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PROPERTIES OF THE ZERO FORCING NUMBER

by

Kayla Owens

A thesis submitted to the faculty of

Brigham Young University

in partial fulfillment of the requirements for the degree of

Master of Science

Department of Mathematics

Brigham Young University

August 2009

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BRIGHAM YOUNG UNIVERSITY

GRADUATE COMMITTEE APPROVAL

of a thesis submitted by

Kayla Owens

This thesis has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

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## ABSTRACT

### PROPERTIES OF THE ZERO FORCING NUMBER

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Master of Science

The zero forcing number is a graph parameter first introduced as a tool for solving the minimum rank problem, which is: Given a simple, undirected graph  $G$ , and a field  $F$ , let  $S(F, G)$  denote the set of all symmetric matrices  $A = [a_{ij}]$  with entries in  $F$  such that  $a_{ij} \neq 0$  if and only if  $ij$  is an edge in  $G$ . Find the minimum possible rank of a matrix in  $S(F, G)$ . It is known that the zero forcing number  $Z(G)$  provides an upper bound for the maximum nullity of a graph. I investigate properties of the zero forcing number, including its behavior under various graph operations.

## ACKNOWLEDGMENTS

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# 1 Introduction and Motivation

The zero forcing number is a graph parameter that was introduced in [AIM08], as a tool for solving the minimum rank problem, which is:

Given a graph  $G$  on  $n$  vertices, and a field  $F$ , define  $S(F, G)$  to be the set of all symmetric matrices  $A$  with entries in  $F$  satisfying the following condition:  $a_{i,j} \neq 0$  if and only if  $ij$  is an edge in  $G$ . There is no restriction on the diagonal entries of  $A$ . Then the *minimum rank of  $G$  over  $F$* ,  $\text{mr}(F, G) = \min\{\text{rank}(A) \mid A \in S(F, G)\}$ . Similarly, we can define the *maximum nullity of  $G$  over  $F$* ,  $\text{M}(F, G)$ . Since  $\text{mr}(F, G) + \text{M}(F, G) = n$ , we see that finding the minimum rank is equivalent to finding the maximum nullity. In general, finding the minimum rank of a graph is an open problem.

The idea of a zero forcing set, though under a different name (“infecting set”), was also introduced in [BG07] and [BM] as it relates to quantum systems. These papers demonstrate how zero forcing sets can be used to learn more about control, relaxation, structure of eigenstates, possible degeneracies, and in some cases, for Hamiltonian identification of many-bodied quantum systems. [BM]

This paper will focus mainly on the zero forcing parameter itself and how it behaves under various graph operations. Then the last section uses zero forcing, along with other tools, to solve the minimum rank problem in the case that the graph is obtained from  $W_5$ , the wheel on 5 vertices, by edge subdivisions. Because of the close relationship between minimum rank and zero forcing sets, results about minimum rank will be used to illustrate properties of the zero forcing number.

## 2 Terminology

This section gives precise definitions for terms we will use throughout the paper. The reader who is familiar with graph theory may safely skip to 2.3.

### 2.1 Graphs

**Definition 2.1.** A *graph*  $G$  is an ordered pair  $(V(G), E(G))$  of a nonempty *vertex* set  $V(G)$  and an *edge* set  $E(G)$  consisting of distinct unordered pairs of distinct elements of  $V(G)$ . We write an edge  $\{x, y\}$  as  $xy$ . The graph is more precisely called a *simple graph*.

**Definition 2.2.** A *trivial graph* is a graph whose edge set is empty.

**Definition 2.3.** An edge  $e = uv$  is said to *join* the vertices  $u$  and  $v$ , and  $u$  and  $v$  are each *incident* to  $e$ . If  $uv \in E(G)$  then  $u$  is *adjacent* to  $v$  and  $u$  is a *neighbor* of  $v$ . The *degree* of  $v$  is the number of neighbors of  $v$ . A degree 1 vertex is also called a *pendant vertex*.

**Definition 2.4.** A *subgraph*  $H$  of  $G$  is a graph with  $V(H) \subset V(G)$  and  $E(H) \subset E(G)$ . A subgraph  $H$  is an *induced subgraph* of  $G$  if, given two vertices  $u, v \in V(H)$ ,  $uv \in E(H)$  if and only if  $uv \in E(G)$ . In this case we say that  $V(H)$  *induces*  $H$  in  $G$ .

**Definition 2.5.** Two subgraphs  $G$  and  $H$  of a graph  $K$  are *disjoint* if their vertex sets are disjoint and there are no edges of  $K$  joining vertices in  $G$  to vertices in  $H$ . A graph is *connected* if it cannot be separated into two disjoint subgraphs.

**Definition 2.6.** A set of vertices is an *independent set* if its vertices are pairwise non-adjacent. A *bipartite* graph is a graph whose vertices can be partitioned into two independent sets.

**Definition 2.7.** A *complete graph*, or *clique* is a graph in which all vertices are pairwise adjacent. The complete graph on  $n$  vertices is notated  $K_n$ . The *clique number* of a graph  $G$ ,  $\omega(G)$ , is the number of vertices of the largest clique in  $G$ .

**Definition 2.8.** A set of subgraphs of  $G$ , each of which is a clique and such that every edge of  $G$  is contained in at least one of these cliques, is called a *clique cover* of  $G$ . The clique cover number of  $G$ , written  $cc(G)$ , is the smallest number of cliques in a clique cover of  $G$ .

**Example 2.9.** Graphs are usually notated pictorially as in figure 1. The circles represent vertices and lines represent edges. This graph contains  $K_3$  as an induced subgraph. The clique cover number is 3.

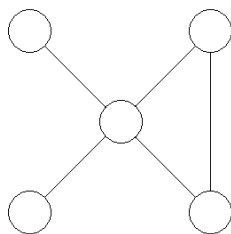


Figure 1: Pictorial representation of a graph

## 2.2 Graph Operations

**Definition 2.10.** If  $G$  is a graph and  $u \in V(G)$ , then the subgraph of  $G$  induced by  $V(G) \setminus \{u\}$  is said to be obtained from  $G$  by *removing a vertex*  $u$ ; it is denoted  $G - u$ . If  $H$  is a subgraph of  $G$  with  $V(H) = V(G)$  and  $E(H) = E(G) \setminus \{e\}$  (where  $e \in E(G)$ ) then  $H$  is obtained from  $G$  by *removing an edge* and we write  $H = G - e$ .

**Definition 2.11.** If  $G$  is a graph and  $uv \in E(G)$ , *subdividing an edge*  $uv$  is the action of creating a new graph from  $G$  by adding a new vertex  $w$ , and adjusting the edge set:  $H = (V(G) \cup \{w\}, (E(G) \setminus \{uv\}) \cup \{uw, vw\})$ . See figure 2

**Definition 2.12.** If  $G$  is a graph and  $uv \in E(G)$ , then  $G/uv$  is the graph obtained from  $G$  by removing  $uv$ , and identifying the vertices  $u$  and  $v$ . If this results in multiple edges

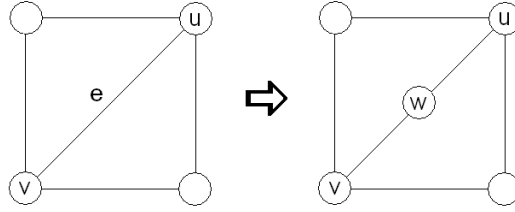


Figure 2: Example of subdividing an edge

between two vertices, we include only one edge, unless otherwise stated. This action is called *contracting an edge  $uv$* . See figure 3. The notation  $G/uv$  could also apply (see Theorems 7.12 and 7.13) to identifying two non-adjacent vertices  $u$  and  $v$ .

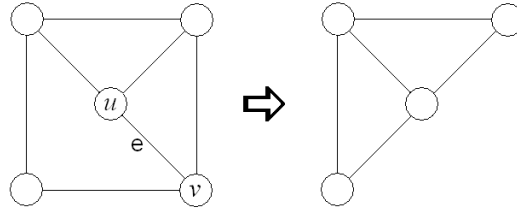


Figure 3: Example of contracting an edge

**Definition 2.13.** A *path* is a graph that can be obtained from  $K_2$  using only edge subdivisions or contractions. The path on  $n$  vertices is denoted  $P_n$ .

**Definition 2.14.** Let  $G$  and  $H$  be disjoint graphs, each with a vertex labeled  $v$ . Then  $G \oplus_v H$  is the graph obtained by identifying the vertex  $v$  in  $G$  with the vertex  $v$  in  $H$ .

### 2.3 Zero Forcing

**Definition 2.15.** *Color-change rule:* If  $G$  is a graph with each vertex colored either white or black,  $u$  is a black vertex of  $G$ , and exactly one neighbor  $v$  of  $u$  is white, then change the color of  $v$  to black. When this rule is applied, we say  $u$  *forces*  $v$ , and write  $u \rightarrow v$ .

**Definition 2.16.** A *zero forcing set* for a graph  $G$  is a subset of vertices  $Z$  such that if initially the vertices in  $Z$  are colored black and the remaining vertices are colored white, the entire graph  $G$  may be colored black by repeatedly applying the color-change rule. The *zero forcing number* of  $G$ ,  $Z(G)$ , is the minimum size of a zero forcing set. Any zero forcing set of order  $Z(G)$  is called a *minimal zero forcing set*.

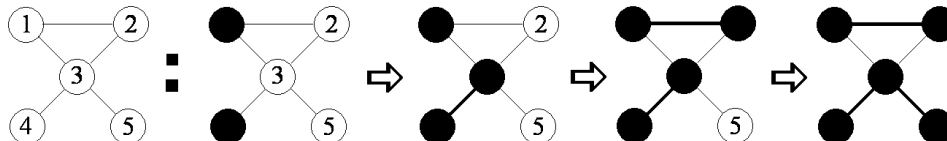


Figure 4: Using the Zero Forcing Rule

Figure 4 gives an example of a zero forcing set, showing explicitly each use of the color change rule. The zero forcing set shown is a minimal zero forcing set.

**Definition 2.17.** Let  $Z$  be a zero forcing set of a graph  $G$ . Construct the derived coloring, recording the forces in the order in which they are performed. This is a *forcing sequence*. A *forcing chain* (for this particular forcing sequence) is a sequence of vertices  $(v_1, v_2, \dots, v_k)$  such that for  $i = 1, \dots, k - 1$ ,  $v_i$  forces  $v_{i+1}$ . A *maximal forcing chain* (for this particular forcing sequence) is a forcing chain that is not a proper subsequence of another forcing chain.

**Example 2.18.** In figure 4, the forcing sequence is  $\{4 \rightarrow 3, 1 \rightarrow 2, 3 \rightarrow 5\}$ . There are two maximal forcing chains:  $(1, 2)$  and  $(4, 3, 5)$ .

### 3 Preliminary Results: Zero forcing and minimum rank

**Observation 3.1.** Let  $Z$  be a zero forcing set for a graph  $G$ . Construct an associated forcing sequence. If there exists a vertex  $v \in Z$  such that  $v \rightarrow u$  does not occur for any

$u$ , then  $Z \setminus \{v\}$  is a zero forcing set for  $G - v$ . Conversely, if  $X$  is a zero forcing set for  $G - v$ , then  $X \cup \{v\}$  is a zero forcing set for  $G$ .

*Proof.* This follows from the observation that we can add (or remove) a black neighbor to (from) any vertex, and the color change rule can still apply.  $\square$

The following definition and two theorems are due to Leslie Hogben [H<sup>+</sup>], and deal with the non-uniqueness of zero forcing sets for non-trivial graphs. This property will be used in various proofs later in the paper.

**Definition 3.2.** Let  $Z$  be a zero forcing set of a graph  $G$ . A *reversal* of  $Z$  is the set of last vertices of the maximal zero forcing chains of a forcing sequence. Note that the order of a reversal of  $Z$  is the same as  $|Z|$ .

**Theorem 3.3.** *If  $Z$  is a zero forcing set of  $G$  then so is any reversal  $W$  of  $Z$ .*

*Proof.* Write the forcing sequence in reverse order, reversing each force (call this the reverse forcing sequence) and let the reversal of  $Z$  for this list be denoted  $W$ . We show the reverse forcing sequence is a valid list of forces for  $W$ . Consider the first force  $u \rightarrow v$  on the reverse forcing sequence. Suppose  $u \rightarrow v$  is not a valid force for  $W$ . Then some neighbor  $w \neq v$  of  $u$  must be white, or in other words,  $w$  is not the last vertex in a maximal forcing chain. Therefore,  $w \rightarrow x$ ,  $x \neq u$  must appear in the original forcing sequence. Since  $u$  was the last vertex forced black, we know that  $u$  must have been white when  $w \rightarrow x$  occurred. But  $w$  has two white neighbors at this point,  $u$  and  $x$ , which contradicts the fact that  $w \rightarrow x$  is a valid force. Therefore,  $u \rightarrow v$  must be a valid force in the reverse forcing sequence. The rest follows by induction.  $\square$

**Theorem 3.4.** *Let  $G$  be a connected graph, with order bigger than one, and let  $ZFS(G)$*

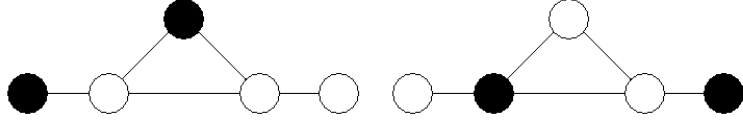


Figure 5: Minimal zero forcing sets are not unique

denote the set of all minimal zero forcing sets of  $G$ . Then

$$\bigcap_{Z \in ZFS(G)} Z = \emptyset$$

*Proof.* Suppose not. Then there exists  $v \in \bigcap_{Z \in ZFS(G)} Z$ . In particular, for each  $Z$  and each reversal  $W$  of  $Z$ ,  $v$  is in both  $Z$  and  $W$ . This means that there is a maximal forcing chain consisting of only  $v$ , or in other words  $v$  does not force any other vertex. Thus for any minimal zero forcing set  $Z$ ,  $Z \setminus \{v\}$  is a zero forcing set for  $G - v$  by Observation 3.1. Construct a forcing sequence. Let  $u \rightarrow w$  be the first force in which the forcing vertex  $u$  is a neighbor of  $v$ . We claim that  $Z' = Z \setminus \{v\} \cup \{w\}$  is a zero forcing set for  $G$ . The forces can proceed until  $u$  is encountered as a forcing vertex. At that time, replace  $u \rightarrow w$  by  $u \rightarrow v$ , and then continue as in the original forcing sequence. Since  $v \notin Z'$ , we have a contradiction.  $\square$

**Corollary 3.5.** *If  $G$  is a non-trivial graph, then  $G$  does not have a unique minimal zero forcing set.*

**Example 3.6.** Figure 5 depicts one graph with 2 possible minimal zero forcing sets.

The following theorem was proved in [AIM08], but I give an alternative proof:

**Theorem 3.7.** *For any graph  $G$ , and any field  $F$ ,  $M(F, G) \leq Z(G)$ .*

Before going into the proof, it is helpful to work through the following example.

**Example 3.8.** Let  $H$  be the graph in Figure 4. Then any matrix  $A$  in  $S(H)$  is a symmetric matrix of the form:

$$\begin{pmatrix} d_1 & a_{12} & a_{13} & 0 & 0 \\ a_{21} & d_2 & a_{23} & 0 & 0 \\ a_{31} & a_{32} & d_3 & a_{34} & a_{35} \\ 0 & 0 & a_{43} & d_4 & 0 \\ 0 & 0 & a_{53} & 0 & d_5 \end{pmatrix}$$

where each labeled  $a_{ij}$  denotes a non-zero entry, and  $a_{ij} = a_{ji}$ . We cross out the columns 1 and 4 (corresponding to the zero forcing set) and also rows 2 and 5 (which are the last vertices in the maximal forcing chains in figure 4). This gives a matrix  $A'$  of the following form:

$$\begin{pmatrix} a_{12} & a_{13} & 0 \\ a_{32} & d_3 & a_{35} \\ 0 & a_{43} & 0 \end{pmatrix}$$

I claim  $A'$  is invertible. The order for the zero forcing gives us an algorithm for establishing the invertibility of  $A'$ . The first force is  $4 \rightarrow 3$ . In the matrix, we look at the row corresponding to vertex 4 and see that the only nonzero entry is in the column corresponding to vertex 3,  $a_{43}$ .

Using cofactor expansion along the row corresponding to vertex 4,  $A'$  is invertible if and only if  $A''$  is invertible, where

$$A'' = \begin{pmatrix} a_{12} & 0 \\ a_{32} & a_{35} \end{pmatrix}$$

$A''$  is obtained from  $A'$  by deleting the row and column containing  $a_{43}$ . The next force in the forcing sequence is  $1 \rightarrow 2$ . Correspondingly, row 1 has exactly one nonzero entry, namely  $a_{12}$ . Similar to above, we cross off the row and column containing  $a_{12}$ .



All that remains in the matrix is  $a_{35}$ .  $A''$  is invertible since  $a_{35}$  is non-zero. But again, this corresponds to the force  $3 \rightarrow 5$ .

This is not a coincidence: the proof for Theorem 3.7 is a generalization of this technique.

*Proof.* Let  $A \in S(F, G)$ .  $M(F, G) \leq Z(G)$  is equivalent to  $n - \text{mr}(F, G) \leq Z(G)$ . I will show this by using a zero forcing set to construct a  $(n - Z(G)) \times (n - Z(G))$  combinatorially invertible submatrix of  $A$ . The result then follows. Let  $Z$  be a zero forcing set for  $G$ . Construct a forcing sequence. Now, let  $A_0$  be the matrix obtained from  $A$  by deleting the columns corresponding to the vertices in  $Z$ , and the rows corresponding to the reversal of  $Z$  (under this forcing sequence). I claim  $A_0$  is invertible. Let  $a \rightarrow b$  be the first force in the forcing sequence. That means that  $a$  has exactly one white neighbor, and all other neighbors are black. The columns corresponding to all black vertices have been deleted, so the row corresponding to  $a$  has exactly one non-zero entry in  $A_0$ . Cross out row  $a$  and column  $b$  to get  $A_1$ . By cofactor expansion,  $\det(A_0) \neq 0 \Leftrightarrow \det(A_1) \neq 0$ . Now, consider the second force in the forcing sequence,  $c \rightarrow d$ . The columns corresponding to  $c$  and all of its neighbors (except  $d$ ) have been crossed out, so there is exactly one non-zero entry left in row  $c$ . Continuing in the same manner establishes the invertibility of  $A_0$ .  $\square$

The following are some results about minimum rank that will be used in future sections.

**Proposition 3.9.** [FH07, Observation 1.6] *If  $G'$  is an induced subgraph of  $G$ , then*

$$\text{mr}(F, G') \leq \text{mr}(F, G).$$

**Proposition 3.10.** [FH07][Observation 1.8] *If  $G$  is a graph, then  $\text{mr}(\mathbb{R}, G) \leq cc(G)$ .*

The following theorem is proved in [BFH04] and [Hsi01]. Field-independent proofs are given in [van08] and [BGL09]:

**Theorem 3.11.** *If  $G, H$  are graphs each containing a vertex  $v$ , then*

$$\text{mr}(F, G \oplus_v H) = \min\{\text{mr}(F, G) + \text{mr}(F, H), \text{mr}(F, G - v) + \text{mr}(F, H - v) + 2\}$$

*Equivalently,*

$$M(F, G \oplus_v H) = \max\{M(F, G) + M(F, H) - 1, M(F, G - v) + M(F, H - v) - 1\}$$

It is also known [Hsi01, Theorem 10] that

**Proposition 3.12.** *If  $v$  is a pendant vertex of a graph  $G$ , then  $\text{mr}(F, G - v) \geq \text{mr}(F, G) - 1$ .*

*Equivalently,  $M(F, G - v) \leq M(F, G)$ .*

The next proposition is a consequence of the preceding two.

**Proposition 3.13.** *If  $G, H$  are graphs each with a pendant vertex  $v$ , then  $M(F, G \oplus_v H) = M(F, G) + M(F, H) - 1$ .*

## 4 Zero Forcing and Path Covers

**Definition 4.1.** Given a graph  $G = (V, E)$ , a *path cover* is a set of disjoint induced paths in  $G$  such that every vertex  $v \in V$  belongs to exactly one path. The *path cover number*  $P(G)$  is the minimum number of paths in a path cover.

**Lemma 4.2.** *Let  $G$  be a graph. If  $X \subset V(G)$  is a zero forcing set for  $G$ , then  $X$  induces a path cover for  $G$ .*

*Proof.* It suffices to show that if  $H$  is the subgraph of  $G$  induced by the vertices of a zero forcing chain, then  $H$  is a path. By construction,  $H$  must contain a path. Suppose that  $H$  is not a path. Then either  $H$  is a cycle, or  $H$  has a vertex  $w$  of degree more than two. If  $H$  is a cycle with more than one vertex, then the first vertex  $u$  (the vertex of  $H$

that appears first in the zero forcing chain) has degree 2 in  $H$ , meaning that  $u$  has two white neighbors and cannot force anything. On the other hand, if some vertex  $w$  in  $H$  has degree at least 3, then  $w$  always has at least two white neighbors. In either case, we have a contradiction.  $\square$

**Corollary 4.3.** *For any graph  $G$ ,  $Z(G) \geq P(G)$ .*

It is possible for  $Z(G)$  and  $P(G)$  to be arbitrarily far apart, even for planar graphs. Consider the example given in figure 6. It is clear that the path cover number of this graph must be 2. Suppose there are  $k$  copies of  $K_4$  in this graph. The black vertices form a zero forcing set of size  $k + 2$ . To see that this is a minimal zero forcing set, note that  $\text{mr}(G) \leq \text{cc}(G)$ . So  $\text{mr}(G) \leq k$ . There are  $2k+2$  vertices in this graph, so we have  $Z(G) \geq M(G) \geq k + 2$ . Therefore we must have  $Z(G) = k + 2$ . So  $Z(G) - P(G) = k$ .

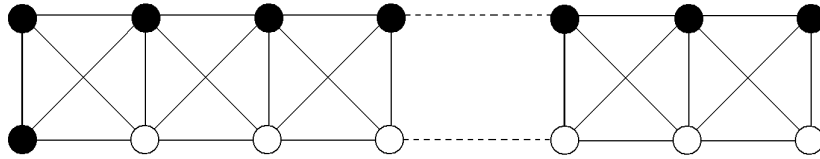


Figure 6:  $Z(G)$  and  $P(G)$  can be arbitrarily far apart

Although the path cover number does not, in general, provide a good estimate of the zero forcing number, the zero forcing number and the path cover number share many properties.

**Definition 4.4.** A path cover  $\mathcal{P}$  of  $G$  is a  $Z$ -induced path cover of  $G$  if  $\mathcal{P}$  can be induced by some zero forcing set of  $G$ . A *minimal  $Z$ -induced path cover* is a  $Z$ -induced path cover associated with a minimal zero forcing set.

It follows that a set  $X$  is a zero-forcing set if and only if there is a  $Z$ -induced path cover of  $G$  associated with  $X$ .

For example, Figure 7 shows one graph with two different path covers. The path cover on the left is  $Z$ -induced, with a corresponding zero forcing set colored black. The reader is invited to verify that the path cover on the right is not  $Z$ -induced.

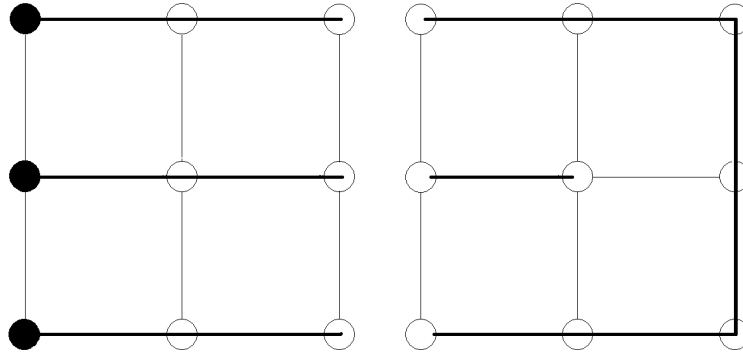


Figure 7: Path covers may or may not be  $Z$ -induced

**Observation 4.5.** *Any superset of a zero forcing set for a graph  $G$  is also a zero forcing set for  $G$ . Equivalently, if  $\mathcal{P}_2$  is a path cover of  $G$  obtained from a  $Z$ -induced path cover  $\mathcal{P}_1$  by splitting 1 or more paths in  $\mathcal{P}_1$ , then  $\mathcal{P}_2$  is also a  $Z$ -induced path cover for  $G$ .*

*Proof.* Let  $Z$  be a zero forcing set for  $G$  and let  $Z' \supset Z$ . Create a forcing sequence associated with  $Z$ . Note that any force in this forcing sequence is also valid if we begin with  $Z'$  colored, with the exception of  $u \rightarrow v$  where  $v \in Z'$  is already black. We simply delete these from the forcing sequence to get a valid forcing sequence associated with  $Z'$ . The observation about path covers follows immediately.  $\square$

The following proposition is an application of Corollary 4.3. The result also follows from the fact that  $M(\mathbb{R}, G) \geq \omega(G) - 1$  [Ban07], but the proof given here relies only on graph theory.

**Proposition 4.6.** *For any graph  $G$ ,  $Z(G) \geq \omega(G) - 1$*

*Proof.* If  $\omega(G) \leq 2$ , this is obvious. If  $\omega(G) = 3$ , then  $G$  cannot be a path, so  $P(G) \geq 2$ . By Corollary 4.3,  $Z(G) \geq P(G) \geq \omega(G) - 1$ . So we may assume that  $\omega(G) \geq 4$ . By way of contradiction, Suppose  $Z(G) \leq \omega(G) - 2$ . Let  $K$  be a clique of size  $\omega(G)$  in  $G$ . Let  $Z$  be a minimal zero forcing set for  $G$ , with associated path cover  $\mathcal{P}$ . Note that  $|\mathcal{P}| \leq \omega(G) - 2$ . Since there are  $\omega(G)$  vertices in  $K$ , and every vertex is contained in some path in  $\mathcal{P}$ , we use a generalized pigeon-hole principle to get two cases.

Case 1: There is some  $P \in \mathcal{P}$  such that  $P$  contains at least 3 vertices of  $K$ . Since these 3 vertices are pairwise adjacent, it follows that the subgraph induced by  $P$  contains a  $K_3$  and cannot be a path. This contradicts the definition of  $\mathcal{P}$ .

Case 2: There exist  $P_1, P_2 \in \mathcal{P}$ , with  $P_1 \neq P_2$ , and  $P_1, P_2$  each contain at least two vertices of  $K$ . Let  $x_1, x_2 \in P_1 \cap K$  and  $y_1, y_2 \in P_2 \cap K$ . Note that since  $x_1, x_2, y_1, y_2$  are distinct and pairwise adjacent, we have that  $y_1$  cannot force  $y_2$  (or  $y_2$  cannot force  $y_1$ ) until both  $x_1$  and  $x_2$  are black. Similarly, at least one of  $x_1, x_2$  must remain white until both  $y_1, y_2$  are black. So  $\mathcal{P}$  must not be a  $Z$ -induced path cover.  $\square$

The last few definitions in this section will be used extensively in Section 6.

**Definition 4.7.** A vertex  $v$  in a graph  $G$  is said to be *Z-terminal* if it occurs as an endpoint in some minimal  $Z$ -induced path cover.

The following is a corollary to Theorem 3.3.

**Corollary 4.8.** *The following statements are equivalent:*

1. *A vertex  $v$  is Z-terminal*
2. *There exists a minimal zero forcing set containing  $v$*
3. *There is a minimal zero forcing set for which  $v$  is the last point in a maximal forcing chain.*

*Proof.*  $2 \rightarrow 1$  and  $3 \rightarrow 1$  are trivial. If  $v$  is  $Z$ -terminal, then by definition, either (2) or (3) must hold. Theorem 3.3 implies that  $2 \leftrightarrow 3$ . Thus,  $1 \rightarrow 2$ ,  $1 \rightarrow 3$ , and the equivalence is established.  $\square$

**Example 4.9.** A pendant vertex is always  $Z$ -terminal. It must appear as an endpoint in any path cover, so, specifically, it must appear as an endpoint in a  $Z$ -induced path cover. This means that when looking for a minimal zero forcing set, you can always choose one pendant vertex to be in that set.

**Definition 4.10.** A vertex  $v$  is *doubly  $Z$ -terminal* if there exists some minimal  $Z$ -induced path cover for which  $v$  appears in a path of length 1. Equivalently, there exists some minimal zero forcing set containing  $v$ , with a forcing sequence such that  $v \rightarrow u$  does not occur for any  $u$ .  $v$  is *simply  $Z$ -terminal* if  $v$  is  $Z$ -terminal but not doubly  $Z$ -terminal.

This terminology will be used in Section 6 and is similar to that used in [BFH05]. This is intentional, since  $Z$ -terminal vertices and terminal vertices play analogous roles. The reader is invited to compare the results in [BFH05] to the results in this paper. Many of the inequalities and formulas proved in the two papers are similar.

**Example 4.11.** Isolated vertices are always doubly  $Z$ -terminal. The pendant vertices in  $P_n$ ,  $n > 1$  are simply  $Z$ -terminal.

In general, it is difficult to check that a vertex is simply  $Z$ -terminal, for it would require proof that for each minimal zero forcing set containing  $v$ , there is no associated path cover for which  $v$  is a singleton.

## 5 Edge Operations

The next couple of sections examine what happens to the zero forcing number under certain graph operations. For the proofs in these sections, we will always assume  $G$  is a connected graph. The theorems also hold for disconnected graphs by the observation that if  $G$  is disconnected, then  $Z(G) = Z(G_1) + Z(G_2) \dots + Z(G_k)$  where  $\{G_i\}$  is the set of connected components of  $G$ . The first such operation is contracting an edge.

**Theorem 5.1.** *Let  $e = uv$  be an edge in a graph  $G$ . Then*

$$Z(G) - 1 \leq Z(G/e) \leq Z(G) + 1$$

*Proof.* Let  $u'$  be the new vertex in  $G/e$  obtained by identifying  $u$  and  $v$ . If  $G = K_2$ , the theorem holds. So we may assume that  $u'$  is not an isolated vertex.

Begin with the first inequality. Let  $X'$  be a minimal zero forcing set for  $G/e$  not containing  $u'$ . This is possible by Theorem 3.4. We will construct a set  $X \supset X'$ , to be defined later, which will be a zero forcing set for  $G$ . Write down a forcing sequence for  $G/e$  associated with  $X'$ . Note that any force in  $G/e$  is valid in  $G$  until  $x \rightarrow u'$  occurs for some vertex  $x$ . For, if  $a \rightarrow b$  occurs, where  $a, b \neq u'$ , then  $a$  has exactly one white neighbor in  $G/e$  and hence has exactly one white neighbor in  $G$ . So this force must be valid in  $G$ . When  $x \rightarrow u'$  occurs (as it will, since  $u' \notin X'$ ), then at that point,  $x$  has exactly one white neighbor in  $G/e$  and hence has at most 2 white neighbors in  $G$ , namely  $u$  and  $v$ . Suppose, without loss of generality, that  $x$  is adjacent to  $u$ . We now define  $X = X' \cup \{v\}$ . Then  $x$  has exactly one white neighbor in  $G$  and we can replace the force  $x \rightarrow u'$  with  $x \rightarrow u$ . Now, if  $u' \rightarrow y$  occurs, then this means that  $u'$  has exactly one white neighbor  $y$  in  $G/e$ . So, in  $G$ , the combined neighborhoods of  $u$  and  $v$  contain exactly one white vertex  $y$ . Then either  $u \rightarrow y$  or  $v \rightarrow y$  is valid, so color  $y$ . After this, the remainder of the

forces in the forcing sequence for  $G/e$  are valid in  $G$ . So  $X$  is a zero forcing set for  $G$  of size  $Z(G/e) + 1$ .

Now, for the second inequality, let  $Z$  be a minimal zero forcing set for  $G$ . We will construct a zero forcing set  $Z' \supset \varphi(Z)$ , where  $\varphi$  is the quotient map  $G \rightarrow G/e$  that identifies the vertices  $u$  and  $v$ . Again, we write down a forcing sequence for  $G$  associated with  $Z$ . By similar logic as above, any force for  $G$  that does not involve  $u$  or  $v$  is also valid in  $G/e$ , provided all previous forces are also valid. So we can follow the same forcing sequence until there is a force involving  $u$  or  $v$ . Suppose a force of the form  $x \rightarrow u$ ,  $x \neq v$ , occurs (the case  $x \rightarrow v$  is similar). Then  $x$  is black and has exactly one white neighbor in  $G$ . Then, in  $G/e$ ,  $x$  has at most one white neighbor, namely  $u'$ . So we can replace  $x \rightarrow u$  with  $x \rightarrow u'$  if needed. If  $u'$  is already black, then we just delete  $x \rightarrow u$  from the forcing sequence. Suppose, then, that  $u \rightarrow v$  occurs ( $v \rightarrow u$  is similar). Then  $u$  is already black, implying that  $u'$  is also already black. This can be deleted from the sequence. Now we need only deal with the possible forces  $u \rightarrow y$  or  $v \rightarrow z$ . Of these two possibilities, suppose without loss of generality that a force of the form  $u \rightarrow y$  occurs first. We will define  $Z' = \varphi(Z) \cup \{y\}$  and delete  $u \rightarrow y$  from the forcing sequence. Now we can keep using the same forcing sequence until we get to a force of the form  $v \rightarrow z$ . At this point, in the graph  $G$ , all neighbors of  $u$  are black (since  $u \rightarrow y$  occurred previously) and all neighbors of  $v$  except  $z$  are black. This means that in  $G/e$  all neighbors of  $u'$  except  $z$  are black. So we can color  $z$  black. We can then follow the same forcing sequence until the whole graph  $G/e$  is colored. So  $Z'$  is a zero forcing set for  $G/e$  with cardinality at most  $Z(G) + 1$ . This proves the theorem.  $\square$

The inequalities in Theorem 5.1 are tight in the sense that equality can be achieved. Figure 8 shows 3 graphs, where each can be obtained from the previous using an edge contraction.  $Z(G_1) = 2$ ,  $Z(G_2) = 3$ , and  $Z(G_3) = 2$ .



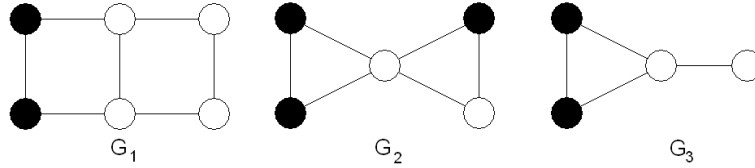


Figure 8: Zero forcing sets and edge contractions

**Theorem 5.2.** *Let  $G$  be a connected graph. If  $e = uv$  is an edge of  $G$ , then*

$$Z(G) - 1 \leq Z(G - e) \leq Z(G) + 1$$

*Proof.* Begin with the inequality on the left. Let  $X$  be a zero forcing set for  $G - e$ . Construct an associated forcing sequence  $\mathcal{S}$ . I will construct a zero forcing set  $X' \supset X$  for  $G$ . Color the vertices of  $X$  black in  $G$ . There are three possible types of forces in  $\mathcal{S}$ :  $a \rightarrow b$  (where  $a \neq u, v$ ),  $u \rightarrow c$ , or  $v \rightarrow d$ . We deal with each one separately. First,  $a \rightarrow b$ . Then, in  $G - e$ ,  $a$  is black and has exactly one white neighbor. Since  $a \neq u, v$ ,  $a$  has exactly the same neighbors in  $G - e$  as in  $G$ . So forces of these type are valid in  $G$ . For the last two types of forces, by symmetry we can assume that a force of the form  $u \rightarrow c$  occurs before  $v \rightarrow d$ . Then we define  $X' = X \cup \{v\}$ . When  $u \rightarrow c$  occurs in  $G - e$ ,  $u$  is black and has exactly one white neighbor in  $G - e$ . Since we colored  $v$  black in  $G$ , then  $u$  has exactly one white neighbor in  $G$ . So we can color  $c$  in  $G$ . At this point,  $u$  and  $v$  are both colored black in  $G$ . Continue forcing until we get to a force of the form  $v \rightarrow d$ . Then, in  $G - e$ , all neighbors of  $v$  except  $d$  are black. Since  $u$  is already black, then  $v$  has exactly one white neighbor in  $G$  so we can color  $d$  black. Then the remainder of the forces in  $G - e$  are valid in  $G$ .

Now, for the inequality on the right: Let  $Z$  be a zero forcing set for  $G$ . Construct a forcing sequence for  $G$  associated with  $Z$ . In this situation, *any* force that is valid in  $G$  is valid in  $G - e$ , with only one exception:  $u \rightarrow v$  (or its symmetric counterpart,  $v \rightarrow u$ ).

Indeed, if  $a \rightarrow b$  occurs, then  $a$  is black and has exactly one white neighbor  $b$ . Then, so long as  $\{a, b\} \neq \{u, v\}$ , then  $a$  and  $b$  are still adjacent and  $b$  is still the unique white neighbor of  $a$ . So we need only deal with the case  $u \rightarrow v$ . We do this quite simply by adding  $v$  to the original set of black vertices. So  $Z \cup \{v\}$  is a zero forcing set for  $G - e$ .  $\square$

Again, equality is possible in each of the inequalities involving deleting an edge. Figure 9 depicts 3 graphs, each obtained from the last by deleting an edge.  $Z(G_1) = 2$ ,  $Z(G_2) = 1$ , and  $Z(G_3) = 2$ .

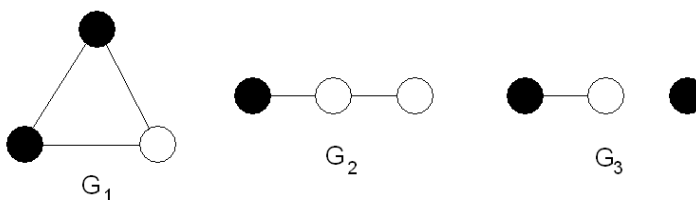


Figure 9: Equality is possible in Theorem 5.2

**Corollary 5.3.** *Let  $G$  and  $e$  be as in Theorem 5.2. If there exists a minimal zero forcing set for  $G$  with a forcing sequence such that neither  $u \rightarrow v$  nor  $v \rightarrow u$  appear, then*

$$Z(G) - 1 \leq Z(G - e) \leq Z(G)$$

*Proof.* This follows trivially from the proof given above. Since  $v \rightarrow u$  and  $u \rightarrow v$  do not occur, we don't need to add anything to the zero forcing set in the proof for the second inequality.  $\square$

The next graph operation we will consider is that of subdividing an edge.

**Theorem 5.4.** *If  $H$  is obtained from  $G$  by subdividing an edge  $e = uv$ , then*

$$Z(G) \leq Z(H) \leq Z(G) + 1.$$

*Proof.* Let  $w$  be the new vertex obtained when the edge is subdivided. Note that  $G$  can be obtained from  $H$  by contracting edge  $vw$ . So by Theorem 5.1,  $Z(H) \leq Z(G) + 1$ . For the other inequality, let  $X$  be a minimal zero forcing set for  $H$ . By Theorem 3.4, we may assume that  $w \notin X$ . Construct the forcing sequence. Now, color the vertices of  $X$  black in  $G$ . Note that every force that is valid in  $H$  is also valid in  $G$  until we get to a force of the form  $u \rightarrow w$  (or  $v \rightarrow w$ , but these two cases are symmetric). We may assume, since  $w$  is of degree 2, that either  $v$  is already black, or the force  $u \rightarrow w$  is followed immediately by  $w \rightarrow v$ . Since  $u \rightarrow w$  is a valid force,  $u$  is black and has precisely one white neighbor in  $H$ , namely  $w$ . Thus, in  $G$ ,  $u$  has at most one white neighbor, namely  $v$ . Color  $v$  if it is white. Delete the force  $w \rightarrow v$  from the sequence if it appears. The remainder of the forces in the forcing sequence for  $H$  are valid in  $G$ . So  $X$  is a zero forcing set for  $G$ .  $\square$

Figure 10 depicts two graphs. The graph on the right is obtained from the graph on the left by subdividing an edge. Note also that the zero forcing number increased when the edge was subdivided. As the following proposition proves, any further subdivisions of this graph do not change the zero forcing number.

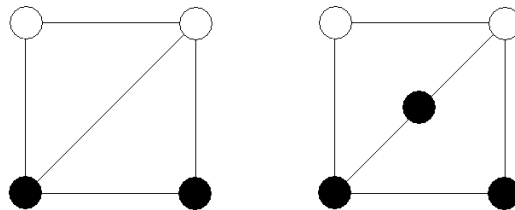


Figure 10: Illustration of Theorem 5.4

**Proposition 5.5.** *If  $H$  is obtained from  $G$  by subdividing an edge incident to a degree 1 or 2 vertex, then  $Z(H) = Z(G)$ .*

*Proof.* Suppose  $u$  has degree 1. Let  $w$  be the new vertex in  $H$  obtained when the edge

$uv$  is subdivided. Then since  $u$  is a pendant vertex,  $u$  must appear as an endpoint in every  $Z$ -induced path cover. So  $u$  is  $Z$ -terminal. Let  $Z$  be a minimal zero forcing set for  $G$  that contains the vertex  $u$ . Without loss of generality, we assume that  $u \rightarrow v$  is the first force. I claim that  $Z$  is also a zero forcing set for  $H$ . We can replace  $u \rightarrow v$  with  $u \rightarrow w, w \rightarrow v$ , then follow the same forcing sequence as in  $G$ . Since  $Z$  is a zero forcing set for  $H$ ,  $Z(G) \geq Z(H)$ . By Theorem 5.4, we have equality.

Now, suppose  $u$  has degree 2, and let  $a$  and  $b$  be neighbors of  $u$ . By Theorem 3.4, let  $X$  be a minimal zero forcing set for  $G$  that does not contain  $u$ . Then somewhere in the forcing sequence, either  $a \rightarrow u$  or  $b \rightarrow u$  occurs. By symmetry, suppose  $a \rightarrow u$  occurs. Since  $u$  has degree 2, it doesn't matter whether the edge  $au$  or the edge  $bu$  is subdivided, the resulting graph is the same. So we assume that the subdivided edge is  $au$ . Then we can replace  $a \rightarrow u$  with  $a \rightarrow w, w \rightarrow u$ . The remainder of  $H$  can be colored using the same forcing sequence as  $G$ . Therefore,  $Z(G) \geq Z(H)$ , so by Theorem 5.4,  $Z(G) = Z(H)$ .  $\square$

## 6 Removing Vertices and Vertex Sums

**Lemma 6.1.** *If  $v$  is  $Z$ -terminal in a graph  $G$ , then  $Z(G - v) \leq Z(G)$ .*

*Proof.* Using corollary 4.8, let  $X$  be a zero forcing set for which  $v$  is a last point in a maximal forcing chain. Then  $v \rightarrow x$  does not occur for any  $x$ . Apply Observation 3.1.  $\square$

**Definition 6.2.** Let  $H$  be an induced subgraph of a graph  $G$ , and let  $\mathcal{P}$  be a path cover of  $G$ . Then the path cover of  $H$  *inherited* from  $\mathcal{P}$ , is the path cover

$$\mathcal{P}_H = \{\text{components of } P \cap H \mid P \in \mathcal{P}\}.$$

Figure 11 depicts a graph with a path cover labeled. The second graph is an induced subgraph of the first, and the inherited path cover is shown. Note that both path covers

are  $Z$ -induced. The following lemma proves that any path cover inherited from a  $Z$ -induced path cover is also  $Z$ -induced. It will be used later in this section.

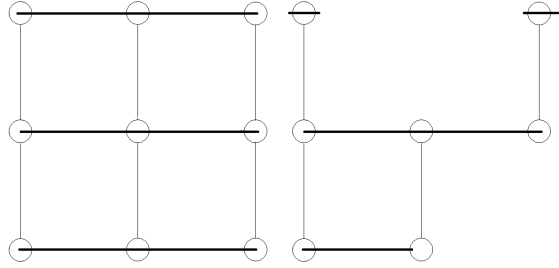


Figure 11: Example of an inherited path cover

**Lemma 6.3.** *Let  $H$  be an induced subgraph of a graph  $G$ , and let  $\mathcal{P}$  be a  $Z$ -induced path cover of  $G$ . Then  $\mathcal{P}_H$  is a  $Z$ -induced path cover of  $H$ , so  $Z(H) \leq |\mathcal{P}_H|$*

*Proof.* If  $H = G$ , this is trivial, so we suppose that  $H \neq G$ . Within each path in  $\mathcal{P}$ , there is a natural ordering established by the forcing sequence. Let  $Y$  be the set of vertices not in  $H$ , and let  $X$ , be the set of “first points” in the paths of  $\mathcal{P}_H$ . Then, by Proposition 4.5,  $X \cup Y$  is also a zero forcing set for  $G$  with associated path cover  $\mathcal{P}_H \cup Y$ . Apply Lemma 3.1 to the vertices of  $Y$ . □

We move on to some results about removing vertices.

**Theorem 6.4.** *Let  $v$  be a vertex of the graph  $G$ . Then*

$$Z(G - v) - 1 \leq Z(G) \leq Z(G - v) + 1$$

*Proof.* We prove the first inequality first. If  $v$  is  $Z$ -terminal in  $G$ , we can apply Lemma 6.1. Otherwise, let  $Y$  be a minimal zero forcing set for  $G$  with associated path cover  $\mathcal{P}$ . Let  $P$  be the element of  $\mathcal{P}$  that contains  $v$ . Divide  $P$  into two paths,  $P_1$  and  $P_2$ , where  $v$  is an endpoint of  $P_2$ . Then  $\bar{\mathcal{P}} = (\mathcal{P} - P) \cup \{P_1\} \cup \{P_2\}$  has order  $|\mathcal{P}| + 1$  and is a

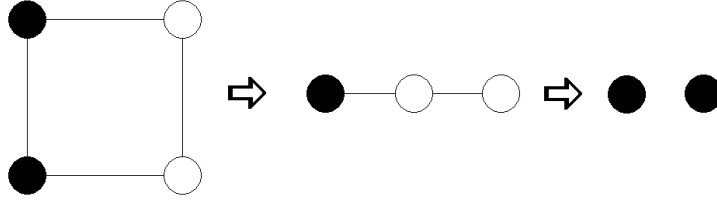


Figure 12: Equality can hold in Theorem 6.4

$Z$ -induced path cover for  $G$  by Proposition 4.5. Then, since  $v$  is an endpoint of a path in  $\bar{\mathcal{P}}$ , we can apply Lemma 6.1 to get a  $Z$ -induced path cover of  $G - v$  of size  $Z(G) + 1$ . Then  $Z(G - v) - 1 \leq Z(G)$ .

The second inequality follows from Observation 3.1. □

Equality can hold in both inequalities in Theorem 6.4, as figure 12 demonstrates. Each graph is obtained from the previous by removing a vertex.

**Corollary 6.5.** *Let  $G$  be a graph, with  $v \in V(G)$ .  $v$  is doubly  $Z$ -terminal if and only if  $Z(G - v) = Z(G) - 1$ .*

*Proof.* Assume  $v$  is doubly  $Z$ -terminal. Choose a minimal  $Z$ -induced path cover  $\mathcal{P}$  such that  $v$  occurs as a singleton. Apply observation 3.1.

Now, assume  $Z(G - v) = Z(G) - 1$ . Let  $X$  be a zero forcing set for  $G - v$ . By Observation 3.1,  $X \cup \{v\}$  is a zero forcing set for  $G$  of size  $|X| + 1 = Z(G)$  in which  $v$  does not need to force anything. Hence  $v$  must be doubly  $Z$ -terminal. □

We put several of the previous results together to get the following:

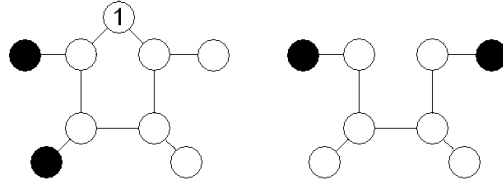


Figure 13: The vertex 1 is not Z-terminal, yet its removal does not raise the zero forcing number

**Theorem 6.6.** *If  $G$  is a graph, with  $v \in V(G)$ , then*

$$Z(G - v) = Z(G) - 1 \text{ if and only if } v \text{ is doubly Z-terminal,}$$

$$Z(G - v) = Z(G) \text{ if } v \text{ is simply Z-terminal.}$$

$$\text{If } Z(G - v) = Z(G) + 1, \text{ then } v \text{ is not Z-terminal.}$$

The converse of this theorem does not hold. That is, there are cases where  $v$  is not Z-terminal in  $G$  and  $Z(G) = Z(G - v)$ . Figure 13 shows a graph where this occurs. First, note that the zero forcing number of this graph is 2. The reader can verify that the vertex labeled 1 is not in any minimal zero forcing set, however, removing vertex 1 does not change the zero forcing number. Black vertices in the picture show one possible zero forcing set for each graph.

Our next objective will be to derive a formula for the zero forcing number of a vertex sum of graphs. We will need several lemmas.

**Theorem 6.7.** *Let  $\{G_i\}_{i=1}^k$  be a family of graphs, each with a vertex labeled  $v$ . If  $v$  is simply Z-terminal in at least 2 of the  $\{G_i\}$ , then  $Z(G_1 \oplus_v G_2 \oplus_v \dots \oplus_v G_k) = \sum_{i=1}^k Z(G_i - v) - 1$ .*

*Proof.* Without loss of generality, suppose  $v$  is simply Z-terminal in  $G_1$  and  $G_2$ . Then, by Corollary 4.8, let  $X_1$  be a minimal zero forcing set for  $G_1$  such that  $v$  is the last

vertex in a maximal forcing chain. Let  $X_2$  be a zero forcing set for  $G_2$  that contains  $v$ . For each  $i \in \{3, 4, \dots, k\}$ , let  $Y_i$  be a minimal zero forcing set for  $G_i - v$ . Then I claim  $X_1 \cup \{X_2 - v\} \cup_{i=3}^k Y_i$  is a zero forcing set for  $G_1 \oplus_v G_2 \oplus_v \dots \oplus_v G_k$ . First, all of  $G_1$  can be forced black because  $v$  is the last point in a maximal forcing chain, so  $v$  never needs to force anything in  $G_1$ .

Then, since  $v$  is black, each of  $G_3, G_4, \dots, G_k$  can be colored black. To see this, note that every vertex of  $G_i$ ,  $i \geq 3$  has the same neighbors in  $G_1 \oplus_v G_2 \oplus_v \dots \oplus_v G_k$  as in  $G_i - v$ , with the possible addition of  $v$  as a neighbor. But  $v$  is black, so that does not change the forcing sequence. Color  $G_3, G_4, \dots, G_k$ . We can then color the remainder of  $G_2$  black.

So this is a valid zero forcing set, and

$$|X_1 \cup \{X_2 - v\} \cup_{i=3}^k Y_i| = Z(G_1) + Z(G_2) - 1 + \sum_{i=3}^k Z(G_i - v) = \sum_{i=1}^k Z(G_i - v) - 1,$$

where the last equality comes from the fact that  $v$  is simply Z-terminal in  $G_1$  and  $G_2$ . So  $Z(G_1 \oplus_v G_2 \oplus_v \dots \oplus_v G_k) \leq \sum_{i=1}^k Z(G_i - v) - 1$ . By Theorem 6.4, equality must hold.  $\square$

**Theorem 6.8.** *If  $G$  and  $H$  are graphs, then  $Z(G \oplus_v H) \geq Z(G) + Z(H) - 1$ , with equality if  $v$  is Z-terminal in  $G$  and in  $H$ .*

Note that this theorem does not need the stronger assumption (used in the previous lemma) that  $v$  is simply Z-terminal in  $G$  and  $H$ .

*Proof.* We first show  $Z(G \oplus_v H) \geq Z(G) + Z(H) - 1$ . Let  $X$  be a minimal zero forcing set for  $G \oplus_v H$ , with associated path cover  $\mathcal{P}$ . By Lemma 6.3 applied to both  $G$  and  $H$ ,  $Z(G) \leq |\mathcal{P}_G|$ , and  $Z(H) \leq |\mathcal{P}_H|$ . Note also that there is exactly one path in  $\mathcal{P}$  containing  $v$ , and hence, there is at most one path that crosses into both  $G$  and  $H$ . So we have by Lemma 6.3,

$$Z(G \oplus_v H) \geq |\mathcal{P}_G| + |\mathcal{P}_H| - 1 \geq Z(G) + Z(H) - 1$$



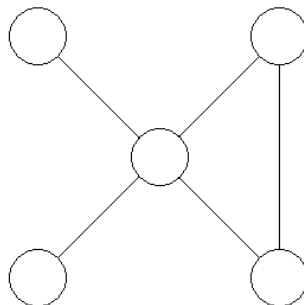


Figure 14: Equality holds in 6.8

Now, suppose that  $v$  is  $Z$ -terminal in  $G$  and in  $H$ . Follow the proof of Theorem 6.7 but do not apply the assumption near the end of the proof that  $v$  is simply  $Z$ -terminal.  $\square$

There are other cases in which equality holds in Theorem 6.8, such as in the graph  $P_3 \oplus_v K_3$  pictured in figure 14. Apply Theorem 6.8 to this graph with  $G = P_3$ ,  $H = K_3$ , and the center vertex  $v$ . The reader can check that  $Z(G) = 1$ ,  $Z(H) = 2$ , and  $Z(G \oplus_v H) = 2$ , but  $v$  is not  $Z$ -terminal in  $P_3$ .

We can combine the results of Theorems 6.8 and 6.4 to get the following:

**Theorem 6.9.** *If  $G, H$  are graphs, each with a vertex labeled  $v$ , then*

$$Z(G \oplus_v H) \geq \max \begin{cases} Z(G) + Z(H) - 1 \\ Z(G - v) + Z(H - v) - 1 \end{cases}$$

Now we establish an upper bound for the zero forcing number of a vertex sum of graphs.

**Lemma 6.10.**

$$Z(G \oplus_v H) \leq \min \begin{cases} Z(G) + Z(H - v) \\ Z(G - v) + Z(H) \end{cases}$$

Further, if the inequality is strict, then  $Z(G) = Z(G-v)$ ,  $Z(H) = Z(H-v)$ , and  $Z(G \oplus_v H) = Z(G) + Z(H) - 1$ .

More generally, if  $\{G_i\}_{i=1}^k$  is a family of graphs, then

$$Z(G_1 \oplus_v G_2 \oplus_v \dots \oplus_v G_k) \leq \min_i \{Z(G_i) + \sum_{j \neq i} Z(G_j - v)\}$$

If the inequality is strict, then

$$Z(G_1 \oplus_v G_2 \oplus_v \dots \oplus_v G_k) = \sum_{i=1}^k Z(G_i - v) - 1$$

*Proof.* We show that  $Z(G \oplus_v H) \leq Z(G) + Z(H - v)$  for any graphs  $G$  and  $H$ . Then  $Z(G \oplus_v H) \leq Z(H) + Z(G - v)$  by symmetry. Let  $X$  be a minimal zero forcing set for  $G$ , and let  $Y$  be a minimal zero forcing set for  $H - v$ . We can apply the zero forcing rule to vertices of  $G$  (independent of  $H$ ) until  $v$  is colored. Then, since  $v$  is colored, Observation 3.1 says  $Y \cup \{v\}$  is a zero forcing set for  $H$ , where the zero forcing rule is never applied to  $v$ . Since  $v$  never forces anything in  $H$ , we can then color all of  $H$  black since it is not necessary for any other neighbors of  $v$  to be black. Then we can color the remainder of  $G$ . So  $X \cup Y$  is a zero forcing set for  $G \oplus_v H$ , and  $Z(G \oplus_v H) \leq |X| + |Y| = Z(G) + Z(H - v)$ . Now, if we let  $G = G_i$  for some fixed  $i$ , and  $H$  be the vertex sum at  $v$  of the graphs  $G_j$ ,  $j \neq i$ , we see that

$$Z(G_1 \oplus_v G_2 \oplus_v \dots \oplus_v G_k) \leq \min_i \{Z(G_i) + \sum_{j \neq i} Z(G_j - v)\}.$$

If the first inequality in the lemma is strict, then apply Theorem 6.9 to conclude

$$\max \begin{cases} Z(G) + Z(H) - 1 \\ Z(G - v) + Z(H - v) - 1 \end{cases} < \min \begin{cases} Z(G) + Z(H - v) \\ Z(G - v) + Z(H) \end{cases} \quad (1)$$

This gives us four inequalities:

$$Z(G - v) + Z(H - v) - 1 < Z(G) + Z(H - v) \quad (2)$$

$$Z(G - v) + Z(H - v) - 1 < Z(G - v) + Z(H) \quad (3)$$

$$Z(G) + Z(H) - 1 < Z(G) + Z(H - v) \quad (4)$$

$$Z(G) + Z(H) - 1 < Z(G - v) + Z(H) \quad (5)$$

From the first inequality,  $Z(G - v) - 1 < Z(G)$ , implying  $Z(G - v) \leq Z(G)$ . Similarly, using the other three inequalities,  $Z(G) \leq Z(G - v)$ ,  $Z(H - v) \leq Z(H)$ , and  $Z(H) \leq Z(H - v)$ . So  $Z(G) = Z(G - v)$  and  $Z(H) = Z(H - v)$ . Substituting this into (1), we get that  $Z(G) + Z(H) - 1 \leq Z(G \oplus_v H) < Z(G) + Z(H)$ . This proves the claim in the case of two graphs in the vertex sum.

For the more general case, recall that  $Z(G_i) \leq Z(G_i - v) + 1$ , and

$$Z(G_1 \oplus_v G_2 \oplus_v \dots \oplus_v G_k) \geq \sum_{i=1}^k Z(G_i - v) - 1.$$

This combined with the inequality already proved yields:

$$\sum_{i=1}^k Z(G_i - v) - 1 \leq Z(G_1 \oplus_v G_2 \oplus_v \dots \oplus_v G_k) \quad (6)$$

$$\leq \min_i \{Z(G_i) + \sum_{j \neq i} Z(G_j - v)\} \quad (7)$$

$$\leq \sum_{i=1}^k Z(G_i - v) + 1 \quad (8)$$

There are 3 inequalities, and the first and the last differ by 2, so one of these must be equality. Suppose inequalities (6) and (7) are both strict. Then

$$Z(G_1 \oplus_v G_2 \oplus_v \dots \oplus_v G_k) = \sum_{i=1}^k Z(G_i - v) \quad (9)$$

$$< \min_i \{Z(G_i) + \sum_{j \neq i} Z(G_j - v)\} \quad (10)$$

So for all  $j$ ,  $Z(G_j - v) < Z(G_j)$ , hence  $Z(G_j - v) = Z(G_j) - 1$ . It follows that  $v$  is doubly  $Z$ -terminal in each  $G_j$ .

I claim that this forces  $Z(G_1 \oplus_v G_2 \oplus_v \dots \oplus_v G_k) = \sum_{i=1}^k Z(G_i - v) + 1$ . I prove the claim by induction on  $k$ . If  $k = 1$  the claim clearly holds. Suppose the claim holds for  $k - 1$ ,  $k \geq 2$ . Let  $H = G_1 \oplus_v G_2 \oplus_v \dots \oplus_v G_{k-1}$  and  $G = G_k$ . By the induction hypothesis, and by Theorem 6.6,  $v$  is doubly  $Z$ -terminal in  $G$  and  $H$ . By Theorem 6.8,  $Z(G \oplus_v H) = Z(G) + Z(H) - 1$ . Since  $v$  is doubly  $Z$ -terminal in  $G$  and  $H$ ,  $Z(G) = Z(G - v) + 1$ , and  $Z(H) = Z(H - v) + 1$ , so

$$Z(G \oplus_v H) = Z(G - v) + Z(H - v) + 1 = \sum_{i=1}^k Z(G_i - v) + 1,$$

contradicting equation 9. Hence, one of inequalities (6) and (7) must be equality. This completes the proof.  $\square$

Now we are ready for the main result, a formula for the zero forcing number of any graph with a cut vertex, in terms of smaller graphs.

**Theorem 6.11.** *Let  $\{G_i\}_{i=1}^k$  be a finite family of graphs, each with a vertex  $v$ . If  $v$  is simply  $Z$ -terminal in 2 or more of the  $\{G_i\}$ , then*

$$Z(G_1 \oplus_v G_2 \oplus_v \dots \oplus_v G_k) = \left( \sum_{i=1}^k Z(G_i - v) \right) - 1$$

*Otherwise,*

$$Z(G_1 \oplus_v G_2 \oplus_v \dots \oplus_v G_k) = \min_i \{ Z(G_i) + \sum_{j \neq i} Z(G_j - v) \}$$

*Proof.* Let  $m = \min\{Z(G_i) + \sum_{j \neq i} Z(G_j - v)\}$ . In light of Lemma 6.10 and Theorem 6.7, it suffices to show that if  $Z(G_1 \oplus_v G_2 \oplus_v \dots \oplus_v G_k) < m$ , then  $v$  must be  $Z$ -terminal in at least 2 of the  $G_i$ . Let  $X$  be a minimal zero forcing set for  $G_1 \oplus_v G_2 \oplus_v \dots \oplus_v G_k$ , and let  $\mathcal{P}$  be an associated path cover. Let  $P$  be that path in  $\mathcal{P}$  that contains  $v$ . Either  $P$  is completely contained in one of the  $G_i$  or  $P$  contains vertices from  $G_j - v$  and  $G_\ell - v$  for some  $j$  and  $\ell$ .

Suppose  $P$  is completely contained in  $G_i$  for some  $i$ . Then we can divide the paths in  $\mathcal{P}$  into  $k$  categories: those completely contained in  $G_i$ , and those completely contained  $G_j - v$  for each  $j \neq i$ . Then  $Z(G \oplus_v H) = |\mathcal{P}_{G_i}| + \sum_{j \neq i} |\mathcal{P}_{G_j - v}| \geq Z(G_i) + \sum_{j \neq i} Z(G_j - v) \geq m$ , which contradicts our initial assumption.

So  $P$  contains vertices from  $G_j - v$  and  $G_\ell - v$  for some  $j$  and  $\ell$ . Therefore,  $v$  is at the end of a path in both  $\mathcal{P}_{G_j}$  and  $\mathcal{P}_{G_\ell}$ . Without loss of generality we may assume that  $j = 1$  and  $\ell = 2$ .

To prove  $v$  is simply  $Z$ -terminal in  $G_1$  and  $G_2$ , we now need to prove that  $\mathcal{P}_{G_1}$  and  $\mathcal{P}_{G_2}$  are induced by *minimal* zero forcing sets. We have, after applying Lemma 6.10:

$$\sum_{i=1}^k Z(G_i - v) - 1 = Z(G_1 \oplus_v G_2 \oplus_v \dots \oplus_v G_k) \quad (11)$$

$$= |\mathcal{P}| \quad (12)$$

$$= |\mathcal{P}_{G_1}| + |\mathcal{P}_{G_2}| - 1 + \sum_{i=3}^k |\mathcal{P}_{G_i - v}| \quad (13)$$

Since  $P$  contains vertices of both  $G_1 - v$  and  $G_2 - v$ , it follows that  $|\mathcal{P}_{G_1}| = |\mathcal{P}_{G_1 - v}|$  and  $|\mathcal{P}_{G_2}| = |\mathcal{P}_{G_2 - v}|$ . Substituting this into 13 yields:

$$\sum_{i=1}^k Z(G_i - v) - 1 = \sum_{i=1}^k |\mathcal{P}_{G_i - v}| - 1 \quad (14)$$

Recalling that  $|\mathcal{P}_{G_i - v}| \geq Z(G_i - v)$ , we must have  $|\mathcal{P}_{G_i - v}| = Z(G_i - v)$  for all  $i$ . Equation 13 then becomes

$$\sum_{i=1}^k Z(G_i - v) - 1 = |\mathcal{P}_{G_1}| + |\mathcal{P}_{G_2}| - 1 + \sum_{i=3}^k Z(G_i - v) \quad (15)$$

In the case of  $G_1$  (the case for  $G_2$  is the same) we have  $Z(G_1 - v) = |\mathcal{P}_{G_1 - v}| = |\mathcal{P}_{G_1}| \geq Z(G_1)$ . Suppose that  $|\mathcal{P}_{G_1}| > Z(G_1)$ . Since  $|\mathcal{P}_{G_1}| = Z(G_1 - v)$ , and  $Z(G_1 - v) \leq Z(G_1) + 1$  we must have  $|\mathcal{P}_{G_1}| = Z(G_1) + 1$ . We can then substitute this into equation (15) to get

$$Z(G_1 \oplus_v G_2 \oplus_v \dots \oplus_v G_k) = Z(G_1) + \sum_{i=2}^k Z(G_i - v) \geq m \quad (16)$$

which contradicts our initial assumption. So we must have  $|\mathcal{P}_{G_1}| = Z(G_1)$ , so  $\mathcal{P}_{G_1}$  is induced by a minimal zero forcing set, and  $v$  is  $Z$ -terminal in  $G_1$ .  $Z(G_1) = Z(G_1 - v)$ , so  $v$  is simply terminal in  $G_1$ . Similarly,  $v$  must also be simply terminal in  $G_2$ .  $\square$

In general, however, it is difficult to determine whether a vertex is simply  $Z$ -terminal. The following are 2 possible ways avoid that:

**Theorem 6.12.** *Let  $G$  and  $H$  be graphs, each with a vertex  $v$ . If  $v$  is  $Z$ -terminal in both  $G$  and  $H$ , then*

$$Z(G \oplus_v H) = Z(G) + Z(H) - 1$$

Otherwise,

$$Z(G \oplus_v H) = \min \begin{cases} Z(G) + Z(H - v) \\ Z(H) + Z(G - v) \end{cases}$$

*Proof.* We examine 3 cases:

Case 1:  $v$  is  $Z$ -terminal in at most one of  $G, H$ . Then by Theorem 6.11,

$$Z(G \oplus_v H) = \min\{Z(G) + Z(H - v), Z(H) + Z(G - v)\}.$$

Case 2:  $v$  is simply  $Z$ -terminal in both  $G$  and  $H$ . Then Theorem 6.11 says that  $Z(G \oplus_v H) = Z(G - v) + Z(H - v) - 1$ . Since  $v$  is simply  $Z$ -terminal, Theorem 6.6 says  $Z(G - v) = Z(G)$  and  $Z(H - v) = Z(H)$ . So  $Z(G \oplus_v H) = Z(G) + Z(H) - 1$ .

Case 3: Without loss of generality  $v$  is doubly  $Z$ -terminal in  $G$ . Then Theorem 6.11 says  $Z(G \oplus_v H) = \min\{Z(G) + Z(H - v), Z(H) + Z(G - v)\}$ . But since  $v$  is doubly  $Z$ -terminal in  $G$ ,  $Z(G) - 1 = Z(G - v)$  (Theorem 6.6). So

$$\begin{aligned} \min\{Z(G) + Z(H - v), Z(H) + Z(G - v)\} &= \min\{Z(G) + Z(H - v), Z(H) + Z(G) - 1\} \\ &\leq Z(G) + Z(H) - 1 \end{aligned}$$

By Theorem 6.8,  $Z(G \oplus_v H) \geq Z(G) + Z(H) - 1$ , so we must have equality.  $\square$

**Corollary 6.13.** *If  $m = \min\{Z(G - v) + Z(H), Z(H - v) + Z(G)\}$ , then*

$$m - Z(G \oplus_v H) \leq 1.$$

*If  $Z(G) \neq Z(G - v)$  or  $Z(H) \neq Z(H - v)$ , then  $Z(G \oplus_v H) = m$*

*Proof.* If  $Z(G \oplus_v H) < \min\{Z(G) + Z(H - v), Z(G - v) + Z(H)\}$ , then the proof of Lemma 6.10 shows that  $Z(G) = Z(G - v)$  and  $Z(H) = Z(H - v)$ , proving the second claim. But then  $Z(G) + Z(H) - 1 = m - 1$ . So  $Z(G \oplus_v H) = m$  or  $m - 1$ .  $\square$

## 7 Applications

It may be of interest to find the zero forcing number of a graph with multiple cut vertices.

We define  $G_1 \oplus_{v_1} G_2 \oplus_{v_2} G_3 \dots \oplus_{v_{k-1}} G_k = (((G_1 \oplus_{v_1} G_2) \oplus_{v_2} G_3) \dots \oplus_{v_{k-1}} G_k)$ .

**Proposition 7.1.** *Let  $G_1, G_2, \dots, G_k$  be graphs. Suppose  $G_i$  and  $G_{i+1}$  each have a  $Z$ -terminal vertex labeled  $v_i$ ,  $i \in \{1, 2, \dots, k - 1\}$ . Then*

$$Z(G_1 \oplus_{v_1} G_2 \oplus_{v_2} G_3 \dots \oplus_{v_{k-1}} G_k) = \left( \sum_{i=1}^k Z(G_i) \right) - (k - 1)$$

*Proof.* This follows by induction on Theorem 6.12.  $\square$

There is a similar result for  $M(G)$ .

**Proposition 7.2.** *Let  $G_1, G_2, \dots, G_k$  be graphs. Suppose  $G_i$  and  $G_{i+1}$  each have a  $Z$ -terminal vertex labeled  $v_i$ ,  $i \in \{1, 2, \dots, k - 1\}$ . Then*

$$M(G_1 \oplus_{v_1} G_2 \oplus_{v_2} G_3 \dots \oplus_{v_{k-1}} G_k) = \left( \sum_{i=1}^k M(G_i) \right) - (k - 1)$$

*Proof.* This follows by induction on Proposition 3.13.  $\square$

**Theorem 7.3.** *Under the same hypotheses as Proposition 7.1, if  $M(F, G_i) = Z(G_i)$  for all  $i$ , then*

$$Z(G_1 \oplus_{v_1} G_2 \oplus_{v_2} G_3 \dots \oplus_{v_{k-1}} G_k) = M(F, G_1 \oplus_{v_1} G_2 \oplus_{v_2} G_3 \dots \oplus_{v_{k-1}} G_k)$$

*The converse holds if all  $v_i$  are pendent vertices in both  $G_i$  and in  $G_{i+1}$ .*

*Proof.* We will prove it for  $k = 2$ . The rest follows from induction. Suppose  $M(G_i) = Z(G_i)$ . Since  $v_1$  is  $Z$ -terminal in both  $G_1$  and  $G_2$ , and the zero forcing number is an upper bound for maximum nullity of any graph over any field, we have  $M(G_1 \oplus_v G_2) \leq Z(G_1 \oplus_v G_2) = Z(G_1) + Z(G_2) - 1 = M(G_1) + M(G_2) - 1 \leq M(G_1 \oplus_v G_2)$ , where the last inequality comes from Theorem 3.11. So we must have equality.

Now, suppose  $M(G_1 \oplus_v G_2) = Z(G_1 \oplus_v G_2)$ . Since  $v_1$  has degree 1 in  $G_1$  and in  $G_2$ , Proposition 3.13 gives:  $M(G_1 \oplus_v G_2) = M(G_1) + M(G_2) - 1$ . So  $M(G_1) + M(G_2) - 1 = Z(G_1) + Z(G_2) - 1$  and since  $Z(G_i) \geq M(G_i)$ , we must then have  $M(G_1) = Z(G_1)$  and  $M(G_2) = Z(G_2)$ .  $\square$

**Example 7.4.** Figure 15 shows a graph in which  $Z(G)$  and  $M(G)$  can be arbitrarily far apart. It is obtained by taking multiple copies of the 5-sun and linking them together.  $Z(5\text{-sun}) = 3$  and  $M(5\text{-sun}) = 2$ . By Proposition 7.1, if  $G$  is the graph depicted, containing  $k$  copies of the 5-sun, then  $Z(G) = 2k + 1$ . Proposition 7.2 implies  $M(G) = k + 1$ . Thus,  $Z(G) - M(G) = k$ , so the difference increases as the number of 5-suns in the graph increases. Interestingly, in this graph,  $Z(G) = P(G)$ , so  $Z(G)$  is, in some sense, as small as we could ask for.

Zero forcing, when combined with other results in the minimum rank problem, can serve as a valuable tool in determining the maximum nullity of an infinite class of graphs. In this section, the minimum rank of any graph obtained by edge subdivisions of the 5-wheel  $W_5$  will be demonstrated.



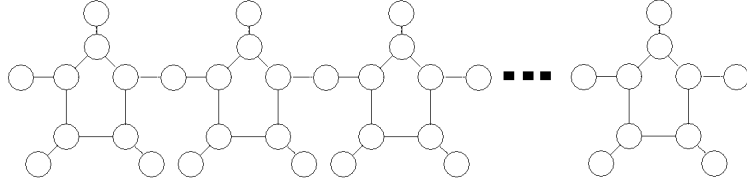


Figure 15: The zero forcing number and maximum nullity can be arbitrarily far apart

First, however, we will need the following regarding the minimum rank of graphs. Several of these results were achieved in collaboration with Dr. Wayne Barrett, and undergraduates Ryan Bowcutt, Seth Gibelyou, and Mark Cutler.

The first observation is well-known, see [FH07].

**Observation 7.5.** *If  $n > 1$ ,  $\text{mr}(K_n) = 1$ ,  $\text{mr}(P_n) = n - 1$ , and  $\text{mr}(C_n) = n - 2$ . If  $T$  is a tree (i.e. a connected acyclic graph) with  $n$  vertices, then  $\text{mr}(T) = n - P(T)$ .*

**Proposition 7.6.** *Let  $F$  be a field, and let  $G$  be a bipartite graph with bipartite sets  $X, Y$ , of cardinality  $m$  and  $n$  respectively. Suppose that no vertex of  $Y$  has degree 1. If  $F = F_2$ , suppose further that each vertex of  $Y$  has even degree. Then  $\text{mr}(F, G) \leq 2m - 2$ .*

*Proof.* We will create the required matrix  $A \in S(F, G)$ . We allow all vertices in  $X$  to precede all vertices in  $Y$ , and all diagonal entries are 0. Let  $p$  denote the characteristic of the field  $F$ . If  $F = F_2$  then let  $A$  be the adjacency matrix of  $G$ . If  $F \neq F_2$ , use the following construction:

For each non-isolated vertex  $v$  in  $Y$ , Let  $d_v$  be the degree of  $v$ , and let each of the first  $d_v - 2$  required nonzero entries in the column corresponding to  $v$  be equal to 1. We then have 2 cases:

Case 1)  $p$  does not divide  $d_v - 1$ . Then let the last two non-zero terms be given by 1 and  $-(d_v - 1)$ , where by convention,  $d - 1 = 1 + 1 + 1 + \dots + 1$ ,  $d - 1$  times.

Case 2)  $p$  divides  $d_v - 1$ . Then let the last two non-zero terms be  $1 + a$  and  $-a$ , where  $a \neq 0, -1$

Following this method of construction for every vertex in  $Y$  defines a matrix  $A$  in  $S(F, G)$ , in which the first  $m$  rows of  $A$  (and, consequently, the first  $m$  columns) sum to 0. Then the rank of this matrix is  $\leq 2m - 2$ .  $\square$

**Definition 7.7.** If  $G$  is a graph, then the *subdivision graph* of  $G$ ,  $\tilde{G}$ , is the graph obtained from  $G$  by subdividing every edge in  $G$ . In particular, note that  $\tilde{G}$  is always bipartite.

**Theorem 7.8.** *Let  $F$  be any field and let  $G = (V, E)$  be a graph on  $n$  vertices that contains the subgraph  $P_n$ . Then  $\text{mr}(F, \tilde{G}) = 2n - 2$ .*

*Proof.* By Proposition 7.6 it suffices to show that  $\text{mr}(F, \tilde{G}) \geq 2n - 2$ . Label the vertices of  $P_n$  consecutively as  $v_1, v_2, \dots, v_n$  and let  $e_{i i+1}$  be the edge incident to  $v_i$  and  $v_{i+1}$ . Let  $y_{i i+1}$  be the new vertex in  $\tilde{G}$  formed by the subdivision of  $e_{i i+1}$ . Then  $v_1, y_{12}, v_2, y_{23}, \dots, v_{n-1}, y_{n-1 n}, v_n$  induce  $P_{2n-1}$  in  $\tilde{G}$ . By Observation 3.9,

$$\text{mr}(F, \tilde{G}) \geq \text{mr}(F, P_{2n-1}) = 2n - 2,$$

which concludes the proof.  $\square$

The next Lemma was in [JLS], but this is an alternative proof.

**Lemma 7.9.** *Let  $F$  be any field, let  $G$  be any graph, and let  $e$  be an edge of  $G$ . Then*

$$\text{mr}(F, G) \leq \text{mr}(F, G_e) \leq \text{mr}(F, G) + 1 \tag{17}$$

$$M(F, G) \leq M(F, G_e) \leq M(F, G) + 1. \tag{18}$$

*Proof.* The inequalities (18) follow immediately from (17) but it is convenient to state both sets of inequalities. Let  $v, w$  be the vertices of  $e$  and let  $u$  be the new vertex in  $G_e$  that is adjacent to  $v$  and  $w$ .

We first prove that  $\text{mr}(F, G) \leq \text{mr}(F, G_e)$ . Let

$$A = \begin{bmatrix} d_1 & a & b & 0^T \\ a & d_2 & 0 & x^T \\ b & 0 & d_3 & y^T \\ 0 & x & y & C \end{bmatrix} \in S(F, G_e)$$

with  $\text{rank } A = \text{mr}(F, G_e)$  and with the first three rows and columns of  $A$  labeled by  $u, v$  and  $w$ . Then  $a, b \neq 0$ . Let  $B$  be the matrix obtained from  $A$  by adding row 1 to row 2 and column 1 to column 2. Then  $\text{rank } B = \text{rank } A$  and  $B(u) \in S(F, G)$ . It follows that

$$\text{mr}(F, G) \leq \text{rank } B(u) \leq \text{rank } B = \text{rank } A = \text{mr}(F, G_e).$$

To prove the second inequality, let

$$A = \begin{bmatrix} d_1 & a & b^T \\ a & d_2 & c^T \\ b & c & B \end{bmatrix} \in S(F, G)$$

with  $\text{rank } A = \text{mr}(F, G)$  and the first two rows and columns of  $A$  labeled by  $v$  and  $w$ .

Then  $a \neq 0$  and

$$A_e = \begin{bmatrix} 0 & 0 & 0 & 0^T \\ 0 & d_1 & a & b^T \\ 0 & a & d_2 & c^T \\ 0 & b & c & B \end{bmatrix} - \begin{bmatrix} a & a & a & 0^T \\ a & a & a & 0^T \\ a & a & a & 0^T \\ 0 & 0 & 0 & 0 \end{bmatrix} \in S(F, G_e).$$

It follows that

$$\text{mr}(F, G_e) \leq \text{rank } A_e \leq \text{rank } A + 1 = \text{mr}(F, G) + 1.$$

□

Our next aim is to give an important case of equality for the first inequality in (18), but first we need some additional results.

**Proposition 7.10.** *Let  $F$  be a field and let  $G$  be a graph with a vertex  $u$  of degree 2. Assume the neighbors  $v, w$  of  $u$  are adjacent, and let  $e = vw$ . Then we have*

a) if  $F \neq F_2$ ,  $\text{mr}(F, G) \leq \text{mr}(F, G - u) + 1$ .

b) if  $F \neq F_2$ ,  $M(F, G - u) \leq M(F, G)$ .

c)  $\text{mr}(F, G) \leq \text{mr}(F, (G - u) \setminus e) + 1$ .

d)  $M(F, (G - u) \setminus e) \leq M(F, G)$ .

*Proof.* Let

$$A = \begin{bmatrix} a & b & x^T \\ b & c & y^T \\ x & y & D \end{bmatrix} \in S(F, G - u)$$

with  $\text{rank } A = \text{mr}(F, G - u)$  and with the first two rows and columns of  $A$  labeled by  $v$  and  $w$ . Since  $F \neq F_2$ , there exists  $d \in F$  such that  $d \neq 0$  and  $d \neq -b$ . Then

$$B = \begin{bmatrix} 0 & 0^T \\ 0 & A \end{bmatrix} + d \begin{bmatrix} J_3 & O \\ O & O \end{bmatrix} \in S(F, G)$$

and  $\text{mr}(F, G) \leq \text{rank } B \leq \text{rank } A + 1 = \text{mr}(F, G - u) + 1$  which proves a). Then b) follows from  $\text{mr}(F, G) = n - M(F, G)$  and  $\text{mr}(F, G - u) = n - 1 - M(F, G - u)$ .

Now let

$$A = \begin{bmatrix} d_1 & 0 & b^T \\ 0 & d_2 & c^T \\ b & c & D \end{bmatrix} \in S(F, (G - u) \setminus e).$$

with  $\text{rank } A = \text{mr}(F, (G - u) \setminus e)$ , and let  $a$  be a nonzero element of  $F$ . Then

$$B = \begin{bmatrix} 0 & 0^T \\ 0 & A \end{bmatrix} + a \begin{bmatrix} J_3 & O \\ O & O \end{bmatrix} \in S(F, G)$$

and as before,  $\text{mr}(F, G) \leq \text{rank } B \leq \text{rank } A + 1 = \text{mr}(F, (G - u) \setminus e) + 1$ , which establishes c). Claim d) follows from c) in the same way that b) follows from a).  $\square$

We need the following definition, extending  $S(F, G)$  to graphs which may have multiple edges, and theorem from [van08].

**Definition 7.11.** Let  $G = (V, E)$  be a multigraph on  $n$  vertices.

If  $F \neq F_2$ , define  $S(F, G)$  as the set of all  $F$ -valued symmetric  $n \times n$  matrices  $A = [a_{ij}]$  with

1.  $a_{ij} = 0$  if  $i \neq j$  and  $i$  and  $j$  are not adjacent,
2.  $a_{ij} \neq 0$  if  $i \neq j$  and  $i$  and  $j$  are connected by exactly one edge,
3.  $a_{ij} \in F$  if  $i \neq j$  and  $i$  and  $j$  are connected by multiple edges, and
4.  $a_{ii} \in F$  for all  $i \in V$ .

If  $F = F_2$ , we define  $S(F_2, G)$  as the set of all  $F_2$ -valued symmetric  $n \times n$  matrices  $A = [a_{ij}]$  with

1.  $a_{ij} \neq 0$  if  $i \neq j$  and  $i$  and  $j$  are connected by an odd number of edges,
2.  $a_{ij} = 0$  and  $i$  and  $j$  are connected by an even number of edges, and
3.  $a_{ii} \in F_2$  for all  $i \in V$ .

Then  $\text{mr}(F, G)$  and  $\text{M}(F, G)$  are defined by the same formulas as at the beginning of the thesis.

**Theorem 7.12.** *[van08] Let  $F$  be a field, let  $G$  be a graph, and let  $u$  be a vertex of degree two in  $G$  with neighbors  $v$  and  $w$ . Then*

$$\text{M}(F, G) = \max\{\text{M}(F, (G - u) + vw), \text{M}(F, (G - u)/vw)\},$$

where the graphs  $(G - u) + vw$  and  $(G - u)/vw$  are understood to include any multiple edges that may occur in doing these graph operations.

We now prove that  $M(F, G)$  is unchanged if an edge adjacent to a degree two vertex or a degree one vertex is subdivided.

**Theorem 7.13.** *Let  $F$  be a field, let  $G = (V, E)$  be a graph, let  $e$  be an edge adjacent to a vertex of degree at most 2, and let  $G_e$  be the graph obtained by subdividing  $e$  once. Then  $M(F, G_e) = M(F, G)$ .*

*Proof.* Let  $u, w$  be the vertices of  $e$ .

**I.**  $\text{degree}(u) = 2$ :

Let  $v$  be the other vertex adjacent to  $u$ . Let  $u'$  be the new vertex in  $G_e$ , By Theorem 7.12 applied to the degree two vertex  $u'$ ,

$$M(F, G_e) = \max\{M(F, (G_e - u') + uw), M(F, (G_e - u')/uw)\}.$$

But  $(G_e - u') + uw = G$ , so

$$M(F, G_e) = \max\{M(F, G), M(F, (G_e - u')/uw)\}. \quad (19)$$

**Case 1.**  $vw \notin E$ . Then  $G$  is an edge subdivison of  $(G_e - u')/uw$ . By Lemma 7.9,  $M(F, G) \geq M(F, (G_e - u')/uw)$  and we conclude that  $M(F, G_e) = M(F, G)$ .

**Case 2.**  $vw \in E$ . Then there are two edges from  $v$  to  $w$  in  $(G_e - u')/uw$ .

Subcase 1.  $F \neq F_2$ . It follows from Definition 7.11 that

$$M(F, (G_e - u')/uw) = \max\{M(F, G - u), M(F, (G - u) \setminus vw)\}.$$

Applying Proposition 7.10 b), d) yields  $M(F, (G_e - u')/uw) \leq M(F, G)$  so by (18),  $M(F, G_e) \leq M(F, G)$ , and by Proposition 7.9, equality holds.

Subcase 2.  $F = F_2$ . By Definition 7.11,  $M(F, (G_e - u')/uw) = M(F, (G - u) \setminus vw)$ . By Proposition 7.10 d), this is less than or equal to  $M(F, G)$ , and again by (18) and Proposition 7.9,  $M(F, G_e) = M(F, G)$ .

**II.**  $\text{degree}(u) = 1$ :

In this case,  $G_e$  is isomorphic to  $G \oplus_u K_2$ . Since the degree of  $u$  in  $G \oplus_u K_2$  is 2,

$$\text{mr}(F, G \oplus_u K_2) = \text{mr}(F, G) + \text{mr}(F, K_2) = \text{mr}(F, G) + 1$$

by Theorem 57 in [BGL09]. Therefore,  $\text{mr}(F, G_e) = \text{mr}(F, G) + 1$  and  $M(F, G_e) = M(F, G)$ .

This completes the proof.  $\square$

**Corollary 7.14.** *Let  $G$  be a graph in which every edge is adjacent to a vertex of degree at most 2, and let  $H$  be a graph obtained by edge subdivisions of  $G$ . Then  $M(F, H) = M(F, G)$ .*

Now, we have enough information to determine the minimum rank of any graph obtained from  $W_5$  using edge subdivisions. We label the wheel on 5 vertices,  $W_5$ , in the following way:

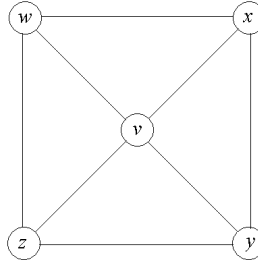


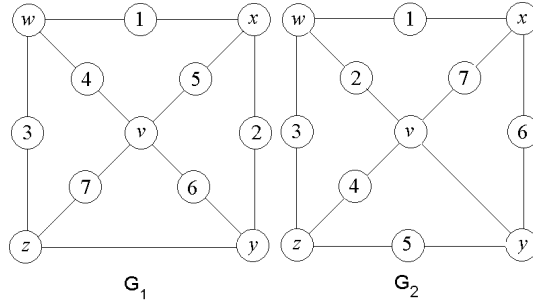
Figure 16: The wheel on 5 vertices,  $W_5$

and note that by Theorem 7.8,  $M(F, \widetilde{W}_5) = 5$ . It is also known that  $M(F, W_5) = 3$ . By Lemma 7.9, and Theorem 7.13, if  $G$  is obtained by subdividing edges of  $W_5$ , then  $3 \leq M(F, G) \leq 5$ .

Let  $\mathcal{I}(W_5)$  be the set of all graphs obtained from  $W_5$  by subdividing each edge at most once. By Theorem 7.13, it suffices to solve the problem for only those graphs in  $\mathcal{I}(W_5)$ .

Up to isomorphism, there are 2 graphs of  $\mathcal{I}(W_5)$  with exactly one edge left unsubdivided. These graphs are pictured below. Since  $\widetilde{W}_5$  can be obtained from either of these

by exactly one subdivision, Lemma 7.9 implies the maximum nullity for either of these graphs must be at least 4. We first consider  $G_1$ .



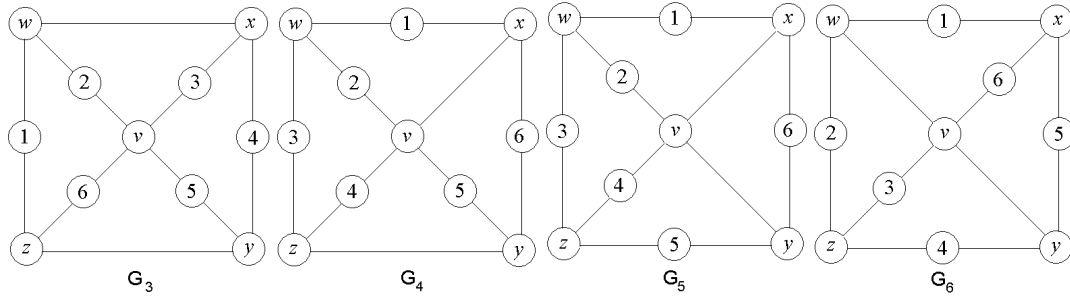
The set  $\{v, 1, 6, 7\}$  forms a zero forcing set for this graph, so

$$M(F, G_1) \leq Z(G_1) \leq 4.$$

Therefore  $M(F, G_1) = 4$ .

Now, we consider  $G_2$ . The set  $\{w, 1, 2, 7\}$  forms a zero forcing set for this graph, so similarly,  $M(F, G_2) = 4$ . It follows that if  $H \in \mathcal{I}(W_5)$  and  $H \neq \widetilde{W}_5$ , then  $M(F, H) \leq 4$ .

There are six graphs in  $\mathcal{I}(W_5)$  with exactly 2 edges left unsubdivided. We presently consider 4 of these and save discussion of the other 2 for later:



$G_3$  has a zero forcing set of size 3, namely  $\{v, 2, 3\}$  Hence

$$3 \leq M(F, G_3) \leq Z(G_3) \leq 3.$$



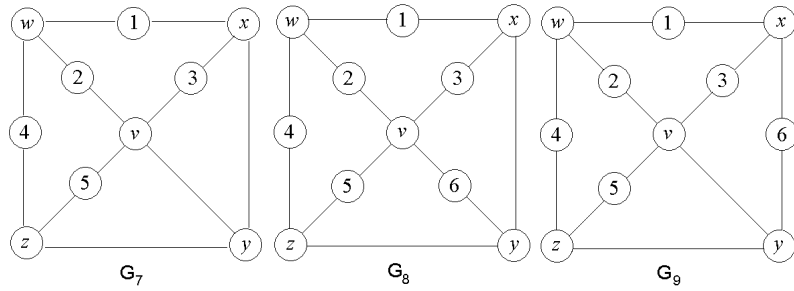
So  $M(F, G_3) = 3$ .

$G_4$ :  $\{w, 1, 2\}$  is a zero forcing set, so similarly  $M(F, G_4) = 3$

$G_5$ :  $\{w, 1, 2\}$ , is a zero forcing set, so  $M(F, G_5) = 3$ .

$G_6$ : A zero forcing set for this graph is  $\{z, 2, 3\}$ , and  $M(F, G_6) = 3$

Note that if  $H \in \mathcal{I}(W_5)$ , and if one of  $G_1, \dots, G_6$  can be obtained by subdividing edges of  $H$ , then  $M(F, H) = 3$ . By examining a table of graphs we see that we have now determined  $M(F, H)$  for all  $H \in \mathcal{I}(W_5)$  of  $W_5$  except the following 3 graphs:



(we have chosen one among many equivalent labelings for each of these graphs). Note that, for all of the graphs considered thus far in this example, we have found maximum nullity independent of our field. Such is not the case for the remaining graphs. We temporarily add the assumption that  $F \neq F_2$ .

Since  $G_1$  is a subdivision of  $G_7$ ,

$$3 \leq M(F, G_7) \leq 4.$$

Applying Proposition 7.6 with  $F \neq F_2$  and  $X = \{v, w, x, z\}$  gives

$$\text{mr}(F, G_7) \leq 6.$$

Hence  $M(F, G_7) \geq 4$ . So  $M(F, G_7) = 4$ .

Since  $G_8$  and  $G_9$  can be obtained from  $G_7$  by subdividing edges, we must also have  $M(F, G_8) = M(F, G_9) = 4$  if  $F \neq F_2$ . We summarize these findings in the following proposition:

**Proposition 7.15.** *Let  $F$  be a field,  $F \neq F_2$ . If  $H$  is a graph obtained from  $W_5$  by subdividing edges, then the following hold:*

$$M(F, H) = 5 \text{ if all edges of } W_5 \text{ have been subdivided}$$

$$M(F, H) = 4 \text{ if } H \text{ is a subdivision of } G_1, G_2, \text{ or } G_7$$

*and not all edges of  $W_5$  have been subdivided*

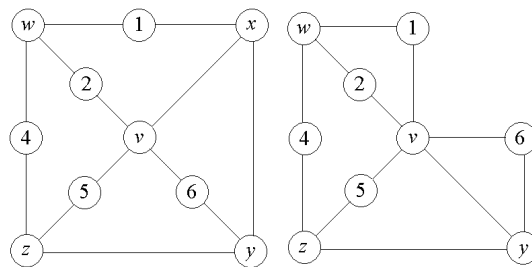
$$M(F, H) = 3 \text{ otherwise}$$

Now, for the case  $F = F_2$ : we have examined 4 of the possible 6 graphs in  $\mathcal{I}(W_5)$  that are obtained from  $W_5$  by subdividing exactly 6 edges. Now we examine the other 2, in turn. First, we look at  $G_8$ :

Using Theorem 7.12, with  $u$  equal to the vertex 3,

$$M(F_2, G_8) = \max\{M(F_2, (G_8 - 3) + vx), M(F_2, (G_8 - 3)/vx)\}.$$

These two graphs are the following:



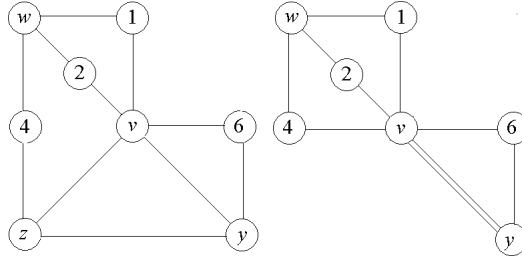
Notice that subdividing  $xy$  in the first graph yields  $G_4$ , so we have  $M(F_2, G_8 - 3) + vx = 3$ .

We use Theorem 7.12 again on the second graph, with  $u$  equal to vertex 5.

$$M(F_2, G_8) = \max\{3, M(F_2, (G_8 - 3)/vx) - 5 + vz\},$$

$$M(F_2, (G_8 - 3)/vx - 5)/vz\}.$$

These two graphs are the following:

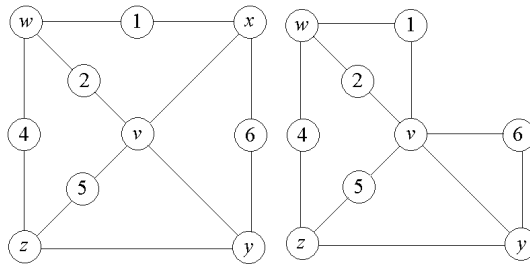


The first has a zero forcing set of  $\{w, 1, 2\}$ , and hence has maximum nullity at most 3. Since we are working over  $F_2$ , we can replace the double edge from  $v$  to  $y$  in the second graph by no edge.  $\{w, 1, 2\}$  is again a zero forcing set, and the maximum nullity over  $F_2$  is at most 3. Therefore,  $M(F_2, G_8) = 3$ .

The final graph we examine is  $G_9$ . Again using Theorem 7.12, with  $u$  equal to the vertex 3,

$$M(F_2, G_9) = \max\{M(F_2, (G_9 - 3) + vx), M(F_2, (G_9 - 3)/vx)\}$$

These two graphs are the following:



We recognize that the second of these is isomorphic to  $(G_8 - 3)/vx$  from above, so we only need consider the first. The first graph has a zero forcing set of  $\{w, 1, 2\}$ , so its

maximum nullity is at most 3. Therefore,

$$M(F_2, G_9) = 3$$

It follows that  $G_1$  and  $G_2$  are the only graphs in  $\mathcal{I}(W_5)$  with maximum nullity 4 over  $F_2$ . We summarize our findings over  $F_2$  as follows:

**Proposition 7.16.** *If  $H$  is a graph obtained from  $W_5$  by subdividing edges, then the following hold:*

$$M(F_2, H) = 5 \text{ if all edges of } W_5 \text{ have been subdivided}$$

$$M(F_2, H) = 4 \text{ if } H \text{ is a subdivision of } G_1 \text{ or } G_2$$

*and not all edges of  $W_5$  have been subdivided*

$$M(F_2, H) = 3 \text{ otherwise}$$

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