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Planar CAT(k) Subspaces

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ABSTRACT

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Let M_k^2 be the complete, simply connected, Riemannian 2-manifold of constant curvature $k \leq 0$. Let E be a closed, simply connected subspace of M_k^2 with the property that every two points in E are connected by a rectifiable path in E. We show that E is CAT(k) under the induced path metric.

Keywords: CAT(k) spaces, Jordan Curve Theorem, nonpositive curvature, convexity

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Chapter 1. Introduction

Part of geometric group theory is the study of CAT(k) spaces, in particular CAT(0) spaces. The Russian geometer A. D. Alexandrov gave a characterization for a geodesic metric space to have curvature bounded above by some $k \in \mathbb{R}$. This characterization describes how "fat" a geodesic triangle can be in the space, and it is now called the CAT(k) inequality (the letters C, A, and T being chosen in honor of Cartan, Alexandrov, and Toponogov). A geodesic triangle which satisfies the CAT(k) inequality is called CAT(k), and a geodesic space in which all geodesic triangles are CAT(k) is called a CAT(k) space.

The CAT(k) inequality gives a surprising amount of information about the overall large-scale geometry of a CAT(k) space, especially for $k \leq 0$. Since for $k < k'$, every CAT(k) space is also a CAT(k') space, a negative value of k gives more information about the space, while a positive value gives less information. In particular, a CAT(k) space with $k < 0$ is in a sense negatively curved, and its geometry inherits some important properties of hyperbolic geometry. Similarly, a CAT(0) space is nonpositively curved, and its geometry inherits some important properties that both Euclidean and hyperbolic geometry have. A CAT(k) space with $k > 0$ may have some positive curvature, and its geometry may reflect some spherical properties, but the upper bound still provides some control on the wildness of the space.

Because of the extra information gained by knowing that a space is CAT(k), an important part of the study of CAT(k) spaces is to provide theorems that recognize whether a given space is CAT(k). One goal of these theorems is to provide a way to more easily construct examples of CAT(k) spaces, by determining when a certain type of construction yields a CAT(k) space. The goal of this paper is to provide another class of easily-constructed examples for CAT(k) spaces with $k \leq 0$. In particular, the main theorem is the following.

Theorem 1.1. *Let M_k^2 be the complete, simply connected, Riemannian 2-manifold of constant curvature $k \leq 0$. Let E be a closed, simply connected subspace of M_k^2 with the property that every two points in E are connected by a rectifiable path in E. Then E is $\mathrm{CAT}(k)$ under the induced path metric.*

The simplest case of this construction is with $k = 0$, where the resultant spaces are "reasonable" subspaces of the Euclidean plane. These spaces are therefore easy to visualize, and they look very CAT(0). However, proving "obvious" theorems in the plane is not always as trivial as one might expect. For example, the Jordan Curve Theorem states that a simple closed curve in the plane bounds exactly two open regions, one of which is bounded, and the curve is precisely the boundary of each region; though the Jordan Curve Theorem looks obvious, it is actually fairly difficult (and long) to prove. The proof we give of Theorem 1.1 relies heavily on the Jordan Curve Theorem.

For Theorem 1.1, one could easily forget the necessity of establishing the equivalence of the obvious notion of angles in the space with the more frequently-used Alexandrov angle (this is proven in Theorem 3.11). Section 3.3 is devoted almost entirely to proving the equivalence and, in fact, represents more than half of the new material (by space consumed, at least) in the paper. On the other hand, the results in Section 3.2, needed to even define the "obvious" angles of Section 3.3, all seem fairly obvious and have simple proofs, yet they were much harder to find (both their statements and their proofs).

See [2] for an alternate treatment of Theorem 1.1 with $k = 0$, and where E is the set of finite-distance points in the homeomorphic image of a closed disk.

Chapter 2. Properties of CAT(k) Spaces

2.1 The Euclidean Plane and the Hyperbolic Plane

Let M_k^2 be the complete, simply connected, Riemannian 2-manifold of constant curvature k. Then M_0^2 is the Euclidean plane $\mathbb{E}^2$ ($\mathbb{R}^2$ with the standard Euclidean metric). For $k < 0$, the space M_k^2 can be obtained from the hyperbolic plane $\mathbb{H}^2 = M_{-1}^2$ by multiplying all distances by $1/\sqrt{-k}$. Thus in many computations for $k \leq 0$, only the cases $k = 0$ and $k = -1$ are necessary.

The following standard results give a way to compute lengths and angles in $\mathbb{E}^2$ and $\mathbb{H}^2$; the angles used here are the standard angles in the spaces. First, recall the Euclidean law of cosines and the Euclidean law of sines.

Proposition 2.1 (Law of Cosines)**.** *Suppose a, b, and c are the side lengths of a Euclidean triangle; let θ be the angle opposite the side of length c. Then*

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

Proposition 2.2 (Law of Sines)**.** *Suppose a, b, and c are the side lengths of a Euclidean triangle; let A, B, and C be the angles opposite the sides of length a, b, and c, respectively. Then*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

In hyperbolic space, we have the following hyperbolic law of cosines. Both [3] and [4] give a proof.

Proposition 2.3 (Hyperbolic Law of Cosines)**.** *Suppose a, b, and c are the side lengths of* *a hyperbolic triangle; let θ be the angle opposite the side of length c. Then*

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \theta.$$

Also, we have the following hyperbolic law of sines (proven in [4]).

Proposition 2.4 (Hyperbolic Law of Sines)**.** *Suppose a, b, and c are the side lengths of a hyperbolic triangle; let A, B, and C be the angles opposite the sides of length a, b, and c, respectively. Then*

$$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}.$$

Another classical fact of Euclidean geometry is that, given any line L and point p not on L, there is a unique line L' parallel to L (i.e., L' does not intersect L) that passes through the point p. Moreover, the metric distance $d(L, L') = \inf_{\substack{x \in L \\ y \in L'}} d(x, y)$ between L and L' equals the distance between p and L', and the line segment from p to the point $q \in L$ closest to p meets both L and L' at an angle of $\pi/2$.

Hyperbolic geometry has parallel (non-intersecting) lines, but they are not uniquely defined by a line and a point, as in Euclidean geometry. However, given a line $L \subset \mathbb{H}^2$ and a point $p \in \mathbb{H}^2 \setminus L$, let K be the line segment from p to the point $q \in L$ closest to p; then there is a unique line L' in $\mathbb{H}^2$ such that $p \in L'$ and the angle between L' and K is $\pi/2$ on both sides. As a result, the metric distance between L and L' equals the distance between p and q. We will use this construction to simplify our arguments later (see Definition 3.3).

Finally, the following result is often useful in constructing CAT(k) spaces. The statement here follows [3, pp. 25-26], where it is also proved; however, here we consider only $k \leq 0$, which simplifies the statement slightly.

Proposition 2.5 (Alexandrov's Lemma)**.** *Consider four distinct points $A, B, B', C \in M_k^2$. Suppose that B and B' lie on opposite sides of the line through A and C.*

Consider the geodesic triangles $\triangle = \triangle(A, B, C)$ *and* $\triangle' = \triangle(A, B', C)$. *Let* α, β, γ *(resp.* α', β', γ'*) be the angles of* $\triangle$ *(resp.* $\triangle'$*) at the vertices* A, B, C *(resp.* A, B', C*). Assume that* $\gamma + \gamma' \geq \pi$. *Then,*

$$d(B, C) + d(B', C) \leq d(B, A) + d(B', A).$$

Let $\overline{\triangle}$ *be a triangle in* M_k^2 *with vertices* $\overline{A}, \overline{B}, \overline{B}'$ *such that* $d(\overline{A}, \overline{B}) = d(A, B)$, $d(\overline{A}, \overline{B}') = d(A, B')$ *and* $d(\overline{B}, \overline{B}') = d(B, C) + d(C, B')$. *Let* $\overline{C}$ *be the point of* $[\overline{B}, \overline{B}']$ *with* $d(\overline{B}, \overline{C}) = d(B, C)$. *Let* $\overline{\alpha}, \overline{\beta}, \overline{\beta}'$ *be the angles of* $\overline{\triangle}$ *at the vertices* $\overline{A}, \overline{B}, \overline{B}'$. *Then,*

$$\overline{\alpha} \geq \alpha + \alpha', \overline{\beta} \geq \beta, \overline{\beta}' \geq \beta' \text{ and } d(A, C) \leq d(\overline{A}, \overline{C});$$

any on equality implies the others, and occurs if and only if $\gamma + \gamma' = \pi$.

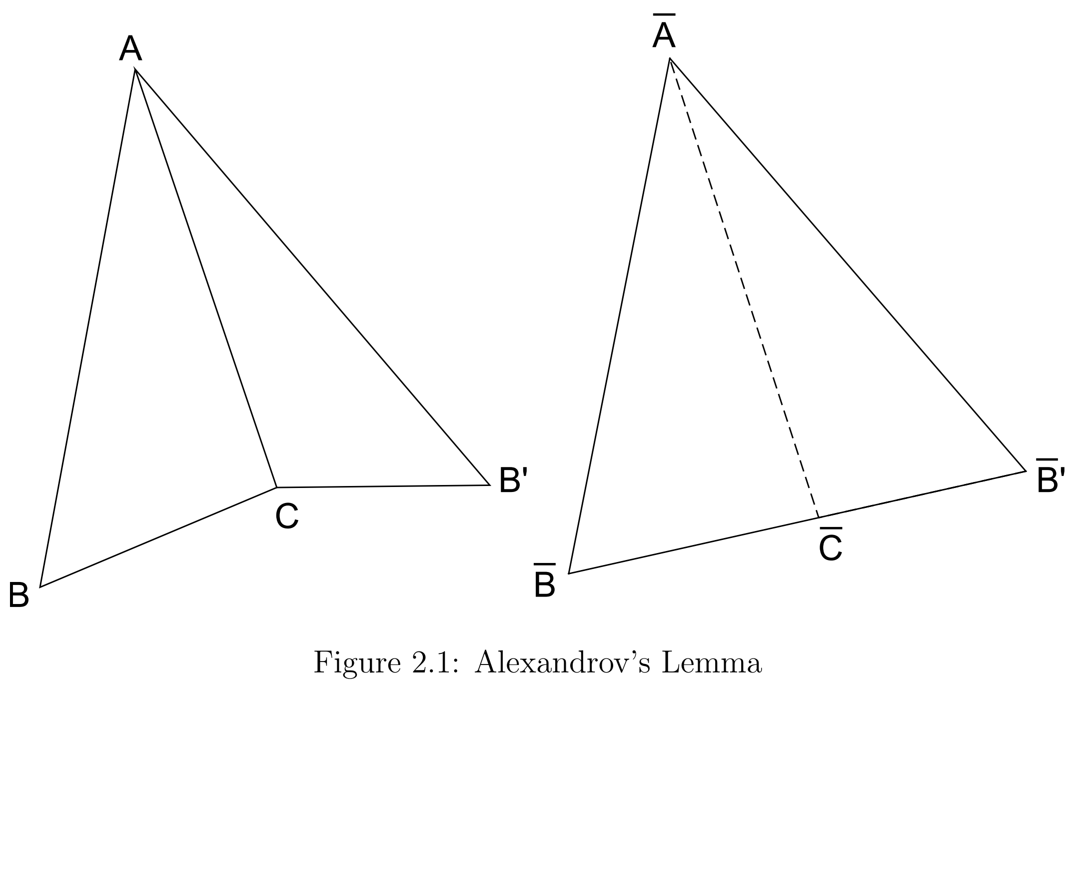

Figure 2.1: Alexandrov's Lemma

2.2 The CAT(k) Condition

Let $\triangle$ be a geodesic triangle (in some metric space) with side lengths a, b, and c. A geodesic triangle in M_k^2 is called a *comparison triangle for* $\triangle$ *in* M_k^2 if the lengths of its sides are a, b, and c. To describe the CAT(k) inequality, we will use the following fact (see [3, p. 24] for a proof).

Proposition 2.6 (Existence of Comparison Triangles in M_k^2)**.** *Let a, b, and c be nonnegative real numbers such that $a \leq b+c$, $b \leq a+c$, and $c \leq a+b$. For every $k \leq 0$, there exists some geodesic triangle in M_k^2 with side lengths a, b, and c. Furthermore, this triangle is unique up to isometry of M_k^2.*

Let $\triangle$ be a geodesic triangle with vertices p, q, and r. Let $\overline{\triangle}$ be a comparison triangle for $\triangle$ in M_k^2 with corresponding vertices $\overline{p}$, $\overline{q}$, and $\overline{r}$. If x lies on the edge between p and q, $\overline{x}$ lies on the edge between $\overline{p}$ and $\overline{q}$, and $d(\overline{x}, \overline{p}) = d(x, p)$, then $\overline{x}$ is called a *comparison point for x on $\overline{\triangle}$.*

Definition 2.7 (CAT(k) Inequality)**.** Let $\triangle$ be a geodesic triangle in the metric space X, and let $\overline{\triangle}$ be the comparison triangle for $\triangle$ in M_k^2. Then $\triangle$ is said to *satisfy the* CAT(k) *inequality* if

$$d(x, y) \leq d(\overline{x}, \overline{y})$$

for all $x, y \in \triangle$ with comparison points $\overline{x}, \overline{y}$, respectively, on $\overline{\triangle}$. If X is a geodesic space in which all geodesic triangles are CAT(k), then X is said to be CAT(k).

The concept of a CAT(k) space is closely related to the Alexandrov angle at the vertex of a geodesic triangle. Let X be a geodesic space, and let $\angle_p^{(k)}(q, r)$ be the angle at $\overline{p}$ in the comparison triangle $\triangle(\overline{p}, \overline{q}, \overline{r})$ in M_k^2 for $\triangle(p, q, r)$. (Note that although the geodesics between p, q, and r may not be unique, the lengths of all geodesics are equal, hence $\angle_p^{(k)}(q, r)$ is also well-defined.) Now suppose $\sigma\colon [0, 1] \to X$ and $\tau\colon [0, 1] \to X$ are constant-speed

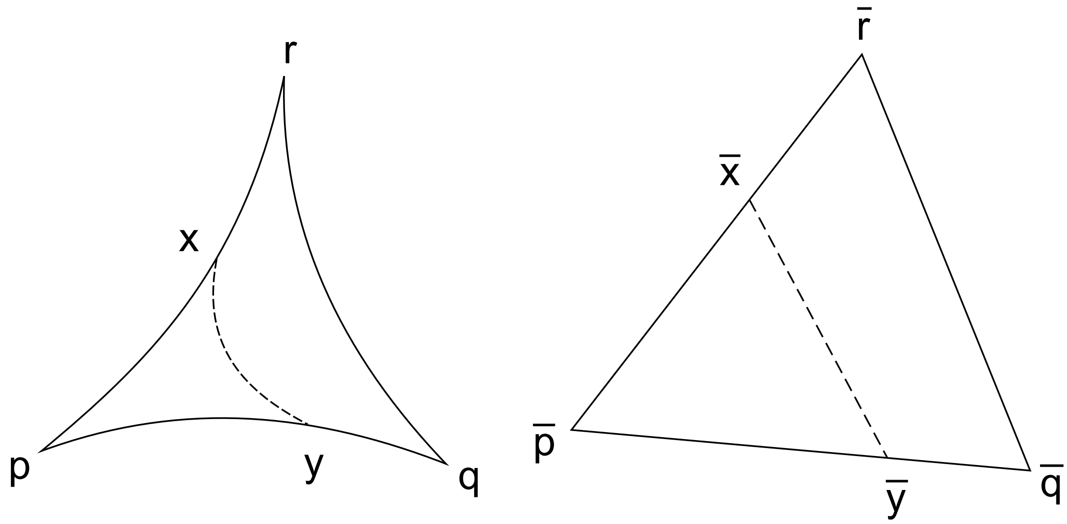

Figure 2.2: The CAT(k) Inequality

geodesic line segments emanating from the point $p \in X$, with $\sigma(1) = q$ and $\tau(1) = r$. The *Alexandrov angle* between σ and τ is defined as

$$\angle_p(\sigma, \tau) = \lim_{\epsilon \to 0} \sup_{0 < t, t' < \epsilon} \angle_p^{(0)}(\sigma(t), \tau(t')).$$

If X is uniquely geodesic, then we say the Alexandrov angle at p between q and r is

$$\angle_p(q, r) = \lim_{\epsilon \to 0} \sup_{0 < t, t' < \epsilon} \angle_p^{(0)}(\sigma(t), \tau(t')).$$

Although the Alexandrov angle is defined only in terms of comparison triangles in $M_0^2 = \mathbb{E}^2$, using comparison triangles in M_k^2 for any k would give the same result (see [3, p. 25]):

Proposition 2.8. *Let X be a geodesic space and suppose $\sigma\colon [0,1] \to X$ and $\tau\colon [0,1] \to X$ are constant-speed geodesic line segments emanating from the point $p \in X$, with $\sigma(1) = q$ and $\tau(1) = r$. Then for any $k \in \mathbb{R}$,*

$$\angle_p(q, r) = \lim_{\epsilon \to 0} \sup_{0 < t, t' < \epsilon} \angle_p^{(k)}(\sigma(t), \tau(t')).$$

The following fact about the Alexandrov angle is often very useful (see [3, p. 10]).

Proposition 2.9 (Triangle Inequality for Angles). *Let X be a metric space and let c, c' and c'' be three geodesic paths in X issuing from the same point p. Then,*

$$\angle(c, c') \leq \angle(c, c'') + \angle(c'', c').$$

The CAT(k) inequality is equivalent to other conditions involving the Alexandrov angles in geodesic triangles (see [3, p. 161-162]). We will use the following one.

Theorem 2.10. *Let X be a geodesic metric space and $k \leq 0$. Then X is* CAT(k) *if and only if $\angle_p(q, r) \leq \angle_p^{(k)}(q, r)$ for every triple of distinct points $p, q, r \in X$.*

For more on CAT(k) spaces, we refer the reader to [3] or [1].

Chapter 3. Planar Subspaces

Our goal in this section is to prove the main result (stated in the introduction):

Theorem (1.1). *Let M_k^2 be the complete, simply connected, Riemannian 2-manifold of constant curvature $k \leq 0$. Let E be a closed, simply connected subspace of M_k^2 with the property that every two points in E are connected by a rectifiable path in E. Then E is* CAT(k) *under the induced path metric.*

Note the following convention.

Convention. We will use the terms *line* and *line segment* to refer to standard geodesic lines and geodesic line segments in M_k^2. We will use *geodesic* and *geodesic segment* to refer to the geodesics and geodesic segments in E under the induced path metric.

3.1 Unique Geodesics

Let E be a closed, simply connected subspace of M_k^2 with the property that every pair of points in E are connected by a rectifiable path in E. Let d be the induced subspace metric and $\bar{d}$ the induced path metric on E. We will write $B_d(p,r)$ and $\overline{B}_d(p,r)$, respectively, for the open and closed balls of radius r about $p \in E$ in the standard metric on M_k^2.

Since E is closed in M_k^2, we know (E,d) is complete. The proofs of the following two more general results are provided for completeness.

Lemma 3.1. *The induced path metric on a complete metric space is complete.*

Proof. Let (X,d) be a complete metric space and $(X,\bar{d})$ be the induced path metric on X. Suppose $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(X,\bar{d})$. Since $\bar{d}(x,y) \geq d(x,y)$ for all $x,y \in X$, we know $\{x_n\}$ is also Cauchy in (X,d). Hence x_n converges under d to some $x \in X$. Now a Cauchy sequence converges if and only if it has a convergent subsequence, so we may assume, by passing to a subsequence if necessary, that $\bar{d}(x_n,x_m) < 2^{-m}$ for all m,n with $n > m$. So for each m there exists a path $c_m\colon [0,1] \to X$ from x_m to x_{m+1} with $l(c_m) \leq 2^{-m}$ by assumption. By linear reparameterization, we have paths $p_m\colon [1-2^{-m+1}, 1-2^{-m}] \to X$ from x_m to x_{m+1} with $l(p_m) \leq 2^{-m}$. Pasting these paths together and setting $p(1) = x$, we have a continuous map $p\colon [0,1] \to X$. Thus p is a path from x_m to x of length at most $\sum_{j=m}^{\infty} 2^{-j} = 2^{-m+1}$, so $\bar{d}(x_m,x) \leq 2^{-m+1}$. Therefore, $x_m \to x$ under $\bar{d}$. □

Corollary 3.2. *Suppose X is a complete metric space, and every two points in X are connected by a rectifiable path. Then the induced path metric on X is geodesic.*

Proof. By definition of path length, every pair of points $x,y \in X$ has approximate midpoints (see [3, p. 164]). Thus X, being complete, is geodesic. □

The following construction gives us a useful notion of parallel lines in M_k^2.

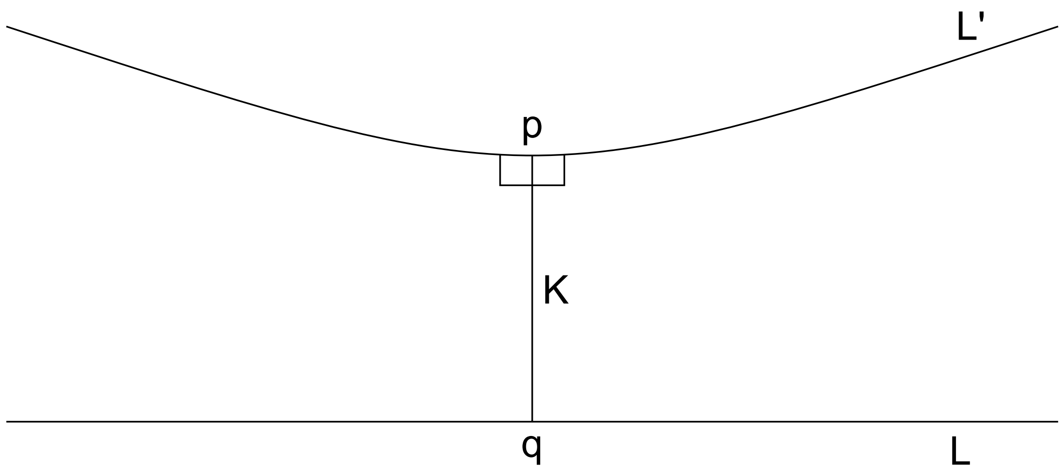

Figure 3.1: The line parallel to L at p

Definition 3.3. Let L be a line in M_k^2 and p be a point in $M_k^2 \setminus L$. Let K be the line segment from p to the point $q \in L$ closest to p. There is a unique line L' in M_k^2 such that $p \in L'$ and the angle between L' and K is $\pi/2$ on both sides. We call L' the *line parallel to* L *at* p and write $\mathrm{par}(L, p)$ for L'.

Note that the metric distance between L and L' equals the distance between p and q in the above definition.

Lemma 3.4. $(E, \bar{d})$ *is uniquely geodesic.*

Proof. Suppose $\sigma \colon [a, b] \to E$ and $\tau \colon [a, b] \to E$ are distinct unit-speed geodesics with $p = \sigma(a) = \tau(a)$ and $q = \sigma(b) = \tau(b)$. Note that since both are unit-speed geodesics, $\sigma(t)$ is in the image of τ if and only if $\sigma(t) = \tau(t)$, and similarly for $\tau(t)$. Since σ and τ are distinct, there is some $t_0 \in (a, b)$ such that $\sigma(t_0) \neq \tau(t_0)$, hence $\sigma(t_0)$ is not in the image of τ. Taking the last $a' \in [a, t_0]$ and the first $b' \in [t_0, b]$ such that $p' = \sigma(a')$ and $q' = \sigma(b')$ are both in the image of τ, we have that $C = \sigma([a', b']) \cup \tau([a', b'])$ is a simple closed curve in E.

Let L be the line in M_k^2 between p' and q'. Let R be the maximum distance from L to C, and let t_1 be the first point of $[a', b']$ such that either $d(\sigma(t_1), L) = R$ or $d(\tau(t_1), L) = R$. We may assume $d(\sigma(t_1), L) = R$. Then, since C is a simple closed curve and $a' < t_1 < b'$, there

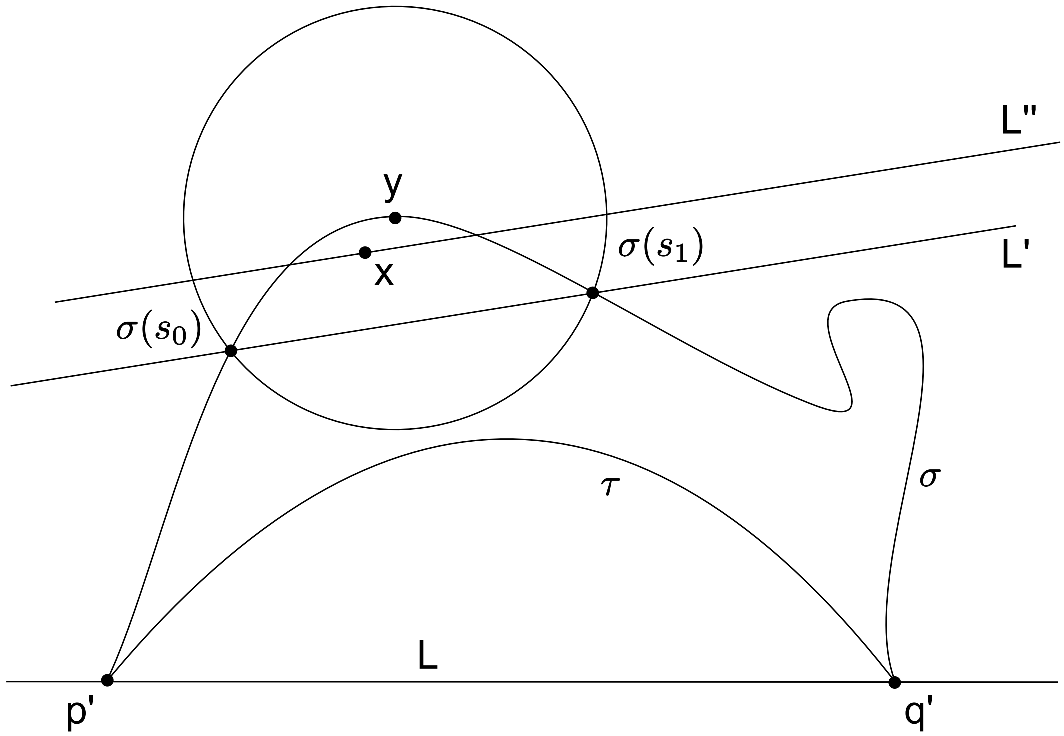

Figure 3.2: Lemma 3.4

is some radius $r > 0$ about $y = \sigma(t_1)$ such that $\overline{B}_d(y, r)$ does not intersect $\tau([a', b'])$. Let A be the connected component of $C \cap \overline{B}_d(y, r)$ containing y, and let $s_0 \in [a', t_1]$ and $s_1 \in [t_1, b']$ satisfy $\sigma([s_0, s_1]) = A$.

Now let L' be the line through $\sigma(s_0)$ and $\sigma(s_1)$. Note that $d(\sigma(s_0), L) < d(y, L)$ and $d(\sigma(s_1), L) \le d(y, L)$ by choice of y, so $y \notin L'$ by convexity of d. By the Jordan curve theorem, y is the limit of points in the interior region D bounded by C. So there is some point $x \in D$ with $d(x, y) < d(x, L')$. Let $L'' = \mathrm{par}(L', x)$; since $d(x, L') = d(L'', L')$, we also have $L' \cap L'' = \varnothing$. Since x is in D, L'' hits C on each side of x; by construction, L'' first hits C inside $\overline{B}_d(y, r)$ in each direction. By choice of r, we therefore have a straight line segment through D between two points on $\sigma([a', b'])$ where σ does not follow the line segment exactly. But $D \subset E$ since (E, d) is simply connected, so this contradicts σ being geodesic. Therefore, $(E, \bar{d})$ is uniquely geodesic. $\square$

3.2 Simple Geodesic Triangles

We will use the following terminology: Call a geodesic triangle $T \subset (E, \bar{d})$ *simple* if $T \subset (E, d)$ is a simple closed curve. For this section, let T be a simple geodesic triangle in $(E, \bar{d})$ with interior (under the standard M_k^2 metric) S and exterior U.

Proposition 3.5. *Let L be a line in M_k^2 that passes through two distinct points p and q that lie on a single edge A of T. Let L_0 be the open line segment between p and q. If L_0 has empty intersection with T then $L_0 \subset U$.*

Proof. Since T is a simple closed curve in (E, d) and (E, d) is simply connected, $S \subset E$. Hence if L_0 has empty intersection with T, we have that L_0 is contained entirely in either S or U. But $L_0 \subset S$ would give us $L_0 \subset E$, and this contradicts the hypothesis that A is the shortest path in E from p to q. Therefore, $L_0 \subset U$. □

Lemma 3.6. *Let L be a line in M_k^2 that passes through the point $p \in T$, where p is not a vertex of T. Let A be the edge of T that contains p. Suppose that $r > 0$ is a radius such that $T \cap B_d(p, r) \subset A$, and let L^- and L^+ be the two components of $L \cap B_d(p, r) \setminus \{p\}$. Then at least one of L^- and L^+ has empty intersection with U. Moreover, if $L^- \cap T = L^- \cap A \neq \varnothing$ then $L^+ \cap U = \varnothing$.*

Proof. First suppose, by way of contradiction, that there exist points $x \in L^- \cap U$ and $y \in L^+ \cap U$. Let $r' > 0$ be some radius with $r' < r$ such that we have both $B_d(x, r') \subset U$ and $B_d(y, r') \subset U$. Now by the Jordan Curve Theorem, $T = \partial S$, so there is some point $q \in S$ close enough to p that $L' = \mathrm{par}(L, q)$ hits points x' in $B_d(x, r')$ and y' in $B_d(y, r')$.

Now L' must be exterior at x' and y', but interior at q; furthermore, q lies between x' and y' on L' by construction. Thus L' must hit T somewhere between x' and q and somewhere between q and y'. Therefore, L' hits T at two points x'' and y'' closest to q (on opposite sides). By hypothesis on the radius r, we must have $x'' \in A$ and $y'' \in A$. Hence L' contains

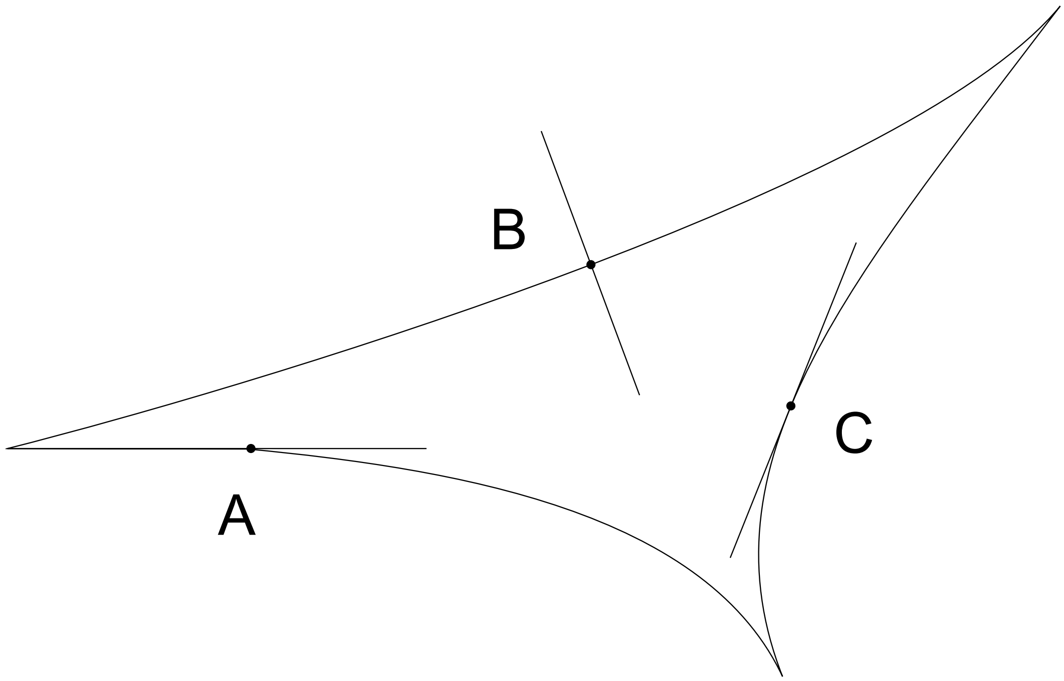

Figure 3.3: Lemma 3.6 allows three types of lines through an edge of a simple triangle: (A) The line intersects the triangle on one side and is locally interior on the other side, (B) the line is locally interior on one side and locally exterior on the other, or (C) the line is locally interior on both sides.

a line segment between two points of A that is completely interior by construction. This contradicts Proposition 3.5, and therefore at least one of L^- and L^+ has empty intersection with U.

Suppose now that there is some point $z \in T \cap L^-$ and some point $w \in U \cap L^+$. Let $r' > 0$ be some radius with $r' < r$ such that we have $B_d(w, r') \subset U$. The Jordan Curve Theorem guarantees points in U arbitrarily close to z, so let $z' \in U$ be close enough to z that the line L'' passing through the points z' and p enters $B_d(w, r')$. But then L'' passes through the point p and has nonempty intersection with U on both sides of p, which contradicts the result of the previous paragraph. Hence L^+ must have empty intersection with U if L^- has nonempty intersection with T. $\square$

Corollary 3.7. *Let p_1, p_2, and p_3 be three distinct points on a single edge A of T. Suppose* *that p_1, p_2, and p_3 lie on a line L in M_k^2, with p_1 and p_3 on opposite sides of p_2. Let L_1 and L_2 be the open line segments from p_1 to p_2 and from p_2 to p_3, respectively. If L_1 and L_2 both have empty intersection with $T \setminus A$, then the arc from p_1 to p_3 along T follows L.*

Proof. Suppose both L_1 and L_2 have empty intersection with $T \setminus A$. Then Proposition 3.5 implies that both L_1 and L_2 must have empty intersection with the interior. Hence Lemma 3.6 gives us that if L_1 has nonempty intersection with U, then L_2 must follow A, so L_2 has nonempty intersection with T, and thus L_1 has empty intersection with U; this is a contradiction, so L_1 must have empty intersection with U. Thus L_1 follows A (i.e., $L_1 \subset A$). Similarly, L_2 must follow A. Therefore, the arc from p_1 to p_3 along T follows L. $\square$

Lemma 3.8. *Suppose the vertices of T are x, y, and z. Let $\triangle'$ be the triangle in M_k^2 with vertices x, y, and z, and let $C \subset M_k^2$ be the convex hull of $\triangle'$. Then T is contained in C.*

Proof. Suppose, by way of contradiction, that $p \in T \setminus C$. Let L be the line passing through x and y. We may assume that p lies in the component of $M_k^2 \setminus L$ that contains no point of C; let H be the closure of this component. Then $H \cap T$ is compact and nonempty, so it contains at least one point p' of maximum distance to L. Let L' be the line parallel to L at p'. Now $L' \cap T$ is compact and nonempty, so let q be a point on $L' \cap T$ of maximum distance to p'.

Since $q \notin C$, q is not a vertex of T. Hence there is a radius $r > 0$ such that $B_d(q, r)$ touches no point of any edge of T other than the one on which q lies. Let L'^- and L'^+ be the two components of $L' \cap B_d(q, r) \setminus \{q\}$. Lemma 3.6 requires both L'^+ and L'^- to be in T since $L' \cap S$ is empty, but this contradicts our choice of q. Therefore, $T \subset C$, and the theorem is proved. $\square$

3.3 Limit Outer Angles

If p, q, and r are distinct point in E, we will call the angle in M_k^2 at p between q and r the *outer angle at p between q and r*, and denote it $A_p(q, r)$. Now suppose $\sigma\colon [0,1] \to E$ and $\tau\colon [0,1] \to E$ are constant-speed geodesic line segments emanating from the point $p \in E$, the images of which intersect only at p, with $\sigma(1) = q$ and $\tau(1) = r$. By Proposition 3.5 and Lemma 3.8, we have that $A_p(\sigma(t), \tau(t'))$ decreases monotonically in both t and t', so the *limit outer angle*

$$A'_p(q, r) = \lim_{t,t' \to 0} A_p(\sigma(t), \tau(t'))$$

is well defined. We will show that the limit outer angle $A'_p(q, r)$ equals the Alexandrov angle $\angle_p(q, r)$.

As before, let T be a simple geodesic triangle in $(E, \bar{d})$ with interior (under the standard M_k^2 metric) S and exterior U; denote the vertices p, q, and r. Also, let $\sigma\colon [0,1] \to E$ and $\tau\colon [0,1] \to E$ be the geodesic line segments from p to q and from p to r, respectively.

Lemma 3.9. *Suppose $A'_p(q, r) < \frac{\pi}{2}$, and τ follows a line L in M_k^2 near p (i.e., $\tau([0, \delta]) \subset L$ for some $\delta > 0$). Then there exists $t_1 > 0$ such that, for any t with $0 < t < t_1$, the line segment from $\sigma(t)$ to L perpendicular to L is contained in $S \cup T$.*

Proof. Since $A_p(\sigma(t), \tau(t'))$ decreases monotonically in both t and t', we may find some $\delta' \in (0, \delta]$ such that $A_p(\sigma(t), \tau(t')) < \frac{\pi}{2}$ for all t and t' with $0 < t, t' \leq \delta'$. Let $D = \overline{B}_d(p, \epsilon)$, where $\epsilon > 0$ is small enough that $D \cap T \subset \sigma([0, \delta']) \cup \tau([0, \delta'])$. Let P be projection in M_k^2 onto L, with domain restricted to the image of σ, and let L^+ be the component of $L \setminus \{p\}$ that has nonempty intersection with the image of τ.

Since $A'_p(q, r) < \frac{\pi}{2}$, there is some $t_0 > 0$ with $C = \sigma([0, t_0]) \subset D$ such that $P(\sigma(t)) \in L^+$ for every t with $0 < t \leq t_0$. Since P is continuous and C is compact, $P(C)$ has some point $q_1 = \sigma(t_1) \in C$ such that $P(q_1)$ attains the maximum distance from p. We further require that t_1 be the smallest such value.

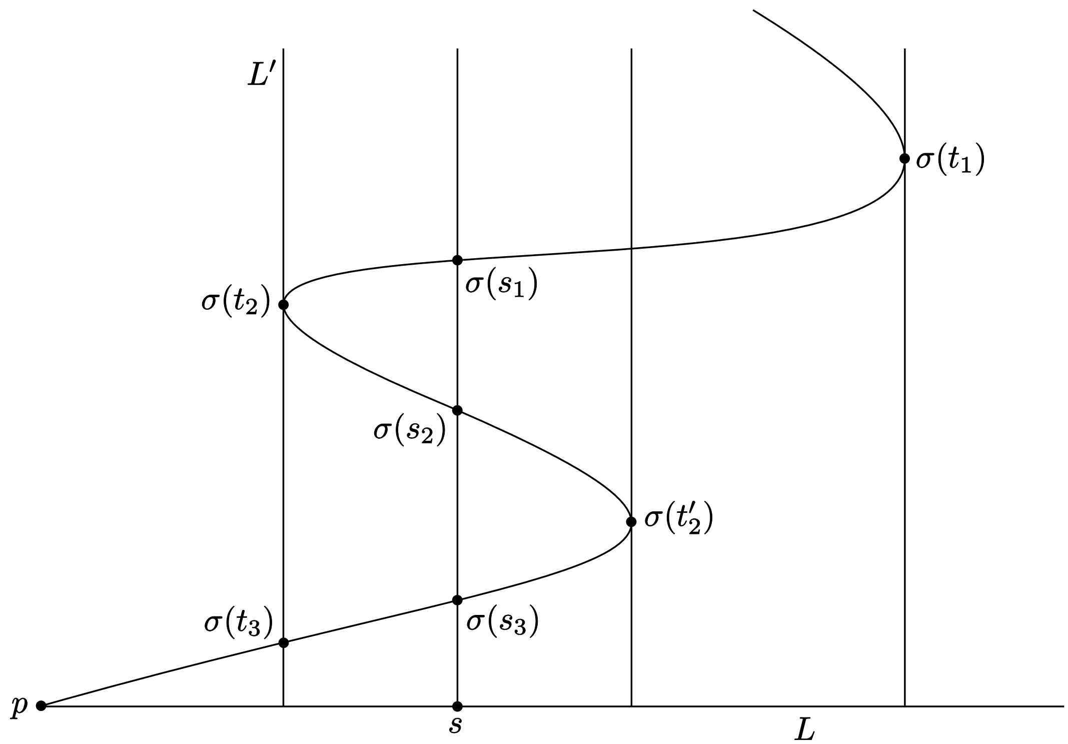

Figure 3.4: Lemma 3.9

Now suppose, by way of contradiction, the line segment L' from $q_2 = \sigma(t_2)$ to $P(q_2)$ contains a point of U for some t_2 with $0 < t_2 < t_1$ (note that $L' \perp L$). Let t_3 be the smallest positive value such that $q_3 = \sigma(t_3)$ lies on L'. If $t_3 = t_2$ then the line segment between q_2 and $P(q_2)$ cuts one of S or U into two components; by Lemma 3.8, it must therefore have interior in S, which contradicts our hypothesis on t_2. Thus $0 < t_3 < t_2$, and L' has nontrivial intersection with U between q_2 and q_3. Hence some t'_2 with $t_3 < t'_2 < t_2$ must have $P(\sigma(t'_2))$ farther from p than $P(q_2) = P(q_3)$. Let $q'_2 = \sigma(t'_2)$, and let s be the midpoint between $P(q_2)$ and $P(q'_2)$. By the intermediate value theorem, there must be some s_1 with $t_2 < s_1 < t_1$ such that $P(s_1) = s$. Similarly, $P^{-1}(s)$ must contain points $\sigma(s_2)$ and $\sigma(s_3)$ with $t'_2 < s_2 < t_2$ and $t_3 < s_3 < t'_2$. Thus these three points lie on a line in M_k^2 (orthogonal to L), so by Corollary 3.7, q_2 and q'_2 must also lie on this line; this is a contradiction, so no such point q_2 can exist.

Therefore, for any t with $0 < t < t_1$, the line segment from $\sigma(t)$ to L perpendicular to L is contained in $S \cup T$. $\square$

Lemma 3.10. *Suppose that $A'_p(q,r) = 0$ and τ follows a line L in M_k^2 near p. Then $\angle_p(q,r) = 0$.*

Proof. For simplicity, we assume $k = 0$ or $k = -1$. Let $\epsilon > 0$ be given. Since $A_p(\sigma(t), \tau(t'))$ decreases monotonically in both t and t', we may find some $\delta > 0$ such that $A_p(\sigma(t), \tau(t')) < \epsilon$ for all t and t' with $\sigma(t), \tau(t') \in \overline{B}_d(p, \delta) \setminus \{p\}$. Replacing δ by a smaller positive constant if necessary, we may assume that every point of T in D is in the image of σ or τ and that the image of τ in D follows L. Let P be the projection from the image of σ onto L, and let t_1 be the point guaranteed by Lemma 3.9.

Let δ' be the distance in M_k^2 from p to $P(\sigma(t_1))$, and note that $0 < \delta' < \delta$. Suppose that q' and r' are points in $B_d(p, \delta') \setminus \{p\}$ along the images of σ and τ, respectively. Let $a = d(p, q')$, $b = d(p, r')$, and $c = d(q', r')$, and let $\phi = A_p(q', r')$. Also let $a' = \bar{d}(p, q')$ and $c' = \bar{d}(q', r')$; note that $a' \geq a$ and $c' \geq c$. Since σ is a geodesic, the path straight from p to $P(q')$ and then straight to q', which stays in E by choice of t_1, must have length at least a'. Hence if $k = 0$ then

$$a' \leq a(\cos\phi + \sin\phi) \leq a(1 + \sin\phi) \leq a(1 + \sin\epsilon) \leq a(1 + \epsilon),$$

and if $k = -1$ then by the hyperbolic law of sines,

$$\sinh a' \leq (\cos\phi + \sin\phi)\sinh a \leq (1 + \epsilon)\sinh a.$$

Now suppose that $c' = c$. By the law of cosines,

$$\cos \angle_p^{(0)}(q', r') = \frac{(a')^2 + b^2 - c^2}{2(a')b} \geq \frac{a^2 + b^2 - c^2}{2(a')b} \geq \frac{a^2 + b^2 - c^2}{2a(1+\epsilon)b} = \frac{1}{1+\epsilon}\cos\phi,$$

and by the hyperbolic law of cosines,

$$\cos \angle_p^{(-1)}(q', r') = \frac{\cosh a' \cosh b - \cosh c}{\sinh a' \sinh b} \geq \frac{\cosh a \cosh b - \cosh c}{(1+\epsilon) \sinh a \sinh b} = \frac{1}{1+\epsilon} \cos \phi.$$

On the other hand, suppose $c' > c$. Note that, by choice of t_1, the geodesic triangle with vertices p, $\sigma(t_1)$, and $P(\sigma(t_1)$ is simple. The interior of this triangle is contained in S, and $q' \neq \sigma(t_1)$. Thus $L' = \mathrm{par}(L, q')$ must be locally interior on one side of q'. Let L'_0 be the segment of L' with q' as one endpoint, interior in S, and other endpoint in T. Let $p' \in T$ be the other endpoint. Since T is a simple triangle, $q' \notin L$, and therefore $p' \notin L$. But $p' \notin \sigma([0, t_1])$, so p' must lie on the line segment from $\sigma(t_1)$ to $P(\sigma(t_1))$. Hence both L'_0 and the line segment from q' to $P(q')$ lie in $S \cup T$. Thus, if $P(q')$ lies between p and r' on the line L, then the line segment from r' to q' is contained in $S \cup T$. Therefore, the outer angle $A_{r'}(p, q')$ is greater than $\frac{\pi}{2}$, and so $a > c$.

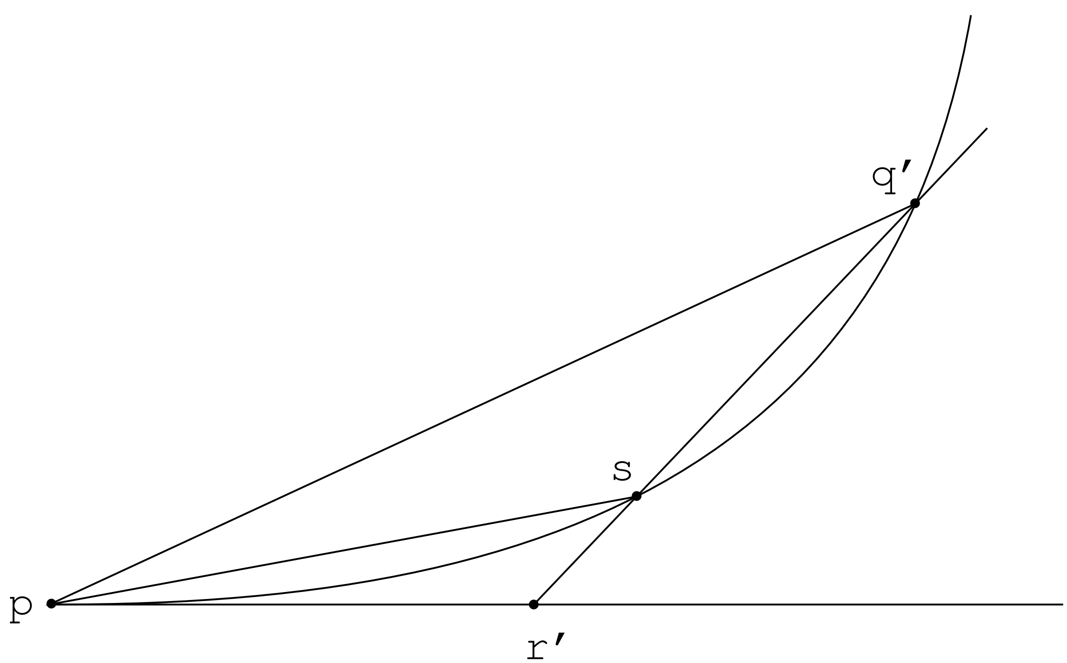

Figure 3.5: Lemma 3.10

Now consider the line segment in M_k^2 from r' to q': It hits T at a first point s (the edge hit is the one between p and q'). Let γ be the path which travels from r' to s along the line segment and then from s to q' along σ. Note that the length $\ell(\gamma)$ of γ is at least c'. Let α be the path that travels in a straight line from p to s and then straight from s to q', and let α' be the path that travels in a straight line from p to s and then from s to q' along σ. Note that $a \leq \ell(\alpha) \leq \ell(\alpha') \leq a'$. Hence $a + c' \leq \ell(\alpha) + \ell(\gamma) = \ell(\alpha') + c \leq a' + c$, and thus $a' - c' \geq a - c$. Therefore, $a > c$ gives us $a' - c' > 0$. Since $a' \geq a > 0$ and $c' \geq c > 0$, we have

$$(a')^{n+1} - (c')^{n+1} = (a' - c') \sum_{k=0}^{n} (a')^k (c')^{n-k} \geq (a - c) \sum_{k=0}^{n} a^k c^{n-k} = a^{n+1} - c^{n+1}$$

for all integers $n \geq 0$. Hence for $k = 0$ we have

$$\cos \angle_p^{(0)}(q', r') = \frac{(a')^2 + b^2 - (c')^2}{2(a')b} \geq \frac{a^2 + b^2 - c^2}{2(a')b} \geq \frac{a^2 + b^2 - c^2}{2a(1+\epsilon)b} = \frac{1}{1+\epsilon} \cos \phi,$$

and for $k = -1$ we have

$$\begin{aligned} \cosh a' - \cosh c' &= \sum_{n=1}^{\infty} \frac{1}{(2n)!} \left((a')^{2n} - (c')^{2n}\right) \\ &\geq \sum_{n=1}^{\infty} \frac{1}{(2n)!} \left(a^{2n} - c^{2n}\right) \\ &= \cosh a - \cosh c. \end{aligned}$$

Hence $\cosh a' - \cosh a \geq \cosh c' - \cosh c$, so the fact that $\cosh b \geq 1$ gives us $\cosh a' \cosh b - \cosh a \cosh b \geq \cosh c' - \cosh c$, and therefore $\cosh a' \cosh b - \cosh c' \geq \cosh a \cosh b - \cosh c$. Thus

$$\cos \angle_p^{(-1)}(q', r') = \frac{\cosh a' \cosh b - \cosh c'}{\sinh a' \sinh b} \geq \frac{\cosh a \cosh b - \cosh c}{(1+\epsilon) \sinh a \sinh b} = \frac{1}{1+\epsilon} \cos \phi.$$

Thus, in either case,

$$\cos \overline{\angle}_p(q', r') \geq \frac{1}{1+\epsilon} \cos \phi,$$

and therefore we obtain $\angle_p(q', r') \leq A'_p(q', r') = 0$ as ϵ tends to zero. This concludes the proof of the lemma. $\square$

Theorem 3.11. *In a simple geodesic triangle T with vertices p, q, and r,*

$$A'_p(q, r) = \angle_p(q, r).$$

Proof. By Proposition 3.5, the rays $R_{1,t}$ from p through $\sigma(t)$ limit monotonically to a ray R_1 as t tends to zero. Similarly, the rays $R_{2,t}$ from p through $\tau(t)$ limit monotonically to a ray R_2 as t tends to zero.

Suppose first that $R_1 \neq R_2$. By construction, R_1 and R_2 are locally contained in $S \cup T$ near p. Let s_1 be the last point of R_1 contained in $S \cup T$. Clearly, $s_1 \in T$; if s_1 lies along σ then s_1 must equal q by Lemma 3.6. Since R_1 is locally contained in $S \cup T$ near p, we have $s_1 \neq p$, and thus s_1 cannot lie along τ. Therefore, s_1 lies along the geodesic arc between q and r. Similarly, the last point s_2 of R_2 that is contained in $S \cup T$ must lie along the geodesic arc between q and r. Note that $\angle_p(s_1, s_2) = A'_p(q, r)$, since both measure the angle between R_1 and R_2.

If σ follows R_1 for some positive distance beyond p, then $\angle_p(q, s_1) = 0$ by definition. On the other hand, if σ does not follow R_1 for any positive distance beyond p, then the geodesic triangle $T_1 = \triangle(p, q, s_1)$ is simple, and $\angle_p(q, s_1) = 0$ by Lemma 3.10. Thus in either case, $\angle_p(q, s_1) = 0$; similarly, $\angle_p(s_2, r) = 0$. Hence

$$\angle_p(q, r) \leq \angle_p(q, s_1) + \angle_p(s_1, s_2) + \angle_p(s_2, r) = \angle_p(s_1, s_2)$$

and

$$\angle_p(s_1, s_2) \leq \angle_p(s_1, q) + \angle_p(q, r) + \angle_p(r, s_2) = \angle_p(q, r)$$

by Proposition 2.9. Therefore $\angle_p(q, r) = \angle_p(s_1, s_2) = A'_p(q, r)$.

Finally, suppose $R_1 = R_2$; note that this gives $A'_p(q, r) = 0$. If σ follows R_1 for some positive distance beyond p, then $\angle_p(q, r) = A'_p(q, r) = 0$ by Lemma 3.10. Thus we may assume, by symmetry, that neither σ nor τ follows R_1 for any positive distance beyond p. Then by construction of $R_1 = R_2$, the last point s of R_1 contained in $S \cup T$ must be along the geodesic arc from q to r. Hence the geodesic triangles $T_1 = \triangle(p, q, s)$ and $T_2 = \triangle(p, s, r)$ are simple, and since $A'_p(q, s) = A'_p(s, r) = 0$ by construction, $\angle_p(q, s) = \angle_p(s, r) = 0$ by Lemma 3.10. Therefore,

$$\angle_p(q, r) \leq \angle_p(q, s) + \angle_p(s, r) = 0,$$

and the theorem is proved. □

The main result result now follows easily.

Proof of **Theorem 1.1.** Note that every geodesic triangle with distinct vertices either can be trimmed to a simple triangle, or it has 0 angle at all 3 vertices. Moreover, this trimming does not decrease the angles at the vertices. So let p, q, and r be the vertices of a simple triangle T. As in the proof of Theorem 3.11, we have two (possibly equal) limit rays R_1 and R_2 from p. Cutting along these rays gives three (not necessarily simple) triangles. The middle triangle has Alexandrov angle at p at most the k-comparison angle, since only the edge opposite p can be longer than the distance in M_k^2. The two outside triangles either are simple, in which case Lemma 3.10 applies, or both edges emanating from p follow the same path for a positive distance; in either case, the Alexandrov angle at p is 0. So by Alexandrov's Lemma, $\angle_p(q, r) \leq \angle_p^{(k)}(q, r)$. Therefore, $(E, \bar{d})$ is CAT(k). □

Bibliography

[1] Werner Ballmann, *Lectures on spaces of nonpositive curvature*, Birkhäuser Verlag, Basel, 1995.

[2] Richard L. Bishop, *The intrinsic geomtery of a Jordan domain*, IEJG **1** (2008), no. 2, 33–39.

[3] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Springer-Verlag, Berlin, 1999.

[4] William P. Thurston, *Three-dimensional geometry and topology*, Springer-Verlag, Berlin, 1999.