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A Lift of Cohomology Eigenclasses of Hecke Operators

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ABSTRACT

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A considerable amount of evidence has shown that for every prime $p \neq N$ observed, a simultaneous eigenvector v_0 of Hecke operators $T(\ell, i),\ i = 1, 2$, in $H^3(\Gamma_0(N), F(0, 0, 0))$ has a "lift" v in $H^3(\Gamma_0(N), F(p-1, 0, 0))$ — i.e., a simultaneous eigenvector v of Hecke operators having the same system of eigenvalues that v_0 has. For each prime $p > 3$ and $N = 11$ and 17, we construct a vector v that is in the cohomology group $H^3(\Gamma_0(N), F(p - 1, 0, 0))$. This is the first construction of an element of infinitely many different cohomology groups, other than modulo p reductions of characteristic zero objects. We proceed to show that v is an eigenvector of the Hecke operators $T(2, 1)$ and $T(2, 2)$ for $p > 3$. Furthermore, we demonstrate that in many cases, v is a simultaneous eigenvector of all the Hecke operators.

Keywords: Serre's Conjecture, Hecke operator, cohomology group, lift, eigenvector

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Contents

Chapter 1. Introduction

The study of Galois representations has been very helpful in obtaining important results in number theory. For instance, in his proof of Fermat's Last Theorem [36], Wiles employed some proven cases of Serre's conjecture on the modularity of Galois representations (see [30] — we will refer to this as simply "Serre's conjecture"). More specifically, and perhaps more importantly due to its implications, the conjecture was used to help prove the modularity of elliptic curves. In fact, this conjecture has played an important role in number theory for over three decades; recently it was proven by Khare and Wintenberger [24], and by Kisin [25, Corollary 0.2].

The question then naturally arises, to what extent can these Galois representations be wielded; i.e., what generalizations of theorems of classical number theory can be proven, and which generalities of Galois representations can be found, especially those applicable to number theory proofs? The introduction to [1] states that the classical law of quadratic reciprocity is interpretable in terms of Galois representations, motivating a search for a more generalized reciprocity law that would connect Galois representations with other mathematical objects (which are, in the case of that paper and this, certain group cohomology classes). One can therefore think of Serre's conjecture as being one on reciprocity. Serre's conjecture is a partial converse to Theorem 6.7 in [14], which states that a classical modular form of some positive weight has an odd two-dimensional Galois representation "attached" (see [7, Def. 1.1]) to it. Ash in [2] conjectured a generalization of this theorem and checked several simple cases. This is, as far as we know, the first formulation of a generalization of this type. The authors of [1] tested this generalization using a homological analogue (for computational purposes), provided a partial verification thereof, and proved a similar conjecture of their own for the cases $p = 5, 7$.

The correspondence observed in the two-dimensional case in Serre's Conjecture naturally prompted the question about whether there is a connection between Hecke eigenclasses and higher-dimensional characteristic p Galois representations. The authors of [7] generalized Serre's conjecture as a connection between Hecke eigenclasses and n-dimensional Galois representations in characteristic p, focusing on the case $n = 3$. A refinement of this generalization, along with a considerable amount of computational evidence supporting it, was presented in [3] (it is this particular refinement to which we refer as "the generalized conjecture"). Since the time of the publication of [3], much more work has been done; several examples in [4], [5], [6], [17], [19], and [29] provide additional evidence, while [6], [15], [17], [18], and [21] give further refinements of the conjecture. The only known proven cases of the generalized conjecture are those presented in [7], which gives a few classes of Galois representations attached to the appropriate eigenclasses. There are about 200 specific examples of these classes shown in [10].

We do not give any further proven cases of the generalized conjecture in this paper, instead focusing on an apparent correlation we have noticed between Hecke eigenclasses in the cohomologies of two related weights predicted by the conjecture for certain Galois representations. Our approach to explaining this correlation is to make it clear enough that, given one eigenclass, we can make a prediction about the other. We attempt to do this by (1) rewriting the eigenclasses in a more readable form, (2) carefully choosing the coset representatives that we use in our calculations, and (3) studying how the eigenclasses depend on these coset representatives for a few small p to get a formula for the general case. Throughout, we use a result of Allison, Ash, and Conrad specialized for our situation, along with a standard isomorphism of cohomology groups, and the natural duality of homology and cohomology to calculate the cohomology.

We use the given eigenclass alluded to above (which we call v_0) to construct an element v in each of infinitely many cohomology groups, finding strong evidence that it corresponds to a Galois representation in the sense of the generalized conjecture. This v is therefore a kind of "lift" of an eigenclass in a related cohomology group. This is the first time of which we are aware that such an element has been constructed, the benefit being that we do not have to calculate the entire cohomology to obtain it. Previous methods of calculating the cohomology involved computing with modules having dimension in the hundreds of thousands, even for primes as small as 101. In contrast, we determine v as a relatively simple function of p.

It turns out that v is also an eigenvector of the Hecke operators $T(2,i)$ for $i = 1, 2$. A proof that v is a Hecke eigenclass of all Hecke operators $T(\ell, i)$ would be of significant importance, because if we can find a Galois representation corresponding to the already-known eigenclass v_0, we likely will be able to find a Galois representation corresponding to its lift v, perhaps leading the way to a great source of potential evidence for the generalized conjecture. In fact, v is an eigenclass for each of the Hecke operators we have observed; unfortunately, we have not been able to conclude that it is for all ℓ. However, because v is in a finite-dimensional space, repeated application of Hecke operators on v eventually yields a Hecke-stable space, guaranteeing the existence of some Hecke eigenclass. Therefore, either v is a Hecke eigenclass, or we have computed a significant portion of the cohomology using Hecke operators, enabling us to find more Hecke eigenclasses. Either result gives us useful elements with which to work to find evidence for the generalized conjecture.

An outline of the paper is as follows: in Chapter 2, we state Serre's conjecture and introduce the generalized conjecture in more detail, noting the comparison between the two and creating the setting for the relation of our work to the problem of establishing the generalized conjecture. More specifically, as the generalized conjecture connects cohomology eigenclasses with Galois representations, we also describe, in Example 2, the representation that appears to correspond to the eigenvector v_0 about which we are concerned.

In Chapter 3, we describe the previously-observed relation between eigenclasses in related weights, some theorems and notation used in our calculations, and the construction of the two eigenclasses in question, v_0 and v. The construction is determined as follows: first we identify an eigenclass $v_0 \in H^3(\Gamma_0(N), F(0,0,0))$ of the Hecke operators. We then find a basis for $H^3(\Gamma_0(N), F(p-1,0,0))$, applying the Hecke operator to each basis element, one by one. Each result is described in terms of the basis, giving us a transformation matrix. We examine the eigenvectors of this matrix for a few small p, finding one having a predictable correlation with v_0, which we call v.

We prove in Chapter 4 that this v is, in a sense, a lift of v_0, by showing that it is in the cohomology $H^3(\Gamma_0(N), F(p-1,0,0))$ for each p. We also prove that it is an eigenvector of the first two Hecke operators.

Finally, the Appendix contains the code to several computer programs which were indispensable as we conducted our research into the solution of this problem.

Chapter 2. Serre's Conjecture and a Generalization

2.1 Serre's Conjecture

In this section we present Serre's conjecture; see [12]. We begin with a few basic preliminaries about modular forms. Refer to [26, pp. 222-7] for more detail.

For $\tau \in \mathbb{C}$ with $\operatorname{Im} \tau > 0$, let $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ act on τ by $M\tau = \dfrac{a\tau + b}{c\tau + d}$.

Definition 1. Let $k \in \mathbb{Z}$, N be a positive integer and $\epsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a homomorphism. Any analytic function f on the upper half plane $\mathcal{H}$ with

$$f(M\tau) = (c\tau + d)^k \epsilon(d) f(\tau)$$

for all $M \in \Gamma_0(N) = \{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in SL_2(\mathbb{Z}) \mid c \equiv 0 \bmod N\}$ and $\tau \in \mathcal{H}$ is called an *unrestricted modular form of weight k, level N, and nebentype* (or *character*) ϵ.

It is easy to see that each unrestricted modular form f is periodic, so that it has a Fourier expansion

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n q^n,$$

where $q = e^{2\pi i \tau}$. If f satisfies certain growth conditions at the cusps (which imply, among other things, that $a_n = 0$ for negative n in this expansion), f is said to be *holomorphic at the cusps* and is a *modular form.* A *normalized* modular form has 1 as its first nonzero Fourier coefficient, and a modular form is said to be an *eigenform* if it is a simultaneous eigenvector for Hecke operators T_ℓ for ℓ prime; see [26, p. 280]. Some examples of modular forms include

$$G_{2k}(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau + n)^{2k}},$$

the *Eisenstein Series* $E_{2k}(\tau) = \dfrac{G_{2k}(\tau)}{2\zeta(2k)}$, and $\Delta = \frac{1}{1728}(E_4^3 - E_6^2)$.

As Serre's Conjecture is a kind of "marriage" of both analytic and algebraic number theory, we now focus on the algebraic side. Let $\bar{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$, and let p be prime.

Definition 2. An n-dimensional Galois representation is a continuous homomorphism $\rho : G_{\mathbb{Q}} \to GL_n(\mathbb{F})$ from $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ (with the Krull topology [28, p. 2]) to the general linear group of invertible $n \times n$ matrices over a field $\mathbb{F}$ of characteristic p (having the discrete topology). If for all $g \in G_{\mathbb{Q}}$ we can write $\rho(g) = \rho_1(g) \oplus \rho_2(g)$ for nontrivial representations ρ_1, ρ_2 of $G_{\mathbb{Q}}$, then ρ is said to be *reducible*.

Given a prime q, let D_q be a decomposition group at q in $G_{\mathbb{Q}}$, and I_q the whole inertia group above q. The *Frobenius* of q is a special generator of D_q/I_q, and is denoted Frob_q [3, p. 522]. A Galois representation ρ is said to be *ramified* at q if I_q does not act trivially on $\mathbb{F}_p$; i.e., if the image of I_q under ρ is nontrivial.

Let ρ be a two-dimensional Galois representation. If the determinant of the matrix to which the complex conjugation map is sent is 1, ρ is called even; otherwise, it is odd. We put the analytic and algebraic sides together with the following

Definition 3. If there exists a normalized eigenform f of weight $k \geq 2$, level N, and character ϵ with Fourier coefficients a_n in $\mathbb{C}$ such that for all ℓ which are unramified for ρ and do not divide Np, the characteristic polynomial of $\rho(\mathrm{Frob}_\ell)$ is congruent to $x^2 - a_\ell x + \ell^{k-1}\epsilon(\ell)$ modulo a prime above p, then ρ is said to be *modular*, and ρ and f are *associated*.

As noted in the Introduction, it was previously shown in [14] that any eigenform f has an associated representation ρ. The general idea of Serre's conjecture is that the converse holds also: any odd irreducible representation ρ as above is modular.

Conjecture 1 (Serre's Conjecture). There exists a normalized mod p eigenform of level

$N(\rho)$, weight $k(\rho)$, and (when char $\mathbb{F} > 3$) character $\epsilon(\rho)$ which is associated to ρ, where $N(\rho)$, $k(\rho)$, and $\epsilon(\rho)$ are defined by a formula of Serre [12, p. 3].

Briefly, the level $N(\rho)$ is the Artin conductor of ρ, with all factors of p removed. The character $\epsilon(\rho)$ is obtained by identifying $\det \rho$ with the Dirichlet character $(\mathbb{Z}/N(\rho)p\mathbb{Z})^\times \to \mathbb{F}_p^\times$, which by the Chinese remainder theorem may be factored into a product of the characters $\epsilon(\rho) : (\mathbb{Z}/N(\rho)\mathbb{Z})^\times \to \mathbb{F}_p^\times$ and $\phi : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{F}_p^\times$. Our work is restricted to the case where $\rho|_{I_p}$ is upper triangularizable with cyclotomic characters on the diagonal, say with exponents a, b. Then

$$k(\rho) = 1 + a + b + (p-1)\min(a,b) + (p-1)\delta,$$

where $\delta = 1$ if ρ is unramified at p or if $\rho|_{I_p}$ is "très ramifié" [12, pp. 4-6]; otherwise $\delta = 0$.

We illustrate Serre's Conjecture with the following

Example 1. Consider the splitting field K of $f(x) = x^3 - x + 1$. We have $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3$. Let $\sigma : S_3 \to GL_2(\mathbb{F}_{23})$ be the homomorphism defined by

$$\sigma((1\ 2)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma((1\ 2\ 3)) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

We then have a Galois representation

$$\rho : G_{\mathbb{Q}} \xrightarrow{\pi} \mathrm{Gal}(K/\mathbb{Q}) \cong S_3 \xrightarrow{\sigma} GL_2(\mathbb{F}_{23}),$$

where π is the natural projection map. For all primes $\ell \neq 23$, the order of the image of

factorization of $f(x)$ mod ℓ	order of $\pi(\mathrm{Frob}_\ell)$	$\mathrm{Tr}(\rho(\mathrm{Frob}_\ell))$
three linear factors	1	2
one linear, one quadratic	2	0
irreducible	3	-1

Table 2.1: Calculation of $\mathrm{Tr}(\rho(\mathrm{Frob}_\ell))$ according to factorization of $f(x)$

ℓ	2	3	5	7	11	13	17	19	23	29	31	37
$\mathrm{Tr}(\rho(\mathrm{Frob}_\ell))$	-1	-1	0	0	0	-1	0	0	*	-1	-1	0
a_ℓ	-1	-1	0	0	0	-1	0	0	1	-1	-1	0

ℓ	41	43	47	53	59	61	67	71	73	79	83	89
$\mathrm{Tr}(\rho(\mathrm{Frob}_\ell))$	-1	0	-1	0	2	0	0	-1	-1	0	0	0
a_ℓ	-1	0	-1	0	2	0	0	-1	-1	0	0	0

Table 2.2: Association of ρ with Δ.

the Frobenius element Frob_ℓ under the projection π depends on the factorization of $f(x)$ modulo ℓ; therefore so does $\mathrm{Tr}(\rho(\mathrm{Frob}_\ell))$. This is illustrated in Table 2.1, and justified by a theorem of Dedekind which relates the cycle structure of Frob_ℓ with the factorization of a polynomial modulo ℓ. (Dedekind's result follows from [27, Thm. 27], which gives the prime decomposition of a prime lying above ℓ in terms of the ideals generated by the prime and the factors of the polynomial modulo ℓ, and from [27, Thm. 33], which relates the splitting of a prime with the orbits of Frob_ℓ as it acts on the cosets of the Galois subgroup fixing the field in which the prime lies. The connection is made by a natural bijection between the cosets and the roots of the polynomial, which preserves the action of the Galois group.)

In this case, ρ is associated with Δ, one of the examples of modular forms mentioned above. Observe Table 2.2, where a_ℓ denotes the coefficient of q^ℓ in the q-expansion of Δ, reduced modulo 23.

2.2 The Generalized Conjecture

We now introduce a generalized version of Serre's conjecture, which we refer to more simply as the "generalized conjecture." This conjecture is based on the refinement of the one made in [7] due to Ash, Doud, and Pollack (Conjecture 3.1 in [3]), and is focused on the case $n = 3$. We proceed exactly as in [3], adopting Definition 2 for an n-dimensional Galois representation, taking care to note that in the n-dimensional case, an *odd* representation ρ "satisfies strict parity" (see Definition 2.10 in [3]).

In the case of $n = 3$, $\Gamma_0(N)$ is the subgroup of matrices in $SL_3(\mathbb{Z})$ whose first row is congruent to $(*, 0, 0)$ modulo N. Let S_N be the subsemigroup of integral matrices in $\mathrm{GL}_n(\mathbb{Q})$ satisfying the same congruence condition and having positive determinant relatively prime to N. Let $\mathcal{H}(N)$ denote the $\bar{\mathbb{F}}_p$-algebra of double cosets $\Gamma_0(N)\backslash S_N/\Gamma_0(N)$. Then $\mathcal{H}(N)$ is a commutative algebra that acts on the cohomology and homology of $\Gamma_0(N)$ with coefficients in any $\bar{\mathbb{F}}_p[S_N]$-module. A double coset is called a Hecke operator when it acts on cohomology or homology. Note that $\mathcal{H}(N)$ contains all double cosets of the form $\Gamma_0(N)D(\ell, k)\Gamma_0(N)$, where ℓ is a prime not dividing N, $0 \leq k \leq 3$, and $D(\ell, k)$ is the diagonal matrix with the first $3 - k$ diagonal entries equal to 1 and the last k diagonal entries equal to ℓ. If viewed as a Hecke operator, the double coset generated by $D(\ell, k)$ is abbreviated $T(\ell, k)$.

Definition 4. Let V be an $\mathcal{H}(pN)$-module and suppose that $v \in V$ is a simultaneous eigenvector for all $T(\ell, k)$ and that $T(\ell, k)v = a(\ell, k)v$ with $a(\ell, k) \in \bar{\mathbb{F}}_p$ for all prime ℓ not dividing pN, and for $0 \leq k \leq 3$. Let $\rho : G_{\mathbb{Q}} \to GL_3(\bar{\mathbb{F}}_p)$ be a representation unramified outside pN and assume that

$$\sum_{k=0}^{3}(-1)^k \ell^{k(k-1)/2} a(\ell, k)X^k = \det(I - \rho(\mathrm{Frob}_\ell)X)$$

for all ℓ not dividing pN. Then we say that ρ is *attached* to v or that v corresponds to ρ [3, p. 523].

We may pull $\epsilon(\rho)$ (the Dirichlet character mentioned in Section 2.1 above) back to S_N by defining it to be the composite character

$$S_N \to (\mathbb{Z}/N(\rho)\mathbb{Z})^\times \to \mathbb{F}_p^\times,$$

where the first map takes a matrix in S_N to its (1,1) entry. Define $\mathbb{F}_\epsilon$ to be the $\bar{\mathbb{F}}_p$-vector space $\bar{\mathbb{F}}_p$ with the action of S_N given by ϵ. Given a $\mathrm{GL}_n(\mathbb{F}_p)$-module V, define

$$V(\epsilon) = V \otimes \mathbb{F}_\epsilon.$$

Since $\Gamma_0(N)$ acts on V by reduction modulo p and S_{pN} acts on $\mathbb{F}_\epsilon$, $V(\epsilon)$ is both a $\Gamma_0(N)$-module and an S_{pN}-module [3, p. 524].

The weights associated to a 3-dimensional Galois representation ρ will be certain irreducible $\mathrm{GL}_3(\mathbb{F}_p)$-modules. The naturality of this follows from the following chain of observations: the Eichler-Shimura theorem (see [31]) relates the space of modular forms of weight k to cohomology with coefficients in $\mathrm{Sym}^g(\mathbb{C}^2)$ with $g = k - 2$. Hence, an eigenform f of level N, nebentype ϵ, and weight k gives rise to a collection of Hecke eigenvalues which, when reduced modulo p, also "occurs" (see [8, Def. 1.2.1(b)]) in $H^1(\Gamma_0(N), V_g(\epsilon))$, where $V_g \cong \mathrm{Sym}^g(\bar{\mathbb{F}}_p^2)$ is the space of two-variable homogeneous polynomials of degree g over $\bar{\mathbb{F}}_p$ with the natural action of $\mathrm{SL}_2(\bar{\mathbb{F}}_p)$. Ash and Stevens have shown in [8] that any system of Hecke eigenvalues occurring in the cohomology of $\Gamma_0(N)$ with coefficients in some $\mathrm{GL}_n(\mathbb{F}_p)$-module also occurs in the cohomology with coefficients in at least one irreducible $\mathrm{GL}_n(\mathbb{F}_p)$-module occurring in a composition series of the original module. Hence, there is some irreducible $\mathrm{GL}_2(\mathbb{F}_p)$-module W such that the system of eigenvalues coming from f also occurs in $H^1(\Gamma_0(N), W(\epsilon))$ [3, p. 525]. Generalizing to the 3-dimensional case, it is natural to study the cohomology of irreducible $\mathrm{GL}_3(\mathbb{F}_p)$-modules. Such a module is denoted as $F(a, b, c)$, where the triple (a, b, c) is *p-restricted*; that is,

$$0 \leq a - b \leq p - 1,$$

$$0 \leq b - c \leq p - 1,$$

$$0 \leq c \leq p - 2$$

$$
\begin{tikzcd}
& F(2(p-1), p-1, 0) & \\
F(p-1, p-1, 0) \arrow[ur] & & F(p-1, 0, 0) \arrow[ul] \\
& F(0, 0, 0) \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Figure 2.1: Alcove Geometry (see [16, p. 423]). Evidence suggests that an eigenclass in the cohomology of any one of these modules has a lift in the cohomology of every module shown here above it.

([16, p. 412]; see also [20, Thm. 6.4b]). There are therefore $p^2(p-1)$ such modules. (The notation reflects the parametrization of irreducible $GL_3(\mathbb{F}_p)$-modules by p-restricted triples described in [16].)

Definition 5. Let (a_1, a_2, a_3) be a triple of integers. When (a_1, a_2, a_3) is p-restricted, we define $F(a_1, a_2, a_3)$ to be the associated irreducible $\mathrm{GL}_3(\mathbb{F}_p)$-module. Denote by $(a_1, a_2, a_3)'$ the set of all p-restricted triples (b_1, b_2, b_3) for which $a_i \equiv b_i \pmod{p-1}$. Then by $F(a_1, a_2, a_3)'$ we mean the set of irreducible $\mathrm{GL}_3(\mathbb{F}_p)$-modules corresponding to triples in $(a_1, a_2, a_3)'$.

Obviously, in certain cases (namely, when some $a_i \equiv a_{i+1} \pmod{p-1}$) there may be more than one triple in $(a_1, a_2, a_3)'$. In this case we interpret any statement concerning $(a_1, a_2, a_3)'$ to mean that the statement is true for some choice of (b_1, b_2, b_3) as in the definition. For example, a statement about $(a, a, 0)'$ is true if it is for either $(a, a, 0)$ or $(a + p - 1, a, 0)$ (or both). In this paper we are primarily concerned with the class of modules represented by $F(0, 0, 0)'$: a statement true for $F(0, 0, 0)$ we wish to prove is also true for $F(p-1, 0, 0)$. Ultimately, we would like to prove that similar statements are true for the other two modules represented by $F(0, 0, 0)'$, namely, $F(p-1, p-1, 0)$ and $F(2(p-1), p-1, 0)$ [3, p. 526]. See Figure 2.1.

Conjecture 2 (Generalized Conjecture for $n = 3$). If $\rho : G_{\mathbb{Q}} \to GL_3(\mathbb{F}_p)$ is an odd Galois representation with level N and nebentype ϵ, and is not the sum of an odd 2-dimensional representation and a character, then ρ is attached to a cohomology class in $H^3(\Gamma_0(N), V(\epsilon))$ for a weight V of the form $F(a, b, c)$, described above [3, pp. 531-532].

Stated vaguely, the generalized claim (for $n = 3$) is that ρ is attached to an eigenclass in $H^3(\Gamma_0(N), V(\epsilon))$. Serre's conjecture is a special case of the generalized conjecture; see [3, Thms. 3.7, 3.8]. In particular, as in Serre's conjecture, the generalized conjecture gives specific predictions about which weights V yield eigenclasses attached to ρ. In this case, if, upon upper-triangularizing $\rho|_{I_p}$, the exponents of the cyclotomic characters on the diagonal are a, b, and c, the weight predicted is $V = F(a-2, b-1, c)$. If, furthermore, ρ is reducible, the relation of its image with respect to certain Levi subgroups further filters the predicted weights (for more detail, see the discussion in [3, pp. 526-531]).

The following example describes the Galois representation that is believed to correspond to an eigenclass $v_0 \in H^3(\Gamma_0(N), F(0, 0, 0))$ about which we are concerned.

Example 2. By Definition 4, if the Galois representation ρ corresponds to a given eigenclass $\beta \in H^3(\Gamma_0(N), F(0, 0, 0))$, then the trace of a Frobenius element at ℓ under the image of ρ is the eigenvalue $a(\ell, 1)$. It is therefore easy to verify, by the use of a computer, a claim that ρ cannot be decomposed into a direct sum of three characters: simply run through all the possibilities of traces of such direct sums, and if the trace is $a(\ell, 1)$ for one in particular, check the trace of that direct sum for higher ℓ, until a case in which they do not match is found. We run this test on $\beta = v_0$ (which we will define in Sections 3.1 and 3.3) and find that in this case ρ cannot be broken up into a direct sum of three characters. We therefore attempt to find a different decomposition of the ρ corresponding to v_0.

From a table of elliptic curves, we find the elliptic curve E of conductor N given by the equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

(For $N = 11, a_1 = 0, a_2 = -1, a_3 = 1, a_4 = -10, a_6 = -20$; for $N = 17, a_1 = 1, a_2 = -1, a_3 = 1, a_4 = -1, a_6 = -14$.) According to [26, Thm. 6.14], we have $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$, where Λ is a lattice of points in $\mathbb{C}$. In the interior of each parallelogram of Λ in the complex plane, there is a $p \times p$ lattice of p-torsion points; therefore, the p-torsion subgroup of this curve over $\mathbb{C}$ is isomorphic to the two-dimensional vector space $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ (see [32, Prop. VI.5.4a, p. 163]).

The group addition law can be described in terms of rational functions with coefficients in $\mathbb{Q}$. As the Galois group $G_{\mathbb{Q}}$ permutes the roots of polynomials over $\mathbb{F}_p$, it commutes with these rational functions and therefore acts linearly on the p-torsion subgroup; hence for all prime $p \nmid N$ we obtain a two-dimensional Galois representation $\rho_{E,p}$ sending $\sigma \in G_{\mathbb{Q}}$ to a matrix in $\mathrm{GL}_2(\mathbb{F}_p)$. From [13, Section 2.2], we have $\mathrm{Tr}(\rho_{E,p}(\mathrm{Frob}_\ell)) = \ell + 1 - \#E(\mathbb{F}_p)$, where Tr is the trace, Frob_ℓ is a Frobenius of ℓ, and $\#E(\mathbb{F}_p)$ is the number of $\mathbb{F}_p$-rational points on the elliptic curve E. By adding ω^2, where ω is the cyclotomic character, we get the three-dimensional representation $\rho = \rho_{E,p} \oplus \omega^2$, and since $\rho_{E,p}$ is similar to

$$\begin{bmatrix} \omega & \\ & 1 \end{bmatrix}$$

when restricted to inertia at p (see [13, Prop. 2.11(c)]), this gives us

$$\rho|_{I_p} \sim \begin{bmatrix} \omega^2 & & \\ & \omega & \\ & & 1 \end{bmatrix} = \begin{bmatrix} \omega^2 & & \\ & \omega^1 & \\ & & \omega^0 \end{bmatrix}$$

and therefore a predicted weight of $F(0, 0, 0)$.

ℓ	2	3	5	13	17	19	23	29	31	37	41
$\mathrm{Tr}(\rho(\mathrm{Frob}_\ell))$	2	1	5	5	0	4	3	1	2	0	0
$a(\ell, 1)$	2	1	5	5	0	4	3	1	2	0	0
$T_2(\rho(\mathrm{Frob}_\ell))$	1	1	2	3	6	5	5	1	3	0	5
$\ell a(\ell, 2)$	1	1	2	3	6	5	5	1	3	0	5

Table 2.3: Comparison of $\mathrm{Tr}(\rho(\mathrm{Frob}_\ell))$ and $T_2(\rho(\mathrm{Frob}_\ell))$ with Hecke eigenvalues of v_0, reduced modulo $p = 7$, for $2 \leq \ell \leq 41$.

Had we instead multiplied $\rho_{E,p}$ by ω and then added the trivial character, we would have obtained $\rho = \omega\rho_{E,p} \oplus 1$, which also corresponds to an eigenclass in $H^3(\Gamma_0(N), F(0,0,0))$. We can show, however, that this second ρ does not actually correspond to v_0 by checking that the relations $\mathrm{Tr}(\rho(\mathrm{Frob}_\ell)) = a(\ell, 1)$ and $T_2(\rho(\mathrm{Frob}_\ell)) = \ell a(\ell, 2)$ (where T_2 is the cotrace function, defined as the sum of all products of pairs of eigenvalues) do not hold for every ℓ. Therefore, $\rho = \rho_{E,p} \oplus \omega^2$ is the only representation that could possibly correspond to v_0, if one corresponds at all.

We calculate $\mathrm{Tr}(\rho(\mathrm{Frob}_\ell)) = \ell^2 + \mathrm{Tr}(\rho_{E,p}(\mathrm{Frob}_\ell))$ and $T_2(\rho(\mathrm{Frob}_\ell)) = \ell + \ell^2\mathrm{Tr}(\rho_{E,p}(\mathrm{Frob}_\ell))$; their comparison with the Hecke eigenvalues of v_0 for $p = 7$ and several ℓ is depicted in Table 2.3. Note that for these ℓ, the conditions of Definition 4 are satisfied, giving evidence that ρ is attached to v_0.

Chapter 3. A Lifted Cohomology Eigenclass

3.1 Computational Evidence

Extensive computational evidence indicates that for every eigenvector

$$v_0 \in H^3(\Gamma_0(N), F(a, a, 0))$$

of the Hecke operators, there is an eigenvector

$$v \in H^3(\Gamma_0(N), F(a + p - 1, a, 0))$$

having the same system of eigenvalues. In this paper $a = 0$, which is the only case we have considered thus far. Ultimately, we would like to be able to prove that this correlation occurs in all cases. With such a proof in hand, then whenever a Galois representation ρ_0 attached to v_0 can be found, this would give us a good idea where to look for a similar Galois representation ρ attached to v. This could also help us specify which weights should be predicted in the conjecture. All of this would be of great service in helping us find additional evidence for the generalized conjecture.

The existence of v_0 is guaranteed by [22, Lemma 1.3.17], which says there must be at least one simultaneous eigenvector of every member of a commuting family of operators, and the Hecke operators are such a family. There may therefore be several eigenvectors, so our choice of v_0 is somewhat arbitrary; Definition 6, at the beginning of Section 3.3, is an explicit description of the v_0 that we actually choose.

We have constructed an element $v \in H^3(\Gamma_0(N), F(p - 1, 0, 0))$ for $p > 3$ and $N = 11$ and 17, and have proved (for every $p \nmid 6N$) that it is an eigenvector of the Hecke operators

$T(2,1)$ and $T(2,2)$. Though we were unable to prove that v is an eigenvector for every Hecke operator, we have nevertheless observed that in the range $3 < p < 47$, v is an eigenvector of all $T(\ell, i)$ such that $\ell \nmid pN$, $i = 1$ or 2, and $N = 11$ or 17, in each case having the same system of eigenvalues as our given v_0.

The method of our approach to this problem involves performing computations of v_0 and v and then comparing the two for several different values of p, to find patterns which we carefully formulate so that we can prove the existence of v for all primes $p > 3$. This necessitates the use of bases for v_0 and v that are easily manageable, the notation for which is described in the following section.

3.2 Notation and Preliminary Theorems

Let p be prime, and let the level N be relatively prime to p. For convenience, we set $t_p = \frac{1}{2}p(p+1)$. The only coefficient modules V we are concerned with in this paper are $F(0,0,0)$ and $F(p-1,0,0)$. By [3, p. 569], bases for these are the sets of monomials in the spaces of homogeneous polynomials over $\bar{\mathbb{F}}_p$ in three variables x, y, z of total degree 0 and $p-1$, respectively. Hence there is only one element in the basis in the former case, and t_p elements in the basis in the latter.

We will not actually calculate the cohomology, but rather the homology, and then employ the natural duality (see [11, Prop. VI.7.1, p. 145]), as was done in [3] (following [9]), to obtain the cohomology. Shapiro's Lemma [35, p. 171] gives

$$H_3(\Gamma_0(N), V) \cong H_3(\mathrm{SL}_3(\mathbb{Z}), \mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} V),$$

where the induced module $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} V$ is defined by

$$\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} V = \{f : \mathrm{SL}_3(\mathbb{Z}) \to V : f(xg) = f(x) \cdot g \text{ for } g \in \Gamma_0(N)\},$$

the dot denoting the (right) action of $\mathrm{SL}_3(\mathbb{Z})$ by left translation on V. In [3, p. 573] the basis for this induced module is expressed in terms of the basis elements v_α of V (monomials in the variables x, y, z, as discussed above) and the coset representatives r_k in $\mathrm{SL}_3(\mathbb{Z})/\Gamma_0(N)$. More precisely, the functions $\phi_{r_k,v_\alpha} : \mathrm{SL}_3(\mathbb{Z}) \to V$ defined by

$$\phi_{r_k,v_\alpha}(x) = \begin{cases} v_\alpha \cdot r_k^{-1}x & \text{if } x \in r_k\Gamma_0(N), \\ 0 & \text{otherwise,} \end{cases}$$

constitute the basis for $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} V$. For each of the two induced modules with which we are concerned, we express a given element as a vector, whose components correspond to the ϕ_{r_k,v_α}. The labelling order of these components is described below. In essence, they are arranged first by coset representative, then by monomial in x, y, z. Since for $V = F(0,0,0)$ there is only one monomial for each coset representative, the components in this case are labelled simply by coset representative. Thus we can assume for the purposes of the following discussion that $V = F(p-1,0,0)$.

Theorem 1. *Suppose $N \neq p$ is prime, and let N' be the unique positive integer less than p such that $NN' \equiv 1$ mod p. Define $m = (NN'-1)/p$. Then each element of $SL_3(\mathbb{Z})/\Gamma_0(N)$ can be represented by the inverse of a matrix having one of the following three forms:*

(i) $$\begin{bmatrix} 1 & i(1-NN') & j(1-NN') \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad i,j \in \{0,...,N-1\}$$

(ii) $$\begin{bmatrix} NN' & 1-NN' & i(1-NN') \\ p(3m+i) & NN' & p(8m^2+3mi) \\ -p & 0 & -2mp+1 \end{bmatrix}, \quad i \in \{0,...,N-1\}$$

(iii) $\begin{bmatrix} NN' & 0 & 1-NN' \\ -p & -2mp+1 & 0 \\ 3mp & 8m^2p & NN' \end{bmatrix}$

Proof. The set of coset representatives is in one-to-one correspondence with distinct points in the projective space

$$\mathbb{P}^2(\mathbb{Z}/N\mathbb{Z}) = \{(a,b,c) \in (\mathbb{Z}/N\mathbb{Z})^3 \setminus \{\mathbf{0}\} : \ (a,b,c) = (\lambda a, \lambda b, \lambda c) \text{ for } \lambda \in (\mathbb{Z}/N\mathbb{Z})^\times\}.$$

This is because $\Gamma_0(N)$ is the stabilizer in $\mathrm{SL}_3(\mathbb{Z})$ of the point $(1,0,0)$ and $\mathbb{P}^2(\mathbb{Z}/N\mathbb{Z})$ is its orbit. Each representative is uniquely characterized by the reduction of its top row modulo N. Since N is prime, each triple (a,b,c), $a \neq 0$, can be "normalized" by multiplying by the inverse of a modulo N, so that those cosets corresponding to these triples can be represented by a triple of the form $(1,*,*)$. There are therefore N^2 such representatives; observe that they are of form 1. A similar normalization is carried out in case $a = 0, b \neq 0$, giving triples of the form $(0,1,*)$. There are N of these representatives, and they fall in the class of form 2. Finally, when $a = 0, b = 0, c \neq 0$, we get the unique triple of the form $(0,0,1)$, which is in the class of form 3. □

Remark 1. Note that the representatives have all been chosen so that they are congruent modulo p to the 3×3 identity matrix. The importance of this choice will be seen in our determination of the action matrices, beginning in Section 4.1.

We index the set of representatives r_k as follows: form 1, $k = Ni + j + 1$ for $0 \leq i, j \leq N-1$; form 2, $k = N^2 + i + 1$ for $0 \leq i \leq N-1$; form 3, $k = N^2 + N + 1$. We therefore likewise index the components of a vector in V, recalling that each coset representative must correspond to t_p (consecutive) components, one for each possible monomial in the variables x, y, z having total degree $p-1$. Considering the exponents α, β, γ of each monomial $x^\alpha y^\beta z^\gamma$, we have ordered the components in the vector as follows: for each fixed α, β increases from 0 to $p-1-\alpha$, and γ decreases from $p-1-\alpha$ to 0. Each monomial of total degree $p-1$ is accounted for as α then increases from 0 to $p-1$. Hence for $\alpha = 0$, there are p monomials; for $\alpha = 1$, there are $p-1$; etc., for a total of $\frac{1}{2}p(p+1) = t_p$ monomials altogether. The relation between the exponent α and the variable $j \in \{1, ..., t_p\}$ indexing the set of monomials is therefore quadratic; in fact, we use a quadratic fitting to find

$$\alpha = \left\lfloor \frac{1}{2}\left(2p+1-\sqrt{4p^2+4p+9-8j}\right)\right\rfloor,$$

where $\lfloor\ \rfloor$ denotes the floor function. For quick reference, we also list the exponents β and γ in terms of α and j:

$$\beta = j + \frac{1}{2}(\alpha^2 - (2p+1)\alpha - 2),$$
$$\gamma = -j - \frac{1}{2}(\alpha^2 - (2p-1)\alpha - 2p - 2) - 1.$$

Alternatively, if two of α, β, γ are known, the third is obtained by simply subtracting the other two from $p-1$. Note that the first in each block of t_p components corresponds to z^{p-1}; the p^{th}, to y^{p-1}; and the t_p^{th}, to x^{p-1}.

Having concluded the outline of our notation, we allow for V to be either of $F(0,0,0)$ or $F(p-1,0,0)$ again, and calculate the desired homology using the following

Theorem 2 (Allison, Ash, and Conrad [1, Thm. 2.1]). *Let $p > 3$ be prime, and let $\mathcal{V}$ be a finite-dimensional vector space over $\mathbb{F}_p$ on which $SL_3(\mathbb{Z})$ acts linearly, with $\cdot$ denoting the action. Then $H_3(SL_3(\mathbb{Z}), \mathcal{V})$ is the subspace of all $v \in \mathcal{V}$ such that*

(i) $v \cdot a = v$,

(ii) $v \cdot b = -v$,

(iii) $v + v \cdot h + v \cdot (h^2) = 0$,

where

$$a = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; b = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; \textit{and } h = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Conditions 1 and 2 are called the *semi-invariant condition* and condition 3 is the *h-condition.*

Proof. See [1, Thm. 2.1]. □

A couple of notes about Theorem 2:

(i) Our statement of the theorem is actually closer to the analogue in [3, p. 568], which states the semi-invariant condition for diagonal matrices and monomial matrices of order 2 in $\mathrm{SL}_3(\mathbb{Z})$. These are equivalent statements because a and b generate the subgroup of such matrices in $\mathrm{SL}_3(\mathbb{Z})$. See also [4, pp. 666-667].

(ii) The condition $p > 3$ is necessary because the statement of the theorem in [1] is for "daggered" homology, which may differ from the usual homology for primes dividing possible orders of elements in $\mathrm{SL}_3(\mathbb{Z})$ (see [1, p. 363]).

Recall that in this paper we are concerned with calculating $H_3(\Gamma_0(N), V)$ in particular, where $V = F(0,0,0)$ or $F(p-1,0,0)$. Since Theorem 2 does not directly calculate homology of this form, we will actually compute $H_3(\mathrm{SL}_3(\mathbb{Z}), \mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} V)$, where $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} V$ is as defined above, whence we obtain $H_3(\Gamma_0(N), V)$ using Shapiro's Lemma, as mentioned before.

3.3 Construction of the Lift

The notation outlined in the previous section helps us express the vectors v_0 and v mentioned in Section 3.1 in a more manageable way. We can do this for specific (small) p and $N = 11$ or 17 by first calculating a basis for $H_3(\mathrm{SL}_3(\mathbb{Z}), \mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} V)$ for $V = F(p-1,0,0)$ using our program `ADP-F(p-1,0,0)` (see Appendix — a similar program is used for $V = F(0,0,0)$). We then apply the Hecke operators to the basis elements using [1, Lemma 9.1], expressing the results in terms of the original basis, which gives us a transformation matrix. When $V = F(0,0,0)$, the eigenvectors of the corresponding transformation matrix include what we will choose to be v_0; when $V = F(p-1,0,0)$, they include the corresponding v. (Comparing systems of eigenvalues, it is easy to determine which of the eigenvectors in weight $F(p-1,0,0)$ is the v which interests us.) We compare the components of v_0 for a few different p, enabling us to define v_0 explicitly in the next definition. (We find that some components involve division by 2, which is taken to mean multiplication by the multiplicative inverse of 2 modulo p. Therefore, we scale by 2 to eliminate denominators.)

Definition 6. Following the indexing outlined in Section 3.2, as i ranges through the N^2+N+1 cosets of $\mathrm{SL}_3(\mathbb{Z})/\Gamma_0(N)$, the components of $2v_0$ for $N = 11$ are as listed in Table 3.1, where $k = -5$. For $N = 17$, the components for $2v_0$ are listed in Table 3.2, where $k = -4$.

Comparing components of v for a few different p, we are able to construct a vector v that we will show is in $H_3(\Gamma_0(N), F(p-1,0,0))$ for every $p > 3$. The description of this construction follows: as described in Section 3.2, we order the components of v into blocks corresponding to coset representatives, with the components of each block corresponding to monomials; thus, v consists of N^2+N+1 blocks of t_p components each. The only nonzero entries in each block are found in the first, p^{th}, and t_p^{th} positions. (For simplicity, we will use the notation $u[i,j]$ to mean the j^{th} component of the i^{th} block of the vector u. Thus,

		$i \bmod N$										
		1	2	3	4	5	6	7	8	9	10	0
$\left\lceil \frac{i}{N} \right\rceil$	1	0	0	-2	-1	1	2	2	1	-1	-2	0
	2	0	0	0	0	0	0	0	0	0	0	0
	3	2	0	0	0	$-k$	$-k$	$-k$	$-k$	0	0	0
	4	1	0	0	0	0	$-k$	$-k$	0	0	0	0
	5	-1	0	k	0	0	0	0	0	0	k	0
	6	-2	0	k	k	0	0	0	0	k	k	0
	7	-2	0	k	k	0	0	0	0	k	k	0
	8	-1	0	k	0	0	0	0	0	0	k	0
	9	1	0	0	0	0	$-k$	$-k$	0	0	0	0
	10	2	0	0	0	$-k$	$-k$	$-k$	$-k$	0	0	0
	11	0	0	0	0	0	0	0	0	0	0	0
	12	0	0	2	1	-1	-2	-2	-1	1	2	0
	13	0										

Table 3.1: Components of $2v_0$ for $N = 11$ (see Definition 6).

		$i \bmod N$																
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	0
$\left\lceil \frac{i}{N} \right\rceil$	1	0	0	-2	-1	0	-1	1	1	2	2	1	1	-1	0	-1	-2	0
	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	3	2	0	0	0	$-k$	0	$-k$	$-k$	$-k$	$-k$	$-k$	$-k$	0	$-k$	0	0	0
	4	1	0	0	0	0	0	$-k$	0	$-k$	$-k$	0	$-k$	0	0	0	0	0
	5	0	0	k	0	0	0	0	0	$-k$	$-k$	0	0	0	0	0	k	0
	6	1	0	0	0	0	0	0	$-k$	$-k$	$-k$	$-k$	0	0	0	0	0	0
	7	-1	0	k	k	0	0	0	0	0	0	0	0	0	0	k	k	0
	8	-1	0	k	0	0	k	0	0	0	0	0	0	k	0	0	k	0
	9	-2	0	k	k	k	k	0	0	0	0	0	0	k	k	k	k	0
	10	-2	0	k	k	k	k	0	0	0	0	0	0	k	k	k	k	0
	11	-1	0	k	0	0	k	0	0	0	0	0	0	k	0	0	k	0
	12	-1	0	k	k	0	0	0	0	0	0	0	0	0	0	k	k	0
	13	1	0	0	0	0	0	0	$-k$	$-k$	$-k$	$-k$	0	0	0	0	0	0
	14	0	0	k	0	0	0	0	0	$-k$	$-k$	0	0	0	0	0	k	0
	15	1	0	0	0	0	0	$-k$	0	$-k$	$-k$	0	$-k$	0	0	0	0	0
	16	2	0	0	0	$-k$	0	$-k$	$-k$	$-k$	$-k$	$-k$	$-k$	0	$-k$	0	0	0
	17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	18	0	0	2	1	0	1	-1	-1	-2	-2	-1	-1	1	0	1	2	0
	19	0																

Table 3.2: Components of $2v_0$ for $N = 17$ (see Definition 6).

the first, p^{th}, and t_p^{th} positions of the i^{th} block of v may be denoted $v[i,1]$, $v[i,p]$, and $v[i,t_p]$, respectively.) To generate these three entries, let i index the set of coset representatives as in Section 3.2. There are six nonzero values assumed by the components of v: ± 1, $\pm k$, and $\pm k/2$, where k is an integer depending on N (for $N = 11$, $k = -5$; for $N = 17$, $k = -4$). Where they appear in v is outlined in Tables 3.3 through 3.8. (The tables for $N = 11$ are shown here; for $N = 17$, they are at the end of this section. Also, for display purposes, we opt to leave the components as they are, instead of scaling by 2 to eliminate denominators, as in the case of v_0 above.) Evidently $v[i,1]$ depends on $\lceil \frac{i}{N} \rceil$ (where $\lceil\ \rceil$ is the ceiling function), and, when $i < N^2$, $v[i,p]$ is a function of i mod N. There may be a similar way to describe $v[i,t_p]$, but thus far a relation more concise than those depicted in Tables 3.5 and 3.8 has eluded us.

$\lceil \frac{i}{N} \rceil$	1	3, 10	4, 9	5, 8	6, 7	12
$v[i,1]$	1	$-k$	$-k/2$	$k/2$	k	-1

Table 3.3: Nonzero $v[i,1]$ for $N = 11$.

i mod N	1	3, 10	4, 9	5, 8	6, 7
$v[i,p]$	-1	k	$k/2$	$-k/2$	$-k$

Table 3.4: Nonzero $v[i,p]$ for $N = 11$ ($i < N^2$). If $i > N^2 + 1$, $v[i,p] = 1$. (If $i = N^2$ or $N^2 + 1$, $v[i,p] = 0$.)

All the remaining entries that are as yet unspecified are left as 0. Programs explicitly generating v are located in the appendix, one for each of the levels 11 and 17. The question may arise as to how all of these vector entries were determined; recall that we observed the same patterns in the cases where p was small enough that we could find v by directly calculating $H_3(\mathrm{SL}_3(\mathbb{Z}), V)$. These patterns of symmetry are especially apparent when the components corresponding to a particular monomial are grouped according to congruence classes of i mod N, N consecutive i, or even the entire table of such components of v.

One apparent relation, which we now describe, connects the first and p^{th} positions of

		$i \bmod N$										
		1	2	3	4	5	6	7	8	9	10	0
$\left\lceil \frac{i}{N} \right\rceil$	1	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
	2	1	0	$-k$	$-k/2$	$k/2$	k	k	$k/2$	$-k/2$	$-k$	0
	3	1	k	0	$k/2$	$-k$	$-k/2$	$-k/2$	$-k$	$k/2$	0	k
	4	1	$k/2$	$-k/2$	0	k	$-k$	$-k$	k	0	$-k/2$	$k/2$
	5	1	$-k/2$	k	$-k$	0	$k/2$	$k/2$	0	$-k$	k	$-k/2$
	6	1	$-k$	$k/2$	k	$-k/2$	0	0	$-k/2$	k	$k/2$	$-k$
	7	1	$-k$	$k/2$	k	$-k/2$	0	0	$-k/2$	k	$k/2$	$-k$
	8	1	$-k/2$	k	$-k$	0	$k/2$	$k/2$	0	$-k$	k	$-k/2$
	9	1	$k/2$	$-k/2$	0	k	$-k$	$-k$	k	0	$-k/2$	$k/2$
	10	1	k	0	$k/2$	$-k$	$-k/2$	$-k/2$	$-k$	$k/2$	0	k
	11	1	0	$-k$	$-k/2$	$k/2$	k	k	$k/2$	$-k/2$	$-k$	0
	12	1	0	$-k$	$-k/2$	$k/2$	k	k	$k/2$	$-k/2$	$-k$	0
	13	-1										

Table 3.5: $v[i, t_p]$ for $N = 11$.

the blocks of v with most of the first N components of v_0, which attain five values, namely, -1, $-1/2$, 0, 1/2, and 1. Where these values appear in v_0 for $1 < j \leq N$ has a bearing with where they appear, scaled by k, in v; specifically, one observes for j in this range that

(i) $v[i, 1] = kv_0[j, 1]$ when $\lceil \frac{i}{N} \rceil = j$, and

(ii) $v[i, p] = -kv_0[j, 1]$ when $i \equiv j \bmod N$ and $i < N^2$.

These relations hold for both $N = 11$ and 17.

For an observation involving $v[i, t_p]$, notice in Tables 3.5 and 3.8 the square matrix having $\frac{1}{2}(N-1)$ rows and columns which ranges from 2 through $\frac{1}{2}(N+1)$ on both axes. The entries in the first row of the matrix are the second through the $\frac{1}{2}(N+1)^{th}$ entries of v_0, scaled by a factor of k. The remaining rows are permutations of these same entries; the permutations seem to depend on N. This matrix, which turns out to be skew-symmetric, is reflected about the lines between $\frac{1}{2}(N+1)$ and $\frac{1}{2}(N+3)$ on both axes, so that the rows and columns for $\frac{1}{2}(N+3), ..., N-1, 0 \pmod N$ mirror the rows and columns for $\frac{1}{2}(N+1), ..., 3, 2$, respectively. So nearly all of the information in the table can be derived from this skew-symmetric matrix, and the rest of it is easily predictable. Thus, if we could ascertain precisely what the aforementioned permutations would be for each N, then in each case the skew-symmetric matrix would follow, and this may lead to a determination of a lift of v_0 for all N.

$\lceil \frac{i}{N} \rceil$	1	3, 16	4, 6, 13, 15	7, 8, 11, 12	9, 10	18
$v[i,1]$	1	$-k$	$-k/2$	$k/2$	k	-1

Table 3.6: Nonzero $v[i,1]$ for $N = 17$.

i mod N	1	3, 16	4, 6, 13, 15	7, 8, 11, 12	9, 10
$v[i,p]$	-1	k	$k/2$	$-k/2$	$-k$

Table 3.7: Nonzero $v[i,p]$ for $N = 17$ ($i < N^2$). If $i > N^2 + 1$, $v[i,p] = 1$. (If $i = N^2$ or $N^2 + 1$, $v[i,p] = 0$.)

		$i \bmod N$																
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	0
$\left\lceil \frac{i}{N} \right\rceil$	1	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
	2	1	0	$-k$	$-k/2$	0	$-k/2$	$k/2$	$k/2$	k	k	$k/2$	$k/2$	$-k/2$	0	$-k/2$	$-k$	0
	3	1	k	0	$k/2$	$-k$	$k/2$	$-k/2$	$-k/2$	0	0	$-k/2$	$-k/2$	$k/2$	$-k$	$k/2$	0	k
	4	1	$k/2$	$-k/2$	0	$k/2$	0	$-k$	k	$-k/2$	$-k/2$	k	$-k$	0	$k/2$	0	$-k/2$	$k/2$
	5	1	0	k	$-k/2$	0	$-k/2$	$k/2$	$k/2$	$-k$	$-k$	$k/2$	$k/2$	$-k/2$	0	$-k/2$	k	0
	6	1	$k/2$	$-k/2$	0	$k/2$	0	k	$-k$	$-k/2$	$-k/2$	$-k$	k	0	$k/2$	0	$-k/2$	$k/2$
	7	1	$-k/2$	$k/2$	k	$-k/2$	$-k$	0	0	$k/2$	$k/2$	0	0	$-k$	$-k/2$	k	$k/2$	$-k/2$
	8	1	$-k/2$	$k/2$	$-k$	$-k/2$	k	0	0	$k/2$	$k/2$	0	0	k	$-k/2$	$-k$	$k/2$	$-k/2$
	9	1	$-k$	0	$k/2$	k	$k/2$	$-k/2$	$-k/2$	0	0	$-k/2$	$-k/2$	$k/2$	k	$k/2$	0	$-k$
	10	1	$-k$	0	$k/2$	k	$k/2$	$-k/2$	$-k/2$	0	0	$-k/2$	$-k/2$	$k/2$	k	$k/2$	0	$-k$
	11	1	$-k/2$	$k/2$	$-k$	$-k/2$	k	0	0	$k/2$	$k/2$	0	0	k	$-k/2$	$-k$	$k/2$	$-k/2$
	12	1	$-k/2$	$k/2$	k	$-k/2$	$-k$	0	0	$k/2$	$k/2$	0	0	$-k$	$-k/2$	k	$k/2$	$-k/2$
	13	1	$k/2$	$-k/2$	0	$k/2$	0	k	$-k$	$-k/2$	$-k/2$	$-k$	k	0	$k/2$	0	$-k/2$	$k/2$
	14	1	0	k	$-k/2$	0	$-k/2$	$k/2$	$k/2$	$-k$	$-k$	$k/2$	$k/2$	$-k/2$	0	$-k/2$	k	0
	15	1	$k/2$	$-k/2$	0	$k/2$	0	$-k$	k	$-k/2$	$-k/2$	k	$-k$	0	$k/2$	0	$-k/2$	$k/2$
	16	1	k	0	$k/2$	$-k$	$k/2$	$-k/2$	$-k/2$	0	0	$-k/2$	$-k/2$	$k/2$	$-k$	$k/2$	0	k
	17	1	0	$-k$	$-k/2$	0	$-k/2$	$k/2$	$k/2$	k	k	$k/2$	$k/2$	$-k/2$	0	$-k/2$	$-k$	0
	18	1	0	$-k$	$-k/2$	0	$-k/2$	$k/2$	$k/2$	k	k	$k/2$	$k/2$	$-k/2$	0	$-k/2$	$-k$	0
	19	-1																

Table 3.8: $v[i, t_p]$ for $N = 17$.

Chapter 4. Properties of the Lift

For the entirety of this chapter, we assume N is 11 or 17, with corresponding v_0 and v as specified in Section 3.3. The proofs and results of this chapter are valid for both values of N; we will make clear mention in case they differ.

4.1 The Lift is in the Cohomology

We now prove that the v constructed in Section 3.3 satisfies the conditions of Theorem 2 above and therefore is in $H_3(\Gamma_0(N), F(p-1,0,0))$ for all $p > 3$ and $N = 11$ and 17. First, some terminology:

Definition 7. The matrix representing the action of a matrix m on V is called the *action matrix* of m on V, where our choice of basis for the action matrix is as described in Section 3.2. When V is clear from the context, we will omit its mention.

We now elaborate upon the computation of these action matrices. (Note that we computed these same action matrices above, for specific p, to calculate the basis for the homology, aiding us in finding a prediction for v. The description that follows entails determining the action matrices and proving a result about v for p in *general*.)

Incidentally, there is no need to go into significant detail about this for the v_0 of Definition 6. As a matter of fact, v_0 is a characteristic 0 object (whereas v is not), which we usually display reduced modulo p. Therefore, v_0 does not actually depend on p, and neither do the action matrices of any $m \in \{a, b, h, h^2\}$ on $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(0,0,0)$, where a, b, and h are as in Theorem 2. A simple calculation then shows that for all p and $N = 11$ and 17, $v_0 \in H^3(\Gamma_0(N), F(0,0,0))$. There may be other vectors satisfying this condition; however, the only one we are concerned with in this paper is this particular v_0, defined in Definition 6.

So we concern ourselves only with the lift v of v_0, where v is as defined in Section 3.3. However, the action matrices of matrices acting on $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(0,0,0)$ are required in our notation for the action matrices of matrices acting on $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(p-1,0,0)$. We determine all these action matrices following [3, p. 573]. These are indexed in the same way as v_0 for $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(0,0,0)$, and as v for $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(p-1,0,0)$. Let A_0, B_0, H_0, and H_0^2 denote the action matrices of a, b, h, and h^2, respectively, on $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(0,0,0)$. (Note that $H_0^2 \neq H_0H_0$.) Each of these, then, has N^2+N+1 rows and columns.

On the other hand, the action matrices of a, b, h, and h^2 on $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(p-1,0,0)$ are all square matrices with $(N^2+N+1)t_p$ rows and columns, since there are t_p monomials for each coset representative. The rows and columns of these action matrices are therefore partitioned into N^2+N+1 blocks of t_p rows and columns. (We will call the resulting blocks of rows and columns "block rows" and "block columns.") Therefore, each action matrix on $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(p-1,0,0)$ is a block matrix, where each block is a $t_p \times t_p$ matrix, and there is a total of $(N^2+N+1)^2$ of them for each action matrix.

There is exactly one nonzero entry, a 1, in each row and column of each of A_0, B_0, H_0, and H_0^2. This follows simply from the uniqueness of a coset representative r_i for a given $x \in \mathrm{SL}_3(\mathbb{Z})$. Similarly, there is exactly one nonzero $t_p \times t_p$ matrix block in each block row and column of the action matrices on $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(p-1,0,0)$. Furthermore,

Lemma 1. If m is a matrix acting on $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(p-1,0,0)$, then each of the nonzero $t_p \times t_p$ matrix blocks in the action matrix of m on $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(p-1,0,0)$ is the same throughout each action matrix.

Proof. Recall from Remark 1 in Section 3.2 that the coset representatives r_i are each congruent modulo p to the 3×3 identity matrix. Because of this, all coset representatives involved in the product in the calculation of the action (detailed in [3, p. 573]) can be eliminated. Hence each action matrix is independent of i. As the block rows are indexed by i, the lemma follows. □

We will call this unique nonzero $t_p \times t_p$ matrix block M_m, where m is a matrix acting on $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(p-1,0,0)$. In this section, $m \in \{a, b, h, h^2\}$. (This block M_m, considered alone, actually represents the action of m on $F(p-1,0,0)$.) For each m, the locations of M_m throughout the larger action matrix coincide with the locations of the nonzero entries in the corresponding action matrix for $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(0,0,0)$. That is, if there is a 1 in the (i,j) entry of the action matrix of m on $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(0,0,0)$, then there is an M_m in the i^{th} block row and j^{th} block column of the action matrix of the action of m on $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(p-1,0,0)$. Therefore, the action matrices for $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(0,0,0)$ and $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(p-1,0,0)$ have the same basic structure, allowing us to employ the following notation: if A is an m-by-n matrix and B is a p-by-q matrix, the Kronecker product $A \otimes B$ is the mp-by-nq block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

(see [23, Def. 4.2.1]). The action matrices of a, b, h, and h^2 on $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(p-1,0,0)$ may therefore be denoted $A_0 \otimes M_a$, $B_0 \otimes M_b$, $H_0 \otimes M_h$, and $H_0^2 \otimes M_{h^2}$, respectively. This notation will prove to be especially useful later on.

Remark 2. Because for each block of t_p entries in v that corresponds to one of the N^2+N+1 coset representatives, there are nonzero values in only the first, p^{th}, and t_p^{th} entries, it is not necessary to determine each M_m in full detail. For our purposes it suffices to determine only the first, p^{th}, and t_p^{th} columns.

The respective sizes of the M_m and v depend on p, and therefore so does our method of calculating the action of m on $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(p-1,0,0)$. Through the following lemma, we are able to alter the calculation to eliminate this dependency, if the M_m are simple enough.

Lemma 2. Let L be a square matrix with N^2+N+1 rows and columns. Suppose that the only nonzero entries in the first, p^{th}, and t_p^{th} columns of a $t_p \times t_p$ matrix M are found in the first, p^{th}, and t_p^{th} rows. Then the calculation $(L \otimes M)v$ may be determined, independent of p, by calculating $(L \otimes M')v'$, where M' is an appropriately chosen 3×3 matrix and v' is the truncation of v obtained by eliminating all but the first, p^{th}, and t_p^{th} entries in each block of t_p components in v corresponding to one of the N^2+N+1 coset representatives. See the accompanying commuting diagram.

$$\begin{tikzcd} v \arrow[r, "L\otimes M"] \arrow[d, "'"'] & (L \otimes M)v \arrow[d, "'"] \\ v' \arrow[r, "L\otimes M'"] & (L \otimes M')v' \end{tikzcd}$$

Proof. Because there are nonzero values in only the first, p^{th}, and t_p^{th} entries in each block of t_p components in v corresponding to one of the N^2+N+1 coset representatives, the calculation of $(L \otimes M)v$ is completely independent of all columns of M except the first, p^{th}, and t_p^{th}. Therefore, we discard all of the columns of M except these three, leaving a $t_p \times 3$ matrix. Since all but the first, p^{th}, and t_p^{th} rows of this resulting matrix are zero, they contribute nothing to the calculation as well, and so we may discard them also, leaving a 3×3 matrix, which we call M'. Then the calculation $(L \otimes M')v'$ is independent of p, since everything in $(L \otimes M)v$ dependent on p has been replaced by something independent of p. After the calculation is carried through, we obtain the original $(L \otimes M)v$ by simply replacing the 0's in all but the first, p^{th}, and t_p^{th} entries in each block of t_p components in v corresponding to one of the N^2+N+1 coset representatives. $\square$

Lemma 3. The vector v constructed in Section 3.3 satisfies the semi-invariant condition of Theorem 2 for $H_3(\Gamma_0(N), F(p-1,0,0))$ for $p > 3$ and $N = 11$ and 17.

Proof. The actions of a and b, up to a change in sign, amount simply to permutations of x, y, and z, so each monomial will be mapped by the action to exactly one other monomial; therefore, in each of M_a and M_b, each row and column will have exactly one nonzero entry. For example, since $a[x,y,z]^T = [z,x,y]^T$, x^{p-1} will map to z^{p-1}, and so we obtain a 1 in the $(1, t_p)$ entry of M_a. (Recall from Section 3.2 that the first, p^{th}, and t_p^{th} positions of each block correspond to z^{p-1}, y^{p-1}, and x^{p-1}, respectively.) Similarly, we find 1's in the $(p, 1)$ and (t_p, p) entries of M_a, and since $b[x,y,z]^T = [y, -x, z]^T$, they appear in the $(1,1), (p, t_p)$, and (t_p, p) entries of M_b. Therefore, in the notation of Lemma 2, we have

$$M'_a = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}, M'_b = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}$$

(all omitted entries are 0). Applying Lemma 2, we reduce the calculation of $v \cdot a$ to $(A_0 \otimes M'_a)v'$ and $v \cdot b$ to $(B_0 \otimes M'_b)v'$. The advantage of this is that the sizes of M'_a, M'_b, and v' are independent of p. We may therefore calculate the respective actions of a and b directly; we use a computer to do so and find that $v \cdot a = v$ and $v \cdot b = -v$. Therefore, v satisfies the semi-invariant condition. □

Theorem 3. *$v \in H_3(\Gamma_0(N), F(p-1, 0, 0))$ for $p > 3$ and $N = 11$ and 17.*

Proof. Having already checked in Lemma 3 that v satisfies the semi-invariant condition of Theorem 2, we need only check the h-condition; i.e., that $v \cdot h + v \cdot (h^2) = -v$. We must therefore determine M_h and M_{h^2} for $p > 3$. We find, by the same method as in the proof of Lemma 3, 1's in the $(1,1)$ and (p, t_p) entries of M_h, and in the $(1,1)$ and (t_p, p) entries of M_{h^2}. In the case of M_h, however, there are several extra 1's in the p^{th} column, whereas for M_{h^2}, they occur in the same rows in the t_p^{th} column. This is because $h[x,y,z]^T = [-y, x-y, z]^T$, so that y^{p-1} maps to $(x-y)^{p-1}$; and likewise, since $h^2[x,y,z]^T = [-x+y, -x, z]^T$, x^{p-1} maps to $(-x+y)^{p-1}$. Since $p > 3$, this is the same polynomial; furthermore, for $0 \leq i \leq p-1$,

$$\binom{p-1}{i} = \frac{(p-1)\cdots(p-i)}{i!} \equiv \frac{(-1)^i i!}{i!} \equiv (-1)^i \bmod p,$$

so each of the coefficients of

$$(x-y)^{p-1} = \sum_{i=0}^{p-1} (-1)^i x^{p-1-i}(-y)^i = \sum_{i=0}^{p-1} x^{p-1-i} y^i$$

is congruent, modulo p, to 1. Therefore, the p^{th} column of M_h is identical to the t_p^{th} column of M_{h^2}.

In summary, M_h and M_{h^2} take on the following forms, where the rows and columns shown are the first, p^{th}, and t_p^{th}:

$$M_h = \begin{bmatrix} 1 & & \\ & 1 & 1 \\ & \vdots & \\ & 1 & \end{bmatrix}, M_{h^2} = \begin{bmatrix} 1 & & \\ & & 1 \\ & & \vdots \\ & 1 & 1 \end{bmatrix}$$

(In this case, the vertical points of ellipsis between the p^{th} and the t_p^{th} rows denote that there are 1's in some, but not necessarily all, of those rows in that particular column; the remainder are 0's. The important thing is that the two columns having the points of ellipsis are identical.)

We cannot apply Lemma 2 to our current situation, so we rewrite $M_h = C_h + D_h$ and

$M_{h^2} = C_{h^2} + D_{h^2}$, where

$$C_h = \begin{bmatrix} 1 & & \\ & & 1 \\ & & \end{bmatrix}, C_{h^2} = \begin{bmatrix} 1 & & \\ & & \\ & 1 & \end{bmatrix},$$

and D_h and D_{h^2} are the respective remainders upon subtracting C_h from M_h and C_{h^2} from M_{h^2} (as before, the rows and columns shown in C_h and C_{h^2} are the first, p^{th}, and t_p^{th} — all other entries in C_h and C_{h^2} are left as 0). We thus have

$$(H_0 \otimes M_h)v = (H_0 \otimes (C_h + D_h))v = (H_0 \otimes C_h)v + (H_0 \otimes D_h)v$$

and

$$(H_0^2 \otimes M_{h^2})v = (H_0^2 \otimes (C_{h^2} + D_{h^2}))v = (H_0^2 \otimes C_{h^2})v + (H_0^2 \otimes D_{h^2})v.$$

Our purpose in doing this is to cancel all terms involving D_h or D_{h^2}, so we now focus on $H_0 \otimes D_h$ and $H_0^2 \otimes D_{h^2}$. If v_i is the vector with $N^2 + N + 1$ entries we obtain by taking the i^{th} entry of each successive block of t_p entries in v (i.e., the j^{th} component of v_i is $v[j, i]$, where j runs through the $N^2 + N + 1$ coset representatives), then we find by direct calculation that $H_0 v_p + H_0^2 v_{t_p} = 0$. But calculating $H_0 v_p + H_0^2 v_{t_p}$ is essentially the same as calculating $(H_0 \otimes D_h)v + (H_0^2 \otimes D_{h^2})v$, since each entry in $H_0 v_p$ and $H_0^2 v_{t_p}$ will be repeated several times in corresponding blocks in $(H_0 \otimes D_h)v$ and $(H_0^2 \otimes D_{h^2})v$, respectively. This is because the only entries in $H_0 \otimes D_h$ and $H_0^2 \otimes D_{h^2}$ relevant (or that contribute a nonzero value) to the multiplication by v are the nonzero entries they have in, respectively, their p^{th} and t_p^{th} columns, all of which are 1. Since they are all in the same column, they are all multiplied by the same component in v, so they all yield the same result. Hence

$(H_0 \otimes D_h)v + (H_0^2 \otimes D_{h^2})v = 0$. This implies that

$$\begin{aligned} v \cdot h + v \cdot (h^2) &= (H_0 \otimes M_h)v + (H_0^2 \otimes M_{h^2})v \\ &= (H_0 \otimes C_h)v + (H_0 \otimes D_h)v + (H_0^2 \otimes C_{h^2})v + (H_0^2 \otimes D_{h^2})v \\ &= (H_0 \otimes C_h)v + (H_0^2 \otimes C_{h^2})v. \end{aligned}$$

We apply Lemma 2 to this last expression on the right, obtaining

$$v \cdot h + v \cdot (h^2) = (H_0 \otimes C'_h)v' + (H_0^2 \otimes C'_{h^2})v'.$$

Since now none of the sizes of the factors of this expression depend on p, we may calculate directly by simply multiplying v' on the left by the matrices $H_0 \otimes C'_h$ and $H_0^2 \otimes C'_{h^2}$ and adding the results together. With the aid of a computer performing all these calculations, we thus find that $v \cdot h + v \cdot (h^2) = -v$. Therefore, the h-condition is satisfied, and $v \in H_3(\Gamma_0(N), F(p-1,0,0))$. □

4.2 The Lift is an Eigenvector of $T(2, i)$

The next question is whether or not v is an eigenvector of the Hecke operators $T(\ell, 1)$, $T(\ell, 2)$, where ℓ is a prime not dividing Np. We have the following result for $\ell = 2$:

Theorem 4. *The vector v as defined in Section 3.3 is an eigenvector of the Hecke operators $T(2,1)$ and $T(2,2)$.*

Proof. It was ascertained in [1, Lemma 9.1] that $v \cdot T(\ell, i) = \sum_j v \cdot q_{\ell,i,j}$ for each i, the $q_{\ell,i,j}$ being 3×3 matrices determined by a method in [34]. We will use this method to directly calculate $T(\ell, i)$, implementing it in the program `LLLMatrixFind` (see Appendix), where we find that there are thirteen such matrices $q_{2,1,j}$ and twelve $q_{2,2,j}$. (Note that these matrices are not necessarily unique; a different choice of matrices will result in relations differing from those shown at the end of this proof. However, because the Hecke operators are well-defined, that v is an eigenvector is independent of this choice.)

The rest of the proof is very similar to the proof of Theorem 3 above. We begin with $T(2,1)$. We determine the action of the thirteen matrices on $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(0,0,0)$ as in the case of the other matrices above, calling the resulting action matrices $Q_{2,1,j}$. Then, for each action matrix of the $q_{2,1,j}$ acting on $\mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} F(p-1,0,0)$, we again have by Lemma 1 a unique nonzero $t_p \times t_p$ matrix (which we call $M_{2,1,j}$) in each "block row" and "column." Remark 2 for $M_m \in \{M_{2,1,j}\}$, where $1 \leq j \leq 13$, also applies.

For a few $M_{2,1,j}$ the first, p^{th}, and t_p^{th} columns each have exactly one nonzero value in the first, p^{th}, and t_p^{th} rows, respectively, and this because the corresponding $q_{2,1,j}$ fix the monomials corresponding to these columns and rows, which are z^{p-1}, y^{p-1}, and x^{p-1}, respectively. We can therefore apply Lemma 2 to these, which have $j = 1, 10$, and 13.

In the remaining ten $M_{2,1,j}$ we find, however, that there are other rows having nonzero entries in the first, p^{th} and t_p^{th} columns. To use Lemma 2, we will need to show that these additional entries cancel. To do this, we determine, for each $j \in S = \{2, 3, ..., 8, 9, 11, 12\}$, the placement of nonzero entries in the first, p^{th}, and t_p^{th} columns of each $M_{2,1,j}$ as a function of p. We demonstrate this for $j = 6$; the rest are calculated similarly. From our program we get $q_{2,1,6}[x, y, z]^T = [x - z, -y - z, 2z]^T$; therefore, in the t_p^{th} column (corresponding to x^{p-1}), we have 1's in all the rows where y does not appear in the monomial. In the p^{th} column (corresponding to y^{p-1}), we get alternating 1's and -1's in the appropriate rows for $y^{p-1-i}z^i$, since reduced modulo p, $(-y-z)^{p-1} = \sum_{i=0}^{p-1}(-1)^i y^{p-1-i} z^i$ (which we calculate as before). In summary, we obtain the following, where the only rows and columns shown are the first, p^{th}, and the t_p^{th}:

$$M_{2,1,6} = \begin{bmatrix} 1 & 1 & 1 \\ & \vdots & \\ & 1 & \\ & & \vdots \\ & & 1 \end{bmatrix}$$

(In this case, the vertical points of ellipsis between the given rows denote that there are nonzero entries in at least some of the intermediary rows in the particular column in which the points appear. These values need not necessarily be 1's.)

As in the proof of Theorem 3, we break apart, for each j, the nonzero $t_p \times t_p$ block matrices $M_{2,1,j}$ as

$$M_{2,1,j} = C_{2,1,j} + D_{2,1,j},$$

where $C_{2,1,j}$ is the $t_p \times t_p$ matrix consisting of the column(s) among the first, p^{th}, and t_p^{th} of $M_{2,1,j}$ having exactly one nonzero entry among the first, p^{th}, and t_p^{th} rows, with all other columns being zero; and $D_{2,1,j}$ is the remainder upon subtracting $C_{2,1,j}$ from $M_{2,1,j}$. (For the purposes of this calculation, we take $C_{2,1,j}$ for $j \notin S$ to be the corresponding $M_{2,1,j}$, so that the resulting $D_{2,1,j} = 0$.) Explicitly, this decomposition for the case $j = 6$ would therefore be

$$C_{2,1,6} = \begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix}, \quad D_{2,1,6} = \begin{bmatrix} & 1 & 1 \\ & \vdots & \\ & 1 & \\ & & \vdots \\ & & 1 \end{bmatrix}.$$

In a manner similar to the way we found the relation between $(H_0 \otimes M_h)v$ and $(H_0^2 \otimes M_{h^2})v$, in the proof of Theorem 3 above, we then calculate $\sum_S (Q_{2,1,j} \otimes D_{2,1,j})v = 0$ using the following relations:

$$[(Q_{2,1,2} \otimes D_{2,1,2}) + (Q_{2,1,3} \otimes D_{2,1,3})]v = 0,$$

$$[(Q_{2,1,4} \otimes D_{2,1,4}) + (Q_{2,1,5} \otimes D_{2,1,5})]v = 0,$$

$$[(Q_{2,1,11} \otimes D_{2,1,11}) + (Q_{2,1,12} \otimes D_{2,1,12})]v = 0,$$

and

$$[(Q_{2,1,6} \otimes D_{2,1,6}) + (Q_{2,1,7} \otimes D_{2,1,7}) + (Q_{2,1,8} \otimes D_{2,1,8}) + (Q_{2,1,9} \otimes D_{2,1,9})]v = 0.$$

This reduces $v \cdot T(2,1)$ from $\sum\limits_j (Q_{2,1,j} \otimes M_{2,1,j})v$ to $\sum\limits_j (Q_{2,1,j} \otimes C_{2,1,j})v$, the advantage of this being that we can now apply Lemma 2. As in the proof of Theorem 3, we then use a computer to calculate the latter expression to be $2v$, for $N = 11$, and therefore v is an eigenvector of $T(2,1)$ with eigenvalue 2. Similarly, for $N = 17$, we find v has eigenvalue 3.

Proving that v is an eigenvector of $T(2,2)$ is also similar: in this case we have twelve matrices $q_{2,2,j}$. Through the same method as before, we first find that we can already apply Lemma 2 to $M_{2,2,1}$, $M_{2,2,9}$ and $M_{2,2,12}$, and among the others we find the following relations:

$$[(Q_{2,2,2} \otimes D_{2,2,2}) + (Q_{2,2,3} \otimes D_{2,2,3})]v = 0,$$

$$[(Q_{2,2,4} \otimes D_{2,2,4}) + (Q_{2,2,5} \otimes D_{2,2,5})]v = 0,$$

$$[(Q_{2,2,10} \otimes D_{2,2,10}) + (Q_{2,2,11} \otimes D_{2,2,11})]v = 0,$$

and

$$[(Q_{2,2,6} \otimes D_{2,2,6}) + (Q_{2,2,7} \otimes D_{2,2,7}) + (Q_{2,2,8} \otimes D_{2,2,8})]v = 0.$$

The eigenvalue of v for $T(2,2)$ for $N = 11$ is -3; for $N = 17$ it is -1. $\square$

Chapter 5. Future Work

We could probably continue to prove v is an eigenvector of $T(\ell, i),\ i = 1, 2$, in the method of Section 4.2 for each individual $\ell \nmid Np$, but the number of action matrices to include in the computations balloons rather quickly. For instance, for $\ell = 3$, there are about 30, and for $\ell = 5$, there are over 100, giving an excessively complicated calculation already. Furthermore, this method of direct calculation does not seem to reveal any promising avenues to follow in proving it for general ℓ.

In an effort to prove it for general ℓ, we have tried looking at the whole computation more explicitly: first we considered the matrices that make up the Hecke action, and then we tried to outline the action exactly. The method of [34] uses a choice of the $\ell^2 + \ell + 1$ coset representatives of $\mathrm{SL}_3(\mathbb{Z})/\Gamma_0(\ell)$ to determine the matrices used in computing the Hecke action. We noticed, in implementing this method in the proof of Theorem 4, that in each individual relation of those used to eliminate the $D_{\ell,i,j}$ terms, the matrices involved all arise from the same coset representative. Proving this could be useful in establishing v as an eigenvector for all Hecke operators $T(\ell, i)$. We were unfortunately not able to do this, as computations were extremely complicated, even in proving small steps along the way.

Another notion is to make use of the bilinear map $\mathrm{H}^n(G, A) \times \mathrm{H}^m(G, B) \to \mathrm{H}^{n+m}(G, C)$ known as the "cup product," as well as perhaps other maps of group cohomology and important pre-established results. This idea stems from a corresponding proof in the two-dimensional case: given $\alpha \in \mathrm{H}^1(\Gamma_0(N), F(0, 0))$, there is a modular form f of weight 2 having the same eigenvalues. Multiplying f by the Eisenstein Series E_{p-1}, we get a modular form of weight $p + 1$, which corresponds to some eigenclass $\beta \in \mathrm{H}^1(\Gamma_0(N), F(p - 1, 0))$. Since by the Claussen-von Staudt theorem, $E_{p-1} \equiv 1 \pmod{p}$, we have $fE_{p-1} \equiv f \pmod{p}$, so β has the same eigenvalues as α. The cup product may help us achieve a similar result in the three-dimensional case.

There is also a possibility of a dimension argument, which seems to require more theoretical evidence than we have yet compiled. In any case, the problem of determining whether v is an eigenvector for all Hecke operators $T(\ell, i)$ appears to be a much more difficult one and may take considerably more work.

For both $N = 11$ and 17 the "eventual" dimension of $H^3(\Gamma_0(N), F(0,0,0))$ — that is, the dimension of the cohomology for almost all p — appears to be 2. The only other prime level for which we have observed this to be the case is 19, thus we may be able to find a prediction for this level. We don't have sufficient evidence to make a prediction, however, largely because the computational power necessary is not readily available. A few computations on a supercomputer could probably reveal a pattern for a prediction and then yield this case fairly easily. Furthermore, with such computational power we may also be able to make a similar prediction about other prime levels with higher "eventual" dimension. This could be a major step in finding a lift for all N, and not just prime N, for a given v_0.

There was another eigenclass in $H^3(\Gamma_0(N), F(0,0,0))$, for which we unsuccessfully tried to find a lift. It is not out of the question, however, that a lift similar to the one described above for the first eigenclass could be found. If so, this could help provide more evidence to the generalized conjecture, which would be enhanced even more if we could also prove that the two lifts are eigenvectors of the Hecke operators for all ℓ.

Among the several Hecke eigenvectors that we found in the cohomology of the higher weight, we found one for $p = 5$, $N = 11$ that appears to correspond to a reducible Galois representation, with its eigenvalues for $T(\ell, 1)$ equal to those for $T(\ell, 2)$. This arises due to the fact that the 5-torsion subgroup of a particular elliptic curve of conductor 11 is nontrivial. There is likely one for $N = 17$ also (the 2-torsion subgroup has order 4 for one elliptic curve, for instance), but our programs were not equipped to verify this. These exceptional eigenclasses may lend information as to why for $N = 11$ we had $k = -5$, whereas for $N = 17$, $k = -4$. This could also help us find a lift for v_0 for all N.

Appendix: Computer Software

In the following GP/PARI [33] program, the first, second, and third entries of `w` correspond to the first, p^{th}, and t_p^{th}, respectively, of each block in v indexed by N^2+N+1 coset representatives. Recall that all other components of v are 0. Division by 2 here means multiplication by the multiplicative inverse of 2 modulo p.

```
N=11;

w=vector(N^2+N+1,X,vector(3));
for(i=1,N,w[i][1]=1);
for(i=N^2+1,N^2+N,w[i][1]=-1);
for(i=N^2+2,N^2+N+1,w[i][2]=1);
for(i=2,N,w[i][3]=-1);
for(i=1,N^2,if(i%N==1,w[i][2]=-1));
for(i=N+1,N^2+1,if(i%N==1,w[i][3]=1));
w[N^2+N+1][3]=-1;

k=-5;
A=-k;B=-k/2;C=k/2;D=k;

for(i=2*N+1,3*N,w[i][1]=A);
for(i=3*N+1,4*N,w[i][1]=B);
for(i=4*N+1,5*N,w[i][1]=C);
for(i=5*N+1,6*N,w[i][1]=D);
for(i=(N-2)*N+1,(N-1)*N,w[i][1]=A);
for(i=(N-3)*N+1,(N-2)*N,w[i][1]=B);
for(i=(N-4)*N+1,(N-3)*N,w[i][1]=C);
for(i=(N-5)*N+1,(N-4)*N,w[i][1]=D);

for(i=1,N^2,if((i%N==6)||(i%N==7),w[i][2]=A));
for(i=1,N^2,if((i%N==5)||(i%N==8),w[i][2]=B));
for(i=1,N^2,if((i%N==4)||(i%N==9),w[i][2]=C));
for(i=1,N^2,if((i%N==3)||(i%N==10),w[i][2]=D));

sA=[14,21,27,30,39,40,48,53,57,66,68,77,81,86,
94,95,104,107,113,120,124,131];
sB=[15,20,28,29,36,43,46,55,60,63,71,74,79,88,
91,98,105,106,114,119,125,130];
sC=[16,19,26,31,35,44,50,51,58,65,69,76,83,84,
90,99,103,108,115,118,126,129];
sD=[17,18,24,33,38,41,47,54,59,64,70,75,80,87,
93,96,101,110,116,117,127,128];
sA=Set(sA);
sB=Set(sB);
sC=Set(sC);
sD=Set(sD);
for(i=1,N^2+N+1,if(setsearch(sA,i),w[i][3]=A));
for(i=1,N^2+N+1,if(setsearch(sB,i),w[i][3]=B));
for(i=1,N^2+N+1,if(setsearch(sC,i),w[i][3]=C));
```

```
for(i=1,N^2+N+1,if(setsearch(sD,i),w[i][3]=D));
```

For N=17, the corresponding program is very similar, the only differences coming as a result of the larger number of blocks to be specified.

```
N=17;

w=vector(N^2+N+1,X,vector(3));
for(i=1,N,w[i][1]=1);
for(i=N^2+1,N^2+N,w[i][1]=-1);
for(i=N^2+2,N^2+N+1,w[i][2]=1);
for(i=2,N,w[i][3]=-1);
for(i=1,N^2,if(i%N==1,w[i][2]=-1));
for(i=N+1,N^2+1,if(i%N==1,w[i][3]=1));
w[N^2+N+1][3]=-1;

k=-4;
A=-k;B=-k/2;C=k/2;D=k;

for(i=2*N+1,3*N,w[i][1]=A);
for(i=3*N+1,4*N,w[i][1]=B);
for(i=5*N+1,6*N,w[i][1]=B);
for(i=6*N+1,7*N,w[i][1]=C);
for(i=7*N+1,8*N,w[i][1]=C);
for(i=8*N+1,9*N,w[i][1]=D);
for(i=(N-2)*N+1,(N-1)*N,w[i][1]=A);
for(i=(N-3)*N+1,(N-2)*N,w[i][1]=B);
for(i=(N-5)*N+1,(N-4)*N,w[i][1]=B);
for(i=(N-6)*N+1,(N-5)*N,w[i][1]=C);
for(i=(N-7)*N+1,(N-6)*N,w[i][1]=C);
for(i=(N-8)*N+1,(N-7)*N,w[i][1]=D);

for(i=1,N^2,if((i%N==9)||(i%N==10),w[i][2]=A));
for(i=1,N^2,if((i%N==7)||(i%N==8)||(i%N==11)||(i%N==12),w[i][2]=B));
for(i=1,N^2,if((i%N==4)||(i%N==6)||(i%N==13)||(i%N==15),w[i][2]=C));
for(i=1,N^2,if((i%N==3)||(i%N==16),w[i][2]=D));

sA=[20,33,39,48,58,63,77,78,93,96,108,115,123,134,138,153,155,170,174,
185,193,200,212,215,230,231,245,250,260,269,275,288,292,305];
sB=[21,23,30,32,41,42,45,46,54,60,61,67,72,74,81,83,88,94,95,101,104,
107,116,119,121,124,133,136,143,144,147,148,160,161,164,165,172,175,
184,187,189,192,201,204,207,213,214,220,225,227,234,236,241,247,248,
254,262,263,266,267,276,278,285,287,293,295,302,304];
sC=[24,25,28,29,38,40,47,49,53,56,65,68,75,76,79,80,87,90,99,102,105,
111,112,118,122,128,129,135,140,142,149,151,157,159,166,168,173,179,
180,186,190,196,197,203,206,209,218,221,228,229,232,233,240,243,252,
255,259,261,268,270,279,280,283,284,296,297,300,301];
sD=[26,27,36,51,59,62,71,84,92,97,106,117,125,132,141,150,158,167,176,
183,191,202,211,216,224,237,246,249,257,272,281,282,298,299];
sA=Set(sA);
sB=Set(sB);
```

```
sC=Set(sC);
sD=Set(sD);
for(i=1,N^2+N+1,if(setsearch(sA,i),w[i][3]=A));
for(i=1,N^2+N+1,if(setsearch(sB,i),w[i][3]=B));
for(i=1,N^2+N+1,if(setsearch(sC,i),w[i][3]=C));
for(i=1,N^2+N+1,if(setsearch(sD,i),w[i][3]=D));
```

The next GP/PARI program, `ADP-F(p-1,0,0)`, calculates a basis for the homology $H_3(\mathrm{SL}_3(\mathbb{Z}), \mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} V)$, where $V = F(p-1,0,0)$, listed as the matrix `kernel`. (A similar program calculates the same for $V = F(0,0,0)$.) The program essentially calculates the actions for the matrices a, b, h, and h^2 (mentioned in Section 3.2), stacks them into one matrix `bigmatrix`, and finds the kernel.

```
p=3;
N=11;

install(FpM_ker,GG);

allocatemem(300000000);

\\Matrix holding all of the matrix kernels
bigmatrix=matrix(3*(N^2+N+1)*p*(p+1)/2,(N^2+N+1)*p*(p+1)/2);

\\Matrices for which to find the action
actingmatrices=vector(2,X,matrix(3,3));
actingmatrices[1][1,3]=1;actingmatrices[1][2,1]=1;
actingmatrices[1][3,2]=1;
actingmatrices[2][1,2]=1;actingmatrices[2][2,1]=-1;
actingmatrices[2][3,3]=1;

\\Form coset reps
for(i=1,p-1,if((N*i)%p==1,Ninv=i));
K=N*Ninv;
m=(K-1)/p;
r=matrix(1,N^2+N+1,X,Y,matrix(3,3));
{for(i=0,N-1,
for(j=0,N-1,
r[1,N*i+j+1][1,1]=1;
r[1,N*i+j+1][2,2]=1;
r[1,N*i+j+1][3,3]=1;
r[1,N*i+j+1][1,2]=i*(1-K);
r[1,N*i+j+1][1,3]=j*(1-K);
r[1,N*i+j+1]=matadjoint(r[1,N*i+j+1])
);
r[1,N^2+i+1][1,1]=K;
r[1,N^2+i+1][1,2]=1-K;
r[1,N^2+i+1][1,3]=i*(1-K);
r[1,N^2+i+1][2,1]=p*(3*m+i);
```

```
r[1,N^2+i+1][2,2]=K;
r[1,N^2+i+1][2,3]=p*(8*m^2+3*i*m);
r[1,N^2+i+1][3,1]=-p;
r[1,N^2+i+1][3,2]=0;
r[1,N^2+i+1][3,3]=-2*m*p+1;
r[1,N^2+i+1]=matadjoint(r[1,N^2+i+1])
);}
r[1,N^2+N+1][1,1]=K;
r[1,N^2+N+1][1,2]=0;
r[1,N^2+N+1][1,3]=1-K;
r[1,N^2+N+1][2,1]=-p;
r[1,N^2+N+1][2,2]=-2*m*p+1;
r[1,N^2+N+1][2,3]=0;
r[1,N^2+N+1][3,1]=p*(3*m);
r[1,N^2+N+1][3,2]=p*(8*m^2);
r[1,N^2+N+1][3,3]=K;
r[1,N^2+N+1]=matadjoint(r[1,N^2+N+1]);

\\Find the action for each of the above matrices
{for(k=1,2,
print(gettime());
g=actingmatrices[k];
action=matrix(p*(p+1)*(N^2+N+1)/2,p*(p+1)*(N^2+N+1)/2);
for(i=1,N^2+N+1,
j=1;
while(((matadjoint(r[1,i])*g*r[1,j])[1,2]%N!=0)||
((matadjoint(r[1,i])*g*r[1,j])[1,3]%N!=0),j=j+1);
xyztrans=matadjoint(r[1,i])*g*r[1,j]*[x,y,z]~;
for(a=0,p-1,
for(b=0,p-1-a,
c=p-1-a-b;
(f(x,y,z)=x^a*y^b*z^c);
F=f(xyztrans[1],xyztrans[2],xyztrans[3]);
for(eye=0,p-1,
for(jay=0,p-1-eye,
kay=p-1-eye-jay;
coef=polcoeff(polcoeff(polcoeff(F,eye,x),jay,y),kay,z)%p;
if(coef!=0,action[((j-1)*p*(p+1)+eye*(2*p-eye+1))/2+jay+1,
((i-1)*p*(p+1)+a*(2*p-a+1))/2+b+1]=coef)
)
)
)
)
);
if(k==1,
for(i=1,(N^2+N+1)*p*(p+1)/2,
action[i,i]=action[i,i]-1;
for(j=1,(N^2+N+1)*p*(p+1)/2,
bigmatrix[i,j]=action[i,j]
)
),
for(i=1,(N^2+N+1)*p*(p+1)/2,
action[i,i]=action[i,i]+1;
for(j=1,(N^2+N+1)*p*(p+1)/2,
bigmatrix[(N^2+N+1)*p*(p+1)/2+i,j]=action[i,j]
```

```
)
)
);
print(k)
);}

\\Now for the h-condition
print(gettime());
h=matrix(3,3);
h[1,2]=-1;h[2,1]=1;h[2,2]=-1;h[3,3]=1;

g=h;
ginv=g^(-1);

action=matrix((N^2+N+1)*p*(p+1)/2,(N^2+N+1)*p*(p+1)/2);0;
{for(i=1,N^2+N+1,
j=1;
while(((matadjoint(r[1,j])*ginv*r[1,i])[1,2]%N!=0)||
((matadjoint(r[1,j])*ginv*r[1,i])[1,3]%N!=0),j=j+1);
xyztrans=matadjoint(r[1,i])*g*r[1,j]*[x,y,z]~;
for(a=0,p-1,
for(b=0,p-1-a,
c=p-1-a-b;
(f(x,y,z)=x^a*y^b*z^c);
F=f(xyztrans[1],xyztrans[2],xyztrans[3]);
for(eye=0,p-1,
for(jay=0,p-1-eye,
kay=p-1-eye-jay;
coef=polcoeff(polcoeff(polcoeff(F,eye,x),jay,y),kay,z)%p;
if(coef!=0,action[((j-1)*p*(p+1)+eye*(2*p-eye+1))/2+jay+1,
((i-1)*p*(p+1)+a*(2*p-a+1))/2+b+1]=coef)
)
)
)
)
);}

g=h^2;
ginv=g^(-1);

actionsquared=matrix((N^2+N+1)*p*(p+1)/2,(N^2+N+1)*p*(p+1)/2);0;
{for(i=1,N^2+N+1,
j=1;
while(((matadjoint(r[1,j])*ginv*r[1,i])[1,2]%N!=0)||
((matadjoint(r[1,j])*ginv*r[1,i])[1,3]%N!=0),j=j+1);
xyztrans=matadjoint(r[1,i])*g*r[1,j]*[x,y,z]~;
for(a=0,p-1,
for(b=0,p-1-a,
c=p-1-a-b;
(f(x,y,z)=x^a*y^b*z^c);
F=f(xyztrans[1],xyztrans[2],xyztrans[3]);
for(eye=0,p-1,
for(jay=0,p-1-eye,
kay=p-1-eye-jay;
coef=polcoeff(polcoeff(polcoeff(F,eye,x),jay,y),kay,z)%p;
```

```
if(coef!=0,actionsquared[((j-1)*p*(p+1)+eye*(2*p-eye+1))/2+jay+1,
((i-1)*p*(p+1)+a*(2*p-a+1))/2+b+1]=coef)
)
)
)
)
);}

action=action+actionsquared;0;
{for(i=1,(N^2+N+1)*p*(p+1)/2,
action[i,i]=action[i,i]+1;
for(j=1,(N^2+N+1)*p*(p+1)/2,
bigmatrix[(N^2+N+1)*p*(p+1)+i,j]=action[i,j]
)
);}

print("bigmatrix put together");

kill(action);
kill(actionsquared);

kernel=FpM_ker(bigmatrix,p);
print(gettime())
```

This GP/PARI program, `LLLMatrixFind`, uses the method of [34] to find matrices $q_{p,i,j}$ to use in calculating the action of the Hecke operators. The "LLL" in the program name refers to the fact that we use the LLL-reduction algorithm to determine optimal vectors to use as replacement "candidates" for columns in the matrices of the original coset representatives. The use of this algorithm is not required in the method of [34], and thus there are many different possibilities for sets of matrices to use to calculate the Hecke action. After we find the matrices, we determine their respective action matrices using the method outlined in the program `ADP-F(p-1,0,0)` above.

```
\\preliminaries

p=2;

R=matrix(2,p^2+p+1);

{for(i=0,p-1,
for(j=0,p-1,R[1,p*i+j+1]=[p,0,0;i,1,0;j,0,1]);
R[1,p^2+i+1]=[1,0,0;0,p,0;0,i,1]
);}
R[1,p^2+p+1]=[1,0,0;0,1,0;0,0,p];
```

```
{for(i=0,p-1,
for(j=0,p-1,R[2,p*i+j+1]=[p,0,0;0,p,0;i,j,1]);
R[2,p^2+i+1]=[p,0,0;i,1,0;0,0,p]
);}
R[2,p^2+p+1]=[1,0,0;0,p,0;0,0,p];

C=matrix(2,p^2+p+1);

{for(j=1,2,
for(i=1,p^2+p+1,
C[j,i]=matadjoint(R[j,i])
)
);}

{candidate(A)=
m=vector(3);
U=qflll(A);
for(n=1,3,
tally=0;
for(j=1,3,
m[j]=A*U;
m[j][j,]=matid(3)[n,];
if(abs(matdet(m[j]))<abs(matdet(A*U)),tally=tally+1);
);
if(tally==3,
return(matid(3)[n,]*U^-1);
break
);
);
}

{reduce(A)=
for(i=1,3,
B=A;
B[i,]=candidate(A);
if(abs(matdet(B))==1,list=concat(list,[B]);count=count+1);
if(abs(matdet(B))>1,reduce(B));
);
}

\\main program

{for(k=1,2,
count=0;
list=[[]];
for(i=1,p^2+p+1,
reduce(C[k,i]);
list[1]=concat(list[1],[count]);
if(i==1,
for(j=1,list[1][i],print(list[j+1]*R[k,i])),
for(j=list[1][i-1]+1,list[1][i],print(list[j+1]*R[k,i],";",count))
)
);
);}
```

Bibliography

[1] Gerald Allison, Avner Ash, and Eric Conrad, *Galois representations, Hecke operators, and the mod-p cohomology of* $GL(3,\mathbb{Z})$ *with twisted coefficients*, Experiment. Math. **7** (1998), no. 4, 361–390.

[2] Avner Ash, *Galois representations attached to mod p cohomology of* $GL(n,\mathbb{Z})$, Duke Math. J. **65** (1992), no. 2, 235–255.

[3] Avner Ash, Darrin Doud, and David Pollack, *Galois representations with conjectural connections to arithmetic cohomology*, Duke Math. J. **112** (2002), no. 3, 521–579.

[4] Avner Ash and David Pollack, *Everywhere unramified automorphic cohomology for* $SL_3(\mathbb{Z})$, Int. J. Number Theory **4** (2008), no. 4, 663–675.

[5] Avner Ash, David Pollack, and Warren Sinnott, A_6*-extensions of* $\mathbb{Q}$ *and the mod p cohomology of* $GL_3(\mathbb{Z})$, J. Number Theory **115** (2005), no. 1, 176–196.

[6] Avner Ash, David Pollack, and Dayna Soares, $SL_3(\mathbb{F}_2)$*-extensions of* $\mathbb{Q}$ *and arithmetic cohomology modulo 2*, Experiment. Math. **13** (2004), no. 3, 298–307.

[7] Avner Ash and Warren Sinnott, *An analogue of Serre's Conjecture for Galois representations and Hecke eigenclasses in the mod p cohomology of* $GL(n,\mathbb{Z})$, Duke Math. J. **105** (2000), no. 1, 1–24.

[8] Avner Ash and Glenn Stevens, *Cohomology of arithmetic groups and congruences between systems of Hecke eigenvalues*, J. Reine Angew. Math. **365** (1986), 192–220.

[9] Avner Ash and Pham Huu Tiep, *Modular representations of* $GL(3,\mathbb{F}_p)$*, symmetric squares, and mod-p cohomology of* $GL(3,\mathbb{Z})$, J. Algebra **222** (1999), no. 2, 376–399.

[10] Jonathan Blackhurst, *Proven cases of a generalization of Serre's Conjecture*, Master's thesis, Brigham Young University, 2006.

[11] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts In Mathematics, vol. 87, Springer-Verlag, New York, 1982.

[12] Henri Darmon, *Serre's conjectures*, CMS Conf. Proc. **17** (1995), 135–153.

[13] Henri Darmon, Fred Diamond, and Richard Taylor, *Fermat's Last Theorem*, Elliptic Curves, Modular Forms, and Fermat's Last Theorem (J. Coates and S. T. Yau, eds.), International Press, 1997, pp. 2–140.

[14] Pierre Deligne and Jean-Pierre Serre, *Formes modulaires de poids* 1 *(French)*, Ann. Sci. École Norm. Sup. **7** (1974), no. 4, 507–530.

[15] Luis Dieulefait and Núria Vila, *On the images of modular and geometric three-dimensional Galois representations*, Amer. J. Math. **126** (2004), no. 2, 335–361.

[16] Stephen R. Doty and Grant Walker, *The composition factors of $F_p[x_1, x_2, x_3]$ as a $GL(3,p)$-module*, Journal of Algebra **147** (1992), 411–441.

[17] Darrin Doud, *Wildly ramified Galois representations and a generalization of a conjecture of Serre*, Experiment. Math. **14** (2005), no. 1, 119–127.

[18] ———, *Supersingular Galois representations and a generalization of a conjecture of Serre*, Experiment. Math. **16** (2007), no. 1, 119–128.

[19] Darrin Doud and Brian Hansen, *Explicit Frobenius calculations supporting a generalization of a conjecture of Serre*, JP J. Algebra Number Theory Appl. **6** (2006), no. 2, 381–398.

[20] J.A. Green, *Polynomial representations of $GL(n)$*, Lecture Notes in Mathematics, vol. 830, Springer-Verlag, New York, 1980.

[21] Florian Herzig, *The weight in a Serre-type conjecture for tame n-dimensional Galois representations*, Duke Math. J. **149** (2009), no. 1, 37–116.

[22] Roger A. Horn and Charles R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1985.

[23] ———, *Topics in matrix analysis*, Cambridge University Press, Cambridge, 1991.

[24] Chandrashekhar Khare and Jean-Pierre Wintenberger, *On Serre's Conjecture for 2-dimensional mod p representations of Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)*, Ann. of Math. (2) **169** (2009), no. 1, 229–253.

[25] Mark Kisin, *Modularity of 2-dimensional Galois representations*, Current Developments in Mathematics, vol. 2005, Int. Press, Somerville, MA, 2007.

[26] Anthony W. Knapp, *Elliptic curves*, Mathematical Notes, vol. 40, Princeton University Press, Princeton, New Jersey, 1992.

[27] Daniel A. Marcus, *Number fields*, Springer-Verlag, New York-Berlin-Heidelberg, 1977.

[28] Jürgen Neukirch, *Class field theory*, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 280, Springer-Verlag, Berlin, 1986.

[29] Michael M. Schein, *Weights in Serre's Conjecture for GL_n via the Bernstein-Gelfand-Gelfand complex*, J. Number Theory **128** (2008), no. 10, 2808–2822.

[30] Jean-Pierre Serre, *Sur les représentations modulaires de degré 2 de Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)*, Duke Math. J. **54** (1987), no. 1, 179–230.

[31] Goro Shimura, *Introduction to the arithmetic theory of automorphic functions*, Kano Memorial Lectures, no. 1, vol. 11, Publications of the Mathematical Society of Japan, Tokyo, 1971.

[32] Joseph Silverman, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 106, Springer-Verlag, New York, 1986.

[33] The PARI Group, Bordeaux, *PARI/GP, version* `2.2.8`, 2003, available from `http://pari.math.u-bordeaux.fr/`.

[34] Bert van Geemen, Wilberd van der Kallen, Jaap Top, and Alain Verberkmoes, *Hecke eigenforms in the cohomology of congruence subgroups of* $SL(3,\mathbb{Z})$, Experimental Mathematics **6** (1997), no. 2, 163–174.

[35] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, 1994.

[36] Andrew Wiles, *Modular elliptic curves and Fermat's Last Theorem*, Ann. of Math. **141** (1995), no. 2, 443–551.