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An Optimal Control and Options Theory Approach to Forecasting and Managing Sustainable Systems

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Abstract: Sustainable management of the natural systems is essential in the presence of anthropogenic and natural disasters. Fisher information based measure and hypothesis have been proposed to quantify the sustainability of natural systems, which are used in this work to formulate management objectives. Since uncertainty is omnipresent in natural systems, its consideration is important. Real options theory deals extensively with the representation and forecasting of uncertainty. The options theory, with the Ito mean reverting process, is used to represent uncertainty, and optimal control theory is used to derive time dependent decisions. The idea is implemented on a three species predator-prey model. The results highlight the effect of uncertainty on system dynamics and decisions. When the decisions are viewed as options available to be exercised, the importance of options theory and control theory in sustainability is evident.

Keywords: *Sustainability; Forecasting; Optimal Control; Real options theory*.

1. INTRODUCTION

Sustainable development, a multifaceted approach to manage the environmental, economic, and social resources, calls for the consideration of long term effects in all the decisions relevant to the society as a whole. Sustainable management of natural systems (e.g. lakes, forests etc.) has assumed importance due to the harmful effects of anthropogenic actions (e.g. global warming) as well as natural disasters (e.g. hurricane Katrina). Being embodied in a multi-disciplinary environment, a suitable mathematical measure of sustainability is essential to quantify sustainability and to communicate successfully amongst various fields. To this effect, Cabezas and Fath [2002] have proposed Fisher information based sustainability hypotheses, with particular focus on natural dynamic ecosystems, allowing the formulation of mathematical objectives of sustainability. To device the management decisions, natural regulation paths exhibited by ecosystems can be used to advantage. Furthermore, since the natural systems are constantly evolving, time dependent decisions to achieve sustainability are more effective. Optimal control theory, commonly used in engineering applications, can be applied to derive such time varying decision profiles. However, consideration of various uncertainties present in natural systems is important for a robust analysis of this problem. The representation of uncertainty is as important as its treatment. Real options theory used in finance extensively deals with uncertainties and proposes various methods to forecast and represent the uncertainties. The effect of these uncertainties on the decisions is important and must be well understood.

This work uses real options theory concepts for uncertainty representation and derives time dependent decisions using optimal control theory in the presence of these uncertainties to achieve FI based sustainability objectives. The application is illustrated using a three species predator-prey model. The next section describes the theoretical basics of this work. Section 3 presents the results for the predator-prey model, while section 4 discusses these results from options theory perspective. The article ends with conclusions in section 5.

2. BACKGROUND THEORY

2.1 FI AS SUSTAINABILITY MATRIX

Cabezas and Fath [2002] have proposed to use Fisher information (FI) [Fisher, 1922] from information theory to derive a measure for the sustainability of a system, and propose the sustainability hypothesis for natural systems. One of the interpretations of FI, relevant for the natural systems, is as a measure of the state of order or organization of a system or phenomenon [Frieden, 1998]. Here, organization refers to the distribution of the states in which the system exists. The central argument in the sustainability hypothesis is that the stability (static or dynamic) of a system is sufficient (but not necessary) for the sustainability of the state of the system. Fisher information, being an indicator of the system stability, is also an indicator of the sustainability of the state of the system. The sustainability hypothesis accordingly states that: the time-averaged Fisher information of a system in a persistent regime does not change with time. Any change in the regime will manifest itself through a corresponding change in Fisher information value [Cabezas and Fath, 2002]. The corollaries to this hypothesis state that increase in the FI of a system ensures that the system is maintaining its state of organization, and they give an idea about the quality of change, if the system is changing its state.

Based on the sustainability hypotheses, from an ecosystem management perspective, two different objectives are formulated: minimization of the Fisher information variance over time, and maximization of time averaged Fisher information.

2.2 ECOSYSTEM REPRESENTATION: FOOD CHAIN MODEL

The natural system considered for this study is represented by a food chain model. It is a three species predator-prey model (Rosenzweing-MacArthur model) which has been extensively used in theoretical ecology [Abrams and Roth, 1994; Gragnani et al., 1998]. The model is given by the following set of differential equations:

$$\frac{dx_1}{dt} = x_1\left[r\left(1-\frac{x_1}{K}\right)-\frac{a_2x_2}{b_2+x_1}\right] \quad (1)$$

$$\frac{dx_2}{dt} = x_2\left[e_2\frac{a_2x_1}{b_2+x_1}-\frac{a_3x_3}{b_3+x_2}-d_2\right] \quad (2)$$

$$\frac{dx_3}{dt} = x_3\left[e_3\frac{a_3x_2}{b_3+x_2}-d_3\right] \quad (3)$$

where, x_1 (prey), x_2 (predator) and x_3 (super-predator) are the population variables of three different species in the food chain, in the ascending order of the position in the chain. r and K are the prey growth rate and the prey carrying capacity, respectively, and a_i, b_i, e_i and d_i, $i = 2, 3$, are the maximum predation rate, half saturation constant, efficiency, and death rate of the predator ($i = 2$) and the super-predator ($i = 3$).

There are various possible sources of uncertainty in this model. In this work, the mortality rate of the predator (d_2) is considered to be uncertain. Since the predator mortality rate is expected to show time dependent variations, such as seasonal variations in mortality, it is modelled as a stochastic process. Real options theory is used to achieve the task of uncertainty representation, which is explained in the next section.

2.3 UNCERTAINTY REPRESENTATION: REAL OPTIONS THEORY

Most investment decisions share three important characteristics in varying degrees. First, the investment is partially or completely "irreversible". In other words, the initial cost of investment is at least partially "sunk"; you cannot recover it all should you change your mind. Second, there is "uncertainty" over the future rewards from the investment. The best you can do is to assess the probabilities of the alternative outcomes that can mean greater or smaller profit (or loss) for your venture. Third, you have some leeway on the "timing" of your investment. You can postpone action to get more information but not with complete certainty. These three characteristics interact to determine the optimal decisions or "options" for investors. A firm with an opportunity is holding an "option" to buy an asset at some future time of its choosing. When a firm makes an irreversible expenditure, it exercises, or "kills", its option to invest. This lost option value is an opportunity cost that must be included as part of the cost of investment. Opportunity cost is highly sensitive to the uncertainty over the future value of the project. In an analogous fashion irreversibility, uncertainty, and timing issues are also important for sustainability [Diwekar, 2003; Dixit and Pindyck, 1994]. Forecasting in the context of sustainability is similar to financial decision making [Diwekar, 2005].

Real options theory presents different ways to represent and forecast uncertainty using the stochastic processes. Wiener process, also known as Brownian motion, is a simple continuous time and continuous state stochastic process. It can be used to model a variety of continuous stochastic processes [Dixit and Pindyck, 1994; Diwekar, 2003]. The Wiener process is represented as:

$$dz = \epsilon_t \sqrt{dt} \tag{4}$$

where, dz is the random variable, and ϵ_t is a normally distributed random variable, with zero mean and unit standard deviation. Random variable dz has the property that the expectation is zero ($E[dz] = 0$) and the variance is dt ($var[dz] = dt$). Using this definition of the Wiener process, the general form of the Ito process is given as:

$$dx = a(x,t)\ dt + b(x,t)\ dz \tag{5}$$

Here, $x(t)$ is the continuous time stochastic variable that is to be modelled, and dz is the Wiener increment, as defined by Eq. (4). $a(x,t)$ (drift parameter) and $b(x,t)$ (variance parameter) are known (nonrandom) functions. Many stochastic processes, such as the simple Brownian motion with drift, the geometric Brownian motion and the mean reverting Ito process are derived from Eq. (5). In this work, the mean reverting Ito process is used to model the predator mortality rate (d_2). The characteristic of the mean reverting Ito process is that, although it models the random variable fluctuations for a short time, in the long run, the variable is drawn back to the mean value. The mean reverting process has been used to model many stochastic variables, such as crude oil and copper prices [Dixit and Pindyck, 1994], relative volatility of non-ideal mixtures [Ulas and Diwekar, 2004] and also the human mortality rate [Diwekar, 2005]. The equation for the mortality rate as a mean reverting process is given as:

$$\frac{dx_4}{dt} = \eta(\bar{x}_4 - x_4) + \frac{\sigma\ \epsilon}{\sqrt{\Delta t}} x_4 \tag{6}$$

Here, x_4 represents the stochastic predator mortality rate, η is the speed of reversion, σ is the constant variance parameter, and $\bar{x}_4$ is the mean mortality rate to which x_4 tends to revert. The expected change in x_4 depends on the difference between x_4 and $\bar{x}_4$. One instance of the predator mortality represented as an Ito process is shown in figure 1 ($\bar{x}_4 = 1$). Eq. (6), along with Eqs. (1)-(3), represent the stochastic tri-trophic food chain model.

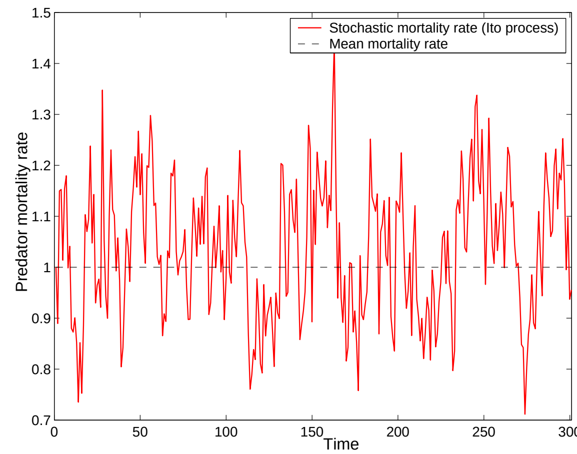

Figure 1. Predator mortality as an Ito process.

2.4 Control Problem Formulation

The food chain model described the preceding section is used to compare various control philosophies proposed in the literature to manipulate the food chain. This includes top-down control (control by manipulating the super-predator mortality) and bottom-up control (control by manipulating the prey carrying capacity). The task is to devise the time dependent profiles of the decision variables, for which optimal control theory is used in this work.

In general, control refers to a closed loop system, where the desired operating point is compared with an actual operating point and a knowledge of difference is fed back to the system. Conventional frequency domain techniques are then used to design a controller. Optimal control problems on the other hand are defined in time domain, and their solution requires establishing an index of performance for the system and designing the course (future) of action so as to optimize the performance index [Diwekar, 1996].

For the deterministic case, Pontryagin's maximum principle can be used to formulate the optimal control problem. For the stochastic case though, methods based on the Ito's lemma need to be used. In this work, the recently proposed stochastic maximum principle is used [Rico-Ramirez et al., 2003; Rico-Ramirez and Diwekar, 2004]. The main advantage of using this approach is that the solution to the partial differential equations in dynamic programming formulation is avoided. Instead, a set of ordinary differential equations needs to be solved as a boundary value problem. The problem formulation is briefly explained below:

Consider a system represented by the following set of differential equations.

$$dx = f(x,u,t)\,dt + g\,dz \tag{7}$$

where, x is the state variable vector of dimension n ($x(t) \in R^n$), and u is the control variable vector of dimension m ($u(t) \in R^m$). The starting condition for the state vector is given by $x(t_0) = x_0$, and the final condition at time T is $x(T)$. Let $1, ..., n_k$ be the set of deterministic states, and $n_{k+1}, ..., n$ be the set of uncertain states. The second part of Eq. (7) models the uncertainty. For deterministic states, the function $g = 0$. In optimal control, there is a time dependent performance index, which, in this case, is represented as:

$$J(t_0) = \int_{t_0}^{T} F(x(t), u(t), t)dt \tag{8}$$

where, F is the function to be optimized over the time interval of $[t_0, T]$. The Hamiltonian for this stochastic case is defined as:

$$H(x,u,t) = F(x,u,t) + \lambda'\, f(x,u,t) + \frac{1}{2}\, g^2\, w \tag{9}$$

where, $\lambda(t)$ is the set of costate or adjoint variables ($\lambda(t) \in R^n$) (the first derivatives of the objective function F with respect to state variables), and λ' represents the matrix transpose. $\omega(t)$ represent the second derivatives of the objective function F with respect to the state variables. This term is included due to the Ito process contribution. The optimal control law is then given by the solution of the following set of equations:

State Equation

$$\dot{x_i} = \frac{\partial H}{\partial \lambda_i} = f \qquad i = 1, ..., n \tag{10}$$

Costate Equation

$$-\dot{\lambda}_i = \frac{\partial H}{\partial x_i} = \frac{\partial f'}{\partial x_i}\lambda_i + \frac{\partial F}{\partial x_i} \qquad i = 1, ..., n \tag{11}$$

$$\frac{dw_j}{dt} = -2\, w_j\, \frac{\partial}{\partial x_j} f_j - \frac{1}{2} w_j \frac{\partial^2}{\partial x_j^2}(g_j^2) - \lambda_j \frac{\partial^2}{\partial x_j^2} f_j \qquad j = n_{k+1}, ..., n \tag{12}$$

Stationarity Condition

$$0 = \frac{\partial H}{\partial u_p} = \frac{\partial F}{\partial u_p} + \frac{\partial f'}{\partial u_p}\lambda \qquad p = 1, ..., m \tag{13}$$

This is a set of $2n + (n - n_k)$ ordinary differential equations (state and costate equations) and m algebraic equations (stationarity condition), and it is solved as a boundary value problem. The boundary values of the state and costate variables depend on the problem specification, while the boundary values for w are given as $w(T) = 0$ [Rico-Ramirez and Diwekar, 2004]. The control trajectory obtained is optimal for the considered objective function and starting conditions.

Due to the complex set of algebraic and ordinary differential equations, analytic solution to the problem is not possible. Hence, numerical technique of steepest ascent of Hamiltonian is used. The next section discusses the important aspects of the results.

3. Results

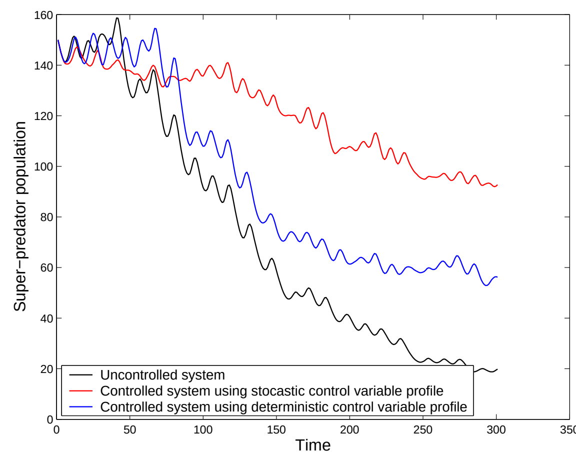

Figure 2. Predator mortality as an Ito process.

The presented problem was solved for various cases where human intervention to maintain sustainability is essential [Shastri and Diwekar, 2006a]. The results indicate that using bottom-up control with FI variance minimization objective ensures system stability and achieves the desired population dynamics in most cases. FI maximization objective causes more severe changes in the dynamics. A quantitative comparison with the results for the deterministic cases (reported in Shastri and Diwekar [2006b]) indicates that uncertainty impacts the relative extent of success or failure of a management option. It is also found that not only the presence of uncertainty, but also the degree of uncertainty, is important to rank various management options.

It is equally important to understand the impact of uncertainty on the decisions. For the stochastic systems, the control profiles are derived by representing the uncertain parameter using an Ito process and then using the stochastic maximum principle from the optimal control theory. By ignoring uncertainty, deterministic methods to derive control profile can be used. However, such an approach should lead to sub-optimal results since the effect of uncertainty on control variable is ignored. This is ascertained by conducting the following simulation study.

The response of the stochastic tri-trophic food chain model exhibiting super-predator extinction, when controlled by prey carrying capacity (bottom-up control) using FI maximization objective is plotted. The following two cases are considered:

- Stochastic system controlled by control variable profile generated using stochastic optimal control theory (stochastic maximum principle) as explained in this work
- Stochastic system controlled by control variable profile generated by using deterministic control theory i.e. not considering uncertainty to derive control profile

The plots for the super-predator population are shown in figure 2. They indicate that super-predator population is elevated much more using the control profile generated by stochastic optimal control theory. Although, using the control profile generated by the deterministic optimal control theory also restricts super-predator extinction, its performance is clearly inferior to the stochastic control variable profile. One can, therefore, conclude that the stochastic control gives a better result as compared to the deterministic control. This trend is observed, to a greater or lesser extent, for other cases too, highlighting the importance of uncertainty incorporation in control problem solution. Figure 3 compares the control variable profiles for the deterministic and stochastic cases. It can be seen that the two control profiles differ, particularly during the initial half of the simulation. The profile generated by stochastic maximum principle is not only better due to the resulting dynamics, but also because its magnitude of fluctuations is less than that for the deterministic control. This emphasizes that the effects of uncertainty on the decisions are important and significant.

4. Discussion

In real options theory, an option exists when the firm has the right but not the obligation to take action. Thus, not only the decisions, but also the timing of the decisions is optimized. As explained in section 2.3, realization of uncertainty can affect decisions and hence options theory allows one to take decisions in the presence of forecasting of these uncertainties.

The results for the stochastic food chain model presented in the previous section can be viewed from this perspective. When the stochastic model was controlled by a control profile derived for a deterministic system, realizations of the uncertain mortality rate were ignored. On the contrary, using the Ito process representation and stochastic maximum principle, the decisions were optimized in the presence of forecast, thereby accounting for the realizations of the uncertain parameter. The decision variable (prey carrying capacity) can be thought of as an option available to maximize the considered benefits. With the stochastic control, these options were optimally utilized by changing the magnitude and timing of the decisions. The decisions will have certain cost associated with them. For example, the bottom-up control in aquatic systems is usually affected by the addition of nutrients in the water body, and will incur certain cost to the regulatory agency. Optimization considering uncertainty forecast ensures that the best use of these resources is made, consequently minimizing the cost of the implementation to achieve the objectives. If the optimization of the cost of these decisions is also an objective, then the proposed way of deriving the

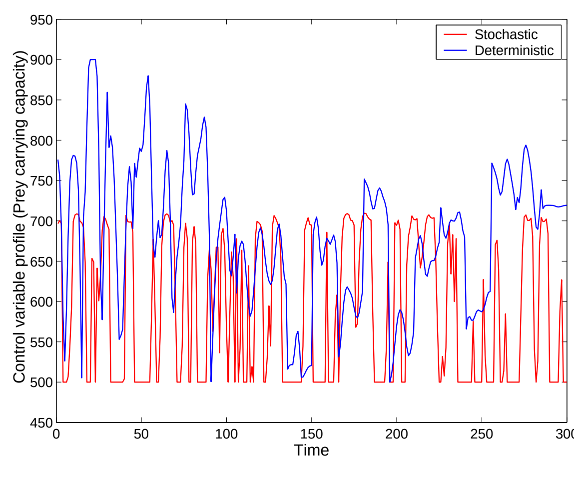

Figure 3. Predator mortality as an Ito process.

control profiles is a better alternative in the light of the options theory.

The uncertainty representation used here is based on finance literature, where uncertainty realizations and their probabilities are known. However, this may not be true for natural systems, where unexpected outcomes with unknown probabilities are possible, leading to unknown values of the decisions taken at the present time. Arrow and Fisher [1974] argue that these unknown values create a quasi-option for the decisions. Conrad [1980] showed that quasi-option value is equivalent to the expected value of information, while option value is equivalent to the expected value of perfect information. In this context, incorporation of quasi-options concepts in future will broaden the scope of this work, particularly for the natural systems. Furthermore, work in finance literature is mostly restricted to linear cases. Here however, the use of stochastic optimal control theory, based on rigorous mathematical concepts, allows one to extend the ideas to natural systems which are often nonlinear in nature.

5. CONCLUSION

Achieving sustainable natural systems through management level decisions is the goal behind the presented work. Since natural systems exhibit many sources of uncertainty, its incorporation in decision making is essential, where appropriate representation of uncertainty is very important. Real options theory deals with the aspects of uncertainty forecasting. In this work, the Ito mean reverting process is used to represent the uncertainty in the food chain model considered for this analysis. Optimal control theory is used to derive the time dependent control profiles. The results indicate that the uncertainties affect not only the resulting system dynamics, but also the decisions considerably. In this wake, the decisions can be viewed as the options with the decision makers, and using the proposed method, the decisions are optimized, analogous to the options theory.

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