2011-05-31

The Weak Cayley Table and the Relative Weak Cayley Table

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A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

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August 2011

Abstract

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In 1896, Frobenius began the study of character theory while factoring the group determinant. Later in 1963, Brauer pointed out that the relationship between characters and their groups was still not fully understood. He published a series of questions that he felt would be important to resolve. In response to these questions, Johnson, Mattarei, and Sehgal developed the idea of a weak Cayley table map between groups. The set of all weak Cayley table maps from one group to itself also has a group structure, which we will call the weak Cayley table group.

We will examine the weak Cayley table group of $AGL(1, p)$ and the dicyclic groups, find a normal subgroup of the weak Cayley table group for a special case with Camina pairs and Semi-Direct products with a normal Hall-π subgroup, and look at some nontrivial weak Cayley table elements for certain p-groups.

We also define a relative weak Cayley table and a relative weak Cayley table map. We will examine the relationship between relative weak Cayley table maps and weak Cayley table maps, automorphisms and anti-automorphisms, characters and spherical functions.

Keywords: Finite Group, Weak Cayley Table, Relative Weak Cayley Table, Camina Pair

Acknowledgments

I would like thank my advisor, Dr. Stephen Humphries, for his patient guidance throughout the research and writing process.

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Chapter 1. Background: Weak Cayley Tables

In 1896 Frobenius and Dedekind corresponded through a series of letters on the problem of factoring the group determinant [Cu, p.50-53]. Previously, characters had been defined for abelian groups, and it was during this time that Frobenius defined characters in a general sense. He chose to do so in such a way that each irreducible factor of the group determinant corresponded to an irreducible character of the group. As part of producing an algorithm to take a character to its corresponding irreducible factor, Frobenius defined k-characters recursively as follows [JS]:

Definition 1.1. Let χ be a character of a finite group G. Let the 1-character, $\chi^{(1)}$, be equal to χ. Then define the *k-character* $\chi^{(k)} : G^k \to \mathbb{C}$ to be the map

$$\begin{aligned} \chi^{(k)}(g_1, g_2, \ldots, g_k) \quad = \quad & \chi(g_1)\chi^{(k-1)}(g_2, \ldots, g_k) \\ & -\chi^{(k-1)}(g_1 g_2, \ldots, g_k) \\ & -\chi^{(k-1)}(g_2, g_1 g_3, \ldots, g_k) \\ & -\vdots \\ & -\chi^{(k-1)}(g_2, \ldots, g_1 g_k). \end{aligned}$$

In particular, the 2-character is defined to be

$$\chi^{(2)}(g_1, g_2) = \chi(g_1)\chi(g_2) - \chi(g_1 g_2).$$

This was the beginning of character and representation theory. Mathematicians began to implement these new ideas to prove powerful results about groups, such as Burnside's pq-theorem.

In 1963, Brauer wrote a paper where he proposed several open questions about the relationship between characters and their groups [Br]. Some of his questions were:

- In addition to the character table, what information is necessary to determine a finite group?

- Given a group G, how much information about the automorphism group $Aut(G)$ of a group can be obtained from the characters of G?

- Given a set of conjugacy classes that form a normal subgroup, is there enough information in the character table to determine if the normal subgroup is abelian?

In response to these questions, Johnson, Mattarei, and Sehgal published a paper in 2000 developing the concept of a weak Cayley table. They were interested in the question, "What properties of a group can be determined by the 1- and 2- characters which cannot be determined by the 1-characters alone?" [JMS]. In this paper, they define a weak Cayley table and proved that knowing the weak Cayley table of a group is equivalent to knowing the 1- and 2-characters of a group.

1.1 Weak Cayley Tables

A weak Cayley table is similar to a multiplication table for a group, only instead of the table containing the product of the two indices, the entries of a weak Cayley table contain the conjugacy class of their product. More specifically, given a finite group G of order n, order the elements of G, index the rows and columns of an $n \times n$ table with the ordered elements; then in the i^{th} row and j^{th} column enter the conjugacy class of ij in G. The resulting table is a *weak Cayley table* for the group. Any two weak Cayley tables for a group G differ only by an ordering of the rows and columns.

Example 1.2. As an example, consider S_3. The conjugacy classes of S_3 are

$$C_1 = \{1\}, C_2 = \{(12), (13), (23)\} \text{ and } C_3 = \{(123), (132)\}.$$

Then a weak Cayley table for S_3 is

	1	(12)	(23)	(13)	(123)	(132)
1	C_1	C_2	C_2	C_2	C_3	C_3
(12)	C_2	C_1	C_3	C_3	C_2	C_2
(23)	C_2	C_3	C_1	C_3	C_2	C_2
(13)	C_2	C_3	C_3	C_1	C_2	C_2
(123)	C_3	C_2	C_2	C_2	C_3	C_1
(132)	C_3	C_2	C_2	C_2	C_1	C_3

When Johnson, Mattarei, and Sehgal defined the weak Cayley table, they proved:

Theorem 1.3 (JMS, Proposition 2.4)**.** *If the irreducible 1- and 2-characters of a group are given, its weak Cayley table can be constructed. Conversely, if the weak Cayley table is given, the irreducible 1- and 2- characters can be calculated.*

Thus by examining weak Cayley tables, we can further understand the relationship between groups and their characters.

1.2 Weak Cayley Table Maps

Weak Cayley tables are not unique to a specific group. For example the two non-abelian non-isomorphic groups of order p^3, where p is an odd prime, have the same weak Cayley table [JMS]. The authors of [JMS] defined a weak Cayley table map to be a bijection between two groups that preserves the weak Cayley table structure. If G_1 and G_2 are two groups, then a *weak Cayley table map* $\phi : G_1 \to G_2$ is a bijection that satisfies two conditions:

(i) $\phi(g^{G_1}) = \phi(g)^{G_2}$

(ii) for every g and h in G_1, $\phi(gh) \sim \phi(g)\phi(h)$.

Where $\sim$ denotes the equivalence relation of conjugacy. We say G_1 and G_2 have *the same weak Cayley table* if there exists such a map.

Example 1.4. Let G_1 and G_2 be the non-isomorphic groups of order p^3 (p an odd prime) with the following presentations:

$$G_1 = \langle a, b, c : a^p = b^p = c^p = 1, b^a = bc \rangle$$

where $\langle c \rangle$ generates the center $Z(G_1)$, and

$$G_2 = \langle x, y, z : x^p = z, x^{p^2} = y^p = z^p = 1, x^y = x^{p+1} \rangle$$

where the center $Z(G_2)$ is $\langle z \rangle$. See [DF, p. 183].

One of the nice properties of G_1 and G_2 is that both groups form a Camina pair over their center.

Definition 1.5. A *Camina pair* is a group G with a normal subgroup H such that the conjugacy classes of G not intersecting H are unions of cosets of H.

For example in G_1 the conjugacy classes are

$$\{1\}, \quad \{c\}, \quad \{c^2\}, \quad \ldots, \quad \{c^{p-1}\},$$

along with classes of the form

$$a^i b^j \{1, c, c^2, \ldots, c^{p-1}\} \text{ for } 0 \leq i \leq p-1,\ 0 \leq j \leq p-1,$$

where i and j are not both equivalent to 0 mod p.

The conjugacy classes for G_2 are

$$\{1\}, \quad \{z\}, \quad \{z^2\}, \quad \ldots, \quad \{z^{p-1}\},$$

together with

$$x^i y^j \{1, z, z^2, \ldots, z^{p-1}\} \text{ for } 0 \leq i \leq p-1,\ 0 \leq j \leq p-1.$$

where i and j are not both equivalent to 0 mod p.

Let the map $\phi : G_1 \to G_2$ be defined by

$$\phi(1) = 1, \quad \phi(a^r) = y^r, \quad \phi(b^s) = x^s, \quad \phi(c^t) = z^{-t}, \quad \text{and} \quad \phi(a^r b^s c^t) = x^s y^r z^{rs-t}.$$

From the above description of the conjugacy classes it is easy to see that ϕ maps G_1 conjugacy classes to G_2 conjugacy classes, thus satisfying condition (i) of a weak Cayley table map.

To meet condition (ii), $\phi(gh)$ must be conjugate to $\phi(g)\phi(h)$ for all g and h in G_1. Let $g = a^r b^s c^t$ and let $h = a^i b^j c^k$ in G_1. Then

$$\begin{aligned} gh &= (a^r b^s c^t)(a^i b^j c^k) \\ &= (a^r b^s)(a^i b^j)c^{t+k} \\ &= a^{r+i} b^{s+j} c^{is+t+k}. \end{aligned}$$

First assume that either $r + i \neq p$ or $s + j \neq p$. Then

$$\begin{aligned} \phi(gh) &= \phi(a^{r+i} b^{s+j} c^{is+t+k}) \\ &= x^{s+j} y^{r+i} z^{(s+j)(r+i)-t-k}, \end{aligned}$$

and

$$\begin{aligned} \phi(g)\phi(h) &= \phi(a^r b^s c^t)\phi(a^i b^j c^k) \\ &= (x^s y^r z^{rs-t})(x^j y^i z^{ij-k}) \\ &= x^{s+j} y^{r+i} z^{rs-t+ij-k+rj}. \end{aligned}$$

Both $\phi(gh)$ and $\phi(g)\phi(h)$ are in the same coset of the center, and therefore must be conjugate to each other since $r + i \neq p$ or $s + j \neq p$.

Next assume that $r + i = p$ and $s + j = p$. Then since gh is a central element, in order for $\phi(gh) \sim \phi(g)\phi(h)$, we need for $\phi(gh) = \phi(g)\phi(h)$. Doing a similar computation, we get that

$$\begin{aligned} \phi(gh) &= \phi(c^{-rs+t+k}) \\ &= z^{rs-t-k}, \end{aligned}$$

and

$$\begin{aligned}
\phi(g)\phi(h) &= \phi(a^r b^s c^t)\phi(a^{-r}b^{-s}c^k) \\
&= (x^s y^r z^{rs-t})(x^{-s}y^{-r}z^{rs-k}). \\
&= x^s y^r x^{-s} y^{-r} z^{2rs-t-k} \\
&= x^{s-s} y^{r-r} z^{-rs} z^{2rs-t-k} \\
&= z^{rs-t-k}.
\end{aligned}$$

Which gives us $\phi(gh) = \phi(g)\phi(h)$, as required. Therefore ϕ is a weak Cayley table map.

In their paper, Johnson, Mattarei, and Sehgal also identified other convenient facts about weak Cayley table maps, several of which are summarized in the following theorem [JMS].

Theorem 1.6 (Johnson, Mattarei, and Sehgal). *Let $\phi : G_1 \to G_2$ be a weak Cayley table map. Then*

(i) $\phi(1_{G_1}) = 1_{G_2}$

(ii) $\phi(x^{-1}) = \phi(x)^{-1}$

(iii) ϕ sends normal subgroups of G_1 to normal subgroups of G_2

(iv) ϕ preserves the cosets of any normal subgroup N of G_1

(v) Any automorphism (or anti-automorphism) of a group G is a weak Cayley table map

(vi) The composition of two weak Cayley table maps is also a weak Cayley table map

(vii) If $g \in G$ is an involution, then $\phi(g)$ is also an involution.

Another interesting fact is that for two non-isomorphic groups, having the same weak Cayley table is a stronger condition than of having the same character table. For example, consider D_8, the dihedral group of order 8 and the Quaterions Q_8. This is a classic example of two non-isomorphic groups possessing the same character table [DF, p. 882]. However, since the number of involutions in both groups are not the same, there is not a bijection between D_8 and Q_8 that preserves inverses, and so no weak Cayley table map between D_8 and Q_8 can exist [JMS].

Proposition 1.7. *The set of weak Cayley table maps from a group G to itself is a group, denoted $WCT(G)$.*

Proof. As stated above, the composition of two weak Cayley table maps is still a weak Cayley table map. Then, since ϕ is a bijection, ϕ^{-1} exists and is also a weak Cayley table map. So $WCT(G)$ is a group. □

Any automorphism or anti-automorphism is called a *trivial weak Cayley table map.* If for some group G, the group $WCT(G)$ consists only of automorphisms or anti-automorphisms then $WCT(G)$ is said to be *trivial.* [Hu] proved that for all $n \geq 1$, $WCT(S_n)$ is trivial and that for all dihedral groups D_{2n}, $WCT(D_{2n})$ is also trivial.

Since any weak Cayley table map $\phi : G_1 \to G_2$ preserves cosets of normal subgroups N of G, we can let $\overline{\phi} : G/N \to G/\phi(N)$ be the map where $\overline{\phi}(gN) = \phi(g)\phi(N)$. Johnson proved that $\overline{\phi}$ is a weak Cayley table map.

Other results from Johnson, Mattarei, and Sehgal's work on weak Cayley table maps are the following:

Theorem 1.8 (JMS, Theorem 3.1)**.** *Let G_1 and G_2 be finite groups and N a normal subgroup in both G_1 and G_2. Suppose further that $G_1/N \cong G_2/N$ and the order of G_i/N is odd. If (G_i, N) forms a Camina pair, then G_1 and G_2 have the same weak Cayley table.*

An example of this theorem would be the two non-isomorphic groups of order p^3 where p is odd, as examined above. Their centers are isomorphic, both groups form Camina pairs with their center, and their quotients are isomorphic and odd ordered, so they meet the criteria of the hypotheses.

The next theorem in [JMS] uses the structure properties of extensions and Camina pairs to create a weak Cayley table map between two non-isomorphic groups.

Definition 1.9. If H, G are groups and N is an abelian group such that

$$1 \to N \to H \to G \to 1$$

is a short exact sequence, we say that H is an *extension of G by N*.

Theorem 1.10 (JMS, Theorem 4.1)**.** *Suppose that G_1 and G_2 have the same weak Cayley table via $\alpha : G_1 \to G_2$. Let H_i be an extension of G_i by the module N, for $i = 1, 2$ where $n^g = n^{\alpha(g)}$ for all $g \in G$ and suppose that (H_1, N) and (H_2, N) are Camina pairs. Finally, having written each H_1 as an extension of N by G_i, suppose that for all involutions $x \in G_1$ we have*

$$(e, x)^2 = (e, \alpha(x))^2.$$

Then H_1 and H_2 have the same weak Cayley tables.

The strong conditions on the theorem above might often force an isomorphism between H_1 and H_2. An interesting case of this is when H_i is a Frobenius group.

Definition 1.11. Let G and N be finite groups, and let G act on N. Then the action of G on N is said to be *Frobenius* if $n^g \neq n$ for all nonidentity elements $n \in N$ and $g \in G$. The group $H = N \rtimes G$ is called a *Frobenius group* if the action of G on N is Frobenius [Is, p.177].

To understand why two Frobenius groups wit the same weak Cayley table must be isomorphic, suppose that the action of G_1 and G_2 on N was Frobenius. Then, with the condition $n^g = n^{\alpha(g)}$ for all $g \in G_1$, we would have

$$\begin{aligned} n^{gh} &= n^{\alpha(gh)} \\ &= (n^g)^h \\ (n^g)^h &= n^{\alpha(g)\alpha(h)} \end{aligned}$$

Therefore the Frobenius property shows that $\alpha(gh) = \alpha(g)\alpha(h)$. Thus $G_1 \cong G_2$ and they both act identically on N, so in the case that the action is Frobenius, H_1 and H_2 in the previous theorem are isomorphic. Since Frobenius groups have interesting properties, the authors of [JMS] also published the following result which eliminates the condition that G_1 and G_2 act the same way on N, thus allowing for the case when there are two non-isomorphic, Frobenius groups.

Theorem 1.12 (JMS, Theorem 4.3)**.** *Suppose that G_1 and G_2 have the same weak Cayley table via $\alpha : G_1 \to G_2$. Let H_i be an extension of G_i by the abelian normal subgroup N, such that the conjugacy classes of H_1 which lie in N are the same as the conjugacy classes of H_2 in N. Suppose that (H_1, N) and (H_2, N) are Camina pairs. Finally, having fixed a representation for each H_i as an extension of G_i by N, suppose that for every involution $x \in G_1$ we have*

$$\begin{aligned} (e, x)^2 &= (e, \alpha(x))^2, \\ n^x &= n^{\alpha(x)} \qquad \text{for all } n \in N. \end{aligned}$$

Then H_1 and H_2 have the same weak Cayley table.

Chapter 2. Weak Cayley group of $AGL(1,p)$

In this chapter, let p be an odd prime and let $G_p = AGL(1,p)$, the group of affine transformations of $A'(\mathbb{F}_p)$. There are two different ways of referring to the elements of the group G_p. One way is where we use the presentation:

$$\langle a, b | a^{p-1} = b^p = 1, a^b = b^r \rangle,$$

where r is a generator for $\mathbb{F}_p^\times$. The other is to represent elements of G_p as a set of matrices of the form $\left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right)$ such that $x \in \mathbb{F}_p^\times$, and $y \in \mathbb{F}_p$. (Notice that $AGL(1,3) \cong S_3$ and $AGL(1,5) \cong F_{20}$.) There is an isomorphism determined by the map $a \mapsto \left(\begin{smallmatrix} r & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $b \mapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then $a^i \mapsto \left(\begin{smallmatrix} r^i & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $b^j \mapsto \left(\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix}\right)$.

Any element in the group can be written as $a^i b^j$ where $0 \le i \le p-2$ and $0 \le j \le p-1$, by using the conjugation $a^{-1}ba = b^r$. So we can simplify any expression using the following identities:

$$\begin{aligned} a^{-1}b^k a &= b^{kr}, \\ a^{-2}ba^2 &= a^{-1}b^r a = b^{r^2}, \\ a^{-s}ba^s &= b^{r^s}, \\ a^{-s}b^k a^s &= b^{kr^s}. \end{aligned}$$

Next let B be the subgroup generated by $\langle b \rangle$ (in the matrix notation $B = \{\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) | y \in \mathbb{F}_p\}$). Then if we take an element $\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) \in B$ and conjugate it by any element $\left(\begin{smallmatrix} w & z \\ 0 & 1 \end{smallmatrix}\right) \in AGL(1,p)$, we get

$$\begin{pmatrix} w^{-1} & -w^{-1}z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} w & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} w^{-1} & -w^{-1}z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} w & z+y \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & w^{-1}y \\ 0 & 1 \end{pmatrix}.$$

Thus B is a normal subgroup and $B - \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$ is a conjugacy class of G_p since we can take w^{-1} to be anything in $\mathbb{F}_p^{\times}$.

Now consider the element $\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where $x \neq 1$. If we conjugate by any element $\left(\begin{smallmatrix} w & z \\ 0 & 1 \end{smallmatrix}\right)$, observe that

$$\begin{pmatrix} w^{-1} & -w^{-1}z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} w & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} w^{-1} & -w^{-1}z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} xw & xz \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} x & w^{-1}z(x-1) \\ 0 & 1 \end{pmatrix}.$$

By conjugating $\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$ over all the elements of G_p, the entry $w^{-1}z(x-1)$ will range over all of $\mathbb{F}_p$. Thus the conjugacy class of $\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is the coset $\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right) B$, which implies that (G_p, B) is a Camina pair. In the terms of the generators a and b, the conjugacy classes of G_p are $\{1\}, B - \{1\}$, and the cosets $a^i B$ for every $1 \leq i \leq p-2$.

Lemma 2.1. *Given any element* $\left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right) \notin B$*, then*

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^{p-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. If $\left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right) \notin B$, then $x \neq 1$. If $y = 0$, then the result follows from $x \in \mathbb{F}_p^{\times}$. If $y \neq 0$ then,

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x^2 & (x+1)y \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} x^3 & (x^2+x+1)y \\ 0 & 1 \end{pmatrix}.$$

An inductive argument shows that for any positive integer k,

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} x^k & (x^{k-1}+x^{k-2}+\cdots+x+1)y \\ 0 & 1 \end{pmatrix}.$$

In particular, let $k = p-1$, then

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^{p-1} = \begin{pmatrix} x^{p-1} & (x^{p-2}+x^{p-3}+\cdots+x+1)y \\ 0 & 1 \end{pmatrix}$$

Using the identity $x^{p-1} = 1$, this becomes

$$\begin{pmatrix} x^{p-1} & (x^{p-2}+x^{p-3}+\cdots+x+1)y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (x^{p-2}+x^{p-3}+\cdots+x+x^{p-1})y \\ 0 & 1 \end{pmatrix}$$

If we consider the upper right entry on both sides of the equation, we notice that

$$\begin{aligned} (x^{p-2}+x^{p-3}+\cdots+x+1)y &= (x^{p-2}+x^{p-3}+\cdots+x+x^{p-1})y \\ &= x(x^{p-2}+x^{p-3}+\cdots+x+1)y. \end{aligned}$$

However $x \neq 1$ and $y \neq 0$, so we must have that $(x^{p-2}+x^{p-3}+\cdots+x+1) = 0$. Therefore

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^{p-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

□

Considering our conjugacy classes, since B has p elements, the conjugacy classes of the form a^iB have size p ($i \neq 0$), and the class $B - \{1\}$ is the unique class of size $p-1$. If

$\varphi \in WCT(G_p)$, then φ is a bijection that preserves conjugacy classes. Thus $\varphi(B) = B$. Also, φ maps the unique class of involutions (the coset $a^{\frac{p-1}{2}}B$) to itself because φ also preserves inverses.

Let $\Phi : WCT(G_p) \to WCT(G_p/B)$ be the map that sends $\varphi \in WCT(G_p)$ to $\overline{\varphi} \in WCT(G_p/B)$.

This map is well-defined, since we have shown that B is fixed by any $\phi \in WCT(G_p)$ and that the other cosets a^iB are permuted by ϕ.

Also $G_p/B \cong \mathbb{F}_p^\times$ is an abelian group and any weak Cayley table map of G_p/B would have to be an automorphism to satisfy condition (ii) in the definition. Thus $WCT(G_p/B) = Aut(G_p/B)$, and Φ is a map from $WCT(G_p)$ to $Aut(G_p/B)$.

Let K be the kernel of Φ. Then K is not trivial, since it contains the automorphism ρ which sends $a \to a, b \to b^s$, for any s that generates $\mathbb{F}_p^\times$. For any $\phi \in K$, $\phi(a^iB) = a^iB$. Therefore ϕ permutes elements inside cosets. Then we can express $\phi(a^ib^j)$ as $a^ib^{\alpha_\phi(i,j)}$ for some function α_ϕ, where for each i, $\alpha_\phi(i, \_) : \mathbb{F}_p \to \mathbb{F}_p$ is an injective function with $\alpha_\phi(0, 0) = 0$ (since $\phi(1) = 1$).

Also ϕ preserves inverses, so $(\phi(a^ib^j))^{-1} = \phi((a^ib^j)^{-1})$.

Now

$$\begin{aligned}(\phi(a^ib^j))^{-1} &= (a^ib^{\alpha_\phi(i,j)})^{-1} \\ &= b^{-\alpha_\phi(i,j)}a^{-i} \\ &= a^{-i}b^{-r^{-i}\alpha_\phi(i,j)},\end{aligned}$$

and

$$\begin{aligned}\phi((a^ib^j)^{-1}) &= \phi(b^{-j}a^{-i}) \\ &= \phi(a^{-i}b^{-r^{-i}j}) \\ &= a^{-i}b^{\alpha_\phi(-i,-r^{-i}j)}.\end{aligned}$$

These two expresssions are equal since α_ϕ preserves inverses, thus for every α_ϕ

$$-r^{-i}\alpha_\phi(i, j) = \alpha_\phi(-i, -r^{-i}j),$$

where $0 \leq i \leq p-2$ and $0 \leq j \leq p-1$.

Lemma 2.2. *The kernel K in $WCT(G_p)$ is the set of all bijections ϕ such that $\phi(a^ib^j) = a^ib^{\alpha_\phi(i,j)}$ where $\alpha_\phi(i,j)$ is an injective function on $\mathbb{F}_p$ to $\mathbb{F}_p$ such that $\alpha_\phi(0,0) = 0$, and*

$$-r^{-i}\alpha_\phi(i,j) = \alpha_\phi(-i, -r^{-i}j)$$

for every $0 \leq i \leq p-2, 0 \leq j \leq p-1$.

Proof. We have already shown that any map in the kernel must satisfy these conditions. All that is left is to show that any map of this form is a weak Cayley table map in K.

So let ψ be a map such that $\psi(a^ib^j) = a^ib^{\alpha_\psi(i,j)}$ where $\alpha_\psi(i,j)$ is an injective function on $\mathbb{F}_p$ to $\mathbb{F}_p$ such that $\alpha_\psi(0,0) = 0$ and $-r^{-i}\alpha_\psi(i,j) = \alpha_\psi(-i, -r^{-i}j)$ for every $0 \leq i \leq p-2$, $0 \leq j \leq p-1$. Since $\psi(a^ib^j) = a^ib^{\alpha_\psi(i,j)}$, we have $\psi(a^iB) = a^iB, \psi(B - \{1\}) = B - \{1\}$, and $\psi(1) = \psi(a^0b^0) = a^0b^{\alpha_\psi(0,0)} = a^0b^0 = 1$. So ψ takes conjugacy classes to the same conjugacy class, which means that it also fixes the cosets of B. Thus if $\psi \in WCT(G_p)$, then $\psi \in K$.

Then given two elements a^ib^j and a^sb^t in G_p, to satisfy condition (ii), we require $\psi(a^ib^ja^sb^t) \sim \psi(a^ib^j)\psi(a^sb^t)$. Now

$$\begin{aligned} \psi(a^ib^ja^sb^t) &= \psi(a^{i+s}b^{r^sj+t}) \\ &= a^{i+s}b^{\alpha_\psi(i+s, r^sj+t)}, \end{aligned}$$

and

$$\begin{aligned} \psi(a^ib^j)\psi(a^sb^t) &= a^ib^{\alpha_\psi(i,j)}a^sb^{\alpha_\psi(s,t)} \\ &= a^{i+s}b^{r^s\alpha_\psi(i,j)+\alpha_\psi(s,t)}. \end{aligned}$$

If $a^s \neq a^{-i}$, then $a^{i+s}b^{\alpha_\psi(i+s,r^sj+t)} \sim a^{i+s}b^{r^s\alpha_\psi(i,j)+\alpha_\psi(s,t)}$ because they belong to the same coset.

If $a^s = a^{-i}$, then

$$a^{i+s}b^{\alpha_\psi(i+s,r^sj+t)} = b^{\alpha_\psi(0,r^{-i}j+t)},$$

and

$$a^{i+s}b^{r^s\alpha_\psi(i,j)+\alpha_\psi(s,t)} = b^{r^{-i}\alpha_\psi(i,j)+\alpha_\psi(-i,t)}.$$

When $b^{r^{-i}\alpha_\psi(i,j)+\alpha_\psi(-i,t)} = 1$, then $r^{-i}\alpha_\psi(i,j) + \alpha_\psi(-i,t) = 0$, which implies

$$-r^{-i}\alpha_\psi(i,j) = \alpha_\psi(-i,t).$$

Above we found $-r^{-i}\alpha_\psi(i,j) = \alpha_\psi(-i,-r^{-i}j)$, which implies $t = -jr^{-i} \mod p$. Then

$$\begin{aligned} b^{\alpha_\psi(0,r^{-i}j+t)} &= b^{\alpha_\psi(0,r^{-i}j-jr^{-i})} \\ &= b^{\alpha_\psi(0,0)} \\ &= 1. \end{aligned}$$

By a similar argument, if $b^{\alpha_\psi(0,r^{-i}j+t)} = 1$, then $b^{r^{-i}\alpha_\psi(i,j)+\alpha_\psi(-i,t)} = 1$.

If $t \neq -jr^{-i}$, then both $b^{\alpha_\psi(0,r^{-i}j+t)}$ and $b^{r^{-i}\alpha_\psi(i,j)+\alpha_\psi(-i,t)}$ are in the conjugacy class $B - \{1\}$. Therefore, $\psi(a^ib^ja^sb^t) \sim \psi(a^ib^j)\psi(a^sb^t)$, and ψ is a weak Cayley table map in K.

□

Thus we can construct elements of K by considering permutations of each of the cosets a^iB $(i \neq 0)$ that preserve inverses. For the class a^mB, where $1 \leq m \leq \frac{p-3}{2}$, its inverse class is equal to $a^{p-m-1}B$. So by choosing any permutation on the elements of a^mB, the corresponding permutation of $a^{p-m-1}B$ is determined. So we can conclude that the kernel of Φ contains $\frac{p-3}{2}$ copies of S_p.

There is also one coset of involutions, $a^{\frac{p-1}{2}}B$, which is sent to itself by any weak Cayley table map, and any permutation on these elements will respect inverses, so this contributes another copy of S_p to the kernel.

Finally the subgroup B contains its own inverses, and so the allowable permutations on B are those that respect the inverses. The Coxeter group of type B on $\frac{p-1}{2}$ elements is the set of permutations which respect those inverses. It is denoted here as $Cox_B(\frac{p-1}{2})$.

The permutations referred to above are all independent of each other and by Lemma 2.2 they can be composed together to give all of K. So we have shown

Lemma 2.3. *The kernel of* Φ *is isomorphic to* $Cox_B(\frac{p-1}{2}) \times S_p \times S_p^{\frac{p-3}{2}}$. $\square$

Next consider any weak Cayley table map ϕ. We can view it as a permutation on the cosets of B composed with an element ψ of the kernel K. Since we know the structure of the kernel, to finish identifying all weak Cayley table maps we need to determine what effect ϕ can have on the cosets of B.

As seen above, the map ϕ will always satisfy $\phi(B) = B$. Further ϕ will take involutions to involutions, so $\phi(a^{\frac{p-1}{2}}B) = a^{\frac{p-1}{2}}B$ since $a^{\frac{p-1}{2}}B$ is the unique class of involutions in G_p. Therefore ϕ only (possibly) permutes the remaining $p-3$ conjugacy classes amongst themselves while preserving inverses.

Condition (ii) of a weak Cayley table map guarantees that ϕ must preserve inverses. Conveniently all the inverses of elements in a^iB lie in the conjugacy class $a^{-i}B$. Therefore ϕ must permute the remaining $p-3$ cosets in such a way that preserves the coset containing the inverses. This gives us another Coxeter group acting on these $p-3$ classes.

However these coset permutations are not completely independent of the permutation of the elements inside of the $p-3$ classes found in the kernel, since the order of these actions matters. So the possibilities for the action of ϕ on the $p-3$ cosets are $S_p^{\frac{p-3}{2}} \rtimes Cox_B(\frac{p-3}{2})$.

Then, how ϕ permutes the elements of B and $a^{\frac{p-1}{2}}B$ are completely independent of the permutations of the other cosets, thus we have:

Theorem 2.4. $WCT(G_p) \cong Cox_B(\frac{p-1}{2}) \times S_p \times (S_p^{\frac{p-3}{2}} \rtimes Cox_B(\frac{p-3}{2}))$. $\square$

Chapter 3. Camina pairs (G, N) with G/N Abelian

In Chapter 2, the kernel of the map Φ played a key role in helping us understand the group $WCT(AGL(1,p))$. The goal of this chapter is to find similar results about a well defined map $\Phi : WCT(G) \to WCT(G/N)$, where (G, N) form a Camina pair. First we start with a lemma.

Lemma 3.1. *If (G, N) is a Camina pair, then G/N is abelian if and only if the conjugacy classes of $G - N$ are cosets Ng.*

Proof. First assume that the conjugacy classes of $G - N$ are of the form Ng. Then for $g, h \in G$,

$$(Nh)^{-1}(Ng)(Nh) = Nh^{-1}gh.$$

Let $g \in G - N$. Then, since $h^{-1}gh \sim g$ and (G, N) is a Camina pair, $h^{-1}gh = ng$ for some $n \in G$. Then

$$(Nh)^{-1}(Ng)(Nh) = Nng = Ng.$$

Thus G/N is abelian.

Next, assume that G/N is abelian. If $g \in G - N$ and $n \in N$, we have that g and ng are in the same coset Ng, and therefor are conjugate, since (G, N) is a Camina pair. Then note that the conjugacy classes off of N are unions of cosets of N. We also observe that the commutator subgroup G' is contained in N since G/N is abelian. However, no class can have size greater than $|G'|$, and so each class is a coset of N. $\square$

Lemma 3.2. *Let (G, H) be a Camina pair. Then for every weak Cayley table map ϕ, $\phi(H) = H$. Thus the map $\Phi : WCT(G) \to WCT(G/H)$ that sends ϕ to $\overline{\phi}$ is a well-defined map.*

Proof. Now ϕ preserves the set of conjugacy classes and sends cosets of H to cosets of $\phi(H)$. Thus if (G, H) is a Camina pair then so is $(G, \phi(H))$. Now if $\phi(H) \neq H$, then there would exist two subgroups H_1 and H_2 of the same order in G such that (G, H_1) and (G, H_2) are both Camina pairs.

Suppose by way of contradiction that there exist two subgroups H_1 and H_2 such that $H_1 \neq H_2$, (G, H_1) and (G, H_2) are Camina pairs, and the order of H_1 equals the order of H_2. Next pick $h \in H_1 - H_2$.

Then h^G is not contained in H_2, which implies h^G is the union of cosets of H_2, i.e. $h^G = \cup H_2 b_i$ for some elements b_i. Then $|h^G| \geq |H_2|$. However $h^G \subseteq H_1 - H_2$ which implies $|h^G| < |H_1|$, which gives

$$|h^G| \geq |H_2| = |H_1| > |h^G|.$$

This is a contradiction, so $H_1 = H_2$.

Thus there can be only one subgroup that forms a Camina pair of size $|H|$. This gives $\phi(H) = H$ for all weak Cayley table maps ϕ.

□

We note that what the above result really proves is

Corollary 3.3. *Let (G, H_1) and (G, H_2) be Camina pairs. Then either $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.*

□

Theorem 3.4. *Let (G, N) be a Camina pair such that G/N is abelian and $N - \{1\}$ is a conjugacy class. Let $\Phi : WCT(G) \to WCT(G/N)$ be the map that sends ϕ to $\overline{\phi}$. Then the kernel of $\Phi : WCT(G) \to WCT(G/N)$ is the set of all bijections from G to G that take inverses to inverses and preserve conjugacy classes (i.e. maps a conjugacy class to itself).*

Proof. Note by Lemma 3.2 that Φ is a well defined map, and by Lemma 3.1 the conjugacy classes of G off of N are the cosets of N. Let K be the kernel of Φ and let L be the set of bijections ψ that satisfy the hypothesis that $\psi(g^G) = g^G$, and $\psi(g^{-1}) = \psi(g)^{-1}$ for all $g \in G$.

First let $\phi \in K$. Then ϕ is a bijection and $\phi(x^{-1}) = \phi(x)^{-1}$ for all $x \in G$. Then $\phi \in K$ implies $\phi(N - \{1\}) = N - \{1\}$ and for $g \notin N$,

$$\begin{aligned} \phi(g^G) &= \phi(Ng) \\ &= \phi(N)\phi(g) \\ &= N\phi(g) \\ &= Ng \\ &= g^G. \end{aligned}$$

Thus any map $\phi \in K$ is also in L.

Next let ψ be in L. We need to show that ψ is a weak Cayley table map. Any map in L will take conjugacy classes to conjugacy classes, thus satisfying condition (i) of a weak Cayley table map. To show that any map $\psi \in L$ also satisfies condition (ii) of a weak Cayley table map, we need to consider three cases.

Case 1: For $g, h \in G$ and $gh \notin N$: Then there exists $n_1, n_2 \in N$ such that $\psi(g) = n_1 g$, $\psi(h) = n_2 h$ and so

$$\psi(g)\psi(h) = n_1 g n_2 h.$$

Then

$$\psi(g)\psi(h) \in NgNh = Ngh.$$

We also have $\psi(gh) \sim \psi(g)\psi(h) = n_1 g n_2 h = n_3 gh$, so that

$$N\psi(gh) = Ngh$$

Note that Ngh is a conjugacy class for $gh \notin N$, so

$$\psi(g)\phi(h) \sim \psi(gh).$$

Case 2:$g, h \in G$, and $gh = 1$. Then $h = g^{-1}$. Moreover since $\psi \in L$, $\psi(g^{-1}) = \psi(g)^{-1}$, so

$$\psi(gh) = \psi(gg^{-1}) = 1 = \psi(g)\psi(g)^{-1} = \psi(g)\psi(h).$$

Case 3: For $g, h \in G$ and $gh \in N - \{1\}$: Then $Ng = Nh^{-1}$ and for some $n \in N$, $g = nh^{-1}$. Then

$$\begin{aligned} \psi(gh) &= \psi(nh^{-1}h) \\ &= \psi(n) \\ &\in N - \{1\}. \end{aligned}$$

Also

$$\begin{aligned} \psi(g)\psi(h) &= \psi(nh^{-1})\psi(h) \\ &\in Nh^{-1}Nh \\ &= N - \{1\}. \end{aligned}$$

Thus, for all $g, h \in G, \psi(gh) \sim \psi(g)\psi(h)$. So ψ is in K.

$\square$

Chapter 4. Camina-Z Groups

In this chapter we generalize the results that we obtained above in the situation where we have a Camina pair (see Theorem 3.4).

Definition 4.1. Given a finite group G and a set π of prime numbers, a *Hall π-subgroup* is a subgroup H such that all primes which divide the order of H are in π and no prime in π divides the index $[G : H]$. [Is, p. 86]

The following is from [Is, p. 87]

Theorem 4.2 (Hall)**.** *Suppose that G is a finite solvable group, and let π be an arbitrary set of primes. Then all Hall π-subgroups of G are conjugate.*

In this section, we will consider groups $G = A \rtimes B$ with $Z = Z(G)$ such that A is a normal Hall π-subgroup and $Ab - Z$ is a conjugacy class for every $b \in B$. We call such a group a *Camina-Z group.*

Example 4.3. An example of a group with these properties is the group

$$G = \langle a, b | a^5 = b^8 = 1, a^b = a^2 \rangle \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_8$$

with $A = \langle a \rangle$ and $B = \langle b \rangle$. One way to examine the conjugacy classes of this group is to consider a part of the weak Cayley table, where the columns are indexed by the elements of A, and the rows are indexed by elements of B. We can then see that the number of conjugacy classes is 10, and that $\langle b^4 \rangle$ is the center. (In this table below, the numbers 1-10 represent different conjugacy classes of G).

	1	a	a^2	a^3	a^4
1	1	3	3	3	3
b	4	4	4	4	4
b^2	5	5	5	5	5
b^3	6	6	6	6	6
b^4	2	7	7	7	7
b^5	8	8	8	8	8
b^6	9	9	9	9	9
b^7	10	10	10	10	10

Here A is a normal Sylow-5 subgroup, and so is a normal Hall $\{5\}$-subgroup, and where $A \cap Z = \{1\}$. Thus for example $b^4 A = \{b^4\} \cup \{b^4 a, b^4 a^2, b^4 a^3, b^4 a^4\}$ is a union of two conjugacy classes.

Theorem 4.4. *Let $G = A \rtimes B$ with $Z = Z(G)$ such that G is a Camina-Z group and A is the normal Hall-π subgroup. Let K be the kernel of*

$$\Phi : WCT(G) \to WCT(G/A).$$

Then K is the set of functions ϕ such that

(i) ϕ is a bijection,

(ii) ϕ preserves conjugacy classes (i.e. maps a conjugacy class to itself),

(iii) ϕ satisfies $\phi(x^{-1}) = \phi(x)^{-1}$,

(iv) $\phi(xz) = \phi(x)z$ for all $x \in G$ and $z \in Z$.

Proof. The map $\Phi : WCT(G) \to WCT(G/A)$ is well defined, since A is a normal Hall-π subgroup, and so it is the unique normal subgroup of its order by Theorem 4.2. Since any weak Cayley table map sends normal subgroups to normal subgroups and is a bijection, the set A must be fixed by every weak Cayley table map. So if ϕ is a weak Cayley table map, $\overline{\phi} \in WCT(G/\phi(A)) = WCT(G/A)$.

Let L be the set of all bijections $\phi : G \to G$ satisfying (ii)-(iv).

If $\phi \in K$, then ϕ is a bijection that preserves classes and respects inverses. Also for $z \in Z(G)$, we have $\phi(z) = \phi(x^{-1}xz) \sim \phi(x)^{-1}\phi(xz)$. Since z is central, $\phi(z)$ is central, and we have

$$\phi(z) = \phi(x)^{-1}\phi(xz)$$

and so

$$\phi(x)\phi(z) = \phi(xz).$$

Because $\phi \in K$, $\phi(z) = z$ and $\phi(x)z = \phi(xz)$. So $\phi \in L$.

Next let $\phi \in L$. Every map in L preserves conjugacy classes, which implies $\phi(Ab) = Ab$ for all $b \in B$. Therefore we can think of ϕ as a permutation on the elements in each Ab. Let $\phi(ab) = \phi_b(a)b$, where $\phi_b \in Sym(A)$. So in order to show that ϕ is a weak Cayley table map, it is sufficient to show that for every $g, h \in G$, $\phi(gh) \sim \phi(g)\phi(h)$. Let $g = a_1b_1$ and $h = a_2b_2$. Then

$$\begin{aligned}\phi(gh) &= \phi(a_1b_1a_2b_2) \\ &= \phi(a_1a_2^{b_1^{-1}}b_1b_2) \\ &= \phi_{b_1b_2}(a_1a_2^{b_1^{-1}})b_1b_2,\end{aligned}$$

and

$$\begin{aligned}\phi(g)\phi(h) &= \phi(a_1b_1)\phi(a_2b_2) \\ &= \phi_{b_1}(a_1)b_1\phi_{b_2}(a_2)b_2 \\ &= \phi_{b_1}(a_1)\phi_{b_2}(a_2)^{b_1^{-1}}b_1b_2.\end{aligned}$$

Here we have two cases:

Case 1: $b_1 b_2 \notin Z$. Then

$$\phi_{b_1 b_2}(a_1 a_2^{b_1^{-1}}) b_1 b_2 \sim \phi_{b_1}(a_1) \phi_{b_2}(a_2)^{b_1^{-1}} b_1 b_2,$$

since they are in the same conjugacy class $Ab_1 b_2$.

Case 2: $b_1 b_2 = z \in Z$. If

$$\phi(a_1 b_1 a_2 b_2) = y$$

for some $y \in Z$, then

$$a_1 b_1 a_2 b_2 = y,$$

since ϕ satisfies (iv). Then

$$a_2 b_2 = (a_1 b_1)^{-1} y,$$

so

$$\begin{aligned} \phi(a_1 b_1)\phi(a_2 b_2) &= \phi(a_1 b_1)\phi((a_1 b_1)^{-1} y) \\ &= \phi(a_1 b_1)\phi(a_1 b_1)^{-1} y \\ &= y. \end{aligned}$$

On the other hand, if

$$\phi(a_1 b_1)\phi(a_2 b_2) = y$$

for some $y \in Z$, then

$$\begin{aligned} \phi(a_2 b_2) &= \phi(a_1 b_1)^{-1} y \\ &= \phi((a_1 b_1)^{-1} y). \end{aligned}$$

Since ϕ is a bijection, we have

$$a_2 b_2 = (a_1 b_1)^{-1} y,$$

so

$$a_1 b_1 a_2 b_2 = y,$$

which implies

$$\phi(a_1b_1a_2b_2) = y.$$

So if either $\phi(a_1b_1a_2b_2)$ or $\phi(a_1b_1)\phi(a_2b_2)$ is central, then both are central and they are equal. If they are both not central, then

$$\begin{aligned} \phi(a_1b_1a_2b_2) &= \phi_{b_1b_2}(a_1a_2^{b_1^{-1}})b_1b_2 \\ &= \phi_{b_1b_2}(a_1a_2^{b_1^{-1}})z, \end{aligned}$$

which is a non-central element in the coset Az. Also

$$\begin{aligned} \phi(a_1b_1)\phi(a_2b_2) &= \phi_{b_1}(a_1)\phi_{b_2}(a_2)^{b_1^{-1}}b_1b_2 \\ &= \phi_{b_1}(a_1)\phi_{b_2}(a_2)^{b_1^{-1}}z, \end{aligned}$$

which is again is a non-central element of Az. Thus we have

$$\phi(a_1b_1a_2b_2) \sim \phi(a_1b_1)\phi(a_2b_2).$$

$\square$

Chapter 5. Dicyclic Groups

Humphries proved a similar result to the following theorem in his paper on Weak Cayley table Groups in 1997 [Hu]. In that paper, he proved that the $WCT(G)$ is trivial for all dihedral groups G. We will prove this is also true for dicyclic groups.

Definition 5.1. A group with the presentation $< a, x | x^{2n} = 1, x^n = a^2, x^a = x^{-1} >$ is called a *dicyclic group*.

Theorem 5.2. *If G_{4n} is a dicyclic group, then $WCT(G_{4n})$ is trivial.*

Proof. Let G_{4n}. If $g \in G_{4n}$ we can write g in the form x^k or ax^k for some k with $0 \leq k \leq n$. Note that the conjugacy classes of G_{4n} are

$$\{1\}, \{a^2 = x^n\}, \{x^i, x^{-i}\} \text{ for } 1 \leq i \leq n-1, \{ax^i | i \text{ is even}\} \text{ and } \{ax^j | j \text{ is odd}\},$$

and each element that has the form ax^k is of order two if $0 < k < n$ [JL, p.420]. Observe that $Aut(G_{4n})$ acts transitively on noncentral involutions since the map

$$a \to ax^k, \qquad x \to x$$

determines an automorphism of G_{4n}. So if we are given $f \in WCT(G_{4n})$, we can assume that $f(a) = a$ by composing with an automorphism.

Given $f \in WCT(G_{4n})$ such that $f(a) = a$, we know that f must send conjugacy classes to classes and so by considering the classes of G_{4n}, we note that $f(x^k) = x^{\alpha(k)}$ for some bijection $\alpha : \mathbb{Z}/(n\mathbb{Z}) \to \mathbb{Z}(n\mathbb{Z})$, and $f(ax^k) = ax^{\beta(k)}$ for some bijection $\beta : \mathbb{Z}/(n\mathbb{Z}) \to \mathbb{Z}(n\mathbb{Z})$. Then the following relations are a result of $f \in WCT(G_{4n})$:

$$\begin{aligned}
x^{\alpha(k+m)} &= f(x^{(k+m)}) \\
&= f(x^k x^m) \\
&\sim f(x^k)f(x^m) = x^{\alpha(k)}x^{\alpha(m)},
\end{aligned}$$

$$\begin{aligned}
x^{\alpha(k)} &= f(x^k) \\
&= f(a^3 a x^k) \\
&\sim a^3 f(ax^k) = a^3 a x^{\beta(k)} = x^{\beta(k)},
\end{aligned}$$

$$\begin{aligned}
x^{\alpha(m-k)} &= f(x^{-k}x^m) \\
&= f(a^4 x^{-k} x^m) \\
&= f(a^3 x^k a x^m) \\
&\sim f(a^3 x^k)f(ax^m) = a^3 x^{\beta(k)} a x^{\beta(m)} = x^{\beta(m)-\beta(k)}.
\end{aligned}$$

Using these equations in conjunction with the structure of the conjugacy classes, we find

$$\begin{aligned}
\alpha(k+m) &= \pm(\alpha(k)+\alpha(m)), \\
\alpha(k) &= \pm\beta(k), \\
\alpha(m-k) &= \pm(\beta(m)-\beta(k))
\end{aligned}$$

for all $k, m \in \mathbb{Z}/(n\mathbb{Z})$. Since $f \in WCT(G_{4n})$, we know $\alpha(0) = 0, \beta(0) = 0, \alpha(-k) = -\alpha(k)$ and $\beta(-k) = -\beta(k)$ for all $k \in \mathbb{Z}/(n\mathbb{Z})$.

Now suppose that $\alpha(1) = r$ for some $r \in \mathbb{Z}/(n\mathbb{Z})$. Since f is a bijection, we know that $gcd(r, n) = 1$. Then $\alpha(-1) = -r$, since $\alpha(-1+1) = \pm(\alpha(-1)+\alpha(1))$. By the same equation, we also know that $\alpha(2) = \pm 2r$. Then, if we consider $\alpha(3) = \alpha(2+1) = \pm(\alpha(2)+\alpha(1)) = \pm(\pm 2r + r)$. Then since α is a bijection, $\alpha(3) \neq \alpha(-1) = -r$, so $\alpha(2) = +2r$. Then $\alpha(-2) = -2r$. So $\alpha(3) = \pm 3r$. By similar reasoning, since $\alpha(4) = \alpha(3+1) = \pm(\alpha(3)+\alpha(1)) = \pm(\pm 3r + r)$, we see that $\alpha(4) = 4r$, and we can continue this to show that $\alpha(k) = kr$ for all k. So α is an automorphism.

Then, since $\alpha(k) = \pm\beta(k)$, we have $\beta(k+m) = \pm(\beta(k)+\beta(m))$ for all $k, m \in \mathbb{Z}/(n\mathbb{Z})$.

Then similar logic as shown above will show that β is also an automorphism, and that $\beta(1) = \pm r$. Now if $\beta(1) = -r$, we can compose β with the inverse map to get $\beta(1) = r$.

Then we have:

$$f(x^k x^m) = x^{\alpha(k+m)} = x^{\alpha(k)} x^{\alpha(m)} = f(x^k) f(x^m);$$

$$f(ax^k x^m) = ax^{\alpha(k+m)} = ax^{\alpha(k)} x^{\alpha(m)} = f(ax^k) f(x^m);$$

$$f(x^k ax^m) = ax^{\alpha(-k+m)} = x^{\alpha(k)} ax^{\alpha(m)} = f(x^k) f(ax^m);$$

$$f(ax^k ax^m) = a^2 x^{\alpha(-k+m)} = ax^{\alpha(k)} ax^{\alpha(m)} = f(ax^k) f(ax^m).$$

Therefore f is then an automorphism, so the $WCT(G_{4n})$ is trivial.

□

Chapter 6. Some Non-trivial Weak Cayley Table Maps

Often it is difficult to find weak Cayley table maps that are not trivial. In this chapter, we are going to define some nontrivial maps for particular groups that are Camina pairs over their center.

In this chapter, given a finite group G and elements $g_1, \ldots, g_r \in G$, we will write $(g_1, g_2, \ldots, g_r)$ to denote the permutation of G which sends g_1 to g_2, g_2 to g_3, and so on. Thus $(g_1, g_2, \ldots, g_r) \in \mathrm{Sym}(G)$.

6.1 Groups with a Camina Pair Structure Over a Center of Order 2

Theorem 6.1. *Let G be a group with center $Z = Z(G) = \langle z \rangle$ of order 2 and (G, Z) is a Camina pair. For $g \in G - Z$ of order 2, let*

$$\phi_g = (g, gz)$$

and for g of order greater than 2 let

$$\phi_g = (g, gz)(g^{-1}, g^{-1}z).$$

Then ϕ_g is a weak Cayley table map for any $g \in G - Z$.

Proof. Let $< z >= Z$. Since (G, Z) is a Camina pair, the conjugacy classes of G are $\{1\}, \{z\}$, and then unions of sets of the form $\{g, gz\}$ for $g \notin G$. By interchanging g and gz, the conjugacy classes of G are preserved, so ϕ_g satisfies condition (i) for the definition of a weak Cayley table map.

To check that ϕ_g satisfies condition (ii), let $x, y \in G$ and consider the cases below. We may also assume that $x, y \neq 1$. Further the cases $x = z$ or the cases $y = z$ are easily checked, so we assume $x, y, \neq z$.

Case 1: $x \notin \{g, gz, g^{-1}, g^{-1}z\}$ and $y \notin \{g, gz, g^{-1}, g^{-1}z, x^{-1}g, x^{-1}gz, x^{-1}g^{-1}, x^{-1}g^{-1}z\}$. Then ϕ_g fixes xy, x and y. Thus $\phi_g(xy) = xy = \phi_g(x)\phi_g(y)$.

Case 2: $x \notin \{g, gz, g^{-1}, g^{-1}z\}$ and $y = g$. Then $xy = xg \neq 1, z$,

$$\begin{aligned}
\phi_g(xy) &= \phi_g(xg) \\
&= xg \text{ or } xgz, \\
\phi_g(x)\phi_g(y) &= \phi_g(x)\phi_g(g) \\
&= xgz \text{ or } xg.
\end{aligned}$$

Then note that $xg \sim xgz$, since G is a Camina pair over Z and $xg \notin Z$. So $\phi_g(xy) \sim \phi_g(x)\phi_g(y)$. Similar reasoning shows this for $y \in \{g, gz, g^{-1}, g^{-1}z\}$.

Case 3: $x \notin \{g, gz, g^{-1}, g^{-1}z\}$ and $y = x^{-1}g$. Then

$$\begin{aligned}
\phi_g(xy) &= \phi_g(xx^{-1}g) \\
&= \phi_g(g) \\
&= gz, \\
\phi_g(x)\phi_g(y) &= \phi_g(x)\phi_g(x^{-1}g) \\
&= xx^{-1}g \\
&= g.
\end{aligned}$$

Since $g \sim gz$, we have $\phi_g(xy) \sim \phi_g(x)\phi_g(y)$. Similar reasoning shows this for y an element of $\{x^{-1}gz, x^{-1}g^{-1}, x^{-1}g^{-1}z\}$.

Case 4: $x = g$ and $y \notin \{g, gz, g^{-1}, g^{-1}z\}$. Then

$$\begin{aligned}\phi_g(xy) &= \phi_g(gy) \\ &= gy, \\ \phi_g(x)\phi_g(y) &= \phi_g(g)\phi_g(y) \\ &= gzy \\ &= gyz.\end{aligned}$$

Whereas $gy \sim gyz$, we observe that $\phi_g(xy) \sim \phi_g(x)\phi_g(y)$. The same argument also works for $x \in \{gz, g^{-1}, g^{-1}z\}$.

Case 5: $x = g$ and $y = g$. Then

$$\begin{aligned}\phi_g(xy) &= \phi_g(g^2) \\ &= g^2, \\ \phi_g(x)\phi_g(y) &= \phi_g(g)\phi_g(g) \\ &= gzgz \\ &= g^2.\end{aligned}$$

So $\phi(xy) = \phi(x)\phi(y)$. This also works for the case when $x \in \{g, gz\}$ and $y \in \{g, gz\}$ or the case when $x \in \{g^{-1}, g^{-1}z\}$ and $y \in \{g^{-1}, g^{-1}z\}$.

Case 6: $x = g$ and $y = g^{-1}$. Then

$$\begin{aligned}\phi_g(xy) &= \phi_g(1) \\ &= 1 \\ \phi_g(x)\phi_g(y) &= \phi_g(g)\phi_g(g^{-1}) \\ &= gzg^{-1}z \\ &= 1.\end{aligned}$$

So $\phi(xy) = \phi(x)\phi(y)$. This final argument also works for $x \in \{g, gz\}$ and $y \in \{g^{-1}, g^{-1}z\}$ or $x \in \{g^{-1}, g^{-1}z\}$ and $y \in \{g, gz\}$. □

Example 6.2. A quick example of such a group is the dihedral group D_8 of order 8, with presentation

$$D_8 = \langle a, b, | a^4 = b^2 = 1, a^b = a^{-1} \rangle.$$

Then the center of D_8 is $Z = \langle a^2 \rangle$, and (D_8, Z) is a Camina pair. Then $|b| = 2$ and $b \notin Z$, so

$$\phi_b = (b, ba^2)$$

is an element of $WCT(D_8)$. Since $WCT(D_8)$ is trivial [Hu], we know that ϕ_b is either an automorphism or anti-automorphism. With some simple computations, one can show that ϕ_b is an anti-automorphism for D_8.

6.2 p-Groups with a Camina Pair Structure

Theorem 6.3. *Let G be a group with cyclic center $\langle z \rangle = Z, |Z| = p$, such that (G, Z) is a Camina pair, G/Z is elementary p-abelian, and let $g \in G$ be noncentral element. Then the map*

$$\phi_g = (g, gz, gz^2, \ldots, gz^{p-1})(g^{-1}, (gz)^{-1}, (gz^2)^{-1}, \ldots, (gz^{p-1})^{-1})$$

is a weak Cayley table map.

Proof. Since G/Z is abelian and $|Z| = p$, we see that $Z = G'$, the commutator subgroup. If $p = 2$ we can use theorem 6.1, so assume p is odd prime, and let

$$C = \{g, gz, gz^2, \ldots, gz^{p-1}\}$$

and

$$K = \{g^{-1}, (gz)^{-1}, (gz^2)^{-1}, \ldots, (gz^{p-1})^{-1}\}.$$

Then C and K are conjugacy classes in G and ϕ_g fixes C, K and all other conjugacy classes. So ϕ_g satisfies condition (i) of the definition of a weak Cayley table map.

The following are some cases to consider to prove the condition (ii) of a weak Cayley table map $\phi_g(xy) \sim \phi_g(x)\phi_g(y)$.

Let $x, y \in G$. The cases where $x \in \langle z \rangle$ or $y \in \langle z \rangle$ are easily checked, we we assume

$x, y \notin \langle z \rangle$.

Case 1: $x \notin C \cup K, y \notin C \cup K$, and $xy \notin C \cup K$. Then ϕ_g fixes x, y and xy, so

$$\phi_g(xy) = xy = \phi_g(x)\phi_g(y).$$

Case 2: $x \notin C \cup K$, and $y = x^{-1}gz^i$. Then $y \notin C \cup K$ and so

$$\begin{aligned}
\phi_g(xy) &= \phi_g(xx^{-1}gz^i) \\
&= \phi_g(gz^i) \\
&= gz^{i+1}, \\
\phi_g(x)\phi_g(y) &= \phi_g(x)\phi_g(x^{-1}gz^i) \\
&= xx^{-1}gz^i \\
&= gz^i.
\end{aligned}$$

Then $gz^{i+1} \sim gz^i$ and so $\phi_g(xy) \sim \phi_g(x)\phi_g(y)$.

Case 3: $x \notin C \cup K$, and $y = x^{-1}g^{-1}z^i$ is a similar argument as above.

Case 4: Then the cases where $y \notin C \cup K$ and $x = gz^iy^{-1}$ or $x = g^{-1}z^iy^{-1}$ are the same as the above, since g, g^{-1} and z all commute.

Case 5: $x = gz^i$, $y \notin C \cup K$, and $y \neq z^k$ or $g^{-2}z^k$. Then $xy \notin C \cup K$, so

$$\begin{aligned}
\phi_g(xy) &= \phi_g(gz^iy) \\
&= gyz^i, \\
\phi_g(x)\phi_g(y) &= \phi_g(gz^i)\phi_g(y) \\
&= gz^{i+1}y.
\end{aligned}$$

Since $gyz^i \sim gyz^{i+1}$, we have $\phi_g(xy) \sim \phi_g(x)\phi_g(y)$.

Case 6: $x = gz^i$, $y = g^{-2}z^k$. Then $xy = g^{-1}z^{i+k}$, so

$$
\begin{aligned}
\phi_g(xy) &= \phi_g(g^{-1}z^{i+k}) \\
&= g^{-1}z^{i+k-1}, \\
\phi_g(x)\phi_g(y) &= \phi_g(gz^i)\phi_g(g^{-2}z^k) \\
&= gz^{i+1}g^{-2}z^k \\
&= g^{-1}z^{i+k+1}.
\end{aligned}
$$

Since $g{-}1z^{i+k-1} \sim g^{-1}z^{i+k+1}$, we have $\phi_g(xy) \sim \phi_g(x)\phi_g(y)$.

Case 7: $x = gz^i$, $y = gz^k$.

$$
\begin{aligned}
\phi_g(xy) &= \phi_g(g^2z^{i+k}) \\
&= g^2z^{i+k}, \\
\phi_g(x)\phi_g(y) &= \phi_g(gz^i)\phi_g(gz^k) \\
&= gz^{i+1}gz^{k+1} \\
&= g^2z^{i+k+2}.
\end{aligned}
$$

Then since $g^2z^{i+k} \sim gz^{i+k+2}$, so $\phi_g(xy) \sim \phi_g(x)\phi_g(y)$.

Case 8: $x = gz^i$, $y = g^{-1}z^k$. Then

$$
\begin{aligned}
\phi_g(xy) &= \phi_g(gg^{-1}z^{i+k}) \\
&= \phi_g(z^{i+k}) \\
&= z^{i+k}. \\
\phi_g(x)\phi_g(y) &= \phi_g(gz^i)\phi_g(g^{-1}z^k) \\
&= gz^{i+1}g^{-1}z^{k-1} \\
&= z^{i+k}.
\end{aligned}
$$

So in this case, $\phi_g(xy) = \phi_g(x)\phi_g(y)$.

Case 9: Then the cases where $x = g^{-1}z^i$ are the same as previous ones.

□

Example 6.4. A group that satisfies these hypotheses is the extraspecial 3-group of of order

27 with exponent 3. Its presentation is given by

$$G_{27} = \langle a, b, c : a^3 = b^3 = c^3 = 1, b^a = bc, a^c = a, b^c = b \rangle.$$

Note that $|G_{27}| = 27$, it has center $Z = \langle c \rangle$, the order of c is 3, (G_{27}, Z) is a Camina pair, and G_{27}/Z is elementary abelian. Then consider the element $a \in G_{27} - Z$. By Theorem 6.2, the map

$$\begin{aligned} \phi_a &= (a, ac, ac^2)(a^{-1}, (ac)^{-1}, (ac^2)^{-1}) \\ &= (a, ac, ac^2)(a^2, a^2c^2, a^2c) \end{aligned}$$

is an element of $WCT(G_{27})$. This map is non-trivial, since if we consider the element $ab \in G_{27}$, we notice

$$\phi_a(ab) = ab,$$

$$\phi_a(a)\phi_a(b) = acb = abc,$$

and

$$\phi_a(b)\phi_a(a) = bac = abc^2$$

which are conjugate but not equal. Thus ϕ_a is a nontrivial element of $WCT(G_{27})$.

6.3 Another map for p-Groups with a Camina Pair Structure

Theorem 6.5. *Let p be an odd prime, and let G be a p-group with cyclic center $< z >= Z, |Z| = p$ such that (G, Z) is a Camina pair, G/Z is elementary abelian, and let $g \in G$ be a noncentral element of order q. Then the map*

$$\begin{aligned} \phi_g = \; & (g, gz, gz^2, \ldots, gz^{p-1})(g^{-1}, (gz)^{-1}, (gz^2)^{-1}, \ldots, (gz^{p-1})^{-1}) \\ & (g^2, g^2z, g^2z^2, \ldots, g^2z^{p-1})(g^{-2}, (g^2z)^{-1}, (g^2z^2)^{-1}, \ldots, (g^2z^{p-1})^{-1}) \\ & \qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ & (g^{\frac{q-1}{2}}, g^{\frac{q-1}{2}}z, g^{\frac{q-1}{2}}z^2, \ldots, g^{\frac{q-1}{2}}z^{p-1})(g^{-\frac{q-1}{2}}, (g^{\frac{q-1}{2}}z)^{-1}, (g^{\frac{q-1}{2}}z^2)^{-1}, \ldots, (g^{\frac{q-1}{2}}z^{p-1})^{-1}) \end{aligned}$$

is a weak Cayley table map.

Proof. Let

$$C = \{g^i, g^i z, g^i z^2, \ldots, g^i z^{p-1} | 1 \leq i \leq \frac{q-1}{2}\},$$

and

$$K = \{g^{-i}, (g^i z)^{-1}, (g^i z^2)^{-1}, \ldots, (g^i z^{p-1})^{-1} \,|\frac{q-1}{2} \leq i \leq p-1\}.$$

Also note that since (G, Z) is a Camina pair and G/Z is elementary p-abelian, conjugacy classes are of the form xZ off of the center.

Note that ϕ_g just permutes the elements of each conjugacy class, so the condition that $\phi_g(x^G) = \phi_g(x)^G$ is met. All that is left is to check to see if $\phi_g(xy) \sim \phi_g(x)\phi_g(y)$ for every x, y in G. There are several cases to check:

Case 1: $x \notin C \cup K$, $y \notin C \cup K$, and $xy \notin C \cup K$. Then ϕ_g fixes x, y and xy, so $\phi_g(xy) = xy = \phi_g(x)\phi_g(y)$.

Case 2: $x \notin C \cup K$ and $y = x^{-1} g^i z^j$ where $g^i z^j \in C$. Then

$$\begin{aligned}
\phi_g(xy) &= \phi_g(x x^{-1} g^i z^j) \\
&= \phi_g(g^i z^j) \\
&= g^i z^{j+1}. \\
\phi_g(x)\phi_g(y) &= x x^{-1} g^i z^{j+1} \\
&= g^i z^{j+1}.
\end{aligned}$$

So $\phi_g(xy) = xy = \phi_g(x)\phi_g(y)$.

Case 3: $x \notin C \cup K$ and $y = x^{-1} g^i z^j$ where $g^i z^j \in K$. Then

$$\begin{aligned}
\phi_g(xy) &= \phi_g(xx^{-1}g^iz^j) \\
&= \phi_g(g^iz^j) \\
&= \phi_g(g^iz^j) \\
&= g^iz^{j-1}, \\
\phi_g(x)\phi_g(y) &= xx^{-1}g^iz^{j-1} \\
&= g^iz^{j-1}.
\end{aligned}$$

So $\phi_g(xy) = xy = \phi_g(x)\phi_g(y)$.

Similarly, for $y \notin C \cup K$ and $x = g^iz^jy^{-1}$, $\phi_g(xy)$ is still conjugate to $\phi_g(x)\phi_g(y)$.

Case 4: $x = g^iz^j$ and $y \notin Z$, $y \notin C \cup K$, and so $xy \notin C \cup K$. Then

$$\begin{aligned}
\phi_g(xy) &= xy \\
&= g^iyz^j, \\
\phi_g(x)\phi_g(y) &= \phi_g(g^iz^j)\phi_g(y) \\
&= g^iz^{j\pm 1}y.
\end{aligned}$$

Then g^iyz^j and $g^iz^{j\pm1}y$ are elements of g^iyZ. Thus $\phi_g(xy) \sim \phi_g(x)\phi_g(y)$.

Case 5: $x = g^iz^j$ and $y = z^k$, which means $xy = g^iz^{j+k}$. So we have that

$$\begin{aligned}
\phi_g(xy) &= \phi_g(g^iz^{j+k}) \\
&= g^iz^{j+k\pm1}, \\
\phi_g(x)\phi_g(y) &= \phi_g(g^iz^j)\phi_g(z^k) \\
&= g^iz^{k\pm1}z^k \\
&= g^iz^{j+k\pm1}.
\end{aligned}$$

Therefore $\phi_g(xy) = \phi_g(x)\phi_g(y)$. The cases where $y = g^iz^j$ and $x \notin C \cup K$ are the same as above.

Case 6: If $x = g^iz^j$ and $y = g^lz^k$, where $l \neq -i$, ϕ_g will keep x, y, and xy in the conjugacy class of $g^{i+l}Z$, so $\phi_g(xy) \sim \phi_g(x)\phi_g(y)$.

Case 7: $x = g^iz^j$ and $y = g^{-i}z^k$. Then $xy = z^{j+k}$. So then we have

$$\begin{aligned}\phi_g(xy) &= \phi_g(z^{j+k})\\ &= z^{j+k},\\ \phi_g(x)\phi_g(y) &= \phi_g(g^i z^j)\phi_g(g^{-i}z^k)\\ &= g^i z^{j\pm1} g^{-i} z^{k\mp1}\\ &= z^{j+k}.\end{aligned}$$

Therefore, $\phi_g(xy) \sim \phi_g(x)\phi_g(y)$.

□

Example 6.6. An example of a group that satisfies the hypothesis of Theorem 6.5 is the extraspecial 5-group of order 125 with exponent 25. Its presentation is given by

$$G_{125} = \langle x, y, z | x^{25} = y^5 = z^5 = 1,\ x^5 = z,\ x^y = xz,\ z \text{ central}\rangle.$$

The center is $Z = \langle z \rangle$, $|z| = 5$, and (G_{125}, Z) is a Camina pair. Then note that since $y \notin Z$, by Theorem 6.3, the map

$$\begin{aligned}\phi_y = \ & (y, yz, yz^2, yz^3, yz^4)(y^{-1}, (yz)^{-1}, (yz^2)^{-1}, (yz^3)^{-1}, (yz^4)^{-1})\\ & (y^2, y^2z, y^2z^2, y^2z^3, y^2z^4)(y^{-2}, (y^2z)^{-1}, (y^2z^2)^{-1}, (y^2z^3)^{-1}, (y^2z^4)^{-1})\\ = \ & (y, yz, yz^2, yz^3, yz^4)(y^4, y^4z^4, y^4z^3, y^4z^2, y^4z)\\ & (y^2, y^2z, y^2z^2, y^2z^3, y^2z^4)(y^3, y^3z^4, y^3z^3, y^3z^2, y^3z)\end{aligned}$$

is a weak Cayley table map. Note that

$$\begin{aligned}\phi_y(xy) &= xy\\ \phi_y(x)\phi_y(y) &= xyz\\ \phi_y(y)\phi_y(x) &= yzx\\ &= xyz^2\end{aligned}$$

are not equal, thus ϕ_y is not an anti-automorphism or automorphism of G_{125}.

Chapter 7. Relative Conjugacy Classes And Relative Weak Cayley Tables

We define a conjugacy classes for an element x of a group G to be the set $\{g^{-1}xg|g \in G\}$. Relative conjugacy classes are similar, only instead of conjugating an element of $x \in G$ over the entire group, we conjugate x only by the elements of a particular subgroup of G. So if H is a subgroup of G, the *relative conjugacy class of* x with respect to H (or the H*-conjugacy class* of x) is the set $\{h^{-1}xh|h \in H\}$. We will use the notation $x \sim_H y$ to mean that x is conjugate to y by an element in H.

This essentially splits some of the conjugacy classes into distinct parts. In particular, the set of relative conjugacy classes will have at least as many elements as the set of conjugacy classes of the group. For example consider the dihedral group of order 8, D_8, with the presentation $\langle a, b|a^4 = b^2 = 1, a^b = a^3\rangle$. Then the conjugacy classes are $\{1\}, \{a^2\}, \{a, a^3\}, \{b, ba^2\}$, and $\{ba, ba^3\}$.

If we let $H = \langle a\rangle$, then the relative conjugacy classes for D_8 with respect to H would be $\{1\}, \{a\}, \{a^2\}, \{a^3\}, \{b, ba^2\}$, and $\{ba, ba^3\}$.

We can use these relative conjugacy classes to define a relative weak Cayley table. This is similar to a weak Cayley table except the entries of the table contain relative conjugacy classes. For example, if we consider the group S_3 with the subgroup $H = \langle(123)\rangle$, then the relative conjugacy classes of S_3 with respect to H are $B_1 = \{1\}, B_2 = \{(12), (13), (23)\}, B_3 = \{(123)\}$ and $B_4 = \{(132)\}$. Then the relative weak Cayley table for S_3 with respect to H is:

	1	(12)	(23)	(13)	(123)	(132)
1	B_1	B_2	B_2	B_2	B_3	B_4
(12)	B_2	B_1	B_4	B_3	B_2	B_2
(23)	B_2	B_3	B_1	B_4	B_2	B_2
(13)	B_2	B_4	B_3	B_1	B_2	B_2
(123)	B_3	B_2	B_2	B_2	B_4	B_1
(132)	B_4	B_2	B_2	B_2	B_1	B_3

7.1 Relative Weak Cayley Table Maps

As with weak Cayley tables, it is convenient to know when two groups with given subgroups have the same relative weak Cayley table. To do so, we will define a map that preserves the weak Cayley table structure. Given two groups G_1, G_2 with subgroups H_1, H_2 respectively, a *relative weak Cayley table map* is a bijection $\phi : G_1 \to G_2$ such that

(i) $\phi(H_1) = H_2$;

(ii) $\phi(x^{H_1}) = (\phi(x))^{H_2}$, for all $x \in G_1$;

(iii) $\phi(xy) \sim_{H_2} \phi(x)\phi(y)$ for all $x, y \in G_1$.

Since this map preserves the structure of the relative H-conjugacy classes, we say two groups with two specified subgroups *have the same relative weak Cayley tables* if there exists a relative weak Cayley table map between the two groups.

Theorem 7.1. *There exists a relative weak Cayley table map between two non-isomorphic groups.*

Proof. Consider the two non-isomorphic non-abelian groups of order p^3. They have presentations

$$G_1 = \langle a, b, c : a^p = b^p = c^p = 1, b^a = bc, a^c = a, b^c = b\rangle;$$
$$G_2 = \langle x, y, z : x^p = z, x^{p^2} = y^p = z^p = 1, x^y = x^{p+1}, x^z = x, y^z = y\rangle;$$

with

$$Z(G_1) = \langle c \rangle;\ Z(G_2) = \langle z \rangle.$$

Let $H_1 = \langle a \rangle$ and $H_2 = \langle y \rangle$.

Then define the map $\phi : (G_1, H_1) \to (G_2, H_2)$ by

$$a^r b^s c^t \to x^s y^r z^{rs-t}$$

where rs is not congruent to 0 mod p.

We will show that ϕ is a relative weak Cayley table map by finding the relative conjugacy classes and comparing several cases that arise.

Notice these two groups are Camina pairs over their centers, so this makes the relative H_i-conjugacy classes easy to compute: for (G_1, H_1):

$$\begin{array}{ccccc}
\{1\}, & \{c\}, & \{c^2\}, & \ldots, & \{c^{p-1}\}, \\
\{a\}, & \{ac\}, & \{ac^2\}, & \ldots, & \{ac^{p-1}\}, \\
\vdots & & & & \\
\{a^i\}, & \{a^i c\}, & \{a^i c^2\}, & \ldots, & \{a^i c^{p-1}\}, \\
\vdots & & & & \\
\{a^{p-1}\} & \{a^{p-1} c\} & \{a^{p-1} c^2\} & \ldots & \{a^{p-1} c^{p-1}\}
\end{array}$$

and then

$$a^i b^j \{1, c, c^2, \ldots, c^{p-1}\}$$

for $0 \le i \le p-1$, $0 < j \le p-1$;

and for (G_2, H_2):

$$
\begin{array}{ccccc}
\{1\}, & \{z\}, & \{z^2\}, & \dots, & \{z^{p-1}\}, \\
\{y\}, & \{yz\}, & \{yz^2\}, & \dots, & \{yz^{p-1}\}, \\
\vdots & & & & \\
\{y^i\}, & \{y^i z\}, & \{y^i z^2\}, & \dots, & \{y^i z^{p-1}\}, \\
\vdots & & & & \\
\{y^{p-1}\}, & \{y^{p-1} z\}, & \{y^{p-1} z^2\}, & \dots, & \{y^{p-1} z^{p-1}\},
\end{array}
$$

and then

$$x^i y^j \{1, z, z^2, \dots, z^{p-1}\}$$

for $0 < i \leq p-1$, $0 \leq j \leq p-1$.

By inspection, it is easy to see that ϕ will send H_1 to H_2, and that ϕ will send H_1-conjugacy classes in G_1 to H_2-conjugacy classes in G_2, which are the first two conditions of a relative weak Cayley table map. The last thing to check is to see if $\phi(\alpha\beta) \sim_{H_2} \phi(\alpha)\phi(\beta)$ for all $\alpha, \beta \in G_1$.

Let $\alpha = a^i b^j c^k$ and $\beta = a^r b^s c^t$. Then there are three cases that can happen which we need to compare to determine if ϕ is a relative weak Cayley table map: when $j + s < p$, $j + s > p$, and $j + s = p$.

Case 1: Let $j + s < p$. Then $\alpha\beta = a^i b^j a^r b^s c^{k+t} = a^{i+r} b^{j+s} c^{k+t+rj}$, so

$$\phi(\alpha\beta) = x^{j+s} y^{i+r} z^{-k-t-rj}.$$

On the other hand $\phi(\alpha) = x^j y^i z^{ij-k}$ and $\phi(\beta) = x^s y^r z^{rs-t}$. Therefore we have

$$
\begin{aligned}
\phi(\alpha)\phi(\beta) &= x^j y^i z^{ij-k} x^s y^r z^{rs-t} \\
&= x^{j+s} y^{i+r} z^{ij-k+rs-t+si}
\end{aligned}
$$

This gives $\phi(\alpha\beta)$ is H_2-conjugate to $\phi(\alpha)\phi(\beta)$.

Case 2: Let $j+s>p$, then $j+s=w+p$ for some $0<w<p$. Thus

$$\phi(\alpha\beta)=x^w y^{i+r} z^{-k-t-rj+1},$$

and

$$\phi(\alpha)\phi(\beta)=x^w y^{i+r} z^{ij-k+rs-t+si+1},$$

both of which are also in the same H_2-conjugacy class.

Case 3: Let $j+s=p$. Then $\phi(\alpha\beta)$ needs to be equal to $\phi(\alpha)\phi(\beta)$ in order to be H_2-conjugate. Note that $j=-s \bmod(p)$.

In this case we have

$$\begin{aligned}\phi(\alpha\beta) &= y^{i+r} z^{-k-t-rj+1}\\ &= y^{i+r} z^{-k-t+rs+1},\end{aligned}$$

and

$$\begin{aligned}\phi(\alpha)\phi(\beta) &= y^{i+r} z^{ij-k+rs-t+si+1}\\ &= y^{i+r} z^{-is-k+rs-t+is+1}\\ &= y^{i+r} z^{-k-t+rs+1}.\end{aligned}$$

Thus $\phi(\alpha\beta)=\phi(\alpha)\phi(\beta)$.

□

Proposition 7.2. *Let G_1, G_2 and G_3 be groups with subgroups H_1, H_2 and H_3 respectively such that there exist relative weak Cayley table maps $\phi : (G_1, H_1) \to (G_2, H_2)$ and $\psi : (G_2, H_2) \to (G_3, H_3)$. Then the map $\psi \circ \phi : (G_1, H_1) \to (G_3, H_3)$ is a relative weak Cayley table map.*

Proof. By the definition of ϕ and ψ, we have

$$\phi(H_1)=H_2,$$

$$\psi(H_2) = H_3$$

and so

$$\psi \circ \phi(H_1) = H_3.$$

Let a, b be elements of G_1. Then if we consider the relative conjugacy class a^{H_1}, we note that

$$\phi(a^{H_1}) = \phi(a)^{H_2},$$

and

$$\psi(\phi(a)^{H_2}) = (\psi \circ \phi(a))^{H_3}.$$

Lastly, consider the relationships

$$\phi(ab) \sim_{H_2} \phi(a)\phi(b),$$

which implies

$$\psi(\phi(ab)) \sim_{H_3} \psi(\phi(a))\psi(\phi(b)).$$

□

Corollary 7.3. *Let G_1 and G_2 have subgroups H_1 and H_2 respectively such that there exists a relative weak Cayley table map $\phi : (G_1, H_1) \to (G_2, H_2)$. Also let $\alpha \in Aut(G_1)$ and $\beta \in Aut(G_2)$. Then $\phi \circ \alpha : (G_1, \alpha^{-1}(H_1)) \to (G_2, H_2)$ and $\beta \circ \phi : (G_1, H_1) \to (G_2, \beta(H_2))$ are also relative weak Cayley table maps.*

Proof. Automorphisms are relative weak Cayley table maps, so by using Proposition 7.2, we can easily see these compositions are relative weak Cayley table maps. □

Chapter 8. Relative Weak Cayley Table Groups

We define the set of relative weak Cayley table maps from G with a subgroup H to itself by $RWCT(G,H)$.

Theorem 8.1. *Let $\phi \in RWCT(G,H)$, then $\phi \in WCT(G)$.*

Proof. By definition if $\phi \in RWCT(G,H)$, then $\phi(xy) \sim_H \phi(x)\phi(y)$ which also implies $\phi(xy) \sim_G \phi(x)\phi(y)$. This implies the second condition of a weak Cayley table map.

Then, let $x' = \phi^{-1}(x)$ and $y' = \phi^{-1}(y)$. Then we know that

$$\phi(x'y') \sim_H \phi(x')\phi(y') = xy$$

also

$$\phi(y'x') \sim_H \phi(y')\phi(x') = yx$$

and $xy \sim_G yx$, which means that $\phi(x'y') \sim_G \phi(y'x')$.

So given any $g \in G, \phi(g) = \phi(gaa^{-1})$ for any $a \in G$. Then $\phi(gaa^{-1}) \sim_G \phi(a^{-1}ga)$ by the above argument. This implies that $\phi(g^G) = \phi(g)^G$, which is the first condition of a weak Cayley table map. Thus $\phi \in WCT(G)$. □

Theorem 8.2. *$RWCT(G,H)$ is a subgroup of $WCT(G)$.*

Proof. Given $\phi, \psi \in RWCT(G,H)$, then $\phi, \psi \in WCT(G)$. Clearly, $\phi \circ \psi$ takes H-conjugacy classes to H-classes. Then given two maps ϕ and ψ in $RWCT(G,H)$, we know from Proposition 7.2 that $\psi \circ \phi$ is also in $RWCT(G,H)$.

Since $WCT(G)$ is a group, for every $\phi \in RWCT(G,H)$, there is a $\phi^{-1} \in WCT(G)$. Since $\phi(H) = H$, $\phi^{-1}(H) = H$, and since ϕ sends H-conjugacy classes to H-conjugacy classes, ϕ^{-1} does the same.

Then let $x' = \phi^{-1}(x)$ and $y' = \phi^{-1}(y)$. Then, since $\phi \in RWCT(G)$, we have that $\phi(x'y') \sim_H \phi(x')\phi(y') = xy$. Since we know that ϕ^{-1} takes H-conjugacy classes to H-conjugacy classes, we know that $\phi^{-1}\phi(x'y') \sim_H \phi^{-1}(xy)$. So $x'y' \sim_H \phi^{-1}(xy)$, which implies that $\phi^{-1}(x)\phi^{-1}(y) \sim_H \phi^{-1}(xy)$. Therefore, $\phi^{-1} \in RWCT(G, H)$, and $RWCT(G, H)$ is a subgroup of $WCT(G)$.

□

Chapter 9. Extensions of Results from [JMS]

Theorem 9.1. *Let G be a group with $|G|$ odd. Let N be an abelian group that is also a G-module and assume the N-conjugacy classes outside of N are Ng. Let G_1 and G_2 be two non-isomorphic groups which are extension of N by G such that (G_1, N) and (G_2, N) are Camina pairs. Then (G_1, N) and $G_2, N)$ have the same relative weak Cayley tables.*

Proof. This result was proven for weak Cayley tables in [JMS] without the assumption that the N-classes outside of N are Ng. We will show that the map that he defined is also a relative weak Cayley table map between (G_1, N) and $G_2, N)$ with this additional condition.

Some notation that [JMS] used was to use the extension structure of G_1 and G_2 to write

$$(n_1, g_1) \circ_i (n_2, g_2) = (n_1 n_2^{g_1^{-1}} f_i(g_1, g_2), g_1 g_2),$$

where $\circ_i$ represents the multiplication in the particular group G_1 or G_2 and f_i is a 2-cocycle in $H^2(G, N)$. One can assume that $f_i(g, e) = f_i(e, g) = e$ for all $g \in G_i$.

They then partitioned $G - \{e\}$ into two subsets, S_1 and S_2, where if $g \in S_1$, then $g^{-1} \in S_2$, and $S_1 \cup S_2 = G$. This is possible since $|G|$ is odd. Using these subsets, they defined the map $\phi : G_1 \to G_2$ as follows:

$$\begin{aligned} \phi(n, e) &= (n, e) && \text{for } n \in N, \\ \phi(n, g) &= (n, g) && \text{for } g \in S_1, \\ \phi((n, g)^{inv(1)}) &= (n, g)^{inv(2)} && \text{for } g \in S_1, \end{aligned}$$

where $inv(i)$ represents the inverse in G_i.

In [JMS], they then went on to prove that this is a weak Cayley table map.

To see that this map is also a relative weak Cayley table map, note that since we have Camina pairs over the abelian subgroup N and that the N-classes outside of N are Ng, the

N-conjugacy classes of G_i $(i = 1, 2)$ are the singleton sets $\{1\}, \{(n, e)\}$ for every $n \in N$, and the cosets Ng for every $g \in G - \{1\}$. It is clear that ϕ preserves the N-conjugacy classes, thus satisfying condition (ii) of a relative weak Cayley table map.

For condition (iii), let $A = \phi((n_1, g_1) \circ_1 (n_2, g_2))$ and let $B = \phi(n_1, g_1) \circ_2 \phi(n_2, g_2)$, and we want to show that A and B are in the the same N-conjugacy class, which would imply that ϕ is a relative weak Cayley table map between G_1 and G_2.

We then need to consider three cases:

Case 1: $g_1 = g_2 = e$. Then $A = (n_1 n_2, e) = B$.

Case 2: $g_1 \neq g_2^{-1}$. Then $A = (m_1, g_1 g_2)$ for some $m_1 \in N$, and $B = (m_2, g_1 g_2)$ for some $m_2 \in N$. Then, since $g_1 g_2 \notin N$, A and B are in the same coset $N g_1 g_2$, so A and B are conjugate by an element of N.

Case 3: $g_2 = g_1^{-1} \neq e$. Assume without loss of generality that $g_1 \in S_1$. Then $(n, g) = (n, e) \circ (e, g)$ for all $n \in N$ and $g \in G$. So then

$$(n, g) \circ (m, g)^{-1} = (n, e) \circ (e, g) \circ (e, g)^{-1} \circ (m, e)^{-1} = (nm^{-1}, e).$$

Which means

$$A = \phi((n_1, g_1) \circ_1 (m, g_1)^{inv(1)}) = \phi((n_1 m^{-1}, e)) = (n_1 m^{-1}, e),$$

and

$$B = \phi(n_1, g_1) \circ_2 \phi((m, g_1)^{inv(1)}) = (n_1, g_1) \circ_2 (m, g_1)^{inv(2)} = (n_1 m^{-1}, e).$$

So $A = B$, which is what was needed for A and B to be N-conjugate.

$\square$

The following lemma shows that the conditions required in Theorem 4.1 from [JMS] (referenced in this paper as Theorem 1.8) force any weak Cayley table to be an automorphism when the action of G_i is Frobenius on N.

Lemma 9.2. *Suppose that G_1 and G_2 have the same weak Cayley table, and $\alpha : G_1 \to G_2$ is a weak Cayley table map. Suppose that H_i is a Frobenius extension of G_i by the module N in such a way that $n^g = n^{\alpha(g)}$ for all g in G_1. Then $G_1 \cong G_2$.*

Proof. Note that all $g, h \in G_1$, we have

$$n^{gh} = n^{\alpha(gh)}.$$

From the group actions we also have:

$$n^{\alpha(gh)} = n^{gh} = (n^g)^h = (n^{\alpha(g)})^{\alpha(h)} = n^{\alpha(g)\alpha(h)}.$$

This means that

$$n^{\alpha(gh)} = n^{\alpha(g)\alpha(h)}.$$

So

$$n^{\alpha(gh)(\alpha(g)\alpha(h))^{-1}} = n.$$

Since the action is Frobenius, this implies that

$$\alpha(gh)(\alpha(g)\alpha(h))^{-1} = 1,$$

and so

$$\alpha(gh) = \alpha(g)\alpha(h),$$

which shows that α is a homomorphism, which means that it is also an isomorphism. □

The following theorem allows us to remove the condition that N be abelian from the statement of Theorem 4.1 of [JMS] if we require that the action of G_i is Frobenius on N.

Theorem 9.3. [Is, p.179] *Let $H_i = N \rtimes G_i$ be a Frobenius group with kernel N. If the order of G_i is even, then G_i has at most one involution and N must be abelian.*

When the order of G_i is odd, the authors of [JMS] comment that in the proof of Theorem 4.1, the fact that N is abelian is not necessary.

Theorem 9.4. *Suppose there exists a weak Cayley table map $\alpha : G_1 \to G_2$. Let H_i be a Frobenius extension of G_i by the normal subgroup N, such that in H_1 and H_2, the relative N-conjugacy classes outside of N are unions of cosets of N. Lastly, for any involution $x \in G_1$, we require*

$$(e, x)^2 = (e, \alpha(x))^2.$$

Then H_1 and H_2 have the same relative weak Cayley table.

Proof. First we note that (H_1, N) and (H_2, N) are Camina pairs, since H_i is a Frobenius extension [Is, pg. 185]. As in the proof of 9.1, write H_1 and H_2 as group extensions. Let I denote the set of involutions of G_1 (if it exists). Next, partition $G_1 - \{e\} - I$ into two subsets S and S^{-1} (where $S^{-1} = \{s^{-1} | s \in S\}$). Then we have that $G_1 = \{e\} \cup I \cup S \cup S^{-1}$ and $G_2 = \{e\} \cup \alpha(I) \cup \alpha(S) \cup \alpha(S^{-1})$.

The map that the authors of [JMS] prove is a weak Cayley table map is

$$\begin{aligned} \phi(n, g) &= (n, \alpha(g)), && \text{for } g \in \{e\} \cup \{x\} \cup S, \\ \phi((n, g)^{-1}) &= (n, \alpha(g))^{-1}, && \text{for } g \in S, \\ \phi((e, x)) &= (e, \alpha(x)), && \text{for } x \in I. \end{aligned}$$

To see this is a relative weak Cayley table map, we need to know the N-conjugacy classes of H_i. Note that by the hypothesis, the N-conjugacy classes contained in H_1 lying in N are the same as those lying in the copy of N in H_2, so ϕ automatically preserves those N-classes. Then the rest of the conjugacy classes are unions of costs $\{(n, g) | n \in N\}$ for a $g \in G_i$. Knowing the N-conjugacy classes, it is clear that ϕ is a bijection that sends N-conjugacy classes to N-conjugacy classes.

Then to show that ϕ is a relative weak Cayley table map, we need to show that $\phi((n,g)(m,h)) \sim_N \phi(n,g)\phi(m,g)$ for all $n,m \in N$ and $g,h \in G$. Let $A = \phi((n,g)(m,h))$ and let $B = \phi(n,g)\phi(m,g)$.

Case 1: $g = h = e$. Then $A = (nm, e) = B$.

Case 2: $g \neq h^{-1}$. Then $A = (m_1, gh)$ for some $m_1 \in N$, and $B = (m_2, gh)$ for some $m_2 \in N$. Then A and B are in the same coset Ng_1g_2, so A and B are conjugate by an element of N.

Case 3: $h = g^{-1} \neq e$. The authors of [JMS] show that for all $g \in G$, $\phi((n,g))^{-1} = \phi((n,g)^{-1})$, and this is the only spot in his proof where he used the action of G on N. If the order of G is even, then in order for $\phi((n,g))^{-1} = \phi((n,g)^{-1})$ to hold, N must be abelian. However, since N is a Frobenius kernel of G, the fact that $|G|$ is even forces N to be abelian. If the order of G is odd, then no assumptions on N are needed to obtain $\phi((n,g))^{-1} = \phi((n,g)^{-1})$.

Therefore, an equivalent statement to $A \sim_N B$ is to show

$$\phi((n,g)(m,h)^{-1}) \sim_N \phi((n,g))\phi((m,h)^{-1}).$$

Consider the two computations:

$$\begin{aligned} \phi((n,g)(m,g)^{-1}) &= \phi((nm^{-1}, e)) \\ &= (nm^{-1}, e) \end{aligned}$$

and

$$\begin{aligned} \phi((n,g))\phi((m,g)^{-1}) &= \phi(n,g)(\phi(m,g))^{-1} \\ &= (n, \alpha(g))(m, \alpha(g))^{-1} \\ &= (n,e)(e,\alpha(g)((m,e)(e,\alpha(g)))^{-1} \\ &= (n,e)(e,\alpha(g)(e,\alpha(g))^{-1}(m^{-1},e) \\ &= (nm^{-1}, e). \end{aligned}$$

These show that $A = B$, and hence conjugate, which means ϕ is a relative weak Cayley table map. $\square$

The authors of [KR] describe a construction of groups that meet the criteria for Theorem 9.4. They start with a semi-direct product of semi-linear maps acting on a finite vector space, and then choose specific subgroups of this semi-direct product.

Example 9.5. [KR, Definition 3.1 and 3.2, pg. 278-279] Let G_n be a group with a cyclic normal subgroup $D_n = \langle d_n \rangle$ of order n with complement $C_{\mu(n)} = \langle c \rangle$, where $\mu(n)$ is the Euler function, i.e., the number of primitive n^{th} roots of unity. The group D_n should be interpreted as the group of n^{th} roots of unity, on which $C_{\mu(n)}$ acts as the Galois group.

By $G_{n,m}$ we denote the subgroup of G_n, where the complement $C_m = \langle c_m \rangle$ is generated by an element of order m dividing $\mu(n)$. In addition, we require that m^2 divides n, and that m and n/m^2 are relatively prime.

A further assumption is that p is a rational prime, such that $\mathbb{F}_{p^m}$ is the smallest field of characteristic p containing all n^{th} roots of unity. Then $M = \mathbb{F}_{p^m}$ is an irreducible $G_{n,m}$-module, on which d_n acts by multiplication with a primitive n^{th} root of unity, and c_m acts as the Frobenius automorphism, i.e., raising to the p^{th} power.

By $G_{n,m,p}$ we denote the semi-direct product $M \rtimes G_{n,m}$. Let $b_m = d_n^{n/m^2}$ be an element of order m^2 in D_n. There is no loss of generality if we assume that $b_m^{c_m} = b_m^{m+1}$. We note that b_m^m lies in the center of $G_{n,m}$.

The authors of [KR] then show that the element

$$(c_m b_m^i)^m = b_m^{im},$$

is central, and then proceed to define the subgroups which meet the criteria of Theorem 9.4.

Let $1 \leq i \leq m-1$ be relatively prime to m, and define the group $H_{n,m,i}$ as a subgroup of $G_{n,m}$ by

$$H_{n,m,i} = \langle d_n^{m^2}, b_m^m, c_m b_m^i \rangle,$$

and put

$$H_{n,m,i,p} = M \rtimes H_{n,m,i},$$

the semi-direct product with the module M.

An example of such groups are the subgroups of $\mathbb{F}_{7^3} \rtimes (\mathbb{F}_{7^3}^* \times C_3)$. Then $d^9 \in \mathbb{F}_{7^3}$ has order 19, $b \in \mathbb{F}_{7^3}$ has order 9, c generates C_3, and

$$H_{n,m,i} = \langle d^9, b^3, cb^i \rangle,$$

for $i = 1, 2$. Then

$$H_{n,m,i,p} = \mathbb{F}_{7^3} \rtimes \langle d^9, b^3, cb^i \rangle.$$

These $H_{n,m,i,p}$ are Frobenius groups with kernel M, and the $H_{n,m,i}$ are isomorphic for all i such that $1 \leq i \leq m-1$ and i is relatively prime to m. Also, for each $1 \leq i \leq m-1$, the orbits of $H_{n,m,i}$ are the same on M [KR, pg. 279]. This is the same as stating that the relative M-classes of $H_{n,m,i,p}$ inside M are the same for all i relatively prime to m, $(1 \leq i \leq m-1)$. Since $H_{n,m,i,p}$, is a Frobenius group with kernel M, $(H_{n,m,i,p}, M)$ is a Camina pair. Thus the groups $H_{n,m,i,p}$ satisfy all the hypotheses of Theorem 9.4. □

Chapter 10. Relative Weak Cayley Table Map Group of $AGL(1, p)$

As in Chapter 2 let $G = AGL(1, p)$ have the presentation

$$G = \langle a, b | a^{p-1} = b^p = 1, a^b = b^r \rangle,$$

and let B be the subgroup $\langle b \rangle$.

Then the B-conjugacy classes of G are the singletons $\{1\}$, $\{b\}$, $\{b^2\}$, $\cdots$ $\{b^{p-1}\}$, and the cosets of the form $a^i B$ where $i \neq 0$.

Since every weak Cayley table map sends B to itself, we can define a map

$$\Psi : RWCT(G, B) \to WCT(G/B)$$

to be the restriction of the map $\Phi : WCT(G, B) \to WCT(G/B)$ (as defined in chapter 2) to the subgroup $RWCT(G, B)$. Then let $K = Ker(\Psi)$.

If $\psi \in K$ and is also a weak Cayley table map, then from Lemma 2.2 in Chapter 2, we know that $\psi(a^i b^j) = a^i b^{\alpha(i,j)}$ where $\alpha(i, j)$ is an injective function on $(F)_p$ to itself such that $\alpha(0, 0) = 0$, and $-r^{-i}\alpha(i, j) = \alpha(-i, -r^{-i}j)$ for every $0 \leq i \leq p-2$ and $0 \leq j \leq p-1$.

Also $\psi|_B$ has to be the identity map on B, since the B-conjugacy classes on B are singletons. Also, since B is abelian, $\psi|_B$ must be an automorphism.

Lemma 10.1. *Let β be an automorphism of B. Then map $\beta^* : G \to G$ which sends $b \to \beta(b)$ and $a^i b \to a^i \beta(b)$ is an automorphism.*

Proof. Let $a^i b^j$, $a^k b^l$ be in G, then using the relations established in Chapter 2, we have:

$$\beta^*(a^i b^j a^k b^l) = \beta^*(a^{i+k} b^{r^i j + l}) = a^{i+k} \beta(b^{r^i j + l}),$$

and

$$\beta^*(a^ib^j)\beta^*(a^kb^l) = a^i\beta(b^j)a^k\beta(b^l) = a^{i+k}\beta(b^j)^{r^i}\beta(b^l) = a^{i+k}\beta(b^{r^ij+l}).$$

□

Similar to chapter 2, we can construct a relative weak Cayley table map in the kernel K by permuting elements of cosets and respecting their inverse cosets. For a^iB, for $1 < i \leq \frac{p-3}{2}$ pick any permutation on the elements of the coset. This determines a permutation of its inverse class. Also there is one coset of involutions, $a^{\frac{p-1}{2}}B$ and any permutation of the elements of this coset will preserve inverses. So from these permutations, we have a subgroups of $RWCT(G, B)$ isomorphic to $S_p \times S_p^{\frac{p-3}{2}}$.

Then, given one of the above maps, we can compose it with an automorphism like those in Lemma 10.1 to get permissable permutations of the elements in B. These give you all of the maps ψ in the kernel K. This means

$$K \cong (S_p \times S_p^{\frac{p-3}{2}}) \rtimes Aut(B).$$

Then for any relative weak Cayley table map, we can view it as a composition of permutations on the nontrivial cosets of B composed with an element of the kernal K. As above in chapter 2, this gives a subgroup of $WCT(G)$ isomorphic to

$$((S_p \times S_p^{\frac{p-3}{2}}) \rtimes Aut(B)) \rtimes Cox_B(\frac{p-3}{2}).$$

Chapter 11. Automorphisms, Anti-Automorphisms and $RWCT(G, H)$

Note that automorphisms always satisfy the requirements for a relative Weak Cayley table map for $G_1 = G_2$, since they are isomorphisms. However, while anti-automorphisms are weak Cayley table maps, they are not always relative weak Cayley table maps.

Example 11.1. Consider the group S_3. The relative-S_2 conjugacy classes are

$$B_1 = \{1\}, \quad B_2 = \{(12)\}, \quad B_3 = \{(13),(23)\}, \quad B_4 = \{(123),(132)\}.$$

Since we can write any anti-automorphism as an automorphism composed with the inverse map, it is sufficient to check if the inverse map is a relative weak Cayley table map.

However, note that the inverse map $\alpha : S_3 \to S_3$ given by $\alpha(g) = g^{-1}$ fails to be a relative weak Cayley table map, since

$$\begin{aligned} \alpha((132)(13)) &= \alpha((12)) \\ &= (12). \end{aligned}$$

However,

$$\begin{aligned} \alpha((132))\alpha((13)) &= (123)(13) \\ &= (23). \end{aligned}$$

Note that (12) is not S_2-conjugate to (23), so the inverse map α fails to be a relative weak Cayley table map, which implies that no anti-automorphisms of S_3 are in $RWCT(S_3, S_2)$.

Theorem 11.2. *Given a group G with a subgroup H, $RWCT(G, H)$ contains the anti-automorphisms if and only if for every $a \notin H, Ha \cap C_G(g) \neq \emptyset$ for all $g \in G$.*

Proof. Since any anti-automorphism can be expressed as an automorphism composed with the inverse map, it is sufficient to find when the inverse map $\alpha : G \to G$ is in $RWCT(G, H)$.

Note that since α permutes H-conjugacy classes, $\alpha \in RWCT(G, H)$ is equivalent to $\alpha(ab) \sim_H \alpha(a)\alpha(b)$ for all $a, b \in G$. Then since

$$\alpha(ab) = (ab)^{-1} = b^{-1}a^{-1},$$

and

$$\alpha(a)\alpha(b) = a^{-1}b^{-1},$$

the statement $\alpha(ab) \sim_H \alpha(a)\alpha(b)$ for all $a, b \in G$ is equivalent to $ab \sim_H ba$ for all $a, b \in G$. This is the same as $(ab)^h = ba$ for some $h \in H$ or $abh = hba$.

If either a or b are in H, then we can find $h \in H$ such that $abh = hba$. The reason for this is if $b \in H$, then take $h = b^{-1}$. Then

$$abh = abb^{-1} = a$$

and

$$hba = b^{-1}ba = a.$$

If $a \in H$, take $h = a$, then

$$abh = aba = hba.$$

So suppose a and b are not in H. Then we note that since $ba = (ab)^a$,

$$\begin{aligned}(ab)^h = ba = (ab)^a &\iff (ab)^{ha^{-1}} = ab \\ &\iff ha^{-1} \in C_G(ab) \\ &\iff Ha^{-1} \cap C_G(g) \neq \emptyset \text{ for all } g \in G, \text{ and all } a \in G - H.\end{aligned}$$

□

An example of a group and a subgroup that satisfies the hypotheses in Theorem 11.2 is the group $S_3 \times C_2$, with the subgroup S_3. Let $C_2 = \langle t \rangle$, then we can write the two cosets of S_3 are itself and S_3t.

Note that S_3t contains the element t, which is central. Certainly $t \in C_{S_3 \times C_2}(g)$ for every $g \in S_3 \times C_2$, so the pair $(S_3 \times C_2,\ S_3)$ fits the criteria of Theorem 11.2, and so anti-automorphisms of $S_3 \times C_2$ are elements of $RWCT(S_3 \times C_2,\ S_3)$.

Chapter 12. Relative Weak Cayley Table Maps and Characters

Let $\phi : (G_1, H_1) \to (G_2, H_2)$ be a relative weak Cayley table map from a group G_1 with subgroup H_1 to a group G_2 with subgroup H_2. Further, let $\chi_{1,i}$ be the irreducible characters of G_1, and let $\chi_{2,i}$ be the irreducible characters of G_2. Let $\psi_{1,i}$ be the irreducible characters of H_1 and let $\psi_{2,i}$ be the irreducible characters of H_2. Then define an action by ϕ on the characters χ_2 of G_2 by $\phi \cdot \chi_2 = \chi_2(\phi(g_1))$ where $g_1 \in G_1$. Thus for every character χ_2 of G_2 we obtain a function $\phi \cdot \chi_2 : G_1 \to \mathbb{C}$. We prove that

Proposition 12.1. *$\phi \cdot \chi_2$ is a character of G_1.*

Proof. Let g_1 be conjugate to k_1 in G_1. Since ϕ is a relative weak Cayley table map we have

$$\phi(g_1) \sim \phi(k_1).$$

Then considering that χ_2 is a character of G_2, we know that

$$\chi_2(\phi(g_1)) = \chi_2(\phi(k_1)).$$

This can be rewritten as $\phi \cdot \chi_2(g_1) = \phi \cdot \chi_2(k_1)$, which means that $\phi \cdot \chi_2$ is constant on the conjugacy classes of G_1, so $\phi \cdot \chi_2$ is a character of G_1. □

Proposition 12.2. *For a relative weak Cayley table map $\phi : (G_1, H_1) \to (G_2, H_2)$, and a character ψ_2 of G_2 as in the above, we have that $\phi \cdot \psi_2$ is an H_1-class function.*

Proof. The proof is very similar to the above proof for $\phi \cdot \chi_2$. Let $g_1 \sim_{H_1} k_1$ in G_1. Then since ϕ is a relative weak Cayley table map,

$$\phi(g_1) \sim_{H_2} \phi(k_1),$$

and

$$\psi(\phi(g_1)) = \psi(\phi(k_1)),$$

which shows $\phi \cdot \psi_2(g_1) = \phi \cdot \psi_2(k_1)$. Therefore $\phi \cdot \psi_2$ is an H_1-class function. □

Theorem 12.3. *The character $\phi \cdot \chi_{2,i}$ is irreducible if and only if $\chi_{2,i}$ is irreducible.*

Proof. Since $\chi_{2,i}$ is irreducible, we know that the inner product

$$(\chi_{2,i}, \chi_{2,i}) = \frac{1}{|G_2|} \sum_{g_2 \in G_2} \chi_{2,i}(g_2)\overline{\chi_{2,i}(g_2)} = 1.$$

Then the inner product of $\phi \cdot \chi_{2,i}$ with itself is

$$\begin{aligned}(\phi \cdot \chi_{2,i}, \phi \cdot \chi_{2,i}) &= \frac{1}{|G_1|} \sum_{g_1 \in G_1} \phi \cdot \chi_{2,i}(g_1)\overline{\phi \cdot \chi_{2,i}(g_1)} \\ &= \frac{1}{|G_1|} \sum_{g_1 \in G_1} \chi_{2,i}(\phi(g_1))\overline{\chi_{2,i}(\phi(g_1))}.\end{aligned}$$

Since ϕ is a relative weak Cayley table map, we can rewrite this in terms of G_2:

$$\begin{aligned}(\phi \cdot \chi_{2,i}, \phi \cdot \chi_{2,i}) &= \frac{1}{|G_2|} \sum_{g_2 \in G_2} \chi_{2,i}(g_2)\chi_{2,i}(\overline{g_2}) \\ &= (\chi_{2,i}, \chi_{2,i}) = 1.\end{aligned}$$

So $\phi \cdot \chi_{2,i}$ is an irreducible character of G_1. The other implication is similar. □

Theorem 12.4. *The H-class function $\phi \cdot \psi_{2,i}$ is irreducible if and only if $\psi_{2,i}$ is irreducible.*

Proof. Assume that $\psi_{2,1}$ is an irreducible H-class function. Then the inner product of $\phi \cdot \psi_{2,i}$ with itself is

$$\begin{aligned}(\phi \cdot \psi_{2,i}, \phi \cdot \psi_{2,i}) &= \frac{1}{|H_1|} \sum_{h_1 \in H_1} \phi \cdot \psi_{2,i}(h_1) \overline{\phi \cdot \psi_{2,i}(h_1)} \\ &= \frac{1}{|H_1|} \sum_{h_1 \in H_1} \psi_{2,i}(\phi(h_1)) \overline{\psi_{2,i}(\phi(h_1))}.\end{aligned}$$

Then, since ϕ maps H_1 into H_2 bijectively, we can rewrite this expression as

$$\begin{aligned}(\phi \cdot \psi_{2,i}, \phi \cdot \psi_{2,i}) &= \frac{1}{|H_2|} \sum_{h_2 \in H_2} \psi_{2,i}(h_2) \overline{\psi_{2,i}(h_2)} \\ &= (\psi_{2,i}, \psi_{2,i}) = 1.\end{aligned}$$

Thus $\phi \cdot \psi_{2,i}$ is an irreducible character of H_1. □

Definition 12.5. Given a group G with a subgroup H, we call the map $\psi : G \to \mathbb{C}$ an *H-class function* of G if ψ is constant on the H-classes.

Theorem 12.6. *A relative weak Cayley table map $\phi : (G_1, H_1) \to (G_2, H_2)$ determines a correspondence between the irreducible characters $\chi_{1,1}, \chi_{1,2}, \ldots, \chi_{1,s}$ of G_1 and $\chi_{2,1}, \chi_{2,2}, \ldots, \chi_{2,s}$ of G_2, and a correspondence between the irreducible H_1-characters $\psi_{1,1}, \psi_{1,2}, \ldots, \psi_{1,r}$ on G_1 and the H_2-characters $\psi_{2,1}, \psi_{2,2}, \ldots, \psi_{2,r}$ on G_2.*

Proof. Above we showed that $\phi \cdot \chi_{2,i}$ is an irreducible character of G_1 obtained from an irreducible character $\chi_{2,i}$ of G_2. Note that G_1 and G_2 have the same number of irreducible characters. Thus it is sufficient to show that if $\chi_{2,i}$ and $\chi_{2,j}$ are two distinct irreducible characters of G_2, then $\phi \cdot \chi_{2,i} \neq \phi \cdot \chi_{2,j}$. This will complete the correspondence required.

If H_2-characters $\chi_{2,i}$ and $\chi_{2,j}$ of G_2 are distinct, then for some element $g_2 \in G_2$, $\chi_{2,i}(g_2) \neq \chi_{2,j}(g_2)$. Further since ϕ is a bijection and $\phi^{-1}(g_2)$ is an element in G_1, we find

$$\begin{aligned} \phi \cdot \chi_{2,i}(\phi^{-1}(g_2)) &= \chi_{2,i}(\phi(\phi^{-1}(g_2))) \\ &= \chi_{2,i}(g_2), \end{aligned}$$

and

$$\begin{aligned} \phi \cdot \chi_{2,j}(\phi^{-1}(g_2)) &= \chi_{2,j}(\phi(\phi^{-1}(g_2))) \\ &= \chi_{2,j}(g_2). \end{aligned}$$

So $\phi \cdot \chi_{2,i}$ and $\phi \cdot \chi_{2,j}$ must be distinct, irreducible characters of G_1.

The same argument on the irreducible H_1-characters and H_2-characters show the correspondence for the subgroups' characters. □

Chapter 13. Overview of Spherical functions

Most of the information in this section came from [Tr]. Spherical functions are very similar to characters. They are functions that are constant on the relative conjugacy classes for a particular subgroup, and they have many of the same properties that characters possess.

Definition 13.1. Let G be a finite group with subgroup H, let χ be a character of G, and let ψ be a character of H. Then the *spherical function* $Y_{\chi\psi} : G \to \mathbb{C}$ is defined as

$$Y_{\chi\psi}(g) = \frac{1}{|H|} \sum_{\sigma \in H} \chi(g\sigma)\psi(\sigma^{-1}).$$

The following properties of spherical fuctions can be found in [Tr]:

(i) $Y_{\chi\psi}(1) = (\chi|_H, \psi)$,

(ii) $Y_{\chi\psi}(g^{-1}) = \overline{Y_{\chi\psi}(g)}$,

(iii) For h in H, $Y_{\chi\psi}(hgh^{-1}) = Y_{\chi\psi}(g)$,

(iv) Let f_χ be the degree of χ and let ξ be another character of G. Then

$$Y_{\chi\psi} * \xi = \delta_{\chi\xi} \frac{1}{f_\chi} Y_{\chi\psi}.$$

where $*$ denotes the convolution. In other words if ϕ_1 and ϕ_2 are functions from G into the complex numbers, then

$$(\phi_1 * \phi_2)(g) = \frac{1}{|G|} \textstyle\sum_{h \in G} \phi_1(gh^{-1})\phi_2(h). \text{ [Ga]}$$

(v) Let ϕ be another character of H, and let $\overline{\phi}$ represent the function of G that vanishes off H, and is equal to $|G : H| \cdot \phi$ on H, then

$$Y_{\chi\psi} * \overline{\phi} = \delta_{\phi\psi} \frac{1}{f_\phi} Y_{\chi\psi},$$

and

$$Y_{\chi\psi} * Y_{\xi\phi} = \delta_{\chi\xi} \delta_{\phi\psi} \frac{1}{f_\chi f_\psi} Y_{\chi\psi}.$$

(vi) The regular representation can be written as:

$$R(g) = \sum_{\chi,\psi} f_\chi f_\psi Y_{\chi\psi}(g).$$

(vii)

Theorem 13.2. [Tr, Theorem 1] *The following are equivalent*

(a) $Y_{\chi\psi}$ *is a G-class function;*

(b) $Y_{\chi\psi}$ *is proportional to* χ*;*

(c) $\chi|_H = c_{\chi\psi} \cdot \psi$.

(viii)

Theorem 13.3. [Tr, Theorem 1′] *The following are equivalent*

(a) $Y_{\chi\psi}$ *vanishes off of* H*;*

(b) $Y_{\chi\psi}$ *is proportional to* $\overline{\psi}$*;*

(c) $\psi^G = c_{\chi\psi} \cdot \chi$.

Chapter 14. Relative Weak Cayley Tables and Spherical Functions

For notation in this chapter let G_1, G_2 be groups with H_1, H_2 as subgroups respectively, such that $\phi : (G_1, H_1) \to (G_2, H_2)$ is a relative weak Cayley table map. Also let $\chi_{1,1}, \chi_{1,2}, \dots, \chi_{1,s}$ be the irreducible characters of G_1, and let $\chi_{2,1}, \chi_{2,2}, \dots, \chi_{2,s}$ be those of G_2. Further let $\psi_{1,1}, \psi_{1,2}, \dots, \psi_{1,s}$ be the irreducible H_1-characters and $\psi_{2,1}, \psi_{2,2}, \dots, \psi_{2,s}$ be those of H_2.

Theorem 14.1. *Let χ_2 be an irreducible character for G_2, and let $\chi_1 = \phi \cdot \chi_2$ be the corresponding irreducible character in G_1. Also let ψ_2 be an irreducible character for H_2, and let $\psi_1 = \phi \cdot \psi_2$ be the corresponding irreducible character in H_1. Then*

$$Y_{\chi_2\psi_2}(g_2) = Y_{\chi_1\psi_1}(\phi^{-1}(g_2))$$

for all $g_2 \in G_2$.

Proof. By the definition of a spherical function

$$\begin{aligned} Y_{\chi_1\psi_1}(\phi^{-1}(g_2)) &= \frac{1}{|H_1|}\sum_{\sigma\in H_1} \chi_1(\phi^{-1}(g_2)\sigma)\psi_1(\sigma^{-1}) \\ &= \frac{1}{|H_1|}\sum_{\sigma\in H_1} \chi_1(\phi^{-1}(g_2)\phi^{-1}(\phi(\sigma)))\psi_1(\sigma^{-1}). \end{aligned}$$

Then because ϕ is a relative weak Cayley table map, $\phi^{-1}(g_2)\phi^{-1}(\phi(\sigma)) \sim_{H_1} \phi^{-1}(g_2\phi(\sigma))$ and since $Y_{\chi_1\psi_1}$ is constant on conjugacy classes, we can rewrite the above equation as:

$$\begin{aligned}
Y_{\chi_1\psi_1}(\phi^{-1}(g_2)) &= \frac{1}{|H_1|}\sum_{\sigma\in H_1}\chi_1(\phi^{-1}(g_2\phi(\sigma)))\psi_1(\sigma^{-1}) \\
&= \frac{1}{|H_1|}\sum_{\sigma\in H_1}\chi_2(\phi(\phi^{-1}(g_2\phi(\sigma))))\psi_2(\phi(\sigma^{-1})) \\
&= \frac{1}{|H_1|}\sum_{\sigma\in H_1}\chi_2(g_2\phi(\sigma))\psi_2(\phi(\sigma^{-1})).
\end{aligned}$$

Since ϕ is a bijection between H_1 and H_2, this becomes

$$\begin{aligned}
&= \frac{1}{|H_2|}\sum_{\sigma\in H_2}\chi_2(g_2\sigma)\psi_2(\sigma^{-1}) \\
&= Y_{\chi_2\psi_2}(g_2).
\end{aligned}$$

□

Chapter 15. An Example of Relative Weak Cayley Table Maps With Spherical Functions

The goal of this Chapter is to illustrate Theorem 14.1 with the non-isomorphic, non-abelian groups of order p^3.

Example 15.1. For a prime p, let G_1 and G_2 be the two non-isomorphic, non-abelian groups of order p^3. As in Chapter 7, let them have the following presentations:

$$G_1 = \langle a, b, c : a^p = b^p = c^p = 1, b^a = bc, a^c = a, b^c = b \rangle;$$

$$G_2 = \langle x, y, z : x^p = z, x^{p^2} = y^p = z^p = 1, x^y = x^{p+1}, x^z = x, y^z = y \rangle.$$

Let $H_1 = \langle a \rangle$ and $H_2 = \langle y \rangle$. In Chapter 7 we defined a bijection $\phi : G_1 \to G_2$ which is a relative weak Cayley table map from (G_1, H_1) to (G_2, H_2). Recall that ϕ was defined to be the map

$$a^r b^s c^t \to x^s y^r z^{rs-t}.$$

Note that G_1 and G_2 have the same character table. The irreducible characters of G_1 are

$$\chi_{1:u,v}(a^r b^s c^t) = \epsilon^{ru+sv},$$

and

$$\xi_{1:u}(a^r b^s c^t) = \begin{cases} 0 & \text{if } a \neq 0 \text{ or } b \neq 0; \\ p\epsilon^{ut} & \text{otherwise;} \end{cases}$$

where $0 \leq u, v \leq p-1$ and ϵ is a primitive p^{th} root of unity [JL, p.301-304]. For G_2 we have

$$\chi_{2:u,v}(x^s y^r z^t) = \epsilon^{ru+sv},$$

and

$$\xi_{1:u}(x^s y^r z^t) = \begin{cases} 0 & \text{if } x \neq 0 \text{ or } y \neq 0; \\ p\epsilon^{ut} & \text{otherwise.} \end{cases}$$

As shown above, we can act on the character $\chi_{2:u,v}$ of G_2 by ϕ to get an irreducible character of G_1:

$$\begin{aligned} \phi \cdot \chi_{2:u,v}(a^r b^s c^t) &= \chi_{2:u,v}(\phi(a^r b^s c^t)) \\ &= \chi_{2:u,v}(x^s y^r z^{-t}) \\ &= \epsilon^{ru+sv} \\ &= \chi_{1:u,v}(a^r b^s c^t). \end{aligned}$$

And if we look at the irreducible characters of H_1 these are just $\psi_{1:j}(a^i) = \epsilon^{ij}$ since H_1 is cyclic of order p. (For H_2 the irreducible characters are $\psi_{2:j}(y^i) = \epsilon^{ij}$.)

Then to examine the spherical functions of G_1 with H_1, we notice

$$\begin{aligned} Y_{\chi_{1:u,v},\psi_{1:j}}(a^r b^s c^t) &= \frac{1}{|H|} \sum_{y^i \in H} \chi_{1:u,v}(a^r b^s c^t a^i) \psi_{1:j}(a^{-i}) \\ &= \frac{1}{p} \sum_{y^i \in H} \chi_{1:u,v}(a^i a^r b^s c^t) \psi_{1:j}(a^{-i}), \end{aligned}$$

since $a^r b^s c^t a^i \sim_{G_1} a^i a^r b^s c^t$, so

$$
\begin{aligned}
Y_{\chi_{1:u,v},\psi_{1:j}}(a^r b^s c^t) &= \frac{1}{p}\sum_{y^i \in H} \chi_{1:u,v}(a^{r+i} b^s c^t)\psi_{1:j}(a^{-i}) \\
&= \frac{1}{p}\sum_{i=1}^{p} \epsilon^{(r+i)u+sv}\epsilon^{-ij} \\
&= \frac{1}{p}\epsilon^{ru+sv}\sum_{i=1}^{p} \epsilon^{i(u-j)}.
\end{aligned}
$$

If $u = j$ this becomes

$$
\begin{aligned}
Y_{\chi_{1:u,v},\psi_{1:u}}(a^r b^s c^t) &= \frac{1}{p}\epsilon^{ru+sv}\sum_{i=1}^{p} \epsilon^{i(0)} \\
&= \epsilon^{ru+sv},
\end{aligned}
$$

and if $u \neq j$ then $\epsilon^{i(u-j)}$ runs over all the roots of unity, so we get

$$
\begin{aligned}
Y_{\chi_{1:u,v},\psi_{1:u}}(a^r b^s c^t) &= \frac{1}{p}\epsilon^{ru+sv}\sum_{i=1}^{p} \epsilon^{i(u-j)} \\
&= \frac{1}{p}\epsilon^{ru+sv}(0)
\end{aligned}
$$

which implies

$$
Y_{\chi_{1:u,v},\psi_{1:u}}(a^r b^s c^t) = 0.
$$

To summarize,

$$
Y_{\chi_{1:u,v},\psi_{1:u}}(a^r b^s c^t) = \begin{cases} \epsilon^{ru+sv} & \text{if } u = j; \\ 0 & \text{otherwise.} \end{cases}
$$

Next if we consider the other characters, $\xi_{1:u}$, on G_1, we notice

$$\begin{aligned} Y_{\xi_{1:u},\psi_{1:j}}(a^r b^s c^t) &= \frac{1}{p}\sum_{y^i \in H} \xi_{1:u}(a^r b^s c^t a^i)\psi_{1:j}(a^{-i}) \\ &= \frac{1}{p}\sum_{y^i \in H} \xi_{1:u}(a^{r+i} b^s c^t)\psi_{1:j}(a^{-i}). \end{aligned}$$

Because of how $\xi_{1:u}$ is defined, when $r + i \neq 0$ we have that $\xi_{1:u}(a^{r+i}b^s c^t) = 0$. Thus

$$\begin{aligned} \frac{1}{p}\sum_{y^i \in H} \xi_{1:u}(a^{r+i} b^s c^t)\psi_{1:j}(a^{-i}) &= \frac{1}{p}\xi_{1:u}(a^{r-r} b^s c^t)\psi_{1:j}(a^r) \\ &= \frac{1}{p}\xi_{1:u}(b^s c^t)\psi_{1:j}(a^r). \end{aligned}$$

Further $\xi_{1:u}(b^s c^t) = 0$ if $s \neq 0$.Therefore if $s \neq 0$,

$$Y_{\xi_{1:u},\psi_{1:j}}(a^r b^s c^t) = 0,$$

and if $s = 0$,

$$\begin{aligned} Y_{\xi_{1:u},\psi_{1:j}}(a^r c^t) &= \frac{1}{p}\xi_{1:u}(c^t)\psi_{1:j}(a^r) \\ &= \frac{1}{p}(p\epsilon^{ut})(\epsilon^{rj}) = \epsilon^{ut+rj}. \end{aligned}$$

To summarize, these spherical functions are of the form

$$Y_{\xi_{1:u,v},\psi_{1:u}}(a^r b^s c^t) = \begin{cases} 0 & \text{if } s \neq 0; \\ \epsilon^{ut+rj} & \text{if } s = 0. \end{cases}$$

The same sort of calculations will result in similar spherical functions for G_2 with H_2.

Lastly we note the inverse of $x^r y^s z^t$ is

$$\phi^{-1}(x^r y^s z^t) = a^s b^r c^{-t}$$

and the action of ϕ on $\chi_{2:u,v}$ and $\psi_{2:j}$ gives

$$\phi \cdot \chi_{2:u,v} = \chi_{1:u,v} \text{ and } \phi \cdot \psi_{2:j} = \psi_{1:j}.$$

Thus $\chi_{1:u,v}$, $\chi_{2:u,v}$, $\psi_{1:j}$, and $\psi_{2:j}$ satisfy the hypothesis to Theorem 14.1. Then notice

$$Y_{\chi_{2:u,v}\psi_{2:u}}(x^r y^s z^t) = \frac{1}{p}\epsilon^{ru+sv}$$

and

$$Y_{\chi_{1:u,v}\psi_{1:u}}(a^s b^r c^{-t}) = \frac{1}{p}\epsilon^{ru+sv}.$$

So we have

$$Y_{\chi_{2:u,v}\psi_{2:u}}(x^r y^s z^t) = Y_{\chi_{1:u,v}\psi_{1:u}}(\phi^{-1}(x^r y^s z^t)),$$

which is the conclusion to Theorem 14.1.

Chapter 16. Questions for further research

Here we list questions for future work.

(i) If H is normal in G what can one say about the subgroup $RWCT(G, H)$ of $WCT(G)$?

(ii) For any subgroup, when is $RWCT(G, H)$ a normal subgroup of $WCT(G)$?

Example 16.1. $RWCT(S_3)$ does not contain anti-automorphisms, then $RWCT(G)$ is strictly contained in $WCT(G)$. Also $WCT(S_3)$ is trivial, meaning it is only composed of automorphisms and anti-automorphisms, which means that $RWCT(G, H)$ has index 2, and therefore is normal.

(iii) Are there any non-trivial groups G such that $WCT(G) = RWCT(G, H)$?

(iv) Are the anti-automorphisms of S_4 in $RWCT(S_4, S_3)$?

(v) Given a subgroup H of G, if $\phi \in RWCT(G, H)$, then we have an induced map $\phi^* : \mathbb{C}G^H \to \mathbb{C}G^H$. What conditions do we need to go the other direction?

(vi) Do there exist non-isomorphic groups G_1, G_2 and $\phi : G_1 \to G_2$ a weak Cayley table map such that for every subgroup H contained in G_1, $\phi(g^H) = \phi(g)^{\phi(H)}$ and $\phi^*(RWCT(G_1, H) = RWCT(G_2, \phi(H))$?

(vii) What other applications do $RWCT(G, H)$ maps have to spherical functions?

(viii) What other conditions are equivalent to the condition: given a group G and a subgroup H, for every $a \notin H, Ha \cap C_G(a)$ is non-empty?

(ix) What conditions are needed for a weak Cayley table map to be a relative weak Cayley table map for some nontrivial subgroup?

(x) If $WCT(G)$ is not trivial, is there always a nontrivial relatively weak Cayley table map for some nontrivial subgroup?

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