>
<td>6</td>
<td>44% loss</td>
<td>99% loss</td>
<td>44% gain</td>
<td>3% gain</td>
</tr>
<tr>
<td>7</td>
<td>67% loss</td>
<td>30% loss</td>
<td>17% loss</td>
<td>56% gain</td>
</tr>
<tr>
<td>8</td>
<td>18% gain</td>
<td>44% gain</td>
<td>25% loss</td>
<td>9% gain</td>
</tr>
<tr>
<td>9</td>
<td>48% loss</td>
<td>41% loss</td>
<td>9% loss</td>
<td>46% gain</td>
</tr>
<tr>
<td>10</td>
<td>77% loss</td>
<td>14% loss</td>
<td>73% gain</td>
<td>86% gain</td>
</tr>
</tbody>
</table>

between the upward or downward energy values. This shows that for these cases, the shear in the middle time/space region did contain the most energy out of the above, middle and below time/space regions, a result aimed for when selecting each middle time/space region. However, there are some instances where the middle up or middle down within a case was not the energy-dominant time/space region. For instance, Case 1 shows there to be the most energy in the middle up time/space region when the middle down time/space region should show the most energy to be consistent with the selection criteria (refer to Table 4.7, column 1 for supposed dominant shear). Case 9 also shows the below down time/space region to contain the most energy when the middle down time/space region was thought to have the most. These findings may have affected the outcome of the results.

Using an average of the same and opposite direction results in Table 4.7, energy losses and gains were calculated for each interaction type assuming the small wave propagated through an inertial shear of 100 meters wavelength. For small waves propagating in the same direction as the background shear, an average relative energy loss of 49% is expected. If the small wave propagates in an opposite direction of the shear, a relative gain of 60% is expected through the interaction.

Generally, internal wave generation is high near the bottom of the ocean and inertia waves are commonly generated near the surface and propagate downward. So, it would be expected that...
the small wave would be propagating upward through a downward propagating inertia wave to consistently take energy from the inertial wave or dissipate (described in the next section). This can be used to estimate enhanced small wave energy losses or gains when there is an inertia wave present.

Figure 4.12: A total energy plot of the energy of the shear within each of the time/space regions and filtered for upward and downward propagation.

Ideally, the same direction interaction results would all lose energy instead of having one gain energy and likewise, the opposite direction interaction results would all gain energy instead of including three interactions where energy is lost. While this is understandably the case with observational results where all possible interactions are not being captured but an effort has been made to discover the reason behind the deviation. Thus, two cases from both the same direction and opposite direction results were examined in more detail to determine the spectral behavior
of the energy before, during and after each interaction as first and third kind encounters behave differently and one may be more prominent.

Figure 4.13: A sequence of plots showing the spectral energy for Case 1 and Case 7 starting with before the interaction in Figure 4.13(a) and moving to after the interaction in Figure 4.13(c).

The first two cases selected were Case 1 and Case 7 within the opposite direction interactions. The results from Case 1 indicate the small wave lost 76% of its energy through the interaction with the background shear when propagating in the opposite direction. In Case 7 though, the small wave gained 71% of its energy through the interaction. With that in mind, Figures 4.13(a)-4.13(c) show the spectral energy comparisons for Case 1 and Case 7 opposite direction interactions where the blue dashed line is Case 1 and the solid green line is Case 7. Figure 4.13(a) shows Case 7 hav-
ing a greater energy magnitude at the lower frequencies or the frequencies associated with third kind encounters. The lower frequency peak is less wide and energy transfers to higher frequencies, indicative of a third kind encounter. This causes an increase in energy in the high frequency wave field.

Figure 4.14: A sequence of plots showing the spectral energy for Case 7 and Case 8 starting with before the interaction in Figure 4.14(a) and moving to after the interaction in Figure 4.14(c).

The last two cases selected were Case 7 and Case 8 within the same direction interactions. The results from Case 7 show the small wave lost 98% of its energy through the interaction with the background shear when propagating in the same direction. In Case 8, the small wave gained 64% of its original energy magnitude through the interaction. Figures 4.14(a)-4.14(c) show the spectral
energy comparisons for Case 7 and Case 8 same direction interactions. Figure 4.14(a) indicates the spectral energy of the small wave before the interaction follows one another closely for both cases. Continuing a similar discussion as commenced for Figures 4.9(a)-4.9(c), the shapes of the spectra change through time or from the first plot to the last. As the low frequency peak of Case 8 shifts from lower to higher frequencies through time, it increases in energy as indicated in the figures. However, increases in energy within the Case 7 spectra are seen mostly at higher frequencies, or frequencies close to those associated with turbulence.

It is shown that the inertia shear has some effect on the propagation of small waves as the two interact. The small wave tends to lose energy when approaching the inertia shear in the same direction it is propagating and gains energy (most of the time) when approaching from an opposite direction of propagation. Ocean topography paralleling that of the Kaena Ridge is assumed to produce similar internal waves and similar results to those found in this research. Therefore, the results of those interactions similar to interaction in the present research but in other regions of the ocean may be estimated.

For each vertical 100 meters of inertial wave present and at the stated conditions, it would be expected the high frequency wavefield would uniformly (while keeping its spectral shape) lose 49% of its initial energy for upward propagating energy and gain 60% of its initial energy due to the inertial wave in addition to common local interactions and other causes of energy transfer. Performing the analysis described within this research on interactions involving waves with ulterior parameters than those found herein may produce results different than those found here. Within same direction interactions, small waves with varying initial amplitudes will be stretched and strained differently within the shear through the interaction producing a variety of results. For opposite direction interactions, with an increase in the phase speed of the inertia wave through an increase or decrease in its frequency or wavenumber respectively, a greater number of third kind encounters would be present. In this scenario, more energy would be expected to be gained through the interaction.

4.2.3 Dissipation Analysis

Using the dissipation estimation as explained in Section 3.3.2, the average dissipation was found for each time/space region shown in Figures 3.11(a) through 3.13(b). To first understand
what results may be expected from the dissipation, the average dissipation for each 50 meters was calculated. A smaller step in depth would be used for the average but when calculating the dissipation the energy at each vertical wavenumber must be found and any step in depth smaller than 50 meters would lose the vertical wavenumber resolution necessary for a sufficiently accurate calculation. The results of this procedure can be seen in Figure 4.15. There is a general trend of increasing average dissipation from the deepest portion of the water column upward.

![Graph showing average dissipation in water column](image)

**Figure 4.15**: A plot showing the average dissipation with depth within the water column.

The barchart shown in Figure 4.16 displays the dissipation values for each time/space region used in the energy analysis described earlier but also that for a 17 day period and the same space regions as before. It can be seen that the dissipation is greatest during the interaction for 8 of the 10 cases within the 1 day results. The average dissipation before and after the interaction in 9
out of 10 cases follows the same trend seen in Figure 4.15 wherein the dissipation decreases with increasing depth in the water column for the 1 day results.

![Dissipation Chart](image)

Figure 4.16: A barchart showing the dissipation before, during and after the interaction within each case and for 1 day and 17 days.

The difference between the 1 and 17 day values are displayed in Figure 4.17. Any negative values here indicates there is less dissipation than the 17 day average value. The majority of the interactions show more dissipation than the 17 day average values for the middle (through the shear) region pointing to the greatest amount of dissipation while the interaction is actually taking place. This may be expected because of the strong refraction of the shear, or the general presence of the shear stretching and straining waves. This will take further study to separate. Recent work on these interactions [31], [34], [48] has shown enhanced wave amplitudes during interactions but breaking and dissipation has not yet been accurately characterized. Additionally, the results from
this dissipation analysis agree, generally, with those from the numerical simulation dissipation analysis in that the dissipation during the interaction is the greatest.

Figure 4.17: A barchart showing the dissipation before, during and after the interaction as a composite of the 1 and 17 day values. Any negative values indicates there is less dissipation than the 17 day average value.

Under the assumptions detailed in this research, the results of the dissipation study show that for the majority of interactions involving a small wave and background shear, there is more dissipation at the lower depths compared to the higher depths (higher in the water column). This may be due to more overall wave activity. Normalizing by the available energy may produce a more accurate assessment of the dissipation taking place. The dissipation is the greatest, though, during the interaction suggesting that the background shear influences the small wave to dissipate the most energy while an interaction takes place. Therefore, in similar conditions or interactions throughout the ocean, similar results may be seen to those detailed in this research.

The relative dissipation expected due to this particular interaction is 25% in addition to common local dissipative mechanisms.
CHAPTER 5. CONCLUSION

5.1 Summary

The breakdown of energy from large to small scales, small scales to large scales or even to dissipation within the ocean was the focus of this effort. Mainly the interaction between two internal waves, one being a background inertia wave, the other a small internal gravity wave, was studied as a likely source of energy transfer and dissipation leading to the breakdown. A portion of the data gathered during the Hawaiian Ocean Mixing Experiment gave an observational data set for observed conditions expected to be present in a wide variety of locations throughout the ocean. These data were broken down and categorized, assuming many thousands of interactions within the data set, into first, second or third kind encounters and analyzed as such and compared to expected (numerical) results.

Possible outcomes from the interaction of this type include an energy transfer between the inertia wave and the gravity wave, no energy transfer, and energy dissipation as a result of the small wave at least partially overturning and breaking. A 2-D non-linear model was used to simulate interactions of this type. The average energy of the small wave before, during and after the interaction was calculated to give an idea of the trend in energy transfer, if any, during these interactions. Multiple first and third kind encounters were analyzed. The dissipation within the small wave before, during and after the interaction was also found. These results were tabulated and analyzed for trends.

The numerical simulation results were then compared to the output from an analysis of observational data. The HOME data provided observational data for the comparison. The data was decomposed using Fourier transforms to uncover the inherent background shear and small wave scales present. Looking at the shear, specific time/space regions were selected wherein an interaction of the type described was expected. The energy within the assumed small wave (high frequency motions) before, during and after the interaction was found and tabulated. From these
energy values, it was determined whether energy was gained or lost from before to after the interaction, suggesting either a transfer of energy or no transfer. Additionally, the dissipation within the same time/space regions were found and tabulated.

5.1.1 Research Goals

- Define the high and low frequency wave fields to find the prominent wave encounter type within the observational data. For each encounter type, a specific result was expected, so in finding the prominent encounter type within the data, the results from other observational data related analyses could be compared to the expected results from the encounter type examination.

- Characterize the energy transfer for various types of encounters. Numerical simulations are run similar to those that would be found at observed conditions. Both first and third kind encounters are simulated initializing the small wave from above and below the shear band to fully cover the possible interactions seen in the ocean. Numerical results are compared to observational results.

- Analyze energy transfer of varying types in observations. Find the prominent shear bands within the observational data to analyze a supposed interaction between a small gravity wave and background inertia shear. The energy before, during and after the interaction is calculated to tell whether an energy transfer is present.

- Estimate dissipation due to an interaction. The dissipation within the same regions used in the previous bullet is found for observed conditions.

5.1.2 Conclusions

The objectives of the project have been met and a discussion will follow. The observational data set was filtered for high and low frequencies. Within the low frequency results, distinct shear bands could be seen repeating approximately once per day. This is an expected result as the low frequencies (shear) are developed from inertial motions with a frequency of twice a day. The shear filtered again was observed to contain upward and downward propagating waves, wherein distinct
shear bands could also be seen. After calculating the group speed for the wave-wave interactions within the data, it was found that a large majority was considered to be first kind encounters. These encounters inherently dictate no energy transfer to the small wave within the interaction. If any energy transfer is present, it suggests a loss to the background from the small wave. There were a small minority of third kind encounters however, the energy associated with these was statistically lower than those of first kind but in looking at the spectra, some third kind encounters cause gains in much of the data. This may explain some of the gains seen in the opposite direction results in the energy analysis.

The numerical results show that for first kind encounters, there was always a decrease in energy through the simulated interaction with a greater decrease for opposite direction interactions. For third kind encounters, same direction interactions experienced a large loss in energy but opposite direction encounters experienced a 200% gain in energy through the interaction. When third kind encounters are present, there is an increase in energy within the higher frequencies within the energy spectrum. The results from the observational analysis that was done are consistent with these from numerical simulations.

Looking at the observational energy analysis, 90% of the same direction interactions studied lost energy through the interaction (some more than others) while 70% of opposite direction interactions saw a gain in the small wave energy through the interaction.

Using an average of the same and opposite direction results in Table 4.7, energy losses and gains were calculated for each interaction type. For small waves propagating in the same direction as the background shear, an average energy loss of 49% is expected. If the small wave propagates in an opposite direction of the shear, a gain of 60% is expected through the interaction.

The dissipation within the interactions shows a majority of the energy being dissipated during the interaction between the small wave and the background shear. However, more energy was projected to be dissipated above the shear band compared to below the shear band. It is assumed that this is due to the dissipation estimation used where the energy dissipated decreases with increasing depth within the water column because energy generally increases with depth.
5.1.3 Future Work

Further steps to expand and use this work could be the following:

• Use a 3D non-linear method of determining dissipation. With a more realistic model to calculate dissipation, although the computational expense may increase, a more accurate model of the occurring phenomena may be represented.

• With the results found through this research, global circulation models may increase in accuracy by accounting for the occurring phenomena. Energy exchanges and existing dissipation otherwise not used in these models can now be incorporated.

• Are there other interactions occurring that are not accounted for? For instance, vortical or turbulent motions are not captured with the methods used within this research. Would there be different results with these included in the models? The results suggest the shear is the cause of energy exchanges but without finding all the phenomena present, other events may be taking place. And, clearly all energy exchange and dissipation are not captured by this analysis. Therefore, other phenomena must be taking place.
REFERENCES


79


Because internal waves only exist when their intrinsic frequency is lower than the natural frequency of the fluid they are in, it is important to be able to calculate the buoyancy frequency. The following is a derivation of the equation for buoyancy frequency. First, the reduced gravity term is introduced. This term is given by the Boussinesq approximation by Equation A.1.

\[ g' = g \frac{\rho_2 - \rho_1}{\rho_o} \tag{A.1} \]

\( \rho_1 \) and \( \rho_2 \) represent the densities at some initial position and a perturbed position (use Figure 1.2 as a reference if needed). \( \rho_o \) is a characteristic or reference density. Using the Boussinesq approximation, \( \rho_o \) could be either the initial density, \( \rho_1 \), or an average of \( \rho_1 \) and \( \rho_2 \) [8]. By convention, the natural frequency of a fluid is given by 'N'. Supposing an increase in density \( \nabla \rho \) through a depth \( H \), the characteristic frequency of motion is \( (g'/H)^{1/2} \) and is approximately the background density gradient \(-d\bar{\rho}/dz\). The term is negative because \( z \) is positive upwards giving a positive result for stably stratified fluids. Equation A.2 shows the characteristic frequency,

\[ N \sim \sqrt[2]{\frac{g'}{H}} \sim \sqrt{-\frac{g}{\rho_o} \frac{d\bar{\rho}}{dz}} \tag{A.2} \]

As predicted by Newton’s Laws, if a fluid volume is displaced vertically through \( \delta z \), Equation A.3 shows the resulting motion.

\[ \rho_0 \frac{d^2 \delta_z}{dt^2} = -\delta_p g \tag{A.3} \]

Rewriting the density difference in terms of \( \delta \rho \), Equation A.4

\[ \delta_p \sim -\frac{d\bar{\rho}}{dz} \delta_z \tag{A.4} \]
Combining Equations A.3 and A.4, the spring equation is given:

\[
\frac{d^2 \delta_c}{dt^2} + N^2 \delta_c = 0 \tag{A.5}
\]

which can be written as

\[
N^2 = -\frac{g}{\rho_0} \frac{d\rho}{dz} \tag{A.6}
\]
APPENDIX B. POLARIZATION RELATIONS

B.1 Polarization Relations

This appendix goes over the derivation of the polarization relations from the Boussinesq equations discussed in Section §3.1.1, the dispersion relation in Section §1.0.3 and continuity from Section §3.1.1.

\[
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \tag{B.1}
\]

\[
\frac{\partial u'}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x} \tag{B.2}
\]

\[
\frac{\partial v'}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial y} \tag{B.3}
\]

\[
\frac{\partial w'}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial z} + \frac{\rho'}{\rho_0} g_z \tag{B.4}
\]

\[
0 = \frac{\partial \rho'}{\partial t} + w \frac{\partial \rho}{\partial z} \tag{B.5}
\]

\[
w' = w_0 \cos(kx + ly + mz - \omega t) \tag{B.6}
\]

\[
0 = \frac{\partial \rho'}{\partial t} + w \frac{\partial \rho}{\partial z} \tag{B.7}
\]

\[
N^2 = -\frac{g}{\rho_0} \frac{\partial \rho}{\partial z} \tag{B.8}
\]
\[ -N^2 \frac{\rho_0}{g} = \frac{\partial \rho}{\partial z} \]  \hspace{1cm} (B.9)

\[ \frac{\partial \rho'}{\partial t} + w_0 N^2 \frac{\rho_0}{g} \cos(kx + ly + mz - \omega t) = 0 \]  \hspace{1cm} (B.10)

\[ \partial \rho' = w_0 N^2 \frac{\rho_0}{g} \cos(kx + ly + mz - \omega t) \partial t \]  \hspace{1cm} (B.11)

\[ \rho' = -\frac{w_0 N^2}{\omega} \frac{\rho_0}{g} \sin(kx + ly + mz - \omega t) \]  \hspace{1cm} (B.12)

\[ \frac{\partial w'}{\partial t} = -1 \frac{\partial \rho'}{\partial z} - \frac{\rho'}{\rho_0} g \]  \hspace{1cm} (B.13)

\[ \frac{\partial w}{\partial t} = -1 \frac{\partial \rho'}{\partial z} + w_0 N^2 \sin(kx + ly + mz - \omega t) \]  \hspace{1cm} (B.14)

\[ w_0 \omega \sin(kx + ly + mz - \omega t) = -1 \frac{\partial \rho'}{\rho_0} \partial z + \frac{w_0}{\omega} N^2 \sin(kx + ly + mz - \omega t) N^2 \]  \hspace{1cm} (B.15)

\[ \left( w_0 \omega - \frac{w_0}{\omega} N^2 \right) \sin(kx + ly + mz - \omega t) = -1 \frac{\partial \rho'}{\rho_0} \partial z \]  \hspace{1cm} (B.16)

\[ \partial \rho' = \rho_0 \left( w_0 \omega - \frac{w_0}{\omega} N^2 \right) \sin(kx + ly + mz - \omega t) \partial z \]  \hspace{1cm} (B.17)

\[ p' = \frac{\rho_0}{m} w_0 \left( \omega - \frac{N^2}{\omega} \right) \cos(kx + ly + mz - \omega t) \]  \hspace{1cm} (B.18)

\[ p' = \frac{\rho_0}{m} w \omega \left( 1 - \frac{N^2}{\omega} \right) \]  \hspace{1cm} (B.19)
\[ p' = \rho_0 \frac{w \omega}{m} \left( 1 - \frac{N^2}{N^2(k^2 + l^2)} \right) \]  
(B.20)

\[ p' = \rho_0 \frac{w \omega}{m} \left( 1 - \frac{k^2 + l^2 + m^2}{(k^2 + l^2)} \right) \]  
(B.21)

\[ p' = \frac{\rho_0}{m} w \omega \left( \frac{-m^2}{k^2 + l^2} \right) \]  
(B.22)

\[ p' = -\rho_0 w \omega \left( \frac{m}{k^2 + l^2} \right) \]  
(B.23)

\[ p' = -\rho_0 \omega \left( \frac{m}{k^2 + l^2} \right) w_0 \cos(kx + ly + mz - \omega t) \]  
(B.24)

\[ \frac{\partial u'}{\partial t} = \frac{-1}{\rho_0} \frac{\partial p'}{\partial x} \]  
(B.25)

\[ \frac{\partial u'}{\partial t} = \frac{-1}{\rho_0} \rho_0 \omega \left( \frac{m}{k^2 + l^2} \right) k \omega_0 \sin(kx + ly + mz - \omega t) \]  
(B.26)

\[ \partial u' = \frac{-1}{\rho_0} \rho_0 \omega \left( \frac{m}{k^2 + l^2} \right) k \omega_0 \sin(kx + ly + mz - \omega t) \partial t \]  
(B.27)

\[ u' = \left( \frac{-mk}{k^2 + l^2} \right) \omega_0 \cos(kx + ly + mz - \omega t) \]  
(B.28)

\[ \frac{\partial v'}{\partial t} = \frac{-1}{\rho_0} \frac{\partial p'}{\partial y} \]  
(B.29)

\[ \frac{\partial v'}{\partial t} = \frac{-1}{\rho_0} \rho_0 \omega \left( \frac{m}{k^2 + l^2} \right) l \omega_0 \sin(kx + ly + mz - \omega t) \]  
(B.30)

\[ \partial v' = \frac{-1}{\rho_0} \rho_0 \omega \left( \frac{m}{k^2 + l^2} \right) l \omega_0 \sin(kx + ly + mz - \omega t) \partial t \]  
(B.31)
\[ v' = \left( \frac{-ml}{k^2 + l^2} \right) \omega_0 \cos(kx + ly + mz - \omega t) \] (B.32)
APPENDIX C. DERIVING DISPERSION RELATION

\[
\frac{\partial}{\partial x}(x - \text{mom}) + \frac{\partial}{\partial y}(y - \text{mom}) \quad (C.1)
\]

\[
\frac{\partial}{\partial t} \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) = -\frac{1}{\rho_0} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) p' \quad (C.2)
\]

Through continuity, we get

\[
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = -\frac{\partial w'}{\partial z} \quad (C.3)
\]

\[
\frac{\partial}{\partial t} \left( -\frac{\partial w'}{\partial z} \right) = -\frac{1}{\rho_0} \nabla_H^2 p' \quad (C.4)
\]

\[
\nabla_H^2 \left( \frac{\partial}{\partial t} \left( \frac{\partial w'}{\partial t} \right) \right) = -\frac{1}{\rho_0} \frac{\partial^2}{\partial t \partial z} \left( \nabla_H^2 p' \right) - \frac{\partial}{\partial t} \nabla_H^2 \left( \frac{p'}{\rho_0} \right) \quad (C.5)
\]

Because

\[
\frac{\partial}{\partial t} \left( \nabla_H^2 \frac{\partial w'}{\partial t} \right) = -\frac{1}{\rho_0} \frac{\partial^2}{\partial t \partial z} \left( \nabla_H^2 p' \right) - \nabla_H^2 \left( N^2 w' \right) \quad (C.6)
\]

Because

\[
\frac{\partial}{\partial t} \left( \nabla_H^2 \frac{\partial w'}{\partial t} \right) = -\frac{\partial^2}{\partial t \partial z} \left( \frac{\partial^2 w'}{\partial t \partial z} \right) - \nabla_H^2 \left( N^2 w' \right) \quad (C.7)
\]

Because

\[
\frac{\partial}{\partial t} \left( \nabla^2 \frac{\partial w'}{\partial t} \right) + \nabla_H^2 \left( N^2 w' \right) = 0 \quad (C.9)
\]